Journal *of* Singularities
Volume 7 (2013), 205-219

received 3 January 2013
in revised form 6 May 2013
DOI: 10.5427/jsing.2013.7k

THE UNIVERSAL ABELIAN COVER OF A GRAPH MANIFOLD

HELGE MØLLER PEDERSEN

Abstract. Complex surfaces singularities with rational homology sphere links play an important role in singularity theory. They include all rational and splice quotient singularities, and in particular in the latter case the universal abelian cover of the link is a key element of the theory. All such links of singularities are graph manifolds, and to a rational homology sphere graph manifold one can associate a weighted tree invariant called splice diagram. It is known that the splice diagram determines the universal abelian cover of the manifold. In this paper we give an explicit method for constructing the universal abelian cover from the splice diagram, which works for most of the graph manifolds in particular for all links of singularities.

1. Introduction

Splice quotient singularities are an important class of normal complex surface singularities with rational homology sphere links ($\mathbb{Q}$HS) recently discovered by Neumann and Wahl (see [NW05b] and [NW05a]). They include all rational and minimally elliptic singularities with $\mathbb{Q}$HS links by work of Okuma [Oku04], and all weighted homogeneous singularities with $\mathbb{Q}$HS links ([Neu83a]). Splice quotient singularities also play an important role in recent works of Némethi and Okuma ([NO08, NO09]). Their analytic structures are defined by the corresponding analytic structures of their universal abelian covers which in turn are given by complete intersection equations called *splice diagram equations*. Although these equations are fairly simple, the topology of the universal abelian cover is in general rather complicated. The aim of this paper is to give a general way to describe it.

Links of normal complex surface singularities belong to a specific class of 3-manifolds called graph manifolds, which are defined as having only Seifert fibered pieces in their JSJ-decompositions, or alternatively having no hyperbolic pieces in their geometric decompositions. If one restricts to $\mathbb{Q}$HS's, then one has a non complete invariant of graph manifolds called splice diagrams. Splice diagrams were original introduced by Eisenbud and Neumann in [EN85] and by Siebermann in [Sie80] for integer homology sphere graph manifolds, and were then later generalized by Neumann and Wahl to $\mathbb{Q}$HS's in [NW02]. In [NW05a], Neumann and Wahl define the splice diagram equations when the splice diagram Γ of M satisfies, what they call the *semigroup condition*. The splice diagram equations define an isolated complete intersection. If M also satisfies the *congruence condition*, they show that there exists a splice quotient singularity whose link is M, and that the link of the isolated complete intersection is the universal abelian cover of M.

Based on the result for links of singularities in [NW05a] Neumann and Wahl conjectured that the splice diagram determines the universal abelian cover for a $\mathbb{Q}$HS graph manifold, even when the graph manifold is not a singularity link. In [Ped10] the following theorem proved this:

Theorem 1.1 ([Ped10]; 6.3)**.** *Let M_1 and M_2 be two $\mathbb{Q}HS$ graph manifolds having the same splice diagram. Let $\widetilde{M}_i \to M_i$ be the universal abelian cover. Then $\widetilde{M}_1$ and $\widetilde{M}_2$ are homeomorphic.*

2000 *Mathematics Subject Classification.* 57M10, 57M27.
Key words and phrases. rational homology sphere, abelian cover.

Graph manifolds are also the 3-manifolds which are boundaries of plumbed 4-manifolds. A very common method to describe a graph manifold M is to give a plumbing diagram of a 4-manifold X, such that $M = \partial X$. Neumann gave a complete calculus for changing X but keeping M fixed in [Neu81]. In section 3 we are going to construct a plumbing diagram of the universal abelian cover.

The proof of Theorem 1.1 consists of inductively constructing the universal abelian cover from the splice diagram, and the purpose of this article is to extract from this proof an algorithm for constructing the topology of the universal abelian cover. The explicit construction given will only work under the assumption that the splice diagram has no edge weight of 0. This assumption is always satisfied if the manifold is a singularity link.

In the proof of Theorem 1.1 one had to extend the notion of splice diagram to a class of orbifolds called *graph orbifolds*, and in [Pedb] the congruence condition is extended to graph orbifolds. Hence if X is a singularity defined by the splice diagram equations associated to the splice diagram Γ, one can use this algorithm to construct a dual resolution diagram, provided that there is a manifold or orbifold satisfying the congruence condition and having Γ as its splice diagram. This is for example always true if Γ only has two nodes (see [Pedb]). Neumann and Wahl conjecture that in fact the splice diagram equations always define the universal abelian cover.

In section 2 we recall the definition of splice diagrams from [Ped10], and their relation with plumbing diagrams, and we state the results needed for the algorithm. In section 3 we describe the algorithm giving the plumbing diagram of the universal abelian cover by performing it on an example, and we give further examples.

Acknowledgements: The auther was supported by the Hungarian Academy of Sciences' *Lendület* LDT program.

2. Splice Diagrams

A *splice diagram* is a weighted tree with no vertices of valence two. By the valence of a vertex we mean the number of adjacent edges. We call vertices of valence greater than two *nodes*. At a node one assigns a sign, and on edges adjacent to nodes one assigns a non negative integer weight.

Let M be a $\mathbb{Q}$HS graph manifold. Let $M = \bigcup_v M_v$ be the JSJ-decomposition of M, that is the unique minimal decomposition of M into Seifert fibered pieces M_v with ∂M_v a union of tori. We associate a splice diagram $\Gamma(M)$ to M by the following procedure:

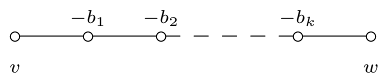
- Take a vertex v for each M_v.
- Connect two vertices v and w by an edge if $M_v \bigcap M_w \neq \emptyset$.
- Add a *leaf*, i.e., a valence one vertex connected by an edge, to a vertex v for each singular fiber of the Seifert fibration of M_v.
- To each vertex v assign the sign of the linking number of two nonsingular fibers of M_v. See Definition 2.1 in [Ped10] for the precise definition of these linking numbers.
- Let v be a node and e an edge adjacent to v. Then the edge weight d_{ve} is determined in the following way. Cut M along the torus T corresponding to e (either a torus of the boundary of M_v or the boundary of a tubular neighborhood of a singular fiber) into the pieces M'_v and M'_{ve}, where $M_v \subset M'_v$. Then glue a solid torus into the boundary of M'_{ve} by identifying a meridian with the image of a fiber of M_v, and call this new closed graph

manifold M_{ve}. Then

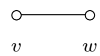
$$d_{ve} := \begin{cases} |H_1(M_{ve})| & \text{if } H_1(M_{ve}) \text{ is finite} \\ 0 & \text{if } H_1(M_{ve}) \text{ is infinite.} \end{cases}$$

A common way to represent graph manifolds is by plumbing diagrams, and we will next describe how to get the splice diagram from a plumbing diagram Δ of M.

To construct the graph structure of $\Gamma(M)$ from Δ one just suppresses all vertices of valence two, i.e., replacing any configuration like



with an edge

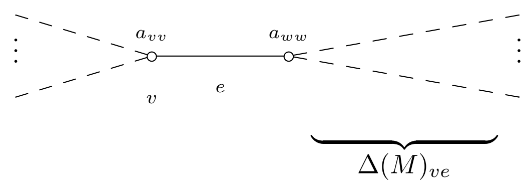


.

Let $A(\Delta)$ be the intersection matrix of the 4 manifold defined by Δ. The edge weights and signs are computed by the following two results.

Lemma 2.1 ([Ped10] 2.1)**.** *Let v be a node in $\Gamma(M)$ and e be an edge at that node. We get the weight d_{ve} by $d_{ve} = |\det(-A(\Delta(M)_{ve}))|$, where $\Delta(M)_{ve}$ is the connected component of $\Delta(M)$ after we remove e, which does not contain v.*

$\Delta(M) =$



Lemma 2.2 ([Ped10] 2.3)**.** *Let v be a node in $\Gamma(M)$. Then the sign ε at v is $\varepsilon = -\operatorname{sign}(a_{vv})$, where a_{vv} is the entry of $A(M)^{-1}$ corresponding to the node v.*

In the algorithm the *rational Euler number* of a Seifert fibered piece of M will play an important role. If M is a closed Seifert fibered manifold, then the rational Euler number e_M is defined by

$$e_M := \sum_{i=0}^{n} \frac{q_i}{p_i},$$

where $(p_0, q_0), \ldots, (p_n, q_n)$ are the unnormalized Seifert invariants (see [NR78]). Notice that we use the opposite choice of orientation when we define the invariants, this is the reason for the sign difference in our formula for e_M compared to the one they use. If we consider M as a plumbed manifold, then the plumbing diagram is star shaped with n strings connected to the central vertex. As explained in [NR78] one can change the Seifert invariants such that $p_0 = 1$ and $p_i > q_i > 0$ for $i = 1, \ldots n$. Then q_0 is the weight at the central vertex, and p_i/q_i for $i = 1, \ldots n$ is the continued fraction

$$[a_{i1}, a_{i2}, \ldots, a_{ik_i}] = a_{i1} - \cfrac{1}{a_{i2} - \frac{1}{a_{i3} - \ldots}},$$

where $-a_{i1}, -a_{i2}, \ldots, -a_{ik_i}$ are the weights along the i'th string of the plumbing diagram leading from the central vertex. In this case the splice diagram of M is also star shaped, it has n leaves with weights $p_1, \ldots, p_n$, and the sign at the node is $-\operatorname{sign}(e_M)$.

We need the rational Euler number e_{M_v} of a Seifert fibered pieces M_v of the JSJ-decomposition of M. Since M_v is not closed, we need additional information to define it. We consider a simple closed curve in each boundary component of M_v. Each curve is the image of a fiber of the Seifert fibered piece on the other side of the corresponding torus. One glues a solid torus in each of the boundary components of M_v, by identifying the simple closed curve with a meridian, and takes e_{M_v} to be the rational Euler number of this closed manifold. If M is given by a plumbing diagram Δ, then the M_v's correspond to the vertices v's with valence ≥ 3 (or vertices with non zero genus). One gets e_{M_v} as the rational Euler number of the starshaped piece containing v, after one removes from Δ all the vertices corresponding to M_w's with $w \neq v$.

The splice diagram itself does not determine neither the rational Euler numbers nor $|H_1(M)|$. But $|H_1(M)|$ and the splice diagram do determine the rational Euler number of any of the Seifert fibered pieces of M, for this result we need the following definition. The edge determinant $D(e)$ associated to an edge e between two nodes v and w is

$$D(e) := r_v r_w - \varepsilon_v \varepsilon_w (\prod_i n_{vi})(\prod_j n_{wj}),$$

where r_v and r_w are the edge weights on e, where ε_v and ε_w are the signs on the nodes and where the n_{vi}'s and n_{wj}'s are the weights adjacent to the nodes not on e.

Proposition 2.3 ([Ped10] 3.4)**.** *Let v be a node in a splice diagram decorated as in Figure 1 below, with $r_i \neq 0$ for $i \neq 1$, and let e_v be the rational Euler number of M_v. Then*

$$e_v = -d\left(\frac{\varepsilon s_1}{N D_1 \prod_{j=2}^k r_k} + \sum_{i=2}^k \frac{\varepsilon_i M_i}{r_i D_i}\right), \tag{1}$$

where $d = |H_1(M)|$, $N = \prod_{j=1}^k n_j$, $M_i = \prod_{j=1}^{l_i} m_{ij}$, and D_i is the edge determinant associated to the edge between v and v_i.

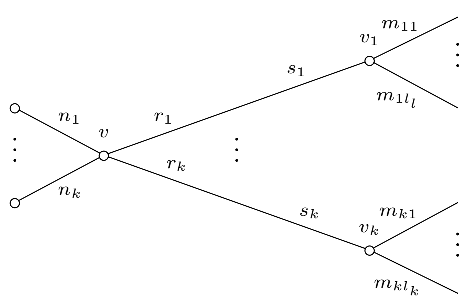


Figure 1 .

Note that this does give a formula for $e_v/|H_1(M)|$ from Γ, which we will need later.

In the algorithm to construct the universal abelian cover of M from $\Gamma(M)$, a number associated to each end of an edge in $\Gamma(M)$ is going to be very important. It is the *ideal generator*, which is constructed in the following way. Let v and w be two vertices of $\Gamma(M)$, then we define the *linking number* l_{vw} of v and w as the product of all edge weights adjacent to but not on the shortest path from v to w. We define l'_{vw} in the same way, except that we omit weights adjacent to v and w. If e is an edge adjacent to v, we let Γ_{ve} be the connected component of $\Gamma(M) - e$ not containing v, and define the following ideal in $\mathbb{Z}$

$$I_{ve} := \langle l'_{vw} |\ w \text{ a leaf in } \Gamma_{ve} \rangle.$$

Then we define the ideal generator $\overline{d}_{ve}$ associated to v and e to be the positive generator of I_{ve}.

Definition 2.4. A splice diagram Γ satisfies the *ideal condition* if the ideal generator $\overline{d}_{ve}$ divides the edge weight d_{ve} for all nodes v and adjacent edges e.

Proposition 2.5. *Let M be a $\mathbb{Q}HS$ graph manifold. Then $\Gamma(M)$ satisfies the ideal condition.*

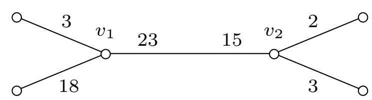
This proposition follows from the following topological description of the ideal generator in Appendix 1 of [NW05a].

Theorem 2.6. *The ideal generator $\overline{d}_{ve}$ equals $|H_1(M/M'_v)|$.*

Remember we defined M'_v, when we constructed $\Gamma(M)$ at the beginning of the present section.

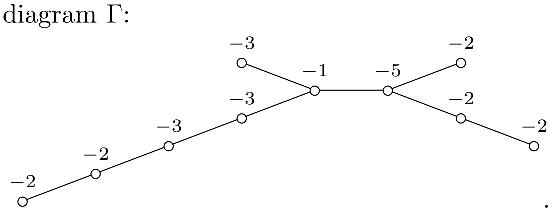
3. Construction of the Universal Abelian Cover: An Example

In this section we explain how the proof of Theorem 1.1 [Ped10] can be used to construct the universal abelian cover $\widetilde{M}$ of a graph manifold M from the splice diagram $\Gamma(M)$. We specify $\widetilde{M}$ by constructing a plumbing diagram Δ for $\widetilde{M}$. To illustrate the construction we use the following example

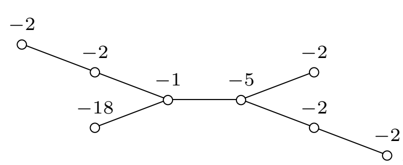
$\Gamma =$ 

.

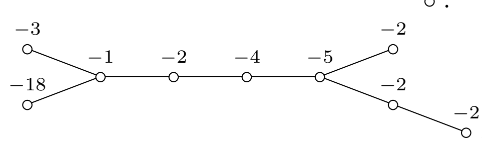
There are four different manifolds which have Γ as their splice diagram, and also several non manifold graph orbifolds. By Theorem 4.1 in [Ped10] Γ is the splice diagram of a singularity link, and [Peda] gives that $\widetilde{M}$ is a rational homology sphere. The example is also interesting, since none of the manifolds having splice diagram Γ satisfy the congruence condition of Neumann and Wahl (see [NW05a]). But there are non manifold orbifolds with splice diagram Γ which satisfy the orbifold congruence condition (see [Pedb]). Below are plumbing diagrams for the four manifolds having splice diagram Γ:



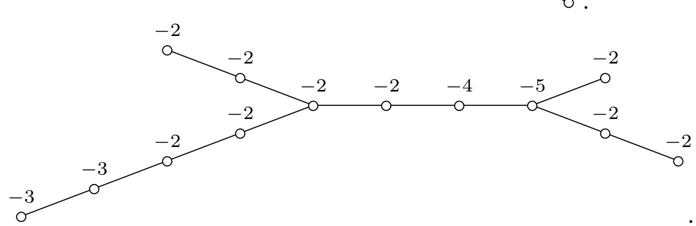
.



.



.

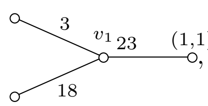


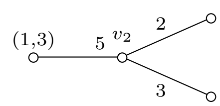
.

The construction of the universal abelian cover is done in two steps. First we construct a one-node splice diagram for each node in Γ, each of these one-node splice diagrams is then used to define a Seifert fibered manifold. We call these Seifert fibered manifolds *the building blocks*. In the second step we take a number of copies of the building blocks, and use the information given by Γ to glue them together to create the universal abelian cover.

3.1. **Constructing the building blocks.** The inductive procedure in the construction of the universal abelian cover consists of taking an edge e between two nodes of Γ, and making a new non connected splice diagram Γ_e, where e has been replaced by two leaves. So starting with the edge called e_1 and going through this process of cutting the edges until we have cut the last edge between two nodes e_{N-1}, we get that $\Gamma_{e_{N-1}}$ is a collection of one-node splice diagrams $\Gamma_{e_{N-1}} = \{\mathscr{G}_i\}_{i=1}^N$. For each of these one-node splice diagrams $\mathscr{G}_i$ one then takes a number of copies of a specific manifold M_i, and uses the information from the Γ_{e_j}'s to glue the pieces together. So the first step is to determine these manifolds $\{M_i\}_{i=1}^N$, which are the building blocks of the universal abelian cover.

First let us describe the Γ_{e_j}'s. Each time we cut an edge e between the nodes w_1 and w_2 in Γ, we divide every edge weight $d_{ve'}$ such that w_1 or w_2 is in $\Gamma_{ve'}$, by the ideal generator $\overline{d}_{w_ie}$ of the edge weight d_{w_ie}, where v is not in Γ_{w_ie}. In our example we only have two edge weights, which have to be divided when we cut along the central edge e namely $d_{v_1e} = 23$ and $d_{v_2e} = 15$, and $\overline{d}_{v_1e} = 1$ and $\overline{d}_{v_2e} = 3$. So the two one-node splice diagrams $\mathscr{G}_1$ and $\mathscr{G}_2$ are

$\mathscr{G}_1 =$ 

, $\mathscr{G}_2 =$ 

.

The pair added to the new leaves, which will be used to describe the gluings, is defined as follows: the first number specifies the order the sequence of cuttings this is, and the second number is the ideal generator associated to the weight before cutting.

Next we want to find the building block M_i associated to each of the $\mathscr{G}_i$'s. To do this we have to separate the $\mathscr{G}_i$'s into two types. The first type consists of the $\mathscr{G}_i$'s that do not have an edge weight of 0, and the second type consists of the $\mathscr{G}_i$'s that have an edge weight of 0. At most one weight adjacent to a node can be 0, since if there were two edge weights of 0 adjacent to a node the edge determinant of any edge with the edge weight 0 would be 0. Then by using The Edge Determinant Equation (Corollary 3.3 of [Ped10]) the Seifert fibration can be extended over the torus corresponding to the edge, hence we would not have cut along this torus in the JSJ-decomposition of M.

In the first case we use the following theorem

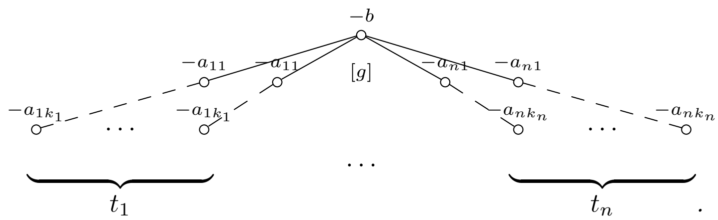
Theorem 3.1. *Let M be a $\mathbb{Q}HS$ orbifold S^1-fibration over a orbifold surface with Seifert invariants $(\alpha_1, \beta_1), \dots, (\alpha_n, \beta_n)$. Then the universal abelian cover of M is the link of the Brieskorn complete intersection $\Sigma(\alpha_1, \dots, \alpha_n)$.*

The way one constructs the manifolds after cutting an edge may result in graph orbifolds instead of just graph manifolds as explained in the proof of 6.3 in [Ped10]. Hence we need this theorem for orbifold S^1-fibrations. Neumann proves this theorem for Seifert fibered manifolds in [Neu83a] and [Neu83b], but the proof given in [Neu83b] also works in the general case of an orbifold S^1-fibration. These theorems assume that the rational Euler number e_M is positive, but if $e_M < 0$ one just composes with an orientation reversing map. Notice that $\alpha_1, \dots, \alpha_n$ are exactly the edge weights of $\Gamma(M)$. The value of the sign ε at the node does not matter, since reversing the orientation of a Seifert fibered manifold only changes the β_i's not the α_i's, and hence only changes the splice diagrams by replacing ε with $-\varepsilon$.

So in our example M_1 is the link of $\Sigma(3,18,23)$, and M_2 is the link of $\Sigma(2,3,5)$.

Next we use the description of the Seifert invariants of $\Sigma(\alpha_1,\ldots,\alpha_n)$ given by Neumann and Raymond in [NR78] to get plumbing diagrams for the M_i's.

Theorem 3.2. *Let M be the link of the Brieskorn complete intersection $\Sigma(\alpha_1,\ldots,\alpha_n)$. A plumbing diagram for M is given by*



The values of g and of the t_i's are given by

$$t_i = \frac{\prod_{j\neq i}(\alpha_j)}{\mathrm{lcm}_{j\neq i}(\alpha_j)} \tag{2}$$

$$g = \tfrac{1}{2}\big(2 + \frac{(n-2)\prod_i \alpha_i}{\mathrm{lcm}_i(\alpha_i)} - \sum_{i=1}^{n} t_i\big). \tag{3}$$

One calculates numbers $p_1,\ldots,p_n$ as

$$p_i = \frac{\mathrm{lcm}_j(\alpha_j)}{\mathrm{lcm}_{j\neq i}(\alpha_j)}, \tag{4}$$

and finds numbers $q_1,\ldots,q_n$ as the smallest no negative solutions to the equations

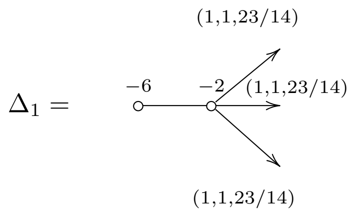
$$\frac{\mathrm{lcm}_j(\alpha_j)}{\alpha_i} q_i \equiv -1(\mod p_i). \tag{5}$$

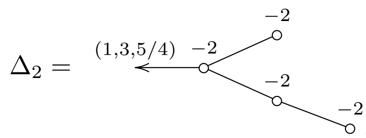
The a_{ij}'s are given by the continued fraction $p_i/q_i = [a_{i1},\ldots,a_{ik_i}]$. If $p_i = 1$ then the string of valence two vertices is empty. Finally b is given by

$$b = \frac{\prod_i \alpha_i + \mathrm{lcm}_k(\alpha_k)\sum_i q_i \prod_{j\neq i}\alpha_j}{(\mathrm{lcm}_k\, \alpha_k)^2}. \tag{6}$$

Before we use this theorem to make a plumbing diagram Δ_i for M_i, notice that we have to remove some solid tori from M_i to make the gluing, so we need to record this data in Δ_i. Some leaves in $\mathscr{G}_i$ have a pair of integers attached. These leaves correspond to the tori in M we cut along when we created $\mathscr{G}_i$. Since M_i is the universal abelian cover of any graph orbifold having splice diagram $\mathscr{G}_i$, several fibers sit above the singular fiber corresponding to these leaves. We have to remove a neighborhood of each of these fibers. So if α_j is an edge weight in $\mathscr{G}_i$ to a leaf with a pair attached, the t_j fibers above the leaf correspond to all the strings with the weights $-a_{j1},\ldots,-a_{jn_j}$. So in the plumbing diagram for M_i we replace these strings by arrows, and add a triple which consists of the pair attached to the leaf and $p_j/q_j = [a_{j1},\ldots,a_{jk_j}]$ to each of the arrows. If the fibers sitting above are non singular, i.e., the set $\{a_{ji}\}$ is empty, we still add t_j arrows and triples, and in this case the third number is $1/0$.

Using this on our example we get the following plumbing diagrams





Notice that the weight of the node is only well define for a closed manifold, so when we remove the solid torus corresponding to an arrow we lose that information. The weight of the nodes are then gotten trough the gluing process in the next section.

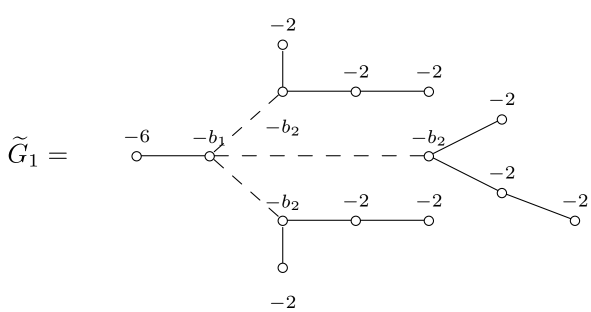
The second case, i.e., when there is an edge weight of 0, is not as easy. The proof of Theorem 6.3 in [Ped10] gives an explicit construction as a gluing of 3-spheres along S^2 boundaries in this case. But it might not be a Seifert fibered manifold, and I have at the present no simple way to find a plumbing diagram for the building blocks in this case. Hence the explicit algorithm does not work in this case.

3.2. **Gluing the building blocks.** The only thing that remains to construct the universal abelian cover is to glue together the building blocks M_i. This will be done by using the plumbing diagrams Δ_i to create a plumbing diagram Δ for $\widetilde{M}$.

We start by taking two of the Δ_i's and create a plumbing diagram G_1. Then we take another of the Δ_i's and glue this to G_1 to create G_2. We continue this process until all the Δ_i's have been used, and then $\Delta = G_{N-1}$ where G_{N-1} is the last created plumbing diagram.

Now the order we glue the Δ_i's together in is important. This is why we added a triple to the arrows. We start by taking Δ_i which has at least one arrow having the triple $(N-1, d_i, r_i)$, where $N-1$ is the highest value for the first number in any triple. We next take Δ_j such that at least one arrow has the triple $(N-1, d_j, r_j)$. By the method we constructed the Δ_i's, there are exactly two graphs Δ_i and Δ_j satisfying respectively these conditions. Then we take d_i copies of Δ_i and d_j copies of Δ_j. We create an intermediate $\widetilde{G}_1$ by removing the arrows with triple $(N-1, d_i, r_i)$ (respectively $(N-1, d_j, r_j)$) on the Δ_i's (respectively Δ_j's), and by replacing these with dashed lines between copies of Δ_i and Δ_j, such that a copy of Δ_i is only connected to a copy of Δ_j once. We also replace the weights at the nodes in the Δ_i piece by an unknown variable b_i and in the Δ_j piece by an unknown variable b_j. This will create a connected weighted graph $\widetilde{G}_1$, with no arrows which have first number in the triple equal to $N-1$.

Let us see how this is done in our example. We only have two Δ_i's, so we start by gluing Δ_1 to Δ_2. The triples are $(1, 1, 23/14)$ and $(1, 3, 5/4)$. So we start by taking one copy of Δ_1 and three copies of Δ_2, replacing each of the arrows in the copy of Δ_1 with a dashed line to one of the copies of Δ_2 replacing its arrow, and replace the weights at the nodes. We get



The next step to create G_1 is to replace the dashed lines by a string of valence two vertices and find the weights at the nodes. We have to compute the number of vertices along the string and their Euler numbers. First by symmetry all the strings will be the same, so we only have to calculate one of them. Likewise the weights at the nodes are the same at the corresponding ends of the identified strings.

Now $G_1 \bigcup (\bigcup_{l \neq i,j} \Delta_l)$ is a plumbing diagram for the non-connected manifold which is the universal abelian cover of any non-connected manifold with splice diagram $\Gamma_{e_{N-2}}$. Hence it is $\Gamma_{e_{N-2}}$ we need to use, when we make the calculations in the following.

Choose nodes v_i and v_j of $\widetilde{G}_1$, which are attached to each other by a dashed line such that v_k comes from a Δ_k piece. First we find the fiber intersection number p, which is also the numerator of the two continued fractions associated to the string. Now $p = f_i \cdot f_j$ where f_k is a fibre in the boundary of M_k. These fibres are gotten as connected components of the preimage of fibres $\tilde{f}_k$ of a graph orbifold $\widetilde{M}$ with splice diagram $\Gamma(\widetilde{M}) = \Gamma_{e_{N-2}}$. This implies that $\pi^{-1}(\tilde{f}_i) \cdot \pi^{-1}(\tilde{f}_j) = d(\tilde{f}_i \cdot \tilde{f}_l)$ where $d = |H_1^{orb}(\widetilde{M})|$. The intersection number $\tilde{f}_i \cdot \tilde{f}_j = |D|/d$ by The Edge Determinant Equation (Corollary 3.3 of [Ped10]), where D is the edge determinant of the corresponding edge in $\Gamma_{e_{N-2}}$. Hence $n_i n_j p = |D|$ where n_k is the number of connected pieces of $\pi^{-1}(f_k)$. We have that $n_k = \deg(\pi|_{M_k})/\deg(\pi|_{f_k})$. Now $\deg(\pi|_{M_k}) = d/d_k$ where d_k is given by the triple attached to the arrow in Δ_k, and $\deg(\pi|_{f_k})$ is calculated in the end of the proof of Theorem 6.3 in [Ped10] to be d/λ_k. Here $\lambda_k = \prod m_j / \operatorname{lcm}(m_1/\overline{d}_1, \dots, m_l/\overline{d}_l)$, where the m_j's are the edge weights adjacent to the node corresponding to v_k in $\Gamma_{e_{N-2}}$, and the $\overline{d}_j$'s are the ideal generators associated to the edges. Putting this together we get the following formula for calculating p:

$$p = \frac{d_i d_j}{\lambda_i \lambda_j} |D|$$

In our example $\Gamma_{e_{N-2}} = \Gamma$, so $|D| = 21$ and $\lambda_1 = \lambda_2 = 3$ and we get that $p = 7$.

To find the complete string and the b_k's we use that there are two different ways to calculate the rational Euler number of the Seifert fibered piece corresponding to a node in G_1. One using G_1 and one given by the splice diagram by a formula derived at the end of the proof of Theorem 6.3 in [Ped10].

From G_1 the rational Euler number e_{v_k} is given by $b_k + \sum_e q_e/p_e$, where the sum is taken over all edges adjacent to v_k (including the dashed lines), and (p_e, q_e) is the Seifert pair associated to the string starting with the edge e. Now there are four types of different edges attached to v_i, and we need to see how to get (p_e, q_e) from each type of the edge. We will first explain how to get (p_e, q_e) for an edge e if e is not a dashed line. Then use this to give an equation relating e_{v_k} to b_k and the Seifert pair associated to the dashed lines, notice that all the dashed lines have the same Seifert pair (p, q_k). We will then calculate e_{v_k} in another way, and use this to get the q_k's and the b_k's.

If e is on a string that ends at a valence one vertex then we get (p_e, q_e) from the continued fraction associated to the string, i.e., $p_e/q_e = [a_{e1}, \dots, a_{ek_e}]$.

If e is on a string that leads to a node (when one makes G_1 these do not exist, but they can be there when we are going to make G_2). We again get the Seifert pair from the continued fraction, this time from the string between v_k and the other node.

If e is an arrow, we get (p_e, q_e) from the triple (n_e, d_e, r_e) attached to the arrow as $p_e/q_e = r_e$.

We can now write the equation relating e_{v_k}, b_k and (p, q_k).

$$e_{v_k} = d'_k \frac{q_k}{p} - b_k + \sum_e \frac{q_e}{p_e}, \tag{7}$$

where the sum is taken over all edges at v_k except the dashed lines, and d'_k is the number of dashed lines at v_k. Notice that if $v_k = v_i$ is a node sitting in a Δ_i piece, then $d'_k = d_j$.

Returning to our example, if we use the leftmost node as v_1, the equation becomes

$$e_{v_1} = 3\frac{q_1}{7} - b_1 + \frac{1}{6}.$$

For one of the rightmost nodes the equation becomes

$$e_{v_2} = \frac{q_1}{7} - b_2 + \frac{1}{2} + \frac{2}{3} = \frac{q_2}{7} - b_1 + \frac{7}{6}.$$

From the end of the proof of Theorem 6.3 in [Ped10] one gets that if v_k sits in a Δ_k piece,

$$e_{v_k} = \frac{\lambda_k^2}{\overline{D}}\tilde{e}_{v_k}/d. \tag{8}$$

Remember we defined λ_k and d when we calculated p in the beginning of this section, and $\overline{D} = \prod_l \overline{d}_l$ where the $\overline{d}_l$'s are the ideal generators of all the edges adjacent to v_k in $\Gamma_{e_{N-2}}$. Notice that there is a mistake in the formula in [Ped10], there one only divides by d_k and not $\overline{D}$. It does not change the result of that article, but it is important when one wants to actually calculate e_{v_k} as we do. Now neither $\tilde{e}_{v_k}$ nor d are determined by $\Gamma_{e_{N-2}}$, but proposition 2.3 gives a formula for $\tilde{e}_{v_k}/d$ only using $\Gamma_{e_{N-2}}$ (the proposition also works for graph orbifolds).

In our example we find that $\lambda_1 = \lambda_2 = 3$, $\tilde{e}_{v_1}/d = -5/378$ and $\tilde{e}_{v_2}/d = -23/126$, so $e_{v_1} = -5/42$ and $e_{v_2} = -23/42$.

Now one finds an equation relating b_k and q_k by combining the equations (7) and (8). Since b_k is an integer this equation gives us an congruence equation $\mod p$ involving q_k as the only unknown. This equation involving q_k might not determine $q_k(\mod p)$, since it is possible that q_k is multiplied by a divisor of p. But the equation involving q_i and the equation involving q_j together with the equation $q_iq_j \equiv -1(\mod p)$ enable us to find q_i and q_j $(\mod p)$, and since we know $0 \leq q_i, q_j < p$ we can determine q_i and q_j.

In our example the equations relating b_1 and q_1 becomes

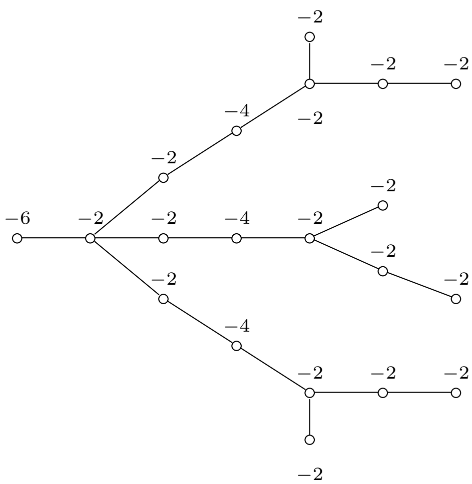
$$b_1 = 3\frac{q_1}{7} + \frac{1}{6} + \frac{5}{42} = 3\frac{q_1}{7} + \frac{2}{7},$$

and the equations relating b_2 and q_2 is

$$b_2 = \frac{q_2}{7} + \frac{7}{6} + \frac{23}{42} = \frac{q_2}{7} + \frac{12}{7}.$$

We get that $q_1 = 4$ and $q_2 = 2$. Remember that p/q_1 is the continued fraction associated to the string replacing the dashed lines when seen from v_1. We find b_k by putting q_k back into the equation we just used. In our example this gives that $b_1 = 2$ and $b_2 = 2$.

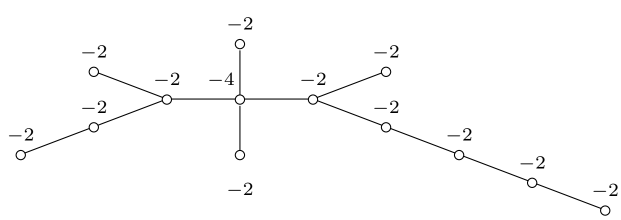
Replacing all the dashed lines with the strings corresponding to the continued fractions and adding the b_i's one gets G_1 from $\widetilde{G}_1$. In our example we get the following plumbing diagram:

$G_1 =$ 

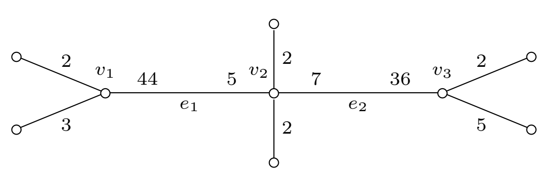
If $\Gamma_{e_{N-2}} \neq \Gamma$, then one adds G_1 to the collection of Δ_i's not used, and one repeats the process by taking the two plumbing diagrams of this collection having arrows whose triples start with $N-2$. One continues this process until all the Δ_i's have been used, and the final G_{N-1} is then a plumbing diagram for the universal abelian cover $\widetilde{M}$ of M.

We will finish by performing the algorithm on a couple of other examples. We will leave the details of the calculation to the readers.

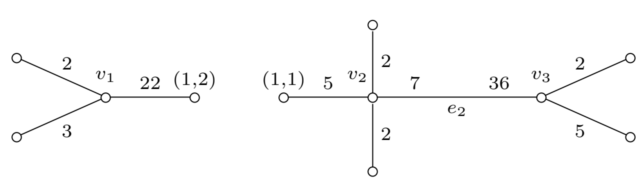
Example 3.3. Let M be the manifold defined by the following plumbing diagram:



Its splice diagram is:

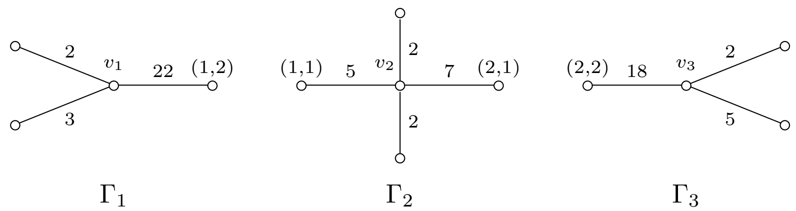
$\Gamma =$ 

If we first cut along the edge called e_1, we get:

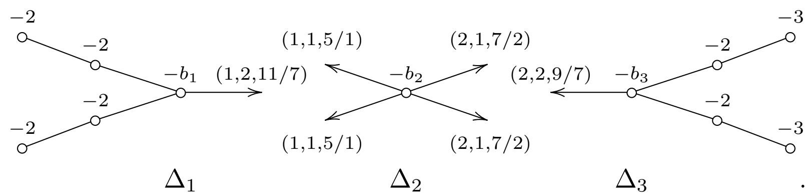
$\Gamma_{e_1} =$ 

and cutting along e_2 gives us:

$\Gamma_{e_2} =$

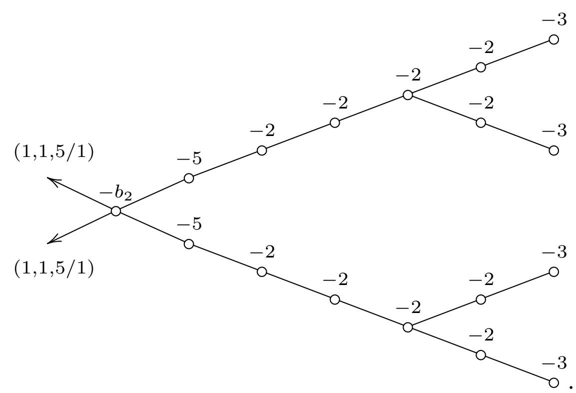


Next one determines the 3 building blocks and gets the following plumbing diagrams:



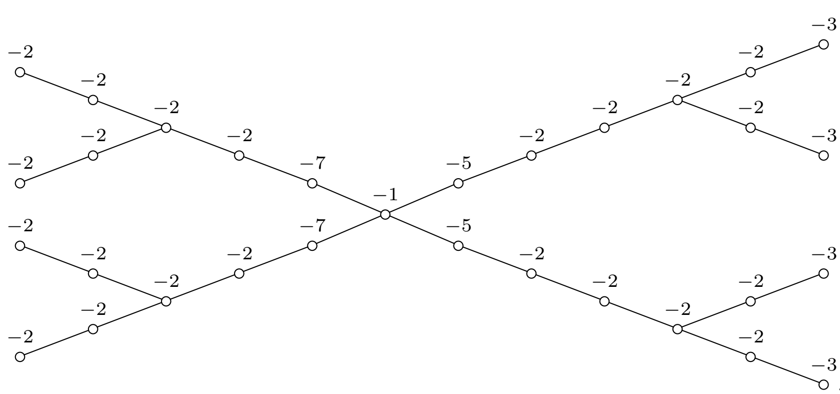
One first glues one copy of Δ_2 to two copies of Δ_3, and gets after calculating the strings and weights at nodes:

$G_1 =$

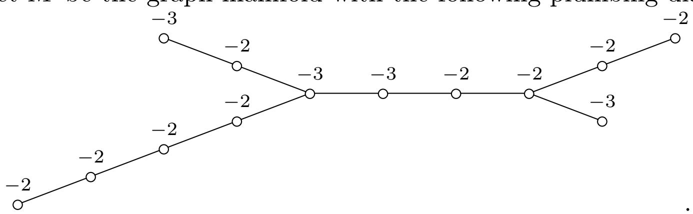


Then gluing two copies of Δ_1 to G_1 and calculating the strings and weights at nodes gives the following plumbing diagram for the universal abelian cover:

$G =$

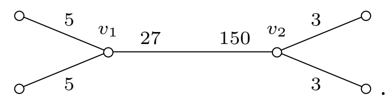


Example 3.4. Let M be the graph manifold with the following plumbing diagram:



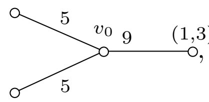
Its splice diagram is:

$\Gamma =$

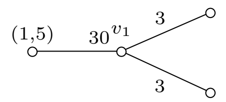


Cutting the edge gives us the one-node splice diagrams:

$\Gamma_1 =$

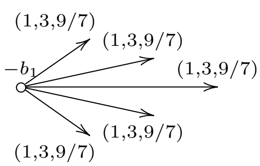


$\Gamma_2 =$

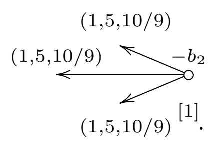


and the building blocks become

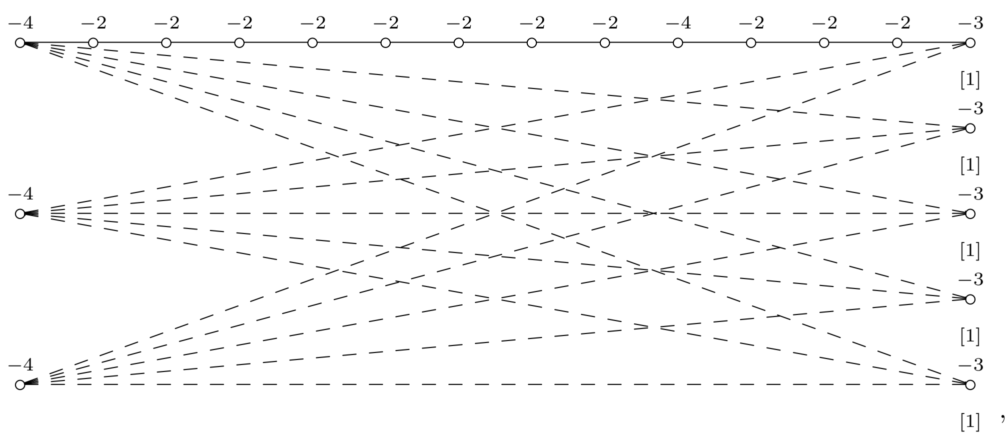
$\Delta_1 =$



$\Delta_2 =$

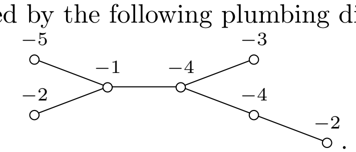


So to create the plumbing diagram G of the universal abelian cover, we glue 3 copies of Δ_1 to 5 copies of Δ_2, we calculate the string and the weights at nodes, and get

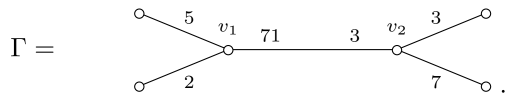


where all the dashed lines represent strings identical to the string at the top. Notice that the graph is not a planar graph. So any intersections between the strings represented by the dashed lines do not represent intersections in G, just crossings arising from a planar projection of G, which is what we see here.

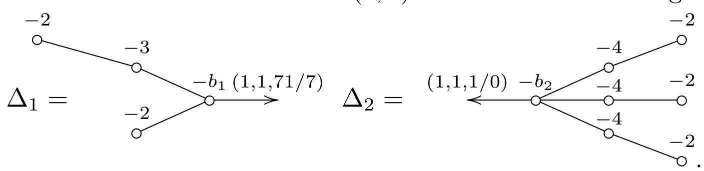
Example 3.5. Let M be defined by the following plumbing diagram



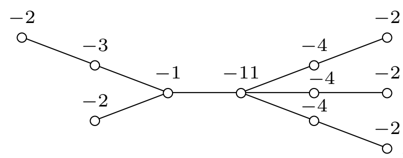
Its splice diagram is:



Since both ideal generators are 1, the one-node splice diagrams Γ_1 and Γ_2 have the same weights as Γ, and the pairs added to the new leaves are $(1,1)$ for both. The building blocks become:



If we use Theorem 3.2 to find the values at the node of the closed Seifert fibered manifolds we got the building blocks from, we will get that at the node in Δ_1 the Euler number is -1 and at the node in Δ_2 the Euler number is -2. Gluing a copy of Δ_1 to a copy of Δ_2 and finding the remaining Euler numbers gives us the following plumbing diagram for the universal abelian cover of M:



Notice that the Euler number of the rightmost node in the universal abelian cover is -11, which is very different of the -2 that was the Euler number of the building block.

References

[EN85] David Eisenbud and Walter Neumann. *Three-dimensional link theory and invariants of plane curve singularities*, volume 110 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1985.

[Neu81] Walter D. Neumann. A calculus for plumbing applied to the topology of complex surface singularities and degenerating complex curves. *Trans. Amer. Math. Soc.*, 268(2):299–344, 1981. DOI: 10.1090/S0002-9947-1981-0632532-8.

[Neu83a] Walter D. Neumann. Abelian covers of quasihomogeneous surface singularities. In *Singularities, Part 2 (Arcata, Calif., 1981)*, volume 40 of *Proc. Sympos. Pure Math.*, pages 233–243. Amer. Math. Soc., Providence, RI, 1983.

[Neu83b] Walter D. Neumann. Geometry of quasihomogeneous surface singularities. In *Singularities, Part 2 (Arcata, Calif., 1981)*, volume 40 of *Proc. Sympos. Pure Math.*, pages 245–258. Amer. Math. Soc., Providence, RI, 1983.

[NO08] András Némethi and Tomohiro Okuma. The Seiberg-Witten invariant conjecture for splice-quotients. *J. Lond. Math. Soc. (2)*, 78(1):143–154, 2008.

[NO09] András Némethi and Tomohiro Okuma. On the Casson invariant conjecture of Neumann-Wahl. *J. Algebraic Geom.*, 18(1):135–149, 2009. DOI: 10.1090/S1056-3911-08-00493-1.

[NR78] Walter D. Neumann and Frank Raymond. Seifert manifolds, plumbing, μ-invariant and orientation reversing maps. In *Algebraic and geometric topology (Proc. Sympos., Univ. California, Santa Barbara, Calif., 1977)*, volume 664 of *Lecture Notes in Math.*, pages 163–196. Springer, Berlin, 1978.

[NW02] Walter D. Neumann and Jonathan Wahl. Universal abelian covers of surface singularities. In *Trends in singularities*, Trends Math., pages 181–190. Birkhäuser, Basel, 2002.

[NW05a] Walter D. Neumann and Jonathan Wahl. Complete intersection singularities of splice type as universal abelian covers. *Geom. Topol.*, 9:699–755, 2005. DOI: 10.2140/gt.2005.9.699.

[NW05b] Walter D. Neumann and Jonathan Wahl. Complex surface singularities with integral homology sphere links. *Geom. Topol.*, 9:757–811, 2005. DOI: 10.2140/gt.2005.9.757.

[Oku04] Tomohiro Okuma. Universal abelian covers of rational surface singularities. *J. London Math. Soc. (2)*, 70(2):307–324, 2004.

[Peda] Helge Møller Pedersen. Rational homology sphere as universal abelian covers. arχiv: 1011.0285v1.

[Pedb] Helge Møller Pedersen. Splice diagram singularities and the universal abelian cover of graph orbifolds. arχiv: 1011.0890.

[Ped10] Helge Møller Pedersen. Splice diagram determining singularity links and universal abelian covers. *Geom. Dedicata*, 150:75–104, 2010.

[Sie80] Laurent Siebenmann. On vanishing of the Rohlin invariant and nonfinitely amphicheiral homology 3-spheres. In *Topology Symposium, Siegen 1979 (Proc. Sympos., Univ. Siegen, Siegen, 1979)*, volume 788 of *Lecture Notes in Math.*, pages 172–222. Springer, Berlin, 1980.

Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, 13-15 Reáltanoda u. 1053 Budapest, Hungary

E-mail address: helge@renyi.hu