Journal *of* Singularities
Volume 9 (2014), 1-26

Proc. of Algebraic Methods in Geometry, Guanajuato, 2011
DOI: 10.5427/jsing.2014.9a

THE HOMOLOGICAL INDEX AND THE DE RHAM COMPLEX ON SINGULAR VARIETIES

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Dedicated to Xavier Gómez-Mont on the occasion of his 60th birthday

Abstract. We discuss several methods of computation of the homological index originated in a paper by X. Gómez-Mont for vector fields given on singular complex varieties. Our approach takes into account basic properties of holomorphic and regular meromorphic differential forms and is applicable in different settings depending on concrete types of varieties. Among other things, we describe how to compute the index in the case of Cohen-Macaulay curves, graded normal surfaces and complete intersections by elementary calculations. For quasihomogeneous complete intersections with isolated singularities, an explicit formula for the index is obtained; it is a direct consequence of earlier results of the author. Indeed, in this case the computation of the homological index is reduced to the use of Newton's binomial formula only.

Introduction

The classical concept of topological index for vector fields with isolated singularities given on 2-dimensional manifolds goes back to H. Poincaré (1887). This notion was generalized to higher-dimensional case by H. Hopf who proved that the total index of a vector field on a closed smooth orientable manifold does not depend on the field and it is equal to the Euler-Poincaré characteristic of the manifold. Since then, many authors studied the index as a topological invariant in different contexts and various settings. However, being purely topological, the original definition of index essentially depends on concrete presentations of vector fields, on the topological and analytical structure of a manifold, on the existence of suitable metrics, etc. That is why when studying varieties with singularities the classical approach does not work perfectly for evident reasons.

A new algebraic concept of the homological index for vector fields on reduced pure-dimensional complex analytic spaces appeared in a work by X. Gómez-Mont [16]; it is easy and well adapted for use in the theory of singular varieties. His main idea is to consider an important algebraic and analytic invariant, the alternating sum of dimensions of the homology groups of the truncated De Rham complex of holomorphic differential forms whose differential is defined by the contraction $\iota_{\mathcal{V}}$ of differential forms along a vector field $\mathcal{V}$ given on a singular complex space. In other terms, this invariant is the Euler-Poincaré characteristic of the contracted De Rham complex. Then it is proved that under certain finiteness assumptions the homological index is equal to the classical *local* topological index of $\mathcal{V}$ *up to a constant* depending on the germ of singular space but not on the vector field. In its turn, the problem of computation of Euler-Poincaré characteristic is reduced to the analysis of the homology or hypercohomology of the contracted De Rham complex.

2010 *Mathematics Subject Classification.* 32S25, 14F10, 14F40, 58K45, 58K70.

Key words and phrases. holomorphic differential forms; contracted De Rham complex; regular meromorphic forms; torsion and cotorsion; generating functions; graded complete intersections; Lebelt resolutions.

Originally the homological index was computed explicitly for vector fields with isolated singularities tangent to a hypersurface embedded in a complex manifold with the use of standard resolvents and properties of spectral sequences [15]. In a paper by the author [6] another method for the calculation of the homological index is described; the main idea of his approach is to compute the homological index with the use of *meromorphic* differential forms defined on the ambient manifold and having logarithmic poles along the given hypersurface. Above all an auxiliary invariant, called the *logarithmic* index, is introduced and studied.

The next example, the case of complete intersections with isolated singularities was considered in [23], [10] with the help of quite a difficult technique and constructions of complicated resolvent or resolutions with detail analysis of spectral sequences, the use of computer algebra systems, and so on.

In the present paper we discuss several methods of computation of the homological index for some types of singular varieties. The key point of our approach lies in the fact that the homology of the contracted De Rham complex can be computed with the use of *meromorphic* differential forms. Among other things we show that in quite a general context the homological index can be naturally described in terms of the contracted complex of meromorphic differential forms or some subcomplexes. For example, in the case of *hypersurfaces* the complex $(\omega_X^\bullet, \iota_\mathcal{V})$ of regular meromorphic differential forms is closely related via the residue map with the contracted complex of *logarithmic* differential forms. In its turn, the latter complex is simply linked to the contracted De Rham complex $(\Omega_X^\bullet, \iota_\mathcal{V})$. Moreover, the Euler-Poincaré characteristics of these three complexes differ by constants which depend on singularities of the space and the vector field (see [6]). More generally, in the case of *normal* singularities it is useful to analyze the complex $(\omega_X^\bullet, \iota_\mathcal{V})$; in the case of *non-normal* singularities one can consider also the complex $(\Delta_X^\bullet, \iota_\mathcal{V})$ of *extendable* (to the normalization) meromorphic differential forms, and so on.

The paper is organized as follows. In the first sections we discuss some basic notions and definitions. Almost all of them are well-known in a more general setting; they are often exploited in many areas of singularity theory, analytic geometry, residue theory, etc. Our aim here is only to unify them and to consider related applications in some special situations. Then some simple methods of computation of the homological index are discussed; they are applied in different situations depending on concrete types of singular varieties. Thus, we show subsequently how to compute the index for Cohen-Macaulay curves, graded normal surfaces and complete intersections of arbitrary dimension. As a curious example, in the case of quasihomogeneous complete intersections with isolated singularities an explicit formula for the homological index is obtained; it is a direct consequence of earlier results by the author [1], [2]. In fact, in such a case the computation of the homological index is readily reduced to the use of Newton's binomial formula only.

I would like to thank all the organizers of Xavierfest at CIMAT Guanajuato for providing me with the opportunity of participating in this exciting meeting. I am also very indebted to Xavier Gómez-Mont for many stimulating conversations, discussions and suggestions concerning the subject of the paper.

1. The contracted De Rham complex

Let X be a complex (or real analytic) space or algebraic variety. Then the sheaves Ω_X^p, $p \geq 0$, of holomorphic (or analytic) differential p-forms on X are defined as follows.

Let $x \in X$ be a closed point. Choose one representative of the germ (X, x) embedded in an open neighborhood U of the origin in $\mathbb{C}^m$. Let $\mathcal{O}_U$ be the sheaf of analytic functions on U, and $\mathcal{I}$ a coherent sheaf of ideals with the zero-set X. Then $(X, \mathcal{O}_X)$ is a closed analytic subspace of U so that $\mathcal{O}_X = \mathcal{O}_U/\mathcal{I}|_X$. Assume that the ideal $\mathcal{I}$ is locally generated by a sequence of functions

$f_1, \dots, f_k$ in $\mathcal{O}_U$, and set

$$\Omega^p_{X,x} = \Omega^p_U / (\sum_{j=1}^k f_j \cdot \Omega^p_U + df_j \wedge \Omega^{p-1}_U)\big|_X.$$

By analogy with non-singular case elements of $\Omega^p_{X,x}$ are usually called germs of (regular) holomorphic forms of degree p on X. The differential d, acting on Ω^p_U, induces the differential on $\Omega^p_{X,x}$; it is denoted by the same symbol. As a result, the family of sheaves Ω^p_X, endowed with the differential d, forms an *increasing* complex $(\Omega^\bullet_X, d)$.

Remark. In the introduction to his famous article [19] A. Grothendieck wrote: "... we can consider the complex of sheaves $\Omega^\bullet_{X/k}$ of regular differentials on X, the differential operator being of course the exterior differentiation." Although this complex was firstly considered and studied already by Poincaré, he called it the *De Rham complex,* and its hypercohomology – the De Rham cohomology of X.

However, along with the structure of a complex, given by the De Rham differentiation, one can endow the family of sheaves of regular differential forms with a structure of a complex in other ways. For example, similarly to the classical theory of differentiation and integration one can associate with the exterior differentiation an important class of *inverse* operators as follows.

Let $\mathrm{Der}(X) = \mathrm{Hom}_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)$ be the sheaf of germs of holomorphic vector fields on X. Given an element $\mathcal{V} \in \mathrm{Der}(X) \cong \mathrm{Hom}_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)$, a canonical action of the interior multiplication (contraction) $\iota_{\mathcal{V}}$ along $\mathcal{V}$ on $\Omega^\bullet_X$ is well-defined. Since $\iota^2_{\mathcal{V}} = 0$, one obtains a *decreasing* complex $(\Omega^\bullet_X, \iota_{\mathcal{V}})$.

Remark. Apparently, J. Carrell and D. Liberman [13] investigated the complex $(\Omega^\bullet_X, \iota_{\mathcal{V}})$ for the first time; they proved in the cited work, that the Hodge numbers $h^{pq}(X)$ of a compact Kähler *manifold* X vanish as soon as the absolute value of the difference $|p-q|$ is greater than the dimension of the zero set of $\mathcal{V}$.

More generally, if X is a complex *manifold*, the sheaves of regular holomorphic forms Ω^p_X are locally *free*. In such a case, the complex $(\Omega^\bullet_X, \iota_{\mathcal{V}})$ is locally isomorphic to the classical Koszul complex. However, if X is a singular complex variety then the corresponding theory is considerably more difficult; the above complex belong to the class of the *generalized* Koszul complexes.

Following [6], in order to avoid an ambiguous terminology we call the complex $(\Omega^\bullet_X, \iota_{\mathcal{V}})$ the *contracted* De Rham complex of X.

2. Meromorphic differential forms

Let $Z \subset X$ be a closed subset, $j\colon X \setminus Z \hookrightarrow X$ be the natural inclusion, and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then there exists a standard exact sequence of $\mathcal{O}_X$-modules

$$0 \longrightarrow \mathcal{H}^0_Z(\mathcal{F}) \longrightarrow \mathcal{F} \longrightarrow j_*j^*\mathcal{F} \longrightarrow \mathcal{H}^1_Z(\mathcal{F}) \longrightarrow 0, \tag{1}$$

where $\mathcal{H}^*_Z(\bullet)$ is the local cohomology functor with supports in the closed subset $Z \subset X$, while j_* and j^* are functors of direct and inverse image, respectively, so that $j_*j^*\mathcal{F} \cong R^0 j_*(\mathcal{F}|_{X\setminus Z})$. One says that $\mathcal{F}$ has support in Z, that is, $\mathrm{Supp}\,(\mathcal{F}) \subseteq Z$, if $\mathcal{H}^0_Z(\mathcal{F}) \longrightarrow \mathcal{F}$ is an isomorphism; $\mathcal{F}$ is said to have no Z-torsion, if $\mathcal{H}^0_Z(\mathcal{F}) = 0$. Usually, the latter local cohomology group is called Z-torsion of $\mathcal{F}$; it is denoted by $\mathrm{Tors}\,(\mathcal{F})$. Similarly $\mathcal{H}^1_Z(\mathcal{F})$ is called Z-cotorsion. For example, $j_*j^*\mathcal{F}$ itself has no Z-torsion. More generally, if $\mathcal{J} \subset \mathcal{O}_X$ is a coherent sheaf of ideals with the zero-locus $Z \subset X$, then for all $i \geq 0$ one has

$$\mathcal{H}^i_Z(\mathcal{F}) = \varinjlim_{\nu} \mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{J}^\nu, \mathcal{F}).$$

Given a morphism of quasi-coherent sheaves $\varrho\colon \mathcal{F} \to \mathcal{G}$, a family of natural morphisms

$$\mathcal{H}^i_Z(\varrho)\colon \mathcal{H}^i_Z(\mathcal{F}) \to \mathcal{H}^i_Z(\mathcal{G}),$$

$i \geq 0$, is well-defined. Suppose also that there is an *extension* $j_*(\varrho)$ of $\varrho|_{X\setminus Z}$ to X such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{H}^0_Z(\mathcal{F}) & \longrightarrow & \mathcal{F} & \longrightarrow & j_*j^*\mathcal{F} & \longrightarrow & \mathcal{H}^1_Z(\mathcal{F}) & \longrightarrow & 0 \\ & & \downarrow \mathcal{H}^0_Z(\varrho) & & \downarrow \varrho & & \downarrow j_*(\varrho) & & \downarrow \mathcal{H}^1_Z(\varrho) & & \\ 0 & \longrightarrow & \mathcal{H}^0_Z(\mathcal{G}) & \longrightarrow & \mathcal{G} & \longrightarrow & j_*j^*\mathcal{G} & \longrightarrow & \mathcal{H}^1_Z(\mathcal{G}) & \longrightarrow & 0. \end{array}$$

is commutative. Then for any complex $\mathcal{L}^\bullet = (\mathcal{L}^\bullet, \partial)$ of sheaves on X such that

$$j_*(\partial)^2 = j_*(\partial^2) = 0$$

the above diagram induces an exact sequence of complexes of $\mathcal{O}_X$-modules:

$$0 \longrightarrow \mathcal{H}^0_Z(\mathcal{L}^\bullet) \longrightarrow \mathcal{L}^\bullet \longrightarrow j_*j^*\mathcal{L}^\bullet \longrightarrow \mathcal{H}^1_Z(\mathcal{L}^\bullet) \longrightarrow 0, \tag{2}$$

since $\mathcal{H}^0_Z(\partial^2) = \mathcal{H}^0_Z(\partial)^2 = 0$, $\mathcal{H}^1_Z(\partial^2) = \mathcal{H}^1_Z(\partial)^2 = 0$.

Let us now take $\mathcal{F} = \Omega^p_X$, $p \geq 0$, and $Z = \operatorname{Sing} X$. Then $j_*j^*\Omega^p_X = j_*\Omega^p_X\big|_{X\setminus Z}$ consists of germs of meromorphic differential p-forms on X with singularities on Z.

Proposition 1. *With the same notations let $\mathcal{V}$ be an element of* $\operatorname{Der}(X)$. *Suppose that the restriction $j^*\mathcal{V} = \mathcal{V}|_{X\setminus Z}$ can be extended (in general, not necessarily uniquely) on X and denote the corresponding contraction, acting on $j_*j^*\Omega^\bullet_X$, by $j_*(\iota_\mathcal{V}) = \iota_{j_*\mathcal{V}}$. Then there exists an exact sequence of decreasing complexes*

$$0 \longrightarrow \mathcal{H}^0_Z(\Omega^\bullet_X) \longrightarrow \Omega^\bullet_X \longrightarrow j_*j^*\Omega^\bullet_X \longrightarrow \mathcal{H}^1_Z(\Omega^\bullet_X) \longrightarrow 0 \tag{3}$$

with differentials $\iota_\mathcal{V}$ and $\iota_{j_*\mathcal{V}}$.

For example, if X is a *normal* variety, then the desirable extension of $\mathcal{V}$ exists. Moreover, there exists also a similar exact sequence of *increasing* complexes with differentials induced by the usual De Rham differentiation d acting on $\Omega^\bullet_X$.

Remark. It should be underlined that, in general, for a coherent sheaf $\mathcal{F}$ the associated quasi-coherent sheaf $j_*j^*\mathcal{F}$ is *non-coherent*. However, there always exists a *coherent* subsheaf $\mathcal{G}_X \subset j_*j^*\mathcal{F}_X$ (as a rule, even not unique) such that $\mathcal{F}_U \cong \mathcal{G}_X|_U$.

The following useful assertion is an easy modification of the well-known statement due to M. Schlessinger (see [30]).

Lemma 1. *Let X be the germ of a complex space, $\mathcal{F}$ a coherent $\mathcal{O}_X$-module, and*

$$\mathcal{F}^\vee = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$$

its dual. Suppose that $Z \subset X$ is a closed subspace, and $\operatorname{depth}_Z X \geq 2$. *Then* $\operatorname{depth}_Z \mathcal{F}^\vee \geq 2$, *so that $\mathcal{H}^0_Z(\mathcal{F}^\vee) = \mathcal{H}^1_Z(\mathcal{F}^\vee) = 0$. Similarly, the condition* $\operatorname{depth}_Z X \geq 1$ *implies* $\operatorname{depth}_Z \mathcal{F}^\vee \geq 1$, *that is, $\mathcal{H}^0_Z(\mathcal{F}^\vee) = 0$.*

Proof. Taking the presentation $0 \to \mathcal{R} \to \mathcal{L} \to \mathcal{F} \to 0$, where $\mathcal{L}$ is a free $\mathcal{O}_X$-module, one gets the dual exact sequence

$$0 \to \mathcal{F}^\vee \to \mathcal{L}^\vee \to \mathcal{Q} \to 0$$

with $\mathcal{Q} \subseteq \mathcal{R}^\vee$. Since $\mathcal{L}$ is free and $\operatorname{depth}_Z \mathcal{O}_X \geq 2$, then $\mathcal{H}^0_Z(\mathcal{L}^\vee) = \mathcal{H}^1_Z(\mathcal{L}^\vee) = 0$. Hence, $\mathcal{H}^0_Z(\mathcal{F}^\vee) = 0$. Analogously, $\mathcal{H}^0_Z(\mathcal{R}^\vee) = 0$; this implies $\mathcal{H}^0_Z(\mathcal{Q}) = 0$. Finally, applying the functor of local cohomology $\mathcal{H}^*_Z(\bullet)$ to the dual exact sequence, one deduces $\mathcal{H}^1_Z(\mathcal{F}^\vee) = 0$. The remaining case $\operatorname{depth}_Z X \geq 1$ is analyzed in the same manner. QED.

Corollary 1. *Let X be a reduced complex space, $Z = \operatorname{Sing} X$. Then the $\mathcal{O}_X$-module* $\operatorname{Der}(X)$ *of vector fields on X has no Z-torsion.*

Proof. By definition, $\operatorname{Der}(X) \cong \operatorname{Hom}_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)$. Since X is reduced, then $\operatorname{codim}(Z, X) \geq 1$ and, hence, $\operatorname{depth}_Z X \geq 1$. QED.

3. Regular meromorphic forms

Let M be a complex manifold, $\dim M = m$, and let $X \subset M$ be an analytical subset in a neighborhood of $x \in U \subset M$ defined by a sequence of functions $f_1, \dots, f_k \in \mathcal{O}_U$ as before. Throughout this section we assume that X is a *Cohen-Macaulay* space and $\dim X = n$. Then

$$\omega^n_X = \operatorname{Ext}^{m-n}_{\mathcal{O}_M}(\mathcal{O}_X, \Omega^m_M)$$

is called the Grothendieck *dualizing* module of X. It is well-known that the dualizing module has *no torsion*, $\operatorname{Tors} \omega^n_X = 0$.

Definition. The coherent sheaf of $\mathcal{O}_X$-modules ω^p_X, $p \geq 0$, is locally defined as the set of germs of meromorphic differential forms ω of degree p on X such that $\omega \wedge \eta \in \omega^n_X$ for any $\eta \in \Omega^{n-p}_X$. In other terms (see [8], [24], [22]),

$$\omega^p_X \cong \operatorname{Hom}_{\mathcal{O}_X}(\Omega^{n-p}_X, \omega^n_X) \cong \operatorname{Ext}^{m-n}_{\mathcal{O}_M}(\Omega^{n-p}_X, \Omega^m_M). \tag{4}$$

Elements of ω^p_X are called *regular meromorphic* differential forms of degree p on X. Some equivalent definitions of these sheaves in terms of Noether normalization and trace (see [24], [8]), in terms of residual currents (see [7]) and others are well-known.

Evidently, $\omega^p_X = 0$ for $p > n$ since $\Omega^{n-p}_X = 0$. Further, $\omega^p_X = 0$ for $p < 0$ because $\Omega^p_X \cong \operatorname{Tors} \Omega^p_X$ for $p > n$ and ω^n_X has no torsion. It is easy also to see that the De Rham differentiation d as well as the contraction $\iota_{\mathcal{V}}$ acting on $\Omega^\bullet_X$ are naturally extended to the family of modules ω^p_X, $0 \leq p \leq n$; they endow this family with structures of *increasing* complex $(\omega^\bullet_X, d)$ or *decreasing* complex $(\omega^\bullet_X, \iota^*_{\mathcal{V}})$, respectively. In particular, the contraction $\iota^*_{\mathcal{V}} \colon \omega^p_X \to \omega^{p-1}_X$ is naturally defined as a *dual* morphism to the action $\iota_{\mathcal{V}}$ on the complex $\Omega^\bullet_X$ in view of presentation (4).

Lemma 2 (see [8])**.** *Let $Z = \operatorname{Sing} X$, $j \colon X \setminus Z \longrightarrow X$. Then there exist natural inclusions $\omega^p_X \subseteq j_* j^* \Omega^p_X$ for all $p \geq 0$. Moreover, if $\operatorname{codim}(Z, X) \geq 1$, then ω^p_X has no Z-torsion; if $\operatorname{codim}(Z, X) \geq 2$, then ω^p_X has no cotorsion for all $p \geq 0$.*

Proposition 2. *If X is a normal space, that is, $c = \operatorname{codim}(\operatorname{Sing} X, X) \geq 2$, then $\omega^p_X \cong j_* j^* \Omega^p_X$ for $p \geq 0$. Thus, a meromorphic form is regular meromorphic on X if and only if it is holomorphic outside the singular subset Z of X. In particular, for $p \geq 0$ the sheaves $j_* j^* \Omega^p_X$ are coherent and exact sequence (3) of complexes transforms in the following way:*

$$0 \longrightarrow \mathcal{H}^0_Z(\Omega^\bullet_X) \longrightarrow \Omega^\bullet_X \longrightarrow \omega^\bullet_X \longrightarrow \mathcal{H}^1_Z(\Omega^\bullet_X) \longrightarrow 0.$$

Moreover, ω^p_X and the bidual sheaves $\Omega^{p\,\vee\vee}_X = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{H}om_{\mathcal{O}_X}(\Omega^p_X, \mathcal{O}_X), \mathcal{O}_X)$ of Zariski differential p-forms are isomorphic for all $0 \leq p < c$, respectively.

Remark. The sheaf ω^0_X contains all the germs of *weakly holomorphic* functions on X, or locally bounded meromorphic functions on X (see [8]); in other terms, it contains meromorphic germs whose preimages are holomorphic on the normalization of X [29]. Furthermore, the sheaves ω^{n-1}_X and $\operatorname{Der}(X)$ are naturally isomorphic.

Evidently, if $X \subset M$, then for any vector field $\mathcal{V} \in \operatorname{Der}(X)$ there exists a holomorphic vector field V given on the ambient manifold M such that $V|_X = \mathcal{V}$. For brevity, we often say that $\mathcal{V}$ has *isolated* singularities when its representative V does. In particular, this implies that such $\mathcal{V}$ has isolated singularities on X.

Proposition 3. *Let $\mathcal{V} \in \mathrm{Der}(X)$ be a vector field with isolated singularities. Then the $\iota_{\mathcal{V}}$-homology groups of the complex $\omega^\bullet_X$ are finite-dimensional vector spaces.*

PROOF. Following [16], let us assume that $M = \mathbb{C}^m$ and the distinguished point

$$x_0 = 0 \in X \subset M$$

is an isolated singularity of the field $\mathcal{V}$, so that $\mathcal{V}(x_0) = 0$. Then $\mathcal{V}(x) \neq 0$ at any point x in a small enough punctured neighborhood of the point x_0. In a suitable neighborhood of x there exists a coordinate system $(t, z'_1, \dots, z'_{m-1})$ such that $\mathcal{V} = \partial/\partial t$. Since $\mathcal{V}(\mathcal{I}) \subseteq (\mathcal{I})\mathcal{O}_{M,0}$, then $X \cong T \times X_0$, where a small disc in t is denoted by T and $X_0 \subseteq M_0 = \mathbb{C}^{m-1}$. Hence, for sheaves of holomorphic forms of degree $p \geq 0$ on X there are isomorphisms

$$\Omega^p_{X,0} \cong \big(\Omega^p_{X_0,0} \oplus \Omega^{p-1}_{X_0,0} \wedge dt\big) \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{C},0};$$

they are readily obtained from consideration of canonical projections $T \times X_0$ on the first and second cofactors and from definition of $\Omega^p_{X,0}$. A similar presentation exists for $j_* j^* \Omega^p_{X,0}$ as well as for $\omega^p_{X,0}$. Finally, on the p-th component of the complex $(\omega^\bullet_{X,0}, \iota^*_{\mathcal{V}})$ one has

$$\mathrm{Ker}(\iota^*_{\partial/\partial t}) \cong \mathrm{Im}\,(\iota^*_{\partial/\partial t}) \cong \big(\omega^p_{X_0,0} \oplus (0)\big) \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{C},0}$$

in virtue of duality. Thus, the corresponding homology groups vanish for all p. If $x_0 \in M \backslash X$, then one can easily get the same conclusion. This implies that $\iota^*_{\mathcal{V}}$-homology groups of the complex $\omega^\bullet_X$ may be non-trivial only at singular points of the field. The coherence of sheaves of regular meromorphic forms and their cohomology implies the statement. QED.

4. THE HOMOLOGICAL INDEX

Let $(\mathcal{L}^\bullet_X, \partial)$ be a (lower and upper) *bounded decreasing* complex of $\mathcal{O}_X$-modules with the differential of degree -1. Assume that all its homology groups $H_i(\mathcal{L}^\bullet_X, \partial)$ are modules of *finite* length, that is, $\ell(H_i(\mathcal{L}^\bullet_X, \partial)) < \infty$ for all $i \in \mathbb{Z}$. Then the Euler-Poincaré characteristic of the complex $(\mathcal{L}^\bullet_X, \partial)$ is defined as follows:

$$\chi(\mathcal{L}^\bullet_X, \partial) = \sum_{i \in \mathbb{Z}} (-1)^i \ell(H_i(\mathcal{L}^\bullet_X, \partial)).$$

REMARK. If all the homology groups of $\mathcal{L}^\bullet$ are *finite-dimensional* vector spaces over the ground field $k = \mathbb{C}$, then

$$\chi(\mathcal{L}^\bullet_X, \partial) = \sum_{i \in \mathbb{Z}} (-1)^i \dim_k H_i(\mathcal{L}^\bullet_X, \partial).$$

Similarly, the Euler-Poincaré characteristic of the stalk of complex $\mathcal{L}^\bullet_{X,x}$ at any point $x \in X$ is well-defined.

Claim 1. *Under the finiteness assumption there exists the following relation:*

$$\chi(\Omega^\bullet_X, \iota_{\mathcal{V}}) = \chi(j_* j^* \Omega^\bullet_X, j_* \iota_{\mathcal{V}}) + \chi(\mathrm{Tors}\,\Omega^\bullet_X, \iota_{\mathcal{V}}) - \chi(\mathcal{H}^1_Z(\Omega^\bullet_X), \iota_{\mathcal{V}}),$$

Moreover, if X is normal then

$$\chi(\Omega^\bullet_X, \iota_{\mathcal{V}}) = \chi(\omega^\bullet_X, \iota^*_{\mathcal{V}}) + \chi(\mathrm{Tors}\,\Omega^\bullet_X, \iota_{\mathcal{V}}) - \chi(\mathcal{H}^1_Z(\Omega^\bullet_X), \iota_{\mathcal{V}}).$$

PROOF. The exact sequence (3) yields two short exact sequences of complexes with differentials induced by the contractions along the vector field and its extension:

$$0 \longrightarrow \mathrm{Tors}\,\Omega^\bullet_X \longrightarrow \Omega^\bullet_X \longrightarrow \widetilde{\Omega}^\bullet_X \longrightarrow 0; \qquad 0 \longrightarrow \widetilde{\Omega}^\bullet_X \longrightarrow j_* j^* \Omega^\bullet_X \longrightarrow \mathcal{H}^1_Z(\Omega^\bullet_X) \longrightarrow 0,$$

where $\widetilde{\Omega}^\bullet_X = \Omega^\bullet_X/\mathrm{Tors}\,\Omega^\bullet_X$ is the quotient complex. Combining the associated long exact sequences of homologies, one obtains the first relation. Next, Proposition 2 implies

$$\chi(j_*j^*\Omega^\bullet_X) = \chi(\omega^\bullet_X);$$

it gives the second relation. QED.

Definition (see [16]). Let $\mathcal{V}$ be a holomorphic vector field given on the germ (X,x) of n-dimensional complex space and let $({}_{\sigma\le n}(\Omega^\bullet_{X,x}), \iota_{\mathcal{V}})$ be the *truncated* contracted De Rham complex of (X,x) :

$$0 \longrightarrow \Omega^n_{X,x} \xrightarrow{\iota_{\mathcal{V}}} \Omega^{n-1}_{X,x} \xrightarrow{\iota_{\mathcal{V}}} \Omega^{n-2}_{X,x} \to \cdots \to \Omega^1_{X,x} \xrightarrow{\iota_{\mathcal{V}}} \Omega^0_{X,x} \cong \mathcal{O}_{X,x} \longrightarrow 0.$$

The Euler-Poincaré characteristic of this complex is called the *homological* index of the vector field at x; it is denoted by $\mathrm{Ind}_{\mathrm{hom},x}(\mathcal{V})$. Thus,

$$\mathrm{Ind}_{\,\mathrm{hom},x}(\mathcal{V}) = \chi({}_{\sigma\le n}(\Omega^\bullet_{X,x}), \iota_{\mathcal{V}}).$$

Remark. In the standard terminology of homological algebra such kind of truncation is usually called the *"stupid"* or *"naïve"* truncation of level n.

Remark. The homological index was originally defined for vector fields on a *reduced pure-dimensional* complex analytic space with finite-dimensional homology groups $H_i(\Omega^\bullet_{X,x}, \iota_{\mathcal{V}})$; if such X and a vector field $\mathcal{V}$ both have isolated singularities at x then the homological index *coincides* with the *local topological (Poincaré-Hopf) index* up to a constant depending on the germ of singular space but not on the vector field (see [16]).

Definition. Let $\mathcal{V}$ be a holomorphic vector field on X and $\iota_{\mathcal{V}}$ the contraction along $\mathcal{V}$. If $(\mathcal{L}^\bullet_X, \partial)$ is equal to one of the described above complexes $(\Omega^\bullet_X, \iota_{\mathcal{V}})$, $(j_*j^*\Omega^\bullet_X, j_*\iota_{\mathcal{V}})$, $(\omega^\bullet_X, \iota^*_{\mathcal{V}})$, $\mathcal{H}^0_Z(\Omega^\bullet_X, \iota_{\mathcal{V}})$, $\mathcal{H}^1_Z(\Omega^\bullet_X, \iota_{\mathcal{V}})$, then we shall often call the corresponding Euler-Poincaré characteristic by holomorphic, meromorphic, regular meromorphic, torsion and cotorsion indices of the vector field $\mathcal{V}$ at $x \in X$, respectively.

It should be noted that the present paper is devoted in the main to the study of the homological index for varieties whose *cotorsion* index is equal to zero, that is, $\chi(\mathcal{H}^1_Z(\Omega^\bullet_X)) = 0$.

Proposition 4. *Assume that (X,x) is the germ of a complex space such that $\Omega^p_{X,x}$ are torsion modules for all $p > n$, that is, $\Omega^p_{X,x} = \mathrm{Tors}\,\Omega^p_{X,x}$. Then*

$$\mathrm{Ind}_{\mathrm{hom},x}(V) = \chi(\Omega^\bullet_{X,x}, \iota_{\mathcal{V}}) - \chi(\mathrm{Tors}\,({}_{\sigma>n}(\Omega^\bullet_{X,x})), \iota_{\mathcal{V}}).$$

Proof.It is enough to compare the Euler-Poincaré characteristics of the contracted $\iota_{\mathcal{V}}$-complexes in the exact sequence

$$0 \longrightarrow {}_{\sigma\le n}(\Omega^\bullet_X) \longrightarrow (\Omega^\bullet_X) \longrightarrow {}_{\sigma>n}(\Omega^\bullet_X) \longrightarrow 0,$$

and then to use an evident isomorphism ${}_{\sigma>n}(\Omega^\bullet_X) \cong \mathrm{Tors}\,({}_{\sigma>n}(\Omega^\bullet_X))$. QED.

Remark. It is well-known that reduced complete intersections with *non-isolated* singularities satisfy the conditions of Proposition 4 (see [17, Proposition 1.11]).

Corollary 2 (cf. [16], (1.4))**.** *Under the same assumptions suppose additionally that (X,x) is an n-dimensional isolated singularity of embedding dimension m. Then*

$$\mathrm{Ind}_{\mathrm{hom},x}(\mathcal{V}) = \chi({}_{\sigma\le n}(\Omega^\bullet_{X,x}), \iota_{\mathcal{V}}) = \chi(\Omega^\bullet_{X,x}, \iota_{\mathcal{V}}) - \sum_{p=n+1}^{m} (-1)^p \dim_k \mathrm{Tors}\,\Omega^p_{X,x}.$$

Proof. Since (X,x) is an isolated singularity, then torsion modules $\operatorname{Tors}\Omega^p_{X,x}$ are finite-dimensional vector spaces for all $p > n$. QED.

Corollary 3. *For an isolated complete intersection singularity of dimension $n \geq 1$ one has*

$$\operatorname{Ind}_{\hom,x}(\mathcal{V}) = \chi(\omega^\bullet_{X,x}, \iota^*_{\mathcal{V}}) + (-1)^n \dim_k \operatorname{Tors}\Omega^n_{X,x}.$$

Proof. In fact, the cotorsion index is equal to zero, that is, $\chi(\mathcal{H}^1_{\{x\}}(\Omega^\bullet_X), \iota_{\mathcal{V}}) = 0$, because in our case there are only two non-trivial cotorsion modules of equal lengths (see [11]). It remains to use Claim 1 and Corollary 2. QED.

Claim 2. *Assume that (X,x) is a quasihomogeneous isolated complete intersection singularity of dimension $n \geq 1$. Then*

$$\dim_k \operatorname{Tors}\Omega^n_{X,x} = \sum_{p=n+1}^{m} (-1)^{p-n-1} \dim_k \Omega^p_{X,x},$$

where m is the embedding dimension of the singularity.

Proof. Let $\mathcal{V}$ be the Euler vector field. First observe that if $n = 1$, then $\iota_{\mathcal{V}}(\operatorname{Tors}\Omega^1_{X,x}) = 0$ in $\mathcal{O}_{X,x}$. Indeed, if $\theta \in \operatorname{Tors}\Omega^1_{X,x}$ then there exists a non-zero divisor $u \in \mathcal{O}_{X,x}$ such that $u\theta \in \sum \mathcal{O}_{\mathbb{C}^m,o} \wedge df_j + (f_1, \ldots, f_{m-1})\mathcal{O}_{\mathbb{C}^m,o}$. Then $\iota_{\mathcal{V}}(u\theta) = u\iota_{\mathcal{V}}(\theta) \in (f)\mathcal{O}_{\mathbb{C}^m,o}$, that is, one has $u\iota_{\mathcal{V}}(\theta) = 0$ in $\mathcal{O}_{X,x}$ and, consequently, $\iota_{\mathcal{V}}(\theta) = 0$ as was required. If $n \geq 2$ then $\operatorname{depth}_x\mathcal{O}_X \geq 2$ and the germ $X \setminus x$ is *connected* (see [18, Corollary 3.9]). In particular, the germ X is *irreducible* and its analytical algebra $\mathcal{O}_{X,x}$ has no zero-divisors, i.e. it is an integral domain. Further, $\operatorname{Tors}\Omega^p_X = 0$ for all $0 < p < n$, $\operatorname{Tors}\Omega^p_X$ are finite-dimensional for $p \geq n$ and $\Omega^p_X \cong \operatorname{Tors}\Omega^p_X$ for $p > n$ (see [17, Proposition 1.11]). Analogously to the above considerations for $n = 1$ one gets $\iota_{\mathcal{V}}(\operatorname{Tors}\Omega^n_{X,x}) = 0$. Since (X,x) is a *contractible* singularity with respect to the Euler vector field then $\chi(_{\sigma\geq n}(\operatorname{Tors}\Omega^\bullet_X), \iota_{\mathcal{V}}) = 0$. It remains to note that the Euler-Poincaré characteristic of a complex, whose terms are finite-dimensional vector spaces, is equal to the alternating sum of dimensions of all its terms (cf. [17, Lemma 5.6]). QED.

Remark. There exist also other cases when a slightly modified version of the Claim 2 is true (see, for example, [28, Satz 3]).

From the above observations it follows that the complex $(\omega^\bullet_{X,x}, \iota^*_{\mathcal{V}})$ may have *non-trivial* homology groups H_i only for $i = 0, 1, \ldots, n$, where n is the dimension of X. That is, in a certain sense the regular meromorphic index is defined in a more intrinsic manner than the homological one, since in the general case the former does not depend on the embedding dimension of the germ X, on the operation of truncation, etc.

5. Cohen-Macaulay curves

Let us consider in detail the case when $(X, \mathfrak{o})$ is the germ of a Cohen-Macaulay curve at the *distinguished* point $\mathfrak{o} \in X$ (for brevity, a singularity) and suppose that $Z = \operatorname{Sing} X = \{\mathfrak{o}\}$. Then $\omega^1_X \ncong j_*j^*\Omega^1_X$ and, moreover, the cotorsion of Ω^1_X, the right term $H^1_{\{\mathfrak{o}\}}(\Omega^1_X)$ of exact sequence (1), is *infinite-dimensional* over $\mathbb{C}$. Let $A = (A, \mathfrak{m})$ be the analytical algebra corresponding to the germ $(X, \mathfrak{o})$. Then A is a Cohen-Macaulay 1-dimensional local ring, $k \cong A/\mathfrak{m}$, $k = \mathbb{C}$ is the ground field and there exists an exact sequence

$$0 \longrightarrow \operatorname{Tors}\Omega^1_A \longrightarrow \Omega^1_A \xrightarrow{c_A} \omega^1_A \longrightarrow {}^{\#}\omega^1_A \longrightarrow 0, \tag{5}$$

where the left and right terms are concentrated at the singularity so that they are *finite-dimensional* vector spaces.

In this case the *dualizing* (or, equivalently, *canonical*) module ω^1_A is contained properly in $\Omega^1_A \otimes_A F/A$, where F is the total ring of fractions of A, c_A is the canonical A-homomorphism of the fundamental class induced by the natural map $\Omega^1_A \to \Omega^1_A \otimes F/A$ (see [20], [25]). By definition, $\mathrm{Ker}(c_A) = \mathrm{Tors}\,\Omega^1_A \cong \mathcal{H}^0_{\mathfrak{m}}(\Omega^1_A)$ is the torsion submodule of Ω^1_A, and $\mathrm{Coker}(c_A) = {}^{\#}\omega^1_A$ is contained in $H^1_{\mathfrak{m}}(\Omega^1_A)$, it is called the ω_A-*cotorsion* of Ω^1_A.

Proposition 5. *Let A be a reduced 1-dimensional analytical algebra and $\mathcal{V} \in \mathrm{Der}_k(A)$ a k-differentiation of A. Then there exists the following commutative diagram with exact rows*

$$\begin{array}{ccccccccc} 0 \longrightarrow & \mathrm{Tors}\,\Omega^1_A & \longrightarrow & \Omega^1_A & \xrightarrow{c_A} & \omega^1_A & \longrightarrow {}^{\#}\omega^1_A & \longrightarrow 0 \\ & & & \downarrow\iota_{\mathcal{V}} & & \downarrow\iota^*_{\mathcal{V}} & \downarrow & \\ 0 & & \longrightarrow & \Omega^0_A \cong A & \longrightarrow & \omega^0_A & \longrightarrow {}^{\#}\omega^0_A & \longrightarrow 0, \end{array} \tag{6}$$

where $\omega^0_A = \mathrm{Hom}_A(\Omega^1_A, \omega^1_A)$, $A \hookrightarrow \omega^0_A$ is the canonical inclusion, ${}^{\#}\omega^0_A \cong \mathrm{Ext}^1_A({}^{\#}\omega^1_A, \omega^1_A)$, the middle left vertical arrow of diagram is the contraction $\iota_{\mathcal{V}}$, acting on Ω^1_A, while the middle term from right is induced by this contraction on $\omega^{\bullet}_A$ in a dual way.

Proof. Since A is, in fact, a Cohen-Macaulay local ring of Krull dimension 1, then

$$\mathrm{Hom}_A(k, \omega^1_A) = 0,$$

and, consequently, $\mathrm{Hom}_A(M, \omega^1_A) = 0$ for any A-module M of finite type with $\mathrm{Supp}\,M \subseteq \{\mathfrak{m}\}$. Applying the functor $\mathrm{Hom}_A(\bullet, \omega^1_A)$ to the following exact sequences subsequently

$$0 \to \mathrm{Tors}\,\Omega^1_A \to \Omega^1_A \to \widetilde{\Omega}^1_A \to 0, \qquad 0 \to \widetilde{\Omega}^1_A \xrightarrow{c_A} \omega^1_A \to {}^{\#}\omega^1_A \to 0, \tag{7}$$

one gets a natural isomorphism $\mathrm{Hom}_A(\widetilde{\Omega}^1_A, \omega^1_A) \cong \mathrm{Hom}_A(\Omega^1_A, \omega^1_A)$, and the first four terms of the long exact sequence

$$0 \to \mathrm{Hom}_A(\omega^1_A, \omega^1_A) \longrightarrow \mathrm{Hom}_A(\Omega^1_A, \omega^1_A) \longrightarrow \mathrm{Ext}^1_A({}^{\#}\omega^1_A, \omega^1_A) \longrightarrow \mathrm{Ext}^1_A(\omega^1_A, \omega^1_A)$$

since supports of $\mathrm{Tors}\,\Omega^1_A$ and ${}^{\#}\omega^1_A$ are contained in the singular point $\{\mathfrak{m}\}$. At last, from [20, 6.1 d)], it follows that $\mathrm{Ext}^1_A(\omega^1_A, \omega^1_A) = 0$, $\mathrm{Hom}_A(\omega^1_A, \omega^1_A) \cong A$, and there exists a canonical exact sequence

$$0 \longrightarrow A \longrightarrow \mathrm{Hom}_A(\Omega^1_A, \omega^1_A) \longrightarrow \mathrm{Ext}^1_A({}^{\#}\omega^1_A, \omega^1_A) \longrightarrow 0, \tag{8}$$

where the inclusion $A \longrightarrow \mathrm{Hom}_A(\Omega^1_A, \omega^1_A)$ is given by the correspondence $1_A \mapsto c_A$ (see details in [25, § 3]). In conclusion, the contraction $\iota_{\mathcal{V}}\colon \Omega^1_A \to \Omega^0_A \cong A$ induces the natural dual mapping $\iota^*_{\mathcal{V}}\colon \omega^1_A \to \omega^0_A$ in view of the definition of $\omega^{\bullet}_A$. It is not difficult to verify, that c_A is *compatible* with the contraction $\iota_{\mathcal{V}}$ and its extension $\iota^*_{\mathcal{V}}$, so that diagram (6), a combination of the latter exact sequence and (5), is, in fact, commutative. QED.

Corollary 4 ([22], (4.4))**.** *Under the same assumptions the lengths of ω-cotorsion modules are equal, that is, $\ell({}^{\#}\omega^1_A) = \ell({}^{\#}\omega^0_A)$, and the index of cotorsion complex is zero, $\chi({}^{\#}\omega^{\bullet}_A) = 0$.*

Remark. In a similar manner one can verify (see [22]) that ${}^{\#}\omega^1_A \cong \mathrm{Ext}^1_A({}^{\#}\omega^0_A, \omega^1_A)$. Really, applying $\mathrm{Hom}_A(\bullet, \omega^1_A)$ to the bottom row of the diagram (6), one gets

$$0 \to \mathrm{Hom}_A(\omega^0_A, \omega^1_A) \longrightarrow \omega^1_A \longrightarrow \mathrm{Ext}^1_A({}^{\#}\omega^0_A, \omega^1_A) \to 0.$$

By definition, the left module of this sequence is isomorphic to $\mathrm{Hom}_A(\mathrm{Hom}_A(\Omega^1_A, \omega^1_A), \omega^1_A)$, while the latter is isomorphic to $\mathrm{Hom}_A(\mathrm{Hom}_A(\widetilde{\Omega}^1_A, \omega^1_A), \omega^1_A) \cong \widetilde{\Omega}^1_A$. As a result, we obtain the second exact sequence (7).

Claim 3. *Let $\widetilde{A}$ be the normalization of a 1-dimensional singularity A in its total ring of fractions F, and let $\mathfrak{C}$ be the conductor of $\widetilde{A}$ in A. Then $\omega^1_{\widetilde{A}} \cong \Omega^1_{\widetilde{A}}$, $\omega^1_{\widetilde{A}} \cong \mathfrak{C} \cdot \omega^1_A$, and $\mathfrak{m} \cdot \omega^0_{\widetilde{A}} \subseteq \mathfrak{C} \cdot \omega^0_A$.*

Proof. The existence of both isomorphisms is well-known (see [25, (3.2)]), the inclusion is evident. QED.

Proposition 6. *Let $(A,\mathfrak{m})$ be a reduced 1-dimensional singularity, and let $\mathcal{V}\in \mathrm{Der}_k(A)$ be a vector field. Then $H_1(\omega^\bullet_A,\iota^*_{\mathcal{V}})=0$.*

Proof. Any vector field $\mathcal{V}$ on the curve singularity A can be extended to its normalization $\widetilde{A}$ (see [14, Lemma 2.33]); this extension is denoted by $\widetilde{\mathcal{V}}\in \mathrm{Der}_k(\widetilde{A})$. Since $\widetilde{\mathcal{V}}(\mathfrak{C})\subseteq\mathfrak{C}$, then $\widetilde{\mathcal{V}}(\mathfrak{m})\subseteq\mathfrak{m}$ (see [14]), and, consequently, $\mathrm{Ker}(\iota_{\widetilde{\mathcal{V}}})=0$. On the other hand, it is well-known that $\omega^1_{\widetilde{A}}\cong\omega^1_A$ and, making use of basic properties of Noether normalization and the definition of ω^1_A, we deduce that $\iota_{\widetilde{\mathcal{V}}}(\omega^1_{\widetilde{A}})=\iota_{\widetilde{\mathcal{V}}}(\mathfrak{C}\cdot\omega^1_A)=\mathfrak{C}\cdot\iota^*_{\mathcal{V}}(\omega^1_A)$. Next, by definition, the conductor $\mathfrak{C}$ is the *maximal* element of the set of ideals of A which are also ideals of the *principal* ideal ring $\widetilde{A}$. It is not difficult to verify, that $\mathfrak{C}=(\theta)\widetilde{A}$, where $\theta\in\mathfrak{m}_A$ is a *non-zero divisor* (see [25, 3.1. b)]). Hence, $\mathrm{Ker}(\iota_{\widetilde{A}})=\mathrm{Ker}(\iota^*_{\mathcal{V}})$; this completes the proof. QED.

Claim 4. *Under the assumptions of Proposition 6 suppose additionally that $\mathcal{V}$ has an isolated singularity on the germ $(X,\mathfrak{o})$. Then*

$$\dim_k H_0(\omega^\bullet_A,\iota^*_{\mathcal{V}})=\dim_k H_0(\Omega^\bullet_A,\iota_{\mathcal{V}})=\dim_k A/J_{\mathfrak{o}}\mathcal{V},$$

where $J_{\mathfrak{o}}\mathcal{V}$ is the ideal of A, generated by the coefficients of the vector field V in a suitable coordinate representation. In particular, we have

$$\chi(\omega^\bullet_A,\iota^*_{\mathcal{V}})=\dim_k A/J_{\mathfrak{o}}\mathcal{V}.$$

Proof. The diagram (6) yields the following exact sequence

$$0\to\mathrm{Ker}({}^\#\omega^1_A\to{}^\#\omega^0_A)\longrightarrow\mathrm{Coker}(\iota_{\mathcal{V}})\longrightarrow\mathrm{Coker}(\iota^*_{\mathcal{V}})\to\mathrm{Coker}({}^\#\omega^1_A\to{}^\#\omega^0_A)\to 0.$$

The difference of lengths of the left and right modules of this sequence *does not depend* on the vector field $\mathcal{V}$; it is equal to the difference of lengths of ${}^\#\omega^1_A$ and ${}^\#\omega^0_A$, which is zero (see [22, (4.4)]). Consequently, the lengths of the two middle modules are equal. Since both modules are concentrated at the singular point, they are vector spaces of the same dimension. It remains to note that $\mathrm{Coker}(\iota^*_{\mathcal{V}})\cong H_0(\omega^\bullet_A,\iota^*_{\mathcal{V}})$, while $\mathrm{Coker}(\iota_{\mathcal{V}})\cong H_0(\Omega^\bullet_A,\iota_{\mathcal{V}})\cong A/J_{\mathfrak{o}}\mathcal{V}$. This completes the proof. QED.

Corollary 5. *Under the same assumptions, one has*

$$\chi(\Omega^\bullet_A)=\chi(\omega^\bullet_A)=\dim_k A/J_{\mathfrak{o}}\mathcal{V},\quad \chi(\sigma_{\le 1}(\Omega^\bullet_A))=\dim_k A/J_{\mathfrak{o}}\mathcal{V}-\dim_k(\mathrm{Tors}\,\Omega^1_A).$$

Proof. Since $H_1(\omega^\bullet_A,\iota^*_{\mathcal{V}})=0$, the diagram (6) implies $\mathrm{Ker}(\iota_{\mathcal{V}}\colon\widetilde{\Omega}^1_A\to A)=0$. Consequently, $\chi(\sigma_{\le 1}(\Omega^\bullet_A))=\chi(\omega^\bullet_A)-\dim_k(\mathrm{Tors}\,\Omega^1_A)=\chi(\Omega^\bullet_A)-\dim_k(\mathrm{Tors}\,\Omega^1_A)$. It remains to use Claim 4. QED.

Thus, the computation of homological index for reduced curves is reduced to the computation of the length of torsion module and dimension of the quotient algebra $A/J_{\mathfrak{o}}\mathcal{V}$.

Proposition 7. *Let A be the dual analytical algebra of the germ of a reduced Gorenstein 1-dimensional singularity and let $\mathcal{V}\in\mathrm{Der}_k(A)$ be a vector field with an isolated singularity. Then*

$$\mathrm{Ind}_{\mathrm{hom},\mathfrak{o}}(\mathcal{V})=\dim_k A/J_{\mathfrak{o}}\mathcal{V}-\tau(A),$$

where $\tau(A)$ is the Tjurina number of the singularity A.

Proof. The local duality [18] implies an equality

$$\dim_k(\mathrm{Tors}\,\Omega^1_A)=\dim_k\mathrm{Ext}^1_A(\Omega^1_A,A).$$

Since A is *reduced* then the latter dimension is equal to $\dim_k T^1(A)$, the Tjurina number of the singularity A. QED.

REMARK. In the general case the dimension of the torsion module can be computed in terms of Noether and Dedekind differentes with the use of a formula from [9]; many papers are devoted to the computation of this invariant for various types of curves and higher-dimensional singularities.

REMARK. In fact, $A \subseteq \omega_A^0$, but, in general, $A \neq \omega_A^0$. To be more precise, let $\pi\colon \widetilde{X} \to X$ be the *normalization*. Then the sheaf ω_X^0 contains the direct image $\pi_*(\mathcal{O}_{\widetilde{X}})$, that is, all the germs of the *weakly holomorphic* functions on X, or, equivalently, the locally bounded meromorphic functions on X (see [8]). In other terms, it contains those meromorphic germs whose preimages are holomorphic on the normalization (cf. [29]).

6. QUASIHOMOGENEOUS CURVES

First recall that if A is a 1-dimensional Gorenstein analytical k-algebra, then $\omega_A^1 \cong A(\eta)$, where $\eta \in \omega_A^1$ is a free generator of the dualizing module. Hence, the exact sequence (8) yields the following inclusion

$$A \longrightarrow \operatorname{Hom}_A(\Omega_A^1, \omega_A^1) \cong \operatorname{Der}_k(A).$$

The image of A does not depend on the generator η; it is denoted by D_A and its elements are called *trivial derivations* (or, equivalently, trivial differentiations) of A over k. Obviously, D_A is a *free* A-module of rank 1.

In the *compete intersection* case the module of trivial differentiations D_A has a canonical generator, the so-called *Hamiltonian* vector field. In this case the defining ideal $\mathcal{I}$ of the singularity is generated by a *regular* sequence of functions $f_1, \dots, f_{m-1} \in \mathcal{O}_U$ (in the notations of Section 1). Next, let Δ be the determinant of the $m \times m$-matrix arising from adjoining to the Jacobian matrix $\|\partial f_j/\partial z_i\|$ the extra row $(\partial/\partial z_1, \dots, \partial/\partial z_m)$, that is,

$$\Delta = \det \begin{pmatrix} \partial/\partial z_1 & \dots & \partial/\partial z_m \\ \partial f_1/\partial z_1 & \dots & \partial f_1/\partial z_m \\ \vdots & \vdots & \vdots \\ \partial f_{m-1}/\partial z_1 & \dots & \partial f_{m-1}/\partial z_m \end{pmatrix}.$$

Then the Hamiltonian vector field H is the cofactor expansion of the determinant Δ along the first row, so that $H(f_j) = 0$ for all $j = 1, \dots, m-1$.

By definition, a commutative ring A is called $\mathbb{Z}$-graded if it decomposes into a direct sum $A = \oplus_{\nu \in \mathbb{Z}} A_\nu$ of abelian groups A_ν such that $A_\nu A_\lambda \subseteq A_{\nu+\lambda}$ for all $\nu, \lambda \in \mathbb{Z}$. The elements of the group A_ν are said to be homogeneous of degree ν. In a similar way one can define graded modules, algebras, etc.

Now assume that for every $j = 1, \dots, m-1$ the defining function f_j of the singularity is *quasihomogeneous* of degree d_j with respect to the weights w_i, $i = 1, \dots, m$. In other terms, the type of homogeneity of the 1-dimensional complete intersection singularity is equal to $(d_1, \dots, d_{m-1}; w_1, \dots, w_m) \in \mathbb{Z}^{m-1} \times \mathbb{Z}^m$ (see [2]). Then the local analytical algebra A, as well as A-modules $\operatorname{Der}_k(A)$, Ω_A^p, ω_A^p, $p \geq 0$, and homology groups of the corresponding contracted complexes are endowed with a natural grading. Thus, the Poincaré *series* or *polynomials* of graded modules are well-defined. Moreover, in this case $\operatorname{Der}_k(A)/D_A$ is a *cyclic* module; it is generated by the Euler vector field (see [2, (6.1)], [25, Satz 2]).

It is clear that the weight of the differential form $\eta = dz_1 \wedge \dots \wedge dz_m / df_1 \wedge \dots \wedge df_{m-1}$ is equal to $-c$, where $c = \sum d_j - \sum w_i$. Hence, there exist natural isomorphisms $\omega_A^1 \cong A[-c]$,

$\omega_A^0 \cong \mathrm{Hom}_A(\Omega_A^1, \omega_A^1) \cong \mathrm{Der}_k(A)[-c]$, and the following identities for Poincaré series:

$$P(\omega_A^1; x) = x^{-c}P(A; x), \qquad P(\omega_A^0; x) = x^{-c}P(\mathrm{Der}_k(A); x).$$

Let us take $\mathcal{V} \in \mathrm{Der}_k(A)_v$ of weight $v \in \mathbb{Z}$. Then the middle column of the diagram (6) gives us an exact sequence:

$$0 \longrightarrow \iota_{\mathcal{V}}^*(\omega_A^1) \longrightarrow \omega_A^0 \longrightarrow H_0(\omega_A^\bullet) \longrightarrow 0.$$

By Proposition 6 one has $H_1(\omega_A^\bullet, \iota_{\mathcal{V}}^*) = 0$, that is, $\mathrm{Ker}(\omega_A^1 \xrightarrow{\iota_{\mathcal{V}}^*} \omega_A^0) = 0$, and, consequently,

$$P(H_0(\omega_A^\bullet, \iota_{\mathcal{V}}^*); x) = P(\omega_A^0; x) - x^v P(\omega_A^1; x) = x^{-c}P(\mathrm{Der}_k(A); x) - x^{v-c}P(A; x).$$

On the other hand, by [2, Proposition 6.1, Theorem 3.2], one has

$$P(\mathrm{Der}_k(A); x) = x^c + P(A; x), \quad P(A; x) = \prod(1 - x^{d_j}) / \prod(1 - x^{w_i}),$$

$$P(\mathrm{Tors}\,(\Omega_A^1); x) = P(H_{\mathfrak{m}}^0(\Omega_A^1); x) =$$

$$= 1 + P(A; x)\,\mathrm{res}_{t=0} t^{-2}(1+t)^{-1} \prod(1 + tx^{w_i}) / \prod(1 + tx^{d_j}) = 1 + P(A; x)(\textstyle\sum x^{w_i} - \sum x^{d_j} - 1).$$

Hence,

$$\begin{aligned} P(H_0(\omega_A^\bullet, \iota_{\mathcal{V}}^*); x) &= 1 + x^{-c}P(A; x) - x^{v-c}P(A; x), \\ \chi(\omega_A^\bullet) &= P(H_0(\omega_A^\bullet, \iota_{\mathcal{V}}^*); 1), \\ \mathrm{Ind}_{\mathrm{hom},\mathfrak{o}}(\mathcal{V}) = \chi({}_{\sigma \le 1}(\Omega_A^\bullet)) &= \chi(\omega_A^\bullet) - P(\mathrm{Tors}\,(\Omega_A^1); 1). \end{aligned}$$

Example 6.1. Let S_5 be the germ of a space curve, defined as the intersection of two quadrics in 3-dimensional space. The type of homogeneity is equal to $(2, 2; 1, 1, 1)$, $c = 1$. Hence,

$$P(A; x) = (1 - x^2)^2/(1 - x)^3 = (1 + x)^2/(1 - x),$$

$$P(\mathrm{Der}_k(A); x) = x + P(A; x) = (1 + 3x)/(1 - x),$$

$$P(H_0(\omega_A^\bullet, \iota_{\mathcal{V}}^*); x) = 1 + (x - x^{v+1})(1 + x)^2/(1 - x) = 1 + x(1 + x)^2(1 + \ldots + x^{v-1}),\ v \ge 1,$$

$$P(\mathrm{Tors}\,(\Omega_A^1); x) = 1 + (3x - 2x^2 - 1)(1 + x)^2/(1 - x) = (2x - 1)(1 + x)^2 = 3x^2 + 2x^3.$$

As a result we get

$$\chi(\omega_A^\bullet, \iota_{\mathcal{V}}^*) = 4v + 1, \quad \mathrm{Ind}_{\mathrm{hom},\mathfrak{o}}(\mathcal{V}) = \chi({}_{\sigma \le 1}(\Omega_A^\bullet), \iota_{\mathcal{V}}) = 4(v - 1),\ v \ge 1.$$

If $v = 0$, that is, $\mathcal{V}$ is the Euler vector field, then $\chi(\omega_A^\bullet, \iota_{\mathcal{V}}^*) = \chi(\Omega_A^\bullet, \iota_{\mathcal{V}}) = 1$, and

$$\mathrm{Ind}_{\mathrm{hom},\mathfrak{o}}(\mathcal{V}) = \chi({}_{\sigma \le 1}(\Omega_A^\bullet)) = -4.$$

Next, if $v = 1$, that is, $\mathcal{V}$ is a combination of the Euler and Hamiltonian vector fields, then $\chi(\omega_A^\bullet, \iota_{\mathcal{V}}^*) = \chi(\Omega_A^\bullet, \iota_{\mathcal{V}}) = 5$, and $\mathrm{Ind}_{\mathrm{hom},\mathfrak{o}}(\mathcal{V}) = \chi({}_{\sigma \le 1}(\Omega_A^\bullet)) = 0$, and so on.

Remark. This example was considered in a different style in [23, (4.4)], where the author recommends to use a computer algebra system for explicit computations.

Example 6.2. Quasihomogeneous and monomial curves. Evidently, every irreducible component of a quasihomogeneous curve has a *monomial* parametrization. For simplicity, let us consider the case of an irreducible curve, that is, X is a monomial curve. Let H be its *value semigroup* so that the local analytical algebra A of the germ X is generated by monomials t^h, $h \in H$, that is, $A \cong k\langle t^H \rangle$ in standard notations. Then one can compute explicitly dimensions of all graded components of the first cotangent cohomology $T^1(A) \cong \mathrm{Ext}_A^1(\Omega_A^1, A)$ in terms of the semigroup (see, for example, [12]).

In the Gorenstein case the local duality implies that $\dim_k \mathrm{Tors}\,\Omega_A^1 = \dim_k T^1(A) = \tau(A)$. Thus, one obtains also the dimension of the torsion module and, consequently, an explicit expression for the index in view of Proposition 7. It should be noted that for quasihomogeneous

Gorenstein curves one has $\dim_k({}^{\#}\omega_A^1) = \mu(A)$, that is, the dimension of the first ω-cotorsion module is equal to the Milnor number of the singularity (see [25, Satz 1]).

7. Normal two-dimensional singularities

Let us now discuss a simple generalization of Proposition 6 and Claim 4 to the higher dimensional case.

Proposition 8. *Let $(A, \mathfrak{m})$ be the local analytical algebra of a reduced isolated singularity of dimension $n \geq 1$ and let $\mathcal{V} \in \mathrm{Der}_k(A)$ be a vector field with an isolated singularity. Then $H_n(\omega_A^\bullet, \iota_\mathcal{V}^*) = 0$.*

Proof. Set $K_A^n = \mathrm{Ker}(\iota_\mathcal{V}^* \colon \omega_A^n \to \omega_A^{n-1})$. The module K_A^n is coherent; it is concentrated at the singular point of a *reduced* singularity A. Hence K_A^n is a finite-dimensional vector space over k. In other words, it is a torsion module, $K_A^n \cong H^0_\mathfrak{m}(K_A^n) \subseteq H^0_\mathfrak{m}(\omega_A^n) = \mathrm{Tors}\,\omega_A^n$. However, in view of basic properties of the dualizing module, ω_A^n has no torsion, that is, $K_A^n = 0$. QED.

Proposition 9. *Suppose that A is a normal Cohen-Macaulay singularity of dimension $n \geq 2$, and $\mathcal{V} \in \mathrm{Der}_k(A)$ is a vector field with an isolated singularity. Then there exists a natural isomorphism $H_0(\widetilde{\Omega}_A^\bullet, \iota_\mathcal{V}) \cong H_0(\omega_A^\bullet, \iota_\mathcal{V}^*)$.*

Proof. Since A is normal, then $\widetilde{\Omega}_A^0 \cong \Omega_A^0 \cong \omega_A^0 \cong A$. Similarly to (6), there exists the following commutative diagram with exact rows

$$\begin{array}{ccccccc}
0 & \longrightarrow & \widetilde{\Omega}_A^2 & \xrightarrow{c_A} & \omega_A^2 & \longrightarrow {}^{\#}\omega_A^2 & \longrightarrow 0 \\
 & & \downarrow\iota_\mathcal{V} & & \downarrow\iota_\mathcal{V}^* & \downarrow & \\
0 & \longrightarrow & \widetilde{\Omega}_A^1 & \xrightarrow{c_A} & \omega_A^1 & \longrightarrow {}^{\#}\omega_A^1 & \longrightarrow 0 \\
 & & \downarrow\iota_\mathcal{V} & & \downarrow\iota_\mathcal{V}^* & \downarrow & \\
0 & \longrightarrow & \Omega_A^0 \cong A & \longrightarrow & \omega_A^0 \cong A & \longrightarrow 0. &
\end{array} \tag{9}$$

The standard Ker-Coker exact sequence associated with the two lower rows looks like this

$$0 \to \mathrm{Ker}(\widetilde{\Omega}_A^1 \to A) \to \mathrm{Ker}(\omega_A^1 \to A) \to {}^{\#}\omega_A^1 \to H_0(\widetilde{\Omega}_A^\bullet, \iota_\mathcal{V}) \to H_0(\omega_A^\bullet, \iota_\mathcal{V}^*) \to 0. \tag{10}$$

Thus, it is enough to show that two right modules in (10) have the same dimension. To prove this, it is convenient to use an equivalent description of regular meromorphic forms in terms of Noether normalization and the trace map. To be more precise (see [22, (2.1)]), for an n-dimensional singularity A a meromorphic differential p-form $\omega \in \Omega_A^p \otimes_A F(A)$ is regular if and only if for any Noether normalization $Q = k\langle T_1, \dots, T_n\rangle \to A$ one has $\mathrm{Tr}_Q^A(\omega \wedge \eta) \in \Omega_Q^n$ for all holomorphic $(n-p)$-form $\eta \in \Omega_A^{n-p}$.

Now let $Q = k\langle T_1, \dots, T_n\rangle \to A$ be a Noether normalization of A, and let $\overline{\mathcal{V}}$ be an extension of the vector field $\mathcal{V}$ to $\Omega_Q^\bullet$. In fact, such (non-trivial) extension always exists because A is Cohen-Macaulay and one can choose suitable regular parameters $(T_1, \dots, T_n)$ from the jacobian ideal $J_\mathfrak{o}\mathcal{V} \subseteq \mathfrak{m}$, generated by $m \geq n+2$ *regular* elements by assumption (if A is quasihomogeneous then one can take homogeneous parameters). Thus, the two lower rows of the diagram (9) transform in the following commutative diagram

$$
\begin{array}{ccccccc}
\Omega^1_Q & \longrightarrow & \Omega^1_A & \xrightarrow{c_A} & \omega^1_A & \xrightarrow{\mathrm{Tr}} & \Omega^1_Q \\
\downarrow \iota_{\overline{\mathcal{V}}} & & \downarrow \iota_{\mathcal{V}} & & \downarrow \iota^*_{\mathcal{V}} & & \downarrow \iota_{\overline{\mathcal{V}}} \\
\Omega^0_Q \cong Q & \longrightarrow & \Omega^0_A \cong A & \xrightarrow{c_A} & \omega^0_A & \xrightarrow{\mathrm{Tr}} & \Omega^0_Q \cong Q \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
Q/J_{\mathfrak{o}}\overline{\mathcal{V}} & \longrightarrow & \mathrm{Coker}(\iota_{\mathcal{V}}) & \longrightarrow & \mathrm{Coker}(\iota^*_{\mathcal{V}}) & \longrightarrow & Q/J_{\mathfrak{o}}\overline{\mathcal{V}}.
\end{array}
\tag{11}
$$

The left and right terms of the bottom row are finite-dimensional vector spaces. On the other hand, it is well-known that the composition of maps of this row is equal to the multiplication by $\mathrm{rank}_Q(A)$ (see [24], [22]). This completes the proof. QED.

REMARK. For normal complete intersections of dimension $n \geq 3$ the proof of Proposition 9 is trivial, since $\Omega^p_A \cong \Omega^p_A{}^{\vee\vee}$ for all $0 \leq p < n-1$. In particular, Ω^1_A (as well as Ω^0_A) has no torsion and *cotorsion*, and the exact sequence (10) splits in two isomorphisms.

Claim 5. *Let A be a quasihomogeneous normal 2-dimensional singularity and let $\mathcal{V} \in \mathrm{Der}_k(A)$ be a vector field with an isolated singularity. Then*

$$\mathrm{Ind}_{\mathrm{hom},\mathfrak{o}}(\mathcal{V}) = \chi(\omega^\bullet_A, \iota^*_{\mathcal{V}}) - \dim_k(\mathrm{Tors}\,\Omega^1_A) + \dim_k(\mathrm{Tors}\,\Omega^2_A).$$

PROOF. First observe that normal 2-dimensional singularities satisfy Serre's conditions R_1 and S_2. In particular, they are isolated and Cohen-Macaulay. Hence, ω^p_A, $p \geq 0$, are well-defined. Further, such singularities may have two non-trivial cotorsion modules only in dimension 1 and 2. Moreover, if A is *graded* then the cotorsion modules are isomorphic: ${}^{\#}\omega^2_A \cong {}^{\#}\omega^1_A$ (see [22, (4.8), Bem. (1)]). Making use of diagram (9), it remains to combine Claim 1 and Proposition 4. QED.

We are able to analyze the 2-dimensional case of an isolated complete intersection singularity similarly to Section 6. Let $(d_1, \ldots, d_{m-2}; w_1, \ldots, w_m)$ be the type of homogeneity of the singularity A, $c = \sum d_j - \sum w_i$. Then the weight of the differential form $\eta = dz_1 \wedge \ldots \wedge dz_m / df_1 \wedge \ldots \wedge df_{m-2}$ is equal to $-c$, and there are natural isomorphisms

$$\omega^2_A \cong A(\eta) \cong A[-c], \quad \omega^1_A \cong \mathrm{Hom}_A(\Omega^1_A, \omega^2_A) \cong \mathrm{Der}_k(A)[-c].$$

Since A is normal, then $\omega^1_A \cong \Omega^1_A{}^{\vee\vee}$ and $\omega^0_A \cong \Omega^0_A{}^{\vee\vee}$ by Proposition 2. In particular, $\omega^0_A \cong A$, and one has the following identities for Poincaré series:

$$P(\omega^2_A; x) = x^{-c} P(A; x), \quad P(\omega^1_A; x) = x^{-c} P(\mathrm{Der}_k(A); x), \quad P(\omega^0_A; x) = P(A; x).$$

Again, by [2, Proposition 6.1], one has $P(\mathrm{Der}_k(A); x) = P(A; x) + x^c P(A; x) - x^c$, and, consequently,

$$P(\omega^1_A; x) = x^{-c} P(A; x) + P(A; x) - 1.$$

Further, since $K^2_A = 0$, one has $\omega^2_A \cong \iota^*_{\mathcal{V}}(\omega^2_A)$. Hence,

$$P(\iota^*_{\mathcal{V}}(\omega^2_A); x) = x^{\nu} P(\omega^2_A; x) = x^{\nu - c} P(A; x).$$

The next step of computations is to consider the following exact sequence

$$0 \to K^1_A \longrightarrow \omega^1_A \xrightarrow{\iota^*_{\mathcal{V}}} A \longrightarrow A/\iota^*_{\mathcal{V}}(\omega^1_A) \to 0,$$

which implies the relations

$$P(K^1_A; x) = P(\omega^1_A; x) - x^{-\nu} P(A; x) + x^{-\nu} P(A/\iota^*_{\mathcal{V}}(\omega^1_A); x),$$

$$P(K^1_A; x) = x^{-c} P(A; x) + P(A; x) - 1 - x^{-\nu} P(A; x) + x^{-\nu} P(A/\iota^*_{\mathcal{V}}(\omega^1_A); x),$$

$$P(K^1_A/\iota^*_{\mathcal{V}}(\omega^2_A); x) = x^{-c} P(A; x) + P(A; x) - 1 - x^{-\nu} P(A; x) + x^{-\nu} P(A/\iota^*_{\mathcal{V}}(\omega^1_A); x) - x^{\nu - c} P(A; x),$$

$$P(H_1(\omega^\bullet_A,\iota^*_{\mathcal{V}});x) = -1 + (1 + x^{-c} - x^{-\nu} - x^{\nu-c})P(A;x) + x^{-\nu}P(A/\iota^*_{\mathcal{V}}(\omega^1_A);x).$$

The last polynomial is equal to $x^{-\nu}P(A/J_{\mathfrak{o}}\mathcal{V};x)$ by Proposition 9, while

$$P(A/J_{\mathfrak{o}}\mathcal{V};x) = P(H_0(\Omega^\bullet_A,\iota_{\mathcal{V}});x).$$

Example 7.1. The germ Q_7 of the intersection of two quadrics in 4-dimensional space. The type of homogeneity is equal to $(2,2;1,1,1,1)$, $c=0$. Hence, by [2, Proposition 6.1, Theorem 3.2],

$$P(A;x) = (1-x^2)^2/(1-x)^4 = (1+x)^2/(1-x)^2,$$
$$P(\mathrm{Der}_k(A);x) = 2P(A;x) - 1 = (1+6x+x^2)/(1-x)^2,$$
$$P(H_1(\omega^\bullet_A,\iota^*_{\mathcal{V}});x) = -1 + (2 - x^{-\nu} - x^{\nu})P(A;x) + x^{-\nu}P(A/J_{\mathfrak{o}}\mathcal{V};x),$$
$$P(\mathrm{Tors}\,(\Omega^2_A);x) = P(H^0_{\mathfrak{m}}(\Omega^2_A);x) = -1 + P(A;x)\,\mathrm{res}_{t=0}t^{-3}(1+t)^{-1}(1+tx)^4)/(1+tx^2) =$$
$$-1 + (1+x)^2(1-2x+3x^2) = 4x^3 + 3x^4,$$

so that $\dim_k \mathrm{Tors}\,(\Omega^2_A) = 7$.

If $v=0$, then $P(A/J_{\mathfrak{o}}\mathcal{V};1) = P(H_0(\omega^\bullet_A,\iota^*_{\mathcal{V}});1) = 1$. Consequently, we obtain

$$P(H_1(\omega^\bullet_A,\iota^*_{\mathcal{V}});x) = -1 + P(A/J_{\mathfrak{o}}\mathcal{V};x) = 0,\ \chi(\omega^\bullet_A,\iota^*_{\mathcal{V}}) = 1,\ \mathrm{Ind}_{\mathrm{hom},\mathfrak{o}}(\mathcal{V}) = \chi({}_{\sigma\le 2}(\Omega^\bullet_A,\iota_{\mathcal{V}})) = 8.$$

Next, if $v=1$, then $P(A/J_{\mathfrak{o}}\mathcal{V};x) = 1+4x+4x^2$. Indeed, there are 10 monomials of degree two in 4 variables and 6 generic relations between them which are given by 4 polynomial coefficients of the vector field and 2 defining equations. In addition, there are 4 variables of degree 1 and the field of constants. Thus,

$$P(H_1(\omega^\bullet_A,\iota^*_{\mathcal{V}});x) = -1 + (2 - x^{-1} - x)(1+x)^2/(1-x)^2 + x^{-1}P(A/J_{\mathfrak{o}}\mathcal{V};x)$$
$$= -1 - x^{-1}(1+x)^2 + x^{-1} + 4 + 4x = -(x^{-1}+3+x) + x^{-1} + 4 + 4x = 1+3x.$$

As a result,

$$\chi(\omega^\bullet_A,\iota^*_{\mathcal{V}}) = 9-4 = 5,\quad \mathrm{Ind}_{\mathrm{hom},\mathfrak{o}}(\mathcal{V}) = \chi({}_{\sigma\le 2}(\Omega^\bullet_A,\iota_{\mathcal{V}})) = \chi(\omega^\bullet_A,\iota^*_{\mathcal{V}}) + \dim_k \mathrm{Tors}\,\Omega^2_A = 5+7 = 12.$$

Remark. This example was investigated in a different manner in [10, Ex.(4.3)], partially with the use of a computer algebra system of symbolic computations.

8. Holomorphic forms on complete intersections

Making use of explicit formulas for modules of holomorphic differential forms (see [2]), in this section the homological index is computed directly for quasihomogeneous isolated complete intersection singularities by another method. In addition, we verify the computational results for curves and surface singularities described above in a slightly different manner.

In the notations of Section 1 suppose that $X = (X,\mathfrak{o})$ is the germ of a reduced complete intersection in a complex manifold M of dimension $m \ge 2$. Thus, the defining ideal $\mathcal{I}$ of $X \subset U$ is generated by a regular sequence of functions $f_1,\dots,f_k$ in $\mathcal{O}_U$, so that $\mathcal{O}_X = \mathcal{O}_U/\mathcal{I}|_X$, and $\dim_{\mathbb{C}} X = m-k = n \ge 1$.

Lemma 3 ([2], Lemma 3.2)**.** *Let $(A,\mathfrak{m})$ be the dual analytical algebra of an isolated complete intersection singularity $(X,\mathfrak{o})$ of dimension $n \ge 1$, $n = m-k \ge 1$, with the type of homogeneity $(d_1,\dots,d_k;w_1,\dots,w_m)$. Then for all $0 \le p \le n$*

$$P(\Omega^p_A;x) = P(A;x)\cdot\mathrm{res}_{t=0}t^{-p-1}\prod(1+tx^{w_i})/\prod(1+tx^{d_j}).$$

Proof. Let X' be an $(n+1)$-dimensional isolated complete intersection singularity and $X = f^{-1}(0)$, where $f\colon X' \to \mathbb{C}$, $f(\mathfrak{o}) = 0$, is a flat holomorphic map such that $f|_{X'-\{\mathfrak{o}\}}$ is regular. In other terms, the singularity X is the *hypersurface section* of X' defined by f (see [27]). Then the sequence of $\mathcal{O}_X$-modules

$$0 \to \Omega^p_X \xrightarrow{\wedge df} \Omega^{p+1}_{X'}/f\Omega^{p+1}_{X'} \longrightarrow \Omega^{p+1}_X \to 0$$

is exact for all $0 \le p \le n-1$ (see [17, Lemma 1.6]). In the quasihomogeneous case we obtain the following relations for Poincaré polynomials:

$$P(\Omega^{p+1}_X;x) = (1-x^d)P(\Omega^{p+1}_{X'};x) - x^d P(\Omega^p_X;x),$$

where $d = \deg f = \deg(\wedge df)$.

First consider the case where the singularity $X = X_k$ can be defined by a regular sequence f_i, $i = 1,\dots,k$, such that for all $j = 1,2,\dots,k$ every germ X_j determined by $f_1,\dots,f_j$ is the *hypersurface section* of X_{j-1} defined by f_j. For convenience of notations set $X_0 = (\mathbb{C}^m, 0)$. In this case we can apply a double induction on k and p similarly to [1]. The general case is reduced to the case of hypersurface sections by arguments in [27, 2.3]. QED.

Remark. In fact, for complete intersections with non-isolated singularities the identity of Lemma 3 is valid for all $0 \le p \le c$, where $c = \operatorname{codim}(\operatorname{Sing} X, X)$.

Proposition 10. *Let A be the local analytical algebra of a reduced n-dimensional isolated singularity, $n \ge 1$, and let $\mathcal{V} \in \operatorname{Der}_k(A)$ be a vector field with an isolated singularity. Then $H_n(\widetilde{\Omega}^\bullet_A, \iota_\mathcal{V}) = 0$.*

Proof. Since $\widetilde{\Omega}^n_A$ has no torsion, all the arguments of Proposition 8, applied for dualizing module ω^n_A, remain valid. Then we conclude that $\iota_\mathcal{V}\colon \widetilde{\Omega}^n_A \to \widetilde{\Omega}^{n-1}_A$ is injective and the n-th homology group vanishes as required. QED.

Corollary 6. *Under the same assumptions one has*

$$\operatorname{Ind}_{\mathrm{hom},\mathfrak{o}}(\mathcal{V}) = \chi({}_{\sigma\le n}(\Omega^\bullet_A), \iota_\mathcal{V}) = \chi(\widetilde{\Omega}^\bullet_A, \iota_\mathcal{V}) + (-1)^n \dim_k \operatorname{Tors}\Omega^n_A.$$

Example 8.1. Let us consider again the germ of space curve S_5 from Section 6 (see Example 6.1). By Lemma 3

$$P(\Omega^1_A;x) = (\textstyle\sum x^{w_i} - \sum x^{d_j})P(A;x) = (3x - 2x^2)P(A;x).$$

Then by [2, Theorem (3.2)] we have

$$P(\operatorname{Tors}(\Omega^1_A);x) = 1 + P(A;x)(\textstyle\sum x^{w_i} - \sum x^{d_j} - 1) = 3x^2 + 2x^3.$$

Hence,

$$P(\widetilde{\Omega}^1_A;x) = P(A;x) - 1, \quad P(\iota_\mathcal{V}(\widetilde{\Omega}^1_A);x) = x^v(P(A;x)-1),$$
$$P(H_0(\widetilde{\Omega}^\bullet_A, \iota_\mathcal{V});x) = P(A;x) - x^v P(A;x) + x^v =$$
$$= (1-x^v)P(A;x) + x^v = x^v + (1+x)^2(1 + x + \dots x^{v-1}),\ v \ge 1.$$

As a result one gets

$$\chi(\Omega^\bullet_A, \iota_\mathcal{V}) = 4v+1, \quad \operatorname{Ind}_{\mathrm{hom},\mathfrak{o}}(\mathcal{V}) = \chi({}_{\sigma\le 1}(\Omega^\bullet_A), \iota_\mathcal{V}) = 4(v-1),\ v \ge 1.$$

If $v = 0$, then $\chi(\Omega^\bullet_A, \iota_\mathcal{V}) = 1$, and $\operatorname{Ind}_{\mathrm{hom},\mathfrak{o}}(\mathcal{V}) = -4$, because $\dim_k \operatorname{Tors}\Omega^1_A = 5$. Further, if $v = 1$, then $\chi(\Omega^\bullet_A, \iota_\mathcal{V}) = 5$, and $\operatorname{Ind}_{\mathrm{hom},\mathfrak{o}}(\mathcal{V}) = 0$.

Remark. Combining observations of two approaches, in this example it is possible to compute the homogeneous structure of two modules of the ω_A-cotorson complex ${}^{\#}\omega^\bullet_A$ explicitly. As was shown before this complex has two non-trivial terms of the same dimension. In fact, they are cyclic A-modules and the module ${}^{\#}\omega^0_A$ is naturally isomorphic to the quotient of $\mathrm{Der}_k(A)$ by the submodule generated by the hamiltonian vector field. The bottom and top rows of diagram (6) imply the following two relations

$$P({}^{\#}\omega^0_A;x) = P(\omega^0_A;x) - P(A;x),$$
$$P({}^{\#}\omega^1_A;x) = P(\omega^1_A;x) - P(\Omega^1_A;x) + P(\mathrm{Tors}\,(\Omega^1_A);x)) = P(\omega^1_A;x) - P(\widetilde{\Omega}^1_A;x),$$

respectively. Hence,

$$P({}^{\#}\omega^0_A;x) = 1 + (x^{-1}-1)P(A;x) = 1 + x^{-1}(1+x)^2 = x^{-1}+3+x,$$
$$P({}^{\#}\omega^1_A;x) = x^{-1}(1+x)^2/(1-x) - (3x+x^2)/(1-x) = x^{-1}(1+2x-2x^2-x^3)/(1-x) = x^{-1}+3+x.$$

To simplify notations, in the sequel we will denote the contraction maps on the families $\Omega^\bullet_A$, $\omega^\bullet_A$ and ${}^{\#}\omega^\bullet_A$ by the same symbol $\iota_{\mathcal{V}}$.

Now we are able to describe the action of $\iota_{\mathcal{V}}$ on the cotorsion complex ${}^{\#}\omega^\bullet_A$ and to compute the homology groups. Obviously, if $v=0$ then this action is an isomorphism. If $v=1$, then

$$P(\mathrm{Ker}(\iota_{\mathcal{V}}\colon {}^{\#}\omega^1_A \to {}^{\#}\omega^0_A);x) = 2+x; \quad P(\mathrm{Coker}(\iota_{\mathcal{V}}\colon {}^{\#}\omega^1_A \to {}^{\#}\omega^0_A);x) = x^{-1}+2,$$

$\dim_k H_1({}^{\#}\omega^\bullet_A) = \dim_k H_0({}^{\#}\omega^\bullet_A) = 3$. If $v=2$, then

$$P(\mathrm{Ker}(\iota_{\mathcal{V}}\colon {}^{\#}\omega^1_A \to {}^{\#}\omega^0_A);x) = 3+x; \quad P(\mathrm{Coker}(\iota_{\mathcal{V}}\colon {}^{\#}\omega^1_A \to {}^{\#}\omega^0_A);x) = x^{-1}+3,$$

$\dim_k H_1({}^{\#}\omega^\bullet_A) = \dim_k H_0({}^{\#}\omega^\bullet_A) = 4$. For all $v \geq 3$ one has $\dim_k H_1({}^{\#}\omega^\bullet_A) = \dim_k H_0({}^{\#}\omega^\bullet_A) = 5$, that is, $\iota_{\mathcal{V}}$ is the zero map.

Example 8.2. In a similar manner for the surface singularity Q_7 from Section 7 one gets

$$P(\Omega^2_A;x) = (6x^2-8x^3+3x^4)P(A;x), \quad P(\Omega^1_A;x) = (4x-2x^2)P(A;x),$$
$$P(\widetilde{\Omega}^2_A;x) = P(\Omega^2_A;x) - P(\mathrm{Tors}\,(\Omega^2_A);x).$$

Further, there are two short exact sequences

$$0 \to \widetilde{\Omega}^2_A \xrightarrow{\iota_{\mathcal{V}}} \Omega^1_A \longrightarrow \Omega^1_A/\iota_{\mathcal{V}}(\Omega^2_A) \to 0, \quad 0 \to \mathrm{Ker}^1_A \longrightarrow \Omega^1_A \xrightarrow{\iota_{\mathcal{V}}} A/\iota_{\mathcal{V}}(\Omega^1_A) \to 0,$$

where $\mathrm{Ker}^1_A = \mathrm{Ker}\,(\iota_{\mathcal{V}}\colon \Omega^1_A \to A)$. They imply two relations

$$P(\iota_{\mathcal{V}}(\Omega^2_A);x) = x^v P(\widetilde{\Omega}^2_A;x), \quad P(\mathrm{Ker}^1_A;x) = P(\Omega^1_A;x) - x^{-v}P(A;x) + x^{-v}P(A/\iota_{\mathcal{V}}(\Omega^1_A);x),$$

respectively. As a result,

$$P(H_1(\widetilde{\Omega}^\bullet_A,\iota_{\mathcal{V}});x) = P(\mathrm{Ker}^1_A;x) - x^v P(\iota_{\mathcal{V}}(\widetilde{\Omega}^2_A);x) =$$
$$= P(\Omega^1_A;x) - x^{-v}P(A;x) + x^{-v}P(A/\iota_{\mathcal{V}}(\Omega^1_A);x) - x^v P(\Omega^2_A;x) + x^v P(\mathrm{Tors}\,(\Omega^2_A);x) =$$
$$= \{(4x-2x^2) - x^{-v} - x^v(6x^2-8x^3+3x^4)\}P(A;x) + x^{-v}P(A/J_{\mathfrak{o}}\mathcal{V};x) + x^v(4x^3+3x^4).$$

If $v=0$, then

$$P(H_1(\widetilde{\Omega}^\bullet_A,\iota_{\mathcal{V}});x) = \{(4x-2x^2) - 1 - (6x^2-8x^3+3x^4)\}P(A;x) + 1 + (4x^3+3x^4).$$

The expression in curly brackets is equal to $\{-2(1-x)^2 + (1-x)^3(1+3x)\}$. Hence,

$$P(H_1(\widetilde{\Omega}^\bullet_A,\iota_{\mathcal{V}});x) = \{-2+(1-x)(1+3x)\}(1+x)^2 + 1 + (4x^3+3x^4) = 0,$$

that is,

$$\chi(\widetilde{\Omega}^\bullet_A,\iota_{\mathcal{V}}) = 1, \quad \mathrm{Ind}_{\mathrm{hom},\mathfrak{o}}(\mathcal{V}) = \chi({}_{\sigma\leq 2}(\Omega^\bullet_A,\iota_{\mathcal{V}})) = 8.$$

Further, if $v=1$, then

$$P(H_1(\widetilde{\Omega}^\bullet_A,\iota_{\mathcal{V}});x) = \{(4x-2x^2)-x^{-1}-x(6x^2-8x^3+3x^4)\}P(A;x)+x^{-1}P(A/J_{\mathfrak{o}}\mathcal{V};x)+x(4x^3+3x^4).$$

The expression in curly brackets transforms in the following way:

$$x^{-1}\{-2(1-x)(1+x-x^2)+(1-x)(1+x+x^2+x^3-5x^4+3x^5)\} = -x^{-1}(1-x)^2(1+2x-x^2-2x^3+3x^4).$$

Hence,

$$P(H_1(\widetilde{\Omega}^\bullet_A,\iota_{\mathcal V});x) = -x^{-1}(1+x)^2(1+2x-x^2-2x^3+3x^4)+x^{-1}P(A/J_{\mathfrak o}\mathcal V;x)+x(4x^3+3x^4)$$
$$= x^{-1}(-1-4x-4x^2+2x^3+2x^4-4x^5-3x^6)+x(4x^3+3x^4)+x^{-1}P(A/J_{\mathfrak o}\mathcal V;x)$$
$$= -x^{-1}(1+4x+4x^2)+2x^2+2x^3+x^{-1}(1+4x+4x^2) = 2x^2+2x^3.$$

That is, $\dim_k H_1(\widetilde{\Omega}^\bullet_A,\iota_{\mathcal V}) = P(H_1(\widetilde{\Omega}^\bullet_A,\iota_{\mathcal V});1) = 4$. As a result,

$$\chi(\widetilde{\Omega}^\bullet_A,\iota_{\mathcal V}) = 9-4=5,\quad \mathrm{Ind}_{\mathrm{hom},\mathfrak o}(\mathcal V) = \chi({}_{\sigma\le 2}(\Omega^\bullet_A,\iota_{\mathcal V})) = 9-4+7=12.$$

In addition, since $\dim_k \Omega^4_A = 1$ and $\dim_k \Omega^3_A = 8$, one concludes $\chi(\Omega^\bullet_A,\iota_{\mathcal V}) = 5$.

For completeness, let us also analyze the homology of the ω_A-cotorson complex ${}^{\#}\omega^\bullet_A$. Two non-trivial terms of this complex has the same dimension, they are cyclic A-modules and the module ${}^{\#}\omega^1_A$ is naturally isomorphic to the quotient of $\mathrm{Der}_k(A)$ by the submodule generated by 4 hamiltonian vector fields. In fact, for 2-dimensional quasihomogeneous isolated complete intersection singularities these two cotorsion modules are isomorphic in view of the canonical isomorphisms ${}^{\#}\omega^1_A \cong H^1_{\mathfrak m}(\Omega^1_A)$ and ${}^{\#}\omega^2_A \cong H^1_{\mathfrak m}(\Omega^2_A) \cong H^1_{\mathfrak m}(\Omega^1_A)$. One can apply the general formulas for Poincaré polynomials of the local cohomology groups from [1] or [2, (3.2)]. However, it is also possible to compute the polynomials directly. To be more precise, similarly to the above example there exist two relations

$$P({}^{\#}\omega^1_A;x) = P(\omega^1_A;x) - P(\Omega^1_A;x),$$
$$P({}^{\#}\omega^2_A;x) = P(\omega^2_A;x) - P(\Omega^2_A;x) + P(\mathrm{Tors}\,(\Omega^2_A);x)).$$

Hence,

$$\begin{aligned}
P({}^{\#}\omega^1_A;x) &= 2P(A;x)-1-P(\Omega^1_A;x) = P(A;x)(2-4x+2x^2)-1 = 1+4x+2x^2,\\
P({}^{\#}\omega^2_A;x) &= P(A;x)\{1-(6x^2-8x^3+3x^4)\}+(4x^3+3x^4)\\
&= (1+x)^2(1-x)(1+3x)+(4x^3+3x^4) = 1+4x+2x^2.
\end{aligned}$$

Again, if $v=0$ then the homology groups of the cotorsion complex ${}^{\#}\omega^\bullet_A$ obviously vanish. If $v=1$, then

$$P(\mathrm{Ker}(\iota_{\mathcal V}\colon {}^{\#}\omega^2_A \to {}^{\#}\omega^1_A);x) = 2x+2x^2;\quad P(\mathrm{Coker}(\iota_{\mathcal V}\colon {}^{\#}\omega^1_A \to {}^{\#}\omega^0_A);x) = 1+3x,$$

and $\dim_k H_2({}^{\#}\omega^\bullet_A) = \dim_k H_1({}^{\#}\omega^\bullet_A) = 4$. If $v=2$, then

$$P(\mathrm{Ker}(\iota_{\mathcal V}\colon {}^{\#}\omega^1_A \to {}^{\#}\omega^0_A);x) = 4x+2x^2;\quad P(\mathrm{Coker}(\iota_{\mathcal V}\colon {}^{\#}\omega^1_A \to {}^{\#}\omega^0_A);x) = 1+4x+x^2$$

and $\dim_k H_2({}^{\#}\omega^\bullet_A) = \dim_k H_1({}^{\#}\omega^\bullet_A) = 6$.

For $v\ge 3$ one has $\dim_k H_2({}^{\#}\omega^\bullet_A) = \dim_k H_1({}^{\#}\omega^\bullet_A) = 7$, that is, the map $\iota_{\mathcal V}$ is identically *zero*.

9. The generating function and homology

Now we will study in some detail the case of quasihomogeneous complete intersections with isolated singularities of arbitrary dimension. The key idea of our approach is based on the fact that the Euler characteristic of a complex of finite-dimensional vector spaces is equal to the alternating sum of their dimensions. In the quasihomogeneous case the *graded components* of the modules of holomorphic differential forms play the role of such spaces. Our method of computation is based on a simplified procedure of "dévissage" applied in [2, (3.3)].

Definition. The generating function of the complex $(\Omega^\bullet_A,\iota_{\mathcal V})$ is defined as follows:

$$G_P((\Omega^\bullet_A,\iota_{\mathcal V});x,y) = \sum_{i\ge 0}(-1)^i P(H_i(\Omega^\bullet_A,\iota_{\mathcal V});x)\,y^i, \tag{12}$$

where $P(H_i(\Omega_A^\bullet, \iota_{\mathcal V}); x)$ are Poincaré polynomials of the corresponding homology groups which are *graded* vector spaces. Evidently, if all homology groups are finite-dimensional, then

$$\chi(\Omega_A^\bullet, \iota_{\mathcal V}) = G_P((\Omega_A^\bullet, \iota_{\mathcal V}); 1, 1).$$

REMARK. Of course, similar generating functions are well-defined for all other complexes considered above; such functions can be considered as variants of χ_y-characteristic of Hirzebruch associated with the "twisted" De Rham cohomology (cf. [2, Introduction]).

Theorem 1. *In the notations of the previous section let* $(A, \mathfrak m)$ *be the local algebra of an* n*-dimensional isolated complete intersection singularity* $(X, \mathfrak o)$ *with the type of homogeneity* $(d_1, \dots, d_k; w_1, \dots, w_m)$, $n = m - k \ge 1$. *Suppose that the weight of* $\mathcal V$ *is equal to* v. *Then*

$$G_P(({}_{\sigma\le n}(\Omega_A^\bullet), \iota_{\mathcal V}); x, x^v) =$$
$$(-1)^n x^{nv} P(A; x)\,\mathrm{res}_{t=0} t^{-n-1}(1+tx^{-v})^{-1} \prod (1+tx^{w_i}) / \prod (1+tx^{d_j}).$$

In particular, $\mathrm{Ind}_{\mathrm{hom},\mathfrak o}(\mathcal V) = G_P(({}_{\sigma\le n}(\Omega_A^\bullet), \iota_{\mathcal V}); 1, 1)$.

PROOF. Let us consider the truncated contracted De Rham complex of A :

$$({}_{\sigma\le n}(\Omega_A^\bullet), \iota_{\mathcal V}) \colon \quad 0 \longrightarrow \Omega_A^n \xrightarrow{\iota_{\mathcal V}} \Omega_A^{n-1} \xrightarrow{\iota_{\mathcal V}} \Omega_A^{n-2} \to \dots \to \Omega_A^1 \xrightarrow{\iota_{\mathcal V}} \Omega_A^0 \cong A \longrightarrow 0.$$

According to Proposition 10, the kernel of the left contraction mapping is the torsion submodule, that is, $H_n({}_{\sigma\le n}(\Omega_A^\bullet), \iota_{\mathcal V}) \cong \mathrm{Tors}\,\Omega_A^n$. Set $\mathrm{Ker}_A^i = \mathrm{Ker}(\iota_{\mathcal V} \colon \Omega_A^i \to \Omega_A^{i-1})$. Then for all $1 \le i < n$ there exist exact sequences

$$0 \longrightarrow \mathrm{Ker}_A^i \longrightarrow \Omega_A^i \xrightarrow{\iota_{\mathcal V}} \iota_{\mathcal V}(\Omega_A^i) \longrightarrow 0$$

and the following relations for Poincaré series:

$$P(\mathrm{Ker}_A^i; x) = P(\Omega_A^i; x) - x^{-v} P(\iota_{\mathcal V}(\Omega_A^i); x),$$
$$P(H_i(\Omega_A^\bullet, \iota_{\mathcal V}); x) = P(\Omega_A^i; x) - x^{-v} P(\iota_{\mathcal V}(\Omega_A^i); x) - P(\iota_{\mathcal V}(\Omega_A^{i+1}); x).$$

In addition, it is clear that

$$P(H_n({}_{\sigma\le n}(\Omega_A^\bullet, \iota_{\mathcal V})); x) = P(\mathrm{Tors}\,(\Omega_A^n); x), \quad P(\iota_{\mathcal V}(\Omega_A^n); x) = x^v P(\Omega_A^n; x) - x^v P(\mathrm{Tors}\,(\Omega_A^n); x),$$
$$P(H_0(\Omega_A^\bullet, \iota_{\mathcal V}); x) = P(A; x) - P(\iota_{\mathcal V}(\Omega_A^1); x).$$

As a result, the n first terms for $i = 0, \dots, n-1$ of the generating function (12) with $y = x^v$ give us the following relations

$$\begin{aligned}
\textstyle\sum_{i=0}^{n-1} (-1)^i x^{iv} P(H_i(\Omega_A^\bullet, \iota_{\mathcal V}); x) \;&=\; \big(P(A; x) - P(\iota_{\mathcal V}(\Omega_A^1); x)\big)\\
&-\; \big(x^v P(\Omega_A^1; x) - P(\iota_{\mathcal V}(\Omega_A^1); x) - x^v P(\iota_{\mathcal V}(\Omega_A^2); x)\big)\\
&+\; \big(x^{2v} P(\Omega_A^2; x) - x^v P(\iota_{\mathcal V}(\Omega_A^2); x) - x^{2v} P(\iota_{\mathcal V}(\Omega_A^3); x)\big) + \dots\\
&=\; \textstyle\sum_{i=0}^{n-1} (-1)^i x^{iv} P(\Omega_A^i; x) + (-1)^n x^{(n-1)v} P(\iota_{\mathcal V}(\Omega_A^n); x)\\
&=\; \textstyle\sum_{i=0}^{n} (-1)^i x^{iv} P(\Omega_A^i; x) + (-1)^{n+1} x^{nv} P(\mathrm{Tors}\,(\Omega_A^n); x).
\end{aligned}$$

Adding then the term of dimension n, one gets

$$G_P({}_{\sigma\le n}(\Omega_A^\bullet, \iota_{\mathcal V}); x, x^v) = \sum_{i=0}^n (-1)^i x^{iv} P(H_i({}_{\sigma\le n}(\Omega_A^\bullet), \iota_{\mathcal V}); x) = \sum_{i=0}^n (-1)^i x^{iv} P(\Omega_A^i; x).$$

Finally, making use of Lemma 3 and elementary transformations, we obtain the following identity

$$G_P(({}_{\sigma\le n}(\Omega_A^\bullet), \iota_{\mathcal V}); x, x^v) = P(A; x) \sum_{p=0}^n (-1)^p x^{pv}\,\mathrm{res}_{t=0} t^{-p-1} \prod (1+tx^{w_i}) / \prod (1+tx^{d_j})$$

which implies the desired formula. QED.

It is possible to represent the obtained expression for the generating function G_P in an explicit form (cf. [2, (3.2)]). Let W_λ be the elementary symmetric polynomials in $y_1, \dots, y_m$ of *weight* $\lambda \geq 0$,

$$\prod_{i=1}^{m}(1 + y_i\zeta) = \sum_{\lambda=0}^{m} W_\lambda(y_1, \dots, y_m)\zeta^\lambda,$$

and let D_λ be the symmetric polynomials in $y_1, \dots, y_k$ of *degree* $\lambda \geq 0$,

$$\prod_{j=1}^{k}(1 + y_i\zeta)^{-1} = \sum_{\lambda=0}^{k}(-1)^\lambda D_\lambda(y_1, \dots, y_k)\zeta^\lambda.$$

Corollary 7. *Under the assumptions of Theorem 1 we have*

$$G_P({}_{\sigma\leq n}(\Omega^\bullet_A, \iota_\mathcal{V}); x, x^v) =$$

$$= \sum_{\lambda_1+\lambda_2+\lambda_3=n} (-1)^{\lambda_2} x^{(n-\lambda_1)v} W_{\lambda_2}(x^{w_1}, \dots, x^{w_m}) D_{\lambda_3}(x^{d_1}, \dots, x^{d_k}) P(A; x).$$

Making use of the expressions from the theorem and corollary above, for the surface germ Q_7 from Example 7.1 one gets immediately

$$G_P(({}_{\sigma\leq 2}(\Omega^\bullet_A), \iota_\mathcal{V}); x, x^v) = \big(1 - x^v(4x - 2x^2) + x^{2v}(6x^2 - 8x^3 + 3x^4)\big)P(A; x).$$

If $v = 1$, then

$$G_P(({}_{\sigma\leq 2}(\Omega^\bullet_A), \iota_\mathcal{V}); x, x) = 1 + 4x + 4x^2 - 2x^3 - 2x^4 + 4x^5 + 3x^6,$$

that is, $\mathrm{Ind}_{\mathrm{hom},\mathfrak{o}}(\mathcal{V}) = \chi({}_{\sigma\leq 2}(\Omega^\bullet_A, \iota_\mathcal{V})) = 9 - 4 + 7 = 12$, as was required.

Comments. 1) The latter formula shows that the homological index is determined completely by the weights of variables, the defining equations and the weight of the vector field; it does not contain the Poincaré polynomials of the module of derivations, the torsion modules, the dualizing module and others except the polynomials of modules of holomorphic differential forms described in Lemma 3.

2) This formula is working correctly without the assumption that the vector field has an isolated singularity on the *ambient* manifold. Indeed, homology groups are finite-dimensional if the *grade* or *depth* of the ideal of A, generated by the coefficients of the vector field, is not less than the dimension of the singularity A. In this case the vector field $\mathcal{V}$ has isolated singularities on the singularity itself.

3) It should be also underlined that in contrast with the formulas for curves and surfaces obtained in Section 6 and Section 7, the homogeneous components of the generating function contain expressions for Poincaré series with shifted *non-canonical* grading since this function is a result of a weighted *convolution* of Poincaré series transcribed in the natural grading.

10. The Lebelt resolutions

Let us now discuss a direct method of computing the homological index in the case of normal complete intersections. This approach is partially based on the construction of a certain subcomplex of the generalized Koszul complex that gives a *free* resolution for exterior powers of a module whose homological dimension does not exceed 1 (see [26]).

Let A be the local analytical algebra of the germ of a reduced singularity of dimension $n \geq 1$ given by an ideal $I = (f_1, \dots, f_k) \subset P = k\langle z_1, \dots, z_m\rangle$, so that $A = P/I$. Then there is a standard exact sequence representing the module of Kähler differentials of k-algebra A as follows

$$0 \longrightarrow \textstyle\int I/I^2 \longrightarrow I/I^2 \xrightarrow{Df} \Omega^1_P \otimes_P A \longrightarrow \Omega^1_A \longrightarrow 0. \tag{13}$$

where $\int I$ is the *primitive ideal* of I, and $Df = \mathrm{Jac}(f)$ is the jacobian matrix associated with the sequence $(f_1, \dots, f_k)$. By definition, the ideal $\int I \subset P$ consists of all $g \in I$, such that $\partial(g) \in I$ for all $\partial \in \mathrm{Der}_k(P)$ (see [31]).

It is well-known that for *complete intersection* germs one has $n = m - k$ and I/I^2 is a *free* A-module of rank k. Moreover, in the case of *reduced* complete intersections we have $\int I/I^2 = 0$ (see [31, §4, ex.(1)]). Hence, the homological dimension of the A-module Ω^1_A is not greater than one. Suppose additionally that A is an *isolated* singularity. According to [26, Folgerung 1(a)], for all A-modules $\Omega^p_A = \wedge^p \Omega^1_A$, $p = 0, 1, \dots, n-1$, there are *free* resolutions

$$\mathcal{L}^k_p \colon 0 \longrightarrow \mathcal{L}^k_{p,p} \longrightarrow \dots \longrightarrow \mathcal{L}^k_{p,r} \longrightarrow \dots \longrightarrow \mathcal{L}^k_{p,1} \longrightarrow \mathcal{L}^k_{p,0} \xrightarrow{\varepsilon} \Omega^p_A \longrightarrow 0, \tag{14}$$

where

$$\mathcal{L}^k_{p,r} = S^r L_0 \otimes \wedge^{p-r} L_1, \quad r = 0, \dots, p, \quad L_0 = I/I^2, \quad L_1 = \Omega^1_P \otimes_P A,$$

and $\varepsilon = \varepsilon^k_{p,0}$ is the quotient map of $\mathcal{L}^k_{p,0}$ to $\mathrm{Coker}\,(\mathcal{L}^k_{p,1} \to \mathcal{L}^k_{p,0}) \cong \Omega^p_A$ for all $p = 0, \dots, n$. It is not difficult to see that

$$\mathcal{L}^k_{p,r} \cong \left(\wedge^{p-r} A^m \right)^{\nu(k,r)}, \quad \text{for all} \quad 0 \le r \le p,$$

where $\nu(k, r) = \binom{k-1+r}{r}$ is the number of homogeneous monomials of degree r in k variables.

EXAMPLE 10.1. As an illustration let us apply this construction to the case of a *quasihomogeneous* isolated complete intersection singularity with type of homogeneity $(d_1, \dots, d_k; w_1, \dots, w_m)$. Then two free A-modules $L_0 \cong \prod A(-d_j)$ and $L_1 \cong \prod A(-w_i)$ of rank k and m, respectively, are endowed with the natural grading. In this grading the resolution $\mathcal{L}^k_p$ is an exact sequence of graded modules connecting by maps whose weights are equal to *zero* (cf. [2, (3.3)]). This property follows from the explicit expressions for differentials $\mathcal{D}^k_{p,r}$ of the Lebelt resolution (see [26]). Next, making use of sequence (14) and elementary transformations, one can deduce the formula of Lemma 3 (and vice versa).

Proposition 11. *Let A be the local analytical algebra of the germ of a normal isolated complete intersection singularity, $\mathcal{V} \in \mathrm{Der}_k(A)$ and $\iota_{\mathcal{V}} \colon \mathcal{L}^k_{p,r} \to \mathcal{L}^k_{p-1,r}$ a family of contraction maps for all admissible p and r. Then a bicomplex $\mathcal{L}^k_{\bullet\bullet}$ is well-defined and there exist natural isomorphisms of the homology groups*

$$H_i(\Omega^\bullet_A, \iota_{\mathcal{V}}) \cong H_i(\mathrm{Tot}(\mathcal{L}^k_{\bullet\bullet})), \quad i = 0, 1, \dots, n-1,$$

where $\mathrm{Tot}(\mathcal{L}^k_{\bullet\bullet})$ is the total complex associated with the bicomplex.

PROOF. It is not difficult to verify that all the differentials of the Lebelt resolutions commute with the contraction map. Hence, $\mathcal{L}^k_{\bullet\bullet}$ is a bicomplex of A-modules. All horizontal homologies of this bicomplex are trivial since its rows are given by the Lebelt resolutions of A-modules Ω^p_A, $p = 0, 1, \dots, n-1$. The desired statement is a direct consequence of basic properties and standard relations between homology and hyperhomology functors. QED.

Claim 6. *If the vector field $\mathcal{V}$ has an isolated singularity then*

$$\dim_k H_n(\sigma_{\le n}(\Omega^\bullet_A), \iota_{\mathcal{V}}) = \dim_k \mathrm{Tors}\,\Omega^n_A = \dim_k {}^{\#}\omega^n_A = \dim_k \mathrm{Ext}^n_A(\Omega^n_A, A) = \dim_k A/F_0(Df),$$

where $F_0(Df)$ is the zeroth Fitting ideal generated by the maximal minors of the jacobian matrix Df.

PROOF. The homology group $H_n(\sigma_{\le n}(\Omega^\bullet_A), \iota_{\mathcal{V}})$ and $\mathrm{Tors}\,\Omega^n_A$ are naturally isomorphic in view of Proposition 10. Further, for isolated complete intersection singularities the dimensions of torsion and cotorsion modules are equal (see [11]). On the other hand, the dimension of $\mathrm{Tors}\,\Omega^n_A$ is equal to $\dim_k \mathrm{Ext}^n_A(\Omega^n_A, A)$ (the latter is computed explicitly in the quasihomogeneous case in [2, (3.2)]), while the dimension of the cotorsion module ${}^{\#}\omega^n_A \cong H^1_{\mathfrak{m}}(\Omega^n_A)$ is nothing but the dimension of $A/F_0(Df)$. QED.

REMARK. From Claim 6 it follows that for an isolated singularity the dimensions of the *highest* homology groups of the complex $({}_{\sigma\le n}(\Omega^{\bullet}_A), \iota_{\mathcal{V}})$ and the torsion module $\operatorname{Tors}\Omega^n_A$ are equal; in the *quasihomogeneous* case this dimension coincides with the Milnor number. It is known at least n (generally speaking, *different*) expressions for Poincaré polynomials of the torsion module and, consequently, for the Milnor number of the singularity (see [1], [2]).

EXAMPLE 10.2. Let us show how one can compute directly homology groups with the use of the bicomplex $\mathcal{L}^k_{\bullet\bullet}$ in the 2-dimensional case. This bicomplex looks like this:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & A^{\binom{k+1}{2}} & \xrightarrow{\mathcal{D}^k_{2,1}} & (A^m)^k & \xrightarrow{\mathcal{D}^k_{2,1}} & \wedge^2 A^m & \xrightarrow{\varepsilon^k_{2,0}} & \Omega^2_A & \longrightarrow & 0 \\
 & & & & \downarrow\iota_{\mathcal{V}} & & \downarrow\iota_{\mathcal{V}} & & \downarrow\iota_{\mathcal{V}} & & \\
 & & 0 & \longrightarrow & A^k & \xrightarrow{\mathcal{D}^k_{1,1}} & A^m & \xrightarrow{\varepsilon^k_{1,0}} & \Omega^1_A & \longrightarrow & 0 \\
 & & & & & & \downarrow\iota_{\mathcal{V}} & & \downarrow\iota_{\mathcal{V}} & & \\
 & & & & 0 & \longrightarrow & A & \xrightarrow{\varepsilon^k_{0,0}} & \Omega^0_A \cong A & \longrightarrow & 0.
\end{array} \tag{15}$$

By the definition above, $\varepsilon^k_{p,0}$ are natural quotient maps $\mathcal{L}^k_{p,0} \longrightarrow \operatorname{Coker}(\mathcal{D}^k_{p,1}) \cong \Omega^p_A$ for all $p = 0, 1, 2$. Next, the middle row is, in fact, equivalent to the truncated exact sequence (13) so that the differential $\mathcal{D}^k_{1,1}$ is given by the jacobian $(k \times m)$-matrix of the defining ideal, while $\mathcal{D}^k_{2,1}(y_1, \ldots, y_k) = \operatorname{jac}(f_1) \wedge y_1 + \ldots + \operatorname{jac}(f_k) \wedge y_k$, and so on (see [26, (2.12), (2.15)]).

In view of Proposition 11 the homology groups of the complex $\Omega^{\bullet}_A$ of dimensions $i = 0, 1$ can be computed as the corresponding homology groups of the total complex $\operatorname{Tot}(\mathcal{L}^k_{\bullet\bullet})$. As a result one has

$$H_0(\Omega^{\bullet}_A) \cong A/\iota_{\mathcal{V}}(A^m), \quad H_1(\Omega^{\bullet}_A) \cong \operatorname{Ker}(\iota_{\mathcal{V}}\colon A^m \to A)/\big(\iota_{\mathcal{V}}(\wedge^2 A^m) + \mathcal{D}^k_{1,1}(A^k)\big).$$

REMARK. In this case all homology groups are *finite-dimensional*, and a simple script for calculations of the index can be readily implemented in any computer system of algebraic calculations similarly to [10], (4.4). It should be underlined that by contrast with the algorithm of [10] one needs expressions for the defining ideal of the singularity and for the coefficients of a vector field only. Explicit (highly non-trivial) expressions for the entries of the structure matrix $C = \| c_{ij} \|$ realizing the *tangency* relation $\mathcal{V}(f) = C(f)$ are no longer required since all calculations are carried out modulo the defining ideal in the local analytical algebra A.

EXAMPLE 10.3. In the case $n \ge 2$ the construction of the Lebelt resolutions implies that all the terms under the polygonal step-line of the lower left corner of the diagram (15) are zeros. Hence, one can described the homology groups of the next dimensions (until $n-1$ inclusively) analogously to the case considered in the Example 10.2, that is,

$$H_2(\Omega^{\bullet}_A) \cong \operatorname{Ker}(\iota_{\mathcal{V}}\colon \wedge^2 A^m \to A^m)/\big(\iota_{\mathcal{V}}(\wedge^3 A^m) + \mathcal{D}^k_{2,1}((A^m)^k)\big).$$

The numerator in the presentation of $H_3(\Omega^{\bullet}_A)$ can be written down as follows

$$\operatorname{Ker}(\iota_{\mathcal{V}}\colon \wedge^3 A^m \to \wedge^2 A^m) + \operatorname{Ker}(\iota_{\mathcal{V}}\colon (A^m)^k \to A^k),$$

while the corresponding denominator is equal to

$$\iota_{\mathcal{V}}(\wedge^4 A^m) + \iota_{\mathcal{V}}((\wedge^2 A^m)^k) + \mathcal{D}^k_{3,1}((\wedge^2 A^m)^k),$$

and so on. As a result, for an *odd* integer $1 \le \ell < n$ the numerator in the presentation of $H_\ell(\Omega^{\bullet}_A)$ is written down in the following way

$$\operatorname{Ker}(\iota_{\mathcal{V}}\colon \wedge^\ell A^m \to A^m) + \operatorname{Ker}(\iota_{\mathcal{V}}\colon \wedge^{\ell-2} (A^m)^k \to (A^m)^k) + \ldots,$$

while the corresponding denominator is equal to

$$\iota_{\mathcal{V}}(\wedge^{\ell+1}A^m)+\iota_{\mathcal{V}}(\wedge^{\ell-1}(A^m)^k)+\ldots+\mathcal{D}^k_{\ell,1}((\wedge^{\ell-1}A^m)^k).$$

Similar presentations also exist for all *even* $0\le\ell<n$.

REMARK. The above example can also be investigated with the use of considerations in the context of the theory of the *generalized* Koszul complex because the length of its homology groups of dimension $0\le i\le n-2$ can be computed explicitly (see a detail review in [21]). To be more precise, these groups are expressed in terms of the symmetric polynomial algebra over the quotient $A/J_{\mathfrak{o}}\mathcal{V}$ similarly to [10]. The non-trivial homology groups of dimension $i=n-1$ are computed analogously to Example 10.2

REMARK. This method shows also that a *necessary* condition under which the homology groups for $i=0,1,\ldots,n-1$ are finite-dimensional is the following: the *depth* of the ideal, generated by the coefficients of the vector field and defining equations, is not less than the dimension of the ambient manifold. In particular, for a Cohen-Macaulay singularity this means that the *grade* of the ideal $J_{\mathfrak{o}}\mathcal{V}$ is equal to the dimension of the singularity (cf. Comment 2 in Section 9).

11. Holomorphic and meromorphic indices

In conclusion we discuss some useful properties and relations between the indexes of complexes of holomorphic and regular meromorphic differential forms. Recall that for isolated complete intersection singularities there is an equality $\chi(\widetilde{\Omega}^\bullet_A,\iota_{\mathcal{V}})=\chi(\omega^\bullet_A,\iota_{\mathcal{V}})$ (see Section 4).

Theorem 2. *For a normal isolated complete intersection singularity of dimension $n\ge 2$ there exist natural isomorphisms*

$$H_i(\Omega^\bullet_A,\iota_{\mathcal{V}})\cong H_i(\omega^\bullet_A,\iota_{\mathcal{V}}),\quad 0\le i\le n-2,$$

and the following exact sequence

$$0\to H_n({}^{\#}\omega^\bullet_A)\to H_{n-1}(\Omega^\bullet_A,\iota_{\mathcal{V}})\to H_{n-1}(\omega^\bullet_A,\iota_{\mathcal{V}})\to H_{n-1}({}^{\#}\omega^\bullet_A)\to 0$$

which induces an equality $\dim_k H_{n-1}(\Omega^\bullet_A,\iota_{\mathcal{V}})=\dim_k H_{n-1}(\omega^\bullet_A,\iota_{\mathcal{V}})$.

PROOF. By Proposition 2 one has $\omega^p_A\cong\Omega^{p\,\vee\vee}_A\cong\Omega^p_A$ for $0\le p<n$. This gives us the required isomorphisms for all $0\le i<n-2$.

On the other hand, in view of Proposition 8 one has $H_n(\widetilde{\Omega}^\bullet_A,\iota_{\mathcal{V}})=H_n(\omega^\bullet_A,\iota_{\mathcal{V}})=0$. Hence the exact sequence of complexes

$$0\longrightarrow\widetilde{\Omega}^\bullet_A\longrightarrow\omega^\bullet_A\longrightarrow{}^{\#}\omega^\bullet_A\longrightarrow 0$$

induces the long exact sequence of higher-dimensional homology groups

$$\begin{aligned}0\to H_n({}^{\#}\omega^\bullet_A)\to H_{n-1}(\Omega^\bullet_A,\iota_{\mathcal{V}})\to H_{n-1}(\omega^\bullet_A,\iota_{\mathcal{V}})\to&\\ \to H_{n-1}({}^{\#}\omega^\bullet_A)\to H_{n-2}(\Omega^\bullet_A,\iota_{\mathcal{V}})\to H_{n-2}(\omega^\bullet_A,\iota_{\mathcal{V}})\to 0.&\end{aligned}\tag{16}$$

Since the dimensions of the two non-trivial cotorsion modules are equal, this implies an equality

$$\dim H_{n-2}(\Omega^\bullet_A,\iota_{\mathcal{V}})-\dim H_{n-1}(\Omega^\bullet_A,\iota_{\mathcal{V}})=\dim H_{n-2}(\omega^\bullet_A,\iota_{\mathcal{V}})-\dim H_{n-1}(\omega^\bullet_A,\iota_{\mathcal{V}}).\tag{17}$$

In addition, we get an inequality $\dim_k H_{n-2}(\Omega^\bullet_A,\iota_{\mathcal{V}})\ge\dim_k H_{n-2}(\omega^\bullet_A,\iota_{\mathcal{V}})$ for evident reasons.

Similarly, the long Ker-Coker sequence, associated with the commutative diagram

$$\begin{array}{ccccccccc}0&\longrightarrow&\widetilde{\Omega}^n_A&\xrightarrow{c_A}&\omega^n_A&\longrightarrow&{}^{\#}\omega^n_A&\longrightarrow&0\\ &&\downarrow\iota_{\mathcal{V}}&&\downarrow\iota_{\mathcal{V}}&&\downarrow&&\\ 0&\longrightarrow&\mathrm{Ker}(\Omega^{n-1}_A\to\Omega^{n-2})&\longrightarrow&\mathrm{Ker}(\omega^{n-1}_A\to\omega^{n-2})&\longrightarrow&{}^{\#}\omega^{n-1}_A,&&\end{array}$$

gives us an exact sequence of finite-dimensional vector spaces

$$0 \longrightarrow H_n({}^{\#}\omega_A^{\bullet}) \longrightarrow H_{n-1}(\Omega_A^{\bullet}, \iota_{\mathcal{V}}) \longrightarrow H_{n-1}(\omega_A^{\bullet}, \iota_{\mathcal{V}}) \longrightarrow H_{n-1}({}^{\#}\omega_A^{\bullet}).$$

Again, since the dimensions of both cotorsion modules are equal, one obtains the inequality $\dim_k H_{n-1}(\Omega_A^{\bullet}, \iota_{\mathcal{V}}) \geq \dim_k H_{n-1}(\omega_A^{\bullet}, \iota_{\mathcal{V}})$.

At last, combining the above two inequalities with relation (17), one gets

$$\dim_k H_{n-2}(\Omega_A^{\bullet}, \iota_{\mathcal{V}}) = \dim_k H_{n-2}(\omega_A^{\bullet}, \iota_{\mathcal{V}}), \quad \dim_k H_{n-1}(\Omega_A^{\bullet}, \iota_{\mathcal{V}}) = \dim_k H_{n-1}(\omega_A^{\bullet}, \iota_{\mathcal{V}}).$$

In view of (16) the first equality implies an isomorphism $H_{n-2}(\Omega_A^{\bullet}, \iota_{\mathcal{V}}) \cong H_{n-2}(\omega_A^{\bullet}, \iota_{\mathcal{V}})$, and the exact sequence of the Theorem. QED.

Remark. For isolated complete intersection singularities we have already proved earlier that $H_n(\Omega_A^{\bullet}, \iota_{\mathcal{V}}) \cong \operatorname{Tors}\Omega_A^{\bullet}$, and $H_n(\widetilde{\Omega}_A^{\bullet}, \iota_{\mathcal{V}}) = H_n(\omega_A^{\bullet}, \iota_{\mathcal{V}}) = 0$. Further, for a vector field $\mathcal{V}$ of weight 1 on a surface Q_7-singularity computations in the above examples give us two Poincaré polynomials for first homology groups:

$$P(H_1(\Omega_A^{\bullet}, \iota_{\mathcal{V}}); x) = 2x^2 + 2x^3, \quad P(H_1(\omega_A^{\bullet}, \iota_{\mathcal{V}}); x) = 1 + 3x.$$

Hence, $H_1(\Omega_A^{\bullet}, \iota_{\mathcal{V}}) \ncong H_1(\omega_A^{\bullet}, \iota_{\mathcal{V}})$, although the dimensions of both homology groups are equal.

Remark. For a normal reduced complete intersection with *non-isolated singularities* the Lebelt resolutions exist for all Ω_A^p, $0 \leq p \leq c-1$, where c is the codimension of the singular subspace. Moreover, there are natural isomorphisms:

$$H_i(\Omega_A^{\bullet}, \iota_{\mathcal{V}}) \cong H_i(\omega_A^{\bullet}, \iota_{\mathcal{V}}), \quad 0 \leq i < c-1.$$

Claim 7. *Let D be the germ of a reduced normal hypersurface and $\mathcal{V}$ a vector field with isolated singularities. Then one has*

$$\chi(\Omega_D^{\bullet}, \iota_{\mathcal{V}}) = \chi(\omega_D^{\bullet}, \iota_{\mathcal{V}}).$$

Proof. If D is the germ of a hypersurface with an *isolated* singularity of dimension $n \geq 1$, then

$$H^{n-1}_{\{\mathfrak{o}\}}(\Omega_D^1) \cong H^0_{\{\mathfrak{o}\}}(\Omega_D^n) \cong \operatorname{Tors}\Omega_D^n, \quad \Omega_D^{n+1} \cong \operatorname{Tors}\Omega_D^{n+1} \cong H^1_{\{\mathfrak{o}\}}(\Omega_D^n)$$

(see [27, Pt. I]) and these modules are, in fact, τ-dimensional vector spaces, where τ is the *Tjurina* number of D. In the case of hypersurfaces with *non-isolated* singularities the statement follows from considerations in [6], where the logarithmic index is introduced and some relations between this index and the index of the contracted complex of regular meromorphic differential forms via the residue map are discussed. QED.

Remark. In fact, the logarithmic index is computed in the ambient space of a singularity in contrast with the regular meromorphic index and the residue map connects both realizations. In a more general context this idea leads to the study of *multi-logarithmic* differential forms [7], their residues and properties of the multi-logarithmic index.

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