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SOLENOIDAL MANIFOLDS

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To Xavier Gómez-Mont who discovers and appreciates beautiful mathematics

Abstract. It is shown that every oriented solenoidal manifold of dimension one is the boundary of a compact oriented solenoidal 2-manifold. For compact solenoidal surfaces one can develop a theory of complex structures parallel to the theory for Riemann surfaces. In particular, there exists a corresponding Teichmüller space. The Teichmüller space of the solenoidal surface $\mathcal{S}$ obtained by taking the inverse limit of all finite pointed covers of a compact surface of genus greater than one is a separable Banach manifold version of the universal Teichmuller space of the upper half plane which is not separable. The commensurability automorphism group of the fundamental group of the surface acts minimally on this solenoidal version of the universal Teichmüller space.

A compact Hausdorff space which is locally homeomorphic to a k-disk cross a compact totally disconnected space is called a *laminar manifold*. If the totally disconnected space is perfect and infinite we call the laminar manifold a *solenoidal manifold*. A laminar manifold is foliated by its path components. We can speak of smooth laminar manifolds, Riemannian laminar manifolds, complex laminar manifolds etc. Such structures are meant to be, respectively, smooth, Riemannian or holomorphic in the leaf direction and continuously varying in the transverse direction.

Examples:

(i) The mapping torus of a homeomorphism of a Cantor set defines a fibration over the circle with fiber the Cantor set. The total space is an oriented solenoidal one-manifold.
(ii) The suspension of a representation of the fundamental group of a closed k-manifold into the group of homeomorphisms of a Cantor set defines a Cantor bundle over the manifold and again the total space is a solenoidal k-manifold.
(iii) The inverse limit of an infinite system of connected finite covers over a closed k-manifold defines a solenoidal k-manifold.

Laminar manifolds may be viewed geometrically in the leaf directions as generalizations of compact manifolds. Manifold concepts and theory can be applied to them. Laminar manifolds may also be viewed in the transverse direction as dynamical systems. The example following Theorem 3 is used seriously in both perspectives.

Theorem 1 (Conversation at IHES with Bob Edwards early 90's)**.** *Any oriented one dimensional solenoidal manifold is the boundary of an oriented solenoidal surface.*

Proof: The argument combines two ideas. First one observes, by choosing enough transversals, that any oriented solenoid is a mapping torus as in example (i). In more detail, choose a finite set of transversals cutting every leaf. By adding more transversals one can be sure that starting at a point on one transversal and going forward (with respect to the orientation) one first meets a different transversal. This picture presents the solenoid as a mapping torus of a homeomorphism on a Cantor set K. Second one writes this homeomorphism of K as a product of g commutators of homeomorphisms of K using the fact that $Homeo(K)$, the group of homeomorphisms of the Cantor set, is equal to its commutator subgroup. This is because R. D. Anderson [1] proved that $Homeo(K)$ is a simple group and in particular it is a perfect group. Now use the fact that the boundary of a surface of genus g with one boundary component is the product of the commutators of the standard generators. Then build using the naturally associated representation of the fundamental group of the surface with boundary into $Homeo(K)$ a compact solenoidal surface with boundary cantor fibred over the surface with boundary. The solenoidal one manifold in question appears over the boundary of the surface.

Problem: Is there a describable cobordism classification of oriented solenoidal manifolds in dimensions $2, 3, \ldots$?

Theorem 2. *In dimension two, the smoothability and the holomorphicity results for compact orientable surfaces extends to orientable laminar surfaces.*

Proof sketch: One may cover a laminar manifold with product charts that have a reasonable size in the leaf direction and a very small size in the transverse direction, moreover they can be chosen to be clopen in the transverse direction by the total discontinuity of the transversals. One sees then that any argument for compact manifolds that is continuous in parameters extends to laminar manifolds. Smoothing a compact surface or realizing it as a complex manifold can be argued to have this continuous form.

Theorem 3 (Candel [3])**.** *For any transversally continuous Riemannian metric on a smooth laminar surface, sometimes both but at least one of the following holds:*

I) the universal cover of every leaf is conformally the disk (compare the example following and Theorem 4);

II) there is a nontrivial tranverse invariant measure (a measure on each transversal so that the germs of transversal holonomy maps along paths are measure preserving).

Sketch of Candel's argument. If I) is not true, some leaf by Ahlfors lemma has a conformal metric with a sequence of neighborhoods of infinity with bounded length boundaries. This leaf is used to define the transverse invariant measure by the 70's argument of Joseph Plante.

Example of a solenoidal surface without transverse invariant measure: Consider the squaring map Q outside the unit disk in the complex plane. On the inverse limit space of the tower of covers defined by iterates of Q, the lift of Q is a bijection defining a properly discontinuous action of Z. The quotient by this action is a solenoidal surface with every leaf hyperbolic and and possessing no transversal invariant measure (cf [5]).

Theorem 4 (Candel and Verjovsky [3], [6])**.** *If every leaf of a laminar Riemannian surface is conformally covered by the disk, then the unique constant curvature minus one metric on each leaf is transversally continuous.*

Remark. To my knowledge H.E. Winkelnkemper was the first person to point out (circa 1976) the very interesting fact that whether a given non compact leaf of a smoothly foliated compact space with a transversally continuous metric has universal cover conformally the disk is independent of the choice of metric and therefore an intrinsic property of the smoothly foliated space. One can show also it is a topological invariant as well.(The hyperbolic plane is not related to the euclidean plane by a homeomorphism which is uniformaly continuous in both directions).

Theorem 5 ([5])**.** *The space of hyperbolic structures on a laminar surface (as in Theorem 4) up to isometries isotopic to the identity has the structure of a separable complex Banach manifold. The metric is the natural Teichmüller metric based on the minimal conformal distortion of a map* *between structures in the fixed isotopy class. The isotopy classes of homeomorphisms preserving a chosen leaf act by isometries of this separable Banach manifold and plays the role of the Teichmuller modular group in the classical case.*

Corollary ([5]): The analog of Riemann's moduli space exists as a Banach orbifold iff this action is appropriately discontinuous (there is a covering by open balls so that under the action only finitely many group elements bring a ball back to intersect itself.)

Consider the solenoidal surface S obtained by taking the inverse limit of all finite pointed covers of a compact surface of genus greater than one and chosen base point. The base points upstairs in the covers determine a point and a distinguished leaf in the inverse limit solenoidal surface.

Theorem 6 (based on Kahn-Marković affirmation of the Ehrenpreis Conjecture)**.** *The space of hyperbolic structures up to isometry preserving the distinguished leaf on this solenoidal surface S is non Hausdorff and any Hausdorff quotient is a point.*

Proof: In affirming the Enrenpreis Conjecture Jeremy Kahn and Vladimir Marković show any two constant negative curvature structures on compact surfaces become almost isometric after taking appropriate finite covers [4]. This shows the group mentioned in Theorem 5 has every orbit dense because it is explained in [5] how every point in the Teichmüller space of Theorem 5 is approximated by a hyperbolic structure on some high finite cover of the inverse system.[These are the transversally locally constant structures of [5]]. The group mentioned in Theorem 5 for this solenoidal surface is the commensurability automorphism group of the fundamental group of any higher genus compact surface (this means all isomorphisms between finite index subgroups). By the affirmation of the Ehrenpreis Conjecture mentioned above this commensurability group acts densely for each transversally locally constant structure in the Teichmüller space of all structures. Since the action is isometric one dense orbit implies all orbits are dense.

References

1. R.D. Anderson *The Algebraic Simplicity of Certain Groups of Homeomorphisms.* American Journal of Mathematics, Vol. 80, No. 4 (Oct., 1958), pp. 955–963 DOI: 10.2307/2372842
2. I. Biswas, S. Nag, D. Sullivan. "Determinant bundles, Quillen metrics and Mumford isomorphisms over the universal commensurability Teichmüller space". *Acta Math.* 176 (1996), no. 2, 145–169.
3. A. Candel. "Uniformization of surface laminations". *Ann. Sci. École Norm. Sup.* (4) 26 (1993), no. 4, 489–516.
4. J. Kahn, V. Marković. "The good pants homology and a proof of the Ehrenpreis conjecture". *arXiv:1101.1330*
5. D. Sullivan. *Linking the universalities of Milnor-Thurston, Feigenbaum and Ahlfors-Bers.* Lisa R. Goldberg et al (ed.), in "Topological methods in modern mathematics". Proceedings of a symposium in honor of John Milnor's sixtieth birthday, held at the State University of New York at Stony Brook, USA, June 1991. Houston, TX: Publish or Perish, Inc. 543-564 (1993).
6. A. Verjovsky. "A uniformization theorem for holomorphic foliations". The Lefschetz centennial conference, Part III (Mexico City, 1984), 233253, *Contemp. Math.*, 58, III, Amer. Math. Soc., Providence, RI, 1987.