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ERRATUM: MILNOR FIBRATIONS AND THE THOM PROPERTY FOR MAPS $f\bar{g}$

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The main theorem in our article [2] is not correct as stated. Presumably, there exist stronger hypotheses under which it does hold. This is the case, for instance, when $n = 2$ and the germs f, g do not have a common branch (see [1, Proposition 1.4]). We thank Adam Parusiński for having pointed out to us this error by sending us two counter-examples. We also thank Mutsuo Oka for having located the mistake in our proof : It comes from the fact that the equation we give in page 147 line -5 is not sufficient to define the tangent space $T_{x_k}G$. In fact, the normal space to it is defined by the two real vectors $\operatorname{grad} u$ and $\operatorname{grad} v$ where $f\bar{g} = u + iv$, while in our calculation we considered only the vector $w_k = 2 \operatorname{grad} u + 2i \operatorname{grad} v$.

Here are the two counter-examples sent to us by A. Parusiński. The first of these was suggested by comments of M. Tibăr. In both examples, the map $f\bar{g}$ has an isolated critical value at $0 \in \mathbb{C}$, but $f\bar{g}$ does not possess the Thom $a_{f\bar{g}}$-property. We reproduce below the arguments given to us by A. Parusiński.

Example 1. Let $f\bar{g}\colon \mathbb{C}^2 \to \mathbb{C}$ be given by $f(z_1, z_2) = z_1 z_2$, $g(z_1, z_2) = z_2$. We have

$$f\bar{g}(z_1, z_2) = z_1 \|z_2\|^2 = x_1(x_2^2 + y_2^2) + iy_1(x_2^2 + y_2^2) = u + iv,$$

$$\operatorname{grad} u = (x_2^2 + y_2^2, 0, 2x_1x_2, 2x_1y_2) \quad \text{and} \quad \operatorname{grad} v = (0, x_2^2 + y_2^2, 2y_1x_2, 2y_1y_2).$$

Thus, the critical locus of $f\bar{g}$ is $Y = \{z_2 = 0\}$ and 0 is the only critical value of $f\bar{g}$. We show that, for the stratification $\{\mathbb{C}^2 \setminus Y, Y\}$, the Thom condition $a_{f\bar{g}}$ fails at every point of Y.

Fix $P = (p, q, 0, 0) \in Y$ and $(a, b) \in \mathbb{R}^2 \setminus 0$ such that $ap + bq = 0$. Let $z = (z_1, z_2)$ tend to P and satisfy $ax_1 + by_1 = 0$. Then, at these points,

$$a \operatorname{grad} u + b \operatorname{grad} v = (x_2^2 + y_2^2)(a, b, 0, 0)$$

and, hence,

$$\frac{a \operatorname{grad} u + b \operatorname{grad} v}{\|a \operatorname{grad} u + b \operatorname{grad} v\|} = \frac{(a, b, 0, 0)}{\|(a, b)\|} ,$$

which contradicts the Thom condition.

In fact, we can deduce from the above arguments that there is no stratification of $f\bar{g}$ satisfying the Thom condition. Indeed, Y, as it is the critical locus, has to be a union of strata for any stratification of $f\bar{g}$. If P is in a stratum open in Y, we may choose points $z = (z_1, z_2)$ that tend to P, are in a stratum open in $\mathbb{C}^2$, and are close to the points considered above. It suffices to suppose that they satisfy $|ax_1 + by_1| \le x_2^2 + y_2^2$, since then

$$a \operatorname{grad} u + b \operatorname{grad} v = (x_2^2 + y_2^2)(a, b, 0, 0) + (ax_1 + by_1)(0, 0, 2x_2, 2y_2),$$

and the second term tends faster to 0 than the first one if $z_2 \to 0$.

Example 2. Consider $f\bar{g}\colon \mathbb{C}^3 \to \mathbb{C}$ given by $f(z_1, z_2, z_3) = z_1(z_2 + z_3^2)$, $g(z_1, z_2, z_3) = z_2$. Write as before $f\bar{g} = u + iv$.

First we determine the critical locus of f. Since $f\overline{g}$ is holomorphic with respect to z_1 and z_3, then for $i = 1, 3$ we have $\frac{\partial(f\overline{g})}{\partial \overline{z_i}} = 0$ and the vectors $(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial y_1})$, $(\frac{\partial v}{\partial x_1}, \frac{\partial v}{\partial y_1})$ are independent if and only if $\frac{\partial(f\overline{g})}{\partial z_1} \neq 0$. The critical locus is then contained in the set with equations

$$\frac{\partial(f\overline{g})}{\partial z_1} = (z_2 + z_3^2)\overline{z}_2 = 0 \quad ; \quad \frac{\partial(f\overline{g})}{\partial z_3} = 2z_1\overline{z}_2 z_3 = 0. \tag{1}$$

The solution set of (1) consists of two components: $\{z_2 = 0\}$ and $\{z_1 = z_2 + z_3^2 = 0\}$. On the second one we have $\frac{\partial u}{\partial x_2} = \frac{\partial u}{\partial y_2} = \frac{\partial v}{\partial x_2} = \frac{\partial v}{\partial y_2} = 0$, and hence the entire component is included in the critical set. We write

$$f\overline{g} = f_1\overline{g} + f_2\overline{g},$$

where $f_1(z_1, z_2, z_3) = z_1 z_2$, $f_2(z_1, z_2, z_3) = z_1 z_3^2$. We write $f_1\overline{g} = u_1 + iv_1$, $f_2\overline{g} = u_2 + iv_2$. On the set $\{z_2 = 0\}$ we have $\frac{\partial u_1}{\partial x_2} = \frac{\partial u_1}{\partial y_2} = \frac{\partial v_1}{\partial x_2} = \frac{\partial v_1}{\partial y_2} = 0$ and hence on this set we consider only the partial derivatives of $f_2\overline{g}$ with respect to x_2, y_2. Since $f_2\overline{g}$ is antiholomorphic with respect to z_2 we get a new set of equations

$$z_2 = 0 \quad \text{and} \quad \frac{\partial(f_2\overline{g})}{\partial \overline{z}_2} = z_1 z_3^2 = 0.$$

This allows us to conclude that

$$Crit(f\overline{g}) = \{z_1 = z_2 + z_3^2 = 0\} \cup \{z_1 = z_2 = 0\} \cup \{z_2 = z_3 = 0\}.$$

Note that 0 is the only critical value of $f\overline{g}$.

Denote $Y = \{z_2 = z_3 = 0\}$. We show that for any stratification of $\mathbb{C}^3$ the Thom condition $a_{f\overline{g}}$ fails at a generic point of Y. Fix $P = (p, q, 0, 0, 0, 0) \in Y$ and $(a, b) \in \mathbb{R}^2 \setminus 0$ such that $ap + bq = 0$. Let $z = (z_1, z_2, z_3)$ tend to P and satisfy

$$|ax_1 + by_1| \leq \|z_2\|^2, \quad \|z_3\| \leq \|z_2\|^4.$$

Then at these points

$$a \operatorname{grad} u_1 + b \operatorname{grad} v_1 = \|z_2\|^2(a, b, 0, 0, 0, 0) + o(\|z_2\|^2)$$

and

$$\|\operatorname{grad} u_2, \operatorname{grad} v_2\| \leq \|z_3\| = o(\|z_2\|^2).$$

Thus we may conclude as in Example 1.

References

[1] A. Pichon, J. Seade, *Fibred multilinks and singularities $f\bar{g}$*. Math. Ann., 342(3), (2008), 487–514. DOI: 10.1007/s00208-008-0234-3

[2] A. Pichon, J. Seade, *Milnor fibrations and the Thom property for maps $f\bar{g}$*, Journal of Singularities vol. 3 (2011), 144-150. DOI: 10.5427/jsing.2011.3i