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JACOBIAN MATES FOR NON-SINGULAR POLYNOMIAL MAPS IN $\mathbb{C}^n$ WITH ONE–DIMENSIONAL FIBERS

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A nuestro maestro Xavier Gómez-Mont, con gratitud

Abstract. Let $(F_2, \dots, F_n) : \mathbb{C}^n \to \mathbb{C}^{n-1}$ be a non-singular polynomial map. We introduce a non-singular polynomial vector field X tangent to the foliation $\mathcal{F}$ having as leaves the fibers of the map $(F_2, \dots, F_n)$. Suppose that the fibers of the map are irreducible in codimension ≥ 2, that the one forms of time associated to the vector field X are exact along the leaves, and that there is a finite set at the hyperplane at infinity containing all the points necessary to compactify the affine curves appearing as fibers of the map. Then, there is a polynomial F_1 (a Jacobian mate) such that the completed map $(F_1, F_2, \dots, F_n)$ is a local biholomorphism. Our proof extends the integration method beyond the known case of planar curves (introduced by Ilyashenko [Ily69]).

1. Introduction and Statement of Results

The topological or analytical classification of non-singular polynomial foliations in $\mathbb{C}^n$ is a very hard problem, even in the lowest dimensional case $n = 2$. See [ACL98], [BT06], [Fer05], [NN02], [Tib07] and references therein.

We study the (holomorphic) polynomial foliations by curves $\mathcal{F}$ in $\mathbb{C}^n$ which can be obtained from the fibers of complex polynomials $F_2, \dots, F_n \in \mathbb{C}[z_1, \dots, z_n]$, chosen in such a way that

$$(1) \qquad \begin{cases} (F_2, \dots, F_n) : \mathbb{C}^n \longrightarrow \mathbb{C}^{n-1} & \text{and} \\ dF_2 \wedge \cdots \wedge dF_n & \text{does not vanish at any } z \in \mathbb{C}^n. \end{cases}$$

The fibers of the map in (1) are nonsingular, but possibly reducible, affine curves that we denote by $\{\mathcal{A}_c\}$. The leaves of $\mathcal{F}$ are the connected components (a unique one generically) of those affine curves. We say that $\mathcal{F}$ is a *non-singular polynomial foliation having* $n-1$ *first integrals.*

As a first step toward a general classification a natural problem is to study topologically or analytically this family of foliations.

An interesting subfamily is as follows. The map $(F_2, \dots, F_n)$ has a *Jacobian mate* when there exists a polynomial $F_1 \in \mathbb{C}[z_1, \dots, z_n]$ such that

$$(2) \qquad \begin{cases} F = (F_1, F_2, \dots, F_n) : \mathbb{C}^n \longrightarrow \mathbb{C}^n & \text{and} \\ dF_1 \wedge dF_2 \wedge \cdots \wedge dF_n = dz_1 \wedge \cdots \wedge dz_n. \end{cases}$$

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Recall that the Jacobian Conjecture in $\mathbb{C}^n$ asserts the existence of the inverse map F^{-1} (which has to be also polynomial).

Given $\mathcal{F}$, where are the obstructions to the existence of F_1?

Note that the singularities of the extended foliation to projective space, still denoted by $\mathcal{F}$, are in the hyperplane at infinity of $\mathbb{C}^n$. In the classification problem one can study the singularities at infinity. Instead, our approach focus on the affine behavior and possible "jumps" in the geometry of the fibres $\{\mathcal{A}_c\}$. By a classical result of S. A. Broughton, see [Bro83], there exists an open Zariski set $U \subset \mathbb{C}^{n-1}$ such that the affine foliation $\mathcal{F}$ is a locally trivial fibration in $(F_2, \ldots, F_n)^{-1}(U)$.

Hence we must consider a priori the existence of atypical fibers (i.e. fibers outside U) of (1) and try to describe the behavior of $\mathcal{F}$. In particular, we point out that an example of (1) having atypical fibers and admitting a F_1, will provide a counterexample for the Jacobian Conjecture.

Another related problems with the existence of a Jacobian mate are the following. First, in the holomorphic category, on Stein manifolds the problem of the existence of F_1 is posed in [For03a] p. 146 and [For03b] p. 96, and it remains open (we thank Filippo Bracci for pointing this out to us). Second, the symmetric problem, i.e. given F_1 how to recognize the existence of $(F_2, \ldots, F_n)$ such that (2) is currently under study for $n \geq 3$, see [FR05] p. 3 or [Kal02].

The main tool that we introduce is a polynomial vector field X depending in an essential way of $\mathcal{F}$. Consider the Jacobian matrix of the map (1)

$$\left(\frac{\partial F_j}{\partial z_i}\right)_{2\leq j\leq n,\ 1\leq i\leq n},$$

and let $A_i(z_1, \ldots, z_n)$ be the determinant of the submatrix obtained after removing the i–th column, then

$$X := \sum_{i=1}^{n} (-1)^{i+1} A_i(z_1, \ldots, z_n) \frac{\partial}{\partial z_i}, \tag{3}$$

obviously X is nowhere zero. If there exists a Jacobian mate F_1, then

$$(F_1, \ldots, F_n)^* \frac{\partial}{\partial w_1} = X. \tag{4}$$

X restricted to any fiber $\mathcal{A}_c$, $c \in \mathbb{C}^{n-1}$, of the map (1), gives a tangent vector field on $\mathcal{A}_c$, that we will denote by X_c. It determines a unique holomorphic one form ω_c on $\mathcal{A}_c$, when we require $\omega_c(X_c) = 1$. Thus, each map $(F_2, \ldots, F_n)$ produces a collection of pairs

$$\{(\mathcal{A}_c,\ X_c) \mid c \in \mathbb{C}^{n-1}\}, \qquad \text{equivalently } \{(\mathcal{A}_c,\ \omega_c)\}. \tag{5}$$

In Section 2, we briefly develop this ideas to make the argument more transparent.

Remark 1. *1. The vector field X defines a singular holomorphic foliation $\mathcal{F}$ by curves in $\mathbb{CP}^n$, such that its singular locus is contained in the hyperplane at infinity $\mathbb{CP}^{n-1}_\infty$.*
2. The polynomial vector field X has $n-1$ polynomial first integrals on $\mathbb{C}^n$, and the leaves of the foliation defined by X in $\mathbb{C}^n$ are given by the curves $\{\mathcal{A}_c \mid c \in \mathbb{C}^{n-1}\}$.
3. The hyperplane $\mathbb{CP}^{n-1}_\infty$ is saturated by leaves of $\mathcal{F}$.

In addition

Remark 2. *Up to multiplication by a non-zero constant, X is the unique non vanishing polynomial vector field giving a trivialization for the tangent line bundle of the non-singular holomorphic foliation $\mathcal{F}$ on $\mathbb{C}^n$.*

Indeed, if a second polynomial vector field Y (providing a trivialization of the tangent line bundle to the foliation) exists, then $X = \lambda Y$, for λ an entire function on $\mathbb{C}^n$, nowhere zero. But λ is clearly polynomial, hence it is necessarily a non-zero constant. Moreover, X is independent on the choice of any polynomial F_1 satisfying (2): it only depends on $(F_2, \ldots, F_n)$. Hence, we can use X to explore the existence of F_1.

The main result about affirmative conditions for the existence of F_1, is the following

Theorem 1. *Let $(F_2, \ldots, F_n) : \mathbb{C}^n \longrightarrow \mathbb{C}^{n-1}$ be a polynomial map such that $dF_2 \wedge \cdots \wedge dF_n$ does not vanish at any point of $\mathbb{C}^n$. Consider X as in (3), and suppose furthermore that:*
(i) The reducible fibers $\{\mathcal{A}_c\} \subset \mathbb{C}^n$ determine an algebraic subset of codimension at least 2.
(ii) For every $c \in \mathbb{C}^{n-1}$, the pairs $(\mathcal{A}_c,\ \omega_c)$ satisfy that

$$\int_\gamma \omega_c = 0, \quad \textit{for every } [\gamma] \in H_1(\mathcal{A}_c, \mathbb{Z}).$$

(iii) There is a finite set $Y \subset \mathbb{CP}^n_\infty$ such that each affine curve $\mathcal{A}_c$ is completed in $\mathbb{CP}^n$ by adding points in Y.
Then, there is a polynomial F_1 such that

$$dF_1 \wedge dF_2 \wedge \cdots \wedge dF_n = \ dz_1 \wedge \cdots \wedge dz_n.$$

Note that the second hypothesis is clearly necessary for ω_c to be an exact one form on the fibers $\mathcal{A}_c$. Concerning the first, it is in fact necessary for the integration method that we use: Example 1 shows a function with a reducible fiber (of codimension one), with zero periods, and such that the function constructed by integration as a candidate for Jacobian mate has a pole on that fiber (see Remark 6).

The third hypothesis, obviously satisfied in the case $n = 2$, is automatically satisfied in case that the map F is surjective. In this case, as F has no critical points, all the fibers are one dimensional, and according to [Ga99] p. 158, they share the same cone at infinity, i.e. all the affine curves are completed by adding the same points at infinity (a finite set). Note that this cone at infinity is defined by the vanishing of the polynomials in the ideal generated by the terms of highest degree of the elements of the ideal generated by the components of the function F. This cone at infinity is contained in, but not necessary equal to, the singular set of the foliation $\mathcal{F}$ extended to projective space.

After proving our result by integration method (see below), we realized that in case F is surjective, it is a consequence of a Theorem of Ph. Bonnet (Theorem 1.5 in [Bon03]). Nevertheless, even in that case, as his approach is algebraic, and our proof extends the integration method beyond the case of planar curves previously known (starting with Ilyashenko [Ily69]), we consider that it can be of interest for the people working in the field. Moreover, with this technique as a fundamental tool, together with some considerations on the degree of the map F and computations of the index of X restricted to the fibers of $(F_2, \ldots, F_n)$ (see the end of this Introduction), we have also obtained some new results on negative conditions for the existence of a jacobian mate. They will be presented in a future work, including the solution in a particular case (see Example 1) of a problem posed by L. Dũng Tráng and C. Weber in [DW94].

1.1. **Method and Structure of the proof.** The proof of Theorem 1 is given in several steps below. Note that, to avoid confusion we use $\mathbb{C}^n_z$ and $\mathbb{C}^{n-1}_w$ to denote the domain and the target in map (1).
Step 1. We construct a polynomial one form of time ω for X on $\mathbb{C}^n_z$. By integration of ω along the irreducible fibers of F, see equation (8), we get a candidate function $\widetilde{F}_1$.
Step 2. We verify that the candidate function is holomorphic on the whole $\mathbb{C}^n_z$, see Proposition

1.
Step 3. We estimate the growth of $\widetilde{F}_1$. This is the hardest step. We will study the growth of $\widetilde{F}_1$ at infinity. We recognize the growth of $|\widetilde{F}_1(z)|$ in a suitable set of complex lines in $\mathbb{C}^n_z$. This requires bounds for: the norm of the end points of the integration paths in (8), see Lemma 1, the norm of the ramification points in the fibers $\{\mathcal{A}_c\}$, see Lemma 2, and the length of the integration paths in (8), see Lemma 3. Thus, $|\widetilde{F}_1(z)|$ has polynomial growth in suitable lines, see Lemma 4.
Step 4. In order to show that $\widetilde{F}_1(z)$ is a polynomial, we make an argument by contradiction, using a property of the growth of entire non-polynomial functions, see Lemma 5 and Proposition 2. We show explicitly that F_1 satisfies $dF_1 \wedge dF_2 \wedge \ldots \wedge dF_n = dz_1 \wedge \ldots \wedge dz_n$.

Concerning the proof of Theorem 1, we point out that the powerful method of integration of one forms ω along the algebraic leaves of a polynomial foliation $\{\mathcal{A}_c\}$ in $\mathbb{C}^2$ to find F_1, was introduced by Yu. Ilyashenko, in his foundational work on the second part of the Hilbert's 16–th problem [Ily69]; see also Yakovenko's article [Yak94], that inspired us when searching for the estimates in Step 3 above. The higher dimensional method of integration of rational one forms ω along the leaves of singular codimension–one foliations in higher dimensional affine and projective manifolds appeared in the work [Muc95] of the third author of this article. In our Theorem 1, the bounds for the integration of one forms along the leaves of an one–dimensional foliation on $\mathbb{C}^n$ is more difficult.

2. Meromorphic maps and vector fields on Riemann surfaces

Let $\mathbb{CP}^1 = \mathbb{C}_w \cup \{\infty\}$ be the projective line, having affine coordinate w. The vector field $\partial/\partial w$ induces a holomorphic vector field in $\mathbb{CP}^1$ having double zero at $\infty \in \mathbb{CP}^1$. Let $\mathcal{L}$ be a compact Riemann surface.

Remark 3. *Let $f : \mathcal{L} \to \mathbb{CP}^1$ be a non-constant meromorphic function. The non-identically zero meromorphic vector field*

$$\frac{\partial}{\partial f} := f^* \left(\frac{\partial}{\partial w} \right)$$

is well defined on $\mathcal{L}$. Moreover, f has canonically associated a meromorphic one form ω, such that the diagram commutes

$$\begin{array}{ccc} & \nearrow & \{\omega\} \\ \{f : \mathcal{L} \to \mathbb{CP}^1\} & & \updownarrow \\ & \searrow & \{X = \frac{\partial}{\partial f}\}, \end{array} \tag{6}$$

X and ω are non-identically zero.

In fact, given f, we define $\omega = df$. The one to one correspondence between meromorphic vector fields and meromorphic one forms is given by the equation $\omega(X) \equiv 1$. This ω is called the one form of time for X, since for $p_0,\ p \in \mathcal{L}$ we have

$$f(p) - f(p_0) = \int_{p_0}^{p} \omega = \left\{ \begin{array}{l} \text{complex time to travel from} \\ p_0 \text{ to } p \text{ under the local flow of } \frac{\partial}{\partial f}. \end{array} \right.$$

The diagram (6) comes from the theory of quadratic differentials, see [Muc02]. The correspondence from ω to f in (6) is elementary.

Remark 4. *A non-identically zero meromorphic one form ω, determines a univalued meromorphic function $f(p) = \int^p \omega$ if and only if the periods and residues of ω vanish, i.e.*

$$\int_\gamma \omega = 0 \qquad \text{for each } [\gamma] \in H_1(\mathcal{L} - \{poles\ of\ \omega\}, \mathbb{Z}).$$

In this case, all the arrows in (6) are bijections.

3. Proof of Theorem 1

Starting from the map $(F_2, \dots, F_n)$ satisfying (1), we get the associated vector field X described by (3) in the Introduction.

3.1. **A candidate function.** For the construction of a polynomial one form of time ω, we show that the one form ω_c on $\mathcal{A}_c$ such that $\omega_c(X_c) = 1$ can be obtained as the restriction to the fiber $\mathcal{A}_c$ of a polynomial one form on $\mathbb{C}^n_z$.

Indeed, as X is never vanishing, recall equation (3), by Hilbert's Nullstellensatz we know that $1 \in (A_1, \dots, A_n)$. Then, there are polynomials $a_1, \dots, a_n \in \mathbb{C}[z_1, \dots, z_n]$ such that

$$1 = a_1 A_1 + \cdots + a_n A_n.$$

These a_i are the coefficients of such an ω.

Observe that if $\{a'_i \mid i = 1, \cdots, n\}$ are polynomials giving another possible way of defining a one form ω' such that $\omega'|_{\mathcal{A}_c} = \omega_c$, then

$$(a_1 - a'_1)A_1 + \cdots + (a_n - a'_n)A_n = 0.$$

Hence, for ω on $\mathbb{C}^n_z$ (as above) and every path γ in $\mathcal{A}_c$ we have

$$\int_\gamma \omega_c = \int_\gamma \omega.$$

The third hypothesis in the statement of theorem asserts that there is a finite set

$$Y = \{\text{the points at infinity of the projective curves } \mathcal{P}_c \mid c \in \mathbb{C}^{n-1}_w\} \subset \mathbb{CP}^{n-1}_\infty,$$

so that we can choose a hyperplane H in $\mathbb{CP}^n$ such that $H \cap Y = \emptyset$. We can also assume that it is not contained in the union of the projective varieties given by the closures of the affine hypersurfaces defined by $A_i = 0$, $i = 1, \dots, n$.

We consider the open set

$$\mathcal{R}^c = \mathbb{C}^n_z - \{\mathcal{A}_c \mid \text{reducible }\}.$$

Every point $z \in \mathcal{R}^c$ is in exactly one affine curve of the family $\{\mathcal{A}_c\}$ which we denote as $\mathcal{A}_{c(z)}$, where $c(z) := (c_2(z), \dots, c_n(z)) := (F_2, \dots, F_n)(z) \in \mathbb{C}^{n-1}_w$. The degree of the projective curve $\mathcal{P}_{c(z)}$ (the projectivization of $\mathcal{A}_{c(z)}$) is

$$d \leq d_2 \cdots d_n, \quad \text{where } \ d_j = (degree\ F_j).$$

We have in addition that $H \cap \mathcal{P}_{c(z)}$ consists of d points in $\mathbb{C}^n_z$, counted with multiplicities, for every $c(z) \in \mathbb{C}^{n-1}_w$. Therefore, we have

$$H \cap \mathcal{A}_{c(z)} = \{p_1(z), \dots, p_d(z)\}.$$

By hypothesis $\mathcal{A}_{c(z)}$ is irreducible, having fixed some point $z \in \mathbb{C}^n_z$, we join z to the above points $p_\ell(z) \in H$, by using smooth paths $\gamma_\ell(z)$, $\ell = 1, \dots, d$. inside the affine curve $\mathcal{A}_{c(z)}$. We observe that

$$\text{(7)} \qquad \begin{array}{rccc} f_c: & \mathcal{A}_{c(z)} & \longrightarrow & \mathbb{C} \\ & z & \mapsto & \sum_{\ell=1}^d \int_{\gamma_\ell(z)} \omega \end{array}$$

is a well–defined holomorphic function, independently of the choices of paths, using the second hypothesis in Theorem 1. Moreover, f_c extends as a meromorphic function on the projective curve $\mathcal{P}_{c(z)}$ and there is a well–defined function

$$\begin{array}{rccc} \widetilde{F}_1: & \mathcal{R}^c \subset \mathbb{C}^n_z & \longrightarrow & \mathbb{C} \\ & z & \mapsto & \sum_{\ell=1}^{d} \int_{\gamma_\ell(z)} \omega. \end{array} \tag{8}$$

3.2. **Holomorphicity of the candidate function.**

Proposition 1. *$\widetilde{F}_1(z)$ is holomorphic on the whole $\mathbb{C}^n_z$.*

Proof. Clearly $\widetilde{F}_1(z)$ is holomorphic along the irreducible curves $\mathcal{A}_{c(z)} \subset \mathcal{R}^c$. Now, we prove the holomorphicity of $\widetilde{F}_1$ at z_0 in the transverse directions to $\mathcal{F}$. We will distinguish two situations, depending on the number of points in the intersection of the leaf and the transversal H.

Case 1. Assume $\mathcal{A}_{c(z_0)} \cap H$ consists exactly of d different points. Thus, the leaf $\mathcal{A}_{c(z_0)}$ is transverse to H.

We may consider without loss of generality, that Σ^{n-1} is a tranversal to $\mathcal{F}$ at z_0, i.e. biholomorphic to some $(n-1)$–dimensional polydisk $\Delta^{n-1}(z_0, \epsilon)$, centered at z_0, embedded in $\mathbb{C}^n_z$ and transversal to $\mathcal{F}$. Now, let z be a point in the transverse directions $j = 2, \ldots, n$, i.e. z is inside the polydisk Σ^{n-1}.

Fixed z_0, we consider the leaf $\mathcal{A}_{c(z_0)}$ and the smooth integration paths $\gamma_\ell(z_0)$, $\ell \in \{1, \ldots, d\}$, inside the leaf $\mathcal{A}_{c(z_0)}$.

Since the foliation $\mathcal{F}$ is non-singular on $\mathcal{A}_{c(z_0)}$, a small variation of z in Σ^{n-1}, a transversal at z_0, induces a small (smooth) variation in the paths of integration in (8). In fact, the holonomy of the foliation $\mathcal{F}$ produces germs of biholomorphisms

$$hol(\gamma_\ell(z_0), \ \cdot\ ) : (\Sigma^{n-1}, z_0) \to (\Sigma^{n-1}_\ell, p_\ell(z_0)) \ , \ \ell \in \{1, \ldots, d\},$$

where each $\Sigma^{n-1}_\ell \subset H$ is a local transversal to the foliation $\mathcal{F}$, given by a small $(n-1)$–dimensional polydisk in the hyperplane H centered at $p_\ell(z_0)$, the end point of the path $\gamma_\ell(z_0)$.

If z moves holomorphically in Σ^{n-1} around z_0, the respective end points of the paths $\gamma_\ell(z)$ move holomorphically in $\Sigma^{n-1}_\ell \subset H$, since the end points are given as the values of the biholomorphism $hol(\gamma_\ell(z_0), z) \in \Sigma^{n-1}_\ell$.

Summing up, the end points of the integration paths in (8) and the one form ω vary holomorphically with z. Thus, $\widetilde{F}_1$ is holomorphic in all directions around z_0, when $\mathcal{A}_{c(z_0)}$ is transverse to H.

Case 2. Assume $\mathcal{A}_{c(z_0)} \cap H$ consists of less than d different points. Thus, the leaf $\mathcal{A}_{c(z_0)}$ is tangent to H at some points.

The set $T = \{z \in \mathbb{C}^n_z \mid \mathcal{A}_{c(z)} \text{ is tangent to } H\}$ is a complex algebraic variety in $\mathbb{C}^n_z$ of codimension least one (T can be empty).

Recall that in (8) the intersection points $\mathcal{A}_{c(z_0)} \cap H$ are taken with multiplicities. It follows that $\widetilde{F}_1(z)$ extends continuously to T and is locally bounded at T. By the Riemann extension Theorem (see [FG02], p. 38), $\widetilde{F}_1(z)$ extends holomorphically to T, hence on $\mathcal{R}^c$.

Finally, the reducible fibers $\{\mathcal{A}_c\} \subset \mathbb{C}^n_z$ determine an algebraic subset of codimension at least 2. By the Second Riemann extension Theorem (see [FG02], p. 151), $\widetilde{F}_1(z)$ extends holomorphically over the points in reducible fibers, hence on the whole $\mathbb{C}^n_z$. □

3.3. **The candidate has polynomial growth.** We will prove that $\widetilde{F}_1(z)$ is a polynomial function. For this we study the growth of $|\widetilde{F}_1(z)|$, when $|z|$ goes to infinity along some lines, this in our goal in this subsection.

Let $\varrho = [m_1 : \ldots : m_n] \in \mathbb{CP}^{n-1}_\infty$ be a non-singular point of the foliation $\mathcal{F}$ in the hyperplane at infinity. We make z go to ϱ in a simple way. Let $(\{t \in \mathbb{C}\} \cup \{\infty\})$ be a projective line, consider the parametrized line

$$z(t) : (\mathbb{C} \cup \{\infty\}) \to \mathbb{CP}^n, \tag{9}$$

$$z(t) = \begin{cases} (m_1 t, \ldots, m_n t) & \text{for } t \in \mathbb{C}, \\ \varrho & \text{for } t = \infty. \end{cases}$$

In all what follows
1) z and $|z|$ will go to infinity as $z = z(t)$,
2) $(F_2, \ldots, F_n)(z(t)) := (c_2(t), \ldots, c_n(t)) := c(t) \in \mathbb{C}^{n-1}_w$,
3) $\mathcal{A}_{c(t)} := \mathcal{A}_{c(z(t))}$ and $H \cap \mathcal{A}_{c(t)} := \{p_1(t), \ldots, p_d(t)\}$.

In order to estimate the growth of

$$|\widetilde{F}_1(z(t))| = \left| \sum_{\ell=1}^{d} \int_{\gamma_\ell(z)} \omega \right|,$$

we will first construct integration paths

$$\gamma_\ell(z(t), s) : [0,1] \to \mathcal{A}_{c(t)}$$

inside the family of curves $\{\mathcal{A}_{c(t)}\}$ and bound their lengths in terms of $|t|$ (see Lemma 3). Note that we are using the notations

$$\gamma_\ell(z) = \gamma_\ell(z(t)) = \gamma_\ell(z(t), s)$$

simultaneously, the dependence on t will be continuous, and smooth on the real variable s. We will bound the growth of ω along the path, from a bound on $|\gamma_\ell(z(t), s)|$ for all the points in the trace of the paths, this is attained in the proof of Lemmas 1, 2.

The construction of the integration paths require formerly, the study of the projections of $\mathcal{A}_{c(t)}$ onto the coordinate axes.

Consider the natural projections $\Pi_i : \mathbb{C}^n_z \to \mathbb{C}_i$, $(z_1, \ldots, z_n) \mapsto z_i$, onto the i–th axis. Obviously, they induce functions for every fixed t,

$$\Pi_i : \mathcal{A}_{c(t)} \to \mathbb{C}_i \,,$$

which are holomorphic branched coverings. Moreover in some special cases for $\mathcal{F}$ these functions can be constant.

Fixing t, and so the fiber $\mathcal{A}_{c(t)}$, and one direction of projection $i \in \{1, \ldots, n\}$ as above, we have two relevant sets of points and their corresponding associated disks in $\mathbb{C}_i$, having radii $r(i,t),\ R(i,t) > 0$ as follows:

The first collection of points and associated disks comes from the i–th projection of $z(t)$ and of the intersection points of $\mathcal{A}_{c(z)}$ with H

$$\{\Pi_i(z(t)), \Pi_i(p_1(t)), \ldots, \Pi_i(p_d(t))\} \subset \Delta(0, r(i,t)) \subset \mathbb{C}_i.$$

The second collection of points is determined by the ramification points of the function $\Pi_i : \mathcal{A}_{c(t)} \to \mathbb{C}_i$, $\{\rho_1(i,t), \ldots, \rho_\beta(i,t)\} \subset \mathcal{A}_{c(t)}$, and its projection to the i–th coordinate:

$$\{\Pi_i(\rho_i(i,t)), \ldots, \Pi_i(\rho_\beta(i,t))\} \subset \Delta(0, R(i,t)) \subset \mathbb{C}_i.$$

The number β depends on $\mathcal{A}_{c(t)}$ and Π_i, but we omit this dependence in the notation. By (b) of Corollary 1 below, β will be constant for large enough t.
So, fixed the i–th direction, our problem is *"for $z(t)$ going to fixed $\varrho \in \mathbb{CP}^{n-1}_\infty$, bound the growth of the radii $r(i,t)$, $R(i,t)$ for all sufficiently large $|t|$"*.

Now, we work in order to estimate of the radious $r(i,t)$.

Lemma 1. *Fixing $i \in \{1,\dots,n\}$, there exists $\xi \in \mathbb{N}$ such that $r(i,t) < |t|^{\xi}$ for large enough t. Moreover, this estimate holds for $z(t)$ going to infinity in the directions ϱ' in a small enough polydisk $\Delta^{n-1}(\varrho,\varepsilon) \subset \mathbb{CP}^{n-1}_{\infty}$ avoiding singularities of $\mathcal{F}$.*

Proof. The intersection points in $H \cap \mathcal{A}_{c(t)}$ are described by the system of algebraic equations in $(z_1,\dots,z_n)$,

$$F_2 - c_2(t) = 0,\ F_3 - c_3(t) = 0,\ \dots,\ F_n - c_n(t) = 0,\ H(z_1,\dots z_n) = 0.$$

For fixed t and i, we want to compute the values of z_i where these intersections appear. The elimination ideal of the above system (see [CLO07] Chapter 3, Section 2, in particular Theorem 3 p. 125), that by definition is

$$\langle F_2 - c_2(t),\dots,F_n - c_n(t), H\rangle \cap \mathbb{C}[z_i],$$

determines the required points.

The elimination procedure, described explicitly in [CLO07] p. 116–117, depends on the choice of a Groebner basis for the ideal of our system of equations (that always exists, see [CLO07] p. 77 Corollary 6). There is a polynomial

$$Q_i(z_i,t) = a_{i,0}(t)z_i^d + a_{i,1}(t)z_i^{d-1} + \cdots + a_{i,d-1}(t)z_i + a_{i,d}(t)$$

describing the position of $\{\Pi_i(p_1(t)),\dots,\Pi_i(p_d(t))\}$ in $\mathbb{C}_i$; here $\{a_{i,\alpha}(t)\}$ are polynomials in t, and d is the degree of the curves $\mathcal{A}_{c(t)}$.

The natural number $\xi(i) = \max_\alpha\{degree(a_{i,\alpha}(t))\}$ depends on the Groebner basis chosen. We can write

$$z_i^d + \frac{a_{i,1}(t)}{a_{i,0}(t)}z_i^{d-1} + \cdots + \frac{a_{i,d}(t)}{a_{i,0}(t)} = 0.$$

Recall that $a_{i,\alpha}(t)/a_{i,0}(t)$ are α–th elementary symmetric functions of the roots. The roots of $Q_i(z_i,t)$ grow at most as $\max_\alpha\{a_{i,\alpha}(t)/a_{i,0}(t)\}$, that is at most like $|t|^{\xi(i)}$, when t goes to infinity. So they are contained in a disk of radius $|t|^{\xi(i)}$.

The computation of the growth is similar for every $i \in \{1,\dots,n\}$. Let us define

$$\xi = (\max_i\{\xi(i)\}) + 1.$$

In addition, for the original point $z(t)$, the norm of the projection $|\Pi_i(z(t))|$ grows linearly, hence $\Pi_i(z(t)) \in \Delta(0,|t|^{\xi})$, for large enough t. The exponent ξ satisfies the assertion in the Lemma.

Finally, the bound is independent on the choice of ϱ' varying in a small enough polydisk $\Delta^{n-1}(\varrho,\epsilon)$, that is the second assertion in the Lemma. □

Now, we get the estimates for the radious $R(i,t)$ of the disks containing all the projections of the ramification points of Π_i restricted to $\mathcal{A}_{c(t)}$.

Lemma 2. *Fixing $i \in \{1,\dots,n\}$, there exists $\kappa \in \mathbb{N}$ such that $R(i,t) < |t|^{\kappa}$ for large enough t. Moreover, this estimate holds for $z(t)$ going to infinity in the directions ϱ' in a small enough polydisk $\Delta^{n-1}(\varrho,\varepsilon) \subset \mathbb{CP}^{n-1}_{\infty}$ avoiding singularities of $\mathcal{F}$.*

Proof. Observe that the ramification points of $\Pi_i : \mathcal{A}_{c(t)} \to \mathbb{C}_i$ come from the vanishing of the i–th coordinate of the vectors in the kernel of the differential of the map (1) at the points in $\mathcal{A}_{c(t)}$, which give the tangent space to $\mathcal{A}_{c(t)}$.

The condition above is given by the vanishing of the determinant of the matrix obtained by adding $(0,\dots,1,\dots,0)$, where the 1 is placed in the i–th column, as the last row to

$$\left(\frac{\partial F_j}{\partial z_i}\right)_{2\le j\le n,\ 1\le i\le n}.$$

This determinant is exactly $A_i(z_1, \ldots, z_n)$ in the definition of the vector field X, (3). Hence the tangencies of $\mathcal{A}_{c(t)}$ with the hyperplanes $\{z_i = \textit{const.}\}$ in $\mathbb{C}^n_z$ are given by the following system of algebraic equations in $(z_1, \ldots, z_n)$

$$F_2 - c_2(t) = 0,\ F_3 - c_3(t) = 0,\ \ldots,\ F_n - c_n(t) = 0,\ A_i = 0.$$

For fixed t, we want to compute the i–th projection of the points where these tangencies appear. The elimination ideal of the above system

$$\langle F_2 - c_2(t), \ldots, F_n - c_n(t), A_i\rangle \cap \mathbb{C}[z_i],$$

determines the smallest algebraic variety containing the i–th projection of the ramification points of $\{\Pi_i(\rho_1(i,t)), \ldots, \Pi_i(\rho_\beta(i,t))\}$.

Using the elimination procedure and the existence of Groebner basis for the ideal as in the proof of Lemma 1, we know that there exists a polynomial

$$P_i(z_i, t) = b_{i,0}(t) z_i^\beta + \cdots + b_{i,\beta-1}(t) z_i + b_{i,\beta}(t)$$

whose roots give the projection of the ramification points above. The degree β is the number of ramification points of Π_i on $\mathcal{A}_{c(z)}$, and it is generically independent of i and t, for large enough t.

We can estimate the growth of the roots of $P_i(z_i, t)$ when t goes to infinity, as we did in the previous Lemma, so that we get a natural number $\kappa(i)$ (depending on the choice of the Grobner basis) such that they are contained in a disk of radius growing like $|t|^{\kappa(i)}$. Let us define

$$\kappa = (\max_i\{\kappa(i)\}) + 1;$$

this exponent provides the estimate in the Lemma. Finally, the bound is independent on the choice of ϱ' varying in a small enough polydisk $\Delta^{n-1}(\varrho, \epsilon)$, that is the second assertion in the Lemma. $\square$

Summing up Lemmas 1 and 2, for the family of fibers $\mathcal{A}_{c(t)}$, we define the exponent $\varsigma := \max\{\xi, \kappa\}$. The n–dimensional polydisk $\Delta^n(0, |t|^\varsigma) \subset \mathbb{C}^n_z$, satisfies the following. The intersection

$$\mathcal{A}_{c(t)} \cap \Delta^n(0, |t|^\varsigma),$$

for large enough t, contains: the original point $z(t)$; the points $p_\ell(t)$, $\ell = 1, \ldots, d$ in $\mathcal{A}_{c(t)} \cap H$; and the ramification points $\rho_j(i,t)$, $j = 1, \ldots, \beta(i,t)$ of the functions $\Pi_i : \mathcal{A}_{c(t)} \to \mathbb{C}_i$, for all $i \in \{1, \ldots, n\}$.

Corollary 1. *There exists some t_0 such that for all $|t| > |t_0|$ the following facts hold.*
a) The intersection $\mathcal{A}_{c(t)} \cap \Delta^n(0, |t|^\varsigma)$ is a path connected Riemann surface.
b) The family of Riemann surfaces

$$\{\mathcal{A}_{c(t)} \cap \Delta^n(0, |t|^\varsigma) \mid |t| > |t_0|\}$$

is topologically trivial respect to t.

Proof. For a), note that there is always a direction such that the projection

$$\Pi_i : (\mathcal{A}_{c(t)} - \Delta^n(0, |t|^\varsigma)) \to \mathbb{C}_i$$

is a non-constant, unramified holomorphic covering. Without loss of generality, we can suppose $i = 1$. We remove from $\mathcal{A}_{c(t)}$ the preimages of the punctured closed disk $\{|z_1| \geq |t|^\varsigma\} \subset \mathbb{C}_1$. These preimages are disjoint punctured disks in $\mathcal{A}_{c(t)}$ (i.e. biholomorphic to $\Delta(0,1) - \{0\}$). Then, $\mathcal{A}_{c(t)} \cap \Delta^n(0, |t|^\varsigma)$ is path connected.

For b), the atypical fibers of each F_j, $j \in \{2, \ldots, n\}$, determine a finite number of hypersurfaces, see [Bro83]. The projective closure of each of them intersects the hyperplane at infinity in a hypersurface. If we choose the point at infinity ϱ, in (9) outside of these hypersurfaces in

$\mathbb{CP}^{n-1}_{\infty}$, then the family $\{\mathcal{A}_{c(t)}\}$ is locally trivial for suitable values of t. The assertion follows. □

Lemma 3. *There exists a point $\varrho \in \mathbb{CP}^{n-1}_{\infty}$, such that for $z(t)$ going to ϱ as in (9), we have a continuous family of smooth paths*

$$\left\{\gamma_\ell(z(t),\,\cdot):[0,1]\to \mathcal{A}_{c(t)}\cap\Delta^n(0,|t|^\varsigma)\,,\ \ell=1,\dots,d\,\right\}_{t\in U},$$

where $U=\mathbb{C}-\overline{\Delta(0,M)}$, satisfying that for every ℓ, $\gamma_\ell(z(t),0)=z(t)$ and $\gamma_\ell(z(t),1)=p_\ell(t)$, as required for the paths in (8), with

$$(\textit{length of }\ \gamma_\ell(z(t),s))<|t|^K,$$

for certain $K\in\mathbb{N}$.

Moreover, the above assertions are valid for $z(t)$ going to infinity in the directions ϱ' in a small enough polydisk $\Delta^{n-1}(\varrho,\varepsilon)\subset\mathbb{CP}^{n-1}_{\infty}$ avoiding singularities of $\mathcal{F}$.

Proof. Let us consider the polynomials A_i in the definition of the the vector field X, (3). Recall that $A_1,\dots,A_n$ do not vanish simultaneously. For a generic choice of $\varrho\in\mathbb{CP}^{n-1}_{\infty}$ and for large enough t, we have

$$A_i(z(t))\neq 0,\ i\in\{1,\dots,n\}.$$

It follows that the initial points of the paths that we are searching for $\gamma_\ell(z(t),s)$ are not ramification points of Π_1, for large enough t. Observe that the projections $\Pi_i|_{\mathcal{A}_{c(t)}}$ are then ramified coverings over $\mathbb{C}_i$.

We also observe that for a generic choice of $\varrho\in\mathbb{CP}^{n-1}_{\infty}$ the paths $\gamma_\ell(z(t),s)$ will have no ramification points of Π_1 as end points $\{p_\ell(t)\}$ for large enough t. To see this, recall that the affine hyperplane determined by H is not contained in the hypersurface $A_i=0$, for any i. Take a point $p\in H$ such that $A_i(p)\neq 0$, for every i. Clearly, the choice of p can be done in such a way that all the points in $F^{-1}(F(p))\cap H$ satisfy the preceding condition.

Take the line through the origin in $\mathbb{C}^n$ determined by p, and let $\varrho\in\mathbb{CP}^{n-1}_{\infty}$ be the corresponding direction. For $z(t)$ going to infinity along this line, we define the algebraic affine surface

$$S^2=\{\mathcal{A}_{c(t)}\mid t\in\mathbb{C}\}$$

given by the union of the fibers $\mathcal{A}_{c(t)}$ intersecting the line $\{z(t)\mid t\in\mathbb{C}\}$. In fact, $(F_2,\dots,F_n)(z(t)):\mathbb{C}\to\mathbb{C}^{n-1}_w$ is a polynomial entire curve $\mathcal{C}$ and its closure $\overline{\mathcal{C}}$ is a rational projective curve in $\mathbb{CP}^{n-1}$. Consider $I(\mathcal{C})=\langle g_1,\dots,g_\nu\rangle$ the affine ideal in $\mathbb{C}[w_2,\dots,w_n]$ describing $\mathcal{C}$ as an algebraic curve. The ideal $(F_2,\dots,F_n)^*I(\mathcal{C})=\langle g_1\circ(F_2,\dots,F_n),\dots,g_\nu\circ(F_2,\dots,F_n)\rangle$ in $\mathbb{C}[z_1,\dots,z_n]$ determines S^2, showing that it is an algebraic surface.

By the conditions imposed in the choice of the direction ϱ along which $z(t)$ goes to infinity, we can assure that $\{A_1=0\}\cap H\cap S^2$ is at most a finite number of points. We get

$$\{A_1=0\}\cap H\cap\mathcal{A}_{c(t)}=\emptyset$$

for large enough t. It follows that the end points $\{p_\ell(t)\}=H\cap\mathcal{A}_{c(t)}$ are not ramification points for large enough t, as we asserted. Observe that this is still the case for ϱ' in a small enough polydisk $\Delta^{n-1}(\varrho,\varepsilon)\subset\mathbb{CP}^{n-1}_{\infty}$.

We take the polydisk $\Delta^n(0,|t|^\varsigma)$ in such a way that it contains $z(t)$, the points $p_1(t),\dots,p_d(t)$, and all the ramification points of the projection Π_1 in $\mathcal{A}_{c(t)}$ (see Lemmas 1 and 2). We will focus on the restricted map

$$\Pi_1:\mathcal{A}_{c(t)}\cap\Delta^n(0,|t|^\varsigma)\to\Delta(0,|t|^\varsigma).$$

Recall that $\mathcal{A}_{c(t)}\cap\Delta^n(0,|t|^\varsigma)$ is path connected (Corollary 1, a). Choose a path from $z(t)$ to $p_\ell(t)$, and project it onto $\Delta(0,|t|^\varsigma)$. Now, choose a smooth path $\gamma_0(s):=z_1(s)$, $0\leq s\leq 1$, in

$\Delta(0,|t|^\varsigma)$ joining $z_1(t)$ to $\Pi_1(p_\ell(t))$, homologous to this projected one, and so that it does not pass through the image of any ramification point of Π_1 on $\mathcal{A}_{c(t)}$. Lifting γ_0 to $\mathcal{A}_{c(t)}$ we have a smooth path

$$\gamma_\ell(z(t),s) = (z_1(s), z_2(z_1(s)), \ldots, z_n(z_1(s))),$$

joining $z(t)$ to $p_\ell(t)$ (we omit the dependence on ℓ and t, in the right term of above notation). Using the Implicit Function Theorem, we have

$$F_j(z_1, z_2(z_1), \ldots, z_n(z_1)) = c_j(t), \quad j = 2, \ldots, n$$

and taking derivatives we get a system of $n-1$ equations

$$\frac{\partial F_j}{\partial z_1} + \frac{\partial F_j}{\partial z_2} z_2' + \cdots \frac{\partial F_j}{\partial z_n} z_n' = 0, \quad j = 2, \ldots n$$

where we write $z_j' = \frac{\partial z_j}{\partial z_1}$. From the system above, we conclude that

$$z_j' = \frac{\widehat{A}_j}{A_1}, \tag{10}$$

where $\widehat{A}_j$ is the minor obtained after replacing the j–th column in the system by $(-\frac{\partial F_2}{\partial z_1}, \ldots, -\frac{\partial F_n}{\partial z_1})$. So we have that $\widehat{A}_j = (-1)^j A_j$, recall (3). If we now derive with respect to s the lifted path $\gamma_\ell(z(t),s)$, we get

$$\dot{\gamma}_\ell(z(t),s) = (\dot{z_1}, z_2'\,\dot{z}_1, \ldots, z_n'\,\dot{z}_1)(s)$$

and

$$(\text{length of } \ \gamma_\ell(z(t),s)) = \int_0^1 |\dot{\gamma}_\ell(z(t),s)| ds =$$

$$= \int_0^1 \left(|\dot{z_1}| \sqrt{1 + |z_2'|^2 + \cdots + |z_n'|^2} \right) ds. \tag{11}$$

As $\{A_1 = 0\}$ and $S^2 \cap H$ are algebraic sets, there exists a number $K_0 \in \mathbb{N}$, such that each lifted path is chosen such that

$$|A_1(\gamma_\ell(z(t),s)| \geq \frac{1}{|t|^{K_0}}$$

going to infinity for all $0 \leq s \leq 1$ and large enough t. Note that this condition can be assured for all the directions ϱ' in a small enough polydisk $\Delta^{n-1}(\varrho, \varepsilon)$ at infinity.

We have from (10) and (11) that

$$(\text{length of } \ \gamma_\ell(z(t),s)) \leq \int_0^1 \left(|\dot{z_1}| \sqrt{1 + \frac{|\widehat{A}_2|^2 + \cdots + |\widehat{A}_n|^2}{\frac{1}{|t|^{K_0}}}} \right) ds.$$

As the determinants $|\widehat{A}_j|$ are products of polynomials of known degrees

$$\left(|\widehat{A}_2|^2 + \cdots + |\widehat{A}_n|^2 \right) \leq |t|^{K_1}$$

for certain $K_1 \in \mathbb{N}$ (for all $0 \leq s \leq 1$) which gives

$$(\text{length of } \ \gamma_\ell(z(t),s)) \leq (\text{length}\, \gamma_0) \cdot |t|^{K_0 + K_1}.$$

We finish by noting that a simple choice of the path γ_0 verifying all the conditions required above can be made inside the disk $\Delta(0,|t|^\varsigma)$, and in such a way that its length is less than twice the diameter of the disk. This ends the proof of Lemma 3, choosing $K > \varsigma + K_0 + K_1$. $\square$

Remark 5. *The estimate for the length in Lemma 3 is inspired by Yakovenko, see* [Yak94]*, who dealt with the case* $n = 2$*. We tried to make the construction transparent by lifting smooth paths not passing through branching points, by means of the Implicit Function Theorem.*

Lemma 4. $|\widetilde{F}_1(z(t))|$ *grows polynomially if* $|z(t)|$ *goes linearly to infinity in the directions determined by* $\varrho' \in \Delta^{n-1}(\varrho, \varepsilon) \subset \mathbb{CP}^{n-1}_\infty$.

Proof. We fix one parametrized complex line $z = z(t)$ as in (9), going to a point at infinity in the polydisk determined in Lemma 3, and bound the growth of $\widetilde{F}_1(z(t))$. Recall that we have

$$|\widetilde{F}_1(z(t))| = \left| \sum_{\ell=1}^{d} \int_{\gamma_\ell(z(t))} \omega \right| \leq \sum_{i=1}^{n} \left| \sum_{\ell=1}^{d} \int_{\gamma_\ell(z(t))} a_i(\gamma_\ell(z(t))) dz_i \right|;$$

where $a_i(z_1, \ldots, z_n)$ are polynomials on $\mathbb{C}^n_z$ defining ω (see the begin of Subsection 3.1), and the notation $\gamma_\ell(z(t))$ omit the dependence on the real parameter s.

We bound the terms in the righthand side for each $a_i dz_i$ and each path $\gamma_\ell(z(t))$, where $i \in \{1, \ldots, n\}$ and $\ell \in \{1, \ldots, d\}$. Note that

$$\left| \int_{\gamma_\ell(z(t))} a_i(z_1, \ldots, z_n) dz_i \right| = \left| \int_0^1 a_i(\gamma_\ell(z(t))) dz_i(\gamma_\ell(z(t))) \right|.$$

Now we use the following bounds that were previously stated. Since $|\gamma_\ell(z(t))| < |t|^\varsigma$ and $a_i(z_1, \ldots, z_n)$ is a polynomial of degree $\delta(i)$ (this degree is not explicit, see Subsection 3.1), the norm $|a_i(\gamma_\ell(z(t)))|$ is bounded by $|t|^{\varsigma+\delta(i)}$. By Lemma 3, the lengths of the paths and their projections $dz_i(\gamma_\ell(z(t)))$ are bounded by $|t|^K$. Finally, if $\delta := \max_i\{\delta(i)\}$, then we can assert that

$$|\widetilde{F}_1(z(t))| < nd|z(t)|^{\varsigma+\delta+K},$$

for large enough t.

Moreover, all the bounds above remain true under variations of ϱ' in a small enough $(n-1)$–dimensional polydisk $\Delta^{n-1}(\varrho, \varepsilon) \subset \mathbb{CP}^{n-1}_\infty$, as asserted in Lemmas 1, 2, and 3. □

3.4. **The candidate is polynomial.** Now, in order to show that $\widetilde{F}_1(z)$ is polynomial we proceed by contradiction. The next result seems to be well known, however we could not find it explicitly in the literature.

Lemma 5. *Let* $\Lambda(z)$ *be an entire non-polynomial function in* $\mathbb{C}^n_z$*. The locus of points* $[m_1 : \ldots : m_n] \in \mathbb{CP}^{n-1}_\infty$ *such that* $|\Lambda(m_1 t, \ldots, m_n t)|$ *grows at most like* $|t|^\rho$*, for large enough* $\rho \in \mathbb{N}$*, is contained in an algebraic subvariety of codimension at least 1 in* $\mathbb{CP}^{n-1}_\infty$.

Proof. As usual, define the multi-index $\nu := (\nu_1, \ldots, \nu_n) \in (\mathbb{N} \cup \{0\})^n$ and its associated degree and monomial as

$$|\nu| := \nu_1 + \ldots + \nu_n, \quad z^\nu := z_1^{\nu_1} \cdots z_n^{\nu_n}.$$

The power series expansion of our entire function is

$$\Lambda(z) = \sum_{|\nu|=0}^{\infty} c_\nu z^\nu.$$

Consider the directions $m := [m_1 : \ldots : m_n]$ such that $|\Lambda(m_1 t, \ldots, m_n t)|$ grows less than $|t|^\rho$, for all sufficiently large $|t|$, where ρ is fixed. For these directions the higher order terms in the series must vanish, i.e.

$$\sum_{s \geq 1} \left(\sum_{|\nu|=\rho+s} c_\nu m^\nu \right) t^{\rho+s} = 0.$$

This equation must be true for sufficiently large $|t|$, in consequence it can be split in a numerable set of equations

$$\sum_{|\nu|=\rho+s} c_\nu m^\nu = 0,\ s \in \mathbb{N}.$$

For fixed s, the corresponding equation is homogeneous of degree $\rho+s$, in the variables $m_1, \ldots, m_n$ of $\mathbb{CP}^{n-1}_\infty$.

Λ is entire but it is not a polynomial, hence it has coefficients $c_\nu \neq 0$ for arbitrarily large $|\nu|$. Take such a ν_0 with $c_{\nu_0} \neq 0$. Then, the homogeneous equation of degree $|\nu_0|$ determines a non-trivial algebraic subvariety $T_{|\nu_0|} \subset \mathbb{CP}^{n-1}_\infty$.

For each ν with $|\nu| = \rho + s$, we have an algebraic subvariety $T_{|\nu|} \subset \mathbb{CP}^{n-1}_\infty$. The set of directions producing growth at most like $|t|^\rho$ is the intersection

$$\bigcap_{|\nu|\geq\rho+1} T_{|\nu|} \subset T_{|\nu_0|},$$

that is the desired algebraic variety. □

Proposition 2. *$\widetilde{F}_1(z)$ is polynomial.*

Proof. By Lemma 4 the restriction $\widetilde{F}_1(m_1t, \ldots, m_nt)$ grows at most like a polynomial in $|t|$ for an open set $\Delta^{n-1}(\varrho, \varepsilon)$ of points in $\mathbb{CP}^{n-1}_\infty$. Assuming that $\widetilde{F}_1(z)$ is a non-polynomial entire function, we get a contradiction, since it must grow slowly in at most a proper algebraic subvariety of points in the hyperplane at infinity, by Lemma 5. Thus, $\widetilde{F}_1$ is a polynomial. □

Let us check the algebraic independence of $\widetilde{F}_1$ with respect to $F_2, \ldots, F_n$. Considering the holomorphic n–form

$$d\widetilde{F}_1 \wedge dF_2 \wedge \cdots \wedge dF_n = \phi(z_1, \ldots, z_n) dz_1 \wedge \cdots \wedge dz_n,$$

ϕ is a nowhere vanishing polynomial. Indeed, by contradiction, let $p \in \mathbb{C}^n_z$ be a point with $\phi(p) = 0$. This says that $d\widetilde{F}_1|_p$ is linearly dependent with $dF_2|_p, \ldots, dF_n|_p$. Then, for $d\widetilde{F}_1|_p$ induces the zero one form on the tangent line $T_p\mathcal{A}_c$ at p. This is a contradiction, since $d\widetilde{F}_1$ is the multiple $d \cdot \omega$ (here $d \geq 1$ is the degree of $\mathcal{A}_c$) and ω is non-zero in every $T_p\mathcal{A}_c$. Hence $\phi \in \mathbb{C}^*$. We define $F_1 := (1/\phi)\widetilde{F}_1$. *The proof of Theorem 1 is done.*

3.5. **Some examples.** For $n = 2$, we show polynomials F_2 satisfying the condition that dF_2 is nowhere zero, having in one case all the periods of ω zero, and with non-zero periods in the other.

Example 1. *A non-singular polynomial with zero periods*

$$F_2(z_1, z_2) = z_1 - z_1^2 z_2.$$

This is the polynomial described by S. A. Broughton, studied in [Bro83], [DW94] and [Dun08], but without considering the residues as we do here. It has irreducible typical fiber $\mathcal{A}_c = \{F_2(z_1, z_2) = c\}$, $c \neq 0$, biholomorphic to $\mathbb{C}^*$. When $\mathcal{A}_c$ is completed with its points at infinity, we have the rational curve: $z_0^2 z_1 - z_1^2 z_2 - c z_0^3 = 0$. It meets the infinity line $z_0 = 0$ at two points, $[0 : 1 : 0]$ and $[0 : 0 : 1]$. The first is a smooth point of the curve, and the second a singular one. Note that we can parametrize our projective curve as

$$\Upsilon[s : \zeta] = [\zeta^2 s : \zeta^3 : \zeta s^2 - cs^3] : \mathbb{CP}^1 \to \mathcal{A}_c,$$

the two distinguished points corresponding to $s = 0$ and to $\zeta = 0$.

In the affine neighbourhood given by $z_1 = 1$, we have that the curve can be parametrized as $\varphi(s) = (s, s^2 - cs^3)$, and so (for $s \neq 0$), we have $(1/s,\ s - cs^2)$ in the original $\mathbb{C}^2 = \{z_0 \equiv 1\}$. Hence, its derivative $(-1/s^2,\ 1 - 2cs)$ (that coincides with the restriction of

$$X = z_1^2 \frac{\partial}{\partial z_1} + (1 - 2z_1 z_2)\frac{\partial}{\partial z_2},$$

to the curve), says that we have the vector field $\varphi^* X = \partial/\partial s$ in $\mathbb{CP}^1$, which is regular at $s = 0$.

Concerning the point $[0 : 0 : 1]$, we have that the curve has a cusp. If $s \neq 0$, and for $\zeta \neq 0$, we note that the image point is $[1 : \zeta : (\zeta - c)/\zeta^2]$, that is the point of affine coordinates $\phi(\zeta) = (\zeta, (\zeta - c)/\zeta^2)$ in the original $\mathbb{C}^2 = \{z_0 \equiv 1\}$. The tangent vector to the affine curve so parametrized is $\phi' = (1, (-\zeta + 2c)/\zeta^3)$. Comparing with the restriction of the vector field X to it, we have that

$$X|_{\phi(\zeta)} = \phi_*(\zeta^2 \frac{\partial}{\partial \zeta}),$$

and the one form such that $\omega(X) = 1$, is written as $d\zeta/\zeta^2$ on the curve. Its period around the pole at $\zeta = 0$ vanishes.

Moreover, if we ignore for a moment the fact that the atypical fiber $\mathcal{A}_0 = \{z_1 = 0\} \cup \{1 - z_1 z_2 = 0\}$ has two irreducible components, and we try to construct $\widetilde{F}_1$ on $\mathbb{C}^2_z - \{F_2 = 0\}$, we get the next result.

Remark 6. *For $F_2(z_1, z_2) = z_1 - z_1^2 z_2$, the candidate function $\widetilde{F}_1$ has a pole in the atypical fiber $\mathcal{A}_0 = \{z_1 z_2 = 1\}$ of F_2.*

Indeed, a global one form of time is

$$\omega = 4z_2^2 dz_1 + (1 + 2z_1 z_2) dz_2,$$

in fact $\omega(X) \equiv 1$. Consider the line $H = \{z_1 - z_2 = 0\}$ transversal to the foliation defined by the fibers of F_2. For each point $z = (z_1, z_2)$, define $c = z_1 - z_1^2 z_2$ and consider the points

$$H \cap \mathcal{A}_c = \{p_1(c), p_2(c), p_3(c)\} = \{\phi(\zeta_1), \phi(\zeta_2), \phi(\zeta_3)\}.$$

For $c \neq 0$, they are determined in the domain of $\phi(\zeta) : \mathbb{C}^* \to \mathcal{A}_c$ by the three roots of the polynomial $\zeta^3 - \zeta + c = 0$. Note that $\phi(\zeta)$ depends on c, but we omit this fact in our notation. In particular since $c = z_1 - z_1^2 z_2$ we have that $\phi(z_1) = (z_1, z_2)$ holds. Following (8), there is a holomorphic function

$$\begin{array}{cccc} \widetilde{F}_1 : & \mathbb{C}^2_z - \mathcal{A}_0 & \longrightarrow & \mathbb{C} \\ & (z_1, z_2) & \mapsto & \sum_{\ell=1}^3 \int_{\gamma_\ell(z_1,z_2)} \omega = \sum_{\ell=1}^3 \int_{z_1}^{\zeta_\ell} \frac{d\zeta}{\zeta^2}. \end{array}$$

We want to study the behavior of $\widetilde{F}_1(z_1, z_2)$ near the atypical fiber $\mathcal{A}_0 := \{1 - z_1 z_2 = 0\}$. For example for $a \neq 0$, we fix $z_1 = a$ and compute

$$\lim_{(a,z_2) \to \mathcal{A}_0} |\widetilde{F}_1(a, z_2)| = \lim_{z_2 \to \frac{1}{a}} |\widetilde{F}_1(a, z_2)|.$$

Note that for $c = 0$,

$$H \cap \mathcal{A}_0 = \{(1, 1), (0, 0), (-1, -1)\} = \{\phi(1), \phi(0), \phi(-1)\}.$$

By using the continuity of the roots of $\zeta^3 - \zeta + c = 0$ as functions of the parameter $c = a - a^2 z_2$ near 0 (equivalently, for z_2 near $1/a$), we obtain that the values $\{\zeta_1(z_2),\ \zeta_2(z_2),\ \zeta_3(z_2)\}$ describing $H \cap \mathcal{A}_c$ remain near $\{1, 0, -1\}$ respectively. We get

$$\lim_{z_2 \to \frac{1}{a}} \left| \sum_{\ell=1}^3 \int_{z_1}^{\zeta_\ell(z_2)} \frac{d\zeta}{\zeta^2} \right| = \left| \int_a^1 \frac{d\zeta}{\zeta^2} + \int_a^0 \frac{d\zeta}{\zeta^2} + \int_a^{-1} \frac{d\zeta}{\zeta^2} \right| = \infty.$$

In fact, in the righthand side the first and third integrals remain bounded when z goes to $1/a$. Hence, $|\widetilde{F}_1(a, z_2)|$ goes to infinity, when z_2 goes to $1/a$. $\widetilde{F}_1(z_1, z_2)$ is a rational function having a pole at the atypical fiber $\{1 - z_1 z_2 = 0\}$. □

Example 2. *A non-singular polynomial with non zero periods*

$$F_2(z_1, z_2) = z_1 - z_1^4 z_2^4.$$

This is also in the classification of polynomials with one critical value and no critical points in [Bod02]. The fiber over 0 is reducible, with a component which is topologically $\mathbb{C}$, and another one which is the Riemann sphere minus several points.

The level curve $\{F_2 = c\}$ corresponds to an octic in $\mathbb{CP}^2$ of equation:

$$z_0^7 z_1 - z_1^4 z_2^4 - c z_0^8 = 0.$$

The curve meets the line at infinity $z_0 = 0$ at the two points $[0 : 1 : 0]$ and $[0 : 0 : 1]$. It is singular at the two and if we look at the affine $\mathbb{C}^2 = \{z_1 \equiv 1\}$ of the first, we have the affine curve $z_0^7(1 - cz_0) - z_2^4 = 0$, that is singular (it has a cusp) at $(0, 0)$, with tangent line $z_2 = 0$. Furthermore, the contact of this tangent with the curve is $\dim_{\mathbb{C}} \frac{\mathcal{O}_{\mathbb{C}^2_0}}{(z_2, z_0^7(1-cz_0)-z_2^4)} = 8$.

On the other hand, if we look at the affine neighbourhood $\{z_2 \equiv 1\}$ of the second point, we see that the affine curve is given by $z_0^7 z_1 - z_1^4 - cz_0^8 = 0$. It is singular at $(0, 0)$ and the tangent is $z_1 = 0$. The contact of the curve and the tangent is $\dim_{\mathbb{C}} \frac{\mathcal{O}_{\mathbb{C}^2_0}}{(z_1, z_0^7 z_1 - z_1^4 - cz_0^8 = 0)} = 7$.

Hence, in order to parametrize we can consider the conics that pass through $(0 : 1 : 0)$, $(0 : 0 : 1)$ and have as tangents at them the lines $z_2 = 0$ and $z_1 = 0$, respectively. The conics fulfilling the conditions are those written as

$$s z_1 z_2 + \zeta z_0^2, \quad [s : \zeta] \in \mathbb{CP}^1.$$

They meet the octic at 16 points, 15 prescribed by the base conditions, and the remaining one giving the parametrization for the curve. Thus, we have

$$\Upsilon[s : \zeta] = [cs^8 + s^4\zeta^4 : (cs^4 + \zeta^4)^2 : s^7\zeta] : \mathbb{CP}^1 \to \mathcal{A}_c.$$

Note that $\Upsilon[0 : 1] = [0 : 1 : 0]$, while we have for points in $\mathbb{CP}^1$ (the roots of $cs^4 + \zeta^4 = 0$) whose image is $[0 : 0 : 1]$, there are four branches of the projective curve through that point.

Proceeding as before, we study the periods of the form ω such that $\omega(X) = 1$ on the level curve $\{F_2 = c\}$. Note that, topologically, this is $\mathbb{CP}^1$ with five points removed. As the affine parametrization is $\varphi(\zeta) = (\zeta^4 + c, \zeta/(\zeta^4 + c))$, we have that

$$X_c := X|_{\{F_2=c\}} = \varphi_* \left((\zeta^4 + c) \frac{\partial}{\partial \zeta} \right), \quad \text{hence} \quad \omega_c(\zeta) = \frac{d\zeta}{\zeta^4 + c}.$$

It is now easy to see that its periods around the finite poles are not zero.

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