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RIEMANN-ROCH THEORY ON FINITE SETS

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Abstract. In [2] M. Baker and S. Norine developed a theory of divisors and linear systems on graphs, and proved a Riemann-Roch Theorem for these objects (conceived as integer-valued functions on the vertices). In [6] and [7] the authors generalized these concepts to real-valued functions, and proved a corresponding Riemann-Roch Theorem in that setting, showing that it implied the Baker-Norine result. In this article we prove a Riemann-Roch Theorem in a more general combinatorial setting that is not necessarily driven by the existence of a graph.

1. Introduction

Baker and Norine showed in [2] that a Riemann-Roch formula holds for an analogue of linear systems defined on the vertices of finite connected graphs. There, the image of the graph Laplacian induces an equivalence relation on the group of *divisors* of the graph, which are integer-valued functions defined on the set of vertices. This equivalence relation is the analogue of linear equivalence in the classical algebro-geometric setting. Gathmann and Kerber [5] later used the Baker-Norine result to prove a Riemann-Roch theorem for tropical curves.

We showed in [6] that the Baker-Norine result implies a generalization of the Riemann-Roch formula to edge-weighted graphs, where the edge weights can be R-valued, where R is an arbitrary subring of the reals; the equivalence relation induced by the image of the edge-weighted graph Laplacian applies equally well to divisors which are R-valued functions defined on the set of vertices. We prove in [7] our version of the R-valued Riemann-Roch theorem from first principles; this gave an independent proof of the Baker-Norine result as well. In [1], Amini and Caporoso develop a Riemann-Roch theory for vertex-weighted graphs over the integers; related work on computing the rank of these divisors can be found in [3] and [4].

The notion of linear equivalence above is induced by the appropriate graph Laplacian acting as a group in the space of divisors, which may be viewed as points in $\mathbb{Z}^n$ (the Baker-Norine case) or more generally R^n, where n is the number of vertices of the graph. A crucial role in this theory is played by a certain set of divisors $\mathcal{N}$, which is a union of orbits of the group action. In this paper, we propose a generalization of this Riemann-Roch formula where no graph structure need be present; however an appropriate set $\mathcal{N}$ must still be defined, having a specified symmetry property. The Baker-Norine proof is combinatorial and relies on properties of the finite graph, where our more general result presented in this paper only assumes this symmetry condition. In the graph case this symmetry condition holds, and therefore this is a generalization of the Baker-Norine result. Potentially our result may apply to many other discrete objects, with the additional generalization that the divisors are R-valued rather than restricted to integer values.

The setup we will use is as follows. Choose a subring R of the reals, and fix a positive integer n. Let V be the group of points in R^n under component-wise addition. If $x \in V$, we we will use the functional notation $x(i)$ to denote the the i-th component of x.

For any $x \in V$, define the *degree* of x as

$$\deg(x) = \sum_{i=1}^{n} x(i).$$

For any $d \in R$, define the subset $V_d \subset V$ to be

$$V_d = \{x \in V \mid \deg(x) = d\}.$$

Note that the subset V_0 is a subgroup; for any d, V_d is a coset of V_0 in V.

Fix the parameter $g \in R$, which we call the *genus*, and choose a set $\mathcal{N} \subset V_{g-1}$. For $x \in V$, define

$$\begin{aligned} x^+ &= \max(x, 0) \\ x^- &= \min(x, 0) \end{aligned}$$

where max and min are evaluated at each coordinate. It follows that

$$x = x^+ + x^- \quad \text{and} \quad x^+ = -(-x)^-.$$

We then define the *dimension* of $x \in V$ to be

$$\ell(x) = \min_{\nu \in \mathcal{N}} \{\deg((x - \nu)^+)\}.$$

This definition of the dimension agrees with the definition given by Baker and Norine in the special application to the graph setting, as we observed in [6].

We can now state our main result.

Theorem 1.1. *Let V be the additive group of points in R^n for a subring $R \subset \mathbb{R}$ and fix $g \in R$. Suppose $\kappa \in V_{2g-2}$, and $\mathcal{N} \subset V_{g-1}$, satisfying the symmetry condition*

$$\nu \in \mathcal{N} \Longleftrightarrow \kappa - \nu \in \mathcal{N}.$$

Then for every $x \in V$,

$$\ell(x) - \ell(\kappa - x) = \deg(x) - g + 1$$

We will give a proof of Theorem 1.1 in §2. In §3, we will give examples of κ and $\mathcal{N}$ (coming from the graph setting) which satisfy the conditions of Theorem 1.1, and show how this Riemann-Roch formulation is equivalent to that given in [7]. Finally in §4, we gives examples that do not arise from graphs.

2. Proof of Riemann-Roch Formula

The dimension of $x \in V$

$$\ell(x) = \min_{\nu \in \mathcal{N}} \{\deg((x - \nu)^+)\}$$

can be written as

$$\ell(x) = \min_{\nu \in \mathcal{N}} \left\{ \sum_{i=1}^{n} \max\{x(i) - \nu(i), 0\} \right\}.$$

If $x(i) \geq \nu(i)$ for each i, $\sum_{i=1}^{n} \max\{x(i) - \nu(i), 0\}$ is the *taxicab* distance from x to ν. Thus, $\ell(x)$ is the taxicab distance from x to the portion of the set $\mathcal{N}$ such that $x \geq \mathcal{N}$, where the inequality is evaluated at each component.

We will now proceed with the proof of the Riemann-Roch formula.

Proof. (Theorem 1.1)

Suppose that $\mathcal{N} \subset V_{g-1}$ and $\kappa \in V$ satisfy the symmetry condition. We can then write

$$\begin{aligned} \ell(\kappa - x) &= \min_{\nu \in \mathcal{N}} \{\deg((\kappa - x - \nu)^+)\} \\ &= \min_{\nu \in \mathcal{N}} \{\deg(((\kappa - \nu) - x)^+)\} \\ &= \min_{\mu \in \mathcal{N}} \{\deg((\mu - x)^+)\}. \end{aligned}$$

Using the identities $x = x^+ + x^-$ and $x^+ = -(x^-)$, we have

$$\begin{aligned}\min_{\mu\in N}\{\deg((\mu-x)^+)\} &= \min_{\mu\in\mathcal{N}}\{\deg((\mu-x)) - \deg((\mu-x)^-)\} \\ &= \min_{\mu\in\mathcal{N}}\{\deg((\mu-x)) + \deg((x-\mu)^+)\}.\end{aligned}$$

Since $\mu \in \mathcal{N}$ we know that $\deg(\mu) = g-1$, thus $\deg(\mu - x) = g - 1 - \deg(x)$ and thus

$$\begin{aligned}\ell(\kappa - x) &= \deg((\mu-x)) + \min_{\mu\in\mathcal{N}}\{\deg((x-\mu)^+)\} \\ &= g - 1 - \deg(x) + \ell(x).\end{aligned}$$

□

Note that $\kappa \in V$ is the analogue to the canonical divisor in the classical Riemann-Roch formula.

3. Graph Examples

Let Γ be a finite, edge-weighted connected simple graph with n vertices. We will assume that Γ has no loops. Let $w_{ij} \in R$ with $w_{ij} \geq 0$ be the weight of the edge connecting vertices v_i and v_j. The no loops assumption is also applied to the edge weights so that $w_{ii} = 0$ for each i. We showed in [7] that such a graph satisfies an equivalent Riemann-Roch formula as in Theorem 1.1.

In this setting, the genus $g = 1 + \sum_{i<j} w_{ij} - n$, and the canonical element κ is defined by $\kappa(j) = \deg(v_j) - 2$. (Here $\deg(v)$ for a vertex v is the sum of the weights of the edges incident to v.) As shown in [7], the set $\mathcal{N} \subset V_{g-1}$ is generated by a set $\{\nu_1, \ldots, \nu_s\}$ as follows.

Fix a vertex v_k and let $(j_1, \ldots, j_n)$ be a permutation of $(1, \ldots, n)$ such that $j_1 = k$. There are then $(n-1)!$ such permutations; for each permutation, we compute a $\nu \in V_{g-1}$ defined by

$$\nu(j_l) = \begin{cases} -1 & \text{if } l = 1 \\ -1 + \sum_{i=1}^{l-1} w_{j_i j_l} & \text{if } l > 1. \end{cases}$$

Each such ν may not be unique; set s to be the number of unique ν's and index this set $\{\nu_1, \ldots, \nu_s\}$. We then define the set $\mathcal{N}$ as

$$\mathcal{N} = \{x \in V \mid x \sim \nu_i \text{ for some } i = 1, \ldots s\}.$$

Here the equivalence relation $\sim$ is induced by the subgroup

$$H =< h_1, h_2, \ldots, h_{n-1} >\subset V_0$$

where each $h_i \in R^n$ is defined as

$$h_i(j) = \begin{cases} \deg(v_i) & \text{if } i = j \\ -w_{ij} & \text{if } i \neq j. \end{cases}$$

Note that H is the edge-weighted Laplacian of Γ.

As an example, consider a two vertex graph Γ with edge weight $w_{12} = p > 0$. Then $g = p - 1$ and $H =< (p, -p) >= \mathbb{Z}(p, -p)$. The set $\mathcal{N} \subset V_{g-1}$ is $\mathcal{N} = \{\nu \mid \nu \sim (p-1, -1)\}$ and $\kappa = (p-2, p-2)$. Figure 1 shows the divisors $x \in \mathbb{R}^2$ for this graph in the plane. The shaded region, which is bounded by the corner points in the set $\mathcal{N}$, represent points x with $\ell(x) = 0$.

We can show directly that $\mathcal{N}$ and κ for the two-vertex graph example satisfy the necessary condition for Theorem 1.1 to hold. If $\kappa - x \in \mathcal{N}$, then $\kappa - x = m(p, -p) + (p-1, -1)$ for some $m \in \mathbb{Z}$. Solving for x, we have

$$\begin{aligned}x &= (p-2, p-2) - m(p, -p) - (p-1, -1) \\ &= (-mp - 1, mp + p - 1) \\ &= (p-1, -1) - (m+1)(p, -p)\end{aligned}$$

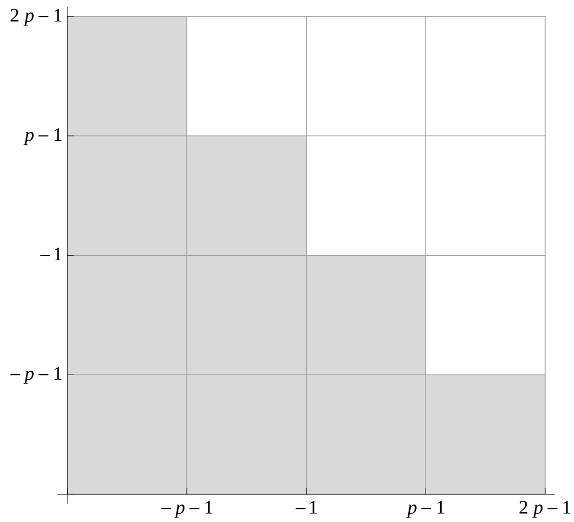

FIGURE 1. Divisors in $\mathbb{R}^2$ for a two-vertex graph with p edges. The shaded region represents points $x \in \mathbb{R}^2$ with $\ell(x) = 0$; for a general point $x \in \mathbb{R}^2$, $\ell(x)$ is the taxicab distance to the shaded region.

and thus $x \in \mathcal{N}$. Similarly, if $\nu \in \mathcal{N}$, it easily follows that $\kappa - \nu \in \mathcal{N}$.

Now consider a three vertex graph with edge weights $w_{12} = p$, $w_{13} = q$ and $w_{23} = r$. The set $\mathcal{N}$ can be generated by $\nu_1 = (-1, p-1, q+r-1)$ and $\nu_2 = (-1, p+r-1, q-1)$; H can be generated by $h_1 = (p+q, -p, -q)$ and $h_2 = (-p, p+r, -r)$. In Figure 2, the region representing points $x \in \mathbb{R}^3$ such that $\ell(x) = 0$ is shown for a three vertex graph with edge weights $p = 1$, $q = 3$ and $r = 4$.

4. Non-graph Examples

The main result of this paper would not be interesting if there were no examples $\mathcal{N}$ and κ that were not derived from graphs.

Theorem 4.1. *There exist $\kappa \in V$ and $\mathcal{N} \subset V_{g-1}$ such that Theorem 1.1 holds where $\mathcal{N}$ is not generated from a finite connected graph.*

Proof. Let $n = 2$ and choose $\mathcal{N} = \{\nu \in G \mid \nu \sim (2, -2)\}$ where $H =< (-4, 4) >$, with $\kappa = (0, 0)$ and $g = 1$. If H were generated from a two vertex graph, using the notation from the previous section we would have $p = 4$. This would require $\kappa = (2, 2)$ with $\mathcal{N}$ generated by $\nu_1 = (3, -1)$.

Since there is no integer m such that $\kappa = (0, 0) = (2, 2) + m(-4, 4)$ (and likewise there is no m such that $\nu_1 = (2, -2) = (3, -1) + m(-4, 4)$), H cannot be generated from a two vertex graph.

Now suppose that $\nu \in \mathcal{N}$. Then $\nu = (2, -2) + m(-4, 4)$ for some integer m, and

$$\begin{aligned} \kappa - \nu &= (0,0) - (2,-2) - m(-4,4) \\ &= (2,-2) - (m-1)(-4,4) \end{aligned}$$

thus $\kappa - \nu \in \mathcal{N}$.

Similarly, if $\kappa - \nu \in \mathcal{N}$, $\kappa - \nu = (2, -2) + m(-4, 4)$ for some integer m, and

$$\begin{aligned} \nu &= \kappa - (2,-2) - m(-4,4) \\ &= (4m-2, -4m+2) \\ &= (2,-2) - (m-1)(-4,4) \end{aligned}$$

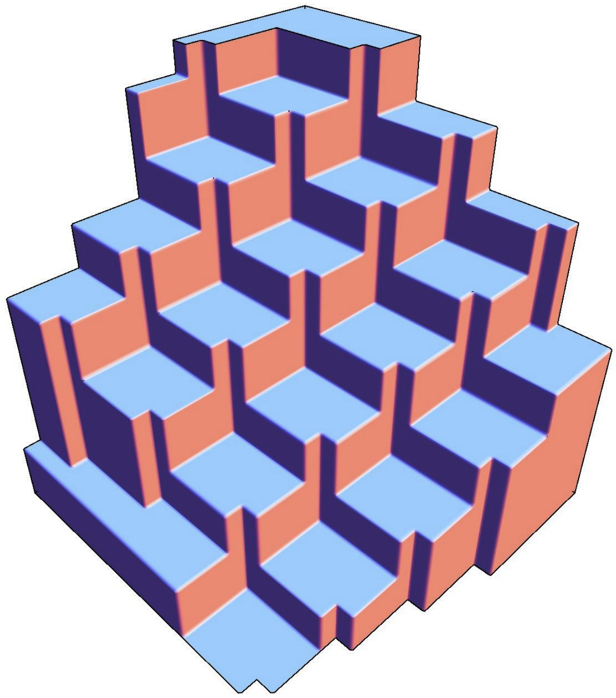

FIGURE 2. Divisors in $\mathbb{R}^3$ for a three-vertex graph edge weights $w_{12} = 1$, $w_{13} = 3$ and $w_{23} = 4$. The solid region represents points $x \in \mathbb{R}^3$ with $\ell(x) = 0$; for a general point $x \in \mathbb{R}^3$, $\ell(x)$ is the taxicab distance to the surface.

thus $\nu \in \mathcal{N}$ and H, κ, $\mathcal{N}$ satisfies Theorem 1.1. □

We include in Figure 3 a representation of the divisors $x \in \mathbb{R}^2$ with $\ell(x) = 0$ for the example used in the proof of Theorem 4.1. The plot is identical to that of a two vertex graph with $p = 4$ but is shifted by -1 in each direction.

It is also possible to produce non-graph examples by using more generators for $\mathcal{N}$. In Figure 4, the divisors $x \in \mathbb{R}^2$ with $\ell(x) = 0$ are shown where $\mathcal{N}$ is generated by two points $\nu_1 = (0, 4)$ and $\nu_2 = (1, 3)$, using $H =< (-3, 3) >$ and $\kappa = (0, 0)$.

As an application to discrete geometry, consider a set of points in $\mathcal{N} \subset \mathbb{R}^n$ along with $\kappa \in \mathbb{R}^n$ and $g \in \mathbb{R}$ which satisfy the conditions of Theorem 1.1. For each $\nu \in \mathcal{N}$, define

$$\mathcal{E}_\nu = \{x \in \mathbb{R}^n \mid x \leq \nu\}$$

and let

$$\mathcal{E} = \bigcup_{\nu \in \mathcal{N}} \mathcal{E}_\nu.$$

The set $\mathcal{E}$ consists of the points $x \in \mathbb{R}^n$ such that $\ell(x) = 0$; more generally, $\ell(x)$ is the taxicab distance from $x \in \mathbb{R}^n$ to $\mathcal{E}$. Theorem 1.1 then gives a lower bound $\ell(x) \geq \deg(x) - g + 1$ as well as an exact formula for $\ell(x)$ using the correction term $\ell(\kappa - x)$. It would be interesting to find and classify different discrete geometric structures, in addition to finite graphs, which admit such $\mathcal{N}$, g, and κ satisifying the symmetry condition of Theorem 1.1.

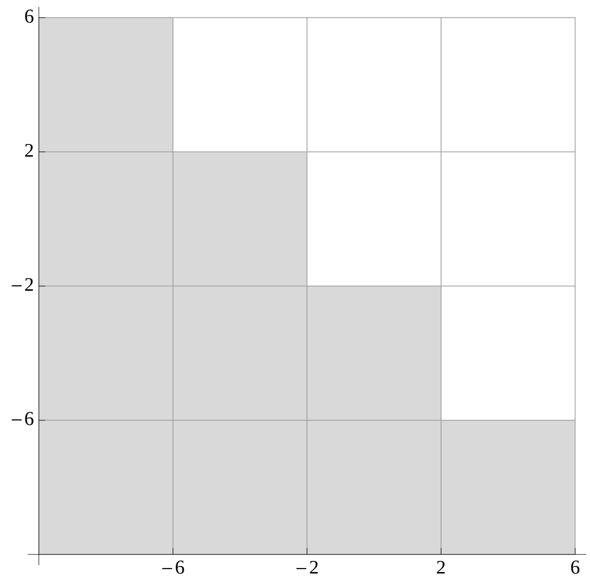

FIGURE 3. Divisors $x \in \mathbb{R}^2$ with $\ell(x) = 0$ for the non-graph example in the proof of Theorem 4.1. Note that this example is identical to the two vertex graph example in Figure 1 with $p = 4$, but shifted by $(-1, -1)$.

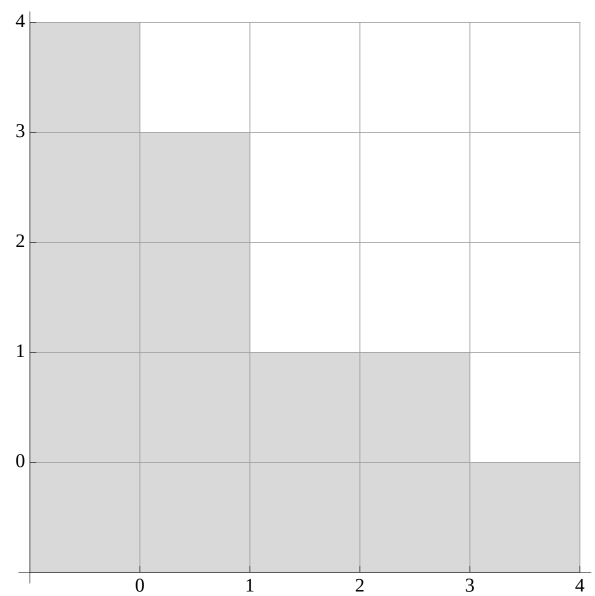

FIGURE 4. Divisors $x \in \mathbb{R}^2$ with $\ell(x) = 0$ for a non-graph example with $\mathcal{N}$ generated by $(0, 4)$ and $(1, 3)$, $H =< (-3, 3) >$ and $\kappa = (0, 0)$.

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