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GSV-INDICES AS RESIDUES

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ABSTRACT. We introduce a local invariant for a vector field v on a complete intersection V with an isolated singularity as the residue of the relevant Chern class of the ambient tangent bundle by a frame consisting of v and some natural meromorphic vector fields associated with defining functions of V. We then show that the residue coincides with the GSV-index as well as the virtual index of v so that it provides another interpretation of these indices. As an application, we give an algebraic formula for the GSV-indices of holomorphic vector fields on singular curves.

In this note we introduce a local invariant of a vector field v on a complete intersection V with an isolated singularity. It is the residue arising from the localization of the relevant Chern class of the ambient tangent bundle by a frame consisting of v and some other vector fields. The last ones are naturally associated to defining functions of V and are holomorphic on and normal to the non-singualr part of V (Definition 2.7 below). Although it is a priori of differential geometric nature, defined in the framework of Chern-Weil theory adapted to the Čech-de Rham cohomology, it is directly related to a topological invariant coming from the obstruction theory (cf. (2.10)).

Historically, there is the so-called GSV-index for a vector field v as above ([6], [13]). It is defined topologically, either using the frame consisting of v and the conjugated gradient vector fields of defining functions or referring to the Milnor fiber. On the differential geometric side, there is the virtual index which is the residue arising from the localization by v of the Chern class of the virtual tangent bundle of V (cf. [11]). It coincides with the GSV-index in the case considered here, however it can be defined in more general settings.

The topological aspect of the residue mentioned in the beginning is that it coincides with the GSV-index (Theorem 3.4) and the differential geometric aspect is that it coincides with the virtual index (Theorem 4.4), so that it provides another interpretation of these indices as well as another way of computing them. On the way we show how topological and differential geometric residues of vector fields on complete intersections interacts.

We then apply the above to the case of holomorphic vector fields on singular curves. A direct computation of the residue taking suitable connections shows an integral representation of the GSV-index (Proposition 5.1), which was given by M. Brunella in [4] by a different approach. This in turn gives an algebraic formula for the GSV-index in this case (Corollary 5.2). The formula is somewhat different from the one in this special case of the general algebraic formula obtained as homological index by X. Gómez-Mont in [5] (see [2] for complete intersections). It is only for the case of curves, however the advantage is that each term of it is expressed as the dimension of the quotient of the ring of holomorphic functions by an ideal generated by a regular sequence.

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1. Preliminaries

We recall localization theory of characteristic classes in the framework of Chern-Weil theory adapted to the Čech-de Rham cohomology, as initiated in [10]. Here we adopt the presentation in [15], see also [18].

Connections. Let M be a C^∞ manifold and E a C^∞ complex vector bundle of rank l on M. We denote by $A^p(M,E)$ the $\mathbb{C}$-vector space of complex valued C^∞ p-forms with coefficients in E on M, i.e., C^∞ sections of the bundle $\bigwedge^p(T^c_{\mathbb{R}}M)^*\otimes E$, where $T^c_{\mathbb{R}}M$ denotes the complexification of the tangent bundle of M. In the case $E=\mathbb{C}\times M$, the trivial line bundle, we denote it by $A^p(M)$ so that it is the space of complex valued p-forms on M.

Recall that a *connection* for E is a $\mathbb{C}$-linear map

$$\nabla : A^0(M,E) \longrightarrow A^1(M,E)$$

satisfying the "Leibniz rule"

$$\nabla(fs) = df\otimes s + f\nabla(s), \quad \text{for} \ \ f\in A^0(M) \ \ \text{and} \ \ s\in A^0(M,E).$$

Note that every vector bundle admits a connection. If ∇ is a connection for E, it induces a $\mathbb{C}$-linear map

$$\nabla : A^1(M,E) \longrightarrow A^2(M,E)$$

satisfying

$$\nabla(\omega\otimes s) = d\omega\otimes s - \omega\wedge\nabla(s), \quad \text{for} \ \ \omega\in A^1(M) \ \ \text{and} \ \ s\in A^0(M,E).$$

The composition

$$K = \nabla\circ\nabla : A^0(M,E) \longrightarrow A^2(M,E)$$

is called the *curvature* of ∇.

The fact that a connection is a local operator allows us to get local representations of it and its curvature by matrices whose entries are differential forms. Thus suppose that ∇ is a connection for E and that E is trivial on an open set U. If $\boldsymbol{e}=(e_1,\dots,e_l)$ is a frame of E on U, we may write

$$\nabla(e_i) = \sum_{j=1}^{l}\theta_{ji}\otimes e_j$$

with θ_{ij} 1-forms on U. We call $\theta=(\theta_{ij})$ the connection matrix with respect to $\boldsymbol{e}$. Also, from the definition we compute to get

$$K(e_i) = \sum_{j=1}^{l}\kappa_{ji}\otimes e_j, \quad \kappa_{ij} = d\theta_{ij} + \sum_{k=1}^{l}\theta_{ik}\wedge\theta_{kj}.$$

We call $\kappa=(\kappa_{ij})$ the curvature matrix with respect to $\boldsymbol{e}$. If $\boldsymbol{e}'=(e'_1\dots,e'_l)$ is another frame of E on U', we have $e'_i=\sum_{j=1}^{l}a_{ji}e_j$ for some C^∞ functions a_{ij} on $U\cap U'$. The matrix $A=(a_{ij})$ is non-singular at each point of $U\cap U'$. If we denote by θ' and κ' the connection and curvature matrices of ∇ with respect to $\boldsymbol{e}'$, we have

$$\theta' = A^{-1}\cdot dA + A^{-1}\theta A \quad \text{and} \quad \kappa' = A^{-1}\kappa A \quad \text{in} \quad U\cap U'. \tag{1.1}$$

Chern forms. Since differential forms of even degrees commute one another with respect to exterior product, we may treat κ above as an ordinary matrix. Thus, for $q = 1, \dots, l$, we define a $2q$-form $\sigma_q(\kappa)$ on U by

$$\det(I_l + \kappa) = 1 + \sigma_1(\kappa) + \cdots + \sigma_l(\kappa),$$

where I_l denotes the identity matrix of rank l. In particular, $\sigma_1(\kappa) = \mathrm{tr}(\kappa)$ and $\sigma_l(\kappa) = \det(\kappa)$. Although $\sigma_q(\kappa)$ depends on the connection ∇, it does not depend on the choice of the frame of E by (1.1) and it defines a global $2q$-form on M, which we denote by $\sigma_q(\nabla)$. An important feature of the forms is that they are closed. We set

$$c^q(\nabla) = \left(\frac{\sqrt{-1}}{2\pi}\right)^q \sigma_q(\nabla)$$

and call it the q-th *Chern form*. The total Chern form is defined by $c^*(\nabla) = 1 + \sum_{q=1}^{l} c^q(\nabla)$ so that locally it is given by

$$c^*(\nabla) = \det\Big(I_l + \frac{\sqrt{-1}}{2\pi}\kappa\Big). \tag{1.2}$$

Note that it is invertible.

If we have two connections ∇_0 and ∇_1 for E, we may construct the *difference form* $c^q(\nabla_0, \nabla_1)$, which is a $(2q-1)$-form with the properties that $c^q(\nabla_1, \nabla_0) = -c^q(\nabla_0, \nabla_1)$ and that

$$d\,c^q(\nabla_0, \nabla_1) = c^q(\nabla_1) - c^q(\nabla_0).$$

In fact the form $c^q(\nabla_0, \nabla_1)$ is constructed as follows. We consider the vector bundle

$$E \times \mathbb{R} \to M \times \mathbb{R}$$

and let $\tilde{\nabla}$ be the connection for it given by $\tilde{\nabla} = (1-t)\nabla_0 + t\nabla_1$, with t a coordinate on $\mathbb{R}$. Then we define

$$c^q(\nabla_0, \nabla_1) = p_* c^q(\tilde{\nabla}),$$

where p_* denotes the integration along the fiber of the projection $p : M \times [0,1] \to M$.

From the above we see that the class $[c^q(\nabla)]$ of the closed $2q$-form $c^q(\nabla)$ in the de Rham cohomology $H^{2q}_d(M)$ depends only on E and not on the choice of the connection ∇. It is the q-th *Chern class* $c^q(E)$ of E.

Remark 1.3. If we use the obstruction theory, the q-th Chern class is defined in the integral cohomology $H^{2q}(M, \mathbb{Z})$. It is shown that the class $c^q(E)$ defined as above is equal to its image by the canonical homomorphism

$$H^{2q}(M, \mathbb{Z}) \longrightarrow H^{2q}(M, \mathbb{C}) \xrightarrow{\sim} H^{2q}_d(M),$$

where the last isomorphism is the de Rham isomorphism (e.g., [18]).

Localization. Let E be a vector bundle of rank l. An r-section of E is an r-tuple $\boldsymbol{s} = (s_1, \dots, s_r)$ of sections of E. A singular point of $\boldsymbol{s}$ is a point where $s_1, \dots, s_r$ fail to be linearly independent. An *r-frame* is an r-section without singularities. An l-frame is simply called a frame, as already used above.

Definition 1.4. Let $\boldsymbol{s} = (s_1, \dots, s_r)$ be a C^∞ r-frame of E on an open set U. We say that a connection ∇ is *trivial with respect to* $\boldsymbol{s}$, or simply $\boldsymbol{s}$*-trivial*, on U, if $\nabla(s_i) = 0,\ i = 1, \dots, r$.

The following is fundamental for the localization we consider:

Proposition 1.5. *If* ∇ *is* $\boldsymbol{s}$*-trivial,*

$$c^q(\nabla) = 0, \quad \textit{for} \ \ q \geq l - r + 1.$$

We explain localization process and the associated residues in the case pertinent to ours. Thus let M be a complex manifold of dimension n and p a point in M. Let $U_0 = M \smallsetminus \{p\}$ and U_1 a neighborhood of p and consider the covering $\mathcal{U} = \{U_0, U_1\}$ of M. We then work in the framework of the Čech-de Rham cohomology of $\mathcal{U}$. Let E be a complex vector bundle of rank l on M. Suppose we have an r-frame $\boldsymbol{s}$ of E on U_0, $r = l - n + 1$. The n-th Chern class $c^n(E)$ of E is represented by the Čech-de Rham cocycle

$$(c^n(\nabla_0), c^n(\nabla_1), c^n(\nabla_0, \nabla_1)),$$

where ∇_i is a connection for E on U_i, $i = 0, 1$. We choose ∇_0 so that it is $\boldsymbol{s}$-trivial. Thus by Proposition 1.5, $c^n(\nabla_0) = 0$ and the cocycle defines a class $c^n(E, \boldsymbol{s})$, called the *localization of $c^n(E)$ by $\boldsymbol{s}$*, in the relative cohomology $H^{2n}(M, M \smallsetminus \{p\}; \mathbb{C})$. This in turn gives rise to the residue $\mathrm{Res}_{c^n}(\boldsymbol{s}, E; p)$ as its image by the Alexander isomorphism

$$H^{2n}(M, M \smallsetminus \{p\}; \mathbb{C}) \xrightarrow{\sim} H_0(\{p\}, \mathbb{C}) = \mathbb{C}.$$

The residue is in fact an integer given by

$$\mathrm{Res}_{c^n}(\boldsymbol{s}, E; p) = \int_R c^n(\nabla_1) - \int_{\partial R} c^n(\nabla_0, \nabla_1),$$

where R is a $2n$-disk around p in U_1.

Exact sequence. Let

$$0 \longrightarrow E'' \xrightarrow{\iota} E \xrightarrow{\varphi} E' \longrightarrow 0 \tag{1.6}$$

be an exact sequence of vector bundles, and ∇'', ∇ and ∇' connections for E'', E and E', respectively. We say that $(\nabla'', \nabla, \nabla')$ is *compatible* with (1.6) if

$$\nabla(\iota \circ s'') = (\mathrm{id} \otimes \iota) \circ \nabla''(s'') \quad \text{and} \quad \nabla'(\varphi \circ s) = (\mathrm{id} \otimes \varphi) \circ \nabla(s)$$

for s'' in $A^0(M, E'')$ and s in $A^0(M, E)$.

The following is proved using the expression (1.2):

Proposition 1.7. *If $(\nabla'', \nabla, \nabla')$ is* compatible *with* (1.6)*,*

$$c^*(\nabla) = c^*(\nabla'') \cdot c^*(\nabla').$$

Remark 1.8. Given connections ∇'' and ∇' for E'' and E', it is possible to construct a connection ∇ for E so that $(\nabla'', \nabla, \nabla')$ is compatible with (1.6). Moreover, this can be done under the assumption that the connections be trivial with respect to appropriate frames.

Virtual bundles. Let E and E' be vector bundles and ∇ and ∇' connections for E and E', respectively. We set $\nabla^\bullet = (\nabla, \nabla')$ and define the total Chern form of the virtual bundle $E - E'$ by

$$c^*(\nabla^\bullet) = c^*(\nabla)/c^*(\nabla').$$

For two pairs of connections $\nabla_0^\bullet = (\nabla_0, \nabla_0')$ and $\nabla_1^\bullet = (\nabla_1, \nabla_1')$, we may define the difference form $c^q(\nabla_0^\bullet, \nabla_1^\bullet)$ with similar properties as before. Namely, letting $\tilde{\nabla}^\bullet = (\tilde{\nabla}, \tilde{\nabla}')$ with

$$\tilde{\nabla} = (1-t)\nabla_0 + t\nabla_1 \quad \text{and} \quad \tilde{\nabla}' = (1-t)\nabla_0' + t\nabla_1',$$

we set

$$c^q(\nabla_0^\bullet, \nabla_1^\bullet) = p_* c^q(\tilde{\nabla}^\bullet).$$

The total Chern class $c^*(E - E')$ of $E - E'$ is the class of $c^*(\nabla^\bullet)$ in the de Rham cohomology $H_d^*(M)$. It is also given by $c^*(E - E') = c^*(E)/c^*(E')$.

2. Residues of vector fields

The localization theory explained in the previous section applies also to the case of singular varieties. We first recall this in the situation relevant to ours. For details we refer to [16], [17] and [18]. Then we define the residue of a vector field on a complete intersection with an isolated singularity using an appropriate frame.

Residues of multi-sections. Let U be a neighborhood of the origin 0 in $\mathbb{C}^m$ and V a subvariety (reduced, but may not be irreducible) of pure dimension n in U. Assume that V contains 0 and that $V \smallsetminus \{0\}$ is non-singular. We take a closed ball B around 0 sufficiently small so that, in particular, $R = B \cap V$ has a cone structure over $\partial R = L$, the link of V at 0 (cf. [12]).

Let E be a C^∞ complex vector bundle of rank l on U and $\boldsymbol{s} = (s_1, \dots, s_r)$ a C^∞ r-frame of E on a neighborhood V' of L in V, $r = l - n + 1$. Then there is a natural localization of the n-th Chern class $c^n(E|_V)$ of $E|_V$ by $\boldsymbol{s}$, which gives rise to a residue, denoted by $\mathrm{Res}_{c^n}(\boldsymbol{s}, E|_V; 0)$. This is given as follows. Let ∇_0 be an $\boldsymbol{s}$-trivial connection for $E|_{V'}$ and ∇_1 a connection for E.

Definition 2.1. The *residue of* $\boldsymbol{s}$ *at* 0 *with respect to* c^n is defined by

$$\mathrm{Res}_{c^n}(\boldsymbol{s}, E|_V; 0) = \int_R c^n(\nabla_1) - \int_{\partial R} c^n(\nabla_0, \nabla_1).$$

Remark 2.2. 1. The definition of the residue above does not depend on the choice of B or the connections involved.

2. In practice, we may assume that E is trivial on U and we may take as ∇_1 the connection trivial with respect to some frame of E. In this case, the first term disappears and we have only an integral on $\partial R = L$.

The fundamental fact is that the residue above coincides with the "topological residue" defined by the obstruction theory. To explain this, we denote by $W_r(\mathbb{C}^l)$ the Stiefel manifold of ordered r-frames in $\mathbb{C}^l$. It is $(2n-2)$-connected and its $(2n-1)$-st homotopy group is naturally isomorphic to $\mathbb{Z}$.

Let us first consider the basic case where $U = V$ and $l = m = n$. Thus $r = 1$ and $\boldsymbol{s}$ consists of a single section s. In this case $L = S^{2n-1}$, a $(2n-1)$-sphere and, if we denote by $h = (h_1, \dots, h_n)$ the components of s with respect to some frame of E, the restriction of h to L defines a map

$$\varphi : L \longrightarrow W_1(\mathbb{C}^n) = \mathbb{C}^n \smallsetminus \{0\}.$$

On the other hand, by appropriate choices of ∇_0 and ∇_1, we may show that $c^n(\nabla_1) = 0$ and $c^n(\nabla_0, \nabla_1) = h^*\beta_n$, where β_n denotes the Bochner-Martinelli form on $\mathbb{C}^n$ (cf. [18, Lemma 3.4.1]). Thus we have

$$\mathrm{Res}_{c^n}(s, E; 0) = \deg \varphi. \tag{2.3}$$

In particular, if $E = TU$, the holomorphic tangent bundle of U, $s = v$ is a vector field and this is the Poincaré-Hopf index $\mathrm{PH}(v, 0)$ of v at 0.

Coming back to the general case, if $V = \bigcup V_i$ is the irreducible decomposition of V, the link L has connected components (L_i) accordingly, each L_i being the link of V_i. The r-frame $\boldsymbol{s}$ defines a map

$$\varphi_i : L_i \longrightarrow W_r(\mathbb{C}^l).$$

Since L_i is a connected real $(2n-1)$-dimensional manifold, we have the degree of φ_i, as an integer. We refer to [18, Theorem 6.3.2] for the following (in [17, Theorem 6.1], we need to assume that V is irreducible):

Lemma 2.4. *We have*

$$\operatorname{Res}_{c^n}(\boldsymbol{s}, E|_V; 0) = \sum \deg \varphi_i.$$

Remark 2.5. 1. In the above E and $\boldsymbol{s}$ may be assumed to be only continuous, as they admit "C^∞ approximations".

2. If E and $\boldsymbol{s}$ are restrictions of holomorphic ones on U, we have an analytic expression of $\operatorname{Res}_{c^n}(\boldsymbol{s}, E|_V; 0)$ as a Grothendieck residue (cf. [16]). Moreover, if V is a complete intersection, or more generally if V admits a smoothing in U, we have an algebraic expression as the dimension of certain analytic algebra (cf. [17]).

Vector fields on complete intersections. Letting U, V and V' be as above, we have an exact sequence

$$0 \longrightarrow TV' \longrightarrow TU|_{V'} \xrightarrow{\pi} N_{V'} \longrightarrow 0, \tag{2.6}$$

where TV' and TU denote the holomorphic tangent bundles of V' and U, and $N_{V'}$ the normal bundle of V' in U.

Let us now assume that V is a complete intersection defined by $f = (f_1, \dots, f_k)$ in U, $k = m - n$. Here we adopt the terminologies in [15, Ch.II, 13] so that V is reduced but may not be irreducible, to make sure.

In a neighborhood of a regular point of f, we may choose $(f_1, \dots, f_k)$ as a part of local coordinates on U so that we have holomorphic vector fields $\frac{\partial}{\partial f_1}, \dots, \frac{\partial}{\partial f_k}$ away from the critical set of f. They are linearly independent and "normal" to the non-singular part of V so that $(\pi(\frac{\partial}{\partial f_1}|_{V'}), \dots, \pi(\frac{\partial}{\partial f_k}|_{V'}))$, which will be simply denoted by $\boldsymbol{\partial}$, form a frame of $N_{V'}$. Here we should note that the restriction means the restriction as a section of the vector bundle TU.

Suppose we have a C^∞ non-singular vector field v on V'. Then the $(k+1)$-tuple of sections

$$\boldsymbol{v} = \Big(v, \frac{\partial}{\partial f_1}\Big|_{V'}, \dots, \frac{\partial}{\partial f_k}\Big|_{V'}\Big)$$

of $TU|_{V'}$ form a $(k+1)$-frame so that we have the residue $\operatorname{Res}_{c^n}(\boldsymbol{v}, TU|_V; 0)$, which we simply call the residue of v:

Definition 2.7. The *residue of* v *at* 0 is defined by

$$\operatorname{Res}(v, 0) = \operatorname{Res}_{c^n}(\boldsymbol{v}, TU|_V; 0).$$

Thus

$$\operatorname{Res}(v, 0) = \int_R c^n(\nabla_1) - \int_{\partial R} c^n(\nabla_0, \nabla_1), \tag{2.8}$$

where ∇_0 is a $\boldsymbol{v}$-trivial connection for $TU|_{V'}$ and ∇_1 a connection for TU.

Remark 2.9. 1. For the frame $\boldsymbol{v}$ above we cannot use the analytic or algebraic expression mentioned in Remark 2.5, 2, even if v admits a holomorphic extension to U, as the vector fields $\frac{\partial}{\partial f_j}$ cannot be extended holomorphically through 0. On the other hand, the topological expression in Lemma 2.4 is still valid:

$$\operatorname{Res}(v, 0) = \sum \deg \varphi_i, \tag{2.10}$$

where φ_i is the map defined by $\boldsymbol{v}$ on each connected component L_i of the link L of V.

2. The above residue is, in some sense, dual to the index for a 1-form introduced in [7].

Proposition 2.11. *If* V *is non-singular at* 0*,*

$$\operatorname{Res}(v, 0) = \operatorname{PH}(v, 0).$$

Proof. In this case, the sequence (2.6) extends to the exact sequence

$$0 \longrightarrow TV \longrightarrow TU|_V \longrightarrow N_V \longrightarrow 0. \tag{2.12}$$

Note that $\mathrm{PH}(v,0)$ is given as the right side of (2.8) with ∇_0 and ∇_1 replaced by a v-trivial connection for TV' and a connection for TV, respectively.

We take a v-trivial connection ∇_0'' for TV', a $\boldsymbol{v}$-trivial connection ∇_0 for $TU|_{V'}$ and the $\boldsymbol{\partial}$-trivial connection ∇_0' for $N_{V'}$ so that $(\nabla_0'',\nabla_0,\nabla_0')$ is compatible with (2.6). Also let ∇_1'' be a connection for TV and ∇_1' the $\boldsymbol{\partial}$-trivial connection for N_V. We take a connection ∇_1 for TU so that $(\nabla_1'',\nabla_1,\nabla_1')$ is compatible with (2.12) (cf. Remark 1.8). Then noting that the total Chern forms satisfy $c^*(\nabla_1) = c^*(\nabla_1'')\cdot c^*(\nabla_1')$ and that $c^*(\nabla_1') = 1$, as ∇_1' is trivial, we have $c^n(\nabla_1) = c^n(\nabla_1'')$. Similarly, from the construction of the difference form and noting that $\nabla_0' = \nabla_1'$ on V', we have $c^n(\nabla_0,\nabla_1) = c^n(\nabla_0'',\nabla_1'')$. □

Remark 2.13. The above may be shown by obstruction theory as well, the essential point being again that $(\frac{\partial}{\partial f_1},\dots,\frac{\partial}{\partial f_k})$ has no singularities on V, if V is non-singular.

In the global case, this type of residues also appear as relative Chern classes. Let us again start with the basic case. Thus let M be the closure of a relatively compact open set of a complex manifold M_1 of dimension n. Suppose ∂M is (piecewise) C^∞ and we have a non-singular vector field v in a neighborhood M' of ∂M in M_1. Let ∇_0 be a v-trivial connection for TM' and ∇_1 a connection for TM_1 and define

$$\mathrm{PH}(v,M) = \int_M c^n(\nabla_1) - \int_{\partial M} c^n(\nabla_0,\nabla_1). \tag{2.14}$$

We may extend v to all of M with possibly a finite number of singularities p_i and using (2.3), we see that

$$\mathrm{PH}(v,M) = \sum \mathrm{PH}(v,p_i). \tag{2.15}$$

Coming back to the situation before, let $f=(f_1,\dots,f_k): U\to\mathbb{C}^k$ and B be as above. We denote by $C(f)$ the set of critical points of f and set $D(f) = f(C(f))$, which is a hypersurface in a neighborhood of the origin 0 in $\mathbb{C}^k$. For t sufficiently near 0, we set $V_t = f^{-1}(t)$, which admits at most isolated singularities $C(f)\cap V_t$, all lying in the interior of B. If t is not in $D(f)$, V_t is non-singular, in fact a Milnor fiber F of f (cf. [12], [8]). Let V_t' be a neighborhood of $R_t = B\cap V_t$ in V_t and v_t a non-singular vector field on V_t'. We set $\boldsymbol{v}_t = (v_t, \frac{\partial}{\partial f_1}|_{V_t'},\dots,\frac{\partial}{\partial f_k}|_{V_t'})$ and define the residue $\mathrm{Res}(v_t,V_t)$ by the formula (2.8) with ∇_0 replaced by a $\boldsymbol{v}_t$-trivial connection for $TU|_{V_t'}$ and R by R_t. The following is proves as Proposition 2.11 :

Proposition 2.16. *If V_t is non-singular,*

$$\mathrm{Res}(v_t,V_t) = \mathrm{PH}(v_t,V_t).$$

3. GSV-index

Let U be a neighborhood of the origin 0 in $\mathbb{C}^{n+k}$ and V a complete intersection in U of dimension n, as in Section 2. Let v be a non-singular continuous vector field on V', a neighborhood in V of the link L of V. For the definition of the GSV-index of v at 0, we adopt the one in [15, Ch.IV, 1]. It is in the spirit of the second definition in [6], involving the Milnor fiber, and is equivalent to the one given in [6] and [13] as the degree of a certain map, *provided that V is irreducible* (see Remark 3.3 below).

Let us consider the situation in the last part of Section 2. Let U' be a neighborhood of L in U. We may assume that U' does not contain critical points of f. Then we have an exact sequence, which extends (2.6), $V' = U' \cap V$:

$$0 \longrightarrow Tf|_{U'} \longrightarrow TU|_{U'} \longrightarrow N|_{U'} \longrightarrow 0, \tag{3.1}$$

where Tf denotes the bundle on $U \smallsetminus C(f)$ of vectors tangent to the fibers of f and N a trivial bundle of rank k on U (cf. Section 4 below). In this situation, $N|_{U'}$ may be thought of as $f^*T\mathbb{C}^k|_{U'}$. Starting from the given non-singular vector field v on V', we may construct a non-singualr vector field $\tilde{v}$ on U' so that it is tangent to V'_t for all t near 0 in $\mathbb{C}^k$. This is done by taking an extension of v to a section of $TU|_{U'}$ and then projecting it to a section of $Tf|_{U'}$ by a splitting of (3.1). Let v_t denote the restriction of $\tilde{v}$ to V'_t. For a regular value t of f we denote V_t by F and v_t by w. Then we have the Poincaré-Hopf index $\mathrm{PH}(w, F)$ (cf. (2.14)).

Definition 3.2. The *GSV-index of* v *at* 0 is defined by

$$\mathrm{GSV}(v, 0) = \mathrm{PH}(w, F).$$

Remark 3.3. 1. The definition does not depend on the choice of the regular value t (cf. the proof of Theorem 3.4 below).

2. If V is irreducible, then L is connected and the above index $\mathrm{GSV}(v, 0)$ coincides with the degree of the map

$$\psi : L \longrightarrow W_{k+1}(\mathbb{C}^{n+k})$$

given as the restriction to L of $(v, \overline{\mathrm{grad}\, f_1}, \dots, \overline{\mathrm{grad}\, f_k})$, where $\overline{\mathrm{grad}\, f_j}$ denotes the complex conjugate of the gradient vector field of f_j: $\overline{\mathrm{grad}\, f_j} = \sum_{i=1}^{n+k} \overline{\frac{\partial f_j}{\partial z_i}} \frac{\partial}{\partial z_i}$ (cf. [6], [13]). This can be shown by the obstruction theory as in [6]. We could also show this by the Chern-Weil theory as Theorem 3.4 below, considering another residue using the frame $(v, \overline{\mathrm{grad}\, f_1}, \dots, \overline{\mathrm{grad}\, f_k})$ instead of $(v, \frac{\partial}{\partial f_1}, \dots, \frac{\partial}{\partial f_k})$.

3. Suppose V is not irreducible and let $V = \bigcup V_i$ be the irreducible decomposition. Note that this happens only if $k \geq n$, as $V \smallsetminus \{0\}$ is assumed to be non-singular. In this case, L has as many connected components (L_i) and it is not appropriate to consider the degree of ψ as above. However, proceeding as Theorem 3.4 below and using Lemma 2.4, we have $\mathrm{GSV}(v, 0) = \sum \deg \psi_i$, with ψ_i the restriction of ψ to L_i.

To further make comments in this situation, we denote the above index by $\mathrm{GSV}(v, V; 0)$. If each V_i is also a complete intersection, restricting v to V_i, we have $\mathrm{GSV}(v, V_i; 0)$ defined as in Definition 3.2 and it is expressed as the degree of a map as above, however the point is that we have to use the defining functions for V_i (not for V) as $(f_1, \dots, f_k)$. For that reason, $\mathrm{GSV}(v, V; 0) \neq \sum_i \mathrm{GSV}(v, V_i; 0)$, in general. For example, in the case $n = k = 1$, denoting V and V_i by C and C_i, we have

$$\mathrm{GSV}(v, C; 0) = \sum_i \mathrm{GSV}(v, C_i; 0) - \sum_{i \neq j} (C_i \cdot C_j)_0,$$

where $(C_i \cdot C_j)_0$ denotes the intersection number of C_i and C_j at 0 (see [15, Ch.V, 5] and references therein).

Let us note that in the beginning of Section 3.2 of [3], V has to be assumed to be irreducible, even in the higher dimensional case, and that in Remark 3.2.2, loc. cit., there are some misplacements of terms in the second displayed formula: it should be read as above with $\mathrm{GSV}(v, C; 0)$ defined as in Definition 3.2.

Here is the main theorem of this section:

Theorem 3.4. *We have*

$$\mathrm{GSV}(v,0)=\mathrm{Res}(v,0).$$

Proof. We compute $\mathrm{Res}(v_t,V_t)$ using (the restriction to V_t of) connections as follows. Let ∇_0 be a $\tilde{v}$-trivial connection for TU' and ∇_1 a connection for TU. Then we have

$$\mathrm{Res}(v_t,V_t)=\int_{R_t}c^n(\nabla_1)-\int_{\partial R_t}c^n(\nabla_0,\nabla_1).$$

which depends continuously on t. For a regular value t, this is $\mathrm{PH}(v_t,V_t)$ (cf. Proposition 2.16), which is an integer (cf. (2.15)). Thus it does not depend on t, since the regular values are dense. For a regular value this is $\mathrm{GSV}(v,0)$, while for $t=0$, this is equal to $\mathrm{Res}(v,0)$. □

Remark 3.5. 1. The above may be shown by the obstruction theory as well.

2. From the above theorem and (2.10), we see that if V is irreducible, we have another expression of the GSV-index as the degree of a map, which involves the vector field v and the holomorphic vector fields $\frac{\partial}{\partial f_j}$. In the case V is not irreducible, we have again from the above theorem and (2.10), $\mathrm{GSV}(v,0)=\sum\deg\varphi_i$ (cf. Remark 3.3, 3 above).

4. Virtual index

The notion of *virtual index* was introduced in [11]. It can be defined for a vector field on a certain type of local complete intersection V. To be a little more precise, let S be a compact set in V and V_1 a neighborhood of S such that $V_1\smallsetminus S$ is in the non-singular part of V. For a C^∞ vector field v non-singular on $V_1\smallsetminus S$, we may define the virtual index $\mathrm{Vir}(v,S)$ of v at S as the residue arising from the localization of the n-th Chern class of the virtual tangent bundle of V by v, $n=\dim V$.

Here we recall the case of isolated singularities. Thus let U, V and V' be as in Section 2. Assume that V is a complete intersection defined by $f=(f_1,\dots,f_k)$ in U. In this case, the bundle map π in (2.6) has an extension

$$\pi:TU|_V\longrightarrow N|_V$$

with N a trivial vector bundle of rank k on U (e.g., [15, Ch.II, 13]). The extension is natural in the sense that N admits a frame $\boldsymbol{\nu}=(\nu_1,\dots,\nu_k)$ extending the frame $\boldsymbol{\partial}=(\pi(\frac{\partial}{\partial f_1}|_{V'}),\dots,\pi(\frac{\partial}{\partial f_k}|_{V'}))$ of $N_{V'}$. We set $\tau_V=TU|_V-N|_V$ and call it the *virtual tangent bundle* of V. Recall that its total Chern class is given by $c^*(\tau_V)=c^*(TU|_V)/c^*(N|_V)$.

Let v be a C^∞ vector field on V'. Then we will see that the n-th Chern class $c^n(\tau_V)$ of τ_V is localized at 0 to give rise to the virtual index $\mathrm{Vir}(v,0)$ of v at 0. In the sequel, we follow the description of [15, Ch.IV, 3].

We take connections ∇, ∇_0 and ∇_0' for TV', $TU|_{V'}$ and $N_{V'}$, respectively, so that
(i) ∇ is v-trivial: $\nabla(v)=0$, and that
(ii) the triple $(\nabla,\nabla_0,\nabla_0')$ is compatible with (2.6).

We set $\nabla_0^\bullet=(\nabla_0,\nabla_0')$. Recall that the total Chern form of the pair $\nabla_0^\bullet$ of connections is defined by $c^*(\nabla_0^\bullet)=c^*(\nabla_0)/c^*(\nabla_0')$. By (ii) above, $c^*(\nabla_0^\bullet)=c^*(\nabla)$ so that by (i),

$$c^n(\nabla_0^\bullet)=c^n(\nabla)=0,$$

which is the key fact for the localization. Let ∇_1 and ∇_1' be connections for TU and N, respectively, and set $\nabla_1^\bullet=(\nabla_1,\nabla_1')$. The total Chern form $c^*(\nabla_1^\bullet)$ of the pair $\nabla_1^\bullet$ is defined as above and $c^n(\nabla_1^\bullet)$ is a $2n$-form on U. Recall that we have also the difference form $c^n(\nabla_0^\bullet,\nabla_1^\bullet)$. Let B and $R=B\cap V$ be as in Section 2.

Definition 4.1. The *virtual index* of v at 0 is defined by

$$\operatorname{Vir}(v,0)=\int_R c^n(\nabla_1^\bullet)-\int_{\partial R} c^n(\nabla_0^\bullet,\nabla_1^\bullet). \tag{4.2}$$

Remark 4.3. 1. If V is non-singular at 0, we have (cf. [15, Ch.IV, Lemma 3.3]):

$$\operatorname{Vir}(v,0)=\operatorname{PH}(v,0).$$

2. In practice we may take as ∇_1 and ∇_1' connections trivial with respect to some frames of TU and N, respectively. In this case, the first term in (4.2) disappears and we have only an integral on $\partial R = L$.

3. If v is the restriction to V' of some holomorphic vector field on U leaving V invariant, this integral can be expressed as a Grothendieck residue relative to V (cf. [11], [15, Ch.IV, (7.3)]). Moreover in this case, we have the "virtual residues" for Chern polynomials of degree n (cf. [15, Ch.IV, 7]), generalizing the Baum-Bott residues for holomorphic vector fields in [1].

Theorem 4.4. *We have*

$$\operatorname{Vir}(v,0)=\operatorname{Res}(v,0).$$

Proof. We take a v-trivial connection ∇ for TV', a $\boldsymbol{v}$-trivial connection ∇_0 for $TU|_{V'}$ and a $\boldsymbol{\partial}$-trivial connection ∇_0' for $N_{V'}$ so that $(\nabla,\nabla_0,\nabla_0')$ is compatible with (2.6) and set $\nabla_0^\bullet=(\nabla_0,\nabla_0')$. Also, let ∇_1 be an arbitrary connection TU and let ∇_1' be the $\boldsymbol{\nu}$-trivial connection for N and set $\nabla_1^\bullet=(\nabla_1,\nabla_1')$. Here we recall that $\boldsymbol{\nu}$ is a frame extending $\boldsymbol{\partial}$.

From $c^*(\nabla_1^\bullet)=c^*(\nabla_1)/c^*(\nabla_1')$ and $c^*(\nabla_1')=1$, we have

$$c^n(\nabla_1^\bullet)=c^n(\nabla_1). \tag{4.5}$$

To find $c^n(\nabla_0^\bullet,\nabla_1^\bullet)$, recall that it is given by integrating $c^n(\tilde\nabla^\bullet)$ over the 1-simplex $[0,1]$, where $\tilde\nabla^\bullet=(\tilde\nabla,\tilde\nabla')$ with $\tilde\nabla=(1-t)\nabla_0+t\nabla_1$ and $\tilde\nabla'=(1-t)\nabla_0'+t\nabla_1'$. Since $\nabla_0'=\nabla_1'$ on V', we have $\tilde\nabla'=\nabla_0'$ and moreover, $c^*(\tilde\nabla')=1$, as ∇_0' is $\boldsymbol{\partial}$-trivial. Thus we have $c^n(\tilde\nabla^\bullet)=c^n(\tilde\nabla)$ exactly as above. Therefore we have $c^n(\nabla_0^\bullet,\nabla_1^\bullet)=c^n(\nabla_0,\nabla_1)$, which together with (4.5) implies the equality. □

Remark 4.6. The above proof is similar to the one for [9, Theorem 4.3]. We note that the latter can also be simplified as above.

From Theorems 3.4 and 4.4, we recover the following equality, which was initially proved in [11], see also [14]:

Corollary 4.7. *We have*

$$\operatorname{Vir}(v,0)=\operatorname{GSV}(v,0).$$

Remark 4.8. As can be seen from the above, we could use, instead of $(\frac{\partial}{\partial f_1},\dots,\frac{\partial}{\partial f_k})$, an arbitrary k-frame of TU to define the residue $\operatorname{Res}(v,0)$ for similar results, as long as it is normal to the non-singular part of V. An advantage of the use of $(\frac{\partial}{\partial f_1},\dots,\frac{\partial}{\partial f_k})$ is, besides its naturalness, that we have some concrete results as shown in the following section.

5. The case of plane curves

Let C be an analytic curve (reduced but may not be irreducible) defined by $f=0$ in a neighborhood U of 0 in $\mathbb{C}^2=\{(z_1,z_2)\}$, containing 0 as a possibly singular point. Also let

$$\tilde v = a_1\frac{\partial}{\partial z_1}+a_2\frac{\partial}{\partial z_2}$$

be a holomorphic vector field on U, possibly singular at 0 and leaving C invariant. The last condition can be rephrased as $\tilde{v}(f) = hf$ for some holomorphic function h. Let v denote the restriction of $\tilde{v}$ to $C' = C \smallsetminus \{0\}$.

We denote by $\mathcal{O}$ the ring of germs of holomorphic functions at 0 in $\mathbb{C}^2$. We may assume that, changing the coordinates of $\mathbb{C}^2$ if necessary, the germs of f and a_1 are relatively prime in $\mathcal{O}$. In this case f and $\frac{\partial f}{\partial z_2}$ are also relatively prime. We set $\partial_i f = \frac{\partial f}{\partial z_i}$. Let L denote the link of C at 0.

Proposition 5.1. *In the above situation,*

$$\mathrm{GSV}(v,0) = \frac{1}{2\pi\sqrt{-1}} \int_L \Big(\frac{da_1}{a_1} - \frac{d(\partial_2 f)}{\partial_2 f} \Big).$$

Proof. By Theorem 3.4, we only need to compute $\mathrm{Res}(v,0) = \mathrm{Res}_{c^n}(\boldsymbol{v}, TU|_C; 0)$ for $\boldsymbol{v} = (v, \frac{\partial}{\partial f}|_{C'})$, as given in (2.8).

Let ∇_0 be the connection for $TU|_{C'}$ trivial with respect to $\boldsymbol{v}$ and ∇_1 the connection for TU trivial with respect to $(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2})$. Then we have $c^1(\nabla_1) = 0$. Now we compute $c^1(\nabla_0, \nabla_1)$. For this, consider the connection $\tilde{\nabla} = (1-t)\nabla_0 + t\nabla_1$ of the bundle $TU|_{C'} \times \mathbb{R}$ on $C' \times \mathbb{R}$. Let θ_0 and θ_1 be the connection matrix of ∇_0 and ∇_1 with respect to the frame $(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2})$. We have $\theta_1 = 0$. We try to find θ_0. We assume that f and a_1 are relatively prime as before. Thus f and $\partial_2 f = \frac{\partial f}{\partial z_2}$ are relatively prime so that (z_1, f) forms a coordinate system on a neighborhood of C' and we may write $\frac{\partial}{\partial f} = (\partial_2 f)^{-1} \frac{\partial}{\partial z_2}$. The matrix A of change of frame from $\boldsymbol{v}$ to $(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2})$ can be computed from $(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}) = \boldsymbol{v}A$ to get

$$A = \frac{\partial_2 f}{a_1} \begin{pmatrix} (\partial_2 f)^{-1} & 0 \\ -a_2 & a_1 \end{pmatrix}.$$

Thus by (1.1), we have

$$\theta_0 = A^{-1} \cdot dA = - \begin{pmatrix} \frac{da_1}{a_1} & 0 \\ * & -\frac{d(\partial_2 f)}{\partial_2 f} \end{pmatrix}.$$

Let $\tilde{\theta}$ and $\tilde{\kappa}$ be the connection and curvature matrices of $\tilde{\nabla}$ with respect to $(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2})$. Then we have $\tilde{\kappa} = d\tilde{\theta} + \tilde{\theta} \wedge \tilde{\theta}$, $\tilde{\theta} = (1-t)\theta_0 + t\theta_1 = (1-t)\theta_0$. The term in $\tilde{\kappa}$ involving dt is $-dt \wedge \theta_0$ so that we have, denoting by p_* the integration along the fiber of the projection $p : C' \times [0,1] \to C'$,

$$c^1(\nabla_0, \nabla_1) = \frac{\sqrt{-1}}{2\pi} p_* \operatorname{tr} \tilde{\kappa} = \frac{1}{2\pi\sqrt{-1}} \operatorname{tr} \theta_0 = -\frac{1}{2\pi\sqrt{-1}} \Big(\frac{da_1}{a_1} - \frac{d(\partial_2 f)}{\partial_2 f} \Big),$$

which proves the proposition. ☐

Corollary 5.2. *In the above situation,*

$$\mathrm{GSV}(v,0) = \dim_{\mathbb{C}} \mathcal{O}/(f, a_1) - \dim_{\mathbb{C}} \mathcal{O}/(f, \partial_2 f).$$

Proof. As f and a_1 are relatively prime, $\Gamma_1 = \{\, z \in U \mid f(z) = 0,\ |a_1(z)| = \varepsilon \,\}$ is a 1-cycle on C' homologous to L, for a small positive number ε. Thus

$$\frac{1}{2\pi\sqrt{-1}} \int_L \frac{da_1}{a_1} = \frac{1}{2\pi\sqrt{-1}} \int_{\Gamma_1} \frac{da_1}{a_1}.$$

Then by the projection formula we have (e.g., [18])

$$\frac{1}{2\pi\sqrt{-1}} \int_{\Gamma_1} \frac{da_1}{a_1} = \Big(\frac{1}{2\pi\sqrt{-1}} \Big)^2 \int_{\Gamma} \frac{df}{f} \wedge \frac{da_1}{a_1},$$

where Γ is the 2-cycle on C' given by $\Gamma = \{\, z \in U \mid |f(z)| = |a_1(z)| = \varepsilon \,\}$. The right side above equals $\dim \mathcal{O}/(f, a_1)$. Similarly for the second term. ☐

Remark 5.3. 1. Proposition 5.1 gives an alternative verification of an integral representation of the GSV-index given in [4, p. 532]. For this, note that we may take $-a_1$ and $\partial_2 f$ as k and g in [4].

2. In fact in [4] the right side of the above formula appears as the difference of the orders of zeros and of poles of a certain vector field on the Milnor fiber. We may also give such an interpretation on the central fiber C as follows. Note that, on $C \smallsetminus \{0\}$, $(a_1 \frac{\partial}{\partial z_1}, \frac{\partial}{\partial f})$ as well as $(v, \frac{\partial}{\partial f})$ is a frame of the holomorphic tangent bundle of $\mathbb{C}^2$ and we may write $\frac{\partial}{\partial f} = \frac{1}{\partial_2 f} \frac{\partial}{\partial z_2}$. Thus the first term above may be thought of as the order of zero of the vector field $a_1 \frac{\partial}{\partial z_1}$ at 0 on C and the second term as the order of pole of the vector field $\frac{\partial}{\partial f}$ at 0 on C.

3. The general algebraic formula in [5] reads, in this particular case,

$$\mathrm{GSV}(v, 0) = \dim_{\mathbb{C}} \mathcal{O}/(f, a_1, a_2) - \dim_{\mathbb{C}} \mathcal{O}/(f, \partial_1 f, \partial_2 f).$$

Compared with the one in Corollary 5.2, the corresponding terms may be different, however the differences are the same.

4. Also in this case, a general integral formula (cf. [11], [15, Ch.IV, Theorem 7.2]) for the virtual index gives

$$\mathrm{GSV}(v, 0) = \frac{1}{2\pi\sqrt{-1}} \int_{\Gamma_1} \Big(\frac{\partial a_1}{\partial z_1} + \frac{\partial a_2}{\partial z_2} - h \Big) \frac{dz_1}{a_1},$$

where Γ_1 is as in the proof of Corollary 5.2 and may be replaced by L, and h a holomorphic function such that $\tilde{v}(f) = hf$.

5. In this case again, the arguments in [16] are still valid, even if $\frac{\partial}{\partial f}$ does not extend through 0. Thus we may use the formula in the case (2), p.285, loc. cit., to directly obtain the formula in Proposition 5.1, noting that the matrix F there is given by ${}^t A^{-1}$, with A as in the above proof. We should note that F becomes meromorphic in this case.

6. It would be an interesting problem to generalize the above formula to the higher dimensional and codimensional case.

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