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ON THE CLASSIFICATION OF RATIONAL SURFACE SINGULARITIES

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ABSTRACT. A general strategy is given for the classification of graphs of rational surface singularities. For each maximal rational double point configuration we investigate the possible multiplicities in the fundamental cycle. We classify completely certain types of graphs. This allows to extend the classification of rational singularities to multiplicity 8. We also discuss the complexity of rational resolution graphs.

INTRODUCTION

The topological classification of complex surface singularities amounts to classifying resolution graphs. Such a graph represents a complex curve on a surface, and the simplest case is when this curve is rational; then the singularity is called rational and the graph in fact determines the analytical type of the singularity up to equisingular deformations.

Classification of singularities tends to lead to long lists, but making them is not a purpose on its own. Sometimes one wants a list to prove statements by case-by-case checking. If the lists become too unwieldy, as in the case on hand, their main use will be to provide an ample supply of examples to test conjectures on. With this objective the most useful description of rational resolution graphs is as a list of parts, together with assembly instructions, guaranteeing that the result is a rational graph. For a special class of rational singularities, those with almost reduced fundamental cycle, such a classification exists [13, 4].

As prototype of our classification and to fix notations we first treat the special case. The fundamental cycle ([1], see also Definition 1.5) can be seen as divisor on the exceptional set of the resolution, with positive coefficients (and it is this divisor which should be rational as non-reduced curve). It is characterised numerically as the minimal positive cycle intersecting each exceptional curve non-positively, and can therefore be computed using the intersection form encoded in the graph. The fundamental cycle is called almost reduced if it is reduced at the non-(-2)'s. So higher multiplicities can only occur on the maximal rational double point (RDP) configurations. The classification splits in two parts: one has to determine the multiplicities on the RDP-configurations and how they can be attached to the rest of the graph. The explicit list of graphs can be found in the paper by Gustavsen [4]. Blowing down the RDP-configurations to rational double point singularities gives the canonical model or RDP-resolution. Its exceptional set can again by described by a graph. Our classification strategy in general is to first find the graphs for the RDP-resolution, and then determine which rational double point (RDP) configurations can occur.

The first new results in this paper are on graphs, where each RDP-configuration is attached to at most one non-reduced non-(-2). The possible graphs for the RDP-resolution are easy to describe, but here a new phenomenon occurs, that not every candidate graph can be realised by a rational singularity. In particular, if the graph contains only one non-(-2), this vertex has multiplicity at most 6 in the fundamental cycle. These considerations apply to all multiplicities, but only for a restricted class of singularities; they cover all singularities of low multiplicity.

Our result extends the classification of rational singularities of multiplicity 4 [14], and allows to recover the classification by Tosun et al. for multiplicity 5 [16].

Multiplicity 6 necessitates the study of RDP-configurations, connecting two non-reduced non-(-2)'s. We first determine the conditions under which the multiplicities in the fundamental cycle become as high as possible. We do this for each RDP-configuration separately. The existence depends on the rest of the graph. Then we use the same computations to treat the case that the non-(-2)'s have multiplicity exactly two. This allows us to complete the classification of rational singularities of multiplicity 6. The same methods work for multiplicity 7 and 8, but we do not treat these cases explicitly, except for one new case, of three non-reduced non-(-2)'s, with which we conclude our classification.

We do not claim that it is feasible to treat all multiplicities with our methods. Our last result, on multiplicity 8, gives a glimpse of what is needed in general. To use induction over the number of non-(-2)'s, one needs detailed knowledge on the graphs for lower multiplicity, and it does not suffice to compute with RDP-configurations separately. We include (at the end of the first section) a non-trivial example of a rational graph, of multiplicity 37; the graph of the canonical model is rather simple. This example comes from a paper by Karras [6], which maybe contains the deepest study of the structure of resolution graphs in the literature. He proves that every rational singularity deforms into a cone over a rational normal curve of the same multiplicity. My main motivation for taking up the classification again lies in the same direction. The ultimate goal is to study the Artin component of the semi-universal deformation. Over this component a simultaneous resolution exists (or, without base change, a simultaneous canonical model). This is one motivation of our classification strategy of first finding the graph for the RDP-resolutions. The analytical type of the total space over the Artin component (up to smooth factors) is an interesting invariant of the singularity. In his thesis [13] Ancus Röhr turned the problem of formats around and defined the format as just this invariant. He showed that the format determines the exceptional set of the canonical model of the singularity. Examples in this paper cast doubt on our earlier conjecture that the converse holds.

RDP-configurations can be of type A, D and E. Our computations show that one cannot reach high multiplicities in the fundamental cycle using configurations of type D and E. With this goal it suffices to look at configurations of type A. Indeed, the picture which arises from our classifications, is that for most purposes it suffices to look at rather simple configurations of type A.

One answer to the question how complex a graph can be is that of Lê and Tosun [10], who take the number of rupture points (vertices with valency at least 3) as measure. We give a simplified proof of their estimate, that this number is bounded by $m-2$, where m is the multiplicity of the singularity. Our argument shows that the highest complexity is attained by graphs with reduced fundamental cycle.

The structure of this paper is as follows. In the first section we review some properties of resolution graphs. The next section gives the classification of singularities with almost reduced fundamental cycle. Section 3 is about complexity in the sense of [10]. Then we discuss the format of a rational singularity, following [13]. Our computations use a special way to compute the fundamental cycle, which we explain in Section 5. The case, where each RDP-configuration is attached to at most one non-reduced non-(-2), is treated in Section 6, while the following section describes RDP-configurations on general graphs. In the final section we complete the classification for multiplicity 6 and treat the case of three non-reduced non-(-2)'s in multiplicity 8.

1. Rational graphs

In this section we review some properties of resolution graphs. References are Artin [1], Wagreich [17] and Wall [18], and for rational singularities in addition Laufer [8].

The topological type of a normal complex surface singularity is determined by and determines the resolution graph of the minimal good resolution [12]. A resolution graph can be defined for any resolution.

Definition 1.1. Let $\pi\colon (M,E)\to(X,p)$ be a resolution of a surface singularity with exceptional divisor $E=\bigcup_{i=1}^r E_i$. The *resolution graph* Γ is a weighted graph with vertices corresponding to the irreducible components E_i. Each vertex has two weights, the self-intersection $-b_i=E_i^2$, and the arithmetic genus $p_a(E_i)$, the second traditionally written in square brackets and omitted if zero. There is an edge between vertices if the corresponding components E_i and E_j intersect, weighted with the intersection number $E_i\cdot E_j$ (only written out if larger than one).

Other definitions, which record more information, are possible: one variant is to have an edge for each intersection point $P\in E_i\cap E_j$, with weight the local intersection number $(E_i\cdot E_j)_P$. These subtleties need not concern us here, as the exceptional divisor of a rational singularity is a simple normal crossings divisor.

We call the vertices of the graph Γ also for E_i. This should cause no confusion. From the context it will be clear whether we consider E_i as vertex or as curve. In fact, we use E_i also in a third sense. The classes of the curves E_i form a preferred basis of $H:=H_2(M,\mathbb{Z})$. Following algebro-geometric tradition the elements of H are called *cycles*. They are written as linear combinations of the E_i.

The resolution graph (as defined above) is also the graph of the quadratic lattice $H:=H_2(M,\mathbb{Z})$, in the sense of [11]. The intersection form on M gives a negative definite quadratic form on H. Let $K\in H^2(M,\mathbb{Z})=H^{\#}$ be the canonical class. It can be written as rational cycle in $H_{\mathbb{Q}}=H\otimes\mathbb{Q}$ by solving the adjunction equations $E_i\cdot(E_i+K)=2p_a(E_i)-2$. The function $-\chi(A)=\frac12 A\cdot(A+K)$, $A\in H$, makes H into a *quadratic lattice* [11, 1.4]. We prefer to work with the genus $p_a(A)=1-\chi(A)$. Note that the genus function determines the intersection form, as

$$p_a(A+B)=p_a(A)+p_a(B)+A\cdot B-1\,.$$

The data (H,p_a) is equivalent to $(H,\{E_i\cdot E_j\},\{p_a(E_i)\})$, encoded in the resolution graph Γ. Sometimes we identify H with the free abelian group on the vertex set of Γ, and talk about cycles on Γ.

Definition 1.2. A cycle $A=\sum a_iE_i$ (in H or $H_{\mathbb{Q}}$) is *effective* or non-negative, $A\geq 0$, if all $a_i\geq 0$. There is a natural inclusion $j\colon H\to H^{\#}$, given by $j(A)(B)=-A\cdot B$ (note the minus sign, because of negative definiteness). A cycle A is *anti-nef*, if $j(A)\geq 0$ in $H^{\#}$, i.e., $A\cdot E_i\leq 0$ for all i. The anti-nef elements in H form a semigroup $\mathcal{E}$ and one writes $\mathcal{E}^+$ for $\mathcal{E}\setminus\{0\}$.

If A is anti-nef, then $A\geq 0$. Indeed, write $A=A_+-A_-$ with A_+, A_- non-negative cycles with no components in common. Then $0\leq -A\cdot A_-=A_-^2-A_+\cdot A_-\leq A_-^2$, so by negative definiteness $A_-=0$. Furthermore, if $A\in\mathcal{E}^+$, then $A\geq E$, where $E=\sum E_i$ is the reduced exceptional cycle. Indeed, if the support of A is not the whole of E, then there exists an E_i intersecting A strict positively, as $A>0$, and E is connected.

Definition 1.3. Given two cycles $A=\sum a_iE_i$, $B=\sum b_iE_i$, their *infimum* is the cycle $\inf(A,B)=\sum c_iE_i$ with $c_i=\min(a_i,b_i)$ for all i. This definition extends to subsets of $\mathcal{E}$.

Lemma 1.4. *Let $\mathcal{W}\subset\mathcal{E}^+$ be a subset. Then* $\inf\,\mathcal{W}\in\mathcal{E}^+$.

Proof. Let $W = \inf \mathcal{W}$. Fix an i and choose $A \in \mathcal{W}$ with a_i minimal. Then

$$0 \geq E_i \cdot A = E_i \cdot (A - W) + E_i \cdot W \geq E_i \cdot W,$$

as $A - W \geq 0$ with coefficient 0 at E_i. So $W \cdot E_i \leq 0$ for all i. As $A \geq E$ for all $A \in \mathcal{W}$, also $W \geq E > 0$. □

Definition 1.5. The *fundamental cycle* Z is the cycle $\inf \mathcal{E}^+$.

In other words, the cycle Z is the smallest cycle such that $E_i \cdot Z \leq 0$ for all i. It can be computed with a *computation sequence* [8]. Start with any cycle Z_0 known to satisfy $Z_0 \leq Z$; one such cycle is E. Let Z_k be computed. If $Z_k \neq Z$, then there is an $E_{j(k)}$ with $Z_k \cdot E_{j(k)} > 0$. Define $Z_{k+1} = Z_k + E_{j(k)}$. Then $(Z - Z_k) \cdot E_{j(k)} < 0$, so $E_{j(k)}$ lies in the support of $Z - Z_k$, giving $E_{j(k)} \leq Z - Z_k$. Therefore $Z_{k+1} \leq Z$.

The fundamental cycle depends of course on the chosen resolution, but in an easily controlled way. Therefore it can be used to define invariants of the singularity [17].

Let $\sigma\colon M' \to M$ be the blow-up in a point of E, with exceptional divisor E_0'. The exceptional divisor of $M' \to X$ is $E' = E_0' + \sum_{i=1}^{r} E_i'$, where the E_i', $i \geq 1$ are mapped onto the E_i. For a cycle $A = \sum a_i E_i$ on M the pull-back $\sigma^* A$ is defined as

$$\sigma^* A = a_0 E_0' + A^{\#} \ , \qquad \text{where } A^{\#} = \sum_{i=1}^{r} a_i E_i' \text{ and } E_0' \cdot \sigma^* A = 0 \ .$$

In fact, a_0 is the multiplicity of A in the point blown up. The main property of the intersection product in this connection is that $\sigma^* A \cdot \sigma^* B = A \cdot B$. This product is then also equal to $\sigma^* A \cdot B^{\#}$.

The canonical cycle on M' satisfies $K' = \sigma^* K + E_0'$. This gives that

$$\sigma^* A \cdot K' = \sigma^* A \cdot (\sigma^* K + E_0') = \sigma^* A \cdot \sigma^* K = A \cdot K$$

and therefore $p_a(\sigma^* A) = p_a(A)$.

Lemma 1.6. *The fundamental cycle Z' on M' is $\sigma^* Z$, the pull back of the fundamental cycle on M.*

Proof. One has $E_0' \cdot Z' = 0$, for otherwise $Z' - E_0'$ is anti-nef. Therefore $Z' = \sigma^* Y$ for some cycle Y and $Y \cdot E_i = \sigma^* Y \cdot \sigma^* E_i = Z' \cdot E_i' \leq 0$, so $Z \leq Y$. On the other hand, $\sigma^* Z \in \mathcal{E}'$, so $\sigma^* Y = Z' \leq \sigma^* Z$. □

Corollary 1.7. *The genus $p_a(Z)$ and degree $-Z^2$ of the fundamental cycle are invariants of the singularity.*

Definition 1.8. The *fundamental genus* of a singularity is the genus $p_a(Z)$ of the fundamental cycle.

A singularity has also an arithmetic genus [17] (the largest value of $p_a(D)$ over all effective cycles D), but this is a less interesting invariant. More important is the geometric genus, which is $h^1(\mathcal{O}_M)$, and also the largest value of $h^1(\mathcal{O}_D)$ over all effective cycles D.

Rational singularities were introduced by Artin [1] using the geometric genus of singularities. He proved the following characterisation, which we take as definition.

Definition 1.9. A normal surface singularity is *rational* if its fundamental genus $p_a(Z)$ is equal to 0.

Artin also proves that the degree $-Z^2$ of the fundamental cycle is equal to the multiplicity m of the singularity. The embedding dimension of X is $m+1$, which is maximal for normal surface singularities of multiplicity m.

Theorem 1.10 (Laufer's rationality criterion)**.** *A resolution graph represents a rational singularity if and only if*

- *each vertex E_i has $p_a(E_i) = 0$,*
- *if a cycle Z_k occurs in a computation sequence and if $Z_k \cdot E_i > 0$, then $Z_k \cdot E_i = 1$.*

For the 'if'-direction it suffices to have the second property for the steps in one computation sequence, starting from a single vertex. The criterion follows from the fact that the genus cannot decrease in a computation sequence, as $p_a(Z_k + E_i) = p_a(Z_k) + p_a(E_i) + Z_k \cdot E_i - 1$.

All irreducible components of the exceptional set have to be smooth rational curves, pairwise intersecting transversally in at most one point. This shows that minimal resolution of a rational singularity is a good resolution.

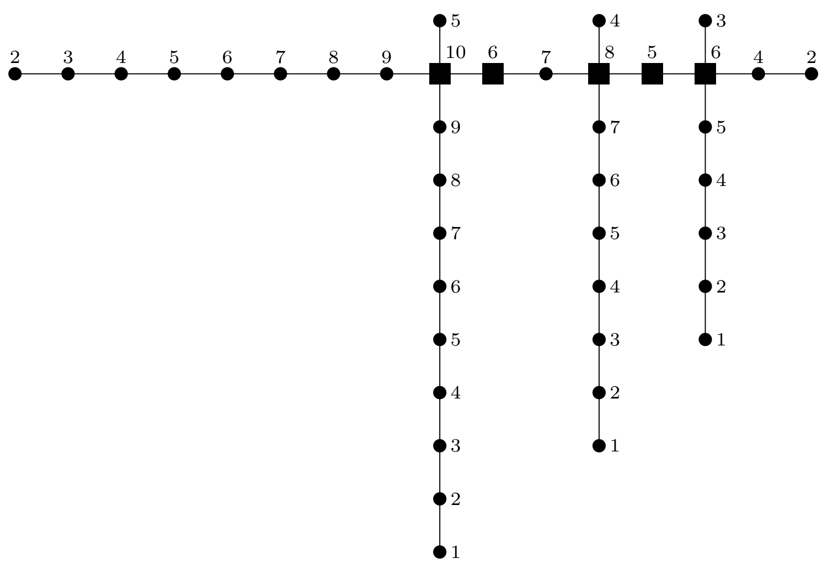
Following Lê–Tosun [10] we call the minimal resolution graph of a rational singularity a *rational graph*. It can be characterised combinatorically as weighted tree (with only vertex weights $-b_i \le -2$), representing a negative definite quadratic form, such that the genus of the fundamental cycle is 0.

The main invariant of a rational graph is its *degree* $-Z^2$. It is related to the *canonical degree* $Z \cdot K$ by $-Z^2 = Z \cdot K + 2$, as $p_a(Z) = 0$. Let $Z = \sum z_i E_i$, $-b_i = E_i^2$. Then

$$Z \cdot K = \sum z_i (b_i - 2) \; .$$

So the degree is determined by the coefficients z_i of the fundamental cycle at non-(-2)-vertices E_i.

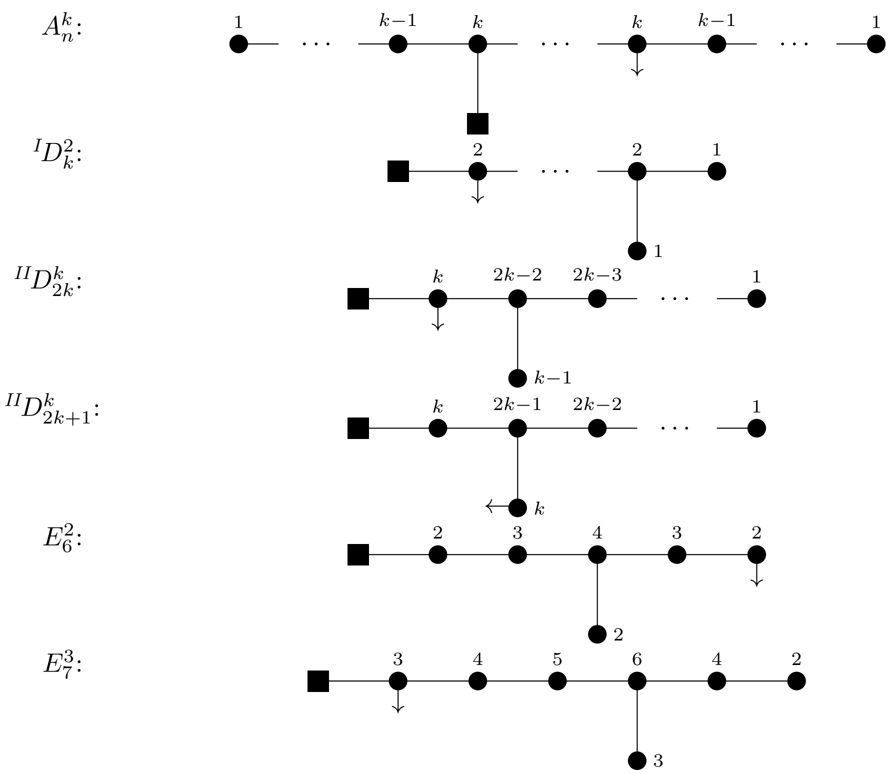
As example of a rational graph we show the one (of degree 37) occurring in the paper of Karras [6]. Every ■ is a (-3)-vertex. The numbers are the coefficients of the fundamental cycle.



2. Almost reduced fundamental cycle

As the lists in the classification become unwieldy, we first treat a simple special case, where only (-2) vertices can have higher multiplicity in the fundamental cycle. Its classification is contained in the thesis of Röhr [13] as part of more general results. The explicit list (Tables 1, 2 and 3) of graphs of RDP-configurations can be found with Gustavsen [4].

TABLE 1. RDP-configurations, attached to one curve



Definition 2.1 ([9]). A rational singularity has an *almost reduced* fundamental cycle if the fundamental cycle $Z = \sum z_i E_i$ on the minimal resolution is reduced at the non-(-2)'s, i.e., $z_i = 1$ if $b_i > 2$.

We also talk about rational graphs with almost reduced fundamental cycle.

One can compute the fundamental cycle starting from the reduced exceptional cycle by only adding curves occurring in rational double point configurations. The computation can be done for each configuration separately. Therefore we start with these configurations.

Theorem 2.2. *A maximal rational double point configuration on a rational graph with almost reduced fundamental cycle occurs in Tables 1, 2 or 3.*

Proof. By rationality at most one vertex in a rational double point configuration can have valency three in the resolution graph. Furthermore, a non-(-2) can only be attached to a vertex with multiplicity one in the fundamental cycle of the rational double point. One then computes for a graph satisfying these restrictions the fundamental cycle. The lists show that all possibilities occur. □

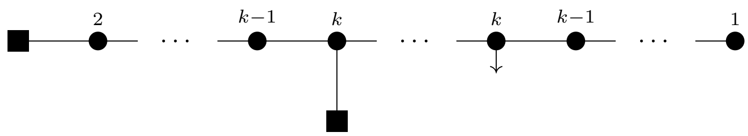
Remark 2.3. The list of configurations attached to two curves is obtained from the list of Table 1 by replacing a vertex with multiplicity one by a non-(-2).

The numbers on the graphs in the Tables indicate the coefficients in the fundamental cycle. The squares are not part of the configuration, but stand for the non-(-2)'s, to which the configuration is attached. The arrow indicates the curve which intersects the fundamental cycle strict negatively.

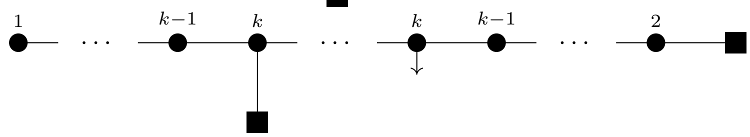
TABLE 2. RDP-configurations, attached to two curves

$A_n^{1,1}$:

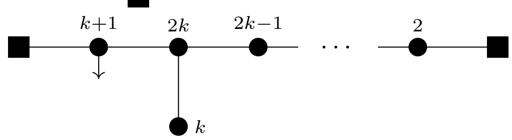
${}^{I}A_n^{2,k}$:



${}^{II}A_n^{k,2}$:



$D_{2k+1}^{k+1,2}$:



$D_{2k}^{k,2}$:

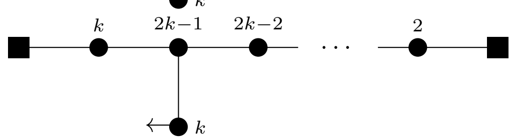
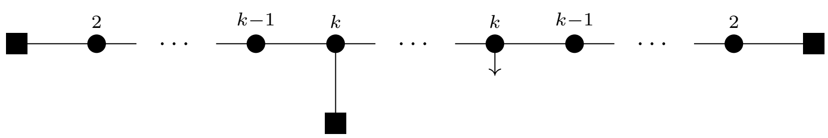


TABLE 3. RDP-configurations, attached to three curves

$A_n^{2,k,2}$:

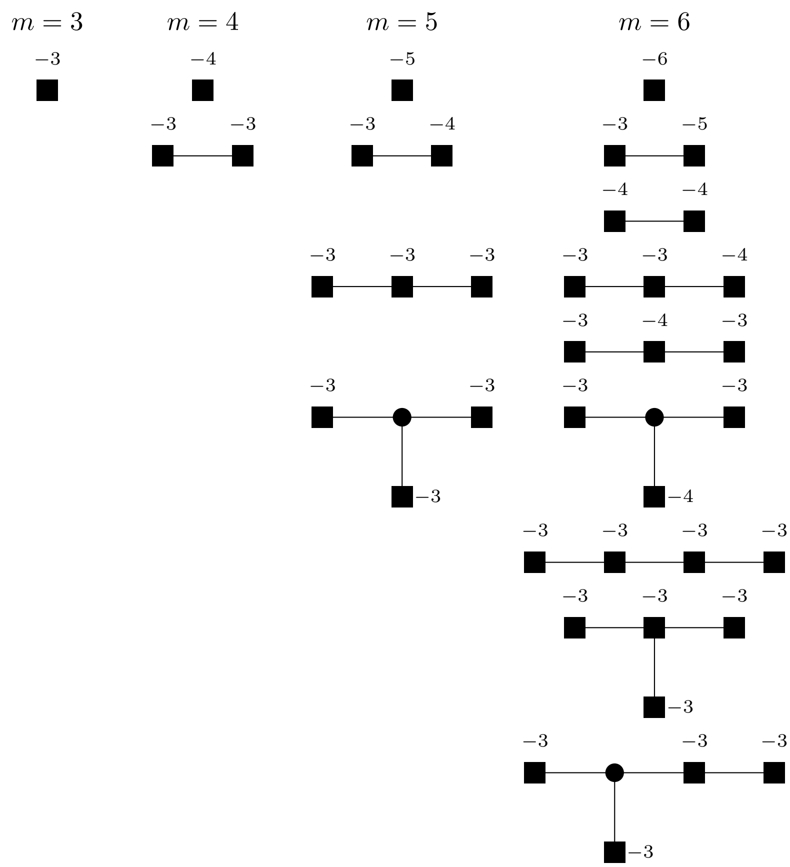


Our notation is a combination of that in [14] and Gustavsen's naming scheme [4], which is based on that of De Jong [5], who gave the list of Table 1, of configurations attached to only one curve. Our ${}^{I}D_k^2$ is called D_k^{I} there. Our upper indices give the multiplicity at the vertices, which are connected to non-(-2)'s. For the D-cases we could do without the upper left I or II, except that D_5^2 can have two meanings.

By blowing down all RDP-configurations on the minimal resolution $M \to X$ one obtains the *canonical model*, or *RDP-resolution*, $\hat{X} \to X$. The only singularities of $\hat{X}$ are rational double points. The reduced exceptional set has two types of singularities, normal crossing of two curves, and three curves intersecting transversally in one point. The last case occurs for an $A_n^{2,k,2}$ -configuration. Again one can form a dual graph $\hat{\Gamma}$, which in this case is a hypertree with edges for the normal crossing points and T-joints for three curves meeting in one point. The canonical model does not determine the multiplicities of the fundamental cycle on the minimal resolution. Therefore we add this multiplicity as second weight (we do not write the weight if it is equal to 1).

We want to draw ordinary graphs. Observe that given a hypertree $\hat{\Gamma}$ for a canonical model, there exists a smallest ordinary tree (i.e., having minimal number of vertices) giving rise to this hypertree: one replaces each T-joint by an $A_1^{2,2,2}$-configuration, i.e., by a single (-2)-vertex.

Table 4. Minimal representatives up to degree 6



In Table 4 we list the graphs of the minimal representatives up to degree $m = 6$. Such a graph has to have an almost reduced fundamental cycle. The necessary and sufficient condition is that for a non-(-2) vertex E_i the sum of its valency $v(i)$ and the number of (-2)'s attached to it, is at most b_i.

Classification (of graphs with almost reduced fundamental cycle)**.** *First classify all hypergraphs of RDP-resolutions with all multiplicities equal to 1, and canonical degree $\sum(b_i - 2) = m - 2$. Each hypertree with b_i at least the valency of E_i occurs. Let then $\hat{\Gamma}$ be such a hypergraph. Replace a T-joint by an $A_n^{2,k,2}$ configuration, replace any number of edges by configurations from Table 2 and attach configurations from Table 1 to vertices, in such a way that the total multiplicity in the fundamental cycle of the neighbours of any vertex v_i does not exceed b_i. The resulting graph is a rational graph with almost reduced fundamental cycle, and all graphs can be obtained this way.*

3. Complexity

Lê and Tosun [10] used the number of rupture points (i.e., vertices with valency at least three, stars in the terminology of [7]) as a measure of the complexity of a rational graph. They showed that it is bounded in terms of the degree $m = -Z^2$ of the graph (that is, the multiplicity of a corresponding rational singularity), more precisely by $m - 2$, if the degree m is at least 3. We give here a simplified proof for a sharpened version. It shows that the most complex graphs are already obtained from singularities with reduced fundamental cycle.

Definition 3.1. The *complexity* of a rational graph is the weighted number of rupture points, where each rupture point is counted with its valency minus two as multiplicity.

Theorem 3.2. *The complexity of a rational graph of degree m at least 3 is at most its canonical degree $m-2$.*

The proof uses the following observation [10, Thm. 8].

Lemma 3.3. *The graph, obtained from a rational graph, by making some vertex weights more negative, is again rational and the fundamental cycle of the new graph is reduced at the changed vertices.*

Proof. We can obtain the new graph as subgraph of the graph of the resolution of the original singularity, blown up in smooth points of the relevant exceptional curves. Its fundamental cycle can be computed by first computing the fundamental cycle of the subgraph. By Laufer's rationality criterion the remaining curves intersect this cycle with multiplicity one. □

Proof of Theorem 3.2. Step 1: reduction to the case of almost reduced fundamental cycle. Consider the cycle Y, which has multiplicity 1 at the non-(-2)'s and multiplicities on the RDP-configurations as in Tables 1, 2 and 3. A vertex E_i with $E_i \cdot Y > 0$ is a non-(-2) and has coefficient $z_i > 1$ in the fundamental cycle. For those E_i we increase b_i by one. By the previous lemma we get the same underlying graph with the same complexity, but with almost reduced fundamental cycle, namely Y. The contribution of E_i to the canonical degree $Z \cdot K$ changes from $z_i(b_i-2)$ to $b_i - 1$ and $(b_i-1) - z_i(b_i-2) = 1 - (z_i-1)(b_i-2) \leq 0$ with equality if and only if $z_i = 2$ and $b_i = 3$. So the degree does not increase.

Step 2: reduction to the case of reduced fundamental cycle. Consider a RDP-configuration, where Z is not reduced. Make the self-intersection of the unique rupture point in the configuration into -3. This increases the canonical degree by 1. For all non-(-2)'s E_j to which the configuration is connected we increase the self-intersection by 1 (decrease b_j by 1). This decreases the canonical degree by at least 1 (here we use that $m > 2$). If b_j was equal to 3, then E_j might be connected to at most one other RDP-configuration, but without rupture point. The result is a longer chain of (-2)'s. Proceeding in this way we obtain without increasing the degree the same underlying graph, but with reduced fundamental cycle.

Step 3. For a graph with reduced fundamental cycle the valency of a vertex is at most b_i. So the complexity is bounded by $\sum(b_i - 2) = Z \cdot K$. □

4. The format of a rational singularity

If a singularity is not a hypersurface, its equations can be written in many ways, some of which have a special meaning. The standard example is the cone over the rational normal curve of degree four, whose equations are the minors of

$$\begin{pmatrix} z_0 & z_1 & z_2 & z_3 \\ z_1 & z_2 & z_3 & z_4 \end{pmatrix} ,$$

but also the 2×2 minors of the symmetric matrix

$$\begin{pmatrix} z_0 & z_1 & z_2 \\ z_1 & z_2 & z_3 \\ z_2 & z_3 & z_4 \end{pmatrix} .$$

In fact, perturbing these matrices gives two different ways of deforming the singularity, leading to the two components of the versal deformation. We say that we can write the total spaces in a determinantal format. In a naive interpretation a format is a way of writing or coding (efficiently) the equations of a singularity. Another point of view is that we have a high-dimensional variety (like the generic determinantal), from which the singularity is derived by specialising the equations. This will lead us to the definition of a format, given by Ancus Röhr [13]. We start with:

Definition 4.1 ([2])**.** Let $Y \subset \mathbb{C}^N$ be a singularity. A germ $X \subset \mathbb{C}^M$ is a *singularity of type* Y, if there exists a map $\phi\colon \mathbb{C}^M \to \mathbb{C}^N$, such that $\phi^*(Y) = X$, which induces a complete intersection morphism $\phi\colon X \to Y$.

For a singularity X of minimal multiplicity (in particular, for a rational surface singularity) of multiplicity at least 3 the existence of a complete intersection morphism $X \to Y$ already implies that X is of type Y [13, 2.4.2]. The singularity Y has the same minimal multiplicity. Indeed, X is cut out by equations with independent linear part, for otherwise the multiplicity increases.

Deformations of type Y of X are obtained by unfolding the map ϕ: for every map

$$\Phi\colon \mathbb{C}^M \times (S,0) \to \mathbb{C}^N,$$

extending ϕ, the map $\pi\colon \Phi^*Y \to (S,0)$ is flat [2, 4.3.4]. In general such deformations will not fill out a component of the deformation space, but one can turn the problem around and start from the total space of the deformation over a smooth component. This total space is then rigid [15, p. 101].

A rational singularity has always a smoothing component with smooth base space. This is the *Artin component*, over which simultaneous resolution exists after base change. This simultaneous resolution is a versal deformation of the resolution M of X. A base change is not needed, if one considers instead deformations of the canonical model $\hat{X} \to X$.

We therefore concentrate on the Artin component. As it is smooth, the singularity X itself is cut out by a regular sequence from the total space Y of the deformation over the Artin component. Therefore the singularity is of type Y. By a result of Ephraim [3] one can write every reduced singularity Y in a unique way (up to isomorphism) as product of a singularity F and a smooth germ of maximal dimension.

Definition 4.2 ([13])**.** The *format* $F(X)$ of a rational surface singularity X is the unique germ F in a decomposition $Y = F \times \mathbb{C}^k$, with k maximal, of the total space Y over the Artin component of X.

Let $\hat{\pi}\colon (\hat{X}, \hat{Z}) \to (X,p)$ be the RDP-resolution of a rational singularity X of multiplicity m; it can be obtained by blowing up a canonical ideal. It gives an embedding of $\hat{X} \hookrightarrow \mathbb{P}^{m-2}_X$ over X and with it an embedding of the exceptional set $\hat{Z} = \hat{\pi}^{-1}(p)$ in $\mathbb{P}^{m-2}$, as arithmetically Cohen-Macaulay scheme of genus 0 and degree $m-2$ [13, 2.6.3]. Röhr calls the cone over $\hat{Z}$ the *canonical cone* of X. One can also obtain $\hat{Z}$ by blowing up a canonical ideal of F. This implies that the canonical cone of a rational surface singularity is determined up to isomorphy by its format. We conjectured that the converse also holds. This would imply that the singularities in Remark 6.8 have the same format.

Röhr proves that quasi-determinantal singularities can be recognised from the resolution graph [13, Satz 4.2.1]. The condition is that the graph contains the graph of a cyclic quotient singularity of the same multiplicity. Equivalently one can say that the graph $\hat{\Gamma}$ of the canonical model is a chain, with everywhere multiplicity 1. The proof is based on a criterion for RDP-configurations to be deformed on the resolution without changing the format [13, Satz 3.3.1]. This criterion also applies to rational singularities with almost reduced fundamental cycle: all RDP-configurations can be deformed away, except $A_1^{2,2,2}$. The graph of the resulting singularity is the minimal tree for the given hypertree $\hat{\Gamma}$. Note that the canonical cone can have moduli, so also the formats. The graph can therefore at most determine an equisingularity class of formats.

5. Computation of the fundamental cycle

In this section we describe, following Röhr [13, 1.3], a special way to compute the fundamental cycle, for a given rational graph. We single out a vertex E_0, which we call *central vertex*. The computation is done in steps, where each time the multiplicity at E_0 is increased by one.

We decompose the complement of a vertex E_0 in a rational graph Γ in irreducible components: $\Gamma \setminus \{E_0\} = \cup_{i=1}^k \Gamma_i$. We suppose that $k > 1$; the case $k = 1$ can be reduced to it by blowing up a point of the curve E_0.

We construct the fundamental cycle inductively. To start with, let $E_0 + Y_i^{(1)}$ be the fundamental cycle on $\{E_0\} \cup \Gamma_i$; as $k > 1$, the support of $Y_i^{(1)}$ is Γ_i: one can compute Z starting from $E_0 + Y_i^{(1)}$, so the coefficient at E_0 is one. Define $Z^{(1)} = E_0 + \sum Y_i^{(1)}$. Then $Z^{(1)} \cdot E_j \leq 0$ for all $j \neq 0$.

Let $Z^{(s)}$ be constructed with $Z^{(s)} \cdot E_j \leq 0$ for all $j \neq 0$, with coefficient s at E_0 and satisfying $Z^{(s)} \leq Z$. If $Z^{(s)} \cdot E_0 \leq 0$, then $Z^{(s)}$ is the fundamental cycle Z. Otherwise, consider the set of vertices $E_{i,j} \in \Gamma_i$ such that $Z^{(s)} \cdot E_{i,j} = 0$ and let $\Gamma_i^{(s+1)}$ be the connected component of this set adjacent to E_0. Let $E_0 + Y_i^{(s+1)}$ be the fundamental cycle on $\{E_0\} \cup \Gamma_i^{(s+1)}$. As $Y_i^{(s+1)} \leq Y_i^{(1)}$, the support of $Y_i^{(s+1)}$ does not contain E_0. Now define

$$Z^{(s+1)} = Z^{(s)} + E_0 + \sum Y_i^{(s)} .$$

Then $Z^{(s+1)} \cdot E_j \leq 0$ for all $j \neq 0$, the coefficient at E_0 is $s+1$ and $Z^{(s+1)} \leq Z$; indeed $Z^{(s+1)}$ can be constructed from $Z^{(s)}$ by first adding E_0 and then continuing in the manner of a computation sequence without ever adding E_0 again.

This construction ends with the fundamental cycle.

If $k = 1$, we blow up a point of the curve E_0, introducing a Γ_2. But this can be avoided, as in fact the same description as above holds, with the only difference that for $k = 1$ the cycle $E_0 + Y_1^{(s)}$ is not the fundamental cycle on $\{E_0\} \cup \Gamma_1^{(s)}$ (in particular, $E_0 + Y_1^{(1)}$ is not the fundamental cycle on Γ), but $Y_1^{(s)}$ is the cycle constructed from $Z^{(s-1)} + E_0$ in the manner of a computation sequence without ever adding E_0.

Let $m_i^{(s)} \leq m_i^{(1)}$ be the coefficient of $Y_i^{(s)}$ at the vertex in Γ_i adjacent to E_0. As $E_0 \cdot Z^{(s)} = 1$ for $s < l$, where $Z^{(l)} = Z$ is the last step of the computation, we have $\sum_i m_i^{(1)} = b_0 + 1$, $\sum_i m_i^{(s)} = b_0$ for $1 < s < l$ and $\sum_i m_i^{(l)} < b_0$.

Example 5.1. Consider an E_6-configuration, connected to a non-(-2) vertex E_0. We compute the $Z^{(s)}$. We only write the multiplicities of E_0 (in boldface) and of the irreducible components of the configuration.

$$\begin{array}{cccccc} \mathbf{1} & 2 & 3 & 4 & 3 & 2 \\ & & & 2 & & \end{array} \qquad \begin{array}{cccccc} \mathbf{2} & 4 & 5 & 6 & 4 & 2 \\ & & & 3 & & \end{array} \qquad \begin{array}{cccccc} \mathbf{3} & 4 & 5 & 6 & 4 & 2 \\ & & & 3 & & \end{array}$$

The sequence $(m_1^{(s)})$ is $(2, 2, 0)$ and therefore an E_6-configuration can only be connected to a curve with multiplicity at most 3. We observe that the same sequence can be obtained from $2A_2^1$, two chains of length two.

6. One non-reduced curve

The goal of this section is to give the elements for the classification of rational graphs, where each RDP-configuration is attached to at most one non-reduced non-(-2)-vertex. We first classify the possible multiplicities at RDP-configurations. These depend only on the multiplicity of the non-(-2), and the computation can again be done for each configuration separately. The candidates for graphs of RDP-resolutions can be found from the graphs with almost reduced fundamental cycle, but not every candidate arises from a rational graph.

Let E_0 be a non-reduced non-(-2), with multiplicity z_0 in the fundamental cycle. According to the previous section, we can compute the fundamental cycle in z_0 steps, each time increasing the multiplicity of E_0 by one. We add cycles with support on the subgraphs Γ_i and each subgraph gives a multiplicity sequence $(m_i^{(1)}, \dots, m_i^{(z_0)})$. These multiplicities satisfy

$$\sum_i m_i^{(1)} = b_0 + 1, \qquad \sum_i m_i^{(s)} = b_0 \text{ for } 1 < s < z_0, \qquad \sum_i m_i^{(z_0)} < b_0.$$

After the first step we add only cycles with support in RDP-configurations intersecting E_0, as all other non-(-2)-curves, intersecting such configurations, have multiplicity one. Each Γ_i contains at most one RDP-configurations adjacent E_0. We include the case that there is no such configuration by calling it $A_0^{1,1}$.

For each RDP-configuration from Tables 1, 2 and 3 we compute the *multiplicity sequence* $(m^{(1)}, \dots, m^{(j)})$. The multiplicities satisfy $m^{(1)} - 1 \le m^{(s)} \le m^{(1)}$ for all $s < j$. We abbreviate a sequence $k, \dots, k$ of l equal multiplicities as k^l. An exponent $l = 0$ means that this factor is absent. If the sequence is infinite, and repeating itself, we underline the repeating section. So in Table 6 the entry $(1^{n+1}, \underline{0, 1^n})$ for $LA_n^{1,1}$ should be read as $(1^{n+1}, 0, 1^n, 0, 1^n, 0, \dots)$. The case $n = 0$, of two non-(-2)'s intersecting each other, is included. The sequence is then $(1, 0, 0, \dots)$.

For configurations between several vertices only one of the non-(-2)'s has higher multiplicity, and we suppose that the other ones have sufficiently negative self-intersection for the graph being rational.

We have to distinguish which of the two or three attached vertices is the non-reduced one. We always draw the graphs as in Tables 2 and 3. In a graph of type ${}^{I}A_n^{2,k}$, ${}^{II}A_n^{k,2}$ or $A_n^{2,k,2}$ the arrowhead (which indicates the curve intersecting the fundamental cycle of the extended configuration negatively) is on the right hand side of the graph. So it makes sense to distinguish between the left, middle or right attached vertex. We denote this by writing an L, M or R before the name. For type D we use L and R.

It is possible to obtain a multiplicity sequence from different configurations or combinations of configurations. We then speak about equivalent configurations. For each configuration we also determine the simplest equivalent combination of configurations.

Proposition 6.1. *The multiplicity sequences and the equivalent configurations for the configurations of Table 1 are as given in Table 5. The different cases arising from the configurations of Table 2 are in Table 6; it gives also the multiplicity at the component attached to the other, reduced non-(-2). If the sequence is infinite, the multiplicity after step s of the computation is given. Table 7 gives the results for $A_n^{2,k,2}$.*

Proof. We do here only the case A_n^k, for $k > 1$, as the other cases involve similar or easier computations. We write $n = lk + r + (k-1)$ with $l \ge 1$ and $0 \le r \le k-1$. This is possible as the number n satisfies $n \ge 2k - 1$. There is a chain of $lk + r - (k-1) = (l-1)k + r + 1$ (-2)-vertices with multiplicity k in $Z^{(1)}$, and the end of this chain not intersecting E_0 intersects $Z^{(1)}$ negatively (when $l = 1$ and $r = 0$ there is only one vertex with multiplicity k; in this case the multiplicity sequence is $(k, 0)$ and the format is kA_1^1, in accordance with the general

TABLE 5.

name	mult sequence	equivalent to
A^1_n	$(\underline{1^n, 0})$	
$A^k_{(l+1)k+r-1},\ r < k-1$	(k^l, r)	$(k-r)A^1_l + rA^1_{l+1}$
A^2_{2l+2}	$(2^l, 1, 1, 2^l, 0)$	
$A^k_{(l+2)k-2},\ k > 2$	$(k^l, k-1, 1)$	$A^1_l + (k-1)A^1_{l+1}$
${}^{I}D^2_k$	$(2, 0)$	$2A^1_1$
${}^{II}D^k_{2k},\ k > 2$	$(k, 0)$	kA^1_1
${}^{II}D^2_5$	$(2, 1, 2, 0)$	$A^1_1 + A^1_3$
${}^{II}D^k_{2k+1},\ k > 2$	$(k, 1)$	$(k-1)A^1_1 + A^1_2$
E^2_6	$(2, 2, 0)$	$2A^1_2$
E^3_7	$(3, 0)$	$3A^1_1$

TABLE 6.

name	mult sequence	other mult	equivalent configuration
$LA^{1,1}_n$	$(1^{n+1}, \underline{0, 1^n})$	$\lceil \frac{n+s}{n+1} \rceil$	
$L^IA^{2,k}_n$	$(2, 1^{n-k}, 0)$	$n-k+2$	$LA^{1,1}_0 + A^1_{n-k+1}$
$M^IA^{2,k}_{(l+1)(k-1)+r}$	$(k, (k-1)^{l-1}, r)$	$\lceil \frac{(l+1)(k-1)+r}{k-1} \rceil$	$LA^{1,1}_0 + (k-1-r)A^1_l + rA^1_{l+1}$
$k > 2$			
$M^{II}A^{k,2}_{(l+1)k+r-2}$	(k^l, r)	2	$LA^{1,1}_{l-1} + rA^1_{l+1} + (k-1-r)A^1_l$
$0 \le r < k-1$			
$M^{II}A^{k,2}_{(l+1)k+k-3}$	$(k^l, k-1, 1)$	3	$LA^{1,1}_{l-1} + (k-1)A^1_{l+1}$
$k > 2,\ l > 1$			
$M^{II}A^{k,2}_{3k-3}$	$(k, k-1, 1)$	3	$LA^{1,1}_1 + A^1_1 + (k-2)A^1_2$
$k > 2$			
$M^{II}A^{2,2}_{2l+1}$	$(2^l, \underline{1^2, 2^{l-1}})$	$\lceil \frac{l+1+s}{l+1} \rceil$	$LA^{1,1}_l + A^1_l$
$R^{II}A^{k,2}_n$	$(2, 1^{k-2}, 0)$	k	$LA^{1,1}_0 + A^1_{k-1}$
$k > 2$			
$R^{II}A^{2,2}_n$	$(2, 0)$	2	$LA^{1,1}_0 + A^1_1$
$LD^{k+1,2}_{2k+1}$	$(k+1, 0)$	2	$LA^{1,1}_0 + kA^1_1$
$RD^{k+1,2}_{2k+1}$	$(2, 1, 1, \dots)$	$\lceil \frac{2k+s}{2} \rceil$	$LA^{1,1}_1 + A^1_1$
$LD^{k,2}_{2k}$	$(k, 1)$	3	$(k-2)A^1_1 + A^1_2 + LA^{1,1}_0$
$k > 2$			
$RD^{k,2}_{2k}$	$(2, 1, 1, \dots)$	$\lfloor \frac{2k+s}{2} \rfloor$	$LA^{1,1}_1 + A^1_1$

formula). The set $\Gamma^{(2)}$ consists of $(l-1)k + r + (k-1)$ vertices. If $l > 1$ this number is at least $2k-1$ and the multiplicities in $Z^{(2)}$ are

$$2, 4, \dots, 2k, 2k, \dots, 2k, 2k-1, 2k-2, \dots, 2, 1 .$$

Table 7.

name	mult seq	mult at L	mult at M	mult at R
$LA_n^{2,k,2}$	$(2,1^{n-k+1},0)$		$n-k+3$	2
$MA_n^{2,k,2}$	$(k,(k-1)^{l-1},r)$	$\lceil\frac{n+1}{k-1}\rceil$		2
$RA_n^{2,k,2}$	$(2,1^{k-2},0)$	2	k	

Here $n=(l+1)(k-1)-1+r$ with $0\le r\le k-2$ and $k>2$.

There are $(l-2)k+r+1$ vertices with multiplicity $2k$ in $Z^{(2)}$. We continue in this way until there are $r+1$ vertices with multiplicity lk in $Z^{(l)}$; all multiplicities are then

$$l,2l,\dots,lk,lk,\dots,lk,lk-1,lk-2,\dots,2,1\ .$$

The set $\Gamma^{(l+1)}$ consists of $r+(k-1)$ vertices (except when $r=0$; then $\Gamma^{(l+1)}$ is empty). We therefore add the multiplicities

$$1,2,\dots,r-1,r,\dots,r,r-1,\dots,2,1,0,\dots,0\ .$$

If $r<k-1$ the sequence stops here, the multiplicity sequence is (k^l,r) and the equivalent configuration is $(k-r)A_l^1+rA_{l+1}^1$. If $r=k-1$ the multiplicities in $Z^{(l+1)}$ are

$$l+1,2(l+1),\dots,(k-1)(l+1),k(l+1)-1,k(l+1)-2,\dots,2,1\ .$$

We add the multiplicities $0,\dots,0,1,\dots,1$. If $k\ge3$ the sequence stops here, the multiplicity sequence is $(k^l,k-1,1)$ and the configuration is equivalent to $A_l^1+(k-1)A_{l+1}^1$. If $k=2$, the sequence continues; as $\Gamma^{(l+3)}$ consists of $n-1$ nodes, the multiplicity sequence is $(2^l,1^2,2^l,0)$. There is no easier equivalent configuration for this A_{2l+2}^2. □

Remark 6.2. The condition $k>2$ in the tables is included to avoid duplications. For example, as $MA_n^{2,2,2}=LA_n^{2,2,2}$, we can assume that $k>2$ for $MA_n^{2,k,2}$.

Remark 6.3. Note that the tables give the maximal multiplicity sequence for each configuration. If the computation stops earlier (due to other configurations), one gets a simpler equivalent singularity.

Corollary 6.4. *Every RDP-configuration, attached to only one vertex, is equivalent to a combination of configurations of type A_n^1 and A_{2l}^2.*

Corollary 6.5. *An RDP-configuration, attached to two or three vertices, of which only one has multiplicity greater than one in the fundamental cycle, is equivalent to a combination of configurations of type A_n^1, A_{2l}^2, $LA_n^{1,1}$ and $LA_n^{2,2,2}$.*

Proof. Table 6 gives the result for configurations between two vertices.

From Table 7 we see that the multiplicities of $LA_n^{2,k,2}$ depend only on $n-k$, so $LA_n^{2,k,2}$ is equivalent to $LA_{n-k+2}^{2,2,2}$. The multiplicities of $RA_n^{2,k,2}$ depend only on k, so we can take the smallest n, which is $2k-3$. In that case the left and right chain of (-2)'s are equally long, so by interchanging L and R we obtain $LA_{2k-3}^{2,k,2}$, which is equivalent to $LA_{k-1}^{2,2,2}$.

For $MA_n^{2,k,2}$ we distinguish between the cases $r=0$ and $0<r\le k-2$. In the first case $\lceil\frac{n+1}{k-1}\rceil=l+1$, while $\lceil\frac{n+1}{k-1}\rceil=l+2$ in the second case. For $r=0$ an equivalent configuration, attached to the vertex v_M, is $MA_l^{2,2,2}+(k-2)A_l^1$ and, for $r>0$, is

$$MA_{l+1}^{2,2,2}+(k-1-r)A_l^1+(r-1)A_{l+1}^1.$$

Finally we note that interchanging M and L makes $MA_n^{2,2,2}$ into $LA_n^{2,2,2}$. □

From an arbitrary rational graph we obtain a graph with almost reduced fundamental cycle and the same underlying graph by making some vertex weights $-b_i < -2$ more negative. This process can also be inverted. The possible candidates for graphs (or hypergraphs) of RDP-resolutions with non-reduced fundamental cycle can be obtained from reduced (hyper)-graphs by replacing a $-(b_i+2)$-vertex by a $-(b_i/z_i+2)$-vertex with multiplicity z_i, but not all graphs can be realised.

Proposition 6.6. *On a rational graph with only one non-(-2) vertex E_0 the multiplicity of E_0 in the fundamental cycle can at most be* 6.

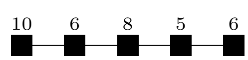
Proof. By Corollary 6.4 it suffices to consider only RDP-configurations of type A_n^1 and A_{2l}^2. If $z_0 > 2$, there is exactly one configuration Γ_i with $m_i^{(2)} = m_i^{(1)} - 1$, so it is either A_1^1 or A_4^2. In the last case $z_0 \le 5$, as A_4^2 gives the sequence $(2,1,1,2,0)$. Suppose now that there is exactly one A_1^1. If $z_0 > 3$, there is exactly one configuration Γ_i with $m_i^{(3)} = m_i^{(2)} - 1 = m_i^{(1)} - 1$, which is either A_2^1 or A_6^2. In the last case $z_0 = 4$, as we combine the sequences $(2,2,1,1,2,2,0)$ and $(1,0,1,0,\dots)$. The sequence of $A_1^1 + A_2^1$ is $(1+1, 0+1, 1+0, 0+1, 1+1, 0+0) = (2,1,1,1,2,0)$, which shows that $z_0 \le 6$. □

Remark 6.7. We realise $z_0 = 6$ for a (-3) with $A_1^1 + A_2^1 + A_4^1 + A_5^1$.

Remark 6.8. With A_4^2 and E_0 a (-3) we can realise $z_0 = 5$ with the configuration $A_4^2 + A_3^1 + A_4^1$. Another way to get $5E_0$ is with $A_1^1 + A_2^1 + 2A_4^1$. It would be interesting to study the formats of the corresponding singularities. We remark that neither is a deformation of the other.

Classification (of graphs, where each RDP-configuration is attached to at most one non-reduced non-(-2))**.** *Start by making a list of all possible hypergraphs $\hat{\Gamma}$ of canonical cones, without edges (or hyperedges) between non-reduced vertices. Given $\hat{\Gamma}$, realise this graph (if possible) in all ways, using only configurations A_n^1 and A_{2l}^2, $A_n^{1,1}$ (including $n = 0$) and $LA_n^{2,2,2}$. Replace (combinations of) RDP-configurations with equivalent ones, as given by the Tables 5, 6 and 7.*

Remark 6.9. The computations so far also can help to compute the fundamental cycle for complicated graphs. As example we return to Karras' graph, given at the end of Section 1. The graph for the canonical model is rather simple. Note also that only configurations of type A_n^1 occur.



We first simplify the graph. The configuration $A_1^1 + A_2^1$ at (-3) on the right implies that its multiplicity is at most 6. Therefore the A_5^1 has no influence on the computation, and we get the same multiplicities, if we remove it and increase the weight (-3) to (-2). We have then a A_4^2 attached to the (-3) of multiplicity 5. The (-3) on the left has multiplicity at most 10 because of $A_1^1 + A_9^1$. Again we can remove the A_9^1 and increase the weight (-3) to (-2). We have then a A_{10}^2 attached to the (-3) of multiplicity 6. By the same argument the A_7^1 at the vertex of multiplicity 8 can be removed, so that we end up with two (-3)-vertices E_1 and E_2 with a $A_3^{2,2}$ in between, an A_{10}^2 attached to E_1 and A_4^2 attached to E_2.

It remains to compute the fundamental cycle for this configuration. This is best done with the rupture point between E_1 and E_2 as central vertex. We give the total multiplicities at each step. The multiplicities of the (-3)'s are in bold face, while those of the central vertex are underlined.

$$\begin{array}{cccccccccccccccc} & & & & & & & & 1 & & & 1 & & 1 & & \\ 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & \mathbf{1} & 1 & \underline{1} & \mathbf{1} & 2 & 2 & 1 \end{array}$$

$$\begin{array}{cccccccccccccccc}
 &&&&&&&& 2 &&& 1 && 2 && \\
1 & 2 & 3 & 4 & 4 & 4 & 4 & 4 & 4 & \mathbf{2} & 2 & \underline{2} & \mathbf{2} & 3 & 2 & 1
\end{array}$$

$$\begin{array}{cccccccccccccccc}
 &&&&&&&& 3 &&& 2 && 2 && \\
1 & 2 & 3 & 4 & 5 & 6 & 6 & 6 & 6 & \mathbf{3} & 3 & \underline{3} & \mathbf{2} & 3 & 2 & 1
\end{array}$$

$$\begin{array}{cccccccccccccccc}
 &&&&&&&& 4 &&& 2 && 2 && \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 8 & \mathbf{4} & 4 & \underline{4} & \mathbf{3} & 4 & 3 & 2
\end{array}$$

$$\begin{array}{cccccccccccccccc}
 &&&&&&&& 5 &&& 3 && 2 && \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \mathbf{5} & 5 & \underline{5} & \mathbf{3} & 4 & 3 & 2
\end{array}$$

$$\begin{array}{cccccccccccccccc}
 &&&&&&&& 5 &&& 3 && 3 && \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \mathbf{5} & 6 & \underline{6} & \mathbf{4} & 6 & 4 & 2
\end{array}$$

$$\begin{array}{cccccccccccccccc}
 &&&&&&&& 5 &&& 4 && 3 && \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \mathbf{5} & 6 & \underline{7} & \mathbf{5} & 6 & 4 & 2
\end{array}$$

$$\begin{array}{cccccccccccccccc}
 &&&&&&&& 5 &&& 4 && 3 && \\
2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & \mathbf{6} & 7 & \underline{8} & \mathbf{5} & 6 & 4 & 2
\end{array}$$

Karras computes in a different way, as his goal is to find a smoothable subcycle.

7. RDP-configurations on general graphs

In this section we determine the maximal multiplicities that most can occur on an RDP-configuration. We continue to compute for each RDP-configuration separately. For some configurations the multiplicities can become arbitrary high, but what actually happens, depends on the rest of the graph. We do not investigate the exact conditions.

The results apply to the classification of graphs, in which two or three non-reduced non-(-2)'s are connected to each other by a single RDP-configuration, but not connected to any other non-reduced non-(-2). In particular, we determine the conditions that the multiplicity of the non-(-2)'s does not exceed two. This suffices to give a complete classification of rational graphs of degree 6. We indicate this in the next section.

We first treat configurations attached to exactly two vertices, both of higher multiplicity. Then there are two vertices E_a and E_b, of self-intersection $-a$ and $-b$, which are connected by a RDP-configuration Δ. The fundamental cycle $E_a + Z_\Delta + E_b$ on $\{E_a\} \cup \Delta \cup \{E_b\}$ is given in Table 2. Let $n_{\Delta,a}$ be the coefficient of the vertex of Δ, adjacent to E_a. Furthermore, let Γ_a be the union of the connected components of the complement of the graph, which are connected to E_a. Let $E_a + Z_a$ be the fundamental cycle on $\{E_a\} \cup \Gamma_a$, let n_a be the sum of the multiplicities of Z_a at the vertices of Γ_a, adjacent to E_a. Define the corresponding objects for E_b.

Definition 7.1. In the above situation E_a is a *bad vertex* if $n_{\Delta,a} + n_a = a + 1$.

We borrow the term *bad* from Tosun, see [10, Definition 3.4] and [16, Definition 3.14], where it is used without multiplicities: Tosun calls a vertex bad if its valency is one less then its vertex weight b. Karras [6] calls it a *basic center*. If $E_i \cdot Z_\Delta = 0$ for every vertex of Δ, then exactly one of E_a and E_b is a bad vertex (in our sense).

7.1. $A_n^{1,1}$**.** We call the two vertices E_L and E_R, and denote the numbers defined above correspondingly; the vertex weight of E_L is $-b_L$, and that of E_R is $-b_R$. Then $n_{\Delta,L} = n_{\Delta,R} = 1$ and there is exactly one bad vertex, which we suppose to be E_L. This means that $n_L = a$, and Γ_L is non-empty. We claim that the multiplicity of Δ in the fundamental cycle can be arbitrarily high. We compute the fundamental cycle with E_L as central vertex. We set $Y_L^{(1)} = Z_L$, $Y_{\Delta,R}^{(1)} = Z_\Delta + E_R + Z_R$. Then $Z^{(1)} = Y_L^{(1)} + E_L + Y_{\Delta,R}^{(1)}$, and E_L is the only vertex with $E_i \cdot Z^{(1)} = 1$. In each next step $Y_L^{(s)} \le Y_L^{(1)}$ and $Y_{\Delta,R}^{(s)} \le Y_{\Delta,R}^{(1)}$. In particular, the multiplicity of the fundamental cycle at E_R does not exceed that at E_L. We describe the case that the computation never stops. For the sum $n_L^{(s)}$ of multiplicities in Γ_L, adjacent to E_L, and the multiplicity $n_{R,\Delta}^{(s)}$ we have then either $n_L^{(s)} = a-1$ and $n_{\Delta,R}^{(s)} = 1$, or $n_L^{(s)} = a$ and $n_{\Delta,R}^{(s)} = 0$. As remarked earlier, we do not investigate the conditions which this assumption imposes on Γ_L and Γ_R.

Let $Z_\Delta = E_1 + \cdots + E_n$ with $E_1 \cdot E_L = 1$ and $E_n \cdot E_R = 1$. Suppose the coefficient of E_R in $Y_{\Delta,R}^{(s)}$ is 1, and the coefficient of E_R in $Z^{(s)}$ is k. If $E_R \cdot Z^{(s)} = -s_k < 0$, then $Y_{\Delta,R}^{(s+1)} = E_1 + \cdots + E_n$, $Y_{\Delta,R}^{(s+2)} = E_1 + \cdots + E_{n-1}$, ..., $y_{\Delta,R}^{(s+n)} = E_1$ and $Y_{\Delta,R}^{(s+n+1)} = \emptyset$. Then $E_R \cdot Z^{(s+n+1)} = -s_k + 1$. We continue by adding only cycles with support on Δ until E_R intersects the total computed cycle trivially. In the next step the coefficient of E_R in the added cycle will again be 1. At this stage the coefficients of the total cycle in the neighbourhood of Δ are as follows.

The coefficient of E_L is $s = k + (n+1)\sum s_i$, the sum of the $n_L^{(j)}$ is

$$(a-1)(n\sum s_i + k - 1) + a(1 + \sum s_i),$$

the coefficient of E_1 is $k + n\sum s_i$, that of E_t is $k + (n+1-t)\sum s_i$, that of E_n is $k + \sum s_i$, the coefficient of E_R is k, and the sum of the multiplicities of the vertices in Γ_R, adjacent to E_R, is $k(b-1) - \sum s_i$.

We remark that the formulas also work, if $n = 0$. This means that $\Delta = \emptyset$ and E_L is adjacent to E_R. Furthermore, if $\sum s_i = 0$, the multiplicities at E_L and E_R are independent of n.

7.2. ${}^{I}A_n^{2,k}$**.** In this case, and also for ${}^{II}A_n^{k,2}$ and $A_n^{2,k,2}$, it is more convenient to compute the fundamental cycle with the rupture point in the chain of (-2)'s as central vertex E_0. We therefore use a slightly different notation, consistent with the description of the computation in Section 5. Let $m_L^{(s)}$, $m_M^{(s)}$ and $m_R^{(s)}$ be the multiplicities in step s at the vertices directly to the left, below or to the right of the central vertex. The non-(-2) vertices are E_L with weight $-b_L$, and E_M with weight $-b_M$.

We have $m_L^{(1)} + m_M^{(1)} + m_R^{(1)} = 3$, $m_L^{(s)} + m_M^{(s)} + m_R^{(s)} \le 2$ for $s > 1$, and the computation stops at the first s where this sum is less than 2.

We start by computing the sequence $(m_R^{(s)})$. We apply Proposition 6.1: as we have a A^1_{n-k+1}-configuration, the sequence is $(1^{n-k+1}, 0, 1^{n-k+1}, 0, \dots)$. So $m_R^{(s)} = 1$ for $s \ne l(n-k+2)$ and $m_R^{(l(n-k+2))} = 0$ for all l.

Next we look at $(m_M^{(s)})$. Let Z_M be the fundamental cycle on the connected component Γ_M of $\Gamma \setminus \{E_0\}$, containing E_M. For the first step $Z^{(1)}$ of the computation we determine the fundamental cycle $Y_M^{(1)}$ on $\{E_0\} \cup \Gamma_M$: it is $E_0 + Z_M$. The condition that E_M is a bad vertex translates into $E_M \cdot Z_M = 1 - k$, so $E_M \cdot Z^{(1)} = 2 - k$. Therefore we put $E_M \cdot Z^{(1)} = 2 - k - t_1$, where $t_1 \ge 0$ with equality if and only if E_M is a bad vertex. In the next steps $Y_M^{(s)}$ is empty. We find that $E_M \cdot Z^{(k+t_1-1)} = 0$, so E_M is in the support of $Y_M^{(k+t_1)}$. We set $E_M \cdot Z^{(k+t_1)} = 2 - k - t_2$, with $t_2 \ge t_1$. Proceeding this way we find the sequence

$$(1, 0^{k+t_1-2}, 1, 0^{k+t_2-2}, 1, 0^{k+t_3-2}, \dots)\ .$$

On the left side $E_L \cdot Z^{(1)} = -s_1$ with $s_1 = 0$ if and only if E_L is a bad vertex. If $s_1 > 0$, then E_L is not contained in the support of $Y_L^{(2)}$, which is the A_{k-2}-configuration between E_L and E_0. We continue in the manner of A^1_{k-2}, until $E_L \cdot Z^{(s_1(k-1)+1)} = 0$ and E_L is in the support of $Y_L^{(s_1(k-1)+2)}$. Then $E_L \cdot Z^{(s_1(k-1)+2)} = -s_2$ with $s_2 \geq s_1$. The sequence is

$$(1, (1^{k-2}, 0)^{s_1}, 1, (1^{k-2}, 0)^{s_2}, 1, (1^{k-2}, 0)^{s_3}, \dots) .$$

Exactly one of E_L and E_M is a bad vertex. If $k = 2$, both E_L and E_M are connected to E_0, so upon relabeling we may assume that the bad vertex is E_L. We first treat the other case, that E_M is the bad vertex. Then $t_1 = 0$, and, as just said, we make the assumption that $k > 2$. To obtain a high multiplicity we need that $m_L^{(s)} + m_M^{(s)} = 1$ for $1 < s < n-k+2$, and $m_L^{(n-k+2)} + m_M^{(n-k+2)} = 2$. We achieve $m_L^{(s)} + m_M^{(s)} = 1$ for $s > 1$ by taking $t_1 = \dots = t_{s_1} = 0$, $t_{s_1+1} = 1$, $t_{s_1+2} = \dots = t_{s_1+s_2} = 0$, $t_{s_1+s_2+1} = 1$, $t_{s_1+s_2+2} = \dots = t_{s_1+s_2+s_3} = 0$, ... The only possibility to get $m_L^{(s)} = m_M^{(s)} = 1$ is by taking $t_{s_1+\dots+s_p+1} = 0$: this gives $s = p+\sum_{i=1}^p s_i(k-1)+k-1$. We therefore put $n-k+2 = p+\sum_{i=1}^p s_i(k-1)+r$ with $r < 1+s_{p+1}(k-1)$. If $r \neq k-1$, the computation stops with $s = n-k+2$. If $r = k-1$, we go one step further, as then $m_L^{(n-k+2)} = m_M^{(n-k+2)} = 1$ and $m_R^{(n-k+2)} = 0$, but $m_L^{(n-k+3)} = m_M^{(n-k+3)} = 0$. So the computation always stops.

Suppose now that E_L is the bad vertex; here $k = 2$ is allowed. In this case the computation need not end. We have $s_i = 0, 1$ for all i. As $m_M^{(p(k-1)+\sum t_i+1)} = 1$, we obtain $m_L^{(s)} + m_M^{(s)} = 1$ for $s > 1$ by taking $s_1 = \dots = s_{t_1} = 0$, $s_{t_1+1} = 1$, $s_{t_1+2} = \dots = s_{t_1+t_2} = 0$, $s_{t_1+t_2+1} = 1$, $s_{t_1+t_2+2} = \dots = s_{t_1+t_2+t_3} = 0$, ... We need $m_L^{(s)} + m_M^{(s)} = 2$ for $s = l(n-k+2)$. This is possible if $p(k-1) + \sum_{i=1}^m t_i + 1 = l(n-k+2)$. In case $l = 1$ we then do not set $s_{\sum t_i+1} = 1$, but continue with $s_{\sum t_i+1} = s_{\sum t_i+2} = \dots = 0$. This gives a shift in the indices of the s_i, which we do not compute here.

7.3. ${}^{II}A_n^{k,2}$**.** In this case the non-(-2) vertices are E_M and E_R. The sequence $(m_L^{(s)})$ is

$$(1^{k-1}, 0, 1^{k-1}, 0, \dots).$$

So $m_L^{(s)} = 1$ for $s \neq lk$ and $m_L^{(lk)} = 0$ for all l. As in the previous case the sequence $(m_M^{(s)})$ is

$$(1, 0^{k+t_1-2}, 1, 0^{k+t_2-2}, 1, 0^{k+t_3-2}, \dots) .$$

We have $E_R \cdot Z^{(1)} = -u_1$ with $u_1 = 0$ if and only if E_R is a bad vertex. If $u_1 > 0$, then E_R is not contained in the support of $Y_R^{(2)}$, which is the A_{n-k} between E_R and E_0. The sequence $(m_R^{(s)})$ is

$$(1, (1^{n-k}, 0)^{u_1}, 1, (1^{n-k}, 0)^{u_2}, 1, (1^{n-k}, 0)^{u_3}, \dots) .$$

The computation stops when $m_L^{(s)} + m_M^{(s)} = 0$, or when $m_R^{(s)} = 0$, except when $m_L^{(s)} + m_M^{(s)} = 2$ for that value of s. We achieve that $m_L^{(s)} + m_M^{(s)} = 1$ for $s > 1$ by taking $t_1 = 0$ and $t_i = 1$ for $i > 1$. It is possible to have $m_L^{(lk)} = m_M^{(lk)} = 1$ for some $l > 1$, while $m_L^{(pk)} + m_M^{(pk)} = 1$ for $p < l$, by setting $t_l = 0$. If $k > 2$, then $m_L^{(lk+1)} = m_M^{(lk+1)} = 0$, so the computation stops at that point. Therefore the computation stops when $m_R^{(s)}$=0, or if $s = lk$, in the next step. The computation never stops if $u_i = 0$ for all i. Note that in that case both E_M and E_R are bad vertices.

If $k = 2$, the situation is a bit different. The sequence $(m_L^{(s)})$ is $(1, 0, 1, 0, \dots)$, $(m_M^{(s)})$ is $(1, 0^{t_1}, 1, 0^{t_2}, 1, 0^{t_3}, \dots)$ and $(m_R^{(s)})$ is

$$(1, (1^{n-2}, 0)^{u_1}, 1, (1^{n-2}, 0)^{u_2}, 1, (1^{n-2}, 0)^{u_3}, \dots) .$$

We always take $t_1 = 0$, and $t_i \leq 1$. By taking suitable consecutive t_i equal to zero we can get $m_L^{(2s)} + m_M^{(2s)} = 2$, with this sum always equal to one for odd indices. It is possible that the computation never stops. If n is odd, we need $u_{2l-1} = 0$ for all l, while the u_{2l} may be arbitrary. If n is even, then $u_i \leq 1$. If $u_i = 0$ then also $u_{i+1} = 0$. If $n = 2$, we see no difference between E_M and E_R, and indeed the sequences $(m_M^{(s)})$ and $(m_R^{(s)})$ are of the same shape.

7.4. $D_{2k+1}^{k+1,2}$**.** The configuration is connected to vertices E_R and E_L. We claim that *the coefficient z_L of E_L in the fundamental cycle can be at most two.* We compute the fundamental cycle with E_L as central vertex. The relevant information on the cycle $Y_{\Delta,R}^{(1)}$ is given in the entry for $RD_{2k+1}^{k+1,2}$ in Table 6. If the coefficient of E_R is s, then the multiplicity of the vertex adjacent to E_L is $m_L^{(1)} = \lfloor \frac{2k+1+s}{2} \rfloor$. We assume that $E_L \cdot Z^{(1)} = 1$. If $s = 2t+1$, then $Y_{\Delta,R}^{(2)} = \emptyset$ and the computation stops with $z_L = 2$ and $z_R = 2t+1$. If $s = 2t+2$, then $\Gamma_{\Delta,R}^{(2)}$ has only an $A_{2k}^{1,1}$-configuration between E_L and E_R, so E_R is not a bad vertex for $Y_{\Delta,R}^{(2)}$ and $m_L^{(2)} = 1$. As $\lfloor \frac{2k+1+s}{2} \rfloor = k+1+t \geq 3$, the computation again stops with $z_L = 2$. Depending on whether $E_R \cdot Z^{(1)} = 0$ or less, $z_R = 2t+3$ or $z_R = 2t+2$.

7.5. $D_{2k}^{k,2}$**.** In this case only one of the vertices E_L and E_R is bad. In the symmetric case $k = 2$ we assume that E_R is the bad vertex. We compute as in the previous case with E_L as central vertex. If the coefficient of E_R in $Y_{\Delta,R}^{(1)}$ is s (with $s > 1$ if and only if E_R is bad), then $m_L^{(1)} = \lfloor \frac{2k+s}{2} \rfloor$. If $s = 2t-1$, then $m_L^{(2)} = 1$. If $k \geq 3$, then $\lfloor \frac{2k+s}{2} \rfloor = k+t-1 \geq k \geq 3$. For $k = 2$ we assumed $s > 1$, so $t > 1$ and again $k+t-1 \geq 3$. So the computation stops with $z_L = 2$, and $z_R = 2t-1$ or $z_R = 2t$. If $s = 2t$, then $Y_{\Delta,R}^{(2)} = \emptyset$ and the computation stops with $z_L = 2$ and $z_R = 2t$.

7.6. $A_n^{2,k,2}$**.** As in the cases ${}^{I}A_n^{2,k}$ and ${}^{II}A_n^{k,2}$ we compute with the rupture point in the $A_n^{2,k,2}$-configuration as central vertex E_0. The sequence $(m_L^{(s)})$ is

$$(1, (1^{k-2}, 0)^{s_1}, 1, (1^{k-2}, 0)^{s_2}, 1, (1^{k-2}, 0)^{s_3}, \dots)\ ,$$

the sequence $(m_M^{(s)})$ is

$$(1, 0^{k+t_1-2}, 1, 0^{k+t_2-2}, 1, 0^{k+t_3-2}, \dots)\ .$$

and finally $(m_R^{(s)})$ is

$$(1, (1^{n-k+1}, 0)^{u_1}, 1, (1^{n-k+1}, 0)^{u_2}, 1, (1^{n-k+1}, 0)^{u_3}, \dots)\ .$$

First suppose E_M is a bad vertex, i.e., $t_1 = 0$. We may assume that $k > 2$. Then E_L is not a bad vertex, $s_1 > 0$, except possibly if n has the lowest possible value $2k-3$, when there is an arrowhead between E_M and E_L at E_0. In that case the chains from E_0 to E_L and E_R are equally long. As $n-k+1 = k-2$, not all three of s_1, t_1 and u_1 are zero, so upon relabeling we may assume also here that $s_1 > 0$. As in the case ${}^{I}A_n^{2,k}$ we find that the computation stops with the first 0 in the sequence $(m_R^{(s)})$, or in the step immediately after. It is however possible that there is no 0 in this sequence; this happens if $u_i = 0$ for all i.

If $t_1 > 0$, then $s_1 = 0$, and if $n = 2k-3$, also $u_1 = 0$. For most values of s we will have $m_L^{(s)} + m_R^{(s)} = 2$, but we want that $m_L^{(s)} + m_R^{(s)} = 1$ for $s = p(k-1) + \sum_{i=1}^{p} t_i + 1$ for all $p \geq 1$. We determine on which places in the sequence $(m_L^{(s)})$ there are zeroes. Let $\sum_{j=1}^{i-1} s_j < r \leq \sum_{j=1}^{i} s_j$. Then the r-th zero is on place $r(k-1)+i$. Similarly the r-th zero in the sequence $(m_R^{(s)})$ is on place $r(n-k+2)+i$, if $\sum_{j=1}^{i-1} u_j < r \leq \sum_{j=1}^{i} u_j$.

If $k=2$, we may upon relabeling assume that $t_1>0$. Then the same description holds. In particular, if $n=1$, we have the sequences

$$(1,0^{s_1},1,0^{s_2},\dots),\quad (1,0^{t_1},1,0^{t_2},\dots),\ \text{and}\ \ (1,0^{u_1},1,0^{u_2},\dots).$$

Once again we stress that we do not investigate, which values of s_i, t_i and u_i are possible.

7.7. **Multiplicity at most two.** In the previous subsections we have tried to make the multiplicity of the fundamental cycle at non-(-2)'s as large as possible. The computations above also tell us when the multiplicity does not exceed two. Now we make the conditions explicit in terms of the multiplicities of the other components of the graph, attached to the two non-(-2)'s. Let E_a be one of these vertices. Then as before Γ_a is the union of connected components, attached to E_a. Let $E_a+Y_a^{(1)}$ be the fundamental cycle on $\{E_a\}\cup\Gamma_a$, and denote by $n_a^{(1)}$ the sum of the multiplicities of $Y_a^{(1)}$ at the vertices of Γ_a, adjacent to E_a. At the stage of the computation of the fundamental cycle, when the multiplicity of E_a has increased to 2, we need the fundamental cycle $E_a+Y_a^{(2)}$ on $\{E_a\}\cup\Gamma_a^{(2)}$, where $\Gamma_a^{(2)}$ is a connected component with vertices satisfying $E_i\cdot(E_a+Y_a^{(1)})=0$; then $E_a^{(2)}$ is the sum of the multiplicities of $Y_a^{(2)}$, adjacent to E_a.

7.7.1. $A_n^{1,1}$. As before we assume that E_L is the bad vertex. The computation with E_L as central vertex should stop at $s=2$, so $E_L\cdot Z^{(1)}=1$ and $E_L\cdot Z^{(2)}\le 0$. As the multiplicity of E_R also should be two, we need $E_R\cdot Z^{(1)}=0$. This gives us

$$\begin{aligned} n_L^{(1)}&=b_L, &\qquad n_R^{(1)}&=b_R-1,\\ n_L^{(2)}&\le b_L-2, &\qquad n_R^{(2)}&\le b_R-1.\end{aligned}$$

7.7.2. ${}^{I}A_n^{2,k}$. First consider the case that E_M is the bad vertex, so $t_1=0$ and $s_1>0$. If $s_1>1$, then the computation stops before the multiplicity z_L becomes two, or z_M becomes at least three. Therefore $s_1=1$. We have the sequences $(1,1^{k-2},0,1,1^{k-2},0,\dots)$ and $(1,0^{k-2},1,0^{k+t_2-2},1,\dots)$. As $n-k+1\ge k$ we have $n-k+2\ge k+1$. The condition that the multiplicities do not exceed two depend on n. If $n-k+2\le 2k-2$, the computation always stops at $s=n-k+2$. If $n-k+2=2k-1$, then we need $t_2\ge 1$ and if $n-k+2\ge 2k$, then we $t_2\ge 2$. Thus

$$\begin{aligned} n_L^{(1)}&=b_L-2, &\qquad n_M^{(1)}&=b_M-k+1,\\ n_L^{(2)}&\le b_L-2, &\qquad n_M^{(2)}&\le\begin{cases} b_M-k+1, & \text{if } n\le 3k-4,\\ b_M-k, & \text{if } n=3k-3,\\ b_M-k-1, & \text{if } n\ge 3k-2.\end{cases}\end{aligned}$$

If E_L is the bad vertex, we have $s_1=0$ and we need $t_1=1$. Furthermore $s_2\ge 1$.

If $n-k+2=k+1$, the computation stops at $s=n-k+2$. Otherwise we need $s_2>1$. This gives

$$\begin{aligned} n_L^{(1)}&=b_L-1, &\qquad n_R^{(1)}&=b_R-k,\\ n_L^{(2)}&\le\begin{cases} b_L-2, & \text{if } n=2k-1,\\ b_L-3, & \text{if } n\ge 2k,\end{cases} &\qquad n_R^{(2)}&\le b_R-k.\end{aligned}$$

7.7.3. ${}^{II}A_n^{k,2}$. If $u_1>0$, so $t_1=0$, the computation stops too early or the coefficient of E_M becomes too high. We need $u_2>0$. The value of t_2 depends again on n. The results also hold for $k=2$.

$$\begin{aligned} n_M^{(1)} &= b_M - k + 1, & n_R^{(1)} &= b_R - 1, \\ n_M^{(2)} &\le \begin{cases} b_M - k + 1, & \text{if } n \le 3k-5, \\ b_M - k, & \text{if } n = 3k-4, \\ b_M - k - 1, & \text{if } n \ge 3k-3, \end{cases} & n_R^{(2)} &\le b_R - 2. \end{aligned}$$

7.7.4. $D^{k+1,2}_{2k+1}$. In the notation of 7.4 we need that $s=2$ and $E_R \cdot Z^{(1)} < 0$. This gives us

$$\begin{aligned} n_L^{(1)} &= b_L - k, & n_R^{(1)} &= b_R - 1, \\ n_L^{(2)} &\le b_L - k, & n_R^{(2)} &\le b_R - 3. \end{aligned}$$

7.7.5. $D^{k,2}_{2k}$. In this case $s \le 2$ and $z_R = 2$. We first assume $k > 2$. This gives two possibilities. If $s = 1$ we obtain

$$\begin{aligned} n_L^{(1)} &= b_L - k + 1, & n_R^{(1)} &= b_R - 2, \\ n_L^{(2)} &\le b_L - k + 1, & n_R^{(2)} &\le b_R - 2 \end{aligned}$$

and for $s = 2$

$$\begin{aligned} n_L^{(1)} &= b_L - k, & n_R^{(1)} &= b_R - 1, \\ n_L^{(2)} &\le b_L - k, & n_R^{(2)} &\le b_R - 2. \end{aligned}$$

The last formula also works for the symmetric case $k = 2$, if we assume that E_R is the bad vertex.

7.7.6. $A^{2,k,2}_n$. We have to determine the conditions that at least two multiplicities become 2, whereas none may become 3. We argue as in the cases ${}^{I}A^{2,k}_n$ and ${}^{II}A^{k,2}_n$. If E_R ($t_1 = 0$) is bad we may assume that $k > 2$. If $s_1 > 1$, the multiplicity of E_L remains 1, which is seen by the absence of the entry for $n_L^{(2)}$:

$$\begin{aligned} n_L^{(1)} &\le b_L - 3, & n_M^{(1)} &= b_M - k + 1, & n_R^{(1)} &= b_R - 1, \\ & & n_M^{(2)} &\le \begin{cases} b_M - k + 1, & \text{if } n \le 3k-6, \\ b_M - k, & \text{if } n \ge 3k-5, \end{cases} & n_R^{(2)} &\le b_R - 2. \end{aligned}$$

If $s_1 = 1$ and $u_1 < 0$ (so $n_R^{(1)} < b_R - 1$), then $n > 2k-3$; for $n = 2k-3$ one has, if necessary, to interchange E_L and E_R. We get

$$\begin{aligned} n_L^{(1)} &= b_L - 2, & n_M^{(1)} &= b_M - k + 1, & n_R^{(1)} &\le b_R - 2. \\ n_L^{(2)} &\le b_L - 2, & m_M^{(2)} &\le \begin{cases} b_M - k + 1, & \text{if } n \le 3k-5, \\ b_M - k, & \text{if } n = 3k-4, \\ b_M - k - 1, & \text{if } n \ge 3k-3, \end{cases} \end{aligned}$$

It is also possible that all three multiplicities are 2:

$$\begin{aligned} n_L^{(1)} &= b_L - 2, & n_M^{(1)} &= b_M - k + 1, & n_R^{(1)} &= b_R - 1. \\ n_L^{(2)} &\le b_L - 2, & n_M^{(2)} &\le \begin{cases} b_M - k + 1, & \text{if } n \le 3k-6, \\ b_M - k, & \text{if } n = 3k-5, \\ b_M - k - 1, & \text{if } n \ge 3k-4, \end{cases} & n_R^{(2)} &\le b_R - 2. \end{aligned}$$

If E_M is not bad, we allow that $k=2$.

$$\begin{aligned} &n_L^{(1)}=b_L-1, && n_M^{(1)}=b_M-k-1, && n_R^{(1)}=b_R-1.\\ &n_L^{(2)}\le b_L-2 && && n_R^{(2)}\le b_R-2. \end{aligned}$$

Also now it is possible that all three multiplicities are 2:

$$\begin{aligned} &n_L^{(1)}=b_L-1, && n_M^{(1)}=b_M-k, && n_R^{(1)}=b_R-1.\\ &n_L^{(2)}\le \begin{cases} B_L-2, & \text{if } n=2k-3,\\ b_L-3, & \text{if } n\ge 2k-2, \end{cases} && n_M^{(2)}\le b_M-k, && n_R^{(2)}\le b_R-2. \end{aligned}$$

8. Low degree

The classification of rational graphs of degree three was given by Artin [1], degree four by the author [14] and degree five by Tosun et al. [16]. In these cases there is at most one non-reduced non-(-2), so the classification can be written using the results of Sections 2 and 6. For degree six one new case arises, with two non-reduced non-(-2)'s; here the results of Subsection 7.7 suffice, as we presently shall make explicit. For degree 7 one can use the same methods; we do not go into detail. Things become more complicated for degree 8, where possibility of three non-reduced non-(-2)'s appears. We classify the occurring graphs in this section.

8.1. **Degree six.** We start with the classification of graphs of canonical models. The ones with reduced fundamental cycle are given in Table 4. From it one can also infer the other possibilities: just replace some vertices with weight $-b$ with a vertex of weight -3 and multiplicity $b-2$, or the (-6) by a (-4) of multiplicity 2. We do not treat all cases, where there is only one non-(-2) with higher multiplicity, but We give partial results for some cases and as example we list the complete classification in the case of highest multiplicity.

The new case in degree 6 is that there are two (-3)-vertices with multiplicity two in the fundamental cycle. The possible configurations are described in Section 7.7. We have to specialise to the case that the vertex weights are 3.

We write $C(m_1,m_2)$ for any combination of RDP-configurations realising the multiplicity sequence (m_1,m_2), and $C(m_1,\le m_2)$ for configurations where the total second multiplicity is at most m_2. The notation $C(0,0)$ stands for the empty configuration. These combinations can be found from Table 5; e.g., $(3,\le 1)$ stands for $2A_1^1+A_n^1$ $(n\ge 1)$, A_5^3, A_6^3, ${}^ID_k^2+A_1^1$, $A_3^2+A_1^1$, $A_4^2+A_1^1$, ${}^{II}D_5^2+A_1^1$, ${}^{II}D_6^3$, ${}^{II}D_7^3$ and E_7^3.

Proposition 8.1. *Suppose the graph of the RDP-resolution consist of two (-3)-vertices, both with multiplicity 2. Then they are connected by one of the RDP-configurations, listed in Table 8 together with the the other configurations at the left and the right vertex.*

Proposition 8.2. *Suppose the graph of the RDP-resolution consist of one (-3)-vertex, with multiplicity 4. The following combinations of RDP-configurations are possible.*

$$\begin{array}{lll} A_4^2+2A_3^1, & A_4^2+A_7^2, & A_1^1+A_6^2+A_{\ge 3}^1,\\ {}^{II}D_5^2+A_6^2, & A_1^1+A_2^1+A_8^2, & A_1^1+A_2^1+A_3^1+A_{\ge 3}^1,\\ {}^{II}D_5^2+A_2^1+A_{\ge 3}^1, & A_1^1+A_{10}^3, & A_1^1+A_2^1+A_7^2. \end{array}$$

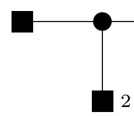
Proof. We argue as in the proof of Proposition 6.6. We first consider only RDP-configurations of type A_n^1 and A_{2k}^2. We need either A_1^1 or A_4^2. As A_4^2 gives the sequence $(2,1,1,2,0)$, we need the sequence $(2,2,2,0)$, so the configuration $2A_3^1$. If there is exactly one A_1^1, we further need A_2^1 or A_6^2. In the first case we can complete with A_8^2 or $A_3^1+A_n^1$ with $n\ge 3$, in the second only with A_n^1, $n\ge 3$. Table 5 gives the possible equivalent configurations. □

TABLE 8.

name	left	right
$A_n^{1,1}$	$C(3,\leq 1)$	$C(2,\leq 2)$
${}^{I}A_3^{2,2}$	$C(2,\leq 1)$	$C(1,\leq 1)$
${}^{I}A_{\geq 4}^{2,2}$,	$C(2,0)$	$C(1,\leq 1)$
${}^{I}A_5^{2,3}$	$C(2,\leq 1)$	$C(0,0)$
${}^{I}A_{\geq 6}^{2,3}$	$C(2,0)$	$C(0,0)$
${}^{I}A_5^{2,3}$	$C(1,\leq 1)$	$(1,\leq 1)$
${}^{I}A_6^{2,3}$	$C(1,\leq 1)$	$C(1,0)$
${}^{I}A_7^{2,4}$	$C(1,\leq 1)$	$C(0,0)$
${}^{I}A_8^{2,4}$	$C(1,\leq 1)$	$C(0,0)$
${}^{II}A_2^{2,2}$	$C(2,\leq 1)$	$C(2,\leq 1)$
${}^{II}A_{\geq 3}^{2,2}$	$C(2,0)$	$C(2,\leq 1)$

name	left	right
${}^{II}A_4^{3,2}$	$C(1,\leq 1)$	$C(2,\leq 1)$
${}^{II}A_5^{3,2}$	$C(1,0)$	$C(2,\leq 1)$
${}^{II}A_6^{4,2}$	$C(0,0)$	$C(2,\leq 1)$
${}^{II}A_7^{4,2}$	$C(0,0)$	$C(2,\leq 1)$
$D_4^{2,2}$	$C(1,\leq 1)$	$C(2,\leq 1)$
$D_5^{3,2}$	$C(1,\leq 1)$	$C(2,0)$
$D_6^{3,2}$	$C(0,0)$	$C(2,\leq 1)$
$D_6^{3,2}$	$C(1,\leq 1)$	$C(1,\leq 1)$
$D_7^{4,2}$	$C(0,0)$	$C(2,0)$
$D_8^{4,2}$	$C(0,0)$	$C(1,\leq 1)$

Next we consider the case that the hypertree for the RDP-resolution has a T-joint. The smallest tree realising it looks as follows.



As drawn, the vertex E_M has multiplicity two. The other cases are also possible, and occur in the classification, but they give basically the same graph.

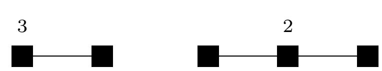
Proposition 8.3. *If the hypertree of the RDP-resolution has a T-joint and E_M is the vertex of higher multiplicity, the configurations*

$MA_3^{2,3,2}+C(1,\leq 1)$, $MA_4^{2,3,2}+C(1,\leq 1)$, $MA_{\geq 5}^{2,3,2}+C(1,0)$, $MA_5^{2,4,2}$, $MA_6^{2,4,2}$, and $MA_{\geq 7}^{2,4,2}$

can be attached to E_M; at E_R an A_n^1 is possible and also at E_L in the symmetric case of minimal $n=2k-3$. To E_R of higher multiplicity the configurations $RA_n^{2,2,2}+C(2,\leq 2)$ and $RA_n^{2,3,2}+C(2,\leq 1)$ can be attached; at E_L an A_n^1 is possible and also at E_M in the case $k=2$. The last possibility is $LA_{\geq 2}^{2,2,2}+C(2,\leq 1)$ with an optional A_n^1 at E_R.

Proof. We use Table 7. The only thing to note is that we stop the computation earlier, at step two, so in the case $LA_{\geq 3}^{2,2,2}$ the multiplicity at E_M does not reach the value $n+1$, but remains 3. □

For two other cases, with the following graphs for the RDP-resolution,



we only show how they can be realised, using configurations of type $LA_n^{1,1}$ for the connection to other (-3)'s, and configurations $C(m_1,\leq m_2)$ and $C(m_1,m_2,\leq m_3)$; as before this notation stands for any combination of configurations, realising a multiplicity sequence.

Proposition 8.4. *Suppose the graph of the RDP-resolution consist of two (-3)-vertices, one with multiplicity 3. This type can be realised by attaching to the curve of multiplicity 3 a combination $LA_0^{1,1}+C(3,3,\leq 2)$, $LA_1^{1,1}+C(3,2,\leq 2)$ or $LA_{\geq 2}^{1,1}+C(3,2,\leq 1)$.*

Two reduced (-3)'s with a (-3) of multiplicity 2 in between can be realised by attaching, to the vertex of multiplicity 2,

$$LA_0^{1,1} + LA_0^{1,1} + C(2, \leq 2),\ LA_{\geq 1}^{1,1} + LA_0^{1,1} + C(2, \leq 1),\ \text{or } LA_{\geq 1}^{1,1} + LA_{\geq 1}^{1,1} + C(2, 0).$$

The three remaining cases are easier.

8.2. **Degree eight.** We consider here only the cases that there are three (-3)-vertices, all with multiplicity two in the fundamental cycle. Either all three are connected by a single $A_n^{2,k,2}$ configuration, or they form a chain. The first possibility is a special case of Section 7.7.6.

Proposition 8.5. *Suppose the graph of the RDP-resolution consist of three (-3)-vertices, all with multiplicity 2, connected by a single $A_n^{2,k,2}$ configuration. Then following values for n and k are possible, with the given other configurations at each vertex.*

name	left	middle	right
$A_1^{2,2,2}$	$C(2, \leq 1)$	$C(1, \leq 1)$	$C(2, \leq 1)$
$A_n^{2,2,2}$	$C(2, 0)$	$C(1, \leq 1)$	$C(2, \leq 1)$
$A_3^{2,3,2}$	$C(2, \leq 1)$	$C(0, 0)$	$C(2, \leq 1)$
$A_n^{2,3,2}$	$C(2, 0)$	$C(0, 0)$	$C(2, \leq 1)$
$A_3^{2,3,2}$	$C(1, \leq 1)$	$C(1, \leq 1)$	$C(2, \leq 1)$
$A_n^{2,3,2}$	$C(1, \leq 1)$	$C(1, 0)$	$C(2, \leq 1)$
$A_5^{2,4,2}$	$C(1, \leq 1)$	$C(0, 0)$	$C(2, \leq 1)$
$A_6^{2,4,2}$	$C(1, \leq 1)$	$C(0, 0)$	$C(2, \leq 1)$

Finally we consider a chain of non-reduced (-3)'s. Let the vertices be called E_L, E_M and E_R. We compute the fundamental cycle as described in Section 5 with E_M as central vertex. The complement $\Gamma \setminus \{E_M\}$ decomposes into the connected components Γ_L and Γ_R, containing respectively E_L and E_R, and the union Γ_M of the remaining components. We consider the multiplicity sequences $(m_L^{(s)}) = (m_L^{(1)}, m_L^{(2)})$, $(m_M^{(1)}, m_M^{(2)})$ and $(m_R^{(1)}, m_R^{(2)})$. We need that $m_L^{(1)} + m_M^{(1)} + m_R^{(1)} = 4$ and $m_L^{(2)} + m_M^{(2)} + m_R^{(2)} \leq 2$. Upon interchanging E_L and E_R we may assume that $m_L^{(1)} \geq m_R^{(1)}$.

Proposition 8.6. *For a chain of three (-3)'s with multiplicity 2 in the fundamental cycle the following multiplicity sequences are possible, when computing with the middle vertex as central vertex.*

$(m_L^{(s)})$	$(m_M^{(s)})$	$(m_R^{(s)})$
$(3, \leq 1)$	$(0, 0)$	$(1, 1)$
$(2, \leq 2)$	$(0, 0)$	$(2, 0)$
$(2, 1)$	$(0, 0)$	$(2, 1)$

$(m_L^{(s)})$	$(m_M^{(s)})$	$(m_R^{(s)})$
$(2, 0)$	$(1, \leq 1)$	$(1, 1)$
$(2, 1)$	$(1, 0)$	$(1, 1)$
$(1, 1)$	$(2, 0)$	$(1, 1)$

The configurations giving the required values for $(m_M^{(1)}, m_M^{(2)})$ can be read off from Table 5. We have $C(1, 0) = A_1^1$, $C(1, 1) = A_n^1$, $n > 1$, and $C(2, 0)$ can be $2A_1^1$, A_3^2 or ${}^I D_k^2$. For $(m_L^{(1)}, m_L^{(2)})$ and $(m_R^{(1)}, m_R^{(2)})$ we use Table 6. It suffices to describe the possible configurations for E_L. The result is given in Table 9.

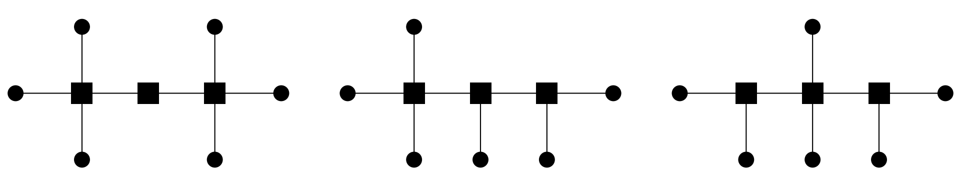
We have to distinguish cases depending on whether E_L is a bad vertex for $\Gamma_L \cup \{E_M\}$ or not. If bad, then the multiplicity of E_L in $Y_L^{(1)}$ is two, and the multiplicity does not increase in the second step. This means that $E_i \cdot Z^{(1)} < 0$ for some vertex E_i on the chain between E_L and E_M.

This is an extra condition, which excludes a number of cases from Table 6. If E_L is not bad, then its multiplicity in $Y_L^{(1)}$ is one, and $E_i \cdot Z^{(1)} = 0$ for all vertices E_i on the chain between E_L and E_M, including E_L. In this case $m_L^{(2)}) \geq 1$.

Table 9.

$(m_L^{(1)}, m_L^{(2)})$	E_L bad	E_L not bad
$(1,1)$		$LA_n^{1,1} + C(2, \leq 2)$
$(2,0)$	$LA_n^{1,1} + C(3, \leq 1)$	
$(2,1)$	$M^{II}A_2^{2,2} + C(2, \leq 1)$ $M^{II}A_3^{2,2} + C(2,0)$ $M^{II}A_4^{3,2} + C(1, \leq 1)$ $M^{II}A_5^{3,2} + C(1,0)$ $M^{II}A_6^{4,2} + C(0,0)$ $M^{II}A_7^{4,2} + C(0,0)$	$L^IA_3^{2,2} + C(1, \leq 1)$ $LD_4^{2,2} + C(1, \leq 1)$ $M^IA_5^{2,3} + C(0,0)$ $LD_6^{3,2} + C(0,0)$
$(2,2)$	$R^{II}A_{\geq 3}^{2,2} + C(2, \leq 2)$ $LD_5^{2,2} + C(1, \leq 1)$ $LD_7^{3,2} + C(0,0)$	$L^IA_{\geq 4}^{2,2} + C(1, \leq 1)$ $M^IA_{\geq 6}^{2,3} + C(0,0)$
$(3,0)$	$L^IA_3^{2,2} + C(2,0)$ $LD_4^{2,2} + C(2, \leq 1)$ $M^IA_5^{2,3} + C(1,0)$ $LD_6^{3,2} + C(1, \leq 1)$ $M^IA_7^{2,4} + C(0,0)$ $LD_8^{4,2} + C(0,0)$	
$(3,1)$	$L^IA_{\geq 4}^{2,2} + C(2,0)$ $R^{II}A_4^{3,2} + C(2,0)$ $RD_5^{3,2} + C(2,0)$ $M^IA_6^{2,3} + C(1,0)$ $M^IA_8^{2,4} + C(0,0)$	$L^IA_5^{2,3} + C(1, \leq 1)$ $RD_6^{3,2} + C(1, \leq 1)$

We give the graphs for the simplest ways to realise a chain of three (-3)'s with multiplicity 2, depending on E_L or E_R being bad. Again it would be interesting to know whether these singularities have the same format.



References

[1] Michael Artin, *On isolated rational singularities of surfaces.* Amer. J. Math. **88** (1966), 129–136. DOI: 10.2307/2373050

[2] Ragnar O. Buchweitz, *Contributions à la théorie des des singularités.* Thèse, Paris 1981.

[3] Robert Ephraim, *Isosingular loci and the Cartesian product structure of complex analytic singularities.* Trans. Amer. Math. Soc. **241** (1978), 357–371. DOI: 10.1090/S0002-9947-1978-0492307-X

[4] Trond Stølen Gustavsen, *On the cotangent cohomology of rational surface singularities with almost reduced fundamental cycle.* Math. Nachr. **279** (2006), 1185–1194. DOI: 10.1002/mana.200310416

[5] Theo de Jong, *Determinantal rational surface singularities.* Compositio Math. **113** (1998), 67–90. DOI: 10.1023/A:1000402614933

[6] Ulrich Karras, *Normally flat deformations of rational and minimally elliptic singularities.* In: Singularities, Part 1 (Arcata, Calif., 1981), Proc. Sympos. Pure Math. **40**, Amer. Math. Soc., Providence, RI, 1983, pp. 619–639.

[7] Henry B. Laufer, *Taut Two-Dimensional Singularities.* Math. Ann. **205** (1973), 131–164. DOI: 10.1007/BF01350842

[8] Henry B. Laufer, *On rational singularities.* Amer. J. Math. **94** (1972), 597–608. DOI: 10.2307/2374639

[9] Henry B. Laufer, *Ambient Deformations for Exceptional Sets in Two-Manifolds.* Invent. math. **55** (1979), 1–36.

[10] Lê Dũng Tráng and Meral Tosun, *Combinatorics of rational singularities.* Comment. Math. Helv. **79** (2004), 582–604.

[11] Eduard Looijenga and Jonathan Wahl, *Quadratic functions and smoothing surface singularities.* Topology **25** (1986), 261–291. DOI: 10.1016/0040-9383(86)90044-3

[12] Walter D. Neumann, *A calculus for plumbing applied to the topology of complex surface singularities and degenerating complex curves.* Trans. Amer. Math. Soc. **268** (1981), 299–344. DOI: 10.1090/S0002-9947-1981-0632532-8

[13] Ancus Röhr, *Formate rationaler Flächensingularitäten.* Diss. Hamburg 1992.

[14] Stevens, Jan. *Partial resolutions of rational quadruple points.* Internat. J. Math. **2** (1991), 205–221. DOI: 10.1142/S0129167X91000144

[15] Stevens, Jan. *Deformations of singularities.* Lecture Notes in Mathematics **1811**. Springer-Verlag, Berlin, 2003.

[16] M. Tosun, A. Ozkan, A. and Z. Oer, *On the classification of rational singularities of surfaces.* Int. J. Pure Appl. Math. **41** (2007), 85-110.

[17] Philip Wagreich, *Elliptic singularities of surfaces.* Am. J. Math. **92** (1970), 419–454. DOI: 10.2307/2373333

[18] C. T. C. Wall, *Quadratic forms and normal surface singularities.* In: Quadratic forms and their applications (Dublin, 1999), Contemp. Math. **272**, Amer. Math. Soc., Providence, RI, 2000, pp. 293–311.

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