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ON RELATIVE INVARIANTS AND DETERMINACY OF PLANE CURVES

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Introduction

In current work on flat singularity theory, I have been led to consider the local invariants of a curve C relative to the divisor D defined by its tangent cone. It turned out in the calculation of these invariants that the fact that D consisted of the tangent cone was not used, only the fact that it was a collection of lines through the origin. This led me to ask whether any restriction at all on D was required. The first object of this note is to show that it is not. The second object is to obtain general estimates for the degree of determinacy of C relative to D.

We begin by recalling some standard notations and ideas of singularity theory. Then we apply them to plane curves, which may be given either by a parametrisation f or by an equation ϕ. Throughout we study only germs (at the origin) of curves, so omit 'germ' from our terminology. We work throughout in the complex analytic framework. All curves will be assumed to be reduced.

Next we will recall the definitions of the invariants $\delta(C)$, $\mu(C)$ and $\tau(C)$ and the calculations of the codimensions $d_e(f,\mathcal{L}) = 2\delta(C)$, $d_e(f,\mathcal{A}) = \tau(C) - \delta(C)$, $d_e(\phi,\mathcal{R}) = \mu(C)$, and $d_e(\phi,\mathcal{K}) = \tau(C)$. We then introduce the relative versions of the above concepts. Our first main result is the calculation of the codimensions in the relative case: the results are

$$d_e(f,\mathcal{L}_D) = 2\delta(C) + C.D, \quad d_e(f,\mathcal{A}_D) = \tau(C\cup D) - \delta(C) - C.D - \tau(D),$$

$$d_e(\phi,\mathcal{R}_D) = \mu(C) + C.D + \mu(D) - 1 - \tau(D), \quad d_e(\phi,\mathcal{K}_D) = \tau(C\cup D) - C.D - \tau(D).$$

The degree of determinacy for C up to right or left equivalence is bounded by the Milnor number $\mu(C)$. In the last section we obtain corresponding bounds for C relative to D; such bounds are also needed for the work on flat singularity theory.

1. Singularities of plane curves

We now recall the methods and notations of singularity theory, following Mather [5] (see e.g. [8]). Write $\mathcal{O}_x$ for the ring of germs of functions on N at x and $\mathfrak{m}_x$ for its maximal ideal. Denote the tangent bundle $\pi_N : TN \to N$ and write θ_N for the set of germs at x of sections of π_N (i.e. vector fields on N); we think of θ_N as the tangent space at the identity to the group $Diff(N,x)$ of germs of diffeomorphisms. Introduce corresponding notations for (P,y).

For $g : (N,x) \to (P,y)$ a map-germ, we consider the diagram

$$\begin{array}{ccc} TN & \xrightarrow{Tg} & TP \\ \pi_N \downarrow & & \pi_P \downarrow \\ N & \xrightarrow{g} & P \end{array},$$

and write θ_g for the set of germs of maps $\xi : N \to TP$ with $\pi_P \circ \xi = g$. Then composition with Tg induces a map $tg : \theta_N \to \theta_g$ which we think of as the tangent map to the action of $\mathcal{R} = Diff(N)$ on $Map(N,P)$ by composition; composition with g induces a map $\omega g : \theta_P \to \theta_g$ tangent to the action of $\mathcal{L} = Diff(P)$. Set $\mathcal{A} := \mathcal{R} \times \mathcal{L}$.

The respective images of tg and ωg are denoted $T\mathcal{R}^e g$ and $T\mathcal{L}^e g$, with sum $T\mathcal{A}^e g$. We write $T\mathcal{R}g := tg(\mathfrak{m}_x.\theta_N)$, $T\mathcal{L}g := \omega g(\mathfrak{m}_y.\theta_P)$ and $T\mathcal{A}g := T\mathcal{R}g + T\mathcal{L}g$. In the classification of map-germs up to $\mathcal{A}$-equivalence, $T\mathcal{A}g$ serves as the tangent space to the $\mathcal{A}$-equivalence class of g, but for unfolding theory we no longer fix the source and target points, so use the extended tangent space $T\mathcal{A}^e g$; similarly for $\mathcal{R}$ and $\mathcal{L}$. We also set $T\mathcal{C}g := g^*\mathfrak{m}_y.\theta_g$, $T\mathcal{K}g := T\mathcal{R}g + T\mathcal{C}g$ and $T\mathcal{K}^e g := T\mathcal{R}^e g + T\mathcal{C}g$.

Following the notation introduced in [8], for any equivalence relation $\mathcal{B}$ on map-germs, I write $d_e(g, \mathcal{B}) := \dim_{\mathbb{C}}(\theta_g / T\mathcal{B}^e g)$. This is the dimension of the miniversal unfolding space for g under $\mathcal{B}$-equivalence.

A plane curve C may be given as $\phi^{-1}(0)$ for an equation $\phi : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ and (if r denotes the number of branches) as the image of a parametrisation $f : \bigcup_{i=1}^r (\mathbb{C}, 0) \to (\mathbb{C}^2, 0)$. From now on, we write $\mathcal{O}_{x,y}$ for the local ring at the origin in $\mathbb{C}^2$, $\mathfrak{m}_{x,y}$ for its maximal ideal and $\theta_{x,y}$ for the set of germs of vector fields, which is a free $\mathcal{O}_{x,y}-$module with basis $\{\partial_x, \partial_y\}$ (where we write ∂_x for $\partial/\partial x$); the corresponding items for $\mathbb{C}$ are denoted $\mathfrak{m}_t \lhd \mathcal{O}_t$ and θ_t. We denote the source variables of f by t_i ($1 \leq i \leq r$), with local rings $\mathcal{O}_{t_i}$ and the constituent maps f_i, and set $\mathcal{O}_T := \bigoplus_i \mathcal{O}_{t_i}$, $\mathfrak{m}_T := \oplus_i \mathfrak{m}_{t_i}$ and $\theta_T := \oplus_i \theta_{t_i}$.

The module θ_f is free over $\mathcal{O}_T$ on ∂_x, ∂_y. The map $tf : \theta_T \to \theta_f$ is the sum of the maps $tf_i : \theta_{t_i} \to \theta_{f_i}$ induced by df_i/dt, and the map $\omega f : \theta_{x,y} \to \theta_T$ agrees on each co-ordinate with the ring homomorphism $f^* : \mathcal{O}_{x,y} \to \mathcal{O}_T$. The local ring of C is defined to be $\mathcal{O}_C := f^*\mathcal{O}_{x,y}$; its integral closure in its quotient ring coincides with $\mathcal{O}_T$; as the kernel of $f^* : \mathcal{O}_{x,y} \to \mathcal{O}_T$ is the ideal $\langle \phi \rangle$, we can also identify $\mathcal{O}_C$ with $\mathcal{O}_{x,y}/\langle \phi \rangle$. The module θ_ϕ is free over $\mathcal{O}_{x,y}$ on a single generator, and we identify it with this ring; $t\phi(\theta_{x,y})$ is the (Jacobian) ideal $\langle \phi_x, \phi_y \rangle$ (where we write ϕ_x for $\partial_x \phi$), and $\phi^*\mathfrak{m}_t.\theta_\phi$ is the ideal $\langle \phi \rangle$.

We say that two curves C and C' are equivalent if there is a local diffeomorphism of $\mathbb{C}^2$ taking C to C': this holds if and only if ϕ, ϕ' are $\mathcal{K}$-equivalent if and only if f, f' are $\mathcal{A}$-equivalent. For an equation ϕ, we also have $\mathcal{R}$-equivalence, and for a parametrisation f have $\mathcal{L}$-equivalence.

The basic invariants of a reduced plane curve C are the number r of branches, the 'double point number' defined as $\delta(C) := \dim(\mathcal{O}_T/\mathcal{O}_C)$, and the Milnor and Tjurina numbers defined respectively by

$$\mu(C) := d_e(\mathcal{R}, \phi), \qquad \tau(C) := d_e(\mathcal{K}, \phi).$$

The following identities are well-known: $\mu(C) = 2\delta(C) - r + 1$ [6], $\delta(C \cup C') = \delta(C) + \delta(C') + C.C'$ and (hence) $\mu(C \cup C') = \mu(C) + \mu(C') + 2C.C' - 1$.

We also have calculations of the codimensions $d_e(\mathcal{L}, f) = 2\delta(C)$ (trivial), and $d_e(\mathcal{A}, f) = \tau(C) - \delta(C)$ [4, Theorem 2.59].

2. Relative singularity theory

We define two curves C, C' to be equivalent relative to a curve D if there is a diffeomorphism of $\mathbb{C}^2$ which preserves D and takes C to C'. The diffeomorphisms which preserve D form a group $Diff_D(\mathbb{C}^2)$, whose tangent space is the module θ_D of 'logarithmic' vector fields tangent to D. The definitions of right- and of left-equivalence of curves C relative to D are obtained by replacing $Diff(\mathbb{C}^2)$ by $Diff_D(\mathbb{C}^2)$ throughout. Each of these fits into the general framework of 'geometric groups' introduced by Damon [3], and we have a general unfolding theory. The tangent spaces for the relative notions of equivalence are obtained from those in the absolute case by replacing $\theta_{\mathbb{C}^2}$ by θ_D throughout: thus $T\mathcal{L}^e_D f := \omega f(\theta_D)$, $T\mathcal{A}^e_D f := T\mathcal{R}^e f + T\mathcal{L}^e_D f$, $T\mathcal{R}^e_D \phi := t\phi(\theta_D)$, $T\mathcal{K}^e_D \phi := T\mathcal{R}^e_D \phi + T\mathcal{C}\phi$ and, for each $\mathcal{B}$, $d_e(g, \mathcal{B}_D) := \dim_{\mathbb{C}}(\theta_g / T\mathcal{B}^e_D g)$.

The case of relative singularity theory when D is a straight line L has been investigated by Arnol'd [1] under the name of 'boundary singularities'.

We seek formulae expressing $d_e(f, \mathcal{L}_D)$, $d_e(f, \mathcal{A}_D)$, $d_e(\phi, \mathcal{R}_D)$ and $d_e(\phi, \mathcal{K}_D)$ in terms of the invariants of C and D. We will also require the (local) intersection number $C.D$, and $\tau(C \cup D)$.

Since any plane curve is a free divisor, the $\mathcal{O}_{x,y}-$module θ_D is free of rank 2. It will be convenient to choose generators for θ_D: write them as

(1) $$\xi_1 = a_1\partial_x + b_1\partial_y, \qquad \xi_2 = a_2\partial_x + b_2\partial_y.$$

By a result of Saito [7] we may take the equation of D as $\psi := a_1b_2 - a_2b_1$.

Proposition 2.1. *We have $d_e(f, \mathcal{L}_D) = 2\delta(C) + C.D$.*

Proof. By definition, $T\mathcal{L}_D^e f$ is the $\mathcal{O}_T-$submodule of $\mathcal{O}_T^2$ generated over $\mathcal{O}_C$ by $(a_1 \circ f, b_1 \circ f)$ and $(a_2 \circ f, b_2 \circ f)$. Hence $\mathcal{O}_C^2/T\mathcal{L}_D^e f$ has the same composition factors as $\mathcal{O}_C/((a_1b_2 - a_2b_1) \circ f).\mathcal{O}_C = \mathcal{O}_C/(\psi \circ f).\mathcal{O}_C$. Thus $d_e(f, \mathcal{L}_D) = \dim(\mathcal{O}_T^2/\mathcal{O}_C^2) + \dim(\mathcal{O}_C/(\psi \circ f).\mathcal{O}_C)$: the first term is equal to $2\delta(C)$ and the second to $\dim(\mathcal{O}_{x,y}/\langle \phi, \psi \rangle)$ and hence to $C.D$. ☐

Lemma 2.2. *The map $\overline{\omega g} : \theta_{\mathbb{C}^2} \to \mathrm{Coker}(tg)$ induced by ωg has kernel θ_D.*

Proof. The kernel in question is the set of vector fields $\xi \in \theta_{\mathbb{C}^2}$ such that $\omega g(\xi) = tg(\eta)$ for some $\eta \in \theta_T$. In particular, at each point of D we must have ξ tangent to D. But this is just the condition defining θ_D. ☐

Now let f, g parametrise C, D respectively, and write h for the pair (f, g), so that h parametrises $C \cup D$. We have

Proposition 2.3. *We have $d_e(f, \mathcal{A}_D) = \tau(C \cup D) - \delta(C) - C.D - \tau(D)$.*

Proof. Since th is the direct sum of tf and tg, we can regard the following as a short exact sequence of chain complexes:

$$\begin{array}{ccccccccc} 0 & \to & 0 & \to & \theta_{\mathbb{C}^2} & = & \theta_{\mathbb{C}^2} & \to & 0 \\ & & \downarrow & & \downarrow \overline{\omega h} & & \downarrow \overline{\omega g} & & \\ 0 & \to & \mathrm{Coker}(tf) & \to & \mathrm{Coker}(tf) \oplus \mathrm{Coker}(tg) & \to & \mathrm{Coker}(tg) & \to & 0 \end{array}.$$

Now the Cokernels of $\overline{\omega h}$ and $\overline{\omega g}$ are $\theta_h/T\mathcal{A}^e h$ and $\theta_g/T\mathcal{A}^e g$ and, by Lemma 2.2, the kernel of $\overline{\omega g}$ is θ_D. Hence the exact homology sequence of the diagram is

$$\theta_D \to \mathrm{Coker}(tf) \to \theta_h/T\mathcal{A}^e h \to \theta_g/T\mathcal{A}^e g \to 0.$$

Thus we have an exact sequence

$$0 \to \theta_f/T\mathcal{A}_D^e(f) \to \theta_h/T\mathcal{A}^e(h) \to \theta_g/T\mathcal{A}^e(g) \to 0,$$

so $\dim(\theta_f/T\mathcal{A}_D^e(f)) = \dim(\theta_h/T\mathcal{A}^e(h)) - \dim(\theta_g/T\mathcal{A}^e(g))$. Here the terms on the right hand side are $\tau(C \cup D) - \delta(C \cup D)$ and $\tau(D) - \delta(D)$, and the result follows on substituting for $\delta(C \cup D)$. ☐

We now consider C as defined by ϕ and introduce the relative Tjurina number

$$\tau_D C := \tau_D \phi := d_e(\phi, \mathcal{K}_D) = \dim(\mathcal{O}_{x,y}/\langle \theta_D \phi, \phi \rangle).$$

Suppose we have three curve-germs C, D_1 and D_2, no two with a common component, with respective equations ϕ, ψ_1 and ψ_2.

Lemma 2.4. *We have exact sequences*

(2) $$0 \to \frac{\mathcal{O}_{x,y}}{\langle \theta_{D_1 \cup D_2}\phi, \phi \rangle} \xrightarrow{Y} \frac{\mathcal{O}_{x,y}}{\langle \theta_{D_2}(\phi\psi_1), \phi\psi_1 \rangle} \xrightarrow{Q} \frac{\mathcal{O}_{x,y}}{\langle \psi_1, \phi\theta_{D_2}(\psi_1) \rangle} \to 0,$$

(3) $$0 \to \frac{\mathcal{O}_{x,y}}{\langle \psi_1, \theta_{D_2}(\psi_1) \rangle} \xrightarrow{X} \frac{\mathcal{O}_{x,y}}{\langle \psi_1, \phi\theta_{D_2}(\psi_1) \rangle} \xrightarrow{P} \frac{\mathcal{O}_{x,y}}{\langle \psi_1, \phi \rangle} \to 0,$$

where Y, X are induced by multiplication by ψ_1, ϕ respectively and Q, P are the projections.

Proof. (i) For $a \in \mathcal{O}_{x,y}$, the condition that $a\psi_1 \in \langle \theta_{D_2}(\phi\psi_1), \phi\psi_1 \rangle$ means that for some $\xi \in \theta_{D_2}$ and $b \in \mathcal{O}_{x,y}$ we have $a\psi_1 = \xi(\phi\psi_1) + b\phi\psi_1$. Since $\xi(\phi\psi_1) = \phi\xi(\psi_1) + \psi_1\xi(\phi)$, it follows that $\xi(\psi_1)$ is divisible by ψ_1, in other words, that $\xi \in \theta_{D_1}$, and hence that $\xi \in \theta_{D_1 \cup D_2}$.

Conversely, if $a = \xi(\phi) + b\phi$ with $\xi \in \theta_{D_1 \cup D_2}$, we have $a\psi_1 = \xi(\phi\psi_1) - \phi\xi(\psi_1) + b\phi\psi_1$, and since $\xi(\psi_1)$ is divisible by ψ_1, this belongs to $\langle \theta_{D_2}(\phi\psi_1), \phi\psi_1 \rangle$.

We have shown that the map Y is well defined and injective. Its cokernel is the quotient of $\mathcal{O}_{x,y}$ by $\langle \psi_1, \theta_{D_2}(\phi\psi_1) \rangle$. Using again the identity $\xi(\phi\psi_1) = \phi\xi(\psi_1) + \psi_1\xi(\phi)$ and absorbing the second term of this sum, we see that this module is the same as $\langle \psi_1, \phi\theta_{D_2}(\psi_1) \rangle$.

(ii) For $a \in \mathcal{O}_{x,y}$, if $a\phi \in \langle \psi_1, \phi\theta_{D_2}(\psi_1) \rangle$, we can write $a\phi = b\psi_1 + \phi\xi(\psi_1)$ for some $b \in \mathcal{O}_{x,y}$ and some $\xi \in \theta_{D_2}$. It follows that $b\psi_1$, and hence b is divisible by ϕ, say $b = c\phi$. Thus $a = c\psi_1 + \xi(\psi_1) \in \langle \psi_1, \theta_{D_2}(\psi_1) \rangle$. The converse is again easy, so the first map exists and is injective; the cokernel is as given. ☐

Proposition 2.5. *We have*

(i) $\tau_{D_2}(D_1 \cup C) = \tau_{D_1 \cup D_2}(C) + \tau_{D_2}(D_1) + D_1.C$, *and*
(ii) $d_e(\phi, \mathcal{K}_D) = \tau_D(C) = \tau(C \cup D) - C.D - \tau(D)$.

Proof. The dimensions of the first two terms in (2) of Lemma 2.4 are $\tau_{D_1 \cup D_2}(C)$ and $\tau_{D_2}(D_1 \cup C)$ respectively: thus $\dim(\mathcal{O}_{x,y}/\langle \psi_1, \phi\theta_{D_2}(\psi_1) \rangle) = \tau_{D_2}(D_1 \cup C) - \tau_{D_1 \cup D_2}(C)$. Since the first term in (3) has dimension $\tau_{D_2}(D_1)$ and the third has dimension $D_1.C$, it follows that this expression is equal to $\tau_{D_2}(D_1) + D_1.C$. This proves (i), and (ii) follows on setting $D_2 = \emptyset$ (and $D_1 = D$). ☐

In fact the apparent extra generality of (i) is spurious: (i) follows from (ii) on substituting for each of the terms $\tau_D C$.

The following turns out to be the most difficult of our four cases; indeed the result is not what I had originally guessed.

It will be convenient to write, for a a function, $[a]$ for the curve defined by $a = 0$ and $[a].[b]$ for the local intersection number of $a = 0$ and $b = 0$, which is equal to $\dim(\mathcal{O}_{x,y}/\langle a, b \rangle)$. We will manipulate such intersection numbers using identities of the forms (i) $[a].[b + ca] = [a].[b]$ and (ii) $[a].[bc] = [a].[b] + [a].[c]$.

Proposition 2.6. *We have $d_e(\phi, \mathcal{R}_D) = C.D + \mu(C) + \mu(D) - 1 - \tau(D)$.*

Proof. Let ξ_1 and ξ_2, as in (1), generate θ_D. Since $\theta_D\phi$ is an ideal with 2 generators, its codimension is equal to the intersection number of the curves they define, hence to $[\xi_1(\phi)].[\xi_2(\phi)]$. We begin by writing

$$[\xi_1(\phi)].[\xi_2(\phi)] = [\xi_1(\phi)].[b_1\xi_2(\phi)] - [\xi_1(\phi)].[b_1].$$

Now manipulate using (i) to reduce the first term to $[\xi_1(\phi)].[(b_1a_2 - b_2a_1)\phi_x] = [\xi_1(\phi)].[\psi\phi_x]$ and the second to $-[a_1\phi_x].[b_1]$. Next use (ii) to obtain

$$[\xi_1(\phi)].[\psi] + [\xi_1(\phi)].[\phi_x] - [a_1].[b_1] - [\phi_x].[b_1] =$$

$$[\xi_1(\phi)].[\psi] + [b_1\phi_y].[\phi_x] - [a_1].[b_1] - [\phi_x].[b_1] = [\xi_1(\phi)].[\psi] + [\phi_y].[\phi_x] - [a_1].[b_1].$$

The second term here is equal to $\mu(C)$.

We pause to establish the

Claim 2.1. *We have $[x].D + [\xi_1(\phi)].D = C.D + [a_1].D$.*

First note that, since ξ_1 is tangent to D, we can set

$$a_1(\alpha_i(t_i), \beta_i(t_i)) = \lambda_i \alpha_i'(t_i) \text{ and } b_1(\alpha_i(t_i), \beta_i(t_i)) = \lambda_i \beta_i'(t_i)$$

for some $\lambda_i(t_i)$, so that

$$\xi_1(\phi)(\alpha_i(t_i), \beta_i(t_i)) = (\lambda_i \alpha_i'(t_i)\phi_x + \lambda_i \beta_i' \phi_y)(\alpha_i(t_i), \beta_i(t_i)) = \lambda_i d\phi(\alpha_i(t_i), \beta_i(t_i))/dt_i.$$

We calculate intersection numbers with $[\psi] = D$ by choosing a parametrisation $(\alpha_i(t_i), \beta_i(t_i))$ for each branch of D, so that $[\psi].[\chi] = \sum_i \mathrm{ord}_{t_i}(\chi(\alpha_i(t_i), \beta_i(t_i)))$. Taking in turn χ equal to x, ϕ, a_1 and $\xi_1(\phi)$, we obtain

$[x].D = \sum_i \mathrm{ord}_{t_i}(\alpha_i(t_i))$,

$C.D = \sum_i \mathrm{ord}_{t_i} \phi(\alpha_i(t_i), \beta_i(t_i))$,

$[a_1].D = \sum_i \mathrm{ord}_{t_i}(\lambda_i \alpha_i'(t_i)) = \sum_i (\mathrm{ord}_{t_i}(\lambda_i) + \mathrm{ord}_{t_i}(\alpha_i(t_i)) - 1)$,

and

$$[\xi_1(\phi)].D = \sum_i \mathrm{ord}_{t_i}(\xi_1(\phi)(\alpha_i(t_i), \beta_i(t_i))) =$$
$$\sum_i \mathrm{ord}_{t_i}(\lambda_i d\phi(\alpha_i(t_i), \beta_i(t_i))/dt_i) = \sum_i (\mathrm{ord}_{t_i}\lambda_i + \mathrm{ord}_{t_i}\phi(\alpha_i(t_i), \beta_i(t_i)) - 1).$$

The claim follows from these four equations.

We also have

$$[a_1].D = [a_1].[b_1 a_2 - b_2 a_1] = [a_1].[b_1 a_2] = [a_1].[b_1] + [a_1].[a_2].$$

Combining this with our Claim, we obtain

$$d_e(\phi, \mathcal{R}_D) = [\xi_1(\phi)].[\psi] + [\phi_y].[\phi_x] - [a_1].[b_1] =$$
$$C.D + [a_1].D - [x].D + \mu(C) - [a_1].[b_1] = C.D - [x].D + \mu(C) + [a_1].[a_2].$$

Since ξ_1, ξ_2 generate θ_D, the coefficients a_1, a_2 generate the ideal $I := \{\alpha \in \mathcal{O}_{x,y} \,|\, \alpha\psi_x \in \langle \psi_y, \psi \rangle\}$. This ideal contains ψ_x and ψ which have no common factor, hence neither do a_1 and a_2, so $[a_1].[a_2] = \dim(\mathcal{O}_{x,y}/\langle a_1, a_2\rangle) = \dim(\mathcal{O}_{x,y}/I)$. We have an exact sequence

$$0 \to \mathcal{O}_{x,y}/I \xrightarrow{\psi_x} \mathcal{O}_{x,y}/\langle \psi_y, \psi\rangle \to \mathcal{O}_{x,y}/\langle \psi_x, \psi_y, \psi\rangle \to 0.$$

The third term has dimension $\tau(D)$; the second has dimension $[\psi_y].[\psi]$, and we have $[\psi].[\psi_y] = [\psi_x].[\psi_y] + [x].[\psi_y]$: we can prove this by the same method as the Claim or appeal to [9, Lemma 6.5.7]. Thus,

$$[a_1].[a_2] = \dim(\mathcal{O}_{x,y}/I) = [\psi_x].[\psi_y] + [x].[\psi_y] - \tau(D) = \mu(D) - \tau(D) + [x].D - 1.$$

The Proposition follows by substituting this in the above formula. □

Corollary 2.7. $d_e(\phi, \mathcal{R}_D) - d_e(\phi, \mathcal{K}_D) = \mu(C \cup D) - 1 - \tau(C \cup D)$.

This follows from Propositions 2.6 and 2.5, and compares with the equation

$$d_e(\phi, \mathcal{R}) - d_e(\phi, \mathcal{K}) = \mu(C) - \tau(C).$$

Corollary 2.8. *We have* $d_e(\phi, \mathcal{K}_D) = d_e(f, \mathcal{A}_D) + \delta(C)$.

This follows from Propositions 2.3 and 2.5, and compares with $d_e(\phi, \mathcal{K}) = d_e(f, \mathcal{A}) + \delta(C)$.

We refer to [4, Theorem 2.59] for a discussion of semi-universal deformations of plane curve-germs, and in particular for the result that the δ constant stratum, which in the case that the central curve is parametrised consists precisely of those curve-germs that can be simultaneously parametrised, has codimension δ in the deformation space, and has a smooth normalisation. Moreover, this normalisation can be identified with the semi-universal deformation of the parametrised curve.

For singularity theory relative to D, we have a deformation of C (given by equations) for which the tangent space to the unfolding space U_e maps isomorphically to $\theta_\phi/T\mathcal{K}^e_D\phi$, and a deformation given by parametrised curves for which the tangent space to the unfolding space U_p maps isomorphically to $\theta_f/T\mathcal{A}^e_D f$. We can construct a map $U_p \to U_e$; its image will certainly lie in the $\delta-$constant part of U_e which, as U_e is certainly versal in the usual sense, is of codimension δ with a smooth normalisation to which our map lifts. We expect this lift to be a (local) isomorphism; the expectation is supported by Corollary 2.8.

3. Determinacy

The theory of determinacy was developed mainly by Mather [5]. We say that f is $m - \mathcal{B}$-determined if any g whose Taylor expansions up to degree m agree with those of f (or equivalently, with $g - f \in \mathfrak{m}_N^{m+1}.\theta_f$) is $\mathcal{B}$-equivalent to f, and f is finitely $\mathcal{B}$-determined if it is $m - \mathcal{B}$-determined for some m. Mather characterised determinacy for $\mathcal{A}$-equivalence, and gave estimates for the degree of determinacy; better estimates can be found in [2]. We recall a key result of that paper, in simplified form.

Theorem 3.1. [2, Theorem 1.9] *Suppose $\mathcal{G}$ a subgroup of $\mathcal{K}$ such that*

(i) *for each s, $J^s\mathcal{G}$ is a closed algebraic subgroup of $J^s\mathcal{K}$,*
(ii) *$J^1\mathcal{G}$ is unipotent, e.g., trivial.*

Then, f is $r - \mathcal{G}$-determined if and only if $\mathfrak{m}^{r+1}.\theta(f) \subseteq T\mathcal{G}f$.

The formulation of this theorem refers only to germs at a single point. However if we consider germs at a finite set (say, with a common target), all the arguments involved go through without other than notational change. We will use this extension without further comment.

For $\mathcal{R}$-equivalence, write J for the Jacobian ideal $\langle \partial_x\phi, \partial_y\phi\rangle$ and recall that $\mu = \dim(\mathcal{O}_{x,y}/J)$. Thus, not all the inclusions

$$\mathfrak{m}^{\mu+1} + J \subseteq \ldots \subseteq \mathfrak{m}^{i+1} + J \subseteq \mathfrak{m}^i + J \ldots \subseteq \mathfrak{m} + J \subseteq \mathcal{O}$$

can be proper, so for some $i \leq \mu$, $\mathfrak{m}^{i+1} + J = \mathfrak{m}^i + J$, so by Nakayama's lemma, $J \supseteq \mathfrak{m}^i \supseteq \mathfrak{m}^\mu$.

We can take $\mathcal{G}$ as the subgroup $\mathcal{R}_1$ of $\mathcal{R}$ of diffeomorphisms with trivial 1-jet. Since $T\mathcal{R}_1\phi = t\phi(\mathfrak{m}^2.\theta_{x,y}) = \mathfrak{m}^2.J \supseteq \mathfrak{m}^{\mu+2}$, ϕ is $(\mu + 1) - \mathcal{R}$-determined. Experiment soon shows that this well-known estimate is usually very poor, though it is best possible for singularities of type A_k.

For $\mathcal{L}$-equivalence, a similar result holds, but can be improved. Let C have branches B_i $(1 \leq i \leq r)$; write $B_i^* := C \setminus B_i$, and set $K_C := \max_i(\mu(B_i) + B_i.B_i^*)$: thus if $r > 1$ we have $K_C < \mu(C)$. Write $m(C)$ for the multiplicity of C: thus, for C not of type A_*, we have $m(C) \geq 3$.

Proposition 3.2. *We have*

(i) *$f^*\mathcal{O}_{x,y} \supseteq \mathfrak{m}_T^{K_C}$, and*
(ii) *if $m(C) \geq 3$, $f^*\mathfrak{m}^2_{x,y} \supseteq \mathfrak{m}_T^{K_C}$.*

Proof. (i) We have $f_i^*\mathcal{O}_{x,y} \supseteq \mathfrak{m}_i^{\mu(B_i)}$ (see e.g. [9, 4.3.3, 6.3.2], thus for each $k \geq \mu(B_i)$ there exists $\alpha_k \in \mathcal{O}_{x,y}$ with $f_i^*\alpha_k$ of order k. If β_i is a defining equation for B_i^*, $f_i^*(\beta_i)$ has order

$B_i.B_i^*$. Hence $f^*(\alpha_k\beta_i)$ vanishes on B_j for $j \neq i$ and has order $k + B_i.B_i^*$ on B_i. As this yields all orders $\geq K_C$, the result follows.

(ii) It will suffice to show that for any $g \in (\mathfrak{m}_{x,y} \setminus \mathfrak{m}_{x,y}^2)$, we have $f^*g \notin \mathfrak{m}_T^{K_C}$. Set $L := g^{-1}(0)$; this is a smooth curve-germ at $(0,0)$. The order of f_i^*g is the intersection number $L.B_i$. We need to show that, for some i, $L.B_i < K_C$.

Suppose the multiplicity sequence for (infinitely near points of) B_i has r instances of $m(B_i)$ followed by an integer $m' < m(B_i)$ (see e.g. [9, 3.5.1] for this sequence). Since the following point in the sequence is proximate to a point other than O_0, it cannot belong to L, so $L.B_i \leq rm(B_i) + m'$; while (see e.g. [9, 6.5.9]) $\mu(B_i) \geq rm(B_i)(m(B_i) - 1) + m'(m' - 1)$. We now distinguish cases.

If $m(B_i) \geq 3$, $L.B_i \leq rm(B_i) + m' \leq \frac{1}{2}\mu(B_i) + 1 < \mu(B_i) \leq K_C$ (here the '+1' is only needed if m' is 1 or 2).

If $m(B_i) = 2$ and $r \geq 2$, $L.B_i \leq rm(B_i) + 1 < \mu(B_i) + B_i.B_i^* \leq K_C$.

If $m(B_i) = 1$ and $r \geq 2$, the mutual orders of contact of L and the B_i are equal to the intersection numbers. Choose i with $L.B_i$ minimum. As the least two of $L.B_i$, $L.B_j$, $B_i.B_j$ are equal, $L.B_i \leq B_i.B_j \leq K_C$, with equality only if $r = 2$. □

Corollary 3.3. *If $m(C) \geq 3$, C is $(K_C - 1) - \mathcal{L}$-determined.*

We apply Theorem 3.1, taking $\mathcal{G}$ to be the group $\mathcal{L}_1$ of left equivalences with trivial 1-jet. Since $T\mathcal{L}_1 f = \omega f(\mathfrak{m}_{x,y}^2.\theta_{x,y})$, it follows from (ii) that if $m(C) \geq 3$, $T\mathcal{L}_1 f \supseteq \mathfrak{m}_T^{K_C}.\theta_T$; the result follows.

If $m(C) = 2$ then either C has type A_{2k-1} for some $k \geq 1$, we have $K_C = k$ and the degree of determinacy is k; or C has type A_{2k}, $K_C = 2k$ and the degree of determinacy is $2k + 1$.

We turn to relative determinacy. We would like to apply Theorem 3.1 taking $\mathcal{G}$ to be the group $Diff_D(\mathbb{C}^2)$ with tangent space θ_D acting on the right on equations and on the left on parametrisations. However this is not always jet unipotent. We thus take $\mathcal{G}$ as the group of diffeomorphisms preserving D and with identity 1-jet, so $T\mathcal{G} = \theta_D \cap \mathfrak{m}_{x,y}^2.\theta_{x,y}$. Set $e_D := \dim(\theta_D/(\theta_D \cap \mathfrak{m}_{x,y}^2.\theta_{x,y}))$.

I conjecture that if D is not weighted homogeneous, then $\theta_D \subset \mathfrak{m}_{x,y}^2.\theta_{x,y}$ (so $e_D = 0$). If D is weighted homogeneous but not of type A_*, then $\theta_D \subset \mathbb{C}.\{x\partial_x + y\partial_y\} + \mathfrak{m}_{x,y}^2.\theta_{x,y}$ (so $e_D = 1$). If D has type A_*, then $e_D = 2$ and if $D = \{y = 0\}$, then $e_D = 4$.

Proposition 3.4. *If C has equation ϕ, the degree of $\mathcal{R}_D$-determinacy of ϕ is at most*

$$\mu(C) + C.D + \mu(D) - 1 - \tau(D) + e_D.$$

Proof. As in the absolute case it follows, using Nakayama's lemma, that $\theta_D\phi \supseteq \mathfrak{m}^k$, where $k = \dim(\mathcal{O}_{x,y}/\theta_D\phi) = d_e(\phi, \mathcal{R}_D)$; by Proposition 2.6, we have $d_e(\phi, \mathcal{R}_D) = \mu(C) + C.D + \mu(D) - 1 - \tau(D)$.

Since $T\mathcal{G}$ is an $\mathcal{O}_{x,y}-$module, the same argument gives $T\mathcal{G}\phi \supseteq \mathfrak{m}^{k+e_D}$. The result now follows from Theorem 3.1. □

Proposition 3.5. *If C has parametrisation f, then f is $(K_{C\cup D} - 1) - \mathcal{L}_D$-determined.*

Proof. Let D have parametrisation g, then $C \cup D$ has parametrisation $h = (f, g)$. We claim that if h is $k - \mathcal{L}$-determined, then f is $k - \mathcal{L}_D$-determined.

For let $j^k f_1 = j^k f$, and set $h_1 := (f_1, g)$. Then h and h_1 have the same $k-$jet, so are $\mathcal{L}$-equivalent. Thus there is a diffeomorphism A of $\mathbb{C}^2$ with $A \circ f = f_1$ and $A \circ g = g$. Hence A is an $\mathcal{L}_D$-equivalence of f and f_1.

By Corollary 3.3, if $m(C \cup D) \geq 3$, we can take k to be $K_{C\cup D} - 1$. The result follows. □

If we set $K_{C,D} := \max_i(\mu(B_i) + B_i.B_i^* + B_i.D)$, then $K_{C\cup D} = \max(K_{C,D}, K_{D,C})$. It seems likely that a direct approach might allow a sharpening of $K_{C\cup D}$ to $K_{C,D}$ above, but this is not useful for our application.

In particular, if $C \cup D$ is reduced, C is finitely determined relative to D in each sense.

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