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ZERO-DIMENSIONAL SYMPLECTIC ISOLATED COMPLETE INTERSECTION SINGULARITIES

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Abstract. We study the local symplectic algebra of the 0-dimensional isolated complete intersection singularities. We use the method of algebraic restrictions to classify these symplectic singularities. We show that there are non-trivial symplectic invariants in this classification.

1. Introduction

The problem of symplectic classification of singular varieties was introduced by V. I. Arnold in [A1]. Arnold showed that the A_{2k} singularity of a planar curve (the orbit with respect to the standard $\mathcal{A}$-equivalence of parameterized curves) split into exactly $2k+1$ symplectic singularities (orbits with respect to the symplectic equivalence of parameterized curves). He distinguished different symplectic singularities by different orders of tangency of the parameterized curve with the *nearest* smooth Lagrangian submanifold. Arnold posed a problem of expressing these new symplectic invariants in terms of the local algebra's interaction with the symplectic structure and he proposed to call this interaction the **local symplectic algebra**. This problem was studied by many authors mainly in the case of singular curves.

In [IJ1] G. Ishikawa and S. Janeczko classified symplectic singularities of curves in the 2-dimensional symplectic space. A symplectic form on a 2-dimensional manifold is a special case of a volume form on a smooth manifold. The generalization of results in [IJ1] to volume-preserving classification of singular varieties and maps in arbitrary dimensions was obtained in [DR]. The orbit of the action of all diffeomorphism-germs agrees with the volume-preserving orbit in the $\mathbb{C}$-analytic category for germs which satisfy a special weak form of quasi-homogeneity e.g. the weak quasi-homogeneity of varieties is a quasi-homogeneity with non-negative weights $\lambda_i \geq 0$ and $\sum_i \lambda_i > 0$.

P. A. Kolgushkin classified stably simple symplectic singularities of parameterized curves in the $\mathbb{C}$-analytic category ([K]).

In [DJZ2] the local symplectic algebra of singular quasi-homogeneous subsets of a symplectic space was explained by the algebraic restrictions of the symplectic form to these subsets. The generalization of the Darboux-Givental theorem ([AG]) to germs of arbitrary subsets of the symplectic space obtained in [DJZ2] reduces the problem of symplectic classification of germs of quasi-homogeneous subsets to the problem of classification of algebraic restrictions of symplectic forms to these subsets. For non-quasi-homogeneous subsets there is one more cohomological invariant apart of the algebraic restriction ([DJZ2], [DJZ1]). The method of algebraic restrictions is a very powerful tool to study the local symplectic algebra of 1-dimensional singular analytic varieties since the space of algebraic restrictions of closed 2-forms to a 1-dimensional singular analytic variety is finite-dimensional ([D]). By this method complete symplectic classifications

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of the $A-D-E$ singularities of planar curves and the S_5 singularity were obtained in [DJZ2]. These results were generalized to other 1-dimensional isolated complete intersection singularities: the S_μ symplectic singularities for $\mu > 5$ in [DT1], the $T_7 - T_8$ symplectic singularities in [DT2] and the $W_8 - W_9$ symplectic singularities in [T].

In this paper we show that some non-trivial symplectic invariants appear not only in the case of singular curves but also in the case of multiple points. We consider the symplectic classification of the 0-dimensional isolated complete intersection singularities (ICISs) in the symplectic space $(\mathbb{C}^{2n}, \omega)$. We need to introduce a symplectic V-equivalence to study this problem since we consider the ideals of function-germs that have not got the property of zeros.

We recall that ω is a $\mathbb{C}$-analytic symplectic form on $\mathbb{C}^{2n}$ if ω is a $\mathbb{C}$-analytic nondegenerate closed 2-form, and $\Phi : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ is a symplectomorphism if Φ is a $\mathbb{C}$-analytic diffeomorphism and $\Phi^*\omega = \omega$.

Definition 1.1. Let $f, g : (\mathbb{C}^{2n}, 0) \to (\mathbb{C}^k, 0)$ be $\mathbb{C}$-analytic map-germs on the symplectic space $(\mathbb{C}^{2n}, \omega)$. f, g are **symplectically V-equivalent** if there exist a symplectomorphism-germ $\Phi : (\mathbb{C}^{2n}, 0, \omega) \to (\mathbb{C}^{2n}, 0, \omega)$ and a $\mathbb{C}$-analytic map-germ $M : (\mathbb{C}^{2n}, 0) \to GL(k, \mathbb{C})$ such that such that $f \circ \Phi = M \cdot g$.

If $\Phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^m, 0)$ is a $\mathbb{C}$-analytic map-germ then for an ideal I in the ring of $\mathbb{C}$-analytic function-germs on $\mathbb{C}^m$ we denote by Φ^*I the following ideal $\{f \circ \Phi : f \in I\}$ in the ring of $\mathbb{C}$-analytic function-germs on $\mathbb{C}^n$. The (symplectic) V-equivalence of map-germs $f = (f_1, \cdots, f_k), g = (g_1, \cdots, g_k) : (\mathbb{C}^{2n}, 0) \to (\mathbb{C}^k, 0)$ corresponds to the following (symplectic) equivalence of finitely-generated ideals $< f_1, \cdots, f_k >$ and $< g_1, \cdots, g_k >$ (see [AVG]).

Definition 1.2. Ideals $< f_1, \cdots, f_k >$ and $< g_1, \cdots, g_k >$ of $\mathbb{C}$-analytic function-germs at 0 on the symplectic space $(\mathbb{C}^{2n}, \omega)$ are **symplectically equivalent** if there exists a symplectomorphism-germ $\Phi : (\mathbb{C}^{2n}, 0, \omega) \to (\mathbb{C}^{2n}, 0, \omega)$ such that $\Phi^* < f_1, \cdots, f_k > = < g_1, \cdots, g_k >$.

In this paper we present the complete symplectic classification of the $I_{a,b}$, I_{2a+1}, I_{2a+4}, I_{a+5}, I_{10}^* singularities. For $n = 1$ all V-orbits coincide with symplectic V-orbits. The situation for $n \geq 2$ is different: the $I_{a,b}$ singularities split into two symplectic V-orbits, the I_{2a+1}, I_{2a+4}, I_{a+5} singularities split into three symplectic orbits and finally I_{10}^* singularity splits into four symplectic V-orbits. The symplectic V-orbits of a map $f = (f_1, \cdots, f_{2n})$ are distinguished by the order of vanishing of a pullback of the germ of the symplectic form to a $\mathbb{C}$-analytic non-singular submanifold M of the minimal dimension such that the ideal of $\mathbb{C}$-analytic function-germs vanishing M is contained in the ideal $< f_1, \cdots, f_{2n} >$ (see Definition 3.2).

To obtain these results we need some reformulation and modification of the method of algebraic restrictions. We present it in Section 2. In Section 3 we give the definitions of discrete symplectic invariants which completely distinguish symplectic V-singularities considered in this paper. We recall basic facts on the classification of V-simple maps in Section 4. In Section 5 we prove the symplectic V-classification theorem for 0-dimensional ICISs (Theorem 5.1).

2. The method of algebraic restrictions for the symplectic V-equivalence.

In this section we present basic facts on the method of algebraic restrictions adapted to the case of the symplectic V-equivalence. The proofs of all results are small modifications of the proofs of analogous results in [DJZ2].

Given a germ at 0 of a non-singular $\mathbb{C}$-analytic submanifold M of $\mathbb{C}^m$ denote by $\Lambda^p(M)$ the space of all germs at 0 of $\mathbb{C}$-analytic differential p-forms on M. By $\mathcal{O}(M)$ denote the ring of $\mathbb{C}$-analytic function-germs on M at 0. Given an ideal I in $\mathcal{O}(M)$ introduce the following subspace of $\Lambda^p(M)$:

$$\mathcal{A}_0^p(I, M) = \{\alpha + d\beta : \quad \alpha \in I\Lambda^p(M), \ \beta \in I\Lambda^{p-1}(M).\}$$

The relation $\omega \in I\Lambda^p(M)$ means that $\omega = \sum_{i=1}^k f_i\alpha_i$, where $\alpha_i \in \Lambda^p(M)$ and $f_i \in I$ for $i = 1, ..., k$.

Definition 2.1. Let I be an ideal of $\mathcal{O}(M)$ and let $\omega \in \Lambda^p(M)$. The **algebraic restriction** of ω to I is the equivalence class of ω in $\Lambda^p(M)$, where the equivalence is as follows: ω is equivalent to $\widetilde{\omega}$ if $\omega - \widetilde{\omega} \in \mathcal{A}_0^p(I, M)$.

Notation. The algebraic restriction of the germ of a p-form ω on M to the ideal I in $\mathcal{O}(M)$ will be denoted by $[\omega]_I$. Writing $[\omega]_I = 0$ (or saying that ω has zero algebraic restriction to I) we mean that $[\omega]_I = [0]_I$, i.e. $\omega \in A_0^p(I, M)$.

Definition 2.2. Two algebraic restrictions $[\omega]_I$ and $[\widetilde{\omega}]_{\widetilde{I}}$ are called **diffeomorphic** if there exists the germ of a diffeomorphism $\Phi : M \to \widetilde{M}$ such that $\Phi^*(\widetilde{I}) = I$ and $[\Phi^*\widetilde{\omega}]_I = [\omega]_I$.

Definition 2.3. The germ of a function, a differential k-form, or a vector field α on $(\mathbb{C}^m, 0)$ is **quasi-homogeneous** in a coordinate system $(x_1, \cdots, x_m)$ on $(\mathbb{C}^m, 0)$ with positive integer weights $(\lambda_1, \cdots, \lambda_m)$ if $\mathcal{L}_E\alpha = \delta\alpha$, where $E = \sum_{i=1}^m \lambda_i x_i \frac{\partial}{\partial x_i}$ is the germ of the **Euler vector field** on $(\mathbb{C}^m, 0)$ and the integer δ is called the quasi-degree.

It is easy to show that α is quasi-homogeneous in a coordinate system $(x_1, \cdots, x_m)$ with weights $(\lambda_1, \cdots, \lambda_m)$ if and only if $F_t^*\alpha = t^\delta\alpha$, where

$$F_t(x_1, \cdots, x_m) = (t^{\lambda_1}x_1, \cdots, t^{\lambda_m}x_m). \tag{2.1}$$

Definition 2.4. A finitely generated ideal I of $\mathcal{O}(\mathbb{C}^m)$ is **quasi-homogeneous** if there exist generators of I which are quasi-homogeneous in the same coordinate system $(x_1, \cdots, x_m)$ on $\mathbb{C}^m$ with the same positive integer weights $(\lambda_1, \cdots, \lambda_m)$.

A map-germ $f = (f_1, \cdots, f_k) : (\mathbb{C}^m, 0) \to (\mathbb{C}^k, 0)$ is **quasi-homogeneous** if function-germs $f_1, \cdots, f_k$ are quasi-homogeneous in the same coordinate system $(x_1, \cdots, x_m)$ on $\mathbb{C}^m$ with the same positive integer weights $(\lambda_1, \cdots, \lambda_m)$.

To prove the generalization of Darboux-Givental theorem suitable for the symplectic V-equivalence of maps or the symplectic equivalence of ideals of function-germs we need the following version of the Relative Poincaré Lemma.

Lemma 2.5. *Let I be a finitely generated quasi-homogeneous ideal in $\mathcal{O}(\mathbb{C}^m)$. If $\omega \in I\Lambda^p(\mathbb{C}^m)$ is closed than there exists $\alpha \in I\Lambda^{p-1}(\mathbb{C}^m)$ such that $\omega = d\alpha$.*

Proof. We use the method described in [DJZ1]. We can find a coordinate system $(x_1, \cdots, x_m)$ on $(\mathbb{C}^m, 0)$ and positive integer weights $(\lambda_1, \cdots, \lambda_m)$ and quasi-homogeneous function-germs $f_1, \cdots, f_k \in \mathcal{O}(\mathbb{C}^m)$ (in this coordinate systems with these weights) such that $I =< f_1, \cdots, f_k >$. Let δ_i be a quasi-degree of f_i for $i = 1, \cdots, k$.

Let F_t be a map defined in (2.1) and let V_t be a vector field along F_t for $t \in [0; 1]$ such that $V_t \circ F_t = F_t'$.

Then we have $F_0^*\omega = 0$ and it implies that

$$\omega = F_1^*\omega - F_0^*\omega = \int_0^1 (F_t^*\omega)'dt = \int_0^1 F_t^*d(V_t\rfloor\omega)dt = d\left(\int_0^1 F_t^*(V_t\rfloor\omega)dt\right).$$

Let $\alpha = \int_0^1 F_t^*(V_t\rfloor\omega)dt$, then $\omega = d\alpha$. But ω belongs to $I\Lambda^p(\mathbb{C}^m)$. It implies that there exist germs of p-forms β_i in $\Lambda^p(\mathbb{C}^m)$ for $i = 1, \cdots, k$ such that $\omega = \sum_{i=1}^k f_i\beta_i$. So we have that

$$\alpha = \int_0^1 F_t^*(V_t\rfloor\sum_{i=1}^k f_i\beta_i)dt = \sum_{i=1}^k f_i \int_0^1 t^{\delta_i} F_t^*(V_t\rfloor\beta_i)dt.$$

Thus α belongs to $I\Lambda^{p-1}(\mathbb{C}^m)$. ∎

The method of algebraic restrictions applied to finitely-generated quasi-homogeneous ideals is based on the following theorem.

Theorem 2.6 (a modification of Theorem A in [DJZ2])**.** *Let I be a finitely generated quasi-homogeneous ideal in $\mathcal{O}(\mathbb{C}^{2n})$.*

1. *If ω_0, ω_1 are germs at 0 of symplectic forms on $\mathbb{C}^{2n}$ with the same algebraic restriction to I then there exists a $\mathbb{C}$-analytic diffeomorphism-germ Φ of $\mathbb{C}^{2n}$ at 0 of the form $\Phi(x) = (x_1 + \phi_1(x), \cdots, x_{2n} + \phi_{2n}(x))$, where $\phi_i \in I$ for $i = 1, \cdots, 2n$, such that $\Phi^*\omega_1 = \omega_0$.*
2. *$\mathbb{C}$-analytic quasi-homogeneous map-germs $f = (f_1, \cdots, f_k), g = (g_1, \cdots, g_k) : (\mathbb{C}^{2n}, 0) \to (\mathbb{C}^k, 0)$ on the symplectic space $(\mathbb{C}^{2n}, \omega)$ are symplectically V-equivalent if and only if algebraic restrictions $[\omega]_{<f_1,\cdots,f_k>}$ and $[\omega]_{<g_1,\cdots,g_k>}$ are diffeomorphic.*

Remark 2.7. It is obvious that if $\Phi(x) = (x_1 + \phi_1(x), \cdots, x_{2n} + \phi_{2n}(x))$ where $\phi_i \in I$ for $i = 1, \cdots, 2n$ then $\Phi^* I = I$

A proof of Theorem 2.6 can be obtain by a small modification of the proof of Theorem A in [DJZ2]. One only needs Lemma 2.5 and the following fact.

Lemma 2.8. *Let I be a finitely generated ideal in $\mathcal{O}(\mathbb{C}^m)$. Let $X_t = \sum_{i=1}^m f_{i,t} \frac{\partial}{\partial x_i}$ for $t \in [0; 1]$ be a family of germs of $\mathbb{C}$-analytic vector fields on $\mathbb{C}^m$ such that $f_{i,t} \in I$ for $i = 1, \cdots, m$.*

If Φ_t for $t \in [0, 1]$ is a family of diffeomorphism-germs of $(\mathbb{C}^m, 0)$ such that

$$\frac{d}{dt}\Phi_t = X_t \circ \Phi_t \tag{2.2}$$

then

$$\Phi_t(x) = (x_1 + \phi_{1,t}(x), \cdots, x_{2n} + \phi_{2n,t}(x)), \tag{2.3}$$

where $\phi_{i,t} \in I$ for $i = 1, \cdots, 2n$.

A sketch of the proof. The map $t \mapsto \Phi_t(x)$ is a solution of ODE $\frac{dy}{dt} = X_t(y)$ with the initial condition $y(0) = x$. So $\Phi_t(x)$ can be obtained as a limit $\lim_{n\to\infty} T^n\Psi$ where $\Psi(t, x) \equiv x$ and $(T\Psi)(t, x) = x + \int_0^t X_s(\Psi(s, x))ds$ is the Picard's operator. It is easy to see that if Ψ has the form (2.3) then $T\Psi$ has the form (2.3) too. The ideal I is finitely generated. Thus Φ_t has also this form. □

Theorem 2.6 reduces the problem of symplectic classification of quasi-homogeneous ideals to the problem of classification of the algebraic restrictions of the germ of the symplectic form to quasi-homogeneous ideals.

The meaning of the zero algebraic restriction is explained by the following theorem.

Theorem 2.9 (a modification of Theorem **B** in [DJZ2])**.** *A finitely generated quasi-homogeneous ideal I of $\mathcal{O}(\mathbb{C}^{2n})$ contains the ideal of $\mathbb{C}$-analytic function-germs vanishing on the germ of a non-singular Lagrangian submanifold of the symplectic space $(\mathbb{C}^{2n}, \omega)$ if and only if the symplectic form ω has zero algebraic restriction to I.*

We now formulate the modifications of basic properties of algebraic restrictions ([DJZ2]). First we can reduce the dimension of the manifold due to the following propositions.

If the ideal I in $\mathcal{O}(\mathbb{C}^m)$ contains an ideal $I(M)$ of function-germs vanishing on a non-singular submanifold $M \subset \mathbb{C}^m$ then the classification of the algebraic restrictions to I of p-forms on $\mathbb{C}^m$ reduces to the classification of the algebraic restrictions to $I|_M = \{f|_M : f \in I\}$ of p-forms on M. At first note that the algebraic restrictions $[\omega]_I$ and $[\omega|_{TM}]_{I|_M}$ can be identified:

Proposition 2.10. *Let I be an ideal in $\mathcal{O}(\mathbb{C}^m)$ which contains an ideal of function-germs vanishing on a non-singular submanifold $M \subset \mathbb{C}^m$ and let ω_1, ω_2 be germs of p-forms on $\mathbb{C}^m$. Then $[\omega_1]_I = [\omega_2]_I$ if and only if $\left[\omega_1|_{TM}\right]_{I|_M} = \left[\omega_2|_{TM}\right]_{I|_M}$.*

The following, less obvious statement, means that the *orbits* of the algebraic restrictions $[\omega]_I$ and $[\omega|_{TM}]_{I|_M}$ also can be identified.

Proposition 2.11. *Let I_1, I_2 be ideals in the ring $\mathcal{O}(\mathbb{C}^m)$, which contain $I(M_1)$ and $I(M_2)$ respectively, where M_1, M_2 are equal-dimensional non-singular submanifolds. Let ω_1, ω_2 be two germs of p-forms. The algebraic restrictions $[\omega_1]_{I_1}$ and $[\omega_2]_{I_2}$ are diffeomorphic if and only if the algebraic restrictions $\left[\omega_1|_{TM_1}\right]_{I_1|_{M_1}}$ and $\left[\omega_2|_{TM_2}\right]_{I_2|_{M_2}}$ are diffeomorphic.*

To calculate the space of algebraic restrictions of germs of 2-forms we will use the following obvious properties.

Proposition 2.12. *If $\omega \in \mathcal{A}_0^k(I, \mathbb{C}^{2n})$ then $d\omega \in \mathcal{A}_0^{k+1}(I, \mathbb{C}^{2n})$ and $\omega \wedge \alpha \in \mathcal{A}_0^{k+p}(I, \mathbb{C}^{2n})$ for any germ of $\mathbb{C}$-analytic p-form α on $\mathbb{C}^{2n}$.*

Then we need to determine which algebraic restrictions of closed 2-forms are realizable by symplectic forms. This is possible due to the following fact.

Proposition 2.13. *Let I be an ideal of $\mathcal{O}(\mathbb{C}^{2n})$. Let r be the minimal dimension of non-singular submanifolds M of $\mathbb{C}^{2n}$ such that I contains the ideal $I(M)$. The algebraic restriction $[\theta]_I$ of the germ of a closed 2-form θ is realizable by the germ of a symplectic form on $\mathbb{C}^{2n}$ if and only if $rank(\theta|_{T_0M}) \geq 2r - 2n$.*

3. Discrete symplectic invariants.

We use discrete symplectic invariants to distinguish symplectic singularity classes. We modify definitions of these invariants introduced in [DJZ2] for the symplectic V-equivalence.

The first invariant is a symplectic multiplicity ([DJZ2]) introduced in [IJ1] as a symplectic defect of a curve.

Let $f : (\mathbb{C}^{2n}, 0) \to (\mathbb{C}^k, 0)$ be the germ of a $\mathbb{C}$-analytic map on the symplectic space $(\mathbb{C}^{2n}, \omega)$.

Definition 3.1. The **symplectic multiplicity** $\mu_{sympl}(f)$ of f is the codimension of the symplectic V-orbit of f in the V-orbit of f.

The second invariant is the index of isotropy [DJZ2].

Definition 3.2. The **index of isotropy** $\iota(f)$ of $f = (f_1, \cdots, f_k)$ is the maximal order of vanishing of the 2-forms $\omega|_{TM}$ over all smooth submanifolds M such that the ideal $< f_1, \cdots, f_k >$ contains $I(M)$.

These invariants can be described in terms of algebraic restrictions.

Proposition 3.3 ([DJZ2])**.** *The symplectic multiplicity of the germ of a quasi-homogeneous map $f = (f_1, \cdots, f_k)$ on the symplectic space $(\mathbb{C}^{2n}, \omega)$ is equal to the codimension of the orbit of the algebraic restriction $[\omega]_{<f_1, \cdots, f_k>}$ with respect to the group of diffeomorphism-germs preserving the ideal $< f_1, \cdots, f_k >$ in the space of the algebraic restrictions of closed 2-forms to $< f_1, \cdots, f_k >$.*

Proposition 3.4 ([DJZ2])**.** *The index of isotropy of the germ of a quasi-homogeneous map $f = (f_1, \cdots, f_k)$ on the symplectic space $(\mathbb{C}^{2n}, \omega)$ is equal to the maximal order of vanishing of closed 2-forms representing the algebraic restriction $[\omega]_{<f_1, \cdots, f_k>}$.*

We will use these invariants to distinguish symplectic singularities.

4. V-simple maps

We recall some results on classification of V-simple germs (for details see [AVG]).

Definition 4.1. The germ $f:(\mathbb{C}^m,0)\to(\mathbb{C}^n,0)$ is said be V**-simple** if its k-jet, for any k, has a neighborhood in the small jet space $J^k_{0,0}(\mathbb{C}^m,\mathbb{C}^n)$ that intersects only a finite number of V-equivalence classes (bounded by a constant independent of k).

Definition 4.2. The p**-parameter suspension of the map-germ** $f:(\mathbb{C}^m,0)\to(\mathbb{C}^n,0)$ is the map germ

$$F:(\mathbb{C}^m\times\mathbb{C}^p,0)\ni(y,z)\mapsto(f(y),z)\in(\mathbb{C}^n\times\mathbb{C}^p,0).$$

Theorem 4.3 (see [AVG])**.** *The V-simple map-germs $(\mathbb{C}^m,0)\to(\mathbb{C}^n,0)$ with $m\geq n$ belong, up to V-equivalence and suspension, to one of the three lists: the $A-D-E$ singularities of map-germs $\mathbb{C}^m\to\mathbb{C}$ (hypersurfaces with an isolated singularity), $S-T-U-W-Z$ singularities of map-germs $\mathbb{C}^3\to\mathbb{C}^2$ (1-dimensional ICISs) and singularities of map-germs $\mathbb{C}^2\to\mathbb{C}^2$ (0-dimensional ICISs) presented in Table 1.*

Notation	Normal form	Restrictions
$I_{a,b}$	(yz,y^a+z^b)	$a\geq b\geq 2$
I_{2a+1}	(y^2+z^3,z^a)	$a\geq 3$
I_{2a+4}	(y^2+z^3,yz^a)	$a\geq 2$
I_{a+5}	(y^2+z^a,yz^2)	$a\geq 4$
I^*_{10}	(y^2,z^4)	-

Table 1. V-simple map-germs $\mathbb{C}^2\to\mathbb{C}^2$.

The normal forms in Table 1 were obtained in [G] by M. Giusti.

5. Symplectic 0-dimensional ICISs

We use the method of algebraic restrictions to obtain a complete classification of singularities presented in Table 1.

Theorem 5.1. *Any map-germ $(\mathbb{C}^{2n},0)\to(\mathbb{C}^{2n},0)$ from the symplectic space $(\mathbb{C}^{2n},\sum_{i=1}^n dp_i\wedge dq_i)$ which is V-equivalent (up to a suitable suspension) to one of the normal forms in Table 1 is symplectically V-equivalent to one and only one of the following normal forms presented in Table 2*

Proof. In the case $n=1$ the proof follows from results in [DR] where it was proved that for quasi-homogeneous singularities in the $\mathbb{C}$-analytic category V-orbits coincide with volume-preserving V-orbits. For general n we present the proof in the case of the I^*_{10} singularity where there are 4 different symplectic singularity classes, and in the case of the I_{a+5} singularity. The proofs in other cases are very similar.

For the I^*_{10} singularity we calculate the space of algebraic restrictions of 2-forms to the ideal $I=<y^2,z^4,x_1,\cdots,x_{2n-2}>$. The ideal generated by $x_1,\cdots,x_{2n-2}$ is contained in I. So by Proposition 2.10 we may consider the following ideal $J=I|_{\{x_1=\cdots=x_{2n-2}=0\}}=<y^2,z^4>$ in the ring $\mathcal{O}(\mathbb{C}^2)$. By Proposition 2.12 germs of 1-forms $d(1/2y^2)=ydy, d(1/4z^4)=z^3dz$ and germs of 2-forms $ydy\wedge dz, z^3dy\wedge dz$ have zero algebraic restriction to J. So any algebraic

Symplectic class	*Normal forms*	*cod*	μ_{sympl}	i
$I^0_{a,b}$, $(n \geq 1)$	$(p_1q_1, p_1^a + q_1^b, p_2, q_2, \cdots, p_n, q_n)$	0	0	0
$I^1_{a,b}$, $(n \geq 2)$	$(p_1p_2, p_1^a + p_2^b, q_1, q_2, p_3, q_3, \cdots, p_n, q_n)$	1	1	∞
I^0_{2a+1}, $(n \geq 1)$	$(p_1^2 + q_1^3, q_1^a, p_2, q_2, \cdots, p_n, q_n)$	0	0	0
I^1_{2a+1}, $(n \geq 2)$	$(p_1^2 + p_2^3, p_2^a, q_1, q_2 + p_1p_2, p_3, q_3, \cdots, p_n, q_n)$	1	1	1
I^2_{2a+1}, $(n \geq 2)$	$(p_1^2 + p_2^3, p_2^a, q_1, q_2, p_3, q_3, \cdots, p_n, q_n)$	2	2	∞
I^0_{2a+4}, $(n \geq 1)$	$(p_1^2 + q_1^3, p_1q_1^a, p_2, q_2, \cdots, p_n, q_n)$	0	0	0
I^1_{2a+4}, $(n \geq 2)$	$(p_1^2 + p_2^3, p_1p_2^a, q_1, q_2 + p_1p_2, p_3, q_3, \cdots, p_n, q_n)$	1	1	1
I^2_{2a+4}, $(n \geq 2)$	$(p_1^2 + p_2^3, p_1p_2^a, q_1, q_2, p_3, q_3, \cdots, p_n, q_n)$	2	2	∞
I^0_{a+5}, $(n \geq 1)$	$(p_1^2 + q_1^a, p_1q_1^2, p_2, q_2, \cdots, p_n, q_n)$	0	0	0
I^1_{a+5}, $(n \geq 2)$	$(p_1^2 + p_2^a, p_1p_2^2, q_1, q_2 + p_1p_2, p_3, q_3, \cdots, p_n, q_n)$	1	1	1
I^1_{a+5}, $(n \geq 2)$	$(p_1^2 + p_2^a, p_1p_2^2, q_1, q_2, p_3, q_3, \cdots, p_n, q_n)$	2	2	∞
I^{*0}_{10}, $(n \geq 1)$	$(p_1^2, q_1^4, p_2, q_2, \cdots, p_n, q_n)$	0	0	0
I^{*1}_{10}, $(n \geq 2)$	$(p_1^2, p_2^4, q_1, q_2 + p_1p_2, p_3, q_3, \cdots, p_n, q_n)$	1	1	1
I^{*2}_{10}, $(n \geq 2)$	$(p_1^2, p_2^4, q_1, q_2 + p_1p_2^2, p_3, q_3, \cdots, p_n, q_n)$	2	2	2
I^{*3}_{10}, $(n \geq 2)$	$(p_1^2, p_2^4, q_1, q_2, p_3, q_3, \cdots, p_n, q_n)$	3	3	∞

TABLE 2. Classification of symplectic 0-dimensional isolated complete intersection singularities, *cod* – codimension of the classes; μ_{sympl}– symplectic multiplicity; i – index of isotropy.

restriction of the germ of a closed 2-forms to J can be presented in the following form $[\omega]_J = A[dy \wedge dz]_J + B[zdy \wedge dz]_J + C[z^2dy \wedge dz]_J$, where $A, B, C \in \mathbb{C}$.

If $A \neq 0$ then we obtain $\Phi^*[\omega]_J = [dy \wedge dz]_J$ by the diffeomorphism-germ of the form $\Phi(y, z) = (y, z(A + 1/2Bz + 1/3Cz^2))$. If $A = 0$ and $B \neq 0$ then we obtain $\Phi^*[\omega]_J = [zdy \wedge dz]_J$ by the diffeomorphism-germ of the form $\Phi(y, z) = (y, z\phi(z))$, where $\phi^2(z) = B + 2/3Cz$. If $A = B = 0$ and $C \neq 0$ then we obtain $\Phi^*[\omega]_J = [z^2dy \wedge dz]_J$ by the diffeomorphism-germ of the form $\Phi(y, z) = (Cy, z)$.

Since the minimal dimension r of the germ of a non-singular submanifold M such that $I(M) \subset I$ is 2 then by Proposition 2.13 for $n = 1$ only the algebraic restriction $[dy \wedge dz]_I$ is realizable by the germ of a symplectic form.

For $n > 1$ all algebraic restrictions are realizable by the following symplectic forms:

$$dy \wedge dz + \sum_{i=1}^{n-1} dx_{2i-1} \wedge dx_{2i}, \tag{5.1}$$

$$zdy \wedge dz + dy \wedge dx_1 + dz \wedge dx_2 + \sum_{i=2}^{n-1} dx_{2i-1} \wedge dx_{2i}, \tag{5.2}$$

$$z^2dy \wedge dz + dy \wedge dx_1 + dz \wedge dx_2 + \sum_{i=2}^{n-1} dx_{2i-1} \wedge dx_{2i}, \tag{5.3}$$

$$dy \wedge dx_1 + dz \wedge dx_2 + \sum_{i=2}^{n-1} dx_{2i-1} \wedge dx_{2i}. \tag{5.4}$$

By a simple change of coordinates we obtain the normal forms in Table 2.

For the I_{a+5} singularity the space algebraic restrictions of germs of closed 2-forms to the ideal $I =< y^2 + z^a, yz^2, x_1, \cdots, x_{2n-2} >$ can calculated in the same way. We obtain that any algebraic restriction of the germs of a closed 2-forms on $\mathbb{C}^2 = \{x_1 = \cdots = x_{2n-2} = 0\}$ to $J = I|_{\{x_1=\cdots=x_{2n-2}=0\}} =< y^2 + z^a, yz^2 >$ can be presented in the following form

$$[\omega]_J = A[dy \wedge dz]_J + B[zdy \wedge dz]_J, \tag{5.5}$$

where $A, B \in \mathbb{C}$.

First assume that $A \neq 0$. Let E denote the germ of the Euler vector field $ay\frac{\partial}{\partial y} + 2z\frac{\partial}{\partial y}$. Then it is easy to check that a flow Φ_t of the germ of a vector field $X = \frac{B}{(a+4)A} zE$ preserves J, $\mathcal{L}_X(Ady \wedge dz) = Bzdy \wedge dz$, $[\mathcal{L}_X(Bzdy \wedge dz)]_J = 0$. Therefore $\Phi_t^*[Ady \wedge dz + tBzdy \wedge dz]_J = [Ady \wedge dz]_J$ for $t \in [0; 1]$ (see [D]). Finally by a linear change of coordinates of the form $(y, z) \mapsto (Cy, Dz)$, where for $C, D \in \mathbb{C}$ such that $C^2 = D^a$ and $CD = A$ we show that if $A \neq 0$ then the algebraic restriction (5.5) is diffeomorphic to $[dy \wedge dz]_J$. By a similar change of coordinates preserving J we show that if $A = 0$ and $B \neq 0$ then the algebraic restriction (5.5) is diffeomorphic to $[zdy \wedge dz]_J$. As in the previous case, for $n = 1$ only $[dy \wedge dz]_I$ can be realizable by the germ of a symplectic form . For $n \geq 2$ algebraic restrictions are realizable by (5.1), (5.2) and (5.4). Normal forms in Table 2 are obtained by an obvious change of coordinates. □

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