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MILNOR FIBERS OF REAL LINE ARRANGEMENTS

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Abstract. We study Milnor fibers of complexified real line arrangements. We give a new algorithm computing monodromy eigenspaces of the first cohomology. The algorithm is based on the description of minimal CW-complexes homotopic to the complements, and uses the real figure, that is, the adjacency relations of chambers. It enables us to generalize a vanishing result of Libgober, give new upper-bounds and characterize the A_3-arrangement in terms of non-triviality of Milnor monodromy.

1. Introduction

The Milnor fiber is a central object in the study of the topology of complex hypersurface singularities. In particular, the monodromy action on its cohomology groups has been intensively studied. Monodromy eigenspaces contain subtle geometric information. For example, for projective plane curves, the Betti numbers of Milnor fiber of the cone detect Zariski pairs [1]. In other words, Betti numbers of Milnor fiber of the cone of a plane curve are not in general determined by local and combinatorial data of singularities.

In the theory of hyperplane arrangements, one of the central problems is to what extent topological invariants of the complements are determined combinatorially. For example, the cohomology ring is combinatorially determined (Orlik and Solomon [16]), while the fundamental group is not (Rybnikov [1, 11]). Between these two cases, local system cohomology groups and monodromy eigenspaces of Milnor fibers recently received a considerable amount of attention.

There are several ways to compute monodromy eigenspaces of the Milnor fiber, especially for line arrangements. One is the topological method developed by Cohen and Suciu [4]. They first give a presentation of the fundamental group of the complement. Then, using Fox calculus, they compute the monodromy eigenspaces. Another approach is the algebraic method, which computes the multiplicities of monodromy eigenvalues as the superabundance of singular points. This approach has recently been well developed, especially for line arrangements having only double and triple points [14].

The purpose of this paper is to develop a topological method of computing Milnor monodromy for complexified real arrangements following Cohen and Suciu. The new ingredient is a recent study of minimal cell structures for the complements of complexified real arrangements [19, 22]. By using the description of twisted minimal chain complexes, we obtain an algorithm which computes monodromy eigenspaces directly from real figures without passing through the presentations of π_1.

The paper is organized as follows. In §2 we recall a few results which are used in this paper. §3 is the main section of the paper. First, in §3.1, we introduce discrete geometric notions, the so-called *k-resonant band* and the *standing wave* on this band. These notions are used in §3.2 for

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the computation of eigenspaces. Several consequences of our algorithm are discussed in §3.3, §3.4 and §3.5. Among other things, we prove that if the arrangement contains more than 6 lines and the cohomological monodromy action (of degree one) is non-trivial, then each line has at least three multiple points (see Corollary 3.24 for a precise statement). Such arrangements have been studied in discrete geometry as "configurations", and several examples are provided in [8, 9]. In §4, we apply our algorithm to arrangements appearing in papers by Grünbaum [8, 9]. We also present several examples and conjectures.

2. Preliminaries

2.1. **Milnor fiber of arrangements.** Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be an affine line arrangement in $\mathbb{R}^2$ with the defining equation $Q_{\mathcal{A}}(x, y) = \prod_{i=1}^n \alpha_i$, where α_i is a defining linear equation for H_i. In this paper, we assume that not all lines are parallel (or equivalently, $\mathcal{A}$ has at least one intersection). The coning $c\mathcal{A}$ of $\mathcal{A}$ is an arrangement of $n+1$ planes in $\mathbb{R}^3$ defined by the equation $Q_{c\mathcal{A}}(x, y, z) = z^{n+1}Q(\frac{x}{z}, \frac{y}{z})$. The line $\{z = 0\} \in c\mathcal{A}$ is called the line at infinity and is denoted by H_∞. The space $\mathsf{M}(\mathcal{A}) = \mathbb{C}^2 \setminus \{Q_{\mathcal{A}} = 0\} = \mathbb{P}^2_{\mathbb{C}} \setminus \{Q_{c\mathcal{A}} = 0\}$ is called the complexified complement. In this article, $\mathcal{A}$ always denotes a line arrangement in $\mathbb{R}^2$ and $c\mathcal{A}$ denotes a line arrangement in $\mathbb{RP}^2$. We call $p \in \mathbb{RP}^2$ a *multiple point* if the multiplicity of $c\mathcal{A}$ at p (that is, the number of lines passing through p) is greater than or equal to 3.

Definition 2.1. $F_{\mathcal{A}} = \{(x, y, z) \in \mathbb{C}^3 \mid Q_{c\mathcal{A}}(x, y, z) = 1)\}$ is called the Milnor fiber of $\mathcal{A}$. The automorphism $\rho : F_{\mathcal{A}} \longrightarrow F_{\mathcal{A}},\ (x, y, z) \longmapsto (\zeta x, \zeta y, \zeta z)$, with $\zeta = \exp(2\pi i/(n+1))$, is called the monodromy action.

The automorphism ρ has order $n+1$. It generates the cyclic group $\langle\rho\rangle \simeq \mathbb{Z}/(n+1)\mathbb{Z}$. The monodromy ρ induces a linear map $\rho^* : H^1(F_{\mathcal{A}}, \mathbb{C}) \longrightarrow H^1(F_{\mathcal{A}}, \mathbb{C})$. Since $(\rho^*)^{n+1}$ is the identity, we have the eigenspace decomposition $H^1(F_{\mathcal{A}}, \mathbb{C}) = \bigoplus_{\lambda^{n+1}=1} H^1(F_{\mathcal{A}}, \mathbb{C})_\lambda$, where $H^1(F_{\mathcal{A}}, \mathbb{C})_\lambda$ is the the set of λ-eigenvectors with eigenvalue $\lambda \in \mathbb{C}^*$. When $\lambda = 1$, $H^1(F_{\mathcal{A}})_1 = H^1(F_{\mathcal{A}})^{\rho^*}$ is the subspace of elements fixed by ρ^*, which is isomorphic to $H^1(F_{\mathcal{A}}/\langle\rho\rangle)$. It is easily seen that the quotient by the monodromy action is $F_{\mathcal{A}}/\langle\rho\rangle \simeq \mathsf{M}(\mathcal{A})$. Therefore, the 1-eigenspace of the first cohomology is combinatorially determined, $H^1(F_{\mathcal{A}})_1 \simeq H^1(\mathsf{M}(\mathcal{A})) \simeq \mathbb{C}^n$. In general, let $\mathcal{L}_\lambda$ be a complex rank one local system associated with a representation

$$\pi_1(\mathsf{M}(\mathcal{A})) \longrightarrow \mathbb{C}^*,\ \gamma_H \longmapsto \lambda,$$

where γ_H is a meridian loop of the line H. Then it is known that

$$H^1(F_{\mathcal{A}})_\lambda \simeq H^1(\mathsf{M}(\mathcal{A}), \mathcal{L}_\lambda). \tag{1}$$

(See [4] for details.)

2.2. **Multinets and Milnor monodromy.** In this section, we recall a relation between the combinatorial structures known as multinets and the eigenvalues of Milnor monodromy. We note that a k-multinet gives a lower bound on the eigenspace.

Definition 2.2. A *k-multinet* on $c\mathcal{A}$ is a pair $(\mathcal{N}, \mathcal{X})$, where $\mathcal{N}$ is a partition of $c\mathcal{A}$ into $k \geq 3$ classes $\mathcal{A}_1, \dots, \mathcal{A}_k$ and $\mathcal{X}$ is a set of multiple points such that

- (i) $|\mathcal{A}_1| = \cdots = |\mathcal{A}_k|$;
- (ii) $H \in \mathcal{A}_i$ and $H' \in \mathcal{A}_j$ $(i \neq j)$ imply that $H \cap H' \in \mathcal{X}$;
- (iii) for all $p \in \mathcal{X}$, $|\{H \in \mathcal{A}_i \mid H \ni p\}|$ is constant and independent of i;
- (iv) for any $H, H' \in \mathcal{A}_i$ $(i = 1, \dots, k)$, there is a sequence $H = H_0, H_1, \dots, H_r = H'$ in $\mathcal{A}_i$ such that $H_{j-1} \cap H_j \notin \mathcal{X}$ for $1 \leq j \leq r$.

The following is a consequence of [7, Theorem 3.11] and [6, Theorem 3.1 (i)]

Theorem 2.3. *Suppose there exists a k-multinet on $c\mathcal{A}$ for some $k \geq 3$ and set $\lambda = e^{2\pi i/k}$. Then*

$$\dim H^1(F_{\mathcal{A}})_\lambda \geq k - 2.$$

2.3. **Twisted minimal cochain complexes.** In this section, we recall the construction of the twisted minimal cochain complex from [19, 20, 21], which will be used for the computation of the right hand side of (1).

A connected component of $\mathbb{R}^2 \setminus \bigcup_{H\in\mathcal{A}} H$ is called a chamber. The set of all chambers is denoted by $\mathsf{ch}(\mathcal{A})$. A chamber $C \in \mathsf{ch}(\mathcal{A})$ is called bounded (resp. unbounded) if the area is finite (resp. infinite). For an unbounded chamber $U \in \mathsf{ch}(\mathcal{A})$, the opposite unbounded chamber is denoted by $U^\vee$ (see [21, Definition 2.1] for the definition; see also Figure 1 below).

Let $\mathcal{F}$ be a generic flag in $\mathbb{R}^2$

$$\mathcal{F} : \emptyset = \mathcal{F}^{-1} \subset \mathcal{F}^0 \subset \mathcal{F}^1 \subset \mathcal{F}^2 = \mathbb{R}^2,$$

where $\mathcal{F}^k$ is a generic k-dimensional affine subspace.

Definition 2.4. For $k = 0, 1, 2$, define the subset $\mathsf{ch}^k_{\mathcal{F}}(\mathcal{A}) \subset \mathsf{ch}(\mathcal{A})$ by

$$\mathsf{ch}^k_{\mathcal{F}}(\mathcal{A}) := \{C \in \mathsf{ch}(\mathcal{A}) \mid C \cap \mathcal{F}^k \neq \emptyset, C \cap \mathcal{F}^{k-1} = \emptyset\}.$$

The set of chambers decomposes into a disjoint union as

$$\mathsf{ch}(\mathcal{A}) = \mathsf{ch}^0_{\mathcal{F}}(\mathcal{A}) \sqcup \mathsf{ch}^1_{\mathcal{F}}(\mathcal{A}) \sqcup \mathsf{ch}^2_{\mathcal{F}}(\mathcal{A}).$$

The cardinality of $\mathsf{ch}^k_{\mathcal{F}}(\mathcal{A})$ is equal to $b_k(\mathsf{M}(\mathcal{A}))$ for $k = 0, 1, 2$.

We further assume that the generic flag $\mathcal{F}$ satisfies the following conditions:

- $\mathcal{F}^1$ does not separate intersections of $\mathcal{A}$,
- $\mathcal{F}^0$ does not separate n-points $\mathcal{A} \cap \mathcal{F}^1$.

Then we can choose coordinates x_1, x_2 so that $\mathcal{F}^0$ is the origin $(0, 0)$, $\mathcal{F}^1$ is given by $x_2 = 0$, all intersections of $\mathcal{A}$ are contained in the upper-half plane $\{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 > 0\}$ and $\mathcal{A} \cap \mathcal{F}^1$ is contained in the half-line $\{(x_1, 0) \mid x_1 > 0\}$.

We set $H_i \cap \mathcal{F}^1$ to have coordinates $(a_i, 0)$. By changing the numbering of lines and the signs of the defining equation α_i of $H_i \in \mathcal{A}$ we may assume that

- $0 < a_1 < a_2 < \cdots < a_n$,
- the origin $\mathcal{F}^0$ is contained in the negative half-plane $H_i^- = \{\alpha_i < 0\}$.

We set $\mathsf{ch}^{\mathcal{F}}_0(\mathcal{A}) = \{U_0\}$ and $\mathsf{ch}^{\mathcal{F}}_1(\mathcal{A}) = \{U_1, \ldots, U_{n-1}, U_0^\vee\}$ so that $U_p \cap \mathcal{F}^1$ is equal to the interval (a_p, a_{p+1}) for $p = 1, \ldots, n-1$. It is easily seen that the chambers $U_0, U_1, \ldots, U_{n-1}$ and $U_0^\vee$ have the following expression:

$$\begin{aligned} U_0 &= \bigcap_{i=1}^{n} \{\alpha_i < 0\}, \\ U_p &= \bigcap_{i=1}^{p} \{\alpha_i > 0\} \cap \bigcap_{i=p+1}^{n} \{\alpha_i < 0\}, \ (p = 1, \ldots, n-1), \\ U_0^\vee &= \bigcap_{i=1}^{n} \{\alpha_i > 0\}. \end{aligned} \tag{2}$$

The notations introduced to this point are illustrated in Figure 1.

Let $\mathcal{L}$ be a complex rank-one local system on $\mathsf{M}(\mathcal{A})$. The local system $\mathcal{L}$ is determined by non-zero complex numbers (monodromy around H_i) $q_i \in \mathbb{C}^*$, $i = 1, \ldots, n$. Fix a square root $q_i^{1/2} \in \mathbb{C}^*$ for each i.

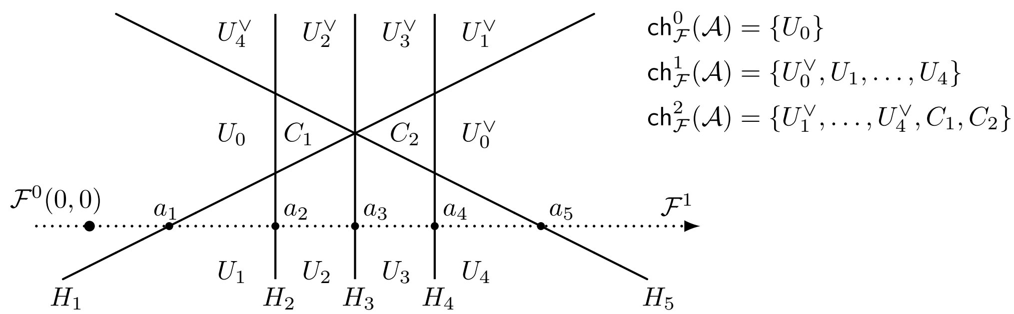

FIGURE 1. Numbering of lines and chambers.

Definition 2.5. (1) For $C, C' \in \mathsf{ch}(\mathcal{A})$, let us denote by $\mathrm{Sep}(C, C')$ the set of lines $H_i \in \mathcal{A}$ which separate C and C'.
(2) Define the complex number $\Delta(C, C') \in \mathbb{C}$ by

$$\Delta(C, C') := \prod_{H_i \in \mathrm{Sep}(C,C')} q_i^{1/2} - \prod_{H_i \in \mathrm{Sep}(C,C')} q_i^{-1/2}.$$

Now we construct the cochain complex $(\mathbb{C}[\mathsf{ch}^\bullet_{\mathcal{F}}(\mathcal{A})], d_{\mathcal{L}})$.

(i) The map $d_{\mathcal{L}} : \mathbb{C}[\mathsf{ch}^0_{\mathcal{F}}(\mathcal{A})] \longrightarrow \mathbb{C}[\mathsf{ch}^1_{\mathcal{F}}(\mathcal{A})]$ is defined by

$$d_{\mathcal{L}}([U_0]) = \Delta(U_0, U_0^\vee)[U_0^\vee] + \sum_{p=1}^{n-1} \Delta(U_0, U_p)[U_p].$$

(ii) $d_{\mathcal{L}} : \mathbb{C}[\mathsf{ch}^1_{\mathcal{F}}(\mathcal{A})] \longrightarrow \mathbb{C}[\mathsf{ch}^2_{\mathcal{F}}(\mathcal{A})]$ is defined by

$$d_{\mathcal{L}}([U_p]) = -\sum_{\substack{C \in \mathsf{ch}^2_{\mathcal{F}}(\mathcal{A}) \\ \alpha_p(C) > 0 \\ \alpha_{p+1}(C) < 0}} \Delta(U_p, C)[C] + \sum_{\substack{C \in \mathsf{ch}^2_{\mathcal{F}}(\mathcal{A}) \\ \alpha_p(C) < 0 \\ \alpha_{p+1}(C) > 0}} \Delta(U_p, C)[C], \text{ (for } p = 1, \dots, n-1),$$

$$d_{\mathcal{L}}([U_0^\vee]) = -\sum_{\alpha_n(C) > 0} \Delta(U_0^\vee, C)[C].$$

Example 2.6. Let $\mathcal{A} = \{H_1, \dots, H_5\}$, and let the flag $\mathcal{F}$ be as in Figure 1. Then

$$d_{\mathcal{L}}([U_0]) = ([U_1], [U_2], [U_3], [U_4], [U_0^\vee]) \begin{pmatrix} q_1^{1/2} - q_1^{-1/2} \\ q_{12}^{1/2} - q_{12}^{-1/2} \\ q_{123}^{1/2} - q_{123}^{-1/2} \\ q_{1234}^{1/2} - q_{1234}^{-1/2} \\ q_{12345}^{1/2} - q_{12345}^{-1/2} \end{pmatrix},$$

$$d_{\mathcal{L}}([U_1], [U_2], [U_3], [U_4], [U_0^\vee]) = ([U_1^\vee], [U_2^\vee], [U_3^\vee], [U_4^\vee], [C_1], [C_2])$$

$$\times \begin{pmatrix} q_{12345}^{1/2} - q_{12345}^{-1/2} & 0 & 0 & 0 & -(q_1^{1/2} - q_1^{-1/2}) \\ q_{125}^{1/2} - q_{125}^{-1/2} & -(q_{15}^{1/2} - q_{15}^{-1/2}) & 0 & q_{1345}^{1/2} - q_{1345}^{-1/2} & -(q_{134}^{1/2} - q_{134}^{-1/2}) \\ q_{1235}^{1/2} - q_{1235}^{-1/2} & 0 & -(q_{15}^{1/2} - q_{15}^{-1/2}) & q_{145}^{1/2} - q_{145}^{-1/2} & -(q_{14}^{1/2} - q_{14}^{-1/2}) \\ 0 & 0 & 0 & q_{12345}^{1/2} - q_{12345}^{-1/2} & -(q_{1234}^{1/2} - q_{1234}^{-1/2}) \\ q_{12}^{1/2} - q_{12}^{-1/2} & -(q_1^{1/2} - q_1^{-1/2}) & 0 & 0 & 0 \\ 0 & 0 & -(q_5^{1/2} - q_5^{-1/2}) & q_{45}^{1/2} - q_{45}^{-1/2} & -(q_4^{1/2} - q_4^{-1/2}) \end{pmatrix}.$$

Theorem 2.7. *Under the above notation,* $(\mathbb{C}[\mathsf{ch}^{\bullet}_{\mathcal{F}}(\mathcal{A})], d_{\mathcal{L}})$ *is a cochain complex and*

$$H^k(\mathbb{C}[\mathsf{ch}^{\bullet}_{\mathcal{F}}(\mathcal{A})], d_{\mathcal{L}}) \simeq H^k(M(\mathcal{A}), \mathcal{L}).$$

See [19, 20, 21] for details.

3. Resonant band algorithm

Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be an arrangement of affine lines in $\mathbb{R}^2$, and let $\mathcal{F}$ be a generic flag as in §2.3.

Fix an integer $k > 1$ with $k|(n+1)$, and set $\lambda = e^{2\pi i/k}$. In this section, we will give an algorithm for computing the λ-eigenspace $H^1(F_{\mathcal{A}})_\lambda$ of the first cohomology of a Milnor fiber.

3.1. Resonant bands and standing waves.

Definition 3.1. A *band* B is a region bounded by a pair of consecutive parallel lines H_i and H_{i+1}.

Each band B includes two unbounded chambers $U_1(B), U_2(B) \in \mathsf{ch}(\mathcal{A})$. By definition, $U_1(B)$ and $U_2(B)$ are opposite each other, $U_1(B)^\vee = U_2(B)$ and $U_2(B)^\vee = U_1(B)$.

Define the adjacency distance $d(C, C')$ between two chambers C and C' to be the number of lines $H \in \mathcal{A}$ that separate C and C', that is,

$$d(C, C') = |\operatorname{Sep}(C, C')|.$$

The distance $d(U_1(B), U_2(B))$ is called the length of the band B.

Remark 3.2. Let $\overline{B}$ be the closure of B in the real projective plane $\mathbb{RP}^2$. $\overline{B}$ intersects H_∞ in one point, $\overline{B} \cap H_\infty$. Each line $H \in \mathcal{A} \cup \{H_\infty\}$ either passes $\overline{B} \cap H_\infty$ or separates $U_1(B)$ and $U_2(B)$. Therefore the length of B is equal to $n + 1 - \operatorname{mult}(\overline{B} \cap H_\infty)$.

Definition 3.3. A band B is called *k-resonant* if the length of B is divisible by k. We denote the set of all k-resonant bands by $\mathrm{RB}_k(\mathcal{A})$.

To a k-resonant band $B \in \mathrm{RB}_k(\mathcal{A})$, we can associate a *standing wave* $\nabla(B) \in \mathbb{C}[\mathsf{ch}(\mathcal{A})]$ on the band B as follows:

$$\begin{aligned}
\nabla(B) &= \sum_{\substack{C \in \mathsf{ch}(\mathcal{A}),\\ C \subset B}} \left(e^{\frac{\pi i d(U_1(B), C)}{k}} - e^{-\frac{\pi i d(U_1(B), C)}{k}} \right) \cdot [C] \\
&= \sum_{\substack{C \in \mathsf{ch}(\mathcal{A}),\\ C \subset B}} \left(\lambda^{\frac{d(U_1(B), C)}{2}} - \lambda^{-\frac{d(U_1(B), C)}{2}} \right) \cdot [C] \\
&= 2i \cdot \sum_{\substack{C \in \mathsf{ch}(\mathcal{A}),\\ C \subset B}} \sin\left(\frac{\pi d(U_1(B), C)}{k} \right) \cdot [C].
\end{aligned} \tag{3}$$

Remark 3.4. Since the length $d(U_1(B), U_2(B))$ of the band B is divisible by k, the coefficients of $[U_1(B)]$ and $[U_2(B)]$ in the linear combination in (3) are zero. Hence the chambers in the summations in (3) run only over bounded chambers contained in B. We also note that exchanging of $U_1(B)$ and $U_2(B)$ affects at most the sign of $\nabla(B)$.

Remark 3.5. To indicate the choice of $U_1(B)$ and $U_2(B)$, we always put the name B of the band in the unbounded chamber $U_1(B)$ (see Figure 2).

3.2. **Eigenspaces via resonant bands.** The map $B \longmapsto \nabla(B)$ can be naturally extended to the linear map

$$\nabla : \mathbb{C}[\mathrm{RB}_k(\mathcal{A})] \longrightarrow \mathbb{C}[\mathsf{ch}(\mathcal{A})]. \tag{4}$$

Theorem 3.6. *The kernel of ∇ is isomorphic to the λ-eigenspace of the Milnor fiber monodromy, that is,*

$$\mathrm{Ker}\,(\nabla : \mathbb{C}[\mathrm{RB}_k(\mathcal{A})] \longrightarrow \mathbb{C}[\mathsf{ch}(\mathcal{A})]) \simeq H^1(F_{\mathcal{A}})_\lambda.$$

In particular, $\dim H^1(F_{\mathcal{A}})_\lambda$ is equal to the number of linear relations among the standing waves $\nabla(B)$, $B \in \mathrm{RB}_k(\mathcal{A})$.

Proof. Let $\mathcal{L}_\lambda$ be the rank-one local system on $\mathsf{M}(\mathcal{A})$ defined by $q_1 = \cdots = q_n = \lambda \in \mathbb{C}^*$ (see §2.1 and §2.3). In this case, $\Delta(C, C')$ depends only on the adjacency distance $d(C, C')$, or more precisely,

$$\Delta(C, C') = \lambda^{\frac{d(C,C')}{2}} - \lambda^{-\frac{d(C,C')}{2}}.$$

Now, we consider the first cohomology group $H^1(\mathbb{C}[\mathsf{ch}^\bullet_{\mathcal{F}}(\mathcal{A})], d_{\mathcal{L}})$ of the twisted minimal cochain complex. The image $d_{\mathcal{L}} : \mathbb{C}[\mathsf{ch}^0_{\mathcal{F}}(\mathcal{A})] \longrightarrow \mathbb{C}[\mathsf{ch}^1_{\mathcal{F}}(\mathcal{A})]$ is generated by

$$d_{\mathcal{L}}([U_0]) = \sum_{p=1}^{n-1} (\lambda^{\frac{p}{2}} - \lambda^{-\frac{p}{2}})[U_p] + \left(\lambda^{\frac{n}{2}} - \lambda^{-\frac{n}{2}}\right)[U_0^\vee].$$

Since $\lambda = e^{2\pi i/k}$ with $k > 1$ and $k|(n+1)$, we have $\lambda^{\frac{n}{2}} - \lambda^{-\frac{n}{2}} = \lambda^{-\frac{n}{2}}(\lambda^n - 1) \neq 0$. Thus the coefficient of $[U_0^\vee]$ in $d_{\mathcal{L}}([U_0])$ is non-zero. Define the subspace V of $\mathbb{C}[\mathsf{ch}^1_{\mathcal{F}}(\mathcal{A})]$ by

$$\begin{aligned} V &= \bigoplus_{p=1}^{n-1} \mathbb{C} \cdot [U_p] \\ &\quad (\simeq \mathrm{Coker}\left(d_{\mathcal{L}} : \mathbb{C}[\mathsf{ch}^0_{\mathcal{F}}(\mathcal{A})] \longrightarrow \mathbb{C}[\mathsf{ch}^1_{\mathcal{F}}(\mathcal{A})]\right)). \end{aligned} \tag{5}$$

Then $H^1(\mathbb{C}[\mathsf{ch}^\bullet_{\mathcal{F}}(\mathcal{A})], d_{\mathcal{L}})$ is isomorphic to $\mathrm{Ker}\left(d_{\mathcal{L}}|_V : V \longrightarrow \mathbb{C}[\mathsf{ch}^2_{\mathcal{F}}(\mathcal{A})]\right)$. It is sufficient to show that $\mathrm{Ker}(d_{\mathcal{L}}|_V) \simeq \mathrm{Ker}\,\nabla$, which will be done in several steps. Suppose that

$$\varphi = \sum_{p=1}^{n-1} c_p \cdot [U_p] \in \mathrm{Ker}(d_{\mathcal{L}}|_V).$$

(i) If H_i and H_{i+1} are not parallel, then $c_i = 0$.

Note that if $j \neq i$, then the chamber $[U_i^\vee]$ does not appear in $d_{\mathcal{L}}([U_j])$. Thus the coefficient of $[U_i^\vee]$ in

$$d_{\mathcal{L}}(\varphi) = \sum_{p=1}^{n-1} c_p \cdot d_{\mathcal{L}}([U_p])$$

is $c_i \cdot \Delta(U_i, U_i^\vee) = c_i(\lambda^{\frac{n}{2}} - \lambda^{-\frac{n}{2}})$. This equals zero if and only if $c_i = 0$.

Now we may assume that $\varphi = \sum_p c_p \cdot [U_p] \in \mathrm{Ker}(d_{\mathcal{L}})$ is a linear combination of $[U_p]$s such that H_p and H_{p+1} are parallel. Suppose that H_i and H_{i+1} are parallel and denote by B_i the band determined by these lines.

(ii) If B_i is not k-resonant, then $c_i = 0$.

In this case, $\Delta(U_i, U_i^\vee) = \lambda^{\frac{d(U_i,U_i^\vee)}{2}} - \lambda^{-\frac{d(U_i,U_i^\vee)}{2}}$. By the assumption that $d(U_i, U_i^\vee)$ is not divisible by k, we have $\Delta(U_i, U_i^\vee) \neq 0$. Since φ is a linear combination of $[U_p]$s with parallel boundaries H_p and H_{p+1}, the term $[U_i^\vee]$ appears only in $d_{\mathcal{L}}([U_i])$, which is equal to $c_i \cdot \Delta(U_i, U_i^\vee)[U_i^\vee]$. Therefore $c_i = 0$.

Finally we may assume that φ is a linear combination of $[U_p]$s such that the boundaries H_p and H_{p+1} are parallel and the length of the corresponding band B_p is divisible by k. In this case, it is straightforward to check that the maps $d_{\mathcal{L}}$ and ∇ are identical. This completes the proof. $\square$

Example 3.7. (A_3-arrangement, $\mathcal{A}(6,1)$ or $\mathcal{B}_6$) The three arrangements in Figure 2 are projectively equivalent, and are respectively called A_3-arrangement, $\mathcal{A}(6,1)$ or $\mathcal{B}_6$. (See §4 for the latter two notations.) We use the left figure to compute $\dim H^1(F_{\mathcal{A}})_\lambda$. (The symbol ∞ indicates that the line at infinity is an element of $\mathcal{A}$.) Since $|c\mathcal{A}| = n+1 = 6$, $k \in \{2,3,6\}$ and we have $\mathrm{RB}_2(\mathcal{A}) = \mathrm{RB}_6(\mathcal{A}) = \emptyset$, $\mathrm{RB}_3(\mathcal{A}) = \{B_1, B_2\}$. By definition, we have

$$\nabla(B_1) = \sqrt{-3}\cdot[C_1] + \sqrt{-3}\cdot[C_2]$$
$$\nabla(B_2) = \sqrt{-3}\cdot[C_1] + \sqrt{-3}\cdot[C_2].$$

Hence we have a linear relation $\nabla(B_1 - B_2) = 0$ and $\dim H^1(F_{\mathcal{A}})_\lambda = 1$ for $\lambda = e^{2\pi i/3}$. (Hence the A_3-arrangement is pure-tone; see Definition 4.1.)

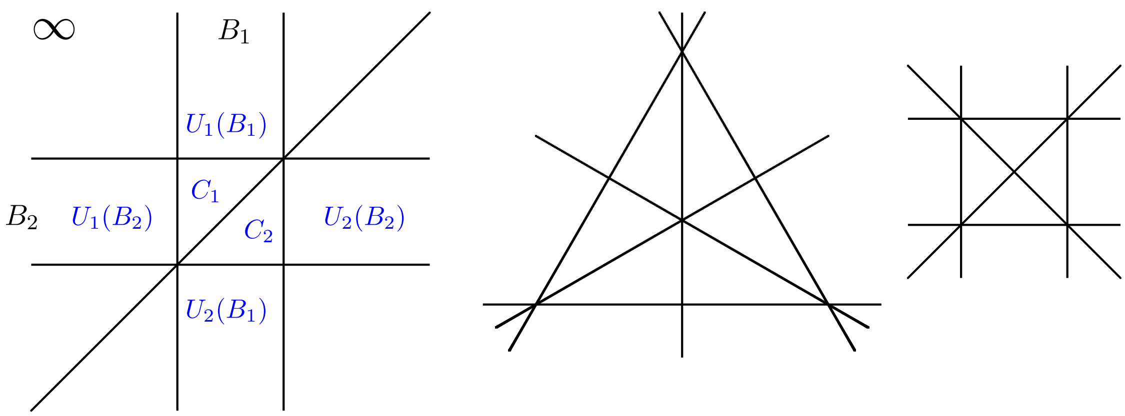

Figure 2. The A_3-arrangement ($= \mathcal{A}(6,1) = \mathcal{B}_6$)

Example 3.8. ($\mathcal{A}(12,2)$ from [8]) Let $\mathcal{A}$ be the line arrangement in Figure 3 (together with the line at infinity). Then $|c\mathcal{A}| = n+1 = 12$. There are seven bands, $B_1, \dots, B_7$. Among them, B_5, B_6 and B_7 have length 7 which is coprime with 12 so we can ignore them. We have $\mathrm{RB}_3(\mathcal{A}) = \{B_1, B_4\}$ and $\mathrm{RB}_2(\mathcal{A}) = \mathrm{RB}_4(\mathcal{A}) = \{B_2, B_3\}$. First consider the case $k=3$. Then

$$\nabla(B_1) = \sqrt{-3}\cdot[C_1] + \dots,$$
$$\nabla(B_4) = \sqrt{-3}\cdot[C_6] + \dots.$$

Since the chamber C_6 is not contained in the band B_1, it does not appear in the linear combination for $\nabla(B_1)$. Hence $\nabla(B_1)$ and $\nabla(B_4)$ are linearly independent. We conclude that $H^1(F_{\mathcal{A}})_\lambda = 0$ for $\lambda = e^{2\pi i/3}$. The cases $k=2$ and $k=4$ are similar. More precisely, since $B_2, B_3 \in \mathrm{RB}_2(\mathcal{A}) = \mathrm{RB}_4(\mathcal{A})$ are parallel and they do not overlap, $\nabla(B_2)$ and $\nabla(B_3)$ are linearly independent. Consequently we have $H^1(F_{\mathcal{A}})_{\neq 1} = 0$ and so the cohomology does not have non-trivial eigenvalues.

The argument used in Example 3.8 is generalized in the next section. See §4 for further examples.

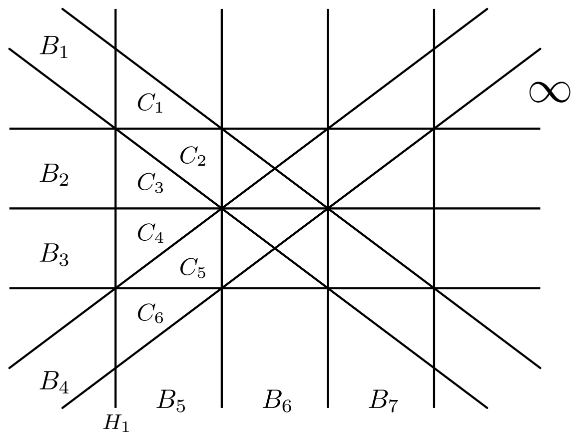

FIGURE 3. $\mathcal{A}(12, 2)$

Remark 3.9. The cohomology of the Milnor fiber $H^1(F_{\mathcal{A}})$ depends only on the projective arrangement $\mathcal{A} \cup \{H_\infty\}$. The change of the line at infinity H_∞ sometimes makes the structure of resonant bands RB_k simpler. This fact will be used in Corollary 3.16.

3.3. **Vanishing.** Fix k and λ as above. We describe some corollaries to Theorem 3.6.

Corollary 3.10. *If* $\mathrm{RB}_k(\mathcal{A}) = \emptyset$*, then* $H^1(F_{\mathcal{A}})_\lambda = 0$.

Proof. Since $\mathbb{C}[\mathrm{RB}_k(\mathcal{A})] = 0$, obviously $\mathrm{Ker}(\nabla : \mathbb{C}[\mathrm{RB}_k(\mathcal{A})] \to \mathbb{C}[\mathsf{ch}(\mathcal{A})]) = 0$. By Theorem 3.6, $H^1(F_{\mathcal{A}})_\lambda = 0$. □

Using the interpretations in Remark 3.2, we have the following.

Proposition 3.11. A band B is k-resonant if and only if $\mathrm{mult}(\overline{B} \cap H_\infty)$ is divisible by k.

Corollary 3.12. *Suppose that there are no points on H_∞ where the multiplicity of*

$$c\mathcal{A} = \mathcal{A} \cup \{H_\infty\}$$

is divisible by k. Then $H^1(F_{\mathcal{A}})_\lambda = 0$.

Proof. By Proposition 3.11, the assumption is equivalent to $\mathrm{RB}_k(\mathcal{A}) = \emptyset$. We then use Corollary 3.10. □

Remark 3.13. Corollary 3.12 is proved by Libgober [13, Corollary 3.5] for more general complex arrangement cases.

For the real case, we obtain a stronger result as follows.

Theorem 3.14. *Suppose that all k-resonant bands are parallel to each other. Then $H^1(F_{\mathcal{A}})_\lambda = 0$.*

Proof. By the assumption, $\mathrm{RB}_k(\mathcal{A}) = \{B_1, \ldots, B_m\}$ consists of parallel bands. Now, the supports of $\nabla(B_1), \ldots, \nabla(B_m)$, that is, the set of chambers appearing in each standing wave, are mutually disjoint. They are obviously linearly independent. (Recall that, in this paper, we assume that the arrangement $\mathcal{A}$ has at least one intersection.) Hence $H^1(F_{\mathcal{A}})_\lambda = 0$. (See Example 3.8.) □

Corollary 3.15. *Suppose that there is at most one point $p \in H_\infty$ such that the multiplicity of $c\mathcal{A} = \mathcal{A} \cup \{H_\infty\}$ at p is divisible by k. Then $H^1(F_{\mathcal{A}})_\lambda = 0$.*

Proof. Again by Proposition 3.11, the assumption is equivalent to $\mathrm{RB}_k(\mathcal{A})$ consisting of parallel bands. We then use Theorem 3.14. □

Let us denote by $H^1(F_{\mathcal{A}})_{\neq 1} = \bigoplus_{\lambda \neq 1} H^1(F_{\mathcal{A}})_\lambda$ the direct sum of non-trivial eigenspaces. The following is immediate from Corollary 3.15 and Remark 3.9.

Corollary 3.16.
(1) *Suppose that $H^1(F_{\mathcal{A}})_\lambda \neq 0$. Then each line $H \in c\mathcal{A} = \mathcal{A} \cup \{H_\infty\}$ has at least two multiple points, such that the multiplicity is divisible by k.*
(2) *Suppose that $H^1(F_{\mathcal{A}})_{\neq 1} \neq 0$. Then each line $H \in c\mathcal{A} = \mathcal{A} \cup \{H_\infty\}$ has at least two multiple points.*

Proof. Let H be such a line. Choose an affine open set in such a way that H is a line at infinity. □

Remark 3.17. We do not know whether Corollary 3.16 holds for complex arrangements. We will prove a stronger result in §3.5.

3.4. **Upper-bound.** Recall that two lines H, H' in the real projective plane $\mathbb{RP}^2$ divide the space into two regions.

Definition 3.18. Let $c\mathcal{A}$ be a line arrangement in the real projective plane $\mathbb{RP}^2$. Then the pair of lines $H_i, H_j \in c\mathcal{A}$ is said to be a *sharp pair* if all intersection points of $c\mathcal{A} \setminus \{H_i, H_j\}$ are contained in one of two regions or lie on $H_i \cup H_j$. (In other words, there are no intersection points in one of the two regions determined by H_i and H_j.)

Example 3.19. A fiber-typer arrangement has a sharp pair of lines.

Example 3.20. In the Pappus arrangement (Figure 7), the line at infinity and the leftmost vertical line form a sharp pair. So do the two boundary lines of the band B_1. Furthermore, all line arrangements appearing in this paper contain sharp pairs of lines. (There also exist arrangements which have no sharp pairs.)

Theorem 3.21. *Assume that the arrangement $c\mathcal{A}$ contains a sharp pair of lines. Then:*
(i) $\dim H^1(F)_\lambda \leq 1$ *for* $\lambda \neq 1$.
(ii) *Suppose that the pair $H_1, H_2 \in c\mathcal{A}$ is sharp. Let $p = H_1 \cap H_2$ be the intersection. If the multiplicity of $c\mathcal{A}$ at p is not divisible by k, then $H^1(F)_\lambda = 0$ for $\lambda = e^{2\pi i/k}$.*

Proof. By the $PGL_3(\mathbb{C})$ action, we may assume that the line at infinity H_∞ and $H_1 = \{x = 0\}$ form a sharp pair and that there are no intersections in the region $\{(x, y) \in \mathbb{R}^2 \mid x < 0\}$ (see Figure 3). The intersection is $p = H_\infty \cap H_1 = \{(0 : 1 : 0)\}$. Let B be a horizontal (that is, non-vertical) band, that does not passing through the point p. Denote by C_B the leftmost bounded chamber in B (e.g., in Figure 3, $C_{B_1} = C_1, C_{B_2} = C_3, C_{B_3} = C_4$ and $C_{B_4} = C_6$).

First, consider the case where the multiplicity of $c\mathcal{A}$ at p is not divisible by k. Then all k-resonant bands are horizontal. Let $B \in \mathrm{RB}_k(\mathcal{A})$. Then

$$\nabla(B) = 2i \sin\left(\frac{\pi}{k}\right) \cdot [C_B] + \cdots, \tag{6}$$

and so $[C_B]$ has a non-zero coefficient. Since C_B is contained in the unique k-resonant band B, $[C_B]$ does not appear in the linear combinations of other k-resonant bands. Hence,

$$\nabla(B), B \in \mathrm{RB}_k(\mathcal{A})$$

are linearly independent. Thus (ii) is proved.

Now we assume that the multiplicity of $\mathcal{A}$ at p is divisible by k. In this case, there are vertical k-resonant bands. Denote by B_{left} the leftmost vertical band (in Figure 3, $B_{left} = B_5$). Suppose that

$$c_{left} \cdot B_{left} + \cdots \in \mathrm{Ker}(\nabla).$$

Let $B \in \mathrm{RB}_k(\mathcal{A})$ be a horizontal k-resonant band. Then, since C_B is contained in only B and B_{left}, the coefficient c_{left} of B_{left} determines the coefficient of B. The coefficients of other vertical k-resonant bands are also determined by those of the horizontal bands. Hence $\mathrm{Ker}(\nabla)$ is at most one-dimensional. $\square$

Example 3.22. Let $\mathcal{A}$ be as in Figure 4, with $|c\mathcal{A}| = 12$. Let $k = 3$. Then

$$\mathrm{RB}_3 = \{B_1^1, B_2^1, B_3^1, B_4^1, B_1^2, B_2^2, B_3^2, B_4^2\}$$

contains eight bands. Suppose that $\sum_{i=1}^{2}\sum_{j=1}^{4} c_{ij}[B_j^i] \in \mathrm{Ker}(\nabla)$. By computing

$$\sum_{i=1}^{2}\sum_{j=1}^{4} c_{ij}\nabla(B_j^i),$$

as in the figure, we conclude that all the coefficients are $c_{ij} = 0$. Hence $H^1(F_{\mathcal{A}})_\lambda = 0$ for $\lambda = e^{2\pi i/3}$. Note that the multiple points on the diagonal line are triple points. If we put the diagonal line at infinity, then $\mathrm{RB}_2 = \mathrm{RB}_4 = \mathrm{RB}_6 = \emptyset$. Therefore,

$$H^1(F_{\mathcal{A}})_{-1} = H^1(F_{\mathcal{A}})_i = H^1(F_{\mathcal{A}})_{e^{2\pi i/6}} = 0$$

by Corollary 3.16.

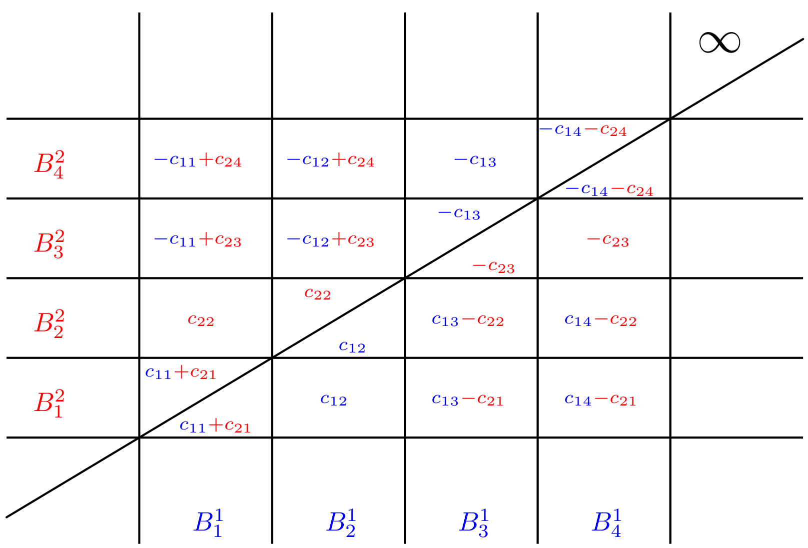

FIGURE 4. Example 3.22

3.5. **A characterization of the A_3-arrangement.** Now we give a characterization of the A_3-arrangement in terms of non-trivial Milnor monodromy.

Theorem 3.23. *Assume that $H^1(F_{\mathcal{A}})_\lambda \neq 0$ with $\lambda = e^{2\pi i/k} \neq 1$, and that the set of k-resonant bands $\mathrm{RB}_k(\mathcal{A})$ consists of at most two directions (this condition is equivalent to H_∞ containing at most two multiple points which have multiplicities divisible by k). Then $c\mathcal{A}$ is equivalent to the A_3-arrangement.*

Proof. If $\mathrm{RB}_k(\mathcal{A})$ consists of one direction, then by Theorem 3.14, $H^1(F_{\mathcal{A}})_\lambda = 0$. Thus we may assume that $\mathrm{RB}_k(\mathcal{A})$ consists of two directions. After a suitable change of coordinates, we assume the following (see Figure 5):

- $\mathrm{RB}_k(\mathcal{A}) = \{B_1^1, B_2^1, \dots, B_p^1, B_1^2, B_2^2, \dots, B_q^2\}$.
- $B_1^1, B_2^1, \dots, B_p^1$ are parallel to the vertical line $x = 0$ and may be expressed as
$$B_i^1 = \{(x, y) \in \mathbb{R}^2 \mid a_i < x < a_{i+1}\}$$
with $a_1 < \cdots < a_{p+1}$. The lines $H_i^1 = \{x = a_i\}$, $i = 1, \dots, p+1$, which are vertical lines, are boundaries of these bands.
- $B_1^2, B_2^2, \dots, B_q^2$ are parallel to the horizontal line $y = 0$ and may be expressed as
$$B_i^2 = \{(x, y) \in \mathbb{R}^2 \mid b_i < y < b_{i+1}\}$$
with $b_1 < \cdots < b_{q+1}$. The lines $H_i^2 = \{y = b_i\}$, $i = 1, \dots, q+1$, which are horizontal lines, are boundaries of these bands.
- Let $\sum_{i=1}^{p} c_{1i} \cdot B_i^2 + \sum_{i=1}^{q} c_{2i} \cdot B_i^2 \in \mathrm{Ker}(\nabla)$ be a non-trivial relation among k-resonant bands.

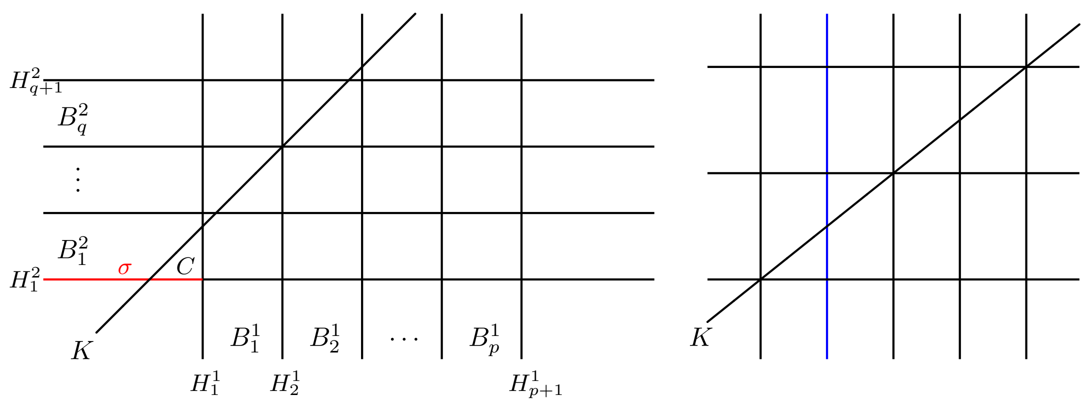

FIGURE 5. Proof of Theorem 3.23

The multiplicity of $c\mathcal{A}$ at $(0:1:0) = \overline{B_i^1} \cap H_\infty$ is $p+2$, which must be divisible by k by Proposition 3.11. Hence p can be expressed as $p = ks - 2$ ($s \in \mathbb{Z}_{>0}$). Similarly, set $q = kt - 2$ ($t \in \mathbb{Z}_{>0}$). So far, together with H_∞, we have $(s+t)k - 1$ lines. The remaining $n+1-(s+t)k+1$ can be expressed as $ku+1$, which in particular cannot be zero. We prove that (1) $u = 0$, (2) $k = 3$, (3) $p = q$, (4) $p = q = 1$, and conclude that $c\mathcal{A}$ is the A_3-arrangement.

(1) We first prove that $u = 0$. Consider the open segment
$$\sigma = \{(x : y : 1) \in \mathbb{R}^2 \mid y = b_1, -\infty < x < a_1\} \subset H_1^2$$

which is bounded by the two points $(1:0:0)$ and $(a_1:b_1:1)$. (See Figure 5.) Let us prove that there are no intersections on σ. Suppose that the line $K\in\mathcal{A}$ intersects σ. The leftmost chamber C in B_1^2 is not contained in the other k-resonant bands and satisfies $d(U_1(B_1^2),C)=1$. Since

$$\nabla(B_1^1)=2i\sin\left(\frac{\pi}{k}\right)\cdot[C]+\cdots,$$

and the coefficient of $[C]$ is non-zero, we have $c_{21}=0$. This implies that

$$c_{11}=c_{12}=\cdots=c_{1p}=0.$$

Then we have $c_{22}=\cdots=c_{2q}=0$. This contradicts the hypothesis that $H^1(F_{\mathcal{A}})_\lambda\neq 0$. This contradiction proves that there are no intersections on the segment σ. Similarly there are no intersections on the seven other similar segments, that is, the boundaries of the four regions

$$\{(x:y:1)\mid x<a_1,y<b_1\},\ \{(x:y:1)\mid x<a_1,y>b_q\},$$
$$\{(x:y:1)\mid x>a_p,y<b_1\},\ \{(x:y:1)\mid x>a_p,y>b_q\}.$$

Thus $K\in\mathcal{A}$ must be one of the two diagonals

$$K_1=\text{the line connecting }(a_1:b_1:1)\text{ and }(a_p:b_q:1),$$
$$K_2=\text{the line connecting }(a_p:b_1:1)\text{ and }(a_1:b_q:1).$$

Hence $ku+1\leq 2$, and we have $u=0$.

(2) Now we prove $k=3$. Using the above notation, we may assume that

$$\mathcal{A}=\{H_1^1,\dots,H_{ks-1}^1,H_1^2,\dots,H_{kt-1}^2,K\},$$

where K is the diagonal line connecting $(a_1:b_1:1)$ and $(a_p:b_q:1)$. Then the point $(a_1:b_1:1)$ has multiplicity 3. The line H_1^1 has exactly two multiple points, $(a_1:b_1:1)$ and $(0:1:0)$. By Corollary 3.16, k is a common divisor of 3 and the multiplicity of $(0:1:0)$. Since $k\neq 1$, we have $k=3$.

(3) If $p\neq q$, then there exists a (either vertical or horizontal) line which intersects the diagonal line K normally (that is, with multiplicity 2, the right-hand side of Figure 5). Then the line has only one multiple point on H_∞ (either $(0:1:0)$ or $(1:0:0)$). This contradicts Corollary 3.16. Hence $p=q$.

(4) If $p=q>1$, then we can prove that $H^1(F_{\mathcal{A}})_{e^{2\pi i/3}}=0$ by an argument similar to Example 3.22. Hence $p=q=1$. This obviously implies that $c\mathcal{A}$ is isomorphic to the A_3-arrangement. □

Corollary 3.24. *Assume that $\mathcal{A}$ is a real arrangement as above, and assume that*

$$|c\mathcal{A}|=n+1\geq 7.$$

If $H^1(F_{\mathcal{A}})_\lambda\neq 0$, then each line $H\in c\mathcal{A}$ passes through at least three multiple points which have multiplicities divisible by k.

Remark 3.25. We do not know whether Theorem 3.23 and Corollary 3.24 hold for complex arrangements.

4. Examples and Conjectures

By the previous result (Corollary 3.24), the Milnor fiber cohomology has non-trivial eigenspaces only when each line has at least three multiple points. Classes of line arrangements known as “simplicial arrangements” and “configurations” provide such examples. In this section, we present examples of non-trivial eigenspaces $H^1(F_{\mathcal{A}})_{\neq 1}\neq 0$.

4.1. **Observation.** As far as the author knows, all examples of real arrangements with

$$H^1(F_{\mathcal{A}})_{\neq 1} \neq 0$$

have the following "pure-tone" property, that is, only the third root of 1 appears with multiplicity one.

Definition 4.1. $\mathcal{A}$ is said to be *pure-tone* if $H^1(F_{\mathcal{A}})_\lambda = 0$ for $\lambda^3 \neq 1$ and $\dim H^1(F_{\mathcal{A}})_\lambda = 1$ for $\lambda = e^{\pm 2\pi i/3}$.

Furthermore, it is observed that all known examples with $H^1(F_{\mathcal{A}})_{\neq 1} \neq 0$ satisfy

- $c\mathcal{A}$ has a sharp pair of lines,
- $c\mathcal{A}$ has a k-multinet structure with $k = 3$.

These two properties imply by Theorem 3.21 and Theorem 2.3 that $\mathcal{A}$ is pure-tone.

4.2. **Simplicial arrangements.** Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be a line arrangement in $\mathbb{R}^2$. Then the projective arrangement $c\mathcal{A} = \mathcal{A} \cup \{H_\infty\}$ in the real projective plane $\mathbb{RP}^2$ is called *simplicial* if each chamber is a triangle. Grünbaum [8] presents a catalogue of known simplicial arrangements with up to 37 lines (see [5] for additional information).

Notation. The symbol ∞ in a figure indicates that the $(n+1)$-st line is H_∞. The notation $\mathcal{A}(n,k)$ comes from [8], which is the k-th simplicial arrangement of n-lines.

Example 3.7 can be generalized in two ways.

Definition 4.2. For a positive integer $n \in \mathbb{Z}_{>0}$, $\mathcal{A}(2n,1)$ is described as follows. Starting with a regular convex n-gon in the Euclidean plane, $\mathcal{A}(2n,1)$ is obtained by taking n lines determined by the sides of the n-gon together with the n-lines of symmetry of that n-gon. $\mathcal{A}(2n,1)$ is a simplicial arrangement of $2n$-lines.

Obviously, the A_3-arrangement is equivalent to $\mathcal{A}(6,1)$.

Example 4.3. Let $c\mathcal{A} = \mathcal{A}(12,1)$ (Figure 6). Then $\mathrm{RB}_3(\mathcal{A}) = \{B_1, \dots, B_7\}$.

$$\begin{array}{rlcccccccccc}
\nabla(B_2) = & \sqrt{-3}(C_1 & & +C_3 & & -C_5 & -C_6 & & +C_8 & & +C_{10}) \\
\nabla(B_3) = & \sqrt{-3}(C_1 & & +C_3 & & & & -C_7 & & -C_9 &) \\
\nabla(B_6) = & \sqrt{-3}(& C_2 & & +C_4 & & & & -C_8 & & -C_{10}) \\
\nabla(B_7) = & \sqrt{-3}(& C_2 & & +C_4 & -C_5 & -C_6 & +C_7 & & +C_9 &)
\end{array}$$

Hence we have a linear relation

$$\nabla(B_2) - \nabla(B_3) + \nabla(B_6) - \nabla(B_7) = 0,$$

and so we have that $\mathcal{A}(12,1)$ is pure-tone.

More generally, using Theorem 2.3 and Theorem 3.21, we can prove that $\mathcal{A}(6m,1)$ is pure-tone. All other examples except for $\mathcal{A}(6m,1)$ in the catalogue [8] (and [5]) satisfy $H^1(F_{\mathcal{A}})_{\neq 1} = 0$. It seems natural to pose the following.

Conjecture 4.4. *Assume that $c\mathcal{A}$ is a simplicial arrangement. Then the following are equivalent.*

(a) $c\mathcal{A} = \mathcal{A}(6m,1)$ *for some* $m > 0$.
(b) $H^1(F_{\mathcal{A}})_{\neq 1} \neq 0$.
(c) $\mathcal{A}$ *is pure-tone.*
(d) $c\mathcal{A}$ *has a k-multinet structure for some* $k \geq 3$.
(e) $c\mathcal{A}$ *has a* 3*-multinet structure.*

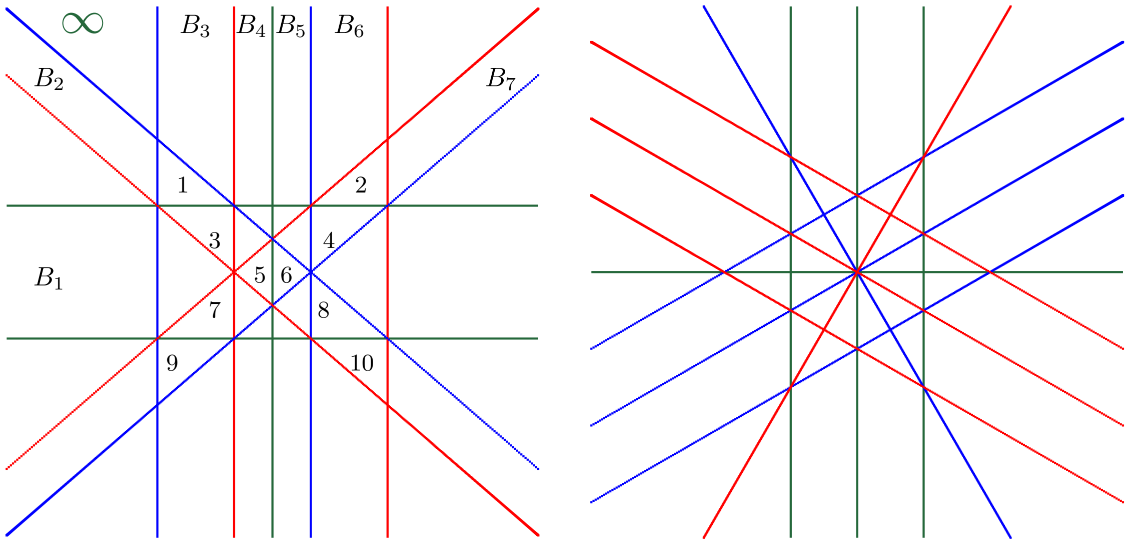

FIGURE 6. $\mathcal{A}(12, 1)$

4.3. **Zoo of non-trivial eigenspaces.**

Example 4.5. Let $c\mathcal{A}$ be the Pappus arrangement (Figure 7), so that $|c\mathcal{A}| = n + 1 = 9$. Let $k = 3$. Then $\mathrm{RB}_3(\mathcal{A}) = \{B_1, B_2, B_3\}$. By the expressions

$$\begin{array}{llllllllll}
\nabla(B_1) = & \sqrt{-3}(C_1 & & +C_3 & & & -C_9 & & -C_{11}) \\
\nabla(B_2) = & \sqrt{-3}(C_1 & +C_2 & +C_3 & +C_4 & -C_8 & -C_9 & -C_{10} & -C_{11}) \\
\nabla(B_3) = & \sqrt{-3}(& C_2 & & +C_4 & -C_8 & & -C_{10} &)
\end{array}$$

there is a unique relation $\nabla(B_1) - \nabla(B_2) + \nabla(B_3) = 0$. Hence the Pappus arrangement is pure-tone.

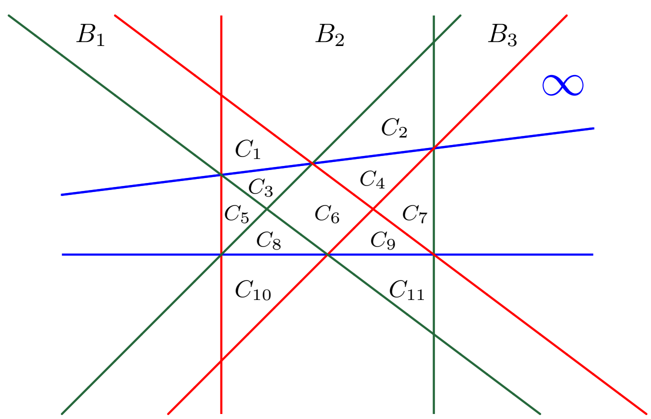

FIGURE 7. Pappus arrangement (Example 4.5)

Example 4.6. (Taken from [9, page 244].) Let $\mathcal{A}$ be as in the right-hand side of Figure 8. Then $|c\mathcal{A}| = n + 1 = 12$ and $\mathrm{RB}_3(\mathcal{A}) = \{B_1, \dots, B_5\}$. There is a unique linear relation

$$\nabla(B_1) - \nabla(B_2) + \nabla(B_3) - \nabla(B_4) = 0$$

(B_5 does not appear). Hence $\mathcal{A}$ is pure-tone.

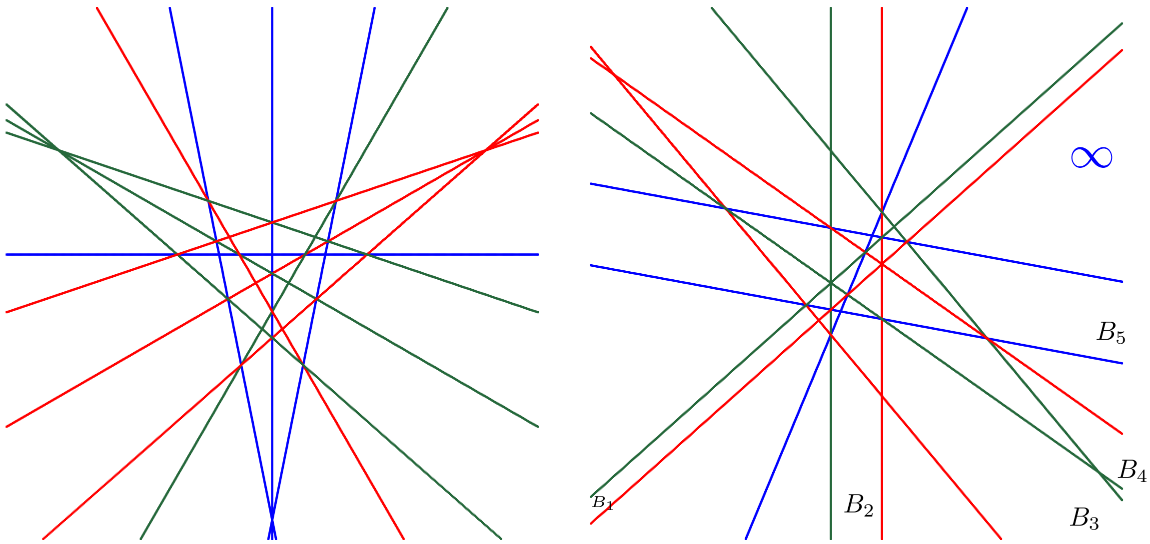

FIGURE 8. Example 4.6

Example 4.7. (Taken from [9, page 244].) Let $\mathcal{A}$ be as in the right-hand side of Figure 9. Then $|c\mathcal{A}| = n + 1 = 12$ and $\mathrm{RB}_3(\mathcal{A}) = \{B_1, \dots, B_4\}$. There is a unique linear relation

$$\nabla(B_1) - \nabla(B_2) + \nabla(B_3) - \nabla(B_4) = 0.$$

Hence $\mathcal{A}$ is pure-tone.

Example 4.8. (Taken from [9, page 44].) Let $\mathcal{A}$ be as in the right-hand side of Figure 10. Then $|c\mathcal{A}| = n + 1 = 15$ and $\mathrm{RB}_3(\mathcal{A}) = \{B_1, \dots, B_7\}$. There is a unique linear relation

$$\nabla(B_1) - \nabla(B_3) + \nabla(B_4) - \nabla(B_6) + \nabla(B_7) = 0$$

(B_2 and B_5 do not appear). Hence $\mathcal{A}$ is pure-tone.

Definition 4.9. For a positive integer $m \in \mathbb{Z}_{>0}$, $\mathcal{B}_{3m}$ is described as follows. Starting with a regular convex $2m$-gon in the Euclidean plane, $\mathcal{B}_{3m}$ is obtained by taking $2m$ lines determined by the sides of the $2m$-gon together with m-diagonal lines connecting opposite vertices. (Note that $\mathcal{B}_6$ is equivalent to the A_3-arrangement, see Figure 2.)

Example 4.10. Using Theorem 2.3 and Theorem 3.21, we can prove that the $\mathcal{B}_{3m}$-arrangement is pure-tone.

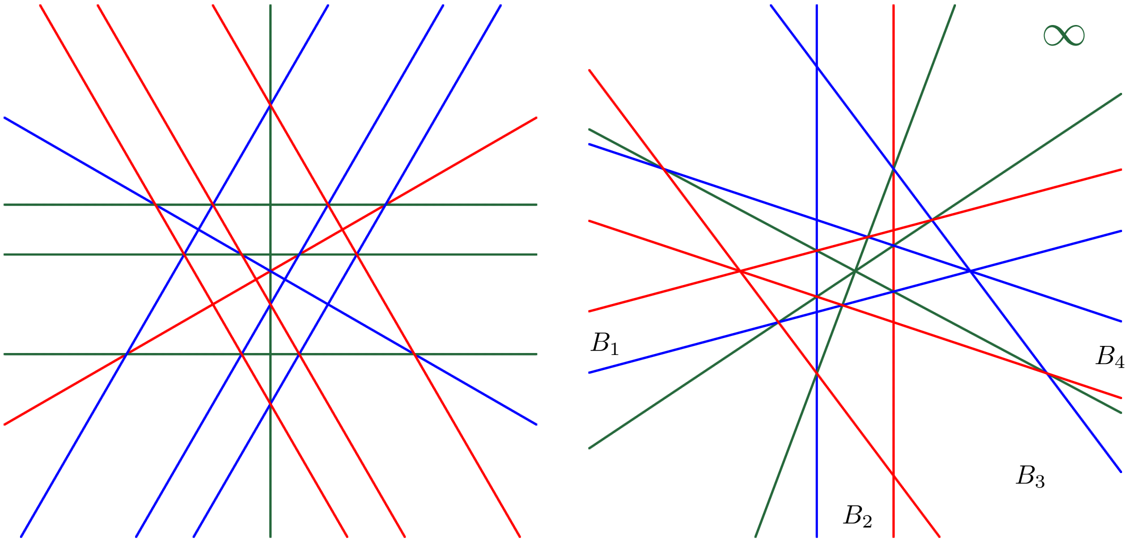

Figure 9. Example 4.7

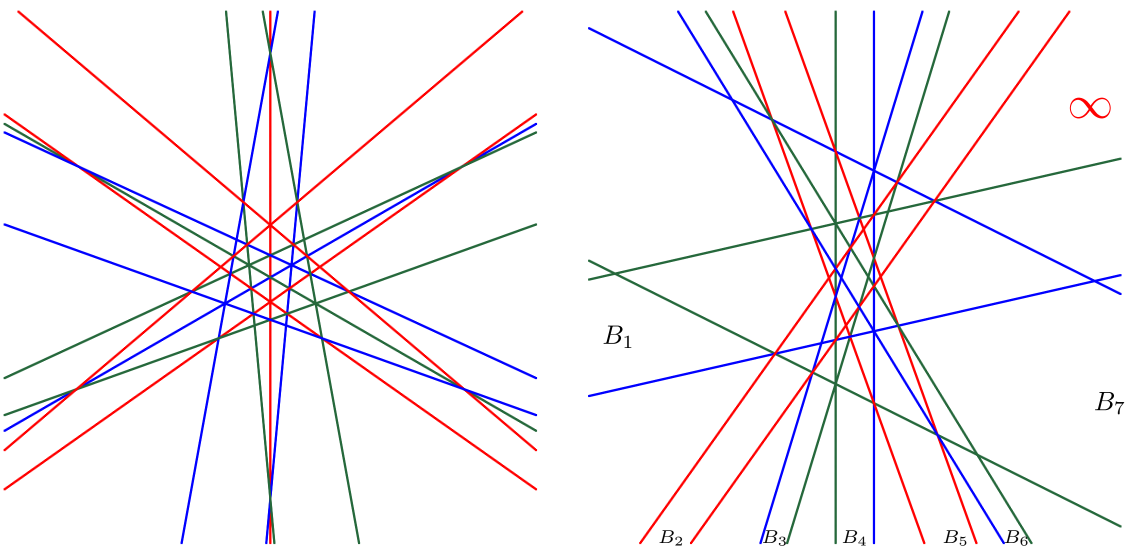

Figure 10. Example 4.8

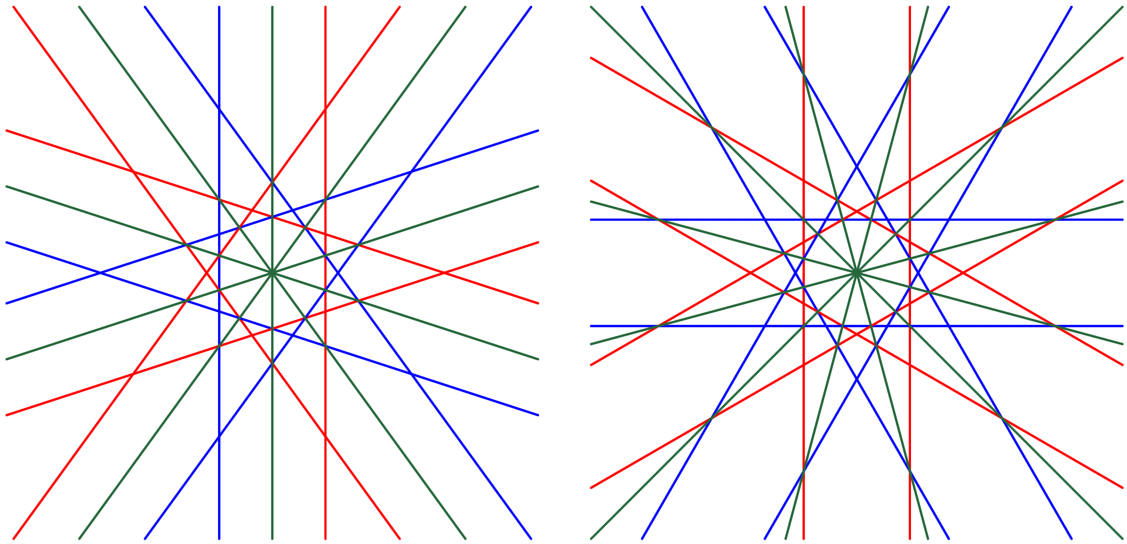

FIGURE 11. $\mathcal{B}_{15}$ and $\mathcal{B}_{18}$

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