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FORMAL AND ANALYTIC NORMAL FORMS OF GERMS OF HOLOMORPHIC NONDICRITIC FOLIATIONS

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To Xavier Gómez-Mont:
source of enthusiasm and creativity!

Abstract. We consider the class $\mathcal{V}_n$ of germs of holomorphic vector fields in $(\mathbb{C}^2, 0)$ with vanishing $(n-1)$-jet at the origin, $n > 1$. For generic germs $\mathbf{v} \in \mathcal{V}_2$ we prove the existence of an analytical orbital normal form whose orbital formal normal form has the form $\mathbf{v}_{c,b}$ given in [ORV4]. Furthermore, fixing one representative $\hat{\mathbf{v}}$ of the analytic class of a germ $\mathbf{v} \in \mathcal{V}_2$ having the y-axis invariant, the corresponding formal normal form $\hat{\mathbf{v}}_{c,\hat{b}}$ is analytic and unique (under strict orbital equivalence). Moreover for generic $\mathbf{v} \in \mathcal{V}_n$, $n \geq 2$ we give a preliminary orbital analytic normal form which is polynomial and of degree at most n in the y-variable.

1. Introduction

The problem of the formal and analytic classification of germs of holomorphic vector fields goes back to Poincaré. He proved that, in the generic situation, such classification relies on the eigenvalues of the linear part of the vector field at the singular point. In such cases, the formal and analytic classification coincides. As it is well-known (see [IY], [ORV3]) the failure of the generic assumptions on the eigenvalues of the linear part of the vector field leads either to simply formal normal forms and complicated analytic ones (and therefore the non coincidence of the formal and analytic classification) or to highly complicated formal and analytic normal forms. In this last situation the formal and analytic classification coincides again: the rigidity phenomena takes place (see [Ce,Mo], [EISV], [M], [Lo1], [Lo2]).

In more complicated situations, when the linear part of the vector field at the singular point is zero (i.e. for degenerated germs of vector fields), the rigidity phenomena takes place again for generic dicritic and nondicritic germs (see [ORV1] and [Vo1] for the classical and orbital rigidity-respectively- of nondicritic germs; [ORV2] for the classical and orbital rigidity of generic dicritic germs of vector fields and [Ca] for orbital rigidity of dicritic germs with higher degeneracies).

In such cases the formal orbital normal form was obtained and Thom's problem on the minimal invariants of the orbital analytic classification of generic dicritic and nondicritic degenerated germs of vector fields was solved (see [ORV2] and [ORV4]). In those works rather simple formal orbital normal forms were obtained and the analytic classification relied in a combination of a finite number of parameters, together with formal invariants related to geometric objects (involutions and separatrices respectively).

The problem on the analyticity of the formal orbital normal forms was solved for generic dicritic germs in [ORV3]. However, the analyticity of the formal orbital normal form for nondicritic generic germs of vector fields given in [ORV4] was still open. In this work we prove the

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analyticity of such normal forms for the generic case: that is, when the formal orbital normal form has quadratic principal part.

For higher degeneracies we give a preliminary orbital analytic normal form (polynomial in the y variable) which does not coincide with the formal orbital normal form given in [ORV4]. We stress that for higher degeneracies one can expect a non coincidence between the formal analytic normal forms. A similar behavior was already observed in the classification of the analytic germs of vector fields with non generic linear part.

As we did in the dicritic case (see [ORV3]), we use surgery of manifolds and Savelev's Theorem for the proof of Theorem 2.1. These ideas were firstly introduced by F.Loray in [Lo2] and [Lo3] for germs at $(\mathbb{C}^2, 0)$ of holomorphic vector fields having a non generic linear term (nilpotent or saddle-node) at the origin.

2. Basic notations.

2.1. Notations.

(1) Let $\mathcal{V}_n$ be the class of holomorphic germs of vector fields in $(\mathbb{C}^2, 0)$ with isolated singularity at the origin, with vanishing $(n-1)$-jet at the origin and non vanishing n-jet, $n \geq 2$.

(2) Given $\mathbf{v} \in \mathcal{V}_n$, we denote by $\mathcal{F}_{\mathbf{v}}$ the germ of foliation generated by $\mathbf{v}$.

(3) Two germs $\mathbf{v}$ and $\mathbf{w}$ in $\mathcal{V}_n$ are *analytically (formally) orbitally equivalent* if there exist an analytic (formal) change of coordinates $H : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ and an analytic function (formal series) $K : (\mathbb{C}^2, 0) \to \mathbb{C}^*$, $(K(0) \neq 0)$ such that $H_*\mathbf{v} = K \cdot \mathbf{w}$, where

$$H_*\mathbf{v}(p) = D_z H \mathbf{v}(z)|_{z=H^{-1}(p)}.$$

(4) The foliations $\mathcal{F}_{\mathbf{v}}, \mathcal{F}_{\mathbf{w}}$ generated by the germs of vector fields $\mathbf{v}, \mathbf{w} \in \mathcal{V}_n$, respectively, are called analytically (formally) equivalent if their corresponding vector fields $\mathbf{v}, \mathbf{w}$ are analytically (formally) orbitally equivalent.

In other words, in the analytic case, if $l_{\mathbf{v},(x,y)} :=$ denotes the leaf through (x, y) of the foliation $\mathcal{F}_{\mathbf{v}}$ then $l_{\mathbf{w},H(x,y)} = H(l_{\mathbf{v},(x,y)})$.

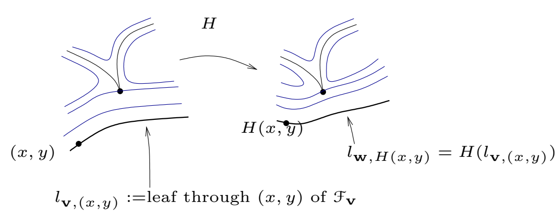

Figure 2.1. analytic equivalence of $\mathcal{F}_{\mathbf{v}}$ and $\mathcal{F}_{\mathbf{w}}$

(5) If the linear part of the germ H is the identity and $K(0) = 1$, we say that the vector fields $\mathbf{v}, \mathbf{w}$ are *strictly* analytically (formally) orbitally equivalent or the foliations $\mathcal{F}_{\mathbf{v}}$, $\mathcal{F}_{\mathbf{w}}$ are *strictly* analytically (formally) equivalent.

(6) In the case when $K \equiv 1$ then the vector fields $\mathbf{v}$, $\mathbf{w}$ are analytically (formally) equivalent.

(7) Let $\mathbf{v} \in \mathcal{V}_n$

$$\mathbf{v} = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y}, \quad P = \sum_{k=n}^{\infty} P_k, \; Q = \sum_{k=n}^{\infty} Q_k \; , \tag{2.1}$$

where P and Q are holomorphic functions and P_k, Q_k are homogeneous polynomials in (x, y) of degree $k,\ k \geq n$, corresponding to the terms of order k of its Taylor expansion at the origin. Let $R_{n+1} := xQ_n - yP_n$.

(8) We say that the germ of vector field $\mathbf{v}$ is *nondicritic* if

$$R_{n+1} \not\equiv 0 \tag{2.2}$$

and if $R_{m+1} \equiv 0$ we say that the germ of vector field $\mathbf{v}$ is *dicritic*.

Remark 2.1. The condition of nondicriticity is generic in $\mathcal{V}_n$ (it is given by the open condition (2.2)) and has finite codimension in the space $\mathcal{V}_n$. On the contrary the dicritic case is nongeneric in $\mathcal{V}_n$. In this work, unless otherwise stated, one will assume the nondicriticity condition (2.2)

2.2. **Main statements and genericity assumptions.** We state the main results of this work. We begin with the genericity assumptions for the first two theorems:

We say that a holomorphic nondicritic germ of vector field $\mathbf{v} \in \mathcal{V}_n$ of the form (2.1) is *generic nondicritic* if it satisfies the following *genericity assumptions*:

- G1. The homogeneous polynomial $R_{n+1} = xQ_n - yP_n$ is of degree $n+1$ and has only simple factors,
- G2. All the characteristic exponents at the singular points of the blown-up foliations are not zero or positive rational.
- G3. At least at one singular point denoted by p_∞ the blown-up foliation $\tilde{\mathcal{F}}_{\mathbf{v}}$ is generated by a non degenerated vector field holomorphically linearizable and its characteristic exponent λ_∞ is different from -1. This implies that in appropriate coordinates the foliation $\tilde{\mathcal{F}}_{\mathbf{v}}$ at the point p_∞ is locally generated by a linear vector field and the quotient of the corresponding eigenvalues is different from -1.

The main goal of this work is to prove under the genericity assumptions G1,G2,G3 the following theorems:

Theorem 2.1. *(Semi polynomial analytic normal form) Each generic nondicritic germ in $\mathcal{V}_n$, $n \geq 2$, is analytically orbitally equivalent to a germ of vector field of the form*

$$\mathbf{v}_{\mathcal{P},\mathcal{Q}}(x, y) = \quad x\mathcal{P}(x, y)\tfrac{\partial}{\partial x} + y\mathcal{Q}(x, y)\tfrac{\partial}{\partial y}, \tag{2.3}$$

with nondicritic singularity at the origin and $\mathcal{P}, \mathcal{Q}$ polynomials of degree at most $n-1$ in the "y" variable with analytic (on x) coefficients.

Theorem 2.2. *(Semipolynomial analytic normal form for $n = 2$) Any generic nondicritic germ of $\mathcal{V}_2$, is analytically orbitally equivalent to a germ of vector field of the form*

$$\mathbf{v}_{an} = x(\mathcal{P}_1 + x^2\beta(x))\frac{\partial}{\partial x} + y(\mathcal{Q}_1 + x^2\beta(x))\frac{\partial}{\partial y} \tag{2.4}$$

where $\mathcal{P}_1(x, y) = ya_0 + b_1x$, $\mathcal{Q}_1(x, y) = y + d_1x$ are homogeneous polynomials of degree 1, and $\beta(x)$ is a holomorphic function in a neighborhood of the origin. For fixed principal part $x\mathcal{P}_1\frac{\partial}{\partial x} + y\mathcal{Q}_1\frac{\partial}{\partial y}$, the function β is unique (and therefore $\mathbf{v}_{an}$) under strict analytic orbital equivalence.

We stress that any nondicritic generic germ $\mathbf{v} \in \mathcal{V}_n$ can be reduced under, rotation and rectification of one of its separatrix, to a germ

$$P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y}\ , \qquad P(0, y) \equiv 0 \tag{2.5}$$

Denote by $\mathcal{V}_n^0$ the class in $\mathcal{V}_n$ of germs satisfying (2.5). Hence, the problem of classification of generic foliations generated by germs in $\mathcal{V}_n$ is reduced to the equivalent one of the classification of generic foliations generated by germs in $\mathcal{V}_n^0$.

We stress that **strict** formal (and analytic) orbital equivalent germs in $\mathcal{V}_n^0$ have the same n-jet at the origin. Therefore the problem of strict formal (and analytic) orbital classification of germs in $\mathcal{V}_n^0$ is transformed to the analogous one in each class

$$\mathcal{V}(\mathbf{v}_0) = \left\{ \mathbf{v} \in \mathcal{V}_n^0 \ : \ j_0^n(\mathbf{v} - \mathbf{v}_0) = 0 \right\}, \tag{2.6}$$

where $\mathbf{v}_0 := P_n \frac{\partial}{\partial x} + Q_n \frac{\partial}{\partial y}$ is called *the principal part* of $\mathbf{v}$ and P_n, Q_n are homogeneous polynomials of degree n, $P_n(0, y) \equiv 0$. Note that in this case the blow-up $\tilde{\mathbf{v}}$ of $\mathbf{v}$ has a singular point p_∞ at infinity, i.e., at $v = 0, y = 0$, where $v = y/x$.

For generic germs (see Remark 2.2) the solutions to the formal orbital classification problem was given in [ORV4]:

Theorem *(on the formal classification of nondicritic vector fields* [ORV4]*) Each generic holomorphic nondicritic germ $\mathbf{v} \in \mathcal{V}_n$, $n > 1$ is formally orbitally equivalent to a formal series $\mathbf{v}_{c,b}$ of the form*

$$\mathbf{v}_{c,b} = \mathbf{v}_0 + \mathbf{v}_c + \mathbf{v}_b, \tag{2.7}$$

where

(1) $\mathbf{v}_0 := P_n \frac{\partial}{\partial x} + Q_n \frac{\partial}{\partial y}$, *$P_n, Q_n$ are homogeneous polynomials of degree n, and $P_n(0, y) \equiv 0$ is a generic principal part.*

(2) $\mathbf{v}_c = -(\mathcal{H}_c)'_y \frac{\partial}{\partial x} + (\mathcal{H}_c)'_x \frac{\partial}{\partial y}$ *is a Hamiltonian vector field with polynomial Hamiltonian*

$$\mathcal{H}_c(x, y) = \sum c_{i,j} x^i y^j, \ 0 \le i \le n-1, \ 0 \le j \le n-1, \ i + j \ge n+2, \tag{2.8}$$

(3) *and $\mathbf{v}_b = b(x, y)(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y})$ is a radial vector field such that $b(x, y) = \sum_{k=0}^{n-2} b_k(x) y^k x^{n-k}$ is a polynomial on the y variable of degree less or equal to $n-2$ whose coefficients, $b_k(x)$, are formal series on x.*

Moreover any two formal series of the form (2.7) that are formally orbitally equivalent to $\mathbf{v}$ and with the same generic principal part $\mathbf{v}_0$, coincide.

Remark 2.2. The genericity assumptions in this theorem are slightly different:

- $\tilde{G}1$. We ask the principal part $\mathbf{v}_0$ to be such that its blow-up has simple singular points (i.e., the homogeneous polynomial $R_{n+1}(x, y) = x\tilde{R}(x, y)$ of degree $n + 1$ has only simple factors, and therefore, in this case, $R_{n+1}(1, u)$ has n simple roots u_j, $j = 1, \dots, n$, the point at infinity p_∞ is also simple)
- $\tilde{G}2$. All the characteristic exponents corresponding to the singular points are not rational numbers.
- $\tilde{G}3$. Within the proof of Theorem 2.1 we ask that for any $k = 2, \dots, n + 1$, a determinant of $2k + 2$ equations to be different from zero (this determinant is a non trivial polynomial on the coefficients of the principal part $\mathbf{v}_0$).

We stress the relevance of $\mathbf{v}_c$ in (2.7): For $\mathbf{v} \in \mathcal{V}(\mathbf{v}_0)$ satisfying the previous genericity assumptions and having nonsolvable projective monodromy group $G_\mathbf{v}$, the tuple $\tau_\mathbf{v} = (\mathbf{v}_c, [G_\mathbf{v}])$ is Thom's invariant on the analytic classification under strict orbital equivalence, where $[G_\mathbf{v}]$ is the class of strict analytic conjugacy of the projective monodromy group $G_\mathbf{v}$ (see [ORV4]).

Remark 2.3. For $n = 2$ the "Hamiltonian" part $\mathbf{v}_c$ in (2.7) is zero. Hence, the strict formal orbital normal form $\mathbf{v}_f := \mathbf{v}_{c,b}$ takes the form:

$$\mathbf{v}_f = (P_2 + x^3 B)\frac{\partial}{\partial x} + (Q_2 + yx^2 B)\frac{\partial}{\partial y} \tag{2.9}$$

where $\mathbf{v}_0 = P_2 \frac{\partial}{\partial x} + Q_2 \frac{\partial}{\partial y}$, P_2, Q_2 are homogeneous polynomials of degree 2, $P_2(0, y) = 0$, $deg_y Q_2 = 2$, and $B(x) = \sum_{k=0}^{\infty} b_k(x)\, x^k$ is a formal power series.

Therefore $\mathbf{v}_f$ in (2.9) is the strict orbital formal normal form for generic nondicritic vector fields in $\mathcal{V}(\mathbf{v}_0)$. As we state in the next theorem (2.9) is, as well, the orbital strict analytic normal form for generic nondicritic vector fields in $\mathcal{V}(\mathbf{v}_0)$.

As an immediate consequence of Theorem 2.2 and considering the generic assumptions G1, $\tilde{\text{G}}1$, G3, and $\tilde{\text{G}}3$, we have the following:

Theorem 2.3. *(Analyticity of the formal normal form for $n = 2$, $\mathbf{v}_f$) For any generic nondicritic germ in $\mathcal{V}_2$, its strict formal orbital normal form $\mathbf{v}_f$ is analytic. Moreover for fixed $\mathbf{v}_0$ the normal form is unique under strict equivalence.*

2.3. **Structure of the work and acknowledgements.** We begin by giving some properties of the foliation generated by the blow-up of a nondicritic germ satisfying the genericity assumptions needed in the proof of Theorem 2.1. In the section 4 we give a sketch of the proof of Theorem 2.1. In section 5 we give an appropriate extension of $\mathbf{v}$, define an auxiliary foliation, suitable biholomorphisms and domains of definition that allow one to use Savelev's Theorem. Further, we analyze the Savelev's diffeomorphism and apply Weierstrass Preparation Theorem. The end of the proof is given in 5.8. On section 6 we prove Theorem 2.2 and as a consequence of it we get Theorem 2.3.

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3. General properties of nondicritic foliations and prenormalized form.

Following [ORV2], we give in this section a geometric description of the nondicritic foliations as well as their simplest properties.

Let $\mathbf{v}$ be a nondicritic germ in $\mathcal{V}_n$. For any $n > 1$ the singular the linear part of $\mathbf{v}$ at the singular point $0 \in \mathbb{C}^2$ is zero; in 3.1 and 3.2 we introduce its blow-up:

3.1. **Blow-up $\mathcal{B}$ of $(\mathbb{C}^2, 0)$.** We recall that the blow-up of a point $0 \in \mathbb{C}^2$ is the 2-dimensional complex manifold $\mathcal{B}$ obtained from the gluing of two copies of $\mathbb{C}^2$ with coordinates (called standard charts) (x, u) and (y, v) by means of the map $\phi : (x, u) \mapsto (y, v) = (xu, u^{-1})$.

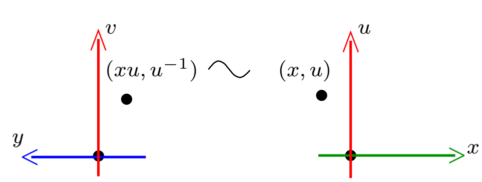

Figure 3.1. Blow-up of $(\mathbb{C}^2, 0)$: $\mathcal{B} = \mathbb{C}^2 \coprod \mathbb{C}^2 / {}_{(y,v)\sim(xu,u^{-1})}$.

The projection $\pi : \mathcal{B} \to (\mathbb{C}^2, 0)$, given in the standard charts by $\pi : (x, u) \mapsto (x, xu)$, $\pi : (y, v) \mapsto (yv, y)$, will be called standard projection as well. The sphere $\mathcal{L} := \pi^{-1}(0) \approx \mathbb{CP}^1$ obtained from the gluing of the regions $\{0\} \times \mathbb{C}$ and $\mathbb{C} \times \{0\}$ by means of $\phi|_{\{0\}\times\mathbb{C}^*}$ will be called the pasted sphere (or the exceptional divisor of the blow-up). The map π is holomorphic and its restriction $\pi|_{\mathcal{B}\setminus\mathcal{L}}$ to the set $\mathcal{B}\setminus\mathcal{L}$ is a biholomorphism whose inverse is denoted by σ and it is: $\sigma := (\pi|_{\mathcal{B}\setminus\mathcal{L}})^{-1}$.

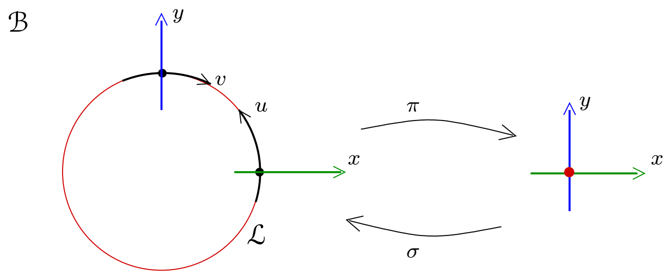

FIGURE 3.2. Blow-up of $(\mathbb{C}^2, 0)$

3.2. **Blow-up of germs of vector fields in $\mathcal{V}_n$.** As it is known, the lifting $\sigma_*\mathbf{v}$ of a germ of vector field $\mathbf{v}$ in $\mathcal{V}_n$ generates, in a neighborhood of the pasted sphere without $\mathcal{L}$, a foliation which can be uniquely extended to $\mathcal{L}$, as a holomorphic foliation $\tilde{\mathcal{F}}_{\mathbf{v}}$ called the blow-up of $\mathcal{F}_{\mathbf{v}}$ at zero (with a finite number of singularities on $\mathcal{L}$, generally speaking). We denote by $\tilde{\mathbf{v}}$ the line field which generates the foliation $\tilde{\mathcal{F}}_{\mathbf{v}}$. We call $\tilde{\mathbf{v}}$ the blow-up of $\mathbf{v}$.

Let $\mathbf{v}$ be a nondicritic germ in $\mathcal{V}_n$. In (x, y)-coordinates, $\mathbf{v}$ has the form (2.1) and the blow-up $\tilde{\mathcal{F}}_{\mathbf{v}}$ of $\mathcal{F}_{\mathbf{v}}$ is given locally, in the standard charts, by the equations

$$\frac{du}{dx} = \frac{xQ(x,ux) - uxP(x,ux)}{x^2P(x,ux)};\qquad \frac{dv}{dy} = \frac{yP(vy,y) - vyQ(vy,y)}{y^2Q(vy,y)}. \tag{3.1}$$

Let $R_{m+1}(x, y) = xQ_m - yP_m, m = n, n+1, \dots$. The condition of nondicriticity $R := R_{n+1} \neq 0$ implies that the blow-up $\tilde{\mathcal{F}}_{\mathbf{v}}$, on the region of definition of the standard chart (x, u), is generated by the vector field $\tilde{\mathbf{v}}_+(x, u) = \tilde{P}_+(x, u)\frac{\partial}{\partial x} + \tilde{Q}_+(x, u)\frac{\partial}{\partial u}$, where

$$\tilde{P}_+(x, u) = x[P_n(1, u) + O(x)], \quad \tilde{Q}_+(x, u) = R_{n+1}(1, u) + O(x), \quad \text{for } x \to 0 . \tag{3.2}$$

In the same way, on the region of definition of the standard chart (y, v), the foliation $\tilde{\mathcal{F}}_{\mathbf{v}}$ is generated by the vector field $\tilde{\mathbf{v}}_-(y, v) = \tilde{P}_-(y, v)\frac{\partial}{\partial y} + \tilde{Q}_-(y, v)\frac{\partial}{\partial v}$ where

$$\tilde{P}_-(y, v) = y[Q_n(v, 1) + O(y)] , \quad \tilde{Q}_-(y, v) = R_{n+1}(v, 1) + O(y) , \quad \text{for } y \to 0 . \tag{3.3}$$

3.3. **Properties of generic germs (Consequences of the genericity assumptions G1, G2, G3).** For any generic nondicritic germ $\mathbf{v} \in \mathcal{V}_n$, the following statements take place:

(1) The germ $\mathbf{v}$ has exactly $n+1$ different separatrices, which are smooth at the origin and have pairwise transversal intersection.
(2) A resolution (see [C-S]) of a generic nondicritic germ $\mathbf{v}$ in $\mathcal{V}_n$ consists exactly of one blow-up.
(3) The corresponding blown-up foliation $\tilde{\mathcal{F}}_{\mathbf{v}}$ has exactly $n+1$ singular points $p_1, \dots, p_{n+1}$ on the divisor $\mathcal{L}_{\mathbf{v}} \sim \mathbb{CP}^1$ and the characteristic exponents $\lambda_1, \dots , \lambda_{n+1}$ are neither zero nor rational positive numbers.

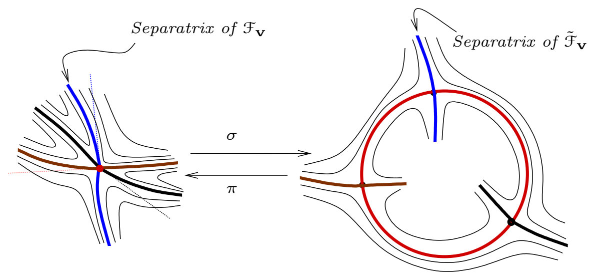

FIGURE 3.3. Phase portrait of a generic nondicritic germ of vector field $\mathbf{v} \in \mathcal{V}_2$ and its blow-up.

(4) **Prenormalized form.** Without loss of generality we will assume that the singular point p_{n+1} is the point at infinity ($p_{n+1} = p_\infty = 0$ in the standard chart (y, v) and thus p_∞ is a nondegenerated singular point of the vector field $\tilde{\mathbf{v}}_-$), and denote by λ_∞ the Camacho-Sad-index with respect to the divisor $\mathcal{L}$. Moreover we assume that the y-axis ($v = 0$) is the separatrix at p_∞ . We stress that such assumptions can be achieved by performing suitable (analytic) change of coordinates. An additional (analytic) change of coordinates allows one to have the x-axis ($y = 0$) as separatrix at the origin (as well as in the (x, u) coordinates).

Hence, we assume in what follows that the vector field $\mathbf{v}$ is written in its *prenormalized form*:

$$\mathbf{v}(x, y) = x\hat{P}(x, y)\frac{\partial}{\partial x} + y\hat{Q}(x, y)\frac{\partial}{\partial y} \ , \tag{3.4}$$

where $\hat{P}(x, y), \hat{Q}(x, y)$ are analytic germs at the origin of order $n-1$, $\hat{P} = \sum_{m=n-1}^{\infty} \hat{P}_m$, $\hat{Q} = \sum_{m=n-1}^{\infty} \hat{Q}_m$, where $\hat{P}_m$, $\hat{Q}_m$ are homogeneous polynomials of degree m.

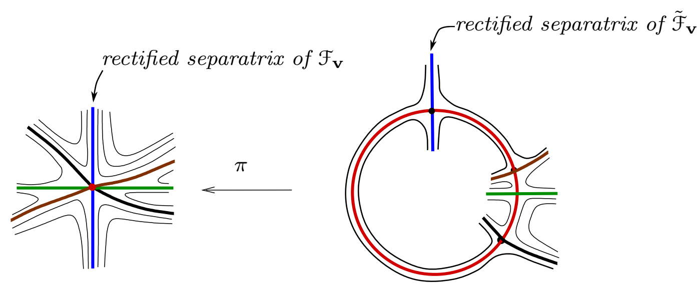

FIGURE 3.4. Phase portrait of a prenormalized nondicritic germ of vector field $\mathbf{v} \in \mathcal{V}_2$ and its blow up.

(5) We stress that the subset $\mathcal{L}_{\mathbf{v}} \setminus \{p_1, ..., p_n, p_\infty\}$ is a leaf of the blown-up foliation $\tilde{\mathcal{F}}_{\mathbf{v}}$. Moreover the polynomial $r(u) = R_{n+1}(1, u)$ has exactly n simple roots; we denote them by $u_1, u_2, ..., u_n$, $r'(u_j) \neq 0$, and their corresponding characteristic exponents which coincide in this case with the Camacho-Sad's index of the foliation $\tilde{\mathcal{F}}_{\mathbf{v}}$ in the singular points with respect to the divisor $\mathcal{L}_{\mathbf{v}}$, λ_j, $\lambda_j = \frac{P_n(1,u_j)}{\frac{\partial R}{\partial y}(1,u_j)} = \frac{p(u_j)}{r'(u_j)}, j = 1, ..., n$, and $\lambda_\infty = -\frac{Q_n(0,1)}{\frac{\partial R}{\partial x}(0,1)}$.

(6) Note that the germ of vector field $\mathbf{v}$ which generates the foliation $\mathcal{F}_{\mathbf{v}}$ has Camacho-Sad's index at the origin with respect to the separatrix $\{x = 0\}$ equal to $1 + \lambda$, where $\lambda = 1/\lambda_\infty$. By the genericity assumption G3 (given in section 2.2) this index is not zero.

4. Sketch of the proof of Theorem 2.1.

Without loss of generality let $\mathbf{v}$ be a generic germ in $\mathcal{V}_n$ written in its prenormalized form (3.3). There exists a cone $\mathcal{C}_{\epsilon_0}$,

$$\mathcal{C}_{\epsilon_0} := \left\{ (x, y) \in \mathbb{C}^2 \ : \ \frac{1}{\epsilon_0}|x| \le |y| \le \epsilon_0 \right\},$$

around the separatrix $\{x = 0\}$ such that, in the blow-up coordinates $(v, y) = (\frac{x}{y}, y)$ the neighborhood $\mathcal{C}_{\epsilon_0}$ takes the form

$$D_{\epsilon_0} \times D_{\epsilon_0} = \{(v, y) \ : \ |v| \le \epsilon_0, \ |y| \le \epsilon_0\}\,.$$

$D_{\epsilon_0} \times D_{\epsilon_0}$ is a neighborhood of the point p_∞ (the origin in the coordinates (v, y)). By the genericity assumptions the blow-up $\tilde{\mathcal{F}}_{\mathbf{v}}$ of $\mathcal{F}_{\mathbf{v}}$ (in the coordinates (v, y)) is locally generated (in a neighborhood of the singular point p_∞) by a linearizable nondegenerated vector field (see generic condition G3.). Hence, for ϵ_0 small enough there exists a biholomorphism G,

$$G : D_{\epsilon_0} \times D_{\epsilon_0} \to (\mathbb{C}^2, 0) \tag{4.1}$$

(preserving the y coordinate) linearizing $\tilde{\mathcal{F}}_{\mathbf{v}}$.

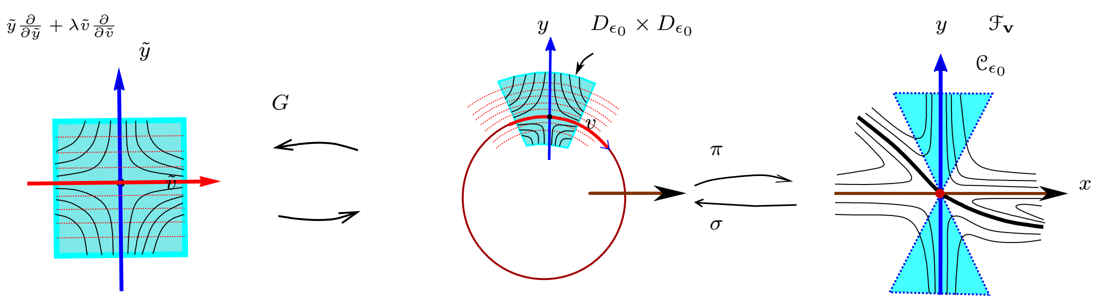

Figure 4.1. Linearizing the foliation $\mathcal{F}_{\mathbf{v}}$ generated by the vector field $\mathbf{v}$ within a conus $\mathcal{C}_{\epsilon_0}$

Let us denote by $\mathbf{v}_\lambda$ the linear vector field such that $\mathbf{v}_\lambda = G_*\tilde{\mathbf{v}}_-$: In the charts $(\tilde{v}, \tilde{y}) := G(v, y)$ the foliation $\tilde{\mathcal{F}}_{\mathbf{v}}$ is thus generated by the vector field:

$$\mathbf{v}_\lambda = \lambda\tilde{v}\frac{\partial}{\partial \tilde{v}} + \tilde{y}\frac{\partial}{\partial \tilde{y}} \tag{4.2}$$

where $\tilde{y} = y$ and $\lambda = \frac{1}{\lambda_\infty}$ is the Camacho-Sad' index of $\mathcal{F}_{\mathbf{v}_\lambda}$ at $(0, 0)$ related to the separatrix $\{v = 0\}$, and λ_∞ is the Camacho-Sad' index of $\mathcal{F}_{\mathbf{v}_\lambda}$ at $(0, 0)$ corresponding to the separatrix $\{y = 0\}$ (the divisor $\mathcal{L}$).

As the vector field $\mathbf{v}_\lambda$ is a linear one, it may be extended to the whole complex manifold $\mathcal{M}$

$$\mathcal{M} := (\mathbb{C} \times D_\epsilon) \sqcup (\mathbb{C} \times D_\epsilon)/_{(\tilde{v},\tilde{y})\sim(\xi=\tilde{v}\tilde{y}\ ,\ \eta=\frac{1}{\tilde{y}}),\ \tilde{y}\neq 0},$$

where $G^{-1}(D_\epsilon \times D_\epsilon) \subset D_{\epsilon_0} \times D_{\epsilon_0}$ for ϵ small enough.

Let $\mathcal{M}_+ := \{(\tilde{v}, \tilde{y}) \in D_\epsilon \times \mathbb{C}\}$, $\quad \mathcal{M}_- := \{(\xi, \eta) \in D_\epsilon \times \mathbb{C}\}$.

On $\mathcal{M}$ the vector field $\mathbf{v}_\lambda$ is defined in (4.2) and straight-forward calculations show that $\mathbf{v}_\lambda$ in $\mathcal{M}_-$ is written as

$$\mathbf{v}_{\lambda+1}(\xi,\eta) = (\lambda+1)\xi\frac{\partial}{\partial\xi} - \eta\frac{\partial}{\partial\eta} \tag{4.3}$$

Therefore there is a foliation on $\mathcal{M}$ defined by the extension of $\mathbf{v}_\lambda$, having no more than two singular points: the $(0,0)$ in coordinates $(\tilde{v},\tilde{y})$ and the $(0,0)$ in the coordinates (ξ,η) . We stress that Camacho-Sad's index at the origin with respect to the y axis is λ, and the respective index at the origin in the charts (ξ,η), $\eta=\frac{1}{y}$ is $-(\lambda+1)$. This means that the self-intersection index of the closure $\{y=0\}$ in $\mathcal{M}$ is -1. Hence, $\mathcal{M}$ is the blow-up of a neighborhood of $(\tilde{v},\tilde{y})=(0,0)$.

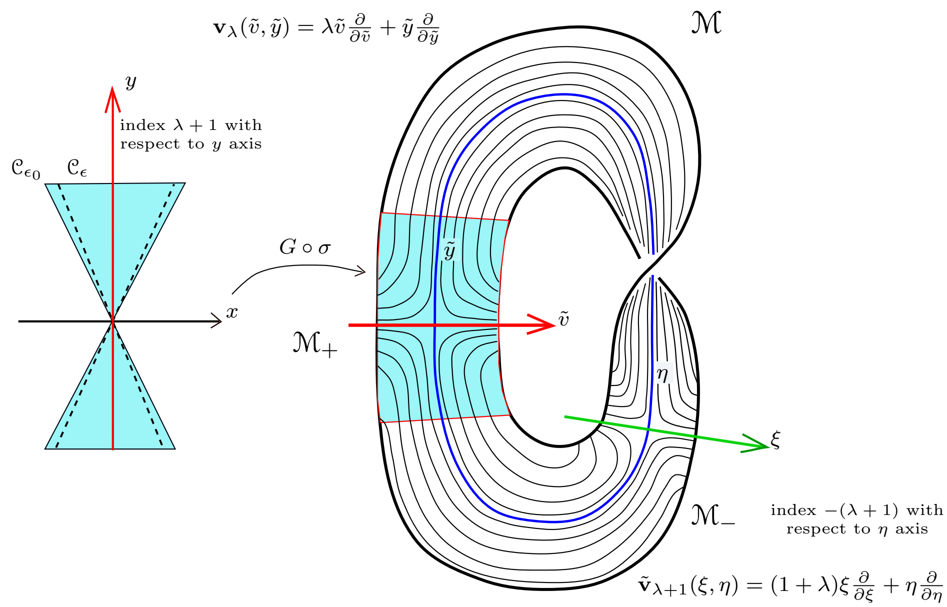

FIGURE 4.2. Extension of the vector field $\mathbf{v}_\lambda$ to $\mathcal{M}$.

We return to the (x,y) coordinates:

Remark that the foliation generated by the vector field $\mathbf{v}$ has Camacho-Sad's index $\lambda+1$ with respect to the y axis. This follows from the correspondence of $\mathcal{F}_\mathbf{v}$ with $\mathcal{F}_{\mathbf{v}_\lambda}$ by means of $G\circ\sigma$.

The next goal is to construct an extension of $\mathcal{F}_\mathbf{v}$. For this purpose we use the vector field $\mathbf{v}_\lambda$ (see (4.2)) and the following construction:

We define, in a neighborhood of the origin in the $(\tilde{v},\tilde{y})$ coordinates, an annulus $\mathcal{A}_\mu\subset\mathcal{M}$,

$$\mathcal{A}_\mu := D_\epsilon\times D_\epsilon\setminus D_\epsilon\times D_{\epsilon'},\quad \epsilon'<\epsilon$$

Let $\mathcal{A}$ be the annulus like domain which is the preimage of $\mathcal{A}_\mu$ under $G\circ\sigma$:

$$\mathcal{A} := (G\circ\sigma)^{-1}(\mathcal{A}_\mu),\quad \mathcal{A}\subset\mathcal{C}_\epsilon\ .$$

We stress that $\mathcal{A}_\mu\subset\mathcal{M}_+\cap\mathcal{M}_-$. Hence by means of $(G\circ\sigma)^{-1}$ we may construct a new manifold by identifying the neighborhood $\mathcal{U}_+$ of the origin in the coordinates (x,y), $\mathcal{A}\subset\mathcal{C}_\epsilon\subset\mathcal{U}_+$, with the open domain $\mathcal{U}_- = D_\epsilon \times D_{\frac{1}{\epsilon}}$ (in the charts (ξ, η)). Namely, we define Φ (see fig 4.3) as the composition

$$\Phi := \beta \circ G \circ \sigma : \mathcal{A} \to \beta(\mathcal{A}_\mu) \subset \mathcal{U}_- \quad , \quad \text{where} \quad \beta(\tilde{v}, \tilde{y}) = (\xi = \tilde{v}\tilde{y}, \eta = 1/\tilde{y}). \tag{4.4}$$

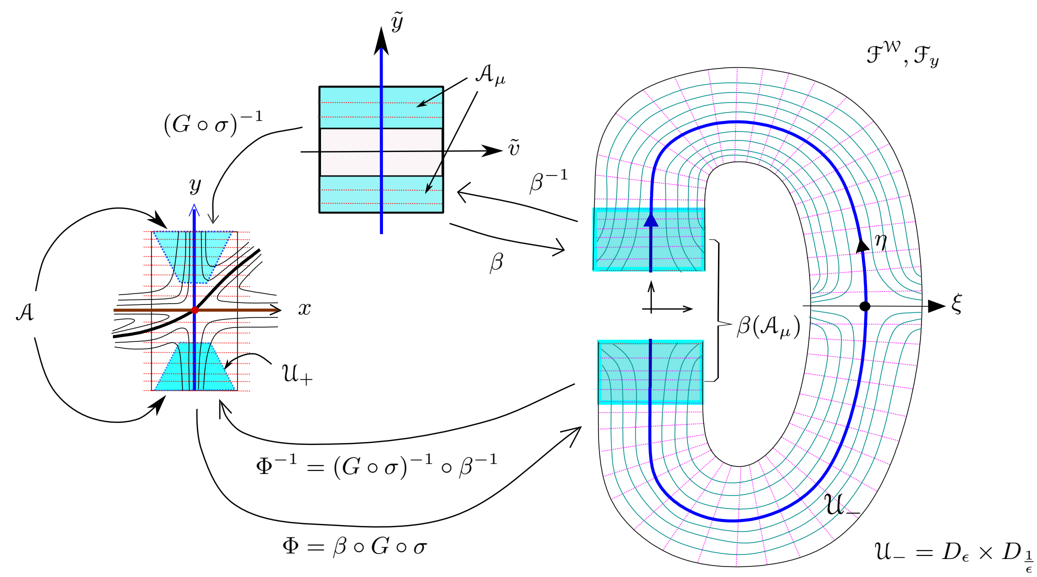

FIGURE 4.3. Extension of the foliation $\mathcal{F}_\mathbf{v}$ to the complex surface $\mathcal{W}$ and foliations $\mathcal{F}^{\mathcal{W}}$ and $\mathcal{F}_y$.

We denote by $\mathcal{W}$ the 2-dimensional complex manifold obtained from the domains $\mathcal{U}_+$ and $\mathcal{U}_-$ and the transition maps $\Phi|_{\mathcal{A}}$, $\Phi^{-1}|_{\Phi(\mathcal{A})}$, $\mathcal{A} \subset \mathcal{U}_+$, $\Phi(\mathcal{A}) \subset \mathcal{U}_-$.

The foliation $\mathcal{F}_\mathbf{v}$ and $\mathcal{F}_{\mathbf{v}_{\lambda+1}}$ defined by the vector fields $\mathbf{v}$ and $\mathbf{v}_{\lambda+1}$ in $\mathcal{U}_+$ and $\mathcal{U}_-$ respectively, define a global foliation $\mathcal{F}^{\mathcal{W}}$ on $\mathcal{W}$ with exactly two singular points O_+ and O_-: the corresponding singular point of $\mathbf{v}$ and $\mathbf{v}_{\lambda+1}$. Remark that the disk $\{0\} \times D_\epsilon \subset \mathcal{U}_+$ is in correspondence under Φ with the disk $\{0\} \times D_{\frac{1}{\epsilon}} \subset \mathcal{U}_-$ $(\eta = \frac{1}{\tilde{y}})$. This defines a Riemann sphere $\mathcal{L}^{\mathcal{W}}$.

Together with the foliation $\mathcal{F}^{\mathcal{W}}$ we consider the foliation $\mathcal{F}_y$ defined by

$$\{y = cst\} \ \ (\eta = cst \in \mathcal{U}_-).$$

This foliation defines a line bundle (the normal bundle over $\mathcal{L}^{\mathcal{W}}$ on $\mathcal{W}$). To know how $\mathcal{L}^{\mathcal{W}}$ is embedded in $\mathcal{W}$ it is sufficient to calculate Camacho-Sad's index for $\mathbf{v}$ at $(0,0)$ with respect to the separatrix $\mathcal{L}^{\mathcal{W}}$. Namely, Camacho-Sad's index of $\mathbf{v}_{\lambda+1}$ at $(0,0)$ with respect to the separatrix $\{x = 0\}$ is $\lambda + 1$. Hence, by Camacho-Sad's Theorem, the self-intersection index $\mathcal{L}^{\mathcal{W}} \cdot \mathcal{L}^{\mathcal{W}} = (\lambda + 1) - (\lambda + 1) = 0$.

By Savelev's Theorem [Sa], there exists a biholomorphism Ψ of a neighborhood of $\mathcal{L}^{\mathcal{W}}$ (in $\mathcal{W}$) to the direct product $(\mathbb{C}, 0) \times \mathbb{CP}^1$ such that $\Psi(\mathcal{L}^{\mathcal{W}}) = \{0\} \times \mathbb{CP}^1$. By reducing, if necessary, the domain of definition in our construction we may assume that Ψ is defined in the whole $\mathcal{W}$, $\Psi : \mathcal{W} \to (\mathbb{C}, 0) \times \mathbb{CP}^1$.

We denote by $\mathcal{F}$ the foliation induced by $\mathcal{F}^{\mathcal{W}}$ under the transformation Ψ, $\mathcal{F} := \Psi(\mathcal{F}^{\mathcal{W}})$. The foliation $\mathcal{F}$ is defined at the direct product $(\mathbb{C}, 0) \times \mathbb{CP}^1$ having singular points $\Psi(O_+)$ and $\Psi(O_-)$. In a neighborhood of $\Psi(O_+)$ the foliation $\mathcal{F}$ is generated by the vector field

$$\mathbf{v}_+ := (\Psi \circ t_+^{-1})_* \mathbf{v}$$

and in a neighborhood of $\Psi(O_-)$ the foliation $\mathcal{F}$ is generated by the vector field

$$\mathbf{v}_- := (\Psi \circ t_-^{-1})_* \mathbf{v}$$

where t_+ and t_- are the natural charts in $\mathcal{W}$ corresponding to the domains $\mathcal{U}_+$ and $\mathcal{U}_-$ respectively ($t_+ : t_+^{-1}(\mathcal{U}_+) \to \mathcal{U}_+$, $t_- : t_-^{-1}(\mathcal{U}_-) \to \mathcal{U}_-$).

Let $G : D_{\epsilon_0} \times D_{\epsilon_0} \to (\mathbb{C}^2, 0)$ be the linearizing biholomorphism defined in the beginning of this section. Then, as we will see in 5.8, Theorem 2.1 is a consequence of the following proposition:

Proposition 4.1. *The linearizing biholomorphism $G : D_{\epsilon_0} \times D_{\epsilon_0} \to (\mathbb{C}^2, 0)$, the coordinate system in $(\mathbb{C}, 0) \times \mathbb{CP}^1$ and the domains used in the construction described along this section may be chosen in such way that $\mathbf{v}_+$ ($\mathbf{v}_-$) is orbitally analytically equivalent to a holomorphic vector field, polynomial with respect to the y variable.*

5. Proof of Proposition 4.1

The proof of the Proposition 4.1 is quite long since we require to give explicit biholomorphisms and domains.

The first step is to show that the linearizing biholomorphism G (linearizing $\tilde{\mathbf{v}}_-$ in $\sigma(\mathcal{C}_\epsilon)$) may be chosen, without loss of generality, as the identity in the y variable.

5.1. **Normalization of the biholomorphism G.** Let G be the biholomorphism at the beginning of section 4. G transforms the leaves of the foliation $\mathcal{F}_{\tilde{\mathbf{v}}}$ into the leaves of the foliation $\mathcal{F}_{\mathbf{v}_\lambda}$ ($\mathcal{F}_{\mathbf{v}_\lambda}$ is the foliation generated by the vector field $\mathbf{v}_\lambda$ -see (4.1)-).

As we wish to have a correspondence between the separatrices $\{v = 0\}$ and $\{y = 0\}$ (of the vector field $\tilde{\mathbf{v}}$), and the separatrices $\{\tilde{v} = 0\}$ and $\{\tilde{y} = 0\}$ of the linear vector field $\mathbf{v}_\lambda$, the biholomorphism G must be written as

$$G(v, y) = (vG_1(v, y), yG_2(v, y)),$$

with $G_j(0, 0) \neq 0, j = 1, 2$.

We stress that the phase curves (cy^λ, y) corresponding to the vector field $\mathbf{v}_\lambda$ are invariant under transformations of the form $\Phi_k(v, y) = (v\, k^\lambda, y\, k)$, $k(0, 0) \neq 0$. For this reason (by performing, if needed, the composition $\Phi_k \circ G$ for an appropriate k) we may assume that the map G has the form

$$G(v, y) = (vg(v, y), y), g(0, 0) = 1. \tag{5.1}$$

To give an explicit expression of the function g we observe that, in a neighborhood of the origin in the coordinates (v, y), the foliation $\mathcal{F}_{\tilde{\mathbf{v}}}$ is defined by the integral curves of the equation:

$$\frac{dv}{dy} = \left.\frac{yP(x, y) - xQ(x, y)}{y^2 Q(x, y)}\right|_{x=vy} \quad ;$$

equivalently, $\tilde{\mathcal{F}}_{\mathbf{v}}$ is defined by the vector field

$$\tilde{\mathbf{v}}_- = C(v, y)\frac{\partial}{\partial v} + y\frac{\partial}{\partial y} \quad ,$$

where

$$C(v, y) = \left.\frac{yP(x, y) - xQ(x, y)}{y}\right|_{x=vy} \tag{5.2}$$

$$\frac{\partial C(v, y)}{\partial v}(0, 0) = \lambda.$$

Recalling that the biholomorphism G satisfies

$$G_* \tilde{\mathbf{v}}_- = \mathbf{v}_\lambda \circ G$$

it follows that

$$vyC(v,y)\,\frac{\partial(vg(v,y))}{\partial v} + vy\frac{\partial g}{\partial y}(v,y) = \lambda vg(vy,y) \tag{5.3}$$

As $\{x=0\}$ is a separatrix of the vector field $\mathbf{v}$, then $P(x,y) = x\hat{P}(x,y)$ and so

$$C(v,y) = v\,c(y) + O(v^2), \quad v \to 0,$$

where

$$c(y) = \left.\frac{y\hat{P}(vy,y) - Q(vy,y)}{Q(vy,y)}\right|_{v=0} \quad , \; c(0) = \lambda\ . \tag{5.4}$$

Hence, from (5.3) we get that, for $v = 0$,

$$\frac{g'_y(0,y)}{g(0,y)} = \frac{-c(y)+\lambda}{y} \quad . \tag{5.5}$$

Thus, $g(0,y)$ is a holomorphic function in a neighborhood of $y = 0$,

$$g(0,y) = exp\left(\int \frac{-c(y)+\lambda}{y}dy\right) \quad , \quad g(0,0) = 1$$

5.2. **Gluing biholomorphism $G\circ\sigma$.** After the rectification of the biholomorphism G introduced in section 4, the composition $G \circ \sigma$ that relates the vector fields $\mathbf{v}$ and $\mathbf{v}_\lambda$ is expressed in terms of the holomorphic function g (in the coordinate charts $(\tilde{v},\tilde{y})$ on $\mathcal{M}$) as

$$G\circ\sigma : (x,y) \mapsto (\tilde{v},\tilde{y}) = (\frac{x}{y}g(x/y,y),y).$$

Recall the change of coordinates β introduced in (4.4), $\beta(\tilde{v},\tilde{y}) = (\tilde{v}\tilde{y}, 1/\tilde{y}) = (\xi,\eta)$.

The composition $\Phi = \beta\circ G\circ\sigma$ is expressed in terms of (x,y) as

$$\Phi(x,y) = (\beta\circ G\circ\sigma)(x,y) = (xg(x/y,y), 1/y) \in \mathcal{U}_- \quad . \tag{5.6}$$

Hence,

$$\Phi_*\mathbf{v}(\xi,\eta) = (\beta\circ G\circ\sigma)_*\mathbf{v}(\xi,\eta) = (1+\lambda)\xi\frac{\partial}{\partial\xi} - \eta\frac{\partial}{\partial\eta} \quad .$$

Moreover, if we define the map $\alpha : D_\epsilon \times D_{\frac{1}{\epsilon}} \to D_\epsilon \times (\mathbb{CP}^1,\infty)$, as $\alpha(\xi,\eta) = (\xi, \frac{1}{\eta}x) = (\xi, y)$, then the composition $\alpha\circ\Phi$ is expressed on (x,y) as

$$(\alpha\circ\Phi)_{(x,y)} = (xg(x/y,y),y) \quad . \tag{5.7}$$

Thus

$$((\alpha\circ\Phi)_*\mathbf{v})\,(\xi,\eta) = (1+\lambda)\xi\partial\xi + y\frac{\partial}{\partial y} \quad . \tag{5.8}$$

We now use (5.7) and (5.8) to understand the consequences of Savelev's biholomorphism Ψ.

5.3. **Properties of the Savelev's biholomorphism and its rectification.** As it was mentioned in section 4, Savelev's Theorem guarantees the existence of a biholomorphism

$$\Psi : \mathcal{W} \to (\mathbb{C}, 0) \times \mathbb{CP}^1.$$

At a first glance we do not know much about Ψ; we need to understand its behavior through the charts on $\mathcal{W}$. To this purpose we recall that $\mathcal{W}$ is the result of the identification of the domains $\mathcal{U}_+$ and $\mathcal{U}_-$. We consider the natural projections: $\Pi_\pm : \mathcal{U}_\pm \to \mathcal{W}$, where $\Pi_\pm(p)$ is the class of the point p in the identifying space $\mathcal{W}$. Let $\tilde{\mathcal{U}}_\pm := \Pi_\pm(\mathcal{U}_\pm)$.

Definition 5.1. *We call* $(\Pi_\pm^{-1}, \tilde{\mathcal{U}}_\pm)$ *the "natural charts" of the complex manifold* $\mathcal{W}$.

Note that $\Pi_\pm^{-1} = t_\pm$ (see section 4).

We stress that $\alpha \circ \Phi$ (see (5.7) is just the change of coordinates of the "normal charts" of $\mathcal{W} : \alpha \circ \Phi = \Pi_-^{-1} \circ \Pi_+$ (see fig. 5.1).

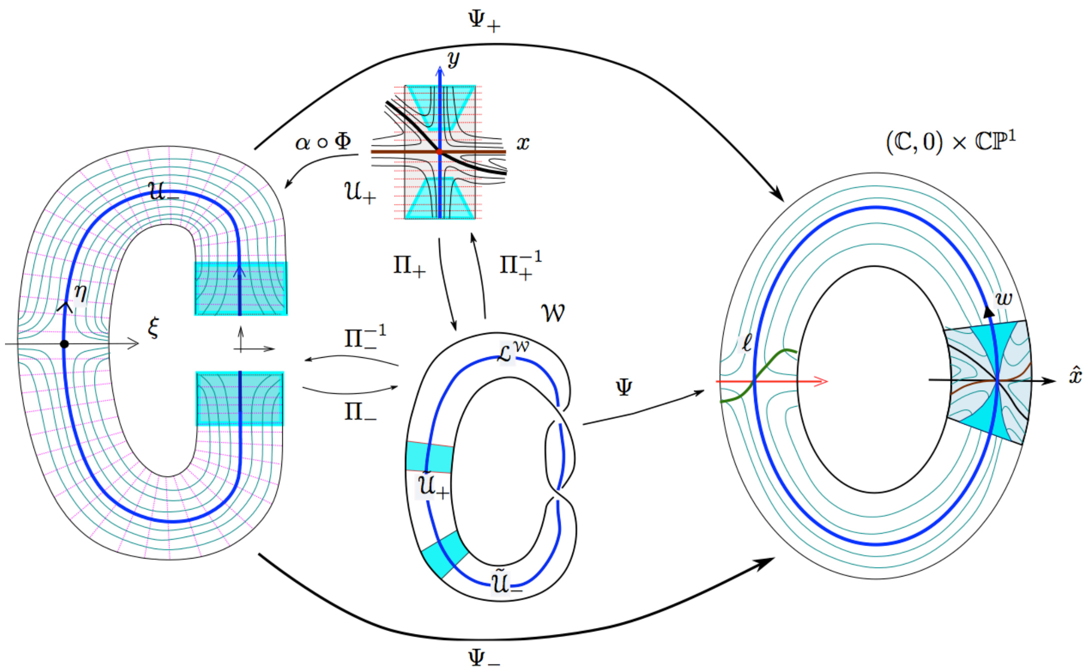

FIGURE 5.1. Savelev's biholomorphism Ψ

Remark that if we define $\Psi_\pm := \Psi \circ \Pi_\pm : \mathcal{U}_\pm \to \Psi(\mathcal{U}_\pm)$, then Ψ_+ and Ψ_- are related by means of $\alpha \circ \Phi$ (where the composition makes sense):

$$\Psi_+ = \Psi_- \circ (\alpha \circ \Phi)\ . \tag{5.9}$$

In order to obtain simple expressions for Ψ_+ and Ψ_- we proceed to give appropriate coordinates in $\hat{\mathcal{W}} := \Psi(\mathcal{W}) = (\mathbb{C}, 0) \times \mathbb{CP}^1$. To this aim we observe that, from Savelev's Theorem we may suppose, without loss of generality, that $\Psi(\mathcal{L}^{\mathcal{W}}) = \{\hat{x} = 0\} \times \mathbb{CP}^1$. Furthermore, we observe that in the charts $\Pi_\pm^{-1}$ the Riemann sphere $\mathcal{L}^{\mathcal{W}}$ is given by $\{\Pi_\pm^{-1} = 0\}$; hence, the restriction of the identifying map $\alpha \circ \Phi = \Pi_-^{-1} \circ \Pi_+$ to the axis $\{x = 0\}$ is the identity. Therefore, it is possible to give coordinates such that

$$\Psi_+ \mid_{x=0} = Id, \quad \Psi_- \mid_{\xi=0} = Id \quad . \tag{5.10}$$

If we consider the image of $\{\eta = 0\}$ under the transformation Ψ_-, $\ell := \Psi_-(\{\eta = 0\})$ we get, from (5.10), that the intersection of ℓ with $\Psi(\mathcal{L}^{\mathcal{W}})$ is transversal in $\Psi_-(\xi, \eta) \mid_{\xi=0,\eta=0}$. Hence if $(\hat{x}, w)$ are the coordinates of $\Psi(\mathcal{W})$, ℓ is expressed as $(\hat{x}, \hat{\gamma}(\hat{x}))$ in a neighborhood of $\hat{x} = 0, w = \infty$. Therefore the curve ℓ may be rectified by means of a Möebius transformation

$$w \mapsto \frac{w}{1 - \hat{\gamma}(\hat{x})w} \ ,$$

so that $\Psi_-(\xi, 0) \in \{w = \infty\}$. Furthermore, under an additional change of coordinates of the form $\hat{x} \mapsto \tilde{\phi}(\hat{x})$ we obtain

$$\Psi_- \mid_{\xi=0} = Id \quad . \tag{5.11}$$

We observe that from (5.9) we get

$$\Psi_+{}^{-1} = (\alpha \circ \Phi)^{-1} \circ \Psi_-{}^{-1}$$

and, as $\alpha \circ \Phi$ is the identity on the second coordinate we get that $\tilde{\Psi}_{+,2} = \tilde{\Psi}_{-,2}$, where

$$\Psi_\pm{}^{-1} = (\tilde{\Psi}_{\pm,1}, \tilde{\Psi}_{\pm,2}).$$

Hence, for small enough fixed $\hat{x}$, the function $\Psi_{\hat{x}}(w) := \tilde{\Psi}_{+,2}(\hat{x}, w)$ may be analytically extended to all $\mathbb{C}$. From (5.11) we get that such extension, which we denote again by $\Psi_{\hat{x}}$, has a pole at $w = \infty$. As Ψ_- is a biholomorphism, then the order of the pole of $\Psi_{\hat{x}}$ is one. Thus, $\Psi_{\hat{x}}$ is a polynomial of degree one on w:

$$\Psi_{\hat{x}}(w) = k(\hat{x})w + \gamma(\hat{x}) \quad ,$$

where k, γ are holomorphic on $\hat{x}$ and $k(0) \neq 0, \gamma(0) = 0$.

In this way the foliation in $\mathcal{U}_+$ given by $\{y = cst\}$ is transformed by the map Ψ_+ to the foliation by curves defined by

$$\begin{aligned} \Psi_+(x, y) &= (\Psi_{+,1}(x, y), \Psi_{+,2}(x, y)) \\ &= (\hat{x}, k(\hat{x})w + \gamma(\hat{x})) \quad . \end{aligned}$$

As $k(0) \neq 0$, $\gamma(0) = 0$, we may define for small enough $\hat{x}$ a rectification biholomorphism

$$\mathbf{r} = \mathbf{r}(\hat{x}, w) = \left(\hat{x}, \frac{w}{k(\hat{x})} - \frac{\gamma(\hat{x})}{k(\hat{x})}\right) \quad ,$$

whose inverse is

$$\mathbf{r}^{-1}(\hat{x}, w) = (\hat{x}, k(\hat{x})\, w + \gamma(\hat{x})) \quad .$$

This biholomorphism sends the curves $k(\hat{x})\, w + \gamma(\hat{x})$, with $w = c$ into the curves $w = c$, $c \in \mathbb{C}$ and fix $w = \infty$.

Finally, as $\mathbf{r} \circ \Psi_+(0, y) = (0, w)$, where $w = w(y)$ is a biholomorphism, we may perform an additional change of coordinates $\mathbf{r}_0(\hat{x}, w) = (\hat{x}, y)$ so that $\mathbf{r}_0 \circ \mathbf{r} \circ \Psi_+(0, y) = (0, y)$. Therefore, in what follows we may assume that

$$\Psi_{+,2}(x, y) \equiv y \equiv \Psi_{-,2}(x, y) \quad . \tag{5.12}$$

Using (5.10) and (5.12) we get

$$\Psi_+(x, y) = (x\alpha_+(x, y), y) \tag{5.13}$$

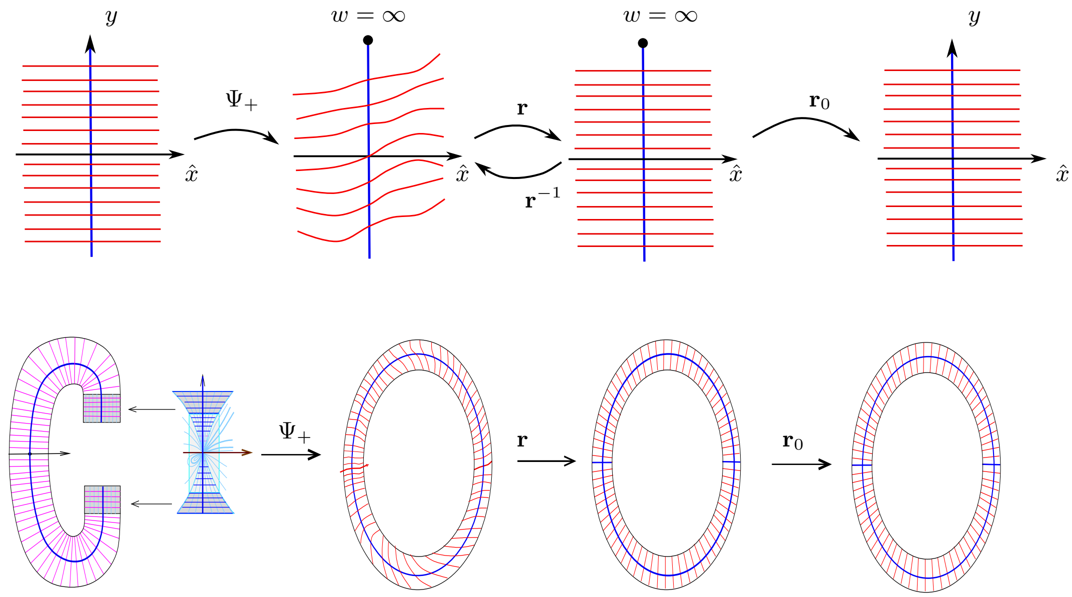

FIGURE 5.2. Rectification process.

$$\Psi_-(x, y) = (\xi\alpha_-(\xi, \eta), \eta) \quad , \tag{5.14}$$

where α_+ is holomorphic in a neighborhood of the disk $\{x = 0, |y| < R\} \subset \mathbb{C} \times \mathbb{CP}^1$, $\alpha_+(0, y) \neq 0$ and α_- is holomorphic in a neighborhood of $\{\xi = 0, |\eta| < R\} \subset \mathbb{C} \times \mathbb{CP}^1$, $\alpha_-(0, \eta) \neq 0$.

5.4. **Asymptotic of $\alpha_\pm$.** By the substitution of the expression (5.7) for $\alpha \circ \Phi$ and the expressions for Ψ_+ and Ψ_- given in (5.13), (5.14) we get

$$(x\alpha_+(x, y), y) = (\xi\alpha_-(\xi, \eta), \eta)_{(x\, g(x/y, y), y)}$$

i.e.

$$(x\alpha_+(x, y), y) = (xg(x/y, y)\alpha_-(xg(x/y, y), y) \quad ;$$

therefore

$$\alpha_+(x, y) = g(x/y, y)\alpha_-(xg(x/y, y), y) \quad . \tag{5.15}$$

Taking limits when $x \to 0$ we get

$$\alpha_+(0, y) = g(0, y)\alpha_-(0, y) \quad . \tag{5.16}$$

From (5.5) we know that $g(0, y)$ is non vanishing and holomorphic in the disk D_ϵ. Hence,

$$\alpha_-(0, y) = \frac{\alpha_+(0, y)}{g(0, y)}$$

is holomorphic in D_ϵ. The function α_- is holomorphic in $D_{\epsilon'} = \{|y| > \epsilon'\} \cup \{\infty\}$ and coincides with $\frac{\alpha_+(0,y)}{g(0,y)}$ in the annulus given by the intersection $D_\epsilon \cap D_{\epsilon'}$. Therefore α_- can be extended to the closure $\bar{\mathbb{C}}$. By Liouville's Theorem we get $\alpha_- \equiv c \equiv \frac{\alpha_+(0,y)}{g(0,y)}$ for a non zero constant c.

Thus

$$\begin{array}{rcl} \alpha_+(x, y) & = & cg(0, y) + O(x) \\ \alpha_-(x, y) & = & c + O(x) \quad . \end{array} \tag{5.17}$$

5.5. **Action of Ψ_+ on the vector field v.** After all the previous constructions we may look to the action of Ψ_+ on the vector field $\mathbf{v}$, and the action of Ψ_- on $(\alpha \circ \Phi)_* \mathbf{v}$.

We denote

$$\mathbf{v}_+ := \Psi_{+*} \mathbf{v} \quad \text{and} \quad \mathbf{v}_- := \Psi_{-*}((\alpha \circ \Phi)_* \mathbf{v}) \quad . \tag{5.18}$$

By construction, $\mathbf{v}_+$ and $\mathbf{v}_-$ generate in their corresponding domain of definition, a complex foliation $\mathcal{F}$ on the complex manifold $\mathcal{W}$. We stress that the definition of $\mathcal{F}$, $\mathbf{v}_0$ and $\mathbf{v}_-$ are in concordance with the definitions introduced at the end of section 4 and in section 5.2.

To get an expression of $\mathbf{v}_\pm(\hat{x}, w) = (P_\pm(\hat{x}, w), Q_\pm(\hat{x}, w))$ we use that ${\Psi_\pm}^{-1}$ may be written as:

$${\Psi_\pm}^{-1} : (\hat{x}, w) \mapsto (\hat{x}\ell_\pm(\hat{x}, w), w) \quad ,$$

where (using (5.13) and (5.17))

$$\ell_+(\hat{x}, w) = ((cg(0, y))^{-1} + O(x) \tag{5.19}$$

$$\ell_-(\hat{x}, w) = c^{-1} + O(x) \quad . \tag{5.20}$$

To get an explicit expression for $P_\pm, Q_\pm$ we recall that $\Psi_\pm(x, y) = (x\alpha_\pm(x, y), y)$, thus

$$\begin{aligned} \mathbf{v}_+(\hat{x}, w) &= D\Psi_+ \,|_{{\Psi_+}^{-1}(\hat{x}, w)} \, \mathbf{v}({\Psi_+}^{-1}(\hat{x}, w)) \\ &= \begin{pmatrix} \left(\alpha_+ + x\frac{\partial \alpha_+}{\partial x}\right)\Big|_{{\Psi_+}^{-1}(\hat{x}, w)} & x \, \frac{\partial \alpha_+}{\partial y}\Big|_{{\Psi_+}^{-1}(\hat{x}, w)} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} P({\Psi_+}^{-1}(\hat{x}, w)) \\ Q({\Psi_+}^{-1}(\hat{x}, w)) \end{pmatrix} \\ &= \begin{pmatrix} \left[\left(\alpha_+ + x\frac{\partial \alpha_+}{\partial x}\right) P + x\frac{\partial \alpha_+}{\partial y} Q\right]_{\left(\hat{x}\ell_+(\hat{x}, w), w\right)} \\ Q\left(\hat{x}\ell_+(\hat{x}, w), w\right) \end{pmatrix} \quad . \end{aligned}$$

Therefore, $\mathbf{v}_+(\hat{x}, w) = (P_+(\hat{x}, w), Q_+(\hat{x}, w))$, where

$$P_+(\hat{x}, w) = \left[\alpha_+ P + x\frac{\partial \alpha_+}{\partial y} Q + O(x^2)\right]_{(\hat{x}\ell_+(\hat{x}, w), w)} \tag{5.21}$$

and

$$Q_+(\hat{x}, w) = Q\left(\hat{x}\ell_+(\hat{x}, w), w\right) \quad , \quad q(w) := Q(0, w) \quad . \tag{5.22}$$

Analogously, we get explicit expressions for $\mathbf{v}_-(\hat{x}, w) = (P_-(\hat{x}, w), Q_-(\hat{x}, w))$

$$\begin{aligned} \mathbf{v}_-(\hat{x}, w) &= D\Psi_- \,|_{{\Psi_-}^{-1}(\hat{x}, w)} \, \mathbf{v}({\Psi_-}^{-1}(\hat{x}, w)) \\ &= \begin{pmatrix} \frac{\partial \xi \alpha_-}{\partial \xi} & \frac{\partial \xi \alpha_-}{\partial \eta} \\ 0 & 1 \end{pmatrix}\Bigg|_{{\Psi_-}^{-1}(\hat{x}, w)} \begin{pmatrix} (\lambda + 1)\hat{x}\ell_-(\hat{x}, w) \\ w \end{pmatrix} \\ &= \begin{pmatrix} (\lambda + 1)\hat{x}\ell_-(\hat{x}, w) \left[\hat{x}\ell_-(\hat{x}, w)\frac{\partial \alpha_-}{\partial \xi} + \alpha_-\right] + \hat{x} \, w \, \ell_-(\hat{x}, w)\frac{\partial \xi \alpha_-}{\partial \eta} \\ w \end{pmatrix} \quad . \end{aligned}$$

Therefore,

$$P_-(\hat{x},w) = \hat{x}\ell_-(\hat{x},w)\left[(\lambda+1)\alpha_-(\hat{x}\ell_-(\hat{x},w),w) + w\frac{\partial\xi\alpha_-}{\partial\eta}\right] + O(\hat{x}^2) \tag{5.23}$$

and

$$Q_-(\hat{x},w) = w \quad . \tag{5.24}$$

5.6. **Locus of functions P_- and Q_-.** At this stage it is important to recall that our goal is to prove that $\mathbf{v}_+$ may be written as a polynomial of degree $n-1$ in w with analytic coefficients depending on the x variable. To this sake we will look to the locus of $P_\pm, Q_\pm$ and then use a slightly modified version of Weierstrass Preparation Theorem.

We begin with the study of the locus of P_-, and Q_-:

From (5.24) we know that $Q_-(\hat{x},w) = w$ does not vanish for $|w| > r$. Moreover, from (5.20) we know that $\ell_-(\hat{x},w) = 1/c + \ell_1(\hat{x},w)$, where ℓ_1 is a holomorphic function on

$$\Delta_- = \{|x| < \delta\} \times \{|w| > r\}\,.$$

In particular, ℓ_1 is holomorphic in the point: $x = 0, w = \infty$. From Cauchy's inequalities it follows that in any polydisk $\Delta'_- = \{|x| < \delta'\} \times \{|w| > r'\}$, $r' > r$, $\delta' < \delta$ the inequality $\left|\frac{\partial\alpha_-}{\partial\eta}(\hat{x},w)\right| \leq \frac{\zeta}{|w|}$ is satisfied for an appropriate constant $\zeta = \zeta(\Delta,\Delta')$. By using expressions (5.17) and (5.20) for α_- and ℓ_- in (5.23) we get:

$$\begin{aligned} P_-(\hat{x},w) &= \hat{x}(1/c + \ell_1(\hat{x},w))((\lambda+1)(c+O(\hat{x})) + O(\hat{x}) \\ &= \hat{x}[(\lambda+1) + O(\hat{x})] \quad . \end{aligned} \tag{5.25}$$

Therefore $\hat{P}_-(\hat{x},w) = \frac{P_-(\hat{x},w)}{\hat{x}}$ is holomorphic and does not vanish in Δ'_-.

5.7. **Locus of functions P_+ and Q_+.** We begin by stating a slightly different version of Weierstrass Preparation Theorem:

Lemma 5.1. *Let $F(x,y)$ be a holomorphic function in the polydisk $\Delta_0 = \{|x| < \delta_0\} \times \{|y| < \epsilon_0\}$ such that the function $F(0,y)$ has, at $y = 0$ a zero of order N. If $F(0,y)$ has no more zeros in the disk $\{|y| < \epsilon_0\}$, then, for any ϵ, $0 < \epsilon < \epsilon_0$, there exist $\delta, 0 < \delta < \delta_0$ and holomorphic functions k, W, defined in $\Delta = \{|x| < \delta\} \times \{|y| < \epsilon\}$ such that*

(1) $F = kW$ *in* Δ
(2) $k \neq 0$ *in* Δ
(3) $W_N(x,y) = W(x,y) = y^N + \sum_{j=0}^{N-1} a_j(x)y^j$, $a_j(0) = 0$.

W_N is known as the Weierstrass polynomial (see Shabat pp.123-126).

Let us consider now the series $Q = Q_n + Q_{n+1} + \cdots$, where Q_j denotes the homogeneous polynomial of degree j in the variables (x,y), $j \geq n$, and $Q_n(x,y) = b_0 y^n + O(x)$. As we did before, let $q(y) = Q(0,y)$. From the genericity assumptions we know that $b_0 \neq 0$. Hence $q(y)$ has at $y = 0$ a zero of order n.

Let $\Delta_0 = \{|\hat{x}| < \delta\} \times \{|w| < \epsilon_0\}$, $\epsilon < \epsilon_0$ such that Q_+ is holomorphic in Δ_0. From (5.22) we have that $q(w) = Q_+(0,w)$ and for any $\hat{\epsilon} \leq \epsilon$ and $\hat{\delta} < \delta$ it is possible to factorize (by Lemma 5.1) $Q_+(\hat{x},w)$ as

$$Q_+(\hat{x},w) = K_Q(\hat{x},w)W_n(\hat{x},w), \tag{5.26}$$

$(\hat{x},w) \in \Delta_+ = \left\{|\hat{x}| < \hat{\delta}\right\} \times \{|w| < \hat{\epsilon}\}$, where $K_Q \neq 0$ at Δ_+ and W_n is the Weierstrass polynomial (of degree n). In particular Q_+ has, for small enough fixed x, exactly n zeros in $\{|w| < \hat{\epsilon}\}$.

We consider now the zeros of $P_+(\hat{x}, w)$ at $\hat{x} = 0$. Recall that the set $\{x = 0\}$ is invariant for the vector field $\mathbf{v}$, and $\{\hat{x} = 0\}$ is invariant for $\Psi_*\mathbf{v}$. Therefore,

$$P(x, y) = x\hat{P}(x, y) \text{ and } P_+(\hat{x}, w) = \hat{x}\hat{P}_+(\hat{x}, w) \quad ,$$

where (see (5.21))

$$\hat{P}_+(\hat{x}, w) = \frac{1}{\hat{x}\ell_+(\hat{x}, w)}\left[\alpha_+ P + x\frac{\partial \alpha_+}{\partial y} Q + O(x^2)\right]_{(\hat{x}\ell_+(\hat{x},w),w)} \quad .$$

Hence, for $\hat{x} = 0$ we get

$$P_+(0, w) = \alpha_+(0, w)\hat{P}(0, w) + \frac{\partial \alpha_+}{\partial y}(0, w)\, Q(0, w) \quad .$$

Moreover, as $\alpha_+(x, y) = cg(0, y) + O(x)$ (see (5.17)), then $\frac{\partial \alpha_+}{\partial y}(x, y) = cg'_y(0, y) + O(x)$. Therefore, $\hat{P}_+(0, w) = cg(0, w)\left[\hat{P}(0, w) + \frac{g'_y(0,w)}{g(0,w)} Q(0, w)\right]$.

From (5.3) follows that $\hat{P}(0, w) = \frac{c(w)q(w)+q(w)}{w}$ (where $q(w) = Q(0, w)$ as before), and from (5.5) $\frac{g'_y(0,w)}{g(0,w)} = \frac{c(w)q(w)+q(w)}{w}$, where $c(0) = 1$. Hence,

$$\begin{aligned} \hat{P}_+(0, w) &= cg(0, w)\left[\tfrac{c(w)q(w)+q(w)}{w} + \tfrac{c(w)q(w)+q(w)}{w} q(w)\right] \\ &= cg(0, w)\left[\tfrac{(\lambda+1)q(w)}{w}\right] \quad . \end{aligned}$$

As $g(0, w)$ satisfies the equation (5.5),

$$g(0, w) = exp\left(\int \frac{-c(y) + \lambda}{y}\right) \quad , \quad and \quad g(0, 0) = 1.$$

Therefore, $g(0, w)$ does not vanish for small enough w. Hence, as $q(w)$ has a zero of order n at $w = 0$, then $\hat{P}_+(0, w)$ has a zero of order $n - 1$ for $|w|$ small enough.

From Lemma 5.1, for small enough δ_1 and ϵ_1, $\Delta_0 = \{|\hat{x}| < \delta_1\} \times \{|w| < \epsilon_1\}$ there exist $K_p(\hat{x}, w)$ and $W_{n-1}(\hat{x}, w)$ such that K_P does not vanish in Δ and W_{n-1} is the Weierstrass polynomial of degree $n - 1$ such that

$$P_+(\hat{x}, w) = K_P(\hat{x}, w)W_{n-1}(\hat{x}, w) \quad . \tag{5.27}$$

In particular, for fixed $\hat{x}$, $|\hat{x}| < \delta_1$, $P_+(\hat{x}, w)$ has exactly $n - 1$ zeros in the disk $|w| < \epsilon_1$.

5.8. **End of the proof of Theorem 2.1.** In 5.7 it was proved that the vector fields

$$\mathbf{v}_+ = \hat{x}\hat{P}_+\frac{\partial}{\partial \hat{x}} + Q_+\frac{\partial}{\partial w} \quad \text{and} \quad \mathbf{v}_- = \hat{x}\hat{P}_-\frac{\partial}{\partial x} + Q_-\frac{\partial}{\partial w}$$

are generators of the same foliation $\mathcal{F}$ of $\hat{\mathcal{W}} = \Psi(\mathcal{W})$. This implies that in the intersection domain of $\mathbf{v}_+$ and $\mathbf{v}_-$ the following equality must take place:

$$\frac{\hat{x}\hat{P}_-}{Q_-} = \frac{\hat{x}\hat{P}_+}{Q_+}$$

Then, for $\hat{x} \neq 0$,

$$\frac{\hat{P}_-}{Q_-} = \frac{\hat{P}_+}{Q_+}$$

and it can be extended to $\hat{x} = 0$. From (5.24), (5.26) and (5.27)

It follows that

$$\frac{\hat{P}_-}{w} = \frac{K_P W_{n-1}}{K_Q W_n} \quad . \tag{5.28}$$

Hence,

$$\frac{\hat{P}_-(\hat{x},w)W_n(\hat{x},w)}{wW_{n-1}(\hat{x},w)} = \frac{K_P(\hat{x},w)}{K_Q(\hat{x},w)} \quad . \tag{5.29}$$

We stress that for small enough $\hat{x}$, the right member of (5.27) is holomorphic in the disk $\{|w| > r\}$ for $r > 0$ (see (5.25)). Moreover, as for small enough $\hat{x}$, $W_n(\hat{x},w)$, $W_{n-1}(\hat{x},w)$ are polynomials on w, $|w| < \epsilon_1$, they can be extended for any w, $|w| > r$.

Therefore, for small enough fixed $\hat{x}$ the left hand side of (5.29) is holomorphic on $|w| > r$. At the same time, the right hand side of (5.29) is holomorphic on $|w| < \epsilon_1$. Hence, as $r > 0$ is arbitrary we can choose $r < \frac{\epsilon_1}{2}$. Then (5.29) is defined in an annulus and has holomorphic extension for $w \in \mathbb{CP}^1$.

Therefore, by Liouville's Theorem (for $\hat{x}$ fixed) it is constant, $\delta = \delta(\hat{x})$:

$$\frac{\hat{P}_-(\hat{x},w)W_n(\hat{x},w)}{wW_{n-1}(\hat{x},w)} = \delta(\hat{x}) \quad .$$

Thus,

$$\frac{\hat{x}\hat{P}_- W_n}{Q_+ W_{n-1}} = \hat{x}\delta(\hat{x})$$

and

$$\frac{\hat{P}_-(\hat{x},w)}{Q_+(\hat{x},w)} = \hat{x}\delta(\hat{x})\frac{W_{n-1}(\hat{x},w)}{\hat{x}W_n(\hat{x},w)} \quad .$$

This last equality implies that the vector field $\mathbf{v}_+$ is proportional (obtained by multiplication by a non vanishing function) to

$$\tilde{\mathbf{v}}_+ = \hat{x}\delta(\hat{x})W_{n-1}(\hat{x},w)\frac{\partial}{\partial\hat{x}} + W_n(\hat{x},w)\frac{\partial}{\partial w}. \tag{5.30}$$

To finish the proof of Theorem 2.1 we stress that by construction $\tilde{\mathbf{v}}_+$ in (5.30) is orbitally analytically equivalent to the original vector field $\mathbf{v}$.

Let $\gamma = \{w = \gamma(\hat{x})\}$ be one separatrix of $\tilde{\mathbf{v}}_+$. The biholomorphism $H(\hat{x},w) = (\hat{x}, w - \gamma(\hat{x}))$ transforms $\tilde{\mathbf{v}}_+$ to a vector field $\hat{\mathbf{v}}_+ = H_*\tilde{\mathbf{v}}_+$ having $\{\hat{w} = 0\}$, $\hat{w} = w - \gamma(\hat{x})$, as a separatrix. hence, the second component of $\hat{\mathbf{v}}_+$ has the form $\hat{W}_+(\hat{x},\hat{w}) = \hat{w}\hat{W}_{n-1}(\hat{x},\hat{w})$, where $\hat{W}_{n-1}(\hat{x},\hat{w})$ is a Weierstrass polynomial of degree $n-1$. Thus, the vector field $\hat{\mathbf{v}}_+$ has all the required properties. This finishes the proof of Theorem 2.1.

6. Analytic normal form for $n = 2$.

In this section we prove Theorem 2.2. As it was already mentioned in the introduction of this work, Theorem 2.2 shows that (after rotation and rectification of one of its separatrices) nondicritic generic germs of vector fields in $(\mathbb{C}^2, 0)$ have analytic strict orbital normal form given by

$$\mathbf{v}_2(x,y) = (P_2 + xB)\frac{\partial}{\partial x} + (Q_2 + yB)\frac{\partial}{\partial y} \quad ,$$

where P_2, Q_2 are homogeneous polynomials of degree 2, $deg_y Q_2 = 2$, $B(x) = x^2 b(x)$ and

$$b(x) = \sum_{k=0}^{\infty} b_k x^k$$

is analytic.

We begin with the preliminary analytic normal form given in Theorem 2.1 for $n = 2$:

$$\hat{\mathbf{v}}(x,y) = x(a(x)y + b(x))\frac{\partial}{\partial x} \; + \; y(c(x)y + d(x))\frac{\partial}{\partial y} \quad , \tag{6.1}$$

where a, b, c, d are holomorphic functions in $(\mathbb{C}, 0)$, $b(0) = d(0) = 0$.

Let us denote $a_0 = a(0), c_0 = c(0), b_1 = b'(0)$ and $d_1 = d'(0)$. From the genericity assumptions given in section 3.3 it follows that

$$a_0 \neq c_0, \; a_0 \neq 0, \; c_0 \neq 0, \; b_1 \neq 0, \; b_1 \neq d_1 \tag{6.2}$$

Remark 6.1. From $a_0 \neq c_0$ and $b_1 \neq d_1$ we get that the polynomial $R_3(1, u)$ has exactly two (different) roots. If $a_0 = 0$ then $\lambda_\infty = -1$. If $b_1 = 0$ there is a characteristic exponent equal to zero. The same happens for $c_0 = 0$ for the characteristic exponent associated to p_∞.

As $c_0 \neq 0$ we may assume that $c \equiv 1$. Indeed, for x small enough we can divide $\hat{\mathbf{v}}$ by $c(x)$. Moreover, by performing if needed the change of coordinates $x \mapsto g(x) = exp\left(\int \frac{a_0}{xa(x)}dx\right)$ (where g is holomorphic since $\operatorname{res}_{x=0} \frac{a_0}{xa(x)} = 1$) we may assume, without loss of generality that the vector field $\hat{\mathbf{v}}$ defined in (6.1) satisfies

$$c \equiv 1, \; a \equiv a_0 \; , \text{ and from (6.2)} \quad a_0 \neq 1 \quad . \tag{6.3}$$

Proposition 6.1. *Let* $\mathbf{v}$ *and* $\mathbf{w}$ *holomorphic vector fields of the form (6.1) satisfying the normalizing conditions (6.3),*

$$\mathbf{v}(x,y) = x(a_0 y + b(x))\tfrac{\partial}{\partial x} \; + \; y(y + d(x))\tfrac{\partial}{\partial y} \quad ,$$

$$\mathbf{w}(x,y) = x(\tilde{a}_0 y + \tilde{b}(x))\tfrac{\partial}{\partial x} \; + \; y(y + \tilde{d}(x))\tfrac{\partial}{\partial y} \quad .$$

The necessary and sufficient conditions for the existence of a holomorphic change of coordinates

$$\begin{gathered} H : (\mathbb{C}, 0) \times \mathbb{CP}^1 \to (\mathbb{C}, 0) \times \mathbb{CP}^1 \\ H : (x, y) \mapsto (\varphi(x), k(x)y) \end{gathered} \tag{6.4}$$

where

$$\varphi(0) = 0; \varphi'(0) = 1; k(0) = 1, \tag{6.5}$$

and such that

$$DH\mathbf{v} = q\,\mathbf{w} \circ H \tag{6.6}$$

where $q = q(x)$ *is an holomorphic function*

$$q(0) = 1, \tag{6.7}$$

is the solvability of the following equations:

$$\begin{gathered} a_0 = \tilde{a}_0 \\ \tilde{b} \circ \varphi = \left(\tfrac{\varphi(x)}{x}\right)^\mu b \end{gathered} \tag{6.8}$$

and

$$(\varphi' \tilde{d} \circ \varphi)(x) = \left(\frac{\varphi(x)}{x}\right)^{\mu+1} \left[\left(\frac{x\varphi'(x)}{\varphi(x)} - 1\right) b(x)\,\mu + d(x)\right] \quad . \tag{6.9}$$

Proof. The substitution of $\mathbf{v}$ and $\mathbf{w}$ and H on (6.6) leads to the equality:

$$\begin{pmatrix} \varphi'(x) & 0 \\ k'(x)y & k(x) \end{pmatrix} \begin{pmatrix} a_0xy + xb(x) \\ y^2 + d(x)y \end{pmatrix} = \begin{pmatrix} q(x)\tilde{a}_0\varphi(x)k(x)\,y + q(x)\varphi(x)\tilde{b}(\varphi(x)) \\ q(x)k^2(x)y^2 + q(x)\tilde{d}(\varphi(x))\,k(x)\,y \end{pmatrix}$$

Therefore,

$$\begin{aligned} \varphi'(x)a_0\,x &= q(x)\tilde{a}_0k(x)\varphi \\ \varphi'(x)b(x)x &= q(x)\varphi(x)\tilde{b}(\varphi(x)) \\ k'(x)a_0x + k(x) &= q(x)k^2(x) \\ k'(x)b(x)x + k(x)d(x) &= q(x)\tilde{d}(\varphi(x))k(x) \end{aligned} \tag{6.10}$$

We stress that condition (6.5) and (6.7) imply that $\tilde{a}_0 = a_0$. Hence, the system of equations (6.10) is equivalent to:

$$\frac{x\varphi'}{k\varphi} = q \tag{6.11}$$

$$x\frac{\varphi'}{\varphi} = q\frac{\tilde{b}\circ\varphi}{b} \tag{6.12}$$

$$q = \frac{a_0xk'}{k^2} + \frac{1}{k} \tag{6.13}$$

$$q\,\tilde{d}\circ\varphi = \frac{k'bx}{k} + d \tag{6.14}$$

By substitution of (6.11) in (6.13) we get

$$\frac{\varphi'}{\varphi} = a_0\frac{k'}{k} + \frac{1}{x} \tag{6.15}$$

The integration of (6.15) yields to an explicit expression of φ:

$$\varphi(x) = x\ (k(x))^{1/\mu}\quad ,$$

where $\mu = 1/a_0$. Equivalently,

$$k(x) = \left(\frac{\varphi(x)}{x}\right)^{\mu} \tag{6.16}$$

Using (6.16) in (6.11) we get

$$\left(\frac{x}{\varphi(x)}\right)^{\mu+1}\varphi'(x) = q(x) \tag{6.17}$$

The substitution of (6.17) in (6.12), and (6.15), (6.16) in (6.14) yields to the pair of equations:

$$(\tilde{b}\circ\varphi)(x) = \left(\frac{\varphi(x)}{x}\right)^{\mu} b(x)$$

$$(\varphi'\tilde{d}\circ\varphi)\,|_x = \left(\frac{\varphi(x)}{x}\right)^{\mu+1}\left[\left(\frac{x\varphi'(x)}{\varphi(x)} - 1\right)b(x)\,\mu + d(x)\right]$$

This proves the Proposition 6.1 □

In what follows we will prove that generic (in the sense G1,G2,G3) germs of vector fields $\mathbf{v} \in \mathcal{V}_n$ always satisfy the conditions (6.8) and (6.9) of Proposition 6.1. This will imply the existence of an analytic (non-strict) change of coordinates taking the germ $\mathbf{v}$ to its analytic normal form.

Let $\mathbf{v} \in \mathcal{V}_n$ be such that $\mathbf{v}$ satisfies that the generic assumptions G1,G2,G3. By Theorem 2.1 and normalizations (6.2) and (6.3) $\mathbf{v}$ is analytically equivalent to a germ $\mathbf{v}_{norm} \in \mathcal{V}_n$ such that

$$\mathbf{v}_{norm} = x(ya_0 + b(x))\frac{\partial}{\partial x} + y(y + d(x))\frac{\partial}{\partial y} \quad . \tag{6.18}$$

We may write b and d in (6.18) as $b(x) = b_1 x + x^2 b_2(x)$ and $d(x) = d_1 x + x^2 d_2(x)$ where b_2 and d_2 are holomorphic germs in $(\mathbb{C}, 0)$.

Remark 6.2. Any generic germ $\mathbf{v}_{norm}$ as in (6.18) satisfies that: $a_0 \neq 0$ and $b_1\mu - d_1 \neq 0$. Indeed, if $b_1\mu - d_1 = 0$ then the characteristic number at p_∞ is -1. This contradicts the generic assumptions. For a_0 see (6.2).

Lemma 6.1. *There exists a change of coordinates H satisfying the conditions of Proposition 6.1, such that $\mathbf{v}_{norm}$ is analytically equivalent to*

$$\mathbf{v}_{an}(x, y) = x(ya_0 + b_1 x + x^2\beta(x))\frac{\partial}{\partial x} + y(y + d_1 x + x^2\beta(x))\frac{\partial}{\partial y} \quad .$$

Proof. We prove the Lemma by making a direct substitution of

$$\tilde{b}(x) = b_1 x + x^2\beta(x) \qquad \text{and} \qquad \tilde{d}(x) = d_1 x + x^2\beta(x)$$

in the equalities (6.8) and (6.9):

$$b_1\varphi + \varphi^2\beta\circ\varphi = \left(\frac{\varphi}{x}\right)^{\mu} b \tag{6.19}$$

$$\varphi' \, \left(d_1\varphi + \varphi^2\beta\circ\varphi\right) = \left(\frac{\varphi}{x}\right)^{\mu+1}\left[d + b\mu\left(\frac{x\varphi'}{\varphi} - 1\right)\right] \quad . \tag{6.20}$$

Multiplying the equation (6.19) by φ' and substracting it from (6.20) we get

$$\varphi'(d_1 - b_1)\varphi = \left(\frac{\varphi}{x}\right)^{\mu}\left[\varphi' \, b(\mu - 1) + (d - b\mu)\frac{\varphi}{x}\right] \quad .$$

Therefore

$$\varphi'\left[xb(\mu - 1) + (b_1 - d_1)\frac{x^{\mu+1}}{\varphi^{\mu-1}}\right] = (\mu b - d)\varphi \quad . \tag{6.21}$$

The substitution in (6.21) of the expression for b and d, and $\varphi(x) = x\Psi(x)$ leads to the equality:

$$x\Psi' + \Psi = \frac{[\mu b_1 - d_1 + (\mu b_2(x) - d_2(x))\, x]\Psi}{(\mu - 1)(b_1 + x b_2(x)) + (b_1 - d_1)\Psi^{1-\mu}} \tag{6.22}$$

Let us define $F(x, \Psi) = \frac{x\Psi' + \Psi}{\Psi}$. Then Ψ is solution of the differential equation

$$\Psi' = \left[\frac{F(x, \Psi) - 1}{x}\right]\Psi \tag{6.23}$$

with initial condition $\Psi(0) = 1$ (see (6.5)).

Together with equation (6.23) we consider the vector field

$$\xi(x, \Psi) = x\frac{\partial}{\partial x} + (F(x, \Psi) - 1)\,\Psi\frac{\partial}{\partial y} \quad . \tag{6.24}$$

Since $\mu b_1 - d_1 \neq 0$ (see Remark 6.1), then the vector field is holomorphic in a neighborhood of the singular point $x = 0, \Psi(0) = 1$:

$$\xi(0, 1) = 0\frac{\partial}{\partial x} + (F(0, 1) - 1)\,\Psi\frac{\partial}{\partial y} = (0, 0) \quad ,$$

where

$$F(0,1) = \frac{\mu b_1 - d_1}{(\mu-1)b_1 + b_1 - d_1} = 1 \ .$$

The eigenvalues of the linearization at the singular point $(0,1)$ of the vector field ξ are $\lambda_1 = 1$ (for $e_1 = \frac{\partial}{\partial x}$) and $\lambda_2 = F'_\Psi(0,1)$ (for the eigenvector e_2 transversal to $\{x=0\}$).

Since $F'_\Psi = (\mu-1)(b_1-d_1)(b_1\mu - d_1)^{-1} \neq 0$, by the Hadamard-Perron's Theorem, there is a smooth separatrix γ at the singular point $(0,1)$ with tangent direction at $(0,1)$ equal to e_2.

This curve is, locally, the graphic of a holomorphic function $\Psi = \Psi(x)$ satisfying equation (6.23) and such that $\Psi(0) = 1$.

We now substitute this function Ψ in (6.19):

$$\varphi^2 \beta \circ \varphi = \left[\Psi^\mu(b_1 x + b_2(x)x^2) - b_1\varphi\right] \ .$$

Therefore

$$\beta \circ \varphi = \frac{x^2}{\varphi^2}\left[\frac{\Psi^\mu(b_1 + x b_2(x))}{x} - \frac{b_1\Psi}{x}\right] \ .$$

Thus,

$$\beta \circ \varphi = \frac{1}{\Psi^2}\left[\Psi^\mu b_2(x) + \frac{b_1(\Psi^\mu - \Psi)}{x}\right] \ ;$$

and since $\Psi(0) = 1$, $\beta \circ \varphi$ is holomorphic in $(\mathbb{C}, 0)$.

We know that $\varphi = x\Psi$, $\Psi(0) = 1$ is holomorphic, thus

$$\beta = \frac{1}{\Psi^2}\left[\Psi\, b_2(x) + b_1\left(\frac{\Psi^\mu - \Psi}{x}\right)\right] \circ \varphi^{-1} \ .$$

is also holomorphic in $(\mathbb{C}, 0)$, and both, φ and β are solutions of equations (6.19) and (6.20). Therefore, by Proposition 6.1, the vector field $\mathbf{v}_{norm}$ is analytically equivalent at the origin to

$$\mathbf{v}_{an} = x(ya_0 + b_1 x + x^2\beta(x))\frac{\partial}{\partial x} + y(y + d_1 x + x^2\beta(x))\frac{\partial}{\partial y} \ .$$

Lemma 6.1 is proved. □

Finally, as $\mathbf{v}$ is analytically equivalent to $\mathbf{v}_{norm}$, then it is also analytically equivalent to

$$\mathbf{v}_{an} = x(ya_0 + b_1 x + x^2\beta(x))\frac{\partial}{\partial x} + y(y + d_1 x + x^2\beta(x))\frac{\partial}{\partial y} \ .$$

Theorem 2.2 is proved.

6.1. **Proof of Theorem 2.3.** To prove Theorem 2.3 we recall that by the Theorem of formal orbital strict classification (see section 2) any generic (in the sense $\tilde{G}1$, $\tilde{G}2$, $\tilde{G}3$) nondicritic germ of vector field $\mathbf{v} \in \mathcal{V}_2$ is formal orbital strict equivalent to a formal vector field $\mathbf{v}_f$

$$\mathbf{v}_f = (P_2 + xB)\frac{\partial}{\partial x} + (Q_2 + yB)\frac{\partial}{\partial y} \tag{6.25}$$

where $\mathbf{v}_0 = P_2\frac{\partial}{\partial x} + Q_2\frac{\partial}{\partial y}$, P_2, Q_2 are homogeneous polynomials of degree 2, $deg_y Q_2 = 2$, $B(x) = x^2 b(x)$, and $b(x) = \sum_{k=0}^{\infty} b_k\, x^k$, $b_k \in \mathbb{C}$, is a formal power series.

We need to prove that if we assume that the singular point at infinity of the blow-up $\tilde{\mathbf{v}}$ of $\mathbf{v}$ is linearizable, then the formal normal form (6.2) is analytic; i.e. we will prove that b is a convergent power series.

By Theorem 2.2 we know that $\mathbf{v}$ is orbitally analytically equivalent (non necessarily strict) to a germ of holomorphic vector field of the form

$$x(\mathcal{P}_1 + x^2\beta(x))\frac{\partial}{\partial x} + y(\mathcal{Q}_1 + x^2\beta(x))\frac{\partial}{\partial y}$$

where $\mathcal{P}_1(x,y) = ya_0 + b_1x$, $\mathcal{Q}_1(x,y) = y + d_1x$, and $\beta(x)$ is a holomorphic function in a neighborhood of the origin.

To finish the proof of Theorem 2.3 we stress that it is always possible to define a linear transformation:

$$L : (x,y) \mapsto (\alpha_0 x, \alpha_1 x + \alpha_2 y) \tag{6.26}$$

such that for appropriate constants a_0, b_1, d_1,

$$\mathbf{v}_{an} = x(ya_0 + b_1x + x^2\beta(x))\frac{\partial}{\partial x} + y(y + d_1x + x^2\beta(x))\frac{\partial}{\partial y}$$

is linearly equivalent to

$$\mathbf{w} = (P_2 + xB)\frac{\partial}{\partial x} + (Q_2 + yB)\frac{\partial}{\partial y} \quad ,$$

where the components of

$$\mathbf{v}_0 = P_2\frac{\partial}{\partial x} + Q_2\frac{\partial}{\partial y}$$

are homogeneous polynomials of degree 2, $P_2(0,y) = 0$, $deg_y Q_2 = 2$, and $B(x) = x^2b(x)$.

The equivalence between $\mathbf{v}$ and $\mathbf{w}$ is strict (orbital and analytic). Then by the uniqueness of the formal normal form under strict orbital equivalence, the formal normal form of $\mathbf{v}$, $\mathbf{v}_f$, and $\mathbf{w}$ must coincide.

Thus, $B(x)$ is analytic and therefore $\mathbf{v}_f$ is analytic too. Theorem 2.3 is proved.

References

[C-S] Camacho, C.; Sad, P. *Invariant varieties through singularities of holomorphic vector fields.* Ann. of Math. (2) 115 (1982), no. 3, 579–595.

[Ca] Calsamiglia-Mendlewicz,G. *Finite determinacy of dicritical singularities in* $(\mathbb{C}^2, 0)$ (Détermination finite de singularités dicritiques dans $(\mathbb{C}^2, 0)$), Annales de l'Institut Fourier, Grenoble, v. 57 (2007), No. 2, 673-691.

[Ce,Mo] Cerveau, D.; Moussu, R. *Groupes d'automorphismes de* $(\mathbb{C}, 0)$ *et équations différentielles* $ydy + \cdots = 0$. (French) Groups of automorphisms of $(\mathbb{C}, 0)$ and differential equations of the form $ydy + \cdots = 0$]. Bull. Soc. Math. France **116** (1988), no. 4, 459–488 (1989).

[EISV] Elizarov, P.M; Ilyashenko, Yu.S; Shcherbakov, A.A. and Voronin, S.M., *Finitely generated groups of germs of one-dimensional conformal mappings, and invariants for complex singular points of analytic foliations of the complex plane*, In Adv. Soviet Math., **14**, Amer. Math. Soc., Providence, RI, 1993. 57–105.

[IY] Ilyashenko, Yu.S; Yakovenko,S; *Lectures on Analytic Differential Equations*, Graduates Studies in Mathematics, V86, AMS, 2007, 605p.

[Lo1] Loray, F. *Rèduction formelle des singularités cuspidales de champs de vecteurs analytiques*, J.Diff.Equations 158 (1999), 152-173. DOI: 10.1016/S0022-0396(99)80021-7

[Lo2] Loray, F. *Analytic normal forms for non-degenerate singular points of planar vector fields*, Preprint, Institute Estudis Catalan, 2003.

[Lo3] Loray, F. *A preparation theorem for codimension-one foliations.* Ann. of Math. (2) 163 (2006), no. 2, 709–722.

[Lo4] Loray, F. *Versal deformation of the analytic saddle-node. Analyse complexe, systèmes dynamiques, sommabilité des séries divergentes et théories galoisiennes. II.* Astérisque No. 297 (2004), 167–187.

[M] Mattei,J.F. *Modules de feuilletages holomorphes singuliers. I. Équisingularité.* (French) [Moduli of singular holomorphic foliations. I. Equisingularity] Invent. Math. **103** (1991), no. 2, 297–325.

[ORV1] Ortiz-Bobadilla,L. Rosales-González,E. Voronin,S.M. *Rigidity theorem for degenerated singular points of germs of holomorphic vector fields in the complex plane.* J. Dynam. Control Systems **7** (2001), no. 4, 553-599.

[ORV2] Ortiz-Bobadilla,L., Rosales-González,E., Voronin,S. *Rigidity theorem for degenerated singular points of germs of dicritic holomorphic vector fields in the complex plane*. Mosc. Math. J. 5 (2005), no. 1, 171-206.

[ORV3] Ortiz-Bobadilla,L., Rosales-González,E., Voronin,S. *Analytic normal forms for germs of dicritic holomorphic vector fields in the complex plane.* Mosc. Math. J. V8 (2008), N 3, 521–545.

[ORV4] Ortiz-Bobadilla,L., Rosales-González,E., Voronin,S. *Thom's problem for generic degenerated singular points of holomorphic foliations in the plane.* Moscow Mathematical Journal, V 12 (2012), no.4, 825-862.

[Sa] Savelev,V. I. *Zero-type imbedding of a sphere into complex surfaces.* (Russian) Vestnik Moskov. Univ. Ser. I Mat. Mekh. 1982, no. 4, 28–32.

[Sha] Shabat,B.V. *Introduction to the Complex Analysis II (Functions of several variables)*, Translations of Math. Monographs of Am. Math. Soc., **V 110** (1991).

[St,Z1] Strozyna,E; Zoladek,H; *The analytic and formal normal forms for the nilpotent singularity*, J.Diff. Equation 179 (2002) 479-537.

[Vo1] Voronin, S.M. *Orbital analytic equivalence of degenerate singular points of holomorphic vector fields on the complex plane.*, Tr. Mat. Inst. Steklova **213** (1997), Differ. Uravn. s Veshchestv. i Kompleks. Vrem., 35–55 (Russian).

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