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ARC SPACES OF cA-TYPE SINGULARITIES

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Let X be a complex variety or an analytic space and $x \in X$ a point. A formal arc through x is a morphism $\phi : \operatorname{Spec} \mathbb{C}[[t]] \to X$ such that $\phi(0) = x$. The set of formal arcs through x – denoted by $\widehat{\operatorname{Arc}}(x \in X)$ – is naturally a (non-noetherian) scheme.

A preprint of Nash, written in 1968 but only published later as [Nas95], describes an injection – called the *Nash map* – from the irreducible components of $\widehat{\operatorname{Arc}}(x \in X)$ to the set of so called *essential divisors.* These are the divisors whose center on every resolution $\pi : X' \to X$ is an irreducible component of $\pi^{-1}(x)$. The *Nash problem* asks if this map is also surjective or not. Surjectivity fails in dimensions ≥ 3 [IK03, dF12] but holds in dimension 2 [FdBP12b].

In all dimensions, the most delicate cases are singularities whose resolutions contain many rational curves. For example, although it is easy to describe all arcs and their deformations on Du Val singularities of type A, the type E cases have been notoriously hard to treat [PS12, Per13].

The first aim of this note is to determine the irreducible components of the arc spaces of cA-type singularities in all dimensions. In Section 1 we prove the following using quite elementary arguments.

Theorem 1. *Let $f(z_1, \dots, z_n)$ be a holomorphic function whose multiplicity at the origin is $m \geq 2$. Let $X := \bigl(xy = f(z_1, \dots, z_n)\bigr) \subset \mathbb{C}^{n+2}$ denote the corresponding cA-type singularity. Assume that $n \geq 1$.*

1. *$\widehat{\operatorname{Arc}}(0 \in X)$ has $(m-1)$ irreducible components $\widehat{\operatorname{Arc}}_i(0 \in X)$ for $0 < i < m$.*
2. *There are dense, open subsets $\widehat{\operatorname{Arc}}_i^{\circ}(0 \in X) \subset \widehat{\operatorname{Arc}}_i(0 \in X)$ such that*

$$\bigl(\psi_1(t), \psi_2(t), \phi_1(t), \dots, \phi_n(t)\bigr) \in \widehat{\operatorname{Arc}}_i^{\circ}(0 \in X)$$

$$\textit{iff } \operatorname{mult} \psi_1(t) = i, \ \operatorname{mult} \psi_2(t) = m - i \textit{ and } \operatorname{mult} f\bigl(\phi_1(t), \dots, \phi_n(t)\bigr) = m.$$

We found it much harder to compute the set of essential divisors and we have results only if $\operatorname{mult}_0 f = 2$. If $\dim X = 3$ then, after a coordinate change, we can write the equation as $(xy = z^2 - u^m)$. Already [Nas95] proved that these singularities have at most 2 essential divisors: an easy one obtained by blowing-up the origin and a difficult one obtained by blowing-up the origin twice. In Section 2 we use ideas of [dF12] to determine the cases when the second divisor is essential. The following is obtained by combining Theorem 1 and Proposition 9.

Example 2. For the singularities $X_m := (xy = z^2 - u^m) \subset \mathbb{C}^4$ the Nash map is not surjective for odd $m \geq 5$ but surjective for even m and for $m = 3$.

Thus the simplest counter example to the Nash conjecture is the singularity

$$(x^2 + y^2 + z^2 + t^5 = 0) \subset \mathbb{C}^4.$$

In higher dimensions our answers are less complete. We describe the situation for the divisors obtained by the first and second blow-ups as above, but we do not control other exceptional divisors. Using Theorem 1 and Proposition 22 we get the following partial generalization of Example 2.

Example 3. Let $g(u_1, \dots, u_r)$ be an analytic function near the origin. Set $m = \operatorname{mult}_0 g$ and let g_m denote the degree m homogeneous part of g. If $m \geq 4$ and the Nash map is surjective for the singularity

$$X_g := \big(xy = z^2 - g(u_1, \dots, u_r)\big) \subset \mathbb{C}^{r+3}$$

then $g_m(u_1, \dots, u_r)$ is a perfect square.

Since we do not determine all essential divisors, the cases when $g_m(u_1, \dots, u_r)$ is a perfect square remain undecided.

On the one hand, this can be interpreted to mean that the Nash conjecture hopelessly fails in dimensions ≥ 3. On the other hand, the proof leads to a reformulation of the Nash problem and to an approach that might be feasible, at least in dimension 3; see Section 5.

In Section 4 we observe that the deformations constructed in Section 1 also lead to an enumeration of the irreducible components of the space of short arcs – introduced in [KN13] – for cA-type singularities.

Question 4 (Arcs on cDV singularities)**.** It is easy to see that Theorem 1 is equivalent to saying that the image of every general arc on X is contained in an A-type surface section of X.

It is natural to ask if this holds for all cDV singularities. That is, let $(0 \in X) \subset \mathbb{C}^n$ be a hypersurface singularity such that $X \cap L^3$ is a Du Val singularity for every general 3-dimensional linear space (or smooth 3–fold) $0 \in L^3 \subset \mathbb{C}^n$.

Let ϕ be a general arc on X. Is it true that there is a 3–fold $L^3 \subset \mathbb{C}^n$ containing the image of ϕ such that $X \cap L^3$ is a Du Val singularity?

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1. Arcs on cA-type singularities

Definition 5 (cA-type singularities)**.** In some coordinates write a hypersurface singularity as

$$X := \big(f(x_1, \dots, x_{n+1}) = 0\big) \subset \mathbb{C}^{n+1}.$$

Assume that X is singular at the origin and let f_2 denote the quadratic part of f. If $\operatorname{mult}_0 f = 2$ then $(f_2 = 0)$ is the tangent cone of X at the origin. We say that X has *cA-type* if $\operatorname{rank} f_2 \geq 2$ and *cA_1-type* if $\operatorname{rank} f_2 \geq 3$. By the Morse lemma, if $\operatorname{rank} f_2 = r$ then we can choose local analytic or formal coordinates y_i such that

$$f = y_1^2 + \cdots + y_r^2 + g(y_{r+1}, \dots, y_{n+1}) \quad \text{where} \quad \operatorname{mult}_0 g \geq 3.$$

In the sequel we also use other forms of the quadratic part if that is more convenient.

Note that by adding 2 squares in new variables we get a map from hypersurface singularities in dimension $n-2$ (modulo isomorphism) to cA-type hypersurface singularities in dimension n (modulo isomorphism). This map is one-to-one and onto; see [AGZV85, Sec.11.1]. Thus cA-type singularities are quite complicated in large dimensions.

We rename the coordinates and write a cA-type singularity as

$$X := \big(xy = f(z_1, \dots, z_n)\big) \subset \mathbb{C}^{n+2}.$$

Thus an arc through the origin is written as

$$t \mapsto \big(\psi_1(t), \psi_2(t), \phi_1(t), \dots, \phi_n(t)\big),$$

where ψ_i, ϕ_j are power series such that $\operatorname{mult}\psi_i, \operatorname{mult}\phi_j \geq 1$ for $i = 1, 2$ and $j = 1, \ldots, n$. We set $\vec{\phi}(t) = \big(\phi_1(t), \ldots, \phi_n(t)\big)$.

A deformation of $\vec{\phi}(t)$ is given by power series $\big(\Phi_1(t,s), \ldots, \Phi_n(t,s)\big)$. Then we compute

$$f\big(\Phi_1(t,s), \ldots, \Phi_n(t,s)\big) \in \mathbb{C}[[t,s]]$$

and try to factor it as

$$\Psi_1(t,s)\Psi_2(t,s) = f\big(\Phi_1(t,s), \ldots, \Phi_n(t,s)\big)$$

where $\Psi_i(t,0) = \psi_i(t)$. Usually $f\big(\Phi_1(t,s), \ldots, \Phi_n(t,s)\big)$ is irreducible, but Newton's method of rotating rulers (Lemma 7 below) says that

$$f\big(\Phi_1(t,s^r), \ldots, \Phi_n(t,s^r)\big)$$

factors for some $r \geq 1$.

6 (Proof of Theorem 1)**.** If $\vec{\phi}(0) = \mathbf{0}$ then $\operatorname{mult} f\big(\vec{\phi}(t)\big) \geq m$. Thus, for every $0 < i < m$ we can choose any $\psi_1(t)$ such that $\operatorname{mult}\psi_1(t) = i$ and then set $\psi_2(t) = \psi_1(t)^{-1} f\big(\vec{\phi}(t)\big)$. This shows that the families $\widehat{\operatorname{Arc}}{}^{\circ}_i(0 \in X)$ are nonempty and open in $\widehat{\operatorname{Arc}}(0 \in X)$.

In order to show that their union is dense, after a linear change of coordinates we may assume that z_1^m appears in f with nonzero constant coefficient.

Set $D := \operatorname{mult}_t f\big(\phi_1(t), \ldots, \phi_n(t)\big)$. Assume first that $D < \infty$ and consider

$$F(t,s) := f\big(\phi_1(t) + st, \phi_2(t), \ldots, \phi_n(t)\big) = \sum_i \frac{\partial^i f}{\partial z_1^i}\big(\vec{\phi}\big) \cdot \frac{(st)^i}{i!}.$$

We know that t^m divides $F(s,t)$ (since $\operatorname{mult}_0 f = m$) and $(st)^m$ appears in F with nonzero coefficient (since z_1^m appears in f with nonzero coefficient). Thus t^m is the largest t-power that divides $F(s,t)$.

Furthermore, t^D is the smallest t-power that appears in F with nonzero constant coefficient. Thus, by Lemma 7 below, there is an $r \geq 1$ such that

$$F(t,s^r) = u(t,s) \prod_{i=1}^{D} \big(t - \sigma_i(s)\big),$$

where $u(0,0) \neq 0$ and $\sigma_i(0) = 0$. Furthermore, exactly m of the σ_i are identically zero.

For $j = 1, 2$ write $\psi_j(t) = t^{a_j} v_j(t)$ where $v_j(0) \neq 0$. Note that $a_1 + a_2 = D$ and

$$u(t,0) = v_1(t)v_2(t).$$

Divide $\{1, \ldots, D\}$ into two disjoint subsets A_1, A_2 such that $|A_j| = a_j$ and they both contain at least 1 index i such that $\sigma_i(t) \equiv 0$. Finally set

$$\Psi_1(t,s) = v_1(t) \cdot \prod_{i \in A_1} \big(t - \sigma_i(s)\big) \quad \text{and} \quad \Psi_2(t,s) = \frac{u(t,s)}{v_1(t)} \cdot \prod_{i \in A_2} \big(t - \sigma_i(s)\big).$$

Then

$$\big(\Psi_1(t,s), \Psi_2(t,s), \phi_1(t) + st, \phi_2(t), \ldots, \phi_n(t)\big)$$

is a deformation of $\big(\psi_1(t), \psi_2(t), \phi_1(t), \ldots, \phi_n(t)\big)$ whose general member is in the rth irreducible component as in Theorem 1.2 iff exactly r of the $\{\sigma_i : i \in A_1\}$ are identically zero.

(This also shows that arcs with $\operatorname{mult}\psi_1(t) \geq m-1$ and $\operatorname{mult}\psi_2(t) \geq m-1$ constitute the intersection of all of the $\widehat{\operatorname{Arc}}_i(0 \in X)$.)

If $D = \infty$, that is, when $f\big(\phi_1(t), \ldots, \phi_n(t)\big)$ is identically zero, we need to perform some similar preliminary deformations first.

First, if both $\psi_1(t), \psi_2(t)$ are identically zero then we can take

$$\big(st, 0, \phi_1(t), \phi_2(t), \dots, \phi_n(t)\big).$$

Hence, up-to interchanging x and y, we may assume that $d := \operatorname{mult} \psi_1(t) < \infty$. Again assuming that z_1^m appears in f with nonzero coefficient, we see that

$$F(t,s) := f\big(\phi_1(t) + st^{d+1}, \phi_2(t), \dots, \phi_n(t)\big)$$

is not identically zero and divisible by t^{d+1}. Thus $F(t,s)/\psi_1(t)$ is holomorphic and divisible by t. Therefore

$$\Big(\psi_1(t), \frac{F(t,s)}{\psi_1(t)}, \phi_1(t) + st^{d+1}, \phi_2(t), \dots, \phi_n(t)\Big)$$

is a deformation of $\big(\psi_1(t), 0, \phi_1(t), \phi_2(t), \dots, \phi_n(t)\big)$ such that

$$\operatorname{mult}_t f\big(\phi_1(t) + st^{d+1}, \phi_2(t), \dots, \phi_n(t)\big) < \infty$$

for $0 < |s| \ll 1$. $\square$

We used Newton's lemma on Puiseux series solutions in the following form.

Lemma 7. *Let $g(x,y) \in \mathbb{C}[[x,y]]$ be a power series. Assume that $m := \operatorname{mult}_0 g(x,0) < \infty$. Then there is an $r \geq 1$ such that one can write $g(x, z^r)$ as*

$$g(x, z^r) = u(x,z) \prod_{i=1}^{m} \big(x - \sigma_i(z)\big)$$

where $u(0,0) \neq 0$ and $\sigma_i(0) = 0$ for every i. The representation is unique, up-to permuting the $\sigma_i(z)$.

Furthermore, if $g(x,y)$ is holomorphic on the bidisc $\overline{\mathbb{D}}_x \times \mathbb{D}_y$ then $u(x,z)$ and the $\sigma_i(z)$ are holomorphic on the smaller bidisc $\overline{\mathbb{D}}_x \times \mathbb{D}_z(\epsilon)$ for some $0 < \epsilon \leq 1$. $\square$

2. Essential divisors on cA_1-type 3-fold singularities

In dimension 3, the only cA_1-type singularities are $X_m := (xy = z^2 - t^m)$ for $m \geq 2$. Already [Nas95, p.37] proved that they have at most 2 essential divisors. We use the method of [dF12, 4.1] to determine the precise count.

Definition 8. Let X be a normal variety or analytic space and E a divisor over X. That is, there is a birational or bimeromorphic morphisms $p : X' \to X$ such that $E \subset X'$ is an exceptional divisor. The closure of $p(E) \subset X$ is called the *center* of E on X; it is denoted by $\operatorname{center}_X E$. If $\operatorname{center}_X E = \{x\}$, we say that E is a divisor over $(x \in X)$.

We say that E is an *essential divisor* over X if for every resolution of singularities $\pi : Y \to X$, $\operatorname{center}_Y E$ is an irreducible component of $\pi^{-1}\big(\operatorname{center}_X E\big)$. (Note that $\pi^{-1} \circ p : X' \dashrightarrow Y$ is regular on a dense subset of E, hence $\operatorname{center}_Y E$ is defined.)

If X is an analytic space, then Y is allowed to be any analytic resolution. If X is algebraic, one gets slightly different notions depending on whether one allows Y to be a quasi-projective variety, an algebraic space or an analytic space; see [dF12]. We believe that for the Nash problem it is natural to allow analytic resolutions.

Proposition 9. *Set $X_m := (xy = z^2 - t^m) \subset \mathbb{C}^4$.*

(1) *If $m \geq 5$ is odd, there are 2 essential divisors.*
(2) *If $m \geq 2$ is even or $m = 3$, there is 1 essential divisor.*

Even in dimension 3, it seems surprisingly difficult to determine the set of essential divisors. A basic invariant is given by the discrepancy.

Definition 10. Let X be a normal variety or analytic space. Assume for simplicity that the canonical class K_X is Cartier. (This holds for all hypersurface singularities.) Let $\pi : Y \to X$ be a resolution of singularities and write

$$K_Y \sim \pi^* K_X + \textstyle\sum_i a(E_i, X) E_i,$$

where the E_i are the π-exceptional divisors. The integer $a(E_i, X)$ is called the *discrepancy* of E_i. (See [KM98, Sec.2.3] for basic references and more general definitions.)

For example, let X be smooth and $Z \subset X$ a smooth subvariety of codimension r. Let $\pi_Z : B_Z X \to X$ denote the blow-up and $E_Z \subset B_Z X$ the exceptional divisor. Then $a(E_Z, X) = r - 1$ and easy induction shows that $a(F, X) \geq r$ for every other divisor whose center on X is Z.

We say that X is *canonical* (resp. *terminal*) of $a(E_i, X) \geq 0$ (resp. $a(E_i, X) > 0$) for every resolution and every exceptional divisor.

For instance, normal cA-type singularities are canonical and a cA-type singularity is terminal iff its singular set has codimension ≥ 3; see [Rei83] for a proof that applies to all cDV singularities or [Kol13, 1.42] for a simpler argument in the cA case.

11 (Resolving X_m)**.** Blow up the origin to get $\pi_1 : X_{m,1} := B_0 X_m \to X_m$. The exceptional divisor is the singular quadric $E_1 \cong (xy - z^2 = 0) \subset \mathbb{P}^3(x, y, z, t)$.

If $m \in \{2, 3\}$ then $B_0 X$ is smooth, hence the only essential divisor is E_1.

For $m \geq 4$ the resulting $B_0 X_m$ has one singular point, visible in the chart

$$(x_1, y_1, z_1, t) := \big(x/t, y/t, z/t, t\big)$$

where the local equation is $x_1 y_1 = z_1^2 - t^{m-2}$. We can thus blow up the origin again and continue. After $r := \lfloor \frac{m}{2} \rfloor$ steps we have a resolution

$$\Pi_r : X_{m,r} \to X_{m,r-1} \to \cdots \to X_{m,1} \to X_m.$$

We get r exceptional divisors $E_r, \dots, E_1$. For $1 \leq c \leq r$ the divisor E_c first appears on $X_{m,c}$. At the unique singular point one can write the local equation as

$$X_{m,c} = \big(x_c y_c = z_c^2 - t^{m-2c}\big) \quad \text{and} \quad E_c = (t = 0).$$

where $(x_c, y_c, z_c, t) := \big(x/t^c, y/t^c, z/t^c, t\big)$.

We thus need to decide which of the divisors $E_1, \dots, E_{\lfloor \frac{m}{2} \rfloor}$ are essential. It is easy to see that E_1 is essential and a direct computation (Lemma 15 below) shows that $E_3, \dots, E_{\lfloor \frac{m}{2} \rfloor}$ are not. (This is actually not needed in order to establish Example 2.) The hardest is to decide what happens with E_2.

Lemma 12. *Notation as above. Then*

1. $a(E_c, X_m) = c$ *for every* c.
2. E_1 *is the only exceptional divisor whose center is the origin and whose discrepancy is* 1.
3. E_1 *appears on every resolution of* X_m *whose exceptional set is a divisor.*
4. *Let* $p : Y \dashrightarrow X_m$ *be any (not necessarily proper) bimeromorphic map from a smooth analytic space* Y *such that* $\operatorname{center}_Y E_1 \subset Y$ *is not empty. Then* $\operatorname{center}_Y E_1$ *is an irreducible component of the exceptional set* $\operatorname{Ex}(p)$.

Proof. The first claim follows from the formula

$$\Pi_r^*\Big(\tfrac{dx \wedge dy \wedge dt}{z}\Big) = t^{-c} \cdot \tfrac{dx_c \wedge dy_c \wedge dt}{z_c}.$$

Let F be any other exceptional divisor whose center is the origin. Then $\operatorname{center}_{X_r} F$ lies on one of the E_c, thus $a(F, X) > a(E_c, X) \geq 1$. (This also proves that X_m is terminal.)

To see (3) set $W_1 := \operatorname{center}_Y E_1 \subset Y$. Let $F_i \subset Y$ be the exceptional divisors and note that, as in [KM98, 2.29],

$$a(E_1, X_m) \geq \bigl(\operatorname{codim}_Y W_1 - 1\bigr) + \textstyle\sum_i \operatorname{mult}_{W_1} F_i \cdot a(F_i, X_m). \tag{12.5}$$

Note that $a(E_1, X_m) = 1$ and $a(F_i, X_m) \geq 1$ for every i. If W_1 is not an irreducible component of $\operatorname{Ex}(p)$ then $W_1 \subset F_i$ form some i and then both terms on the right hand side of (12.5) are positive, a contradiction. □

13 (Small resolutions and factoriality of X_m)**.** If $m = 2a$ is even, then X_m has a small resolution obtained by blowing up either $(x = z - t^a = 0)$ or $(x = z + t^a = 0)$. The resulting blow-ups $Y_{2a}^{\pm} \subset \mathbb{C}^4_{xyzt} \times \mathbb{P}^1_{uv}$ are defined by the equations

$$Y_{2a}^{\pm} := \operatorname{rank} \begin{pmatrix} x & z \pm t^a & u \\ z \mp t^a & y & v \end{pmatrix} \leq 1 \tag{13.1}$$

We show that X_m does not have small resolutions if m is odd. More generally, let

$$X_f := \bigl(xy = f(z,t)\bigr) \subset \mathbb{C}^4$$

be an isolated cA-type singularity. Write $f = \prod_j f_j$ as a product of irreducibles. The f_j are distinct since the singularity is isolated. Set $D_j := (x = f_j = 0)$. By [Kol91, 2.2.7] the local divisor class group is

$$\operatorname{Div}\bigl(0 \in X_f\bigr) = \bigl(\textstyle\sum_j \mathbb{Z}[D_j]\bigr) / \textstyle\sum_j [D_j]. \tag{13.2}$$

In particular, X_f is factorial iff f is irreducible.

This formula works both algebraically and analytically. If we are interested in the affine variety X_f, then we consider factorizations of f in the polynomial ring. If we are interested in the complex analytic germ X_f, then we consider factorizations of f in the ring of germs of analytic functions. Thus, for example,

$$(xy = z^2 - t^2 - t^3) \subset \mathbb{C}^4$$

is algebraically factorial, since $z^2 - t^2 - t^3$ is an irreducible polynomial, but it is not analytically factorial, since

$$z^2 - t^2 - t^3 = \bigl(z - t\sqrt{1+t}\bigr)\bigl(z + t\sqrt{1+t}\bigr).$$

Thus if m is odd then X_m is factorial (both algebraically and analytically) and it does not have small resolutions; see Lemma 17 below for stronger results.

Lemma 14. *If m is even then there is a divisorial resolution whose sole exceptional divisor is birational to E_1. Thus the only essential divisor is E_1.*

Proof. The $m = 2$ case was noted in Paragraph 11, hence we may assume that $m = 2a \geq 4$.

There are 2 ways to obtain such resolutions. First, we can blow up the exceptional curve in either of the $Y_{2a}^{\pm}$ as in (13.1).

Alternatively, we first blow up the origin to get $B_0 X_m$ which has one singular point with local equation $x_1 y_1 = z_1^2 - t_1^{2a-2}$ and then blow up

$$D^+ := (x_1 = z_1 + t_1^{a-1} = 0)$$

or

$$D^- := (x_1 = z_1 - t_1^{a-1} = 0).$$

□

Lemma 15. [Nas95, p.37] *The divisors $E_3, \dots, E_r$ are not essential.*

Proof. If m is even, this follows from Lemma 14, but for the proof below the parity of m does not matter.

If $2b \geq a \geq 0$ and $m \geq a$ then $(u, v, w, t) \mapsto (ut, vt^{a+1}, wt^{b+1}, t) = (x, y, z, t)$ defines a birational map

$$g(a, b, m) : Z_{abm} := \left(uv = w^2 t^{2b-a} - t^{m-2-a}\right) \to X_m.$$

Note that $\operatorname{Ex}\big(g(a, b, m)\big) = (t = 0)$ is mapped to the origin and Z_{abm} is smooth along the v-axis, save at the origin.

If $1 \leq c \leq m/2$ then $(x_c, y_c, z_c, t) \mapsto (x_c t^c, y_c t^c, z_c t^c, t) = (x, y, z, t)$ defines a birational map

$$h(c, m) : X_{m,c} := \left(x_c y_c = z_c^2 - t^{m-2c}\right) \to X_m.$$

By composing we get a birational map $g(a, b, m)^{-1} \circ h(c, m) : Y_c \dashrightarrow Z_{abm}$ given by

$$(x_c, y_c, z_c, t) \mapsto (x_c t^{c-1}, y_c t^{c-a-1}, z_c t^{c-b-1}, t) = (u, v, w, t)$$

which is a morphism if $c \geq a+1, b+1$. If $c = a+1$ and $c > b+1$ then we have

$$(x_c, y_c, z_c, t) \mapsto (x_c t^{c-1}, y_c, z_c t^{c-b-1}, t) = (u, v, w, t)$$

which maps E_c to the v-axis.

If $c \geq 3$ then by setting $a = c-1, b = c-2$ we get a birational morphism

$$p(c, m) := g(c, c-1, m)^{-1} \circ h(c, m)$$

given by

$$(x_c, y_c, z_c, t) \mapsto (x_c t^c, y_c, z_c t, t) = (u, v, w, t).$$

Note that

$$p(c, m) : Y_c = \left(x_c y_c = z_c^2 - t^{m-2c}\right) \to \left(uv = w^2 t^{c-2} - t^{m-c}\right) = Z_{c,c-1,m}$$

maps E_c onto the v-axis. Thus E_c is not essential for $c \geq 3$. □

Lemma 16. *If $m \geq 5$ is odd then E_2 is essential.*

Proof. We follow the arguments in [dF12, 4.1]. Let $p : Y \to X_m$ be any resolution and set $Z := \operatorname{center}_Y E_2 \subset Y$. Since X_m is factorial (here we use that m is odd), $\operatorname{Ex}(p)$ has pure dimension 2 by Lemma 17.2.

Assume to the contrary that Z is not a divisor. Using that $a(E_2, X_m) = 2$, (12.5) implies that Z is a curve, there is a unique exceptional divisor $F \subset Y$ that contains Z, F is smooth at general points of Z and $a(F, X_m) = 1$.

If $p(F)$ is a curve then Z is an irreducible component of $p^{-1}(0)$. The remaining case is when $p(F) = 0$, thus $F = E_1$ by Lemma 12.2.

Since t vanishes along E_2 with multiplicity 1, it also vanishes along Z with multiplicity 1. Since p^*x, p^*y, p^*z, p^*t all vanish along E_1, the rational functions $p^*(x/t), p^*(y/t), p^*(z/t)$ are regular generically along Z. Thus $p_1 := \pi_1^{-1} \circ p : Y \dashrightarrow X_{m,1}$ is a morphism generically along Z. Note that our E_2 is what we would call E_1 if we started with $X_{m,1}$. Applying Lemma 12.4 to $p_1 : Y \dashrightarrow X_{m,1}$ we see that Z is an irreducible component of $\operatorname{Ex}(p_1)$. Since m is odd, $X_{m,1}$ is analytically factorial by Paragraph 13, hence Z is a divisor by Lemma 17.2 below. This is a contradiction. □

Lemma 17. *Let X, Y be normal varieties or analytic spaces and $g : Y \to X$ a birational or bimeromorphic morphism. Then the exceptional set $\operatorname{Ex}(g)$ has pure codimension 1 in Y in the following cases.*

(1) *Y is an algebraic variety and X is $\mathbb{Q}$-factorial.*
(2) *$\dim Y = 3$ and X is analytically locally $\mathbb{Q}$-factorial.*

Proof. The algebraic case is well known; see for instance the method of [Sha74, Sec.II.4.4].

If $\dim Y = 3$ and $\operatorname{Ex}(g)$ does not have pure codimension 1 then it has a 1-dimensional irreducible component $C \subset Y$. After replacing X by a suitable neighborhood of $g(C) \in X$ we may assume that there is a divisor $D_Y \subset Y$ such that $\operatorname{Ex}(g) \cap D_Y$ is a single point of C and $g|_{D_Y}$ is proper. Thus $D_X := g(D_Y)$ is a divisor on X. If mD_X is Cartier then so is $g^*(mD_X)$ hence its support has pure codimension 1 in Y. On the other hand, $\operatorname{Supp}\big(g^*(mD_X)\big) = \operatorname{Ex}(g) \cup D_Y$ does not have pure codimension 1. (Note that there are many possible choices for D_Y; the resulting D_X determine an algebraic equivalence class of divisors.) □

Somewhat surprisingly, the analog of Lemma 17.2 fails in dimension 4.

Example 18. Let $W \subset \mathbb{P}^4$ be a smooth quintic 3–fold and $C \subset W$ a line whose normal bundle is $\mathcal{O}(-1) + \mathcal{O}(-1)$. Let $X \subset \mathbb{C}^5$ denote the cone over W with vertex 0; it is analytically locally factorial by [Gro68, XI.3.14].

The exceptional divisor of the blow-up $B_0X \to X$ can be identified with W; let $C \subset B_0X$ be our line. Its normal bundle is $\mathcal{O}(-1) + \mathcal{O}(-1) + \mathcal{O}(-1)$.

Blow up the line C to obtain $B_C B_0 X \to B_0 X$. Its exceptional divisor is $E \cong \mathbb{P}^1 \times \mathbb{P}^2$. One can contract E in the other direction to obtain $g : Y \to X$.

By construction, $\operatorname{Ex}(g)$ is the union of $\mathbb{P}^2$ and of a 3-fold obtained from W by flopping the line C. The two components intersect along a line.

This completes our analysis of 3–dimensional cA_1-type singularities. Our study of the higher dimensional cases relies on a deeper understanding of the proof of Lemma 17.2 for

$$X_c := \big(xy = z^2 - ct^m\big),$$

where $c \neq 0$.

The reader may wish to jump to Section 3 and return to this point once formula (22.3) shows why the question answered in Proposition 19 is of interest.

Let $g_c : Y_c \to X_c$ be a proper birational or bimeromorphic morphism and $E_c \subset \operatorname{Ex}(g_c)$ a 1-dimensional irreducible component.

The proof of Lemma 17.2 associates to E_c an algebraic equivalence class of non-Cartier divisors on X_c. Thus m has to be even by Paragraph 13.

If $m = 2a$ is even then the divisor class group is $\operatorname{Div}(X_c) \cong \mathbb{Z}$. The two possible generators correspond to $(x = z - \sqrt{c}t^a = 0)$ and $(x = z + \sqrt{c}t^a = 0)$. Starting with E_c we constructed a divisor $D_c \subset X_c$ which is a nontrivial element of $\operatorname{Div}(X_c)$. Thus $[D_c]$ is a positive multiple of either $(x = z - \sqrt{c}t^a = 0)$ or $(x = z + \sqrt{c}t^a = 0)$. Hence, to $E_c \subset Y_c$ we can associate a choice of $\sqrt{c}$.

This may not be very interesting for a fixed value of c (since many other choices are involved) but it turns out to be quite useful when c varies.

Proposition 19. *Let $g(u_1, \dots, u_r, v)$ be a holomorphic function for $u_i \in \mathbb{C}$ and $|v| < \epsilon$ such that $g(u_1, \dots, u_r, 0)$ is not identically zero. For $m \geq 4$ set*

$$X := \big(xy = z^2 - v^m g(u_1, \dots, u_r, v)\big) \subset \mathbb{C}^{r+4}.$$

Let $\pi : Y \to X$ be a birational or bimeromorphic morphism. Assume that there is an irreducible component $Z \subset \operatorname{Ex}(\pi)$ that dominates $(x = y = z = v = 0) \subset X$, has codimension ≥ 2 in Y and such that $\pi|_Z : Z \to (x = y = z = v = 0)$ has connected fibers.

Then m is even and $g(u_1, \dots, u_r, 0)$ is a perfect square.

Proof. For general $\mathbf{c} = (c_1, \dots, c_r) \in \mathbb{C}^r$ the repeated hyperplane section

$$X(\mathbf{c}) := \big(xy = z^2 - v^m g(\mathbf{c}, v)\big) \subset \mathbb{C}^4$$

has an isolated singularity at the origin and we get a proper birational or bimeromorphic morphism $\pi(\mathbf{c}) : Y(\mathbf{c}) \to X(\mathbf{c})$ where $Y(\mathbf{c}) \subset Y$ is the preimage of $X(\mathbf{c})$.

Furthermore, $Z(\mathbf{c}) := Z \cap Y(\mathbf{c})$ is an irreducible component of $\operatorname{Ex}\bigl(\pi(\mathbf{c})\bigr)$ and has codimension ≥ 2 in $Y(\mathbf{c})$.

Thus, as we noted above, $m = 2a$ is even and our construction gives a function

$$(c_1, \dots, c_r) \mapsto \text{a choice of } \sqrt{g(c_1, \dots, c_r, 0)}.$$

It is clear that this function is continuous on a Zariski open set $U \subset \mathbb{C}^r$. Therefore $g(u_1, \dots, u_r, 0)$ is a perfect square. □

Remark 20. Conversely, assume that m is even and $g(u_1, \dots, u_r, 0) = h^2(u_1, \dots, u_r)$ is a square. Write the equation of X as

$$xy = z^2 - v^m\bigl(h^2(u_1, \dots, u_r) + vR(u_1, \dots, u_r, v)\bigr)$$

Over the open set $X^0 \subset X$ where $h \neq 0$, change coordinates to $w := h^{-2}v$. (Equivalently, blow up $(v = h = 0)$ twice.) Then

$$D := \bigl(x = z - w^{m/2}h^{m+1}\sqrt{1 + wR(u_1, \dots, u_r, h^2 w)}\bigr)$$

is a globally well defined analytic divisor. Blowing it up gives a bimeromorphic morphism $X_D \to X$ whose exceptional set over X^0 has codimension 2.

It seems that even if X is algebraic, usually X_D is not an algebraic variety.

3. Essential divisors on cA_1-type singularities

In higher dimensions cA_1-type singularities are more complicated and their resolutions are much harder to understand. There is no simple complete answer as in dimension 3.

In the previous Section, the key part was to understand the exceptional divisors that correspond to the first 2 blow-ups. These are the 2 divisors that we understand in higher dimensions as well.

21 (Defining E_1 and E_2)**.** In order to fix notation, write the equation as

$$X := \bigl(xy = z^2 - g(u_1, \dots, u_r)\bigr) \subset \mathbb{C}^{r+3}. \tag{21.1}$$

Set $m := \operatorname{mult}_0 g$ and let $g_s(u_1, \dots, u_r)$ denote the homogeneous degree s part of g. In a typical local chart the 1st blow-up $\sigma_1 : X_1 := B_0 X \to X$ is given by

$$x_1 y_1 = z_1^2 - \bigl(u'_r\bigr)^{-2} g(u'_1 u'_r, \dots, u'_{r-1} u'_r, u'_r) \tag{21.2}$$

where $x = x_1 u'_r, y = y_1 u'_r, z = z_1 u'_r,\ u_1 = u'_1 u'_r, \dots, u_{r-1} = u'_{r-1} u'_r$ and $u_r = u'_r$. The exceptional divisor is the rank 3 quadric

$$E_1 := \bigl(x_1 y_1 - z_1^2 = 0\bigr) \subset \mathbb{P}^{r+2}. \tag{21.3}$$

Note also that

$$\begin{aligned}\bigl(u'_r\bigr)^{-2} g(u'_1 u'_r, \dots, u'_{r-1} u'_r, u'_r) &= \\ = \bigl(u'_r\bigr)^{m-2}\Bigl(g_m(u'_1, \dots, u'_{r-1}, 1) + u'_r g_{m+1}(u'_1, \dots, u'_{r-1}, 1) + \cdots\Bigr).\end{aligned} \tag{21.4}$$

From this we see that, for $m \geq 4$, the blow-up X_1 is singular along the closure of the linear space

$$L := (x_1 = y_1 = z_1 = u'_r = 0), \tag{21.5}$$

X_1 has terminal singularities and a general 3-fold section has equation

$$x_1 y_1 = z_1^2 - \bigl(u'_r\bigr)^{m-2}\Bigl(g_m(c_1, \dots, c_{r-1}, 1) + u'_r g_{m+1}(c_1, \dots, c_{r-1}, 1) + \cdots\Bigr).$$

Blowing up the closure of L we obtain X_2 with exceptional divisor E_2. As in Lemma 12 we compute that

(6) $a(E_1, X) = r$,
(7) $a(E_2, X) = r + 1$,
(8) $a(F, X) \geq r + 1$ for every other exceptional divisor whose center on X is the origin and
(9) the pull-backs of the u_i vanish along E_1, E_2 with multiplicity 1.

The key computation is the following.

Proposition 22. *Notation as above and assume that $m \geq 4$.*

(1) *E_1 is an essential divisor.*
(2) *E_2 is an essential divisor iff $g_m(u_1, \dots, u_r)$ is not a perfect square.*

Proof. By (21.6) and (21.8), E_1 has the smallest discrepancy among all divisors over X whose center on X is the origin. Thus E_1 is essential by Proposition 24.

If E_2 is non-essential then there is a resolution $\pi : Y \to X$ and an irreducible component $W \subset \operatorname{Supp} \pi^{-1}(0)$ such that $Z := \operatorname{center}_Y E_2 \subsetneq W$. By (21.9), the $\pi^* u_i$ vanish at a general point of Z with multiplicity 1. Since the $\pi^* u_i$ vanish along W, this implies that $\operatorname{Supp} \pi^{-1}(0)$ is smooth at a general point of Z. In particular, W is the only irreducible component of $\operatorname{Supp} \pi^{-1}(0)$ that contains Z and W is smooth at general points of Z. Therefore the blow-up $B_W Y$ is smooth over the generic point of Z. So, if we replace Y by a suitable desingularization of $B_W Y$, we get a situation as before where, in addition, W is a divisor.

The $\pi^* u_i$ are local equations of W at general points of Z and $\pi^* x, \pi^* y, \pi^* z$ all vanish along W. Thus the rational functions

$$\pi^*(x/u_r), \pi^*(y/u_r), \pi^*(z/u_r), \pi^*(u_1/u_r), \dots, \pi^*(u_{r-1}/u_r),$$

are all regular at general points of Z. Hence the birational map $\sigma_1^{-1} \circ \pi : Y \to B_0 X = X_1$ is a morphism at general points of Z. Furthermore, $\sigma_1^{-1} \circ \pi$ maps W birationally to $E_1 \subset X_1$ and it is not a local isomorphism along Z since Y is smooth but X_1 is singular along the center L of E_2. Thus Z is an irreducible component of $\operatorname{Ex}(\sigma_1^{-1} \circ \pi)$. Since $E_2 \to L$ has connected fibers, all the assumptions of Proposition 19 are satisfied by the equation of the blow-up

$$x_1 y_1 = z_1^2 - (u'_r)^{m-2} \Big(g_m(u'_1, \dots, u'_{r-1}, 1) + u'_r g_{m+1}(u'_1, \dots, u'_{r-1}, 1) + \cdots \Big). \tag{22.3}$$

Thus m is even and $g_m(u'_1, \dots, u'_{r-1}, 1)$ is a perfect square. Since it is a dehomogenization of $g_m(u_1, \dots, u_{r-1}, u_r)$, the latter is also a perfect square.

The converse follows from Remark 20. □

Definition 23. For $(x \in X)$ let min-discrep$(x \in X)$ be the infimum of the discrepancies $a(E, X)$ where E runs through all divisors over X such that $\operatorname{center}_X E = \{x\}$. (It is easy to see that either min-discrep$(x \in X) \geq -1$ and the infimum is a minimum or min-discrep$(x \in X) = -\infty$; cf. [KM98, 2.31]. We do not need these facts.)

Proposition 24. *Let $(x \in X)$ be a canonical singularity and E a divisor over X such that* $\operatorname{center}_X E = \{x\}$ *and $a(E, X) < 1 +$* min-discrep$(x \in X)$. *Then E is essential.*

Proof. Let F be any non-essential divisor over X whose center on X is the origin. Thus there is a resolution $\pi : Y \to X$ and an irreducible component $W \subset \operatorname{Supp} \pi^{-1}(x)$ such that $Z := \operatorname{center}_Y F \subsetneq W$. Let E_W be the divisor obtained by blowing up $W \subset Y$. As we noted in Definition 10,

$$a(E_W, Y) = \operatorname{codim}_Y W - 1 \quad \text{and} \quad a(F, Y) \geq \operatorname{codim}_Y Z - 1 \geq \operatorname{codim}_Y W. \tag{24.1}$$

Write $K_Y = \pi^* K_X + D_Y$ where D_Y is effective since X is canonical and note that

$$a(E_W, X) = a(E_W, Y) + \operatorname{mult}_W D_Y \quad \text{and} \quad a(F, X) \geq a(F, Y) + \operatorname{mult}_Z D_Y. \tag{24.2}$$

Since $\operatorname{mult}_Z D_Y \geq \operatorname{mult}_W D_Y$, we conclude that

$$a(F, X) \geq 1 + a(E_W, X) \geq 1 + \text{min-discrep}(x \in X). \tag{24.3}$$

Thus any divisor E with $a(E, X) < 1 + \text{min-discrep}(x \in X)$ is essential. □

4. Short arcs

Let $\mathbb{D} \subset \mathbb{C}$ denote the open unit disk and $\overline{\mathbb{D}} \subset \mathbb{C}$ its closure. The open (resp. closed) disc of radius ϵ is denoted by $\mathbb{D}(\epsilon)$ (resp. $\overline{\mathbb{D}}(\epsilon)$). If several variables are involved, we use a subscript to indicate the name of the coordinate.

25 (Short arcs)**.** [KN13] Let X be an analytic space and $p \in X$ a point. A *short arc* on $(p \in X)$ is a holomorphic map $\phi(t) : \overline{\mathbb{D}}_t \to X$ such that $\operatorname{Supp} \phi^{-1}(p) = \{0\}$.

The space of all short arcs is denoted by $\operatorname{ShArc}(p \in X)$. It has a natural topology and most likely also a complex structure that, at least for isolated singularities, locally can be written as the product of a finite dimensional complex space and of a complex Banach space; see [KN13, Sec.11] for details.

A *deformation* of short arcs is a holomorphic map $\Phi(t, s) : \overline{\mathbb{D}}_t \times \mathbb{D}_s \to X$ such that

$$\Phi(t, s_0) : \overline{\mathbb{D}}_t \to X$$

is a short arc for every $s_0 \in \mathbb{D}_s$. Equivalently, if $\operatorname{Supp} \Phi^{-1}(p) = \{0\} \times \mathbb{D}_s$.

In general the space of short arcs has more connected components than the space of formal arcs. As a simple example, consider arcs on $(xy = z^m) \subset \mathbb{C}^3$. For $0 < i < m$ the deformations

$$(t, s) \mapsto \big(t^i (t+s)^{m-i}, t^{m-i}(t+s)^i, t(t+s)\big) \tag{25.1}$$

show that the arc (t^m, t^m, t^2) is in the closure of the families $\widehat{\operatorname{Arc}}^{\circ}_i(0 \in X)$, provided we work in the space of formal arcs. However, (25.1) is *not* a deformation of short arcs and (t^m, t^m, t^2) is a typical member of a new connected component of $\operatorname{ShArc}\big(0 \in (xy = z^m)\big)$.

By contrast, adding one more variable kills this component. For example, starting with the arc $(t^m, t^m, t^2, 0)$ on $(xy = z^m) \subset \mathbb{C}^4$, we have deformations of short arcs

$$(t, s) \mapsto \big(t^i (t+s)^{m-i}, t^{m-i}(t+s)^i, t(t+s), ts\big). \tag{25.2}$$

This example turns out to be typical and it is quite easy to modify the deformations in the proof of Theorem 1 to yield the following.

Theorem 26. *Let $X = (xy = f(z_1, \dots, z_n) \subset \mathbb{C}^{n+2}$ be a cA-type singularity. Assume that $n \geq 2$ and $m := \operatorname{mult}_0 f \geq 2$.*

Then $\operatorname{ShArc}(0 \in X)$ has $(m-1)$ irreducible components as in Theorem 1.2.

It is not always clear if a deformation $\Phi(t, s)$ is short or not. There is, however, one case when this is easy, at least over a smaller disc $\mathbb{D}_s(\epsilon) \subset \mathbb{D}_s$.

Lemma 27. *Let $\Phi(t, s) = \big(\Phi_1(t, s), \dots, \Phi_r(t, s)\big)$ be a deformation of arcs on $X \subset \mathbb{C}^r$. Assume that $\Phi(t, 0)$ is short and $\Phi_i(t, s)$ is independent of s and not identically zero for some i. Then $\Phi(t, s_0) : \overline{\mathbb{D}}_t \to X$ is short for $|s_0| \ll 1$.*

Proof. By assumption $\Phi(*, s_0)^{-1}(p) \subset \Phi_i(*, s_0)^{-1}(p) = \Phi_i(*, 0)^{-1}(p)$ for every $s_0 \in \mathbb{D}_s$, thus there is a finite subset $Z = \Phi_i(*, 0)^{-1}(p) \subset \overline{\mathbb{D}}_t$ such that

$$\Phi^{-1}(0) \subset Z \times \mathbb{D}_s \quad \text{and} \quad \Phi^{-1}(0) \cap (s = 0) = \{(0, 0)\}.$$

Since $\Phi^{-1}(0)$ is closed, this implies that

$$\Phi^{-1}(0) \cap \big(\overline{\mathbb{D}}_t \times \mathbb{D}_s(\epsilon)\big) \subset \{0\} \times \mathbb{D}_s(\epsilon) \quad \text{for } 0 < \epsilon \ll 1. \qquad \square$$

28 (Proof of Theorem 26)**.** At the very beginning of the proof of Theorem 1, after a linear change of coordinates we may assume that z_1^m appears in f with nonzero coefficient and ϕ_2 is not identically zero. Then the construction gives a deformation of short arcs by Lemma 27.

The deformations at the end of the proof were written to yield short arcs. $\square$

5. A revised version of the Nash problem

As we saw, the Nash map is not surjective in dimensions ≥ 3. In this section we develop a revised version of the notion of essential divisors. This leads to a smaller target for the Nash map, so surjectivity should become more likely. Our proposed variant of the Nash problem at least accounts for all known counter examples.

We start with a reformulation of the original definition of essential divisors.

29. Let Y be a complex variety and $Z \subset Y$ a closed subset. Let $\widehat{\operatorname{Arc}}(Z \subset Y)$ denote the scheme of formal arcs $\phi : \operatorname{Spec} \mathbb{C}[[t]] \to Y$ such that $\phi(0) \in Z$.

An easy but key observation is the following.

29.1. If Y is smooth, then the irreducible components of $\widehat{\operatorname{Arc}}(Z \subset Y)$ are in a natural one–to–one correspondence with the irreducible components of Z.

We say that a divisor E over Y is *essential* for $Z \subset Y$ if E is obtained by blowing up one of the irreducible components of Z. (For each irreducible component $Z_i \subset Z$, the blow-up $B_Z Y$ contains a unique divisor that dominates Z_i.)

The definition of essential divisors can now be reformulated as follows.

29.2. Let $(x \in X)$ be a singularity. A divisor E is *essential* for $(x \in X)$ if E is essential for $\big(\operatorname{Supp} \pi^{-1}(x) \subset Y\big)$ for every resolution $\pi : Y \to X$.

In order to refine the Nash problem, we need to understand singular spaces for which the analog of (29.1) still holds.

Definition 30 (Sideways deformations)**.** Let X be a variety (or an analytic space) and

$$\phi : \operatorname{Spec} \mathbb{C}[[t]] \to X$$

a formal arc such that $\phi(0) \in \operatorname{Sing} X$. A *sideways deformation* of ϕ is a morphism

$$\Phi : \operatorname{Spec} \mathbb{C}[[t, s]] \to X$$

such that

$$\Phi^* I_{\operatorname{Sing} X} \supset (t, s)^m \quad \text{for some } m \geq 1,$$

where $I_{\operatorname{Sing} X} \subset \mathcal{O}_X$ is the ideal sheaf defining $\operatorname{Sing} X$.

If Φ comes from a convergent arc $\Phi^{\mathrm{an}} : \mathbb{D}_t \times \mathbb{D}_s \to X$ then this is equivalent to assuming that for every $0 \neq |s_0| \ll 1$ the nearby arc $\Phi^{\mathrm{an}}(t, s_0)$ maps $\mathbb{D}_t(\epsilon)$ to $X \setminus \operatorname{Sing} X$ for some $0 < \epsilon \leq 1$.

We say that $(x \in X)$ is *arc-wise Nash-trivial* if every general arc in $\widehat{\operatorname{Arc}}(x \in X)$ has a sideways deformation. (By [FdBP12a], this implies that every arc in $\widehat{\operatorname{Arc}}(x \in X)$ has a sideways deformation.)

Comment 31. If $(x \in X)$ is an isolated singularity with a small resolution $\pi : X' \to X$ then every arc has a sideways deformation. We can lift the arc to X' and there move it away from the π-exceptional set. This is not very interesting and the notion of essential divisors captures this phenomenon.

To exclude these cases, we are mainly interested in arc-wise Nash-trivial singularities that do not have small modifications. If arc-wise Nash-trivial singularities are log terminal then assuming analytic $\mathbb{Q}$-factoriality captures this restriction, but in general one needs to be careful of the difference between analytic $\mathbb{Q}$-factoriality and having no small modifications.

Also, in the few examples of which we know, general arcs of every irreducible component of $\widehat{\mathrm{Arc}}(x \in X)$ have sideways deformations. If there are singularities where sideways deformations exist only for some of the irreducible components, the following outline needs to be suitably modified.

The main observation is that, for the purposes of the Nash problem, $\mathbb{Q}$-factorial arc-wise Nash-trivial singularities should be considered as good as smooth points. The first evidence is the following straightforward analog of (29.1).

Lemma 32. *Let Y be a complex space with isolated, arc-wise Nash-trivial singularities. Let $Z \subset Y$ a closed subset that is the support of an effective Cartier divisor. Then the irreducible components of $\widehat{\mathrm{Arc}}(Z \subset Y)$ are in a natural one–to–one correspondence with the irreducible components of Z.* □

If Z has lower dimensional irreducible components, the situation seems more complicated, but, at least in dimension 3, the following seems to be the right generalization of (29.1).

Conjecture 33. *Let Y be a 3–dimensional complex space with isolated, $\mathbb{Q}$-factorial, arc-wise Nash-trivial singularities. Let $Z \subset Y$ be a closed subset. Then the irreducible components of $\widehat{\mathrm{Arc}}(Z \subset Y)$ are in a natural one–to–one correspondence with the union of the following two sets.*

(1) *Irreducible components of Z.*
(2) *Irreducible components of $\widehat{\mathrm{Arc}}(p \in Y)$, where $p \in Y$ is any singular point such that $p \in Z$ and $\dim_p Z \leq 1$.*

Definition 34. With the above assumptions, a divisor over Y is *essential* for $Z \subset Y$ if it corresponds to one of the irreducible components of $\widehat{\mathrm{Arc}}(Z \subset Y)$, as enumerated in Conjecture 33.1–2.

Definition 35. Let $(x \in X)$ be a 3–dimensional, normal singularity. A divisor E over X is called *very essential* for $(x \in X)$ if E is essential for $\big(\mathrm{Supp}\, \pi^{-1}(x) \subset Y\big)$ for every proper bimeromorphic morphism $\pi : Y \to X$ where Y has only isolated, $\mathbb{Q}$-factorial, arc-wise Nash-trivial singularities. (As in Definition 8, it is better to allow Y to be an analytic space.)

It is easy to see that the Nash map is an injection from the irreducible components of $\widehat{\mathrm{Arc}}(x \in X)$ into the set of very essential divisors. One can hope that there are no other obstructions.

Problem 36 (Revised Nash problem)**.** Is the Nash map surjective onto the set of very essential divisors for normal 3-fold singularities?

As a first step, one should consider the following.

Problem 37. In dimension 3, classify all $\mathbb{Q}$-factorial, arc-wise Nash-trivial singularities.

Hopefully they are all terminal and a complete enumeration is possible. The papers [Hay05a, Hay05b] contain several results about partial resolutions of terminal singularities.

We treat two easy cases next. A positive solution of Question 4 would imply that all isolated, 3-dimensional cDV singularities are arc-wise Nash-trivial.

Theorem 38. *Let* $(0 \in X)$ *be a cA-type singularity such that* $\dim \operatorname{Sing} X \leq \dim X - 3$. *Then all arcs in* $\widehat{\operatorname{Arc}}^{\circ}_i(0 \in X)$ *(as in Theorem 1.2) have sideways deformations.*

Proof. We use the notation of the proof of Theorem 1.

Since $\operatorname{mult} f\big(\phi_1(t), \dots, \phi_n(t)\big) = m$, we see that $\operatorname{mult} \phi_j(t) = 1$ for at least one index j. We may assume that $j = 1$ and $\phi_1(t) = t$. Thus, after the coordinate change $z_i \mapsto z_i - \phi_i(z_1)$ for $i = 2, \dots, n$ and an additional general linear coordinate change among the $z_2, \dots, z_n$ we may assume that

(1) $\phi_1(t) = t$,
(2) $\phi_j(t) \equiv 0$ for $j > 1$,
(3) $\big(xy = g(z_1, z_2)\big) \subset \mathbb{C}^4$ has an isolated singularity at the origin and $g(z_1, z_2)$ is divisible neither by z_1 nor by z_2 where $g(z_1, z_2) = f(z_1, z_2, 0, \dots, 0)$.

By Lemma 7 there is an $r \geq 1$ such that

$$g(t, s^r) = u(t, s) \prod_{i=1}^{m} \big(t - \sigma_i(s)\big).$$

Since $g(z_1, z_2)$ is not divisible by z_1, none of the σ_i are identically zero. Since $g(t, s)$ has an isolated critical point at the origin and is not divisible by s, $g(t, s^r)$ also has an isolated critical point at the origin. Thus all the $\sigma_i(s)$ are distinct.

As before, for $j = 1, 2$ write $\psi_j(t) = t^{a_j} v_j(t)$ where $v_j(0) \neq 0$. Note that $a_1 + a_2 = m$ and $u(t, 0) = v_1(t) v_2(t)$.

Divide $\{1, \dots, m\}$ into two disjoint subsets A_1, A_2 such that $|A_j| = a_j$. Finally set

$$\Psi_1(t, s) = v_1(t) \cdot \prod_{i \in A_1} \big(t - \sigma_i(s)\big) \quad \text{and} \quad \Psi_2(t, s) = \frac{u(t, s)}{v_1(t)} \cdot \prod_{i \in A_2} \big(t - \sigma_i(s)\big).$$

Then

$$\big(\Psi_1(t, s), \Psi_2(t, s), t, s^r, 0, \dots, 0\big)$$

is a sideways deformation of $\big(\psi_1(t), \psi_2(t), t, 0, \dots, 0\big)$. □

The opposite happens for quotient singularities.

Proposition 39. *Let* $(0 \in X) := \mathbb{C}^n / G$ *be an isolated quotient singularity. Then arcs with a sideways deformation are nowhere dense in* $\widehat{\operatorname{Arc}}(0 \in X)$.

Proof. Let $\Phi : \operatorname{Spec} \mathbb{C}[[t, s]] \to X$ be a sideways deformation of an arc $\phi(t) = \Phi(t, 0)$. By the purity of branch loci, Φ lifts to an arc $\tilde{\Phi} : \operatorname{Spec} \mathbb{C}[[t, s]] \to \mathbb{C}^n$. In particular, $\phi : \operatorname{Spec} \mathbb{C}[[t]] \to X$ lifts to $\tilde{\phi} : \operatorname{Spec} \mathbb{C}[[t]] \to \mathbb{C}^n$.

By [KN13], such arcs constitute a connected component of $\operatorname{ShArc}(0 \in X)$. We claim, however, that these arcs do not cover a whole irreducible component of $\widehat{\operatorname{Arc}}(0 \in X)$.

It is enough to show the latter on some intermediate cover of X. The simplest is to use $(0 \in Y) := \mathbb{C}^n / C$ where $C \subset G$ is any cyclic subgroup.

Set $r := |C|$, fix a generator $g \in C$ and diagonalize its action as

$$(x_1, \dots, x_n) \mapsto \big(\epsilon^{a_1} x_1, \dots, \epsilon^{a_n} x_n\big),$$

where ϵ is a primitive rth root of unity. Thus Y is the toric variety corresponding to the free abelian group $N = \mathbb{Z}^n + \mathbb{Z}\big(a_1/r, \dots, a_n/r\big)$ and the $\Delta = \big(\mathbb{Q}_{\geq 0}\big)^n$. The Nash conjecture is true for toric singularities and by [IK03, Sec.3] the essential divisors are all toric and correspond to interior vectors of $N \cap \Delta$ that can not be written as the sum of an interior vector of $N \cap \Delta$ and of a nonzero vector of $N \cap \Delta$. In our case, all such vectors are of the form $\big(\overline{ca_1}/r, \dots, \overline{ca_n}/r\big)$ for $c = 1, \dots, r-1$ where $\overline{ca_i}$ denotes remainder mod r.

Arcs that lift to $\mathbb{C}^n$ correspond to the vector $(1, \dots, 1)$, which is not minimal. In fact $(1, \dots, 1) = \big(\overline{a_1}/r, \dots, \overline{a_n}/r\big) + \big(\overline{(r-1)a_1}/r, \dots, \overline{(r-1)a_n}/r\big)$. □

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