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STRONG TOPOLOGICAL INVARIANCE OF THE MONODROMY GROUP AT INFINITY FOR QUADRATIC VECTOR FIELDS

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ABSTRACT. In this work we consider foliations on $\mathbb{C}\mathrm{P}^2$ which are generated by quadratic vector fields on $\mathbb{C}^2$. Generically these foliations have isolated singularities and an invariant line at infinity. We say that the monodromy groups at infinity of two such foliations having the same singular points at infinity are strongly analytically equivalent provided there exists a germ of a conformal mapping at zero which conjugates the monodromy maps defined *along the same loops* on the infinite leaf.

The object of this paper is to show that topologically equivalent generic foliations from this class must have, after an affine change of coordinates, their monodromy groups at infinity strongly analytically conjugated.

As a corollary it is proved that any two such generic and sufficiently close foliations can only be topologically conjugated if they are affine equivalent. This improves, in the case of quadratic vector fields, the main result of [2] which claims that two generic, topologically equivalent and sufficiently close foliations are affine equivalent provided the conjugating homeomorphism is close enough to the identity map.

1. INTRODUCTION

It is a well known result that every polynomial vector field on $\mathbb{C}^2$ can be analytically extended to a line field on $\mathbb{C}\mathrm{P}^2$. In this paper we will consider holomorphic foliations on $\mathbb{C}\mathrm{P}^2$ which in a *fixed* affine chart are generated by quadratic vector fields.

1.1. Holomorphic foliations from the class $\mathcal{A}_2$.

Definition 1. *Let $\mathbb{I}$ be a line on $\mathbb{C}\mathrm{P}^2$ which will be fixed throughout this paper. The space $\mathcal{A}_n$ is defined to be the class of all foliations on $\mathbb{C}\mathrm{P}^2$ generated by a polinomial vector field of degree n in the affine chart $\mathbb{C}^2 \approx \mathbb{C}\mathrm{P}^2 \setminus \mathbb{I}$ and having only isolated singularities.*

Having fixed this affine chart, the space $\mathcal{A}_n$ can naturally be embedded in the projective vector space of polynomial vector fields of degree at most n; two such vector fields generate the same foliation if and only if they differ only by a scalar multiple.

In this work we will deal exclusively with the class of foliations $\mathcal{A}_2$. Let $\mathcal{A}'_2$ be the subclass of foliations from $\mathcal{A}_2$ which have the line at infinity $\mathbb{I}$ invariant and exactly three singularities on $\mathbb{I}$. The space $\mathcal{A}'_2$ is Zariski open in $\mathcal{A}_2$.

Definition 2. *Two foliations $\mathcal{F}, \widetilde{\mathcal{F}} \in \mathcal{A}_2$ are topologically equivalent provided there exists a homeomorphism $\mathcal{H}\colon \mathbb{C}\mathrm{P}^2 \to \mathbb{C}\mathrm{P}^2$ which preserves the orientation both on the leaves and on $\mathbb{C}\mathrm{P}^2$ and brings the leaves of the first foliation to those of the second one. In such case we will say that the two foliations are topologically conjugated by the homeomorphism $\mathcal{H}$. The foliations are said to be affine equivalent if $\mathcal{H}$ is an affine transformation.*

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Let $\mathcal{F}$, $\widetilde{\mathcal{F}}$ in $\mathcal{A}_2'$ be two foliations having the same singular locus at infinity $\Sigma = \mathrm{Sing}(\mathcal{F}) \cap \mathbb{I}$. Choose a base point b on the infinite leaf $\mathcal{L}_{\mathcal{F}} = \mathbb{I} \setminus \Sigma$ and consider, for each element on the fundamental group $\gamma \in \pi_1(\mathbb{I} \setminus \Sigma, b)$, the monodromy transformations Δ_γ and $\widetilde{\Delta}_\gamma$ corresponding to the foliations $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ respectively.

Definition 3. *We say that the monodromy groups at infinity $G_{\mathcal{F}}$ and $G_{\widetilde{\mathcal{F}}}$ of two foliations having the same singular set at infinity are strongly analytically equivalent provided there exists a germ h of a conformal mapping at zero such that*

$$h \circ \Delta_\gamma = \widetilde{\Delta}_\gamma \circ h$$

for any element γ of the fundamental group of the infinite leaf.

1.2. Main result.

Theorem 1. *If two generic foliations from the class $\mathcal{A}_2$ with the same singular points at infinity are topologically equivalent and the conjugacy fixes these singular points then their monodromy groups at infinity are strongly analytically equivalent.*

This property is called strong topological invariance of the monodromy group at infinity. Note that for any two topologically equivalent foliations in $\mathcal{A}_2'$ we can always assume, after an affine change of coordinates, that both foliations have the same singular points at infinity and that the conjugating homeomorphism preserves these singular points.

Previously the following invariance property was known:

Proposition 1 ([5])**.** *If two foliations from $\mathcal{A}_2'$ with non-solvable monodromy group at infinity are topologically equivalent and have the same singular points at infinity then for any set of generators γ_1, γ_2 of the fundamental group of the infinite leaf there exists another set of generators ρ_1, ρ_2 and an analytic germ $h\colon (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ such that*

$$h \circ \Delta_{\gamma_i} = \widetilde{\Delta}_{\rho_i} \circ h, \quad i = 1, 2.$$

Contrary to this proposition, Theorem 1 claims that the second set of generators can be chosen to coincide with the original set of generators. As an important corollary of Theorem 1 we obtain the following result:

Theorem 2. *A generic foliation from the class $\mathcal{A}_2$ has a neighborhood in this class such that any other foliation in this neighborhood which is topologically equivalent to the first foliation must be affine equivalent to the original foliation.*

Note that in the above theorem no assumptions are being made about the conjugating homeomorphism. This property was introduced in [3] and is called *ideal rigidity*. However, it was stated as an *unknown property* for polynomial foliations.

1.3. Sketch of the proofs.

A topological equivalence between two generic foliations from the class $\mathcal{A}_2$ having the same singular points at infinity restricts to a homeomorphism $H\colon \mathcal{L}_{\mathcal{F}} \to \mathcal{L}_{\widetilde{\mathcal{F}}}$ from the infinite leaf onto itself. Theorem 1 is proved by studying the isomorphisms that such homeomorphism induces on the fundamental group and first homology group of the infinite leaf. It will be shown that if the conjugacy preseves the singular points at infinity then the induced isomorphism on homology is the identity map and therefore an inner automorphism is induced on the fundamental group. From this fact we will easily deduce that the monodromy groups at infinity are strongly analytically equivalent.

Notice that Theorem 1 is stated only for generic foliations from the class $\mathcal{A}_2$. The proof of Theorem 1 cannot be carried out in a similar way for the classes $\mathcal{A}_n$ with $n > 2$ due to an algebraic obstruction; if the fundamental group of the infinite leaf is free on more than two generators a trivial action on homology[1] does not imply that the action on the fundamental group is an inner automorphism. The fact that the action on fundamental group is an inner automorphism is the key ingredient in the proof of Theorem 1.

Ideal rigidity is very closely related to a property called *absolute rigidity* which was introduced in [2], yet in Theorem 2 there are no restrictions on the conjugating homeomorphism.

Definition 4. *A foliation $\mathcal{F} \in \mathcal{A}_n$ is absolutely rigid in the class $\mathcal{A}_n$ provided there exists a neighborhood $U \subseteq \mathcal{A}_n$ of $\mathcal{F}$ and a neighborhood $\mathcal{U}$ of the identity $id\colon \mathbb{CP}^2 \to \mathbb{CP}^2$ in the space Homeo$(\mathbb{CP}^2)$ of homeomorphisms of $\mathbb{CP}^2$ onto itself such that every foliation $\mathcal{F}' \in U$ topologically conjugated to $\mathcal{F}$ by a homeomorphism $\mathcal{H} \in \mathcal{U}$ is affine equivalent to $\mathcal{F}$.*

Proposition 2 ([2])**.** *A generic foliation from the class $\mathcal{A}_n$ is absolutely rigid.*

In the proof of Proposition 2 the closeness of the topological conjugacy to the identity homeomorphism is required in order to guarantee that the monodoromy groups at infinity are strongly analytically equivalent. In the case of quadratic vector fields, in virtue of Theorem 1, such hypothesis can be dropped and so Theorem 2 is deduced.

1.4. **Genericity assumptions.** Consider the following properties for a foliation $\mathcal{F} \in \mathcal{A}_2'$:
(i) The monodromy group at infinity $G_{\mathcal{F}}$ is non-solvable;
(ii) The characteristic numbers of the singular points at infinity are pairwise different;
(iii) All singularities of $\mathcal{F}$ are hyperbolic;
(iv) Foliation $\mathcal{F}$ has no algebraic leaves except for the infinite line.

The genericity of conditions (i) and (iv) is discussed in [5]. Foliations having pairwise different characteristic numbers form a complex Zariski open subset of $\mathcal{A}_2'$ and the set of foliations with hyperbolic singularities determines a real Zariski subset of $\mathcal{A}_2'$.

It is proved in [5] that non-solvable groups of germs are topologically rigid, hence condition (i) is sufficient to prove Theorem 1. For Theorem 2 all conditions (i)–(iv) are assumed.

2. Induced automorphisms on the fundamental group and first homology group

In the following constructions we will consider foliations with *close* tuples of singular points at infinity yet not necessarily equal.

Let $\mathcal{F} \in \mathcal{A}_2'$ be a generic foliation and let $\Sigma = \{a_1, a_2, a_3\}$ be its singular locus at infinity. Let D_1, D_2, D_3 be open disks on $\mathbb{I}$ centered at a_1, a_2, a_3 respectively with pairwise disjoint closures and define $D = \cup D_i$. Let b be an arbitrary point in $\mathbb{I} \setminus \overline{D}$.

Denote by $\widetilde{U}$ the set of those foliations in $\mathcal{A}_2'$ with the property of having their singularities at infinity on D and having exactly one singularity on each D_i.

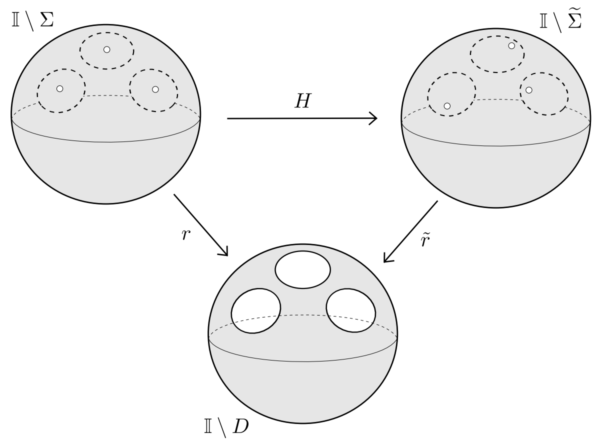
Definition 5. *Denote by $\mathcal{T}_{\mathcal{OP}}(\mathcal{F}, b)$ the set of all pairs $(\widetilde{\mathcal{F}}, \mathcal{H})$ in the product $\widetilde{U} \times$ Homeo$(\mathbb{CP}^2)$ such that $\mathcal{H}$ is a topological conjugacy between $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ that fixes the point b.*

Choose any $(\widetilde{\mathcal{F}}, \mathcal{H}) \in \mathcal{T}_{\mathcal{OP}}(\mathcal{F}, b)$. The foliation $\mathcal{F}$, and so does $\widetilde{\mathcal{F}}$, has a unique algebraic leaf; the punctured infinite line. This implies that the homeomorphism $\mathcal{H}\colon \mathbb{CP}^2 \to \mathbb{CP}^2$ preserves the line $\mathbb{I}$ and maps bijectively the singular set $\Sigma = \mathrm{Sing}(\mathcal{F}) \cap \mathbb{I}$ onto $\widetilde{\Sigma} = \mathrm{Sing}(\widetilde{\mathcal{F}}) \cap \mathbb{I}$.

From now on if $\mathcal{H}$ is a homeomorphism from $\mathbb{CP}^2$ onto itself which preserves the infinite line $\mathbb{I}$ we shall denote by H its restriction $H = \mathcal{H}|_{\mathbb{I}}$.

If $\Sigma \neq \widetilde{\Sigma}$ the fundamental groups $\pi_1(\mathbb{I} \setminus \Sigma, b)$ and $\pi_1(\mathbb{I} \setminus \widetilde{\Sigma}, b)$ do not coincide. However, both surfaces $\mathbb{I} \setminus \Sigma$ and $\mathbb{I} \setminus \widetilde{\Sigma}$ deformation retract onto $\mathbb{I} \setminus D$ and thus both the fundamental groups $\pi_1(\mathbb{I} \setminus \Sigma, b)$ and $\pi_1(\mathbb{I} \setminus \widetilde{\Sigma}, b)$ are naturally isomorphic to the group $\pi_1(\mathbb{I} \setminus D, b)$.

[1]See Section 2 for the corresponding definitions.



In fact, for every loop γ on $\mathbb{I}\setminus\Sigma$ based on b we can assume, without loss of generality, that it is contained in $\mathbb{I}\setminus D$ and so it can be regarded indistinctly as an element of any of the groups $\pi_1(\mathbb{I}\setminus\Sigma,b)$, $\pi_1(\mathbb{I}\setminus\widetilde{\Sigma},b)$, $\pi_1(\mathbb{I}\setminus D,b)$. The concept of strong analytic equivalence for the monodromy groups can naturally be extended for pairs of foliations whose singularities at infinity are *close enough*. In particular, this can be done for foliations $\mathcal{F}$, $\widetilde{\mathcal{F}}$ if $\widetilde{\mathcal{F}}$ belongs to the neighborhood $\widetilde{U}$ constructed above.

Definition 3'. *Let $\widetilde{\mathcal{F}}\in\widetilde{U}$. We say that the monodromy groups at infinity $G_{\mathcal{F}}$ and $G_{\widetilde{\mathcal{F}}}$ are strongly analytically equivalent provided there exists a germ h of a conformal mapping at zero such that*

$$h\circ\Delta_\gamma=\widetilde{\Delta}_\gamma\circ h$$

for any element γ of the fundamental group $\pi_1(\mathbb{I}\setminus D,b)$.

We are now going to define *the action that H has on the fundamental group* by assigning to each pair $(\widetilde{\mathcal{F}},\mathcal{H})\in\mathcal{T}_{\mathcal{OP}}(\mathcal{F},b)$ an element of the automorphism group of the group $\pi_1(\mathbb{I}\setminus D,b)$ in the following way:

Let $r\colon\mathbb{I}\setminus\Sigma\to\mathbb{I}\setminus D$ and $\tilde{r}\colon\mathbb{I}\setminus\widetilde{\Sigma}\to\mathbb{I}\setminus D$ be the retractions mentioned above. Since they are homotopy equivalences they induce isomorphisms

$$r_*\colon\pi_1(\mathbb{I}\setminus\Sigma,b)\to\pi_1(\mathbb{I}\setminus D,b)\qquad\text{and}\qquad\tilde{r}_*\colon\pi_1(\mathbb{I}\setminus\widetilde{\Sigma},b)\to\pi_1(\mathbb{I}\setminus D,b)$$

on the fundamental groups. The homeomorphism $H|_{\mathbb{I}\setminus\Sigma}$ also induces an isomorphism

$$H_*\colon\pi_1(\mathbb{I}\setminus\Sigma,b)\to\pi_1(\mathbb{I}\setminus\widetilde{\Sigma},b).$$

There exists a unique group automorphism $\Phi(H)\colon\pi_1(\mathbb{I}\setminus D,b)\to\pi_1(\mathbb{I}\setminus D,b)$ which makes the following diagram commutative:

$$\begin{array}{ccc}\pi_1(\mathbb{I}\setminus\Sigma,b) & \xrightarrow{\;H_*\;} & \pi_1(\mathbb{I}\setminus\widetilde{\Sigma},b)\\ {\scriptstyle r_*}\downarrow & & \downarrow{\scriptstyle \tilde{r}_*}\\ \pi_1(\mathbb{I}\setminus D,b) & \xrightarrow[\Phi(H)]{} & \pi_1(\mathbb{I}\setminus D,b)\end{array}$$

Thus we get a well defined map

$$\Phi\colon\mathcal{T}_{\mathcal{OP}}(\mathcal{F},b)\to\operatorname{Aut}(\pi_1(\mathbb{I}\setminus D,b)).$$

Here $\mathrm{Aut}(\pi_1(\mathbb{I}\setminus D, b))$ denotes the automorphism group of $\pi_1(\mathbb{I}\setminus D, b)$. For the sake of simplicity we shall write $\Phi(H)$ instead of $\Phi(\widetilde{\mathcal{F}}, \mathcal{H})$.

2.1. **Inner automorphisms of the fundamental group.** Let $(\widetilde{\mathcal{F}}, \mathcal{H}) \in \mathcal{T}_{\mathcal{OP}}(\mathcal{F}, b)$ and suppose $\Phi(H) = id$. For any germ of cross-section Γ at b transversal to the leaves of $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ there exists (Proposition 1) an analytic germ

$$h\colon (\Gamma, b) \to (\Gamma, b)$$

induced by $\mathcal{H}$ that conjugates the monodromy groups in the following way

$$h \circ \Delta_{\gamma_i} = \widetilde{\Delta}_{\rho_i} \circ h, \quad i = 1, 2$$

where ρ_i is defined by the composition $\rho_i = H \circ \gamma_i$ and γ_1, γ_2 are canonical generators of $\pi_1(\mathbb{I}\setminus D, b)$. But the condition $\Phi(H) = id$ implies that the loops ρ_i are homotopic to the corresponding γ_i and so the monodromy groups are strongly analytically equivalent.

The following lemma shows that we can also deduce the strong analytic equivalence of the monodromy groups in the case when the action on the fundamental group is an inner automorphism, not necessarily trivial.

Lemma 1. *If $(\widetilde{\mathcal{F}}, \mathcal{H}) \in \mathcal{T}_{\mathcal{OP}}(\mathcal{F}, b)$ and $\Phi(H)$ is an inner automorphism on $\pi_1(\mathbb{I}\setminus D, b)$ then the monodromy groups $G_{\mathcal{F}}$ and $G_{\widetilde{\mathcal{F}}}$ are strongly analytically equivalent.*

Proof. Let $(\widetilde{\mathcal{F}}, \mathcal{H}) \in \mathcal{T}_{\mathcal{OP}}(\mathcal{F}, b)$ and suppose $\Phi(H)$ is an inner automorphism; namely, there exists an element $\lambda \in \pi_1(\mathbb{I}\setminus D, b)$ such that for any $\gamma \in \pi_1(\mathbb{I}\setminus D, b)$

$$\Phi(H)(\gamma) = \lambda \cdot \gamma \cdot \lambda^{-1}.$$

Since the curve $H \circ \gamma$ is homotopic to $\Phi(H)(\gamma)$ for any $\gamma \in \pi_1(\mathbb{I}\setminus D, b)$ there exists an analytic germ $h\colon (\Gamma, b) \to (\Gamma, b)$ such that

$$h \circ \Delta_\gamma = \widetilde{\Delta}_{\lambda\cdot\gamma\cdot\lambda^{-1}} \circ h.$$

This implies

$$h \circ \Delta_\gamma = \widetilde{\Delta}_{\lambda^{-1}} \circ \widetilde{\Delta}_\gamma \circ \widetilde{\Delta}_\lambda \circ h,$$

and so

$$h_0 \circ \Delta_\gamma = \widetilde{\Delta}_\gamma \circ h_0,$$

where h_0 is defined to be $h_0 = \widetilde{\Delta}_\lambda \circ h$. □

2.2. **Induced action on homology.** In an analogous way, moving on to the first homology group, we are now going to define a map

$$\begin{aligned} \eta : \mathcal{T}_{\mathcal{OP}}(\mathcal{F}, b) &\longrightarrow \mathrm{Aut}(H_1(\mathbb{I}\setminus D; \mathbb{Z})) \\ (\widetilde{\mathcal{F}}, \mathcal{H}) &\longmapsto \eta(H) \end{aligned}$$

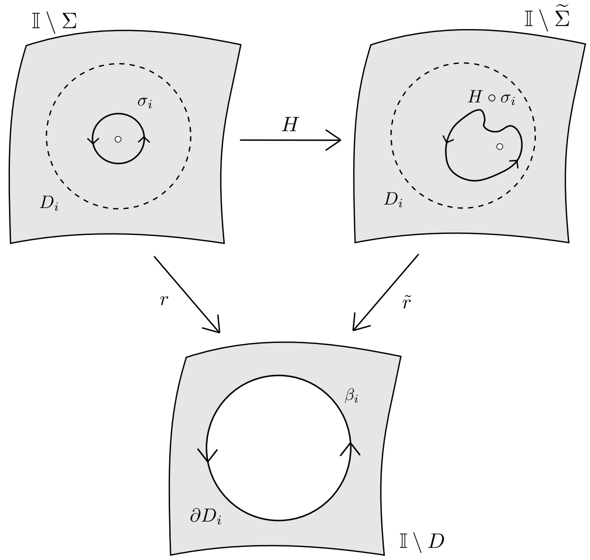
such that $\eta(H)$ is the only automorphism which makes the following diagram commutative:

$$\begin{array}{ccc} H_1(\mathbb{I}\setminus \Sigma; \mathbb{Z}) & \xrightarrow{\;H_*\;} & H_1(\mathbb{I}\setminus \widetilde{\Sigma}; \mathbb{Z}) \\ \downarrow r_* & & \downarrow \tilde{r}_* \\ H_1(\mathbb{I}\setminus D; \mathbb{Z}) & \xrightarrow[\eta(H)]{} & H_1(\mathbb{I}\setminus D; \mathbb{Z}) \end{array}$$

Lemma 2. *Let* $(\widetilde{\mathcal{F}}, \mathcal{H}) \in \mathcal{T}_{\mathcal{OP}}(\mathcal{F}, b)$. *Then* $\eta(H) = id$ *provided that* $H(a_i) \in D_i$ *for each* $i = 1, 2, 3$.

Proof. Let us choose 1-cycles[2] $\sigma_1, \sigma_2 \colon \Delta^1 \to \mathbb{I} \setminus \Sigma$ in such a way that they make up a canonical set of generators of the group $H_1(\mathbb{I} \setminus \Sigma; \mathbb{Z})$ and $\sigma_i(\Delta^1) \subseteq D_i$, $H(\sigma_i(\Delta^1)) \subseteq D_i$.

Define now $\beta_i = r \circ \sigma_i$. In this way β_1, β_2 is a canonical set of generators of the group $H_1(\mathbb{I} \setminus D; \mathbb{Z})$ which satisfies $\beta_i(\Delta^1) \subseteq \partial D_i$.



$H(\sigma_i(\Delta^1)) \subseteq D_i$ and so $\tilde{r} \circ H \circ \sigma_i(\Delta^1) \subseteq \partial D_i$. Therefore $(\tilde{r} \circ H)_* \sigma_i$ must be homologous to an integer multiple of β_i. This implies that the automorphism $\eta(H)$ can be expressed as

$$\eta(H)(\beta_1) = m\beta_1, \quad \eta(H)(\beta_2) = n\beta_2.$$

for some integers m, n.

On the other hand, the composition $\tilde{r} \circ H \colon \mathbb{I} \setminus \Sigma \to \mathbb{I} \setminus D$ is a homotopy equivalence and so it induces an isomorphism on the homology group. Thus $m\beta_1$ and $n\beta_2$ generate $H_1(\mathbb{I} \setminus D; \mathbb{Z})$. This is only possible if $m, n = \pm 1$, i.e. $(\tilde{r} \circ H)_* \sigma_i \simeq \pm \beta_i$, $i = 1, 2$. But both $\tilde{r}$ and H are orientation preserving maps and so we conclude that $(\tilde{r} \circ H)_* \sigma_i \simeq \beta_i$ and thus $\eta(H)$ is the identity automorphism. □

3. Proof of the main results

3.1. **Proof of Theorem 1.**

Proof of Theorem 1. Suppose $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ are generic foliations having the same singular points at infinity, are topologically conjugated by a homeomorphism $\mathcal{H}$ and this topological conjugacy preserves the singular points at infinity. Without loss of generality we can assume it also preserves the base point b. Therefore $(\widetilde{\mathcal{F}}, \mathcal{H}) \in \mathcal{T}_{\mathcal{OP}}(\mathcal{F}, b)$ and clearly the condition $H(a_i) \in D_i$ is satisfied. By Lemma 2 the action on homology $\eta(H)$ is the identity automorphism.

By Hurewicz Theorem $H_1(\mathbb{I} \setminus D; \mathbb{Z})$ is naturally isomorphic to the abelianization of $\pi_1(\mathbb{I} \setminus D, b)$. Let $q \colon \pi_1(\mathbb{I} \setminus D, b) \to H_1(\mathbb{I} \setminus D; \mathbb{Z})$ be the canonical projection. Through q every automorphism f on $\pi_1(\mathbb{I} \setminus D, b)$ descends to a unique automorphism on $H_1(\mathbb{I} \setminus D; \mathbb{Z})$. This assignment gives

[2] Δ^1 is the standard 1-simplex $\Delta^1 = \{(t_0, t_1) \in \mathbb{R}^2 \mid t_0 + t_1 = 1 \text{ and } t_1, t_2 \geq 0\}$.

raise to a natural and surjective homomorphism $T\colon \mathrm{Aut}(\pi_1(\mathbb{I}\setminus D, b)) \to \mathrm{Aut}(H_1(\mathbb{I}\setminus D;\mathbb{Z}))$ such that $\forall f \in \mathrm{Aut}(\pi_1(\mathbb{I}\setminus D, b))$ the diagram commutes:

$$\begin{array}{ccc} \pi_1(\mathbb{I}\setminus D, b) & \xrightarrow{\;f\;} & \pi_1(\mathbb{I}\setminus D, b) \\ q\big\downarrow & & \big\downarrow q \\ H_1(\mathbb{I}\setminus D;\mathbb{Z}) & \xrightarrow[T(f)]{} & H_1(\mathbb{I}\setminus D;\mathbb{Z}) \end{array}$$

Moreover, the kernel of such homomorphism consists precisely on those automorphisms on $\pi_1(\mathbb{I}\setminus D, b)$ which are inner automorphisms [3] [4]; i.e. $\mathrm{Ker}(T) = \mathrm{Inn}(\pi_1(\mathbb{I}\setminus D, b))$.

The homeomorphism H satisfies $q \circ \Phi(H) = \eta(H) \circ q$,

$$\begin{array}{ccc} \pi_1(\mathbb{I}\setminus D, b) & \xrightarrow{\;\Phi(H)\;} & \pi_1(\mathbb{I}\setminus D, b) \\ q\big\downarrow & & \big\downarrow q \\ H_1(\mathbb{I}\setminus D;\mathbb{Z}) & \xrightarrow[\eta(H)]{} & H_1(\mathbb{I}\setminus D;\mathbb{Z}) \end{array}$$

and therefore $\eta(H) = T(\Phi(H))$. Since $\eta(H) = id$ then $\Phi(H) \in \mathrm{Ker}(T)$ and so is an inner automorphism on $\pi_1(\mathbb{I}\setminus D, b)$. By Lemma 1 the monodromy groups $G_{\mathcal{F}}$ and $G_{\widetilde{\mathcal{F}}}$ are strongly analytically equivalent. □

3.1.1. *A remark about conjugating homeomorphisms.* Theorem 1 has been proved above by exhibiting explicitly a conformal germ $h_0\colon (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ that conjugates the monodromy groups. In fact, this germ can be realized as the transverse component of a global topological conjugacy between $\mathcal{F}$ and $\widetilde{\mathcal{F}}$. Namely, we have the following lemma:

Lemma 3. *Let $(\widetilde{\mathcal{F}}, \mathcal{H}) \in \mathcal{T}_{\mathcal{OP}}(\mathcal{F}, b)$ and choose a cross-section Γ at b transversal to the leaves of $\mathcal{F}$ and $\widetilde{\mathcal{F}}$. If $H(a_i) \in D_i$ for each $i = 1, 2, 3$ then there exists another topological conjugacy $\mathcal{H}_0\colon \mathbb{C}\mathrm{P}^2 \to \mathbb{C}\mathrm{P}^2$ between $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ such that its transverse component*

$$h_0 = \mathcal{H}_0{}^{\pitchfork}_{b}\colon (\Gamma, b) \longrightarrow (\Gamma, b)$$

yields a strong analytic equivalence between the monodromy groups $G_{\mathcal{F}}$ and $G_{\widetilde{\mathcal{F}}}$;

$$h_0 \circ \Delta_\gamma = \widetilde{\Delta}_\gamma \circ h_0$$

for any element $\gamma \in \pi_1(\mathbb{I}\setminus D, b)$.

Proof. We have a topological conjugacy $\mathcal{H}$ that satisfies $H(a_i) \in D_i$. By Lemma 2 the action on homology $\eta(H)$ is trivial and so $\Phi(H)$ is an inner automorphism on $\pi_1(\mathbb{I}\setminus D, b)$. By the same arguments used on Section 2.1 there is an analytic germ

$$h\colon (\Gamma, b) \to (\Gamma, b)$$

induced by $\mathcal{H}$ (its transverse component at b) and an element $\lambda \in \pi_1(\mathbb{I}\setminus D, b)$ that the monodromy groups $G_{\mathcal{F}}$ and $G_{\widetilde{\mathcal{F}}}$ are conjugated in the following way

$$h \circ \Delta_\gamma = \widetilde{\Delta}_{\lambda^{-1}} \circ \widetilde{\Delta}_\gamma \circ \widetilde{\Delta}_\lambda \circ h,$$

[3] This statement would not hold if $\pi_1(\mathbb{I}\setminus D, b)$ was a free groups of rank grater than two. This fact is precisely the obstruction for proving the same result in the case of polynomial vector fields of degree $n > 2$.

therefore

$$(\widetilde{\Delta}_\lambda \circ h) \circ \Delta_\gamma = \widetilde{\Delta}_\gamma \circ (\widetilde{\Delta}_\lambda \circ h).$$

Suppose we can find a homeomorphism $\widetilde{\mathcal{H}} : \mathbb{C}\mathrm{P}^2 \to \mathbb{C}\mathrm{P}^2$ that self-conjugates $\widetilde{\mathcal{F}}$, preserves the cross-section Γ and such that its transverse component at b

$$\widetilde{\mathcal{H}}_b^{\pitchfork} : (\Gamma, b) \to (\Gamma, b)$$

coincides with the germ $\widetilde{\Delta}_\lambda$. Then the composition $\mathcal{H}_0 = \widetilde{\mathcal{H}} \circ \mathcal{H}$ would yield a topological conjugacy between $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ whose transverse component at b

$$h_0 = \widetilde{\mathcal{H}}_b^{\pitchfork} \circ h = \widetilde{\Delta}_\lambda \circ h$$

strongly conjugates the monodromy groups $G_{\mathcal{F}}$ and $G_{\widetilde{\mathcal{F}}}$. Such a homeomorphism $\widetilde{\mathcal{H}}$ can easily be constructed in the following way: Consider the monodromy map $\widetilde{\Delta}_\lambda$. Recall that holonomy transformations along a path are defined as a finite composition of correspondence maps

$$\Delta_j : (\tau_j, p_j) \to (\tau_{j+1}, p_{j+1})$$

where τ_j, τ_{j+1} are cross-sections at points p_j, p_{j+1} that lay on the same leaf and belong to a same flow box. We can assume this correspondence maps are given by the time-one map of a constant (in the appropriate coordinates) vector field. If the flow box is sufficiently small we can extend such vector field to a smooth (real C^∞) vector field tangent to the leaves of $\widetilde{\mathcal{F}}$ that vanishes outside a compact neighborhood of the flow box. The time-one map of this new vector field is a homeomorphism $\mathcal{H}_j : \mathbb{C}\mathrm{P}^2 \to \mathbb{C}\mathrm{P}^2$ which preserves the foliation $\widetilde{\mathcal{F}}$, maps the cross-section (τ_j, p_j) to the cross-section (τ_{j+1}, p_{j+1}) and the restriction

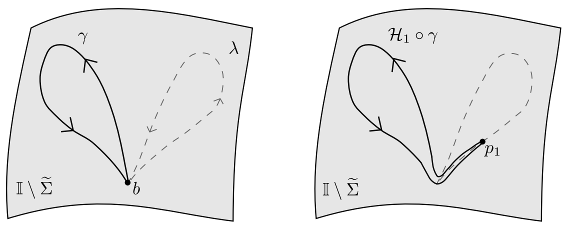
$$\mathcal{H}_j|_{(\tau_j, p_j)} : (\tau_j, p_j) \to (\tau_{j+1}, p_{j+1})$$

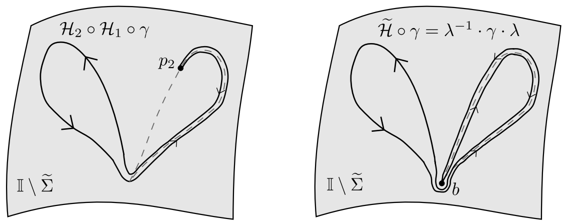
coincides with the correspondence map Δ_j.

The composition of all of the homeomorphisms $\mathcal{H}_j$ will be a homeomorphism $\widetilde{\mathcal{H}}$ which self-conjugates foliation $\widetilde{\mathcal{F}}$ and whose transverse component at b, by construction, coincides with the monodromy map $\widetilde{\Delta}_\lambda$.

Lemma 3 is now proved. □

Remark 1. The homeomorphism $\widetilde{\mathcal{H}}$ constructed above is isotopic to the identity map on $\mathbb{C}\mathrm{P}^2$. Its restriction to the infinite leaf is a map isotopic to the identity and such isotopy is obtained by *sliding* the base point b along the closed loop λ. For any loop $\gamma \in \pi_1(\mathbb{I} \setminus D, b)$ the composition $\widetilde{H} \circ \gamma$ turns out to be homotopic to the loop $\lambda^{-1} \cdot \gamma \cdot \lambda$.





This action is exactly inverse to the one induced by the original conjugacy between $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ and so the composition $\widetilde{\mathcal{H}} \circ \mathcal{H}$ has a trivial action on the fundamental group $\pi_1(\mathbb{I} \setminus D, b)$; this is, $(\widetilde{\mathcal{F}}, \mathcal{H}_0) \in \mathcal{T}_{\mathcal{OP}}(\mathcal{F}, b)$ and $\Phi(H_0) = id$.

3.2. **Topological invariance of the characteristic numbers of the singular points.** On this section we shall define the neighborhood U of $\mathcal{F}$ in $\mathcal{A}_2$ that is claimed to exist in Theorem 2. Its defining property being that if $(\widetilde{\mathcal{F}}, \mathcal{H}) \in \mathcal{T}_{\mathcal{OP}}(\mathcal{F}, b)$ and $\widetilde{\mathcal{F}} \in U$ then the homeomorphism H satisfies $H(a_i) \in D_i$ for $i = 1, 2, 3$. Whenever this situation happens we will say that H preserves the numbering of the singular points at infinity.

Let us denote by $\lambda_1, \lambda_2, \lambda_3$ the characteristic numbers of the singular points a_1, a_2, a_3 respectively. Given any other foliation $\widetilde{\mathcal{F}} \in \widetilde{U}$, denote by $a_i(\widetilde{\mathcal{F}})$ the unique singularity that $\widetilde{\mathcal{F}}$ has on the disk D_i. Let us denote by $\lambda(a_i(\widetilde{\mathcal{F}}))$ the characteristic number of the singularity $a_i(\widetilde{\mathcal{F}})$ corresponding to the foliation $\widetilde{\mathcal{F}}$. We shall keep writing a_i and λ_i instead of $a_i(\mathcal{F})$ and $\lambda(a_i(\mathcal{F}))$.

Let $M\colon \widetilde{U} \to \mathbb{C}^3$ be the map $M(\widetilde{\mathcal{F}}) = (\lambda(a_1(\widetilde{\mathcal{F}})), \lambda(a_2(\widetilde{\mathcal{F}})), \lambda(a_3(\widetilde{\mathcal{F}})))$. Since the characteristic numbers $\lambda_1, \lambda_2, \lambda_3$ are pairwise different there exists $\epsilon > 0$ such that if $j \neq k$ then $|\lambda_j - \lambda_k| \geq 2\epsilon$. Denote by V_i the disk $V_i = \{z \in \mathbb{C} \mid |\lambda_i - z| < \epsilon\}$ and let $U = M^{-1}(V_1 \times V_2 \times V_3)$. The map M is continuous (in fact, it is algebraic [1]) so U is an open neighborhood of $\mathcal{F}$ contained in $\widetilde{U}$.

Lemma 4. *If $\mathcal{F}$ is a generic foliation then for any other foliation $\widetilde{\mathcal{F}} \in U$ topologically conjugated to $\widetilde{\mathcal{F}}$ by a homeomorphism $\mathcal{H}\colon \mathbb{CP}^2 \to \mathbb{CP}^2$ the homeomorphism H preserves the numbering of the singular points at infinity; this is, for every $i = 1, 2, 3$ $H(a_i) \in D_i$.*

Proof. Choose $\widetilde{\mathcal{F}} \in U$ topologically conjugated to $\mathcal{F}$ by $\mathcal{H}\colon \mathbb{CP}^2 \to \mathbb{CP}^2$. The genericity conditions imposed on $\mathcal{F}$ imply that the characteristic numbers of the singularities at infinity are topological invariants in the following sense [1]: if $\mathcal{H}$ is a topological conjugacy between $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ then $\lambda(H(a_i)) = \lambda_i$. Additionally, from the definition of U it follows that

$$|\lambda(a_j(\widetilde{\mathcal{F}})) - \lambda_k| < \epsilon \quad \text{if } j = k$$

$$|\lambda(a_j(\widetilde{\mathcal{F}})) - \lambda_k| \geq \epsilon \quad \text{if } j \neq k,$$

which implies $H(a_i) = a_i(\widetilde{\mathcal{F}})$ for each $i = 1, 2, 3$; this is, H preserves the numbering of the singular points at infinity. □

3.3. **Ideal rigidity of foliations from the class $\mathcal{A}_2$.** In order to conclude that generic foliations from the class $\mathcal{A}_2$ are ideally rigid we will use a modified version of Proposition 2 which appears in [3].

Definition 6. *Let $S = \{a_1, ..., a_{n+1}\} \subseteq \mathbb{I}$ be a finite set of $n+1$ distinct points; $D_1, ..., D_{n+1}$ a collection of $n+1$ disjoint open disks covering S; $D = \cup D_i$ and $b \in \mathbb{I} \setminus D$.*

A homeomorphism $H\colon \mathbb{I} \to \mathbb{I}$ is called homotopically trivial over $\mathbb{I} \setminus D$ if $H(b) = b$, for each point a_i its image $H(a_i)$ belongs to the same disk D_i and the images $H(\alpha_i)$ of the segments $\alpha_i = [b, a_i]$ connecting the base point b with each point a_i are homotopic to the corresponding *segments α_i in the class of homotopy with the fixed endpoint b and free endpoint $a_{i,t} \in D$ restricted to the respective disk.*

A homeomorphism is said to be homotopically trivial without specifying the system of disks, if it is homotopically trivial over *some* system of disks.

Definition 7. *A foliation $\mathcal{F} \in \mathcal{A}'_n$ will be called reasonably rigid if there exists a neighborhood U of it in $\mathcal{A}_n$ such that any foliation $\mathcal{F}' \in U$ topologically equivalent to $\mathcal{F}$ is affine equivalent to $\mathcal{F}$ provided that the topological equivalence between $\mathcal{F}$ and $\mathcal{F}'$ induces a homotopically trivial homeomorphism of the infinite line $\mathbb{I}$ onto itself.*

Proposition 3 ([3])**.** *A generic foliation from the class $\mathcal{A}'_n$ is reasonably rigid.*

We now prove Theorem 2.

Proof of Theorem 2. Let $\mathcal{F} \in \mathcal{A}_2$ be a generic foliation. Let U be the neighborhood of $\mathcal{F}$ constructed in Section 3.2. Since foliation $\mathcal{F}$ is reasonably rigid there exists a neighborhood U' of it in $\mathcal{A}_2$ such that any foliation $\mathcal{F}' \in U'$ topologically equivalent to $\mathcal{F}$ is affine equivalent to $\mathcal{F}'$ provided that the topological equivalence between $\mathcal{F}$ and $\mathcal{F}'$ induces a homotopically trivial homeomorphism.

Suppose now that $\widetilde{\mathcal{F}} \in U \cap U'$ is topologically equivalent to $\mathcal{F}$. Without loss of generality we can suppose this conjugacy preserves the base point b. Since $\widetilde{\mathcal{F}} \in U$ the topological conjugacy $\mathcal{H}$ preserves the numbering of the singular points at infinity and, according to Lemma 3 and Remark 1, we can also suppose that the topological conjugacy satisfies $\Phi(H) = id$. This condition is equivalent to H being a homotopically trivial homeomorphism. Since $\widetilde{\mathcal{F}} \in U'$ we conclude that both foliations are affine equivalent. □

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