EQUIVARIANT CHERN CLASSES AND LOCALIZATION THEOREM

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ABSTRACT. For a complex variety with a torus action we propose a new method of computing Chern-Schwartz-MacPherson classes. The method does not apply resolution of singularities. It is based on the Localization Theorem in equivariant cohomology.

This is an extended version of the talk given in Hefei in July 2011.

Equivariant cohomology is a powerful tool for studying complex manifolds equipped with a torus action. The Localization Theorem of Atiyah and Bott and the resulting formula of Berline-Vergne allow to compute global invariants, for example invariants of singular subsets, in terms of some data attached to the fixed points of the action. We will concentrate on the equivariant Chern-Schwartz-MacPherson classes. The global class is determined by the local contributions coming from the fixed points. On the other hand, the sum of the local contributions divided by the Euler classes is equal to zero in an appropriate localization of equivariant cohomology. Especially for Grassmannians we obtain interesting formulas with nontrivial relations involving rational functions. We discuss the issue of positivity: the local equivariant Chern class may be presented in various ways, depending on the choice of generating circles of the torus. For some choices we find that the coefficients of the presentation are nonnegative. Also the coefficients in the Schur basis are nonnegative in many examples, but it turns out that not always.

We begin with §1 which contains a review of the results concerning the equivariant fundamental class of an invariant subvariety in a smooth G-manifold M. The first two chapters are valid for any algebraic group, but further we will consider only torus actions. The equivariant fundamental class lives in the equivariant cohomology $H_G^*(M)$. The invariant subvarieties contained in a vector space V, on which the torus $G = T = (\mathbb{C}^*)^n$ acts linearly, are of particular interest. If the weights of the torus action are nonnegative then the equivariant fundamental class is a nonnegative combination of monomials in $H_T^*(V) = \mathbb{Q}[t_1, t_2, \ldots, t_n]$.

In §2 we discuss the equivariant version of the Chern-Schwartz-MacPherson class, denoted by c^T, which is a refinement of the equivariant fundamental class. To give the precise definition, following [33], one has to introduce equivariant homology. Eventually we will assume that the variety is contained in a smooth manifold. By Poincaré duality it is enough for our purpose to consider the equivariant Chern-Schwartz-MacPherson classes as the elements of equivariant cohomology of the ambient space.

The main tool of the equivariant cohomology for a torus action is the Localization Theorem of Atiyah-Bott or Berline-Vergne formula. It says that the equivariant cohomology class can be read from certain data concentrated at the fixed points of the action. The precise formulation of the Localization Theorem is recalled in §3.

The section §4 is a kind of interlude for fun. We give some examples of calculations based on the Localization Theorem for torus acting on projective spaces and Grassmannians. For example we show how the formula for Gysin map for the Grassmann bundle immediately follows from the Laplace determinant expansion.

Next, in §5 we discuss equivariant Chern-Schwartz-MacPherson classes of toric varieties. From the Localization Theorem we deduce that the equivariant class of an orbit $c^T(Tx)$ is equal to the fundamental class of its closure $\overline{[Tx]}$. Exactly the same formula holds in the nonequivariant setting, by [7].

The section §6 is important for whole inductive procedure of computations of equivariant Chern classes. The key statement is the following:

Theorem 1. *Suppose that X is a T-variety, not necessarily smooth, contained in a T-manifold M. Let $p \in X$ be an isolated fixed point. Then the zero degree of the class $c^T(X)$ restricted to $\{p\}$ is Poincaré dual of the product of weights appearing in the tangent representation $\mathbf{T}_p M$.*

It is also convenient to consider a version of the Localization Theorem in which we express the global cohomology class by its restriction to an arbitrary submanifold containing the fixed point set. The main example is the projective space $\mathbb{P}^n$. The class which we want to compute is the equivariant Chern-Schwartz-MacPherson class of the projective cone over a subvariety in $\mathbb{P}^{n-1}$. In §7 from partial localization we deduce the following:

Proposition 2. *Suppose that X is a T-invariant cone in a linear representation V of T. Let $h = c_1^T(\mathcal{O}_{\mathbb{P}(V)}(1)) \in H_T^2(\mathbb{P}(V))$ be the equivariant Chern class. If*

$$c^T(\mathbb{P}(X)) = \Big(\sum_{i=0}^{\dim(V)-1} b_i(t)h^i \Big) \cap [\mathbb{P}(V)] \in H_*^T(\mathbb{P}(V))$$

for $b_i(t) \in H_T^*(pt)$ then

$$c^T(X) = (b_0(t) + e_0) \cap [V] \in H_*^T(V)$$

where e_0 is the product of weights appearing in the representation V.

Even a seemingly trivial application of this result (discussed in §8) is meaningful. If $T = \mathbb{C}^*$ acts by scalar multiplication on $\mathbb{C}^n$, then T-invariant subvariety is just a cone in $\mathbb{C}^n$. The characteristic classes of cones were already considered by Aluffi and Marcolli. In their paper [4] there was given a formula for the Chern-Schwartz-MacPherson class of an open affine cone in $\mathbb{P}^n$. It is not a coincidence, that their computation agrees with our result about the equivariant class in $H_T^*(\mathbb{C}^n)$:

Proposition 3. *Suppose X a cone in $\mathbb{C}^n$. Let $x = c_1(\mathcal{O}_{\mathbb{P}^{n-1}}(1)) \in H^2(\mathbb{P}^{n-1})$ and let $t \in H_T^2(pt)$ be the generator corresponding to the identity character. If*

$$c_{SM}(\mathbb{P}(X)) = \Big(\sum_{i=0}^{n-1} a_i x^i \Big) \cap [\mathbb{P}^{n-1}] \in H_*(\mathbb{P}^{n-1})$$

then

$$c^T(X) = \Big(\sum_{i=0}^{n-1} a_i t^i + t^n \Big) \cap [\mathbb{C}^n] \in H_*^T(\mathbb{C}^n).$$

The result follows from the previous one since the equivariant Chern class $h = c_1^T(\mathcal{O}_{\mathbb{P}^{n-1}}(1))$ is equal to

$$1 \otimes x - t \otimes 1 \in H_T^*(\mathbb{P}^{n-1}) = H_T^*(pt) \otimes H^*(\mathbb{P}^{n-1}).$$

By the product property of equivariant Chern-Schwartz-MacPherson classes we obtain for free „Feynman rule" for the polynomial G_X introduced [4, Lemma 3.10].

In the next section §9 we propose a new method of computing equivariant Chern-Schwartz-MacPherson classes which does not involve resolution of singularities. It is based on the fact, that the sum of the equivariant Chern-Schwartz-MacPherson classes localized at fixed points and divided by Euler classes is equal to zero, except from the zero degree. A similar observation was already made by Féher and Rimányi in [16, §8.1] for computation of Thom polynomials. On the other hand the zero degree of the

equivariant Chern class is given by the result of §7 (stated before as Theorem 1). This way often we can compute local Chern classes by induction on the depth of the singularity.

Our main example in §10 is the determinant variety, the subset of square matrices $n \times n$ defined by the equation $\det = 0$. We study its compactification, the Schubert variety of codimension one in $\mathrm{Grass}_n(\mathbb{C}^{2n})$. We discuss computational problems appearing for that example. The concrete formula for the equivariant Chern class is a huge sum of fractions. Surprisingly all the difficulties lie in simplifying that expression. We compute the equivariant Chern class for the determinant variety for $n \leq 4$. It turns out that it is a nonnegative combination of monomials with suitable choice of generators of $H_T^2(pt)$. This supports the conjecture of Aluffi-Mihalcea that the Chern-MacPherson-Schwartz class of the Schubert varieties are effective. On the other hand for $n = 4$ the local equivariant Chern-Schwartz-MacPherson class expanded in the Schur basis has negative coefficients in few places. We present the result of calculations in §12.

The connection of our local formulas with the calculations of [5] and [23] is not clear. The formula for the global class can be read from the local contributions by Theorem 11. Nevertheless the shape of this relation seems to be combinatorially nontrivial due to presence of the denominators.

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1. EQUIVARIANT FUNDAMENTAL CLASS

Let M be a complex manifold and $X \subset M$ a closed complex subvariety. The *fundamental class* of X, which is the Poincaré dual of the cycle defined by X is denoted by

$$[X] \in H^{2\,\mathrm{codim}(X)}(M)\,.$$

When the ambient manifold M is contractible, for example when M is an affine space, there is no use of $[X]$ since the cohomology of M is trivial. An interesting situation appears when an algebraic group G acts on M and X is preserved by the action. In that case there is an equivariant fundamental class of X which belongs to the equivariant cohomology of M

$$[X] \in H_G^{2\,\mathrm{codim}(X)}(M).$$

Now even if M is contractible we obtain a remarkable invariant of the pair (M, X). For contractible M its equivariant cohomology coincides with the equivariant cohomology of a point

$$H_G^{2\,\mathrm{codim}(X)}(M) \simeq H_G^{2\,\mathrm{codim}(X)}(pt)$$

and the cohomology of a point *is the ring of characteristic classes for* G. In particular

- if $G = (\mathbb{C}^*)^n$ then $H_G^*(pt) = \mathbb{Q}[t_1, t_2, \ldots, t_n]$
- if $G = GL_n$ then $H_G^*(pt) = \mathbb{Q}[\sigma_1, \sigma_2, \ldots, \sigma_n] = \mathbb{Q}[t_1, t_2, \ldots, t_n]^{\Sigma_n}$
- in general the ring of characteristic classes coincides with the invariants of the Weyl group acting on the characteristic classes for the maximal torus

$$H_G^*(pt) = H_T^*(pt)^W.$$

(We consider here only cohomology with rational coefficients.)

For a torus $G = T$ we identify $H_T^*(pt)$ with the polynomial algebra spanned by characters of T, i.e.

$$H_T^*(pt) = \mathbb{Q}[T^\vee] = \bigoplus_{k=0}^\infty Sym^k[T^\vee \otimes \mathbb{Q}]\,.$$

A character $\lambda : T \to \mathbb{C}^*$ corresponds to an element of $H_T^2(pt)$.

We will briefly recall the construction of equivariant cohomology in §2. The reader can find its basic properties in [36]. For a review of equivariant cohomology in algebraic geometry see e.g. [18]. An extended discussions of different names for the equivariant fundamental class can be found in [9, §2.1].

For $G = GL_n$ equivariant cohomology and the equivariant fundamental classes $[X] \in H^*_{GL_n}(pt)$ has turned out to be an adequate tool for studying the Thom polynomials of singularities of maps. Here X is a set of singular jets in the space of all jets of maps. Its equivariant fundamental class $[X]$ is the universal characteristic class which describes cohomological properties of singular loci of maps. In the last decade there appeared a series of papers by Rimányi and his collaborators (starting from [37]) and Kazarian (see e.g. [25]). Powerful tools allowing effective computations were developed and some structure theorems were stated. The geometric approach to equivariant cohomology leads to positivity results [34, 31, 32]. The source of these results is the following principle:

Theorem 4. *If $X \subset \mathbb{C}^N$ is a cone in a polynomial representation of GL_n, then $[X]$ is a nonnegative combination of Schur functions.*

The examples of polynomial representations are the following: the natural representation, its tensor products, symmetric products, exterior products and in general quotients of the sums of tensor products. The Schur functions constitute a basis of the ring of characteristic classes

$$H^*_{GL_n}(pt) = H^*(\mathrm{Grass}_n(\mathbb{C}^\infty))$$

corresponding to the decomposition of the infinite Grassmannian into Schubert cells. For an algebraic treatment of Schur functions see [28].

A version of Theorem 4 holds for G being a product of the general linear groups. We will be interested in torus actions. Theorem 5 stated in [35] reduces to:

Theorem 5. *Let $T = (\mathbb{C}^*)^n$ and let $t_1, t_2, \ldots, t_n \in \mathrm{Hom}(T, \mathbb{C}^*)$ be the characters corresponding to the decomposition of T into the product. Suppose $V = \bigoplus V_\lambda$ is a representation of T such that each weight λ appearing in V is a nonnegative combination of t_i's. Let $X \subset V$ be a variety preserved by T-action. Then the equivariant fundamental class $[X] \subset H^*_T(V) = \mathbb{Q}[t_1, t_2, \ldots, t_n]$ is a polynomial with nonnegative coefficients.*

2. Equivariant Chern class

Our goal is to study more delicate invariants of subvarieties in representations of algebraic groups, the invariants which are refinements of the equivariant fundamental class. In most of the interesting cases the subvarieties to study are singular. Our first choice is the equivariant version of the Chern-Schwartz-MacPherson classes. We recall that the usual Chern-Schwartz-MacPherson classes, introduced in [29] and denoted by c_{SM}, live in homology, they are Poincaré duals of the Chern classes of the tangent bundle when the variety is smooth. These classes are functorial in a certain sense, and therefore usually they are computed via resolution of singularities.

The equivariant version of Chern-Schwartz-MacPherson classes was developed by Ohmoto [33]. To define these classes one has to recall the Borel construction of the equivariant cohomology. Let G be a topological group. Denote by $EG \to BG = EG/G$ the universal principal G-bundle. This bundle is defined up to G-equivariant homotopy. For a topological G-space the equivariant cohomology is by definition the cohomology of the associated X-bundle $EG \times^G X$. Now we apply this construction to G being an algebraic reductive group and X a complex algebraic G-variety. With the exclusion of the case of the trivial group, EG does not admit a finite dimensional model. Instead, EG always has an approximation by algebraic G-varieties, see [39]. For example if $G = \mathbb{C}^*$, then $EG = \mathbb{C}^\infty - \{0\}$ and $BG = \mathbb{P}^\infty$. It can be approximated by $U = \mathbb{C}^n - \{0\}$ with $U/G = \mathbb{P}^{n-1}$. In general as the approximation of EG we take an open set U in a linear representation V of G satisfying

- U is G-invariant,

- G acts freely on U and the action admits a geometric quotient,
- $V - U$ has a sufficiently large codimension in V.

If we are interested only in the cohomology classes of degrees bounded by d we take an approximation with $2 \operatorname{codim}(V - U) > d + 1$. Then

$$H_G^k(X) = H^k(U \times^G X)$$

for $k \le 2d$. The equivariant Chern classes of a smooth G-variety coincides with the equivariant Chern class of the tangent bundle. By Borel construction it is the usual Chern class of the tangent bundle to fibers of the fibration $EG \times^G X \to BG$. Using an approximation it can be written as

$$c^G(X) = p^* c(U/G)^{-1} \cup c(U \times^G X) \in H^*(U \times^G X) \simeq H_G^*(X) \,,$$

where $p : U \times^G X \to U/G$ is the projection. If X is singular we apply the same formula with obvious modifications. First of all $U \times^G X$ is singular and we have the homology Chern-Schwartz-MacPherson class

$$c_{SM}(U \times^G X) \in H_*^{BM}(U \times^G X) \,.$$

(The superscript BM stands for Borel-Moore homology.) We are forced to use less known equivariant homology $H_*^G(X)$, [12, 14], which can be defined via approximation:

$$H_k^G(X) = H_{k+2 \dim(U/G)}^{BM}(U \times^G X)$$

for $2n - k < 2 \operatorname{codim}(V - U) - 1$, i.e. for $k > 2n - 2 \operatorname{codim}(V - U) + 1$.

Definition 6. *The equivariant Chern-Schwartz-MacPherson class of X is defined by the formula*

$$c^G(X) = p^* c(U/G)^{-1} \cap c_{SM}(U \times^G X) \in H_{*+2 \dim(U/G)}^{BM}(U \times^G X) \simeq H_*^G(X) \,.$$

The definition can be extended to the equivariant constructible functions on X.

Note that $H_*^G(X)$ can have nontrivial negative degrees, but the equivariant Chern-Schwartz-MacPherson class lives in $H_{\ge 0}^G(X)$.

We will not use the long name *equivariant Chern-Schwartz-MacPherson classes*. Hopefully saying just *equivariant Chern classes* in the context of possibly singular algebraic G-varieties or constructible functions will not lead to any confusion. Additionally we will always write $c^G(\mathbb{1}_X)$ instead of $c^G(X)$. Later, in §§9-10, where we compute the equivariant Chern classes of a subvariety in a smooth manifold M, for convenience we skip the cap-product $\cap [M]$ in the notation identifying $H_*^T(M)$ with $H_T^{2 \dim(M) - *}(M)$.

The definition of equivariant Chern classes is in fact irrelevant. All what we need follows from the formal properties.

- **Normalization:** if X is smooth, then $c^G(\mathbb{1}_X)$ is Poincaré dual of the usual equivariant Chern class of the tangent bundle.
- **Functoriality:** for a G-constructible function α and a proper G-map $f : X \to Y$ we have $c^G(f_* \alpha) = f_* c^G(\alpha)$.
- **Product formula:** if X is G-variety and Y is G'-variety, then

$$c^{G \times G'}(\mathbb{1}_{X \times Y}) = c^G(\mathbb{1}_X) \otimes c^{G'}(\mathbb{1}_Y)$$

under the Künneth isomorphism $H_*^{G \times G'}(X \times Y) \simeq H_*^G(X) \otimes H_*^{G'}(Y)$. In particular when X is a trivial G-space, then

$$c^G(\mathbb{1}_X) = 1 \otimes c_{SM}(\mathbb{1}_X) \in H_*^G(X) \simeq H_G^{-*}(pt) \otimes H_*^{BM}(X) \,.$$

- **Functoriality with respect to G:** Let $\phi : G' \to G$ be a group homomorphism and X a G-space. The induced map $\phi^* : H_*^G(X) \to H_*^{G'}(X)$ sends $c^G(\mathbb{1}_X)$ to $c^{G'}(\mathbb{1}_X)$.

All five properties easily follow from the corresponding properties of the usual Chern-Schwartz-MacPherson classes. The equivariant Chern class carries more information than nonequivariant Chern-Schwartz-MacPherson class. There is a natural map $H_*^G(X) \to H_*^{BM}(X)$ which is induced by the inclusion of the trivial group into G. It transports the equivariant Chern class to the nonequivariant one.

Let us focus on the case when $G = T$ is a torus and V is a complex linear representation. The equivariant homology $H_*^T(V)$ is a free rank one module over $H_T^*(pt)$ generated by $[V] \in H_{2\dim(V)}^T(V)$. The action of a character $\lambda \in H_T^2(pt)$ lowers the degree by 2. By Poincaré duality we have the isomorphisms

$$H_{2k}^T(V) \simeq H_T^{2(\dim(V)-k)}(V) \simeq H_T^{2(\dim(V)-k)}(pt) \simeq Sym^{\dim(V)-k}(T^\vee \otimes \mathbb{Q}) \,.$$

We start with the basic example.

Example 7. Let V be a complex linear representation of a torus T. Suppose that V decomposes as the sum of the weight spaces

$$V = \bigoplus_\lambda V_\lambda \,.$$

Then

$$c^T(\mathbb{1}_V) = \Big(\prod_\lambda (1 + \lambda)^{\dim(V_\lambda)} \Big) \cap [V] \in H_*^T(V) \simeq Sym^{\dim(V)-*}(T^\vee \otimes \mathbb{Q})$$

and

$$c^T(\mathbb{1}_{\{0\}}) = [\{0\}] = \Big(\prod_\lambda \lambda^{\dim(V_\lambda)} \Big) \cap [V] \in H_0^T(V) \simeq Sym^{\dim(V)}(T^\vee \otimes \mathbb{Q}) \,.$$

The last formula follows from covariant functoriality.

Now let us see what the equivariant Chern class means for conical sets in affine spaces.

Example 8. Let $T = \mathbb{C}^*$ acts on $\mathbb{C}^n$ by scalar multiplication. Consider a nonempty cone $X \subset \mathbb{C}^n$. We will compute its equivariant Chern class with respect to the action of T. Denote by $\mathbb{P}(X) \subset \mathbb{P}^{n-1}$ the projectivization of X. Let $h = c_1(\mathcal{O}(1)) \in H^2(\mathbb{P}^{n-1})$ and let $t \in H_T^2(pt)$ be the element corresponding to the identity character. Suppose that

$$c_{SM}(\mathbb{1}_{\mathbb{P}(X)}) = (a_0 + a_1 h + \cdots + a_{n-1}h^{n-1}) \cap [\mathbb{P}^n] \in H_*(\mathbb{P}^n) \,.$$

We will show in §8 that the equivariant Chern class of the cone is equal to

$$c^T(\mathbb{1}_X) = (a_0 + a_1 t + \cdots + a_{n-1}t^{n-1} + t^n) \cap [\mathbb{C}^n] \in H_*^T(\mathbb{C}^n) \,.$$

This formula agrees with computation of Aluffi-Marcolli who calculated the invariant of conical sets defined as the Chern-Schwartz-MacPherson class of the constructible function $\mathbb{1}_X$ considered not in $\mathbb{C}^n$ but in $\mathbb{P}^n$.

Now suppose $G = T = (\mathbb{C}^*)^n$ is acting on a vector space V as in Theorem 5. We pose a question:

Question. *When does $c^T(\mathbb{1}_X) \in H_*^T(V) \simeq \mathbb{Q}[t_1, t_2, \ldots, t_n]$ have nonnegative coefficients?*

This is a special property of X since in general the answer is negative. If the equivariant Chern classes are effective, i.e. represented by an invariant cycle, then the answer is positive. Also it is easy to find a counterexample: if $T = \mathbb{C}^*$ acts on $V = \mathbb{C}^n$ by scalar multiplication and let X be a cone over a curve of genus $g > 1$ and of degree d. Then

$$c^T(\mathbb{1}_X) = ([X] + 2(1-g)t^{n-1} + t^n) \cap [\mathbb{C}^n] = d[\mathbb{C}^2] + 2(1-g)[\mathbb{C}^1] + [\mathbb{C}^0]$$

is a counterexample. On the other hand we have a bunch of positive examples: local equivariant Chern classes have nonnegative coefficients for

- toric singularities (see Corollary 18),
- generic hyperplane arrangements with a small number of hyperplanes [2],
- banana Feynman motives [3].

3. LOCALIZATION THEOREM

For the moment we leave the question of positivity. Our current goal is to develop a calculus which would allow to compute equivariant Chern classes avoiding resolution of singularities. Our main tool is the Localization Theorem for torus action. The topological setup is the following: suppose the torus $T = (S^1)^n$ or $(\mathbb{C}^*)^n$ acts on a compact space M (decent enough, e.g. equivariant CW-complex). The equivariant cohomology $H_T^*(M)$ is a module over equivariant cohomology of the point

$$H_T^*(pt) = \mathbb{Q}[t_1, t_2, \ldots t_n].$$

The following theorem goes back to Borel.

Theorem 9 ([36], [6]). *The restriction to the fixed set*

$$\iota^* : H_T^*(M) \longrightarrow H_T^*(M^T)$$

becomes an isomorphism after localizing in the multiplicative set generated by the nontrivial characters

$$S = T^\vee - 0 \subset H_T^2(pt).$$

If M is a manifold, then the inverse of the restriction map is given by the Atiyah-Bott/Berline-Vergne formula. To explain that let us fix a notation. We decompose the fixed point set into components $M^T = \bigsqcup_{\alpha \in A} M_\alpha$. Each M_α is a manifold and denote by $e_\alpha \in H_T^*(M_\alpha) = H^*(M_\alpha) \otimes \mathbb{Q}[t_1, t_2, \ldots t_n]$ the equivariant Euler class of the normal bundle. The following map is the inverse of the restriction to the fixed points

(1)
$$
\begin{array}{ccc}
S^{-1} H_T^*(M^T) \;=\; \bigoplus_{\alpha \in A} S^{-1} H_T^*(M_\alpha) & \xrightarrow{\;\simeq\;} & S^{-1} H_T^*(M) \\[2mm]
\{x_\alpha\}_{\alpha \in A} & \mapsto & \sum_{\alpha \in A} \iota_{\alpha *}\left(\frac{x_\alpha}{e_\alpha}\right),
\end{array}
$$

where $\iota_\alpha : M_\alpha \hookrightarrow M$ is the inclusion. The key point in the formula (1) is that the Euler class e_α is *invertible* in $S^{-1} H_T^*(M_\alpha)$.

Remark 10. Note that if M is a smooth compact algebraic variety and the action of the torus is algebraic, then $H_T^*(M)$ is a free module over $H_T^*(pt)$, so we do not kill any class inverting nontrivial characters.

I other words we can state the theorem:

Theorem 11 ([6, 15]). *Let M be an algebraic variety with algebraic torus action, then with the previous notation*

$$x = \sum_{\alpha \in A} \iota_{\alpha *}\left(\frac{x_{|M_\alpha}}{e_\alpha}\right) \in H_T^*(M).$$

Therefore we can say that x is a sum of local contributions. Although one has to understand that this statement is a bit misleading. In fact it is not possible to extract individual summands in $H_T^*(M)$. This can be done only in the localized ring. How a single fixed point component contributes to the global class is obscured by the weights of the tangent representation.

Furthermore consider the push-forward, i.e. the integration along M

$$p_* = \int_M : H_T^*(M) \to H_T^{*-2\dim(M)}(pt)$$

where $p : M \to pt$ is the constant map. Another form of the Localization Theorem allows to express the integration along M by integrations along components of the fixed point set.

Theorem 12 (Berline-Vergne [10]). *For $x \in H_T^*(M)$ the integral can be computed by summation of local contributions*

$$(2) \qquad \int_M x = \sum_{\alpha \in A} \int_{M_\alpha} \frac{x_{|M_\alpha}}{e_\alpha}.$$

In particular, when the fixed point set is discrete $M^T = \{p_0, p_1, \ldots p_n\}$ then the Euler class is the product of weights

$$e_p = \prod_{\lambda \in \Lambda} \lambda^{\dim(V_\lambda)} \in H_T^*(\{p\}) \in \mathbb{Q}[t_1, t_2, \ldots, t_n],$$

provided that $\mathbf{T}_p M$, the tangent space at p is the sum of weight spaces

$$\mathbf{T}_p M = \bigoplus_{\lambda \in \Lambda} V_\lambda.$$

The integral along M is equal to the sum of fractions:

$$\int_M a = \sum_{p \in M^T} \frac{a_{|p}}{e_p}.$$

Remark 13. The Berline-Vergne formula (2) can be formulated for singular spaces embedded into a smooth manifold. The local factor $\frac{1}{e_\alpha}$ is replaced by $\frac{[X]_{|M_\alpha}}{e_\alpha}$, see [15, 9]. There is a generalization of the Theorem (11) for equivariant homology (or Chow groups) of singular spaces, but one needs an additional assumption allowing to define ι^*, [15, Proposition 6].

4. SOME CALCULI OF RATIONAL FUNCTIONS

Before examining equivariant Chern classes of Schubert varieties let us look closer at some computations based on the Localization Theorem for Grassmannians. Let us start with the projective space $M = \mathbb{P}^n$ with the standard torus $T = (\mathbb{C}^*)^{n+1}$ action. The fixed point set is discrete and consists of coordinate lines

$$M^T = \{p_0, p_1, \ldots p_n\}.$$

The tangent space at the point p_k decomposes into one dimensional representations:

$$\mathbf{T}_{p_k} M = \bigoplus_{\ell \neq k} \mathbb{C}_{t_\ell - t_k}.$$

The Euler class is equal to

$$e_{p_k} = \prod_{\ell \neq k} (t_\ell - t_k).$$

Let us integrate powers of $c_1 := c_1(\mathcal{O}(1))$. Of course

$$\int_{\mathbb{P}^n} c_1^m = \begin{cases} 0 & \text{for} \quad m < n \\ 1 & \text{for} \quad m = n \end{cases}$$

Applying Berline-Vergne (2) formula we get the identity

$$(3) \qquad \sum_{k=0}^n \frac{(-t_k)^m}{\prod_{\ell \neq k}(t_\ell - t_k)} = \begin{cases} 0 & \text{for} \quad m < n \\ 1 & \text{for} \quad m = n, \end{cases}$$

which is not obvious at the first sight. For example we encourage the reader to compute by hand the sum

$$\frac{t_0^2}{(t_1-t_0)(t_2-t_0)(t_3-t_0)} + \frac{t_1^2}{(t_0-t_1)(t_2-t_1)(t_3-t_1)} + \frac{t_2^2}{(t_0-t_2)(t_1-t_2)(t_3-t_2)} + \frac{t_3^2}{(t_0-t_3)(t_1-t_3)(t_2-t_3)}.$$

This is exactly the expression (3) for $m = 2$, $n = 3$. Replacing

$$t_0 = 0\,, \quad t_1 = 1\,, \quad t_2 = 2\,, \ \ldots\,, \ t_n = n$$

(i.e. specializing to a subtorus) the sum (3) is equal to

$$\sum_{k=0}^{n} \frac{(-1)^{m+k} k^m}{k!(n-k)!}$$

Multiplying by $n!$ we obtain

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^{m+k} k^m = \begin{cases} 0 & \text{for} \quad m < n \\ n! & \text{for} \quad m = n\,. \end{cases}$$

which is a good exercise for students.

The integral of higher powers of c_1 is even more interesting: Let us see what we do get for $m > n$? For example $n = 2$, $m = 4$ we have

$$\frac{t_0^4}{(t_1 - t_0)(t_2 - t_0)} + \frac{t_1^4}{(t_0 - t_1)(t_2 - t_1)} + \frac{t_2^4}{(t_0 - t_2)(t_1 - t_2)}$$

It takes some time to check that the sum is equal to

$$t_0^2 + t_1^2 + t_2^2 + t_0 t_1 + t_0 t_2 + t_1 t_2,\,.$$

In terms of the elementary symmetric functions it is equal to

$$\sigma_1^2 - \sigma_2\,.$$

Proposition 14. *In general*

$$(-1)^k \int_{\mathbb{P}^n} c_1^{n+k}$$

is equal to the Schur function S_k (which corresponds to the Segre class of vector bundles).

Proof. By Jacobi-Trudy formula (which is the definition of the Schur function)

$$S_k(t_0, t_1, \ldots, t_n) = \frac{\begin{vmatrix} t_0^{n+k} & t_0^{n-1} & t_0^{n-2} & \cdots & t_0^1 & 1 \\ t_1^{n+k} & t_1^{n-1} & t_1^{n-2} & \cdots & t_1^1 & 1 \\ t_2^{n+k} & t_2^{n-1} & t_2^{n-2} & \cdots & t_2^1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ t_n^{n+k} & t_n^{n-1} & t_0^{n-2} & \cdots & t_n^1 & 1 \end{vmatrix}}{\prod_{i<j}(t_i - t_j)}\,.$$

To prove the proposition it is enough to use Laplace expansion with respect to the first column and watch carefully the signs. □

Remark 15. It is wiser to use the dual Grassmannian of hyperplanes, then one gets rid of the factor $(-1)^k$. The general formula with positive signs for Grassmannians is given by Theorem 16.

We will have a look now at the calculus on $\mathrm{Grass}_m(\mathbb{C}^n)$. The fixed point set consists of coordinate subspaces:

$$\mathrm{Grass}_m(\mathbb{C}^n)^T = \{p_\lambda \,:\, \lambda = (\lambda_1 < \lambda_2 < \cdots < \lambda_m),\ 1 \le \lambda_1,\ \lambda_m \le n\}$$

The tangent space at the fixed point p_λ decomposes into distinct line representations of T:

$$\mathbf{T}_{p_\lambda} \mathrm{Grass}_m(\mathbb{C}^n) = \bigoplus_{k\in\lambda, \ell\notin\lambda} \mathbb{C}_{t_\ell - t_k}\,.$$

The Euler class is equal to

$$e_{p_\lambda} = \prod_{k \in \lambda,\ \ell \notin \lambda} (t_\ell - t_k).$$

Let us integrate a characteristic class of the tautological bundle $\mathcal{R}_m$. Suppose that the class $\phi(\mathcal{R}_m)$ is given by a symmetric polynomial in Chern roots $W(x_1, x_2, \ldots, x_m)$. Then

$$\int_{\mathrm{Grass}_m(\mathbb{C}^n)} \phi(\mathcal{R}_m) = \sum_\lambda \frac{W(t_i : i \in \lambda)}{\prod_{k \in \lambda,\ \ell \notin \lambda} (t_\ell - t_k)}$$

It looks like a rational function, but we obtain a polynomial in t_i's of degree $deg(W) - \dim(\mathrm{Grass}_m(\mathbb{C}^n))$. This expression can be written as the iterated residue[1]

(4) $$\frac{1}{m!} Res_{z_1 = \infty} Res_{z_2 = \infty} \ldots Res_{z_m = \infty} \frac{W(z_1, z_2, \ldots z_m) \prod_{i \neq j} (z_i - z_j)}{\prod_{i=1}^n \prod_{j=1}^m (t_i - z_j)},$$

see [8]. Of course if $deg(W) < \dim(\mathrm{Grass}_m(\mathbb{C}^n)) = (n-m)m$, then

$$\sum_\lambda \frac{W(t_i : i \in \lambda)}{\prod_{k \in \lambda,\ \ell \notin \lambda} (t_\ell - t_k)} = 0$$

If $deg(W) = \dim(\mathrm{Grass}_m(\mathbb{C}^n))$, then we get a constant. For example for $W = c_1^{\dim(\mathrm{Grass}_m(\mathbb{C}^n))} = \big(- (x_1 + x_2 + \cdots + x_m)\big)^{(n-m)m}$ we obtain the degree of the Plücker embedding $\mathrm{Grass}_m(\mathbb{C}^n) \subset \mathbb{P}(S^m(\mathbb{C}^n))$ (or the volume of $\mathrm{Grass}_m(\mathbb{C}^n)$). According to Hook Formula [19, §4.3]

$$deg(\mathrm{Grass}_m(\mathbb{C}^n)) = \frac{(m(n-m))!}{\prod_{(i,j) \in \lambda} h(i,j)},$$

where $h(i,j)$ denotes the length of the hook with vertex at $(i,j) \in \lambda$ contained in the rectangle $m \times (n-m)$. For $\mathrm{Grass}_3(\mathbb{C}^7)$ the hook lengths are the following

6	5	4	3
5	4	3	2
4	3	2	1

Hence the degree is equal to

$$\frac{12!}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 462.$$

It would be interesting to find an immediate connection of the Hook formula and the residue method given by the formula (4).

Let us now formulate a generalization of Proposition 14. For a partition $I = (i_1 \geq i_2 \geq i_n)$ the Schur function is defined by Jacobi-Trudy formula [28, §I.3]

$$S_I(t_1, t_2, \ldots, t_n) = \frac{\begin{vmatrix} t_1^{n-1+i_1} & t_1^{n-2+i_2} & t_1^{n-3+i_3} & \cdots & t_1^{1+i_{n-1}} & t_1^{i_n} \\ t_2^{n-1+i_1} & t_2^{n-2+i_2} & t_2^{n-3+i_3} & \cdots & t_2^{1+i_{n-1}} & t_2^{i_n} \\ t_3^{n-1+i_1} & t_3^{n-2+i_2} & t_3^{n-3+i_3} & \cdots & t_3^{1+i_{n-1}} & t_3^{i_n} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ t_n^{n-1+i_1} & t_n^{n-2+i_2} & t_n^{n-3+i_3} & \cdots & t_n^{1+i_{n-1}} & t_n^{i_n} \end{vmatrix}}{\prod_{1 \leq i < j \leq n} (t_i - t_j)}.$$

---

[1]More general formulas were found recently by Magdalena Zielenkiewicz for Grassmannians of all classical groups.

The definition of Schur function is extended to characteristic classes of vector bundles. Expanding the determinant with respect to the first block column containing $m \times m$ minors we find the formula for push-forward:

Theorem 16. *Consider the quotient bundle $\mathcal{Q}$ and the tautological bundle $\mathcal{R}$ over $\mathrm{Grass}_m(\mathbb{C}^n)$. Let $J = (j_1 \geq j_2 \geq \cdots \geq j_{n-m})$ and $K = (k_1 \geq k_2 \geq \cdots \geq k_m)$ be partitions. Suppose $j_{n-m} - m \geq k_1$. Then*

$$\int_{\mathrm{Grass}_m(\mathbb{C}^n)} S_J(\mathcal{Q}) S_K(\mathcal{R}) = S_I(t_1, t_2, \ldots, t_n),$$

where $I = (j_1 - m \geq j_2 - m \geq \cdots \geq j_{n-m} - m \geq k_1 \geq k_2 \geq \cdots \geq k_m)$.

A suitable modifications of Theorem 16 can be easily formulated for the partitions not satisfying the inequality $j_{n-m} - m \geq k_1$. The integral is equal up a sign to the Schur function for another partition or it is zero. By the splitting principle Theorem 16 implies the corresponding statement for Grassmannian bundles over any base, not necessarily over the classifying space BT. This way we obtain a proof of the Gysin homomorphism formula [24], [20, §4.1].

The equivariant Schubert calculus was studied by a number of authors: Knutson–Tao [26], Laksov–Thorup [27], Gatto–Santiago [21] and others. Some formulas can be obtained by taking residue at infinity [8, 9]. Concluding this section I would like to say that it seems that still the calculus of *rational* symmetric functions is not developed enough. In §10 we will present a method of computation of equivariant Chern classes of Schubert varieties. Unfortunately I do not know (maybe except Theorem 16) a tool which would allow us to simplify the expressions which appear in computation.

5. Toric varieties

We keep in mind that our purpose is to compute equivariant Chern classes. From the Localization Theorem it follows that equivariant Chern classes are determined by local equivariant Chern classes belonging to the homologies of the components of M^T. In the beginning let us consider the toric varieties, which are quite easy, but unfortunately not very general from our point of view.

Theorem 17. *Let X be a toric variety. Consider the cycle Ξ_X which is equal to the sum of the closures of orbits. Then Ξ_X represents the equivariant Chern class $c^T(\mathbb{1}_X) \in H_*^T(X)$.*

Proof. First we consider the case when X is a smooth toric variety. If $X = \mathbb{C}^1$ with the standard action of $T = \mathbb{C}^*$ then, indeed, the equivariant Chern class is equal $[\mathbb{C}] + [0] = [\Xi_X]$. By Whitney formula and the product property of sets the statement holds for $X = \mathbb{C}^n$ with the standard action of $T = (\mathbb{C}^*)^n$. Every smooth toric variety locally looks like $\mathbb{C}^n$ with the standard action of the torus, therefore the equation $c^T(1_X) = [\Xi_X]$ holds locally, i.e. after restriction to each fixed point. Let X be a complete smooth toric variety. Then $H_*^T(X)$ is free over $H_T^*(pt)$. By the Localization Theorem $c^T(\mathbb{1}_X) = \Xi_X$ holds globally. The noncomplete case follows since any smooth toric variety can be compactified equivariantly.

The singular case can be deduced as usual by functoriality. One sees that for smooth toric varieties the equivariant Chern class of the constructible function supported by a single orbit is exactly the fundamental class of the closure of that orbit without boundary cycles. The equality is preserved by the push-forward. $\square$

Note that the theorem holds in equivariant homology of X and we do not have to use any embedding into a smooth manifold. The non-equivariant case was proven by Ehlers and Barthel-Brasselet-Fieseler [7] and it also follows immediately from [1].

The cycle representing the equivariant Chern class of a toric variety is effective. Therefore for the embedded case by Theorem 5 we have the corollary:

Corollary 18. *Let V be a representation of T. Suppose an affine T-variety X (possibly singular) is embedded equivariantly into V. If the weights of the torus acting on V are nonnegative then the coefficients of $c^T(\mathbb{1}_X) \in H_T^*(V) = \mathbb{Q}[t_1, t_2, \ldots, t_n]$ are nonnegative.*

The situation described in the Corollary 18 appears when

$$X = X_\sigma = Spec(\mathbb{C}(\sigma^\vee \cap N))$$

is presented in the usual way: the embedding into

$$V = Spec(\mathbb{C}[x_1, x_2, \ldots, x_n])$$

is given by a choice of the generators of the semigroup $\sigma^\vee \cap N$, see [17, §1.3].

Remark 19. All the singularities of the Schubert varieties in Grassmannians of planes $Grass_2(\mathbb{C}^n)$ are toric. Therefore the local equivariant Chern classes are nonnegative combinations of monomials for a suitable choice of a basis of $H_T^2(pt)$.

The global positivity of Chern classes of Schubert varieties in $Grass_2(\mathbb{C}^n)$ seems not to follow automatically. Except from the case of projective spaces only the Schubert varieties with isolated singularities (the partitions $(n-3, k)$ for $k \leq n-4$, according to the standard convention) are toric. Nevertheless it was shown in [5, §4.3] that the nonequivariant Chern classes of Schubert varieties in $Grass_2(\mathbb{C}^n)$ are indeed effective.

6. Equivariant Chern class of degree zero

The following Theorem 20 is the key to the inductive procedure for computing equivariant Chern classes. The theorem says that the degree zero component of the equivariant Chern class localized at a fixed point does not depend seriously on the set itself, but only on wether the point belongs to the set or not.

Theorem 20. *Suppose that X is a T-variety, not necessarily smooth, contained in a T-manifold M. Let $p \in X$ be an isolated fixed point. Then the degree zero component of the class $c^T(\mathbb{1}_X)$ restricted to $\{p\}$ is Poincaré dual to the product of weights appearing in the tangent representation $\mathbf{T}_p M$*

$$\left(c^T(\mathbb{1}_X)_{(0)}\right)_{|p} = e_p \cap [p].$$

By additivity of equivariant Chern classes it follows that if $p \notin X$ then $(c^T(\mathbb{1}_X)_{(0)})_{|p}) = 0$.
The core of the proof is the basic equation of Euler characteristics

$$\chi(X) = \chi(X^T).$$

Nevertheless the argument demands some formal manipulations. First of all we note the following fact.

Proposition 21. *Let N be a complete manifold with a torus action. Let us decompose the fixed point set $N^T = \sqcup_{\alpha \in A} N_\alpha$ into connected components. Let $i_\alpha : N_\alpha \to N$ be the inclusion. The equivariant cohomology top Chern class of N is equal to the sum*

$$(5) \qquad c_{top}^T(N) = \sum_{\alpha \in A} (i_\alpha)_* \left(c_{top}(N_\alpha)\right) \in H_T^{2\dim(N)}(N).$$

Dually we have

$$(6) \qquad c^T(\mathbb{1}_N)_{(0)} = \sum_{\alpha \in A} (i_\alpha)_* \left(c_{SM}(\mathbb{1}_{N_\alpha})_{(0)}\right) \in H_0^T(N).$$

Proof. The proof is the straightforward application of the Theorem (11) since

$$i_\alpha^*(c_{top}^T(N)) = e_\alpha \cdot c_{top}(N_\alpha) \in H_T^*(N_\alpha) = H_T^*(pt) \otimes H^*(N_\alpha) \,.$$

□

Proof of Theorem 20. Denote by $\iota_p : \{p\} \to M$ the inclusion of the point. We will argue that for any equivariant constructible function $\alpha : M \to \mathbb{Z}$ the equality holds

$$\iota_p^*(c^T(\alpha)) = \alpha(p) \, e_p \cap [p] \in H_{-2\dim(M)}^T(\{p\}) \,.$$

It is enough to show that statement for M complete and the constructible function of the shape $\alpha = f_*(\mathbb{1}_N)$ for an equivariant map $f : N \to M$ from a smooth complete variety N. (We can assume that N is smooth by the usual argument which is available thanks to equivariant completion [38] and equivariant resolution of singularities [11].) It remains to prove that

(7) $$\iota_p^* f_*(c^T(\mathbb{1}_N)_{(0)}) = \chi(f^{-1}(p)) \, e_p \cap [p] \in H_{-2\dim(M)}^T(\{p\}) \,.$$

Let i_α be as in Proposition 21 and $f_\alpha = f \, i_\alpha : N_\alpha \to M$. We compute the push-forward of the zero degree component:

$$f_* c^T(\mathbb{1}_N)_{(0)} = \sum_{\alpha \in A} f_*(i_\alpha)_* \big(c_{SM}(\mathbb{1}_{N_\alpha})_{(0)} \big)$$

(8) $$= \sum_{\alpha \in A} (f_\alpha)_* \big(c_{SM}(\mathbb{1}_{N_\alpha})_{(0)} \big) \in H_0^T(M) \,.$$

Let $B \subset A$ be the set of components of N^T which are mapped to p. Then

$$\iota_p^* \big(f_* c^T(\mathbb{1}_N)_{(0)} \big) = \iota_p^* \big(\sum_{\beta \in B} (f_\beta)_* c(\mathbb{1}_{N_\beta})_{(0)} \big)$$

$$= \iota_p^* \sum_{\beta \in B} \chi(N_\beta) \, [p]$$

(9) $$= \sum_{\beta \in B} \chi(N_\beta) \, e_p \cap [p] \,.$$

We conclude that the equation (7) holds because $\chi(f^{-1}(p)) = \chi(f^{-1}(p)^T)$ and $f^{-1}(p)^T = \bigsqcup_{\beta \in B} N_\beta$.
□

7. PARTIAL LOCALIZATION

There exists the following modification of the localization formula: we can replace M^T by any invariant submanifold or even arbitrary invariant subset) Y containing the fixed point set M^T. Then the restriction map

$$H_T^*(M) \to H_T^*(Y)$$

becomes an isomorphism after inversion of nontrivial characters S. Also the Berline-Vergne formula holds, but it makes sense only for Y being a submanifold. Suppose that $Y = Y_1 \sqcup \{p\}$. It follows that for any $x \in H_T^*(M)$ we have

(10) $$\frac{x_{|p}}{e_p} + \int_{Y_1} \frac{x_{|Y_1}}{e_{Y_1}} = 0$$

for degree smaller than $\dim(M)$. We will apply this formula for Poincaré dual of $c^T(\mathbb{1}_X)$. The integral of the zero degree Chern-Schwartz-MacPherson class (which corresponds to the top degree of the cohomology class) is equal to the Euler characteristic and the same holds for the equivariant Chern class by

the commutativity of the diagram:

$$c^T(\mathbb{1}_X)_{(0)} \quad \in \quad H_0^T(M) \quad \longrightarrow \quad H_0(M) \quad \ni \quad c(X)_{(0)}$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$\int_M c^T(\mathbb{1}_X)_{(0)} \quad \in \quad H_0^T(pt) \quad \xrightarrow{\simeq} \quad H_0(pt) \quad \ni \quad \chi(X)\,.$$

We apply the partial localization and we find that

(11)
$$\frac{(c^T(X)_{(0)})_{|p}}{e_p} + \int_{Y_1} \frac{(c^T(X)_{(0)})_{|Y_1}}{e_{Y_1}} = \chi(X)\,.$$

Here e_{Y_1} is the equivariant Euler class of the normal bundle of Y_1. (Of course it may be of different degrees over distinct components of Y_1.)

Example 22. The partial localization allows us to compute the equivariant Chern class of the affine cone over a projective variety. Suppose T acts on $\mathbb{C}^n$ with nonzero weights

$$w_1, w_2, \ldots, w_n\,.$$

First recall that the equivariant cohomology ring of $\mathbb{P}^{n-1}$ is the quotient of the polynomial algebra

$$H_T^*(pt)[h] = \mathbb{Z}[t_1, t_2, \ldots, t_n, h]$$

by the relation

$$\prod_{i=1}^n (h + w_i) = 0.$$

Using the elementary symmetric functions σ_i the relation takes form

(12)
$$\sum_{i=0}^n \sigma_i(w_\bullet) h^{n-i} = 0\,.$$

Let $X \subset \mathbb{C}^n$ be a nonempty T-invariant cone and $\mathbb{P}(X) \subset \mathbb{P}^{n-1}$ its projectivization. We consider $X = \overline{X} - \mathbb{P}(X)$ as a constructible set in $\mathbb{P}^n$ and we will compute its equivariant Chern class in $H_T^*(\mathbb{P}^n)$. In this example we skip the Poincaré duals in the notation. Denote by $\iota : \mathbb{P}^{n-1} \to \mathbb{P}^n$ the inclusion. The equivariant Chern class of X restricted to $\mathbb{P}^{n-1}$ is equal to

$$\iota^* c^T(\mathbb{1}_X) \quad = \iota^* c^T(\mathbb{1}_{\overline{X}}) - \iota^* \iota_* c^T(\mathbb{1}_{\mathbb{P}(X)})$$

$$= (1 + h) \cdot c^T(\mathbb{1}_{\mathbb{P}(X)}) - h \cdot c^T(\mathbb{1}_{\mathbb{P}(X)})$$

$$= c^T(\mathbb{1}_{\mathbb{P}(X)})\,.$$

Suppose that the equivariant Chern class of $\mathbb{P}(X)$ is written as

$$c^T(\mathbb{1}_{\mathbb{P}(X)}) = \sum_{i=0}^{n-1} b_i(t) h^i \in H_T^*(\mathbb{P}^{n-1})$$

for some polynomials $b_i(t) \in H_T^*(pt)$ of degree $\leq n - 1 - i$. To compute the local equivariant Chern class at 0 we will apply the formulas (10) and (11) to $M = \mathbb{P}^n$, $Y = \{0\} \cup \mathbb{P}^{n-1}$ and $Y_1 = \mathbb{P}^{n-1}$. We compute

$$\int_{\mathbb{P}^{n-1}} \frac{c^T(\mathbb{1}_X)}{e_{Y_1}} = \int_{\mathbb{P}^{n-1}} \sum_{i=0}^{n-1} b_i(t) h^{i-1}\,.$$

Except from $i = 0$ the summands are integral (belong to $H_T^*(\mathbb{P}^{n-1})$) and they are of degree smaller than $n - 1$. Therefore

$$\int_{\mathbb{P}^{n-1}} \frac{c^T(\mathbb{1}_X)}{e_{Y_1}} = \int_{\mathbb{P}^{n-1}} \frac{b_0(t)}{h} \,.$$

An easy calculation using (12) shows that the inverse Euler class of the normal bundle to $\mathbb{P}^{n-1}$ is equal to

$$h^{-1} = -\sum_{i=1}^{n} \frac{\sigma_{n-i}(w_\bullet)}{\sigma_n(w_\bullet)} h^{i-1} \,.$$

Hence

$$\int_{\mathbb{P}^{n-1}} \frac{c^T(\mathbb{1}_X)}{e_{Y_1}} = -\int_{\mathbb{P}^{n-1}} b_0(t) \sum_{i=1}^{n} \frac{\sigma_{n-i}(w_\bullet)}{\sigma_n(w_\bullet)} h^{i-1} = -\frac{b_0(t)}{\sigma_n(w_\bullet)}$$

By the formulas (10) and (11) and since $\sigma_n(w_\bullet) = e_p$ we find that

$$\frac{c^T(\mathbb{1}_X)_{|p}}{e_p} - \frac{b_0(t)}{e_p} = \chi(X) = 1 \,.$$

Therefore

$$c^T(\mathbb{1}_X)_{|p} = b_0(t) + e_0 \in H_T^*(\{p\})$$

We obtain the following result:

Proposition 23. *Suppose that X is a nonempty T-invariant cone in a linear representation V of T. Let $h = c_1^T(\mathcal{O}_{\mathbb{P}(V)}(1))$ be the equivariant Chern class. If*

(13) $$c^T(\mathbb{1}_{\mathbb{P}(X)}) = \Big(\sum_{i=0}^{\dim(V)-1} b_i(t) h^i \Big) \cap [\mathbb{P}(V)] \in H_*^T(\mathbb{P}(V))$$

then

$$c^T(\mathbb{1}_X) = (b_0(t) + e_0) \cap [V] \in H_*^T(V)$$

where e_0 is the Euler class of the representation V.

Proof. First note, that restriction $H_T^*(V) \to H_T^*(pt)$ is an isomorphism. We apply the calculation of the previous example. The degree of b_0 is at most $\dim(V) - 1$, therefore it does not interfere with e_p, which is homogeneous of degree $\dim(V)$. $\qquad\square$

8. Conical sets in an affine space

We come back to the Example 8 of §2 which was the starting point of our interest in equivariant Chern classes. In [4] there was defined an invariant of a conical set $X \subset \mathbb{C}^n$. It is equal to the Chern-Schwartz-MacPherson class of X considered as a constructible set in $\mathbb{P}^n$. This Chern class

$$c_{SM}(\mathbb{1}_X) \in H^*(\mathbb{P}^n)$$

is expressed via the Chern class of the projectivization. The calculation is based on the following formula:

Proposition 24 ([3, Prop 5.2]). *Let $X \subset \mathbb{C}^n$ be a nonempty conical set. Let $\overline{X} = X \cup \mathbb{P}(X)$ be the closure of X in $\mathbb{P}^n$. Let $x = c_1(\mathcal{O}_{\mathbb{P}^{n-1}}(1))$ and $\tilde{x} = c_1(\mathcal{O}_{\mathbb{P}^n}(1))$. Suppose that*

(14) $$c_{SM}(\mathbb{1}_{\mathbb{P}(X)}) = \Big(\sum_{i=0}^{n-1} a_i x^i \Big) \cap [\mathbb{P}^{n-1}] \in H_*(\mathbb{P}^{n-1})$$

then

(15) $$c_{SM}(\mathbb{1}_{\overline{X}}) = \Big((1 + \tilde{x})\big(\sum_{i=0}^{n-1} a_i \tilde{x}^i \big) + \tilde{x}^n \Big) \cap [\mathbb{P}^n] \in H_*(\mathbb{P}^n)$$

It follows that

$$(16) \qquad c_{SM}(\mathbb{1}_X) = \Big(\sum_{i=0}^{n-1} a_i \tilde{x}^i + \tilde{x}^n \Big) \cap [\mathbb{P}^n] \in H_*(\mathbb{P}^n)$$

It seems natural to look at the conical sets from the point of view of equivariant cohomology. Let $T = \mathbb{C}^*$ acts on $\mathbb{C}^n$ by scalar multiplication.

Proposition 25. *Under assumption of Proposition 24*

$$(17) \qquad c^T(\mathbb{1}_X) = \Big(\sum_{i=0}^{n-1} a_i t^i + t^n \Big) \cap [\mathbb{C}^n] \in H_*^T(\mathbb{C}^n)$$

Proof. Assume that the usual, nonequivariant Chern class of $\mathbb{P}(X)$ satisfies the formula (14). To apply Proposition 23 we have to express the equivariant Chern class $c^T(\mathbb{1}_{\mathbb{P}(X)}) = 1 \otimes c_{SM}(\mathbb{1}_{\mathbb{P}(X)})$ by $h = c_1^T(\mathcal{O}_{\mathbb{P}(X)}(1))$. The point is that the torus T acts on the fibers of the tautological bundle $\mathcal{O}(-1)$ with weight equal to one, therefore the equivariant Chern class of $\mathcal{O}_{\mathbb{P}^{n-1}}(1)$ is equal to

$$h = 1 \otimes x - t \otimes 1 = x - t$$

under the identification

$$H_T^*(\mathbb{P}^{n-1}) = \mathbb{Q}[t] \otimes H^*(\mathbb{P}^{n-1}) = H^*(\mathbb{P}^{n-1})[t].$$

Hence the equivariant Chern class of $\mathbb{1}_{\mathbb{P}(X)}$ can be written as

$$c^T(\mathbb{1}_{\mathbb{P}(X)}) = \Big(\sum_{i=0}^{n-1} a_i(h+t)^i \Big) \cap [\mathbb{P}^{n-1}] = \Big(\sum_{i=0}^{n-1} \sum_{j=0}^{i} \binom{i}{j} a_i t^{i-j} h^j \Big) \cap [\mathbb{P}^{n-1}].$$

Here the coefficient $b_0(t)$ of the expression (13) is equal to

$$b_0(t) = \sum_{i=0}^{n-1} a_i t^i.$$

By Proposition 23 we obtain the claim.   □

We see that formally the Chern-MacPherson-Schwartz class of $\mathbb{1}_X$ in $\mathbb{P}^n$ and the equivariant Chern class in $\mathbb{C}^n$ satisfy the same formula. The equivariant approach has the advantage that we have for free the Chern class of the product

$$c^{T \times T}(\mathbb{1}_{X \times Y}) = c^T(\mathbb{1}_X) \times c^T(\mathbb{1}_Y).$$

Further we can restrict the Chern class of the product via diagonal inclusion $T \hookrightarrow T \times T$ to obtain $c^T(\mathbb{1}_{X \times Y})$. With the original approach the proof of the above property was a bit demanding, see [4, Lemma 3.10]

9. COMPUTING EQUIVARIANT CHERN CLASSES WITHOUT RESOLUTION OF SINGULARITIES

Below we sketch a method of computing the equivariant Chern class of a T-invariant singular variety not using a resolution of singularities. The calculi will be done in equivariant cohomology and we will omit the Poincaré duality in the notation.

Assume that the fixed point set of the action of the torus on a complex manifold M is discrete. For a given class $x \in H_T^k(M)$ of degree $k < 2 \dim(M)$ the integral $\int_M x$ vanishes. By the Localization

Theorem also the sum $\sum_{p \in M^T} \frac{x_{|p}}{e_p}$ has to vanish. In particular if $x = c^T(\mathbb{1}_X)$, then except from the zero degree

(18)
$$\sum_{p \in M^T} \frac{c^T(\mathbb{1}_X)_{|p}}{e_p} = 0 \,.$$

This relation between local equivariant Chern classes allows in many cases to compute them inductively. Suppose $M^T = \{p_0, p_1, \ldots, p_N\}$ and assume that we know all local equivariant Chern classes for $p_1, p_2, \ldots, p_N$. Then

(19)
$$c^T(\mathbb{1}_X)_{p_0} = -\sum_{i=1}^{N} \frac{e_{p_0}}{e_{p_i}} c^T(\mathbb{1}_X)_{p_i}$$

except from the zero degree. For Grassmannians the quotient $\frac{e_{p_0}}{e_{p_i}}$ simplifies remarkably.

The zero component of the local equivariant Chern class is easy. If $p \in X^T$ then by Theorem 20 this class is equal to the Euler class at the point p

(20)
$$(c^T(\mathbb{1}_X)_{(0)})_p = e_p \in H_T^{2 \dim(M)}(pt).$$

In fact this statement is the crucial point for computation. Any other equivariant characteristic class satisfies the relation 19. The condition fixing the zero equivariant Chern class and vanishing for the degrees higher than the dimension of the ambient space makes the equivariant Chern class unique.
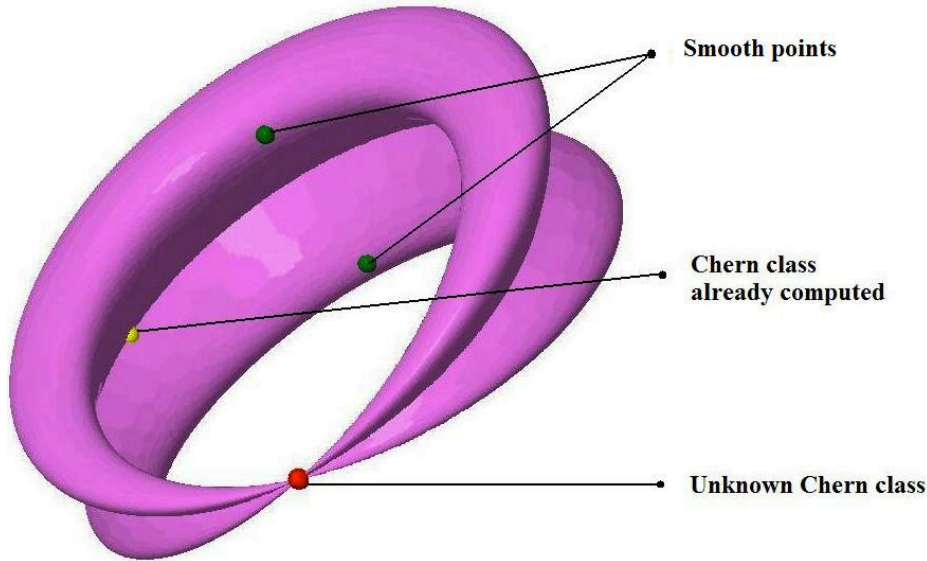
Computation of the local equivariant Chern classes

Of course the inductive step of computation can be applied when for a given singularity one can find a compact variety for which this singularity is the only deepest one. If $X \subset \mathbb{C}^m$ is a cone then taking the closure of X in $\mathbb{P}^n$ will not introduce new singularities. In the next section we present another situation, when the compactifying variety is the Grassmannian.

10. COMPUTATION OF LOCAL EQUIVARIANT CHERN CLASS OF THE DETERMINANT VARIETY

Let us compute the local equivariant Chern class of the variety

$$\Omega_1^o(n) = \{\phi \in \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^n) \ : \ \det(\phi) = 0\}\,.$$

Its compactification in $Grass_n(\mathbb{C}^{2n})$ is the Schubert variety of codimension one

$$\Omega_1(n) = \{W \ : \ W \cap \langle \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n \rangle \neq 0\}\,.$$

We will apply the method sketched above. Let us start with $n = 2$. The canonical neighborhood of the point $p_{1,2}$ in $\mathrm{Grass}_2(\mathbb{C}^4)$ is identified with

$$\mathrm{Hom}(span(\varepsilon_1, \varepsilon_2), span(\varepsilon_3, \varepsilon_4))$$

and the variety $\Omega_1(2)$ intersected with this neighbourhood is exactly $\Omega_1^o(2)$. The corresponding elements of $\Omega_1(2)$ are the planes spanned by the row-vectors of the matrix

$$\begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{pmatrix}\,.$$

The equation of $\Omega_1(2)$ is

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 0\,.$$

Before performing computations let us draw the Goresky-Kottwitz-MacPherson graph ([22, Th. 7.2]) for $M = \mathrm{Grass}_2(\mathbb{C}^4)$ with the variety $\Omega_1(2)$ displayed.
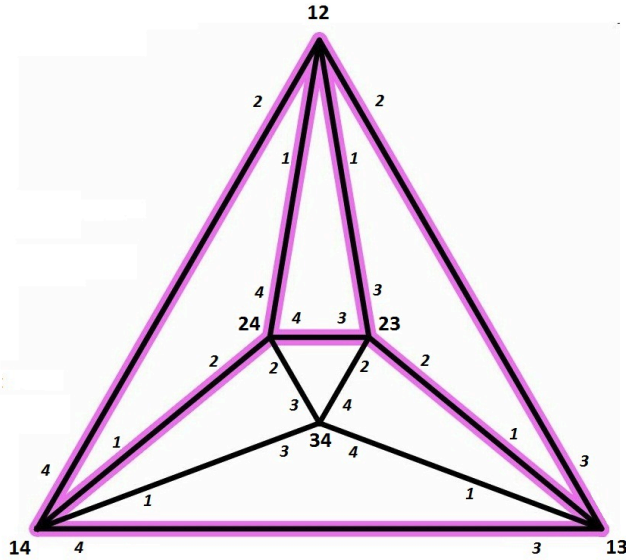
Schubert variety Ω_1 in $\mathrm{Grass}_2(\mathbb{C}^4)$.

The numbers attached to the edges indicate the weights of the T actions along the one dimensional orbits. For example at the point $p_{1,3}$ in the direction towards $p_{1,2}$ the action is by the character $t_2 - t_3$. The variety $\Omega_1(2)$ is singular at the point $p_{1,2}$ and it is smooth at the remaining points. For example at the point $p_{1,3}$ the coordinates are

$$\begin{pmatrix} 1 & a & 0 & b \\ 0 & c & 1 & d \end{pmatrix}$$

and the equation of $\Omega_1(2)$ is $b = 0$. For that point the local equivariant Chern class is equal to

$$(t_4 - t_1)(1 + t_2 - t_1)(1 + t_2 - t_3)(1 + t_4 - t_3).$$

The summand in the formula (18) is the following

$$\frac{(t_4 - t_1)(1 + t_2 - t_1)(1 + t_2 - t_3)(1 + t_4 - t_3)}{(t_4 - t_1)(t_2 - t_1)(t_2 - t_3)(t_4 - t_3)} =$$

$$= \left(1 + \frac{1}{t_2 - t_1}\right)\left(1 + \frac{1}{t_2 - t_3}\right)\left(1 + \frac{1}{t_4 - t_3}\right)$$

We sum up the contribution coming from the fixed points $p_{1,3}$, $p_{1,4}$, $p_{2,3}$, $p_{2,4}$, simplify and multiply by $-(t_3 - t_1)(t_4 - t_1)(t_3 - t_2)(t_4 - t_2)$. We obtain

$$(t_3 + t_4 - t_1 - t_2) \hspace{7cm} deg = 1$$

$$(t_3 + t_4 - t_1 - t_2)^2 \hspace{6.5cm} deg = 2$$

$$(t_3 + t_4 - t_1 - t_2)(2t_1t_2 - t_1t_3 - t_2t_3 - t_1t_4 - t_2t_4 + 2t_3t_4) \hspace{1.5cm} deg = 3$$

$$-4(t_3 - t_1)(t_4 - t_1)(t_3 - t_2)(t_4 - t_2) \hspace{4cm} deg = 4$$

The terms of degree < 4 coincide with the equivariant Chern class of $\Omega_1(2)$ localized at the point $p_{1,2}$. The result is symmetric in two groups of variables: $\{t_1, t_2\}$ and $\{t_3, t_4\}$. The coefficients of the expansion in the basis of the Schur functions

$$c^T(\mathbb{1}_{\Omega_1}) = \sum a_{I,J} S_I(-t_1, -t_2) \cdot S_J(t_3, t_4)$$

has the following coefficients:

	0	1	11	2	21	22
0		1	1	1	2	1
1	1	1	3	1	1	
11	1	3		1		
2	1	1	1			
21	2	1				
22	1					

Computations of the equivariant Chern class $\Omega_1(3) \subset \mathrm{Grass}_3(\mathbb{C}^6)$ can be continued without problems by the same method. At the points of the type p_I with $|I \cap \{1, 2, 3\}| = 1$ the variety is smooth, while at the points p_I with $|I \cap \{1, 2, 3\}| = 2$ the singularity is of the type $\Omega_1(2)_{p_{1,2}} \times \mathbb{C}^5$. We write the sum of fractions according to the rule (19) and simplify. For example the expression which has to be simplified to compute the degree one is the following:

$$-\frac{(s_3 - t_1)(s_3 - t_2)(s_1 - t_3)(s_2 - t_3)}{(s_3 - s_1)(s_3 - s_2)(t_1 - t_3)(t_2 - t_3)}(s_3 - t_3) \; + \; sym. +$$

(21)

$$\frac{(s_3 - t_1)(s_3 - t_2)(s_1 - t_3)(s_2 - t_3)}{(s_1 - s_3)(s_2 - s_3)(t_3 - t_1)(t_3 - t_2)}(s_1 + s_2 - t_1 - t_2) \; + \; sym.$$

(Here $s_1 = t_4$, $s_2 = t_5$, $s_3 = t_6$.) The given summands are the contributions coming from the points $p_{3,4,5}$ and $p_{1,2,6}$. Of course the sum is equal to the fundamental class

$$[\Omega_1] = s_1 + s_2 + s_3 - t_1 - t_2 - t_3$$

(which may be computed in another way). This example shows how a complicated rational functions may in fact lead to a simple result. The difficulty lies in simplifying that expression. Higher degree terms are much more complex. We write the final result in the Schur basis

$$c^T(\mathbb{1}_{\Omega_1}) = \sum a_{I,J} S_I(-t_1, -t_2, -t_3) \cdot S_J(s_1, s_2, s_3).$$

The coefficients are the following:

	0	1	11	2	111	21	3	211	31	22	311	221	32	321	222	33	331	322	332	333
0	1	2	2	4	5	1	9	3	4	6	9	3	8	4	1	3	6	3	1	
1	1	4	8	5	12	12	2	19	5	8	8	16	4	8	10	1	2	4	1	
11	2	8	12	9	16	16	3	18	6	8	6	10	4	4			1	1		
2	2	5	9	4	11	9	1	13	2	5	3	10	1	2	5			1		
111	4	12	16	11	8	16	4			6	10			4		1				
21	5	12	16	9	16	14	2	15	3	5	3	5	1	1						
3	1	2	3	1	4	2		3		1		2				1				
211	9	19	18	13		15	3		3	5			1							
31	3	5	6	2	6	3		3		1		1								
22	4	8	8	5	10	5	1	5	1		1									
311	6	8	6	3		3			1											
221	9	16	10	10		5	2		1											
32	3	4	4	1	4	1		1												
321	8	8	4	2		1														
222	4	10		5			1													
33	1	1	1			1														
331	3	2	1																	
322	6	4		1																
332	3	1																		
333	1																			

We note that all the coefficients are nonnegative.

While computing the equivariant Chern class of $\Omega_1(4) \subset \mathrm{Grass}_4(\mathbb{C}^8)$ appears a problem with the size of the expressions, since $\dim(\mathrm{Grass}_4(\mathbb{C}^8)) = 16$ and $\dim(T) = 8$. In a polynomial of degree 15 in 8 variables there are

$$490\,314 \text{ monomials.}$$

The expression is a sums of 68 fractions with factors $t_i - t_j$ in denominators. We might have used another compactification of $\mathbb{C}^{16}$, e.g. the projective space $\mathbb{P}^{16}$. There are less fixed points, but the denominators are more complicated. They are of the form $\prod[(t_i - t_j) - (t_k - t_\ell)]$.

One practical solution appears naturally. The fixed points can be divided into groups with $|I \cap \{1, 2, \ldots, n\}|$ fixed. Let $f_k(u_\bullet, v_\bullet)$ be the expression for the local equivariant Chern class of $\Omega_1(k)$ with $u_\bullet = (t_1, t_2, \ldots, t_k)$ and $v_\bullet = (t_{k+1}, t_{k+2}, \ldots, t_{2k})$. The local equivariant Chern $c^T(\mathbb{1}_{\Omega_1(n)})|_{p_{1,2,\ldots,n}}$ class can by computed by the formula (19), which becomes

$$(22) \qquad -\sum_{k=1}^{n-1} \sum_{\substack{I \subset \{1, 2, \ldots, 2n\} \\ |I| = n,\ |I \cap \{1, 2, \ldots, n\}| = k}} \frac{e_{p_{1,2,\ldots,n}}}{e_{p_I}} f_k(I) g_k(I),$$

where $f_k(I)$ depends on the two group of variables

$$u_\bullet = t_{I \cap \{1,2,\ldots,n\}} \quad \text{and} \quad v_\bullet = t_{\{n+1,n+2,\ldots,2n\} \setminus I}$$

and $g_k(I)$ is the equivariant Chern class of the singular stratum of the type $\Omega_1(k)$. The factors in the quotients $\frac{e_{p_{1,2,\ldots,n}}}{e_{p_I}}$ cancel out partially and miraculously all the summands for a fixed k turn out to be integral. For $n = 3$ and degree one the summands are given by the formula (21).

Such a division of fixed points has a geometric meaning. In fact we deal with the partial localization (see §7). Consider the action of the subtorus $\mathbb{C}^*$ acting on $\mathbb{C}^{2n}$ with weight 1 on the first n coordinates and with the weight -1 on the remaining coordinates. Then the fixed point set decomposes into disjoint

union of the products of the Grassmannians:

$$\mathrm{Grass}_n(\mathbb{C}^{2n})^{\mathbb{C}^*} = \bigsqcup_{k=0}^{n} \mathrm{Grass}_k(\mathbb{C}^n) \times \mathrm{Grass}_{n-k}(\mathbb{C}^n).$$

The summand for $k = 0$ consists of one point

$$\{0\} \oplus \langle \varepsilon_{n+1}, \varepsilon_{n+2}, \dots, \varepsilon_{2n} \rangle,$$

which does not belong to $\Omega_1(n)$, while for $k = n$ we have

$$\langle \varepsilon_1, \varepsilon_2, \dots, \varepsilon_n \rangle \oplus \{0\},$$

the point which we are concerned with. Let $\mathcal{R}_k$ and $\mathcal{Q}_k$ be the tautological and the quotient bundles over $\mathrm{Grass}_k(\mathbb{C}^n)$. The result of the sum (22) is equal to

$$-(-1)^{n-k} \sum_{k=1}^{n-1} \int_{\mathrm{Grass}_k(\mathbb{C}^n) \times \mathrm{Grass}_{n-k}(\mathbb{C}^n)} [f_k(\mathcal{R}_k, \mathcal{Q}_{n-k}) \cdot \bar{g}_k],$$

where $\bar{g}_k$ is (up to multiplication by a certain Euler class) the equivariant Chern class of the stratum of the singularity type $\Omega_1(k)$. Precisely

$$\bar{g}_k = e(\mathcal{Q}_k^* \otimes \mathcal{Q}_{n-k}) \cdot e(\mathcal{R}_k^* \otimes \mathcal{R}_{n-k}) \cdot$$

$$\cdot c(\mathcal{R}_k^* \otimes \mathcal{Q}_k) \cdot c(\mathcal{R}_{n-k}^* \otimes \mathcal{Q}_{n-k}) \cdot c(\mathcal{Q}_k \otimes \mathcal{R}_{n-k}^*).$$

Using Fubini theorem we do not have to simplify a large expression in one step and we arrive to the result relatively quickly. Also knowing the Schur expansion of the functions f_k one can apply Theorem 16.

11. GKM-relations

Less time-consuming method of computation of the local equivariant Chern class is based on the relation discovered by Chen-Skjelbred [13], called GKM-relations after the rediscovery in [22]. These relation allow us to determine the local equivariant Chern class at the point p_I knowing only the local equivariant Chern classes at the neighbouring points in the GKM-graph. This is so since

$$c^T(\mathbb{1}_V)_{|p_I} \equiv c^T(\mathbb{1}_V)_{|p_J} \text{ modulo } (t_i - t_j),$$

whenever

$$J = (I - \{i\}) \cup \{j\}.$$

Again the method works for all the degrees smaller then the dimension of the Grassmannian, since the intersection of the ideals $(t_i - t_j)$ is contained in the degree greater or equal to the dimension of the Grassmannian. That is so for any GKM-space. Now the problem of simplifying huge rational function is replaced by solving a relatively small system of linear equations.

12. The result for $\mathrm{Grass}_4(\mathbb{C}^8)$

Let us write the local equivariant Chern class in the Schur basis

$$c^T(\mathbb{1}_{\Omega_1(4)})_{|p_{1,2,3,4}} = \sum a_{I,J} S_I(-t_1, -t_2, -t_3, -t_4) \cdot S_J(t_5, t_6, t_7, t_8).$$

Just to quench readers curiosity we show here the most interesting fragment of the table of coefficients.

It is hard not to have impression that there should be a way of writing down this equivariant Chern class in a compact way. For example the equivariant Chern class of the tangent bundle written in the Schur basis is as much complicated as ours, but it is just $c(\mathrm{Hom}(\mathcal{R}_n, \mathcal{Q}_n))$.

It turns out that the local equivariant Chern class of $\Omega_1(4)$ is a positive combination of monomials in $-t_1, -t_2, -t_3, -t_4, t_5, t_6, t_7, t_8$. As one can see it is *not* a positive combinations of products of Schur functions. Fortunately we do not have a contradiction with the conjecture of Aluffi and Mihalcea [5] which says that the Chern-Schwartz-MacPherson classes are effective. Note that the Schubert varieties

are only T-invariant, and the Theorem 4 does not apply. Instead we have a freedom with choosing the basis of weights. The local equivariant Chern class is a polynomial in $u_{i,j} = t_i - t_j$. To write $c^T(\mathbb{1}_X)$ in a unique way we chose a spanning tree of the full graph with vertices $1, 2, \ldots, 2n$. The edge between i and j (with the orientation forced by the partition) corresponds to the generator $t_j - t_i$. Some choices lead to an expression with nonnegative coefficients.

<center>Positive monomial bases for $\mathrm{Grass}_2(\mathbb{C}^4)$</center>
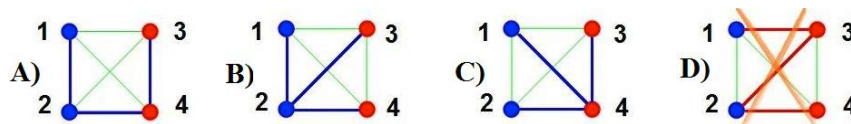
A) $t_2 - t_1,\quad t_4 - t_2,\quad t_3 - t_4$
B) $t_2 - t_1,\quad t_3 - t_2,\quad t_4 - t_2$
C) $t_4 - t_1,\quad t_4 - t_2,\quad t_4 - t_3$
D) $t_3 - t_1,\quad t_3 - t_2,\quad t_4 - t_2$ this is not a positive basis

The positivity condition for a graph is the following:

- Characters of the tangent representation are nonnegative sums of base elements.

That in fact supports the conjecture of Aluffi and Mihalcea in a stronger, equivariant version.

The original, nonequivariant version was checked by B. Jones [23] for cells in $\mathrm{Grass}_m(\mathbb{C}^n)$ for $m \le 3$. In his computations equivariant cohomology and the Localization Theorem was used to compute the push-forward of classes from a resolution.

13. FURTHER DIRECTIONS OF WORK

Several goals have not been reached so far. The most obvious directions of further work would be:

- deduce positivity results,
- study global equivariant Chern classes of Schubert varieties and open cells,
- in particular relate our computations to the determinant formulas of [5] and the combinatorial interpretation of [30],
- develop a suitable calculus of symmetric rational functions to handle expressions appearing in the Berline-Vergne formula for Grassmannians.

We hope to realize this program in future.

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