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# ON THE COBORDISM GROUPS OF COORIENTED, CODIMENSION ONE MORIN MAPS

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Abstract. We compute cobordism groups of fold maps, cusp maps, and more general Morin maps of oriented $n$-dimensional manifolds to $R^{n+1}$. The results for fold maps in dimensions $n \leq 10$ are complete. In general we express the results through the stable homotopy groups of spheres and that of the infinite projective space.

## Introduction

We consider cobordism groups of maps of $n$-dimensional manifolds into $R^{n+1}$ such that they have at most:

(a) $\Sigma^{1,0}$ type singularities, i.e., fold maps (in Part 1),
(b) $\Sigma^{1,1}$ type singularities, i.e., cusp maps (in Part 2),
(c) $\Sigma^{1_r}$-type singularities (in Part 3).

We express these groups completely through the stable homotopy groups of spheres and those of the infinite projective space in case a) and modulo some small prime components for the cases b) and c).

The main tools of the computation are:

(1) the classifying spaces of the cobordisms of maps with a given set of allowed singularities, see [RSz], [Sz3],
(2) a fibration connecting these classifying spaces, the so-called key bundle in [Sz3], [T],
(3) identification of the boundary map in the homotopy exact sequence of the above mentioned fibration with a well-studied map in homotopy theory, namely the so called Kahn–Priddy map.

## Part 1. Cobordism of fold maps and the Kahn–Priddy map

### 1.1. Formulation of the result

Let us denote by $\operatorname{Cob}\Sigma^{1,0}(n)$ the cobordism group of cooriented, codimension 1 fold maps of closed, smooth, $n$–dimensional manifolds in $R^{n+1}$ (see [Sz3]).

(A fold map may have only $\Sigma^{1,0}$–type (or $A_1$–) singular points, see [AGV].)

**Theorem A.**

(a) $\operatorname{Cob}\Sigma^{1,0}(n)$ *is a finite Abelian group.*
(b) *Its odd torsion part is isomorphic to that of the* $n^{th}$ *stable homotopy group of spheres, i.e., for any odd prime* $p$, $\operatorname{Cob}\Sigma^{1,0}(n)_p \approx \pi^s(n)_p$, *where the lower index* $p$ *denotes the* $p$*–primary part. The isomorphism is induced by the natural forgetting map* $\pi^s(n) \to \operatorname{Cob}\Sigma^{1,0}(n)$.
(c) *Its* 2*–primary part is isomorphic to the kernel of the Kahn–Priddy homomorphism* [KP]*:*

$$\lambda_*\colon \pi^s_{n-1}(RP^\infty) \longrightarrow \pi^s(n-1).$$

**Remark.** The group $\pi^s_{n-1}(RP^\infty)$ is a 2–primary group and $\lambda_*$ is onto the 2–primary part of $\pi^s(n-1)$, see [KP].

**Corollary.** $\operatorname{Cob}\Sigma^{1,0}(n) \approx \pi^s(n)_{\{\text{odd torsion part}\}} \oplus \operatorname{Ker}\big(\lambda^*\colon\ \pi^s_{n-1}(RP^\infty) \longrightarrow \pi^s(n-1)\big)$.

### 1.2. The Kahn–Priddy map ([KP], [K])

Let us consider the composition of the following maps:

a) $RP^{q-1} \hookrightarrow O(q)$. A line $L \subset R^q$, $[L] \in RP^{q-1}$ is mapped into the reflection in its orthogonal hyperplane.

b) $O(q) \hookrightarrow \Omega^q S^q$ maps $A \in O(q)$ to the map

$$\begin{array}{ccccc} S^q = R^2 \cup \infty & \longrightarrow & S^q = R^q \cup \infty & \text{defined as} \\ x & \longrightarrow & A(x) & \text{for } \ x \in R^q \ \text{ and} \\ \infty & \longrightarrow & \infty. & \end{array}$$

Take the adjoint of the composition map $RP^{q-1} \longrightarrow \Omega^q S^q$. It is a map

$$\lambda\colon\ \Sigma^q RP^{q-1} \longrightarrow \Omega^q S^q.$$

If $n < q$, then the homotopy groups $\pi_{q+n}(\Sigma^q\, RP^{q-1})$ and $\pi_{q+n+1}(S^{q+1})$ are stable, and

$$(*) \qquad \pi^s_n(RP^{q-1}) \approx \pi^s_n(RP^\infty).$$

The Kahn–Priddy homomorphism $\lambda_*\colon\ \pi^s_n(RP^\infty) \longrightarrow \pi^s(n)$ is the homomorphism induced by $\lambda$ in the stable homotopy groups (precomposed with the isomorphism $(*)$).

**Theorem 1** (Kahn–Priddy [KP])**.** *$\lambda_*$ is onto the 2–primary component of $\pi^s(n)$.*

### 1.3. Koschorke's interpretation of $\lambda_*$

Ulrich Koschorke gave a very geometric description of the Kahn–Priddy homomorphism through the so-called "figure 8 construction". Given an immersion of an $(n-1)$–dimensional (unoriented) manifold $N^{n-1}$ into $R^n$ the figure 8 construction associates with it an immersion of an oriented $n$–dimensional manifold $M^n$ into $R^{n+1}$ as follows:

Let us consider the composition $N^{n-1} \looparrowright R^n \hookrightarrow R^{n+1}$.

This has normal bundle of the form $\varepsilon^1 \oplus \zeta^1$, where $\varepsilon^1$ is the trivial line bundle (the $(n+1)^{\text{th}}$ coordinate direction in $R^{n+1}$) and $\zeta^1$ is the normal line bundle of $N^{n-1}$ in $R^n$.

Let us put a figure 8 in each fiber of $\varepsilon^1 \oplus \zeta^1$ symmetrically with respect to the reflection in the fiber $\zeta^1$. Choosing these figures 8 smoothly their union gives the image of an immersion of an oriented $n$–dimensional manifold $M^n$ into $R^{n+1}$. (Clearly $M^n$ is the total space of the circle bundle $S(\varepsilon^1 \oplus \zeta^1)$ over $N^{n-1}$.)

This construction gives a map $8_*\colon\ \pi^s_n(RP^\infty) \longrightarrow \pi^s_n(n)$. Indeed, the cobordism group of immersion of unoriented $(n-1)$–dimensional manifolds in $R^n$ is isomorphic to $\pi^s_n(RP^\infty)$, and that of oriented $n$–dimensional manifolds in $R^{n+1}$ is $\pi^s(n)$.

Since the figure 8 construction respects the cobordism relation (i.e. it associates to cobordant immersions such ones) we obtain a map of the cobordism groups.

**Theorem 2** (Koschorke, [K, Theorem 2.1])**.** *The maps $\lambda_*$ and $8_*$ coincide.*

This theorem of Koschorke will be the main tool in the computation of the cobordism groups of fold maps.

## 1.4. Generalities on the cobordisms of singular maps

In [Sz3] we considered cobordism groups of singular maps with a given set $\tau$ of allowed local forms. (Such a map was called a $\tau$–map.) The cobordism group of (cooriented) $\tau$–maps of $n$–dimensional manifolds in Euclidean space was denoted by $\mathrm{Cob}_\tau(n)$. A classifying space $X_\tau$ has been constructed for $\tau$–maps with the property that its homotopy groups are isomorphic to the groups $\mathrm{Cob}_\tau(n)$.

An ancestor of the spaces $X_\tau$ was the classifying space for the cobordism groups of immersions. Namely given a vector bundle $\xi^k$ we denote by $\mathrm{Imm}^\xi(n)$ the cobordism group of immersions of $n$–manifolds in $R^{n+k}$ such that the normal bundle is induced from $\xi$. There is a classifying space $Y(\xi)$ such that

$$\pi_{n+k}(Y(\xi)) \approx \mathrm{Imm}^\xi(n).$$

Namely $Y(\xi) = \Gamma(T\xi)$, where $T\xi$ denotes the Thom space of the bundle $\xi$, and $\Gamma = \Omega^\infty S^\infty$. (This follows by a slight modification from [W].)

Next we recall the so-called "key bundle", that is the main tool in handling cobordism groups of singular maps.

Let $\tau$ be a list of allowed local forms, and let $\eta$ be a maximal element in it. (The set of local forms has a natural partial ordering, $\eta$ is greater than $\eta'$ if an isolated $\eta$–germ (at the origin) has an $\eta'$–point arbitrarily close to the origin.)

Let $\tau'$ be $\tau \setminus \{\eta\}$ (i.e. we omit the maximal element $\eta$).

Note that the stratum of $\eta$ points is immersed. We have established in [Sz3] that there is a universal bundle – denoted by $\widetilde{\xi}_\eta$ – for the normal bundles of $\eta$–strata from which these normal bundles always can be induced (with the smallest possible structure group).

In particular to the cobordism class $[f]$ of a $\tau$–map $f\colon M^n \longrightarrow R^{n+k}$ we can associate the element in $\mathrm{Imm}^{\widetilde{\xi}_\eta}(m)$ represented by the restriction of $f$ to its $\eta$–stratum. (Here $m$ is the dimension of the $\eta$–stratum.) Hence a homomorphism $\mathrm{Cob}_\tau(n) \longrightarrow \mathrm{Imm}^{\widetilde{\xi}_\eta}(m)$ arises. Both these groups are homotopy groups (of $X_\tau$ and $\Gamma T\widetilde{\xi}_\eta$ respectively). It turns out that this map is induced by a map of the classifying spaces $X_\tau \longrightarrow \Gamma T\widetilde{\xi}_\eta$. Moreover the latter is a Serre fibration with (homotopy) fiber $X_{\tau'}$. (This was shown in [Sz3] using some nontrivial homotopy theory. Terpai in [T] gave an elementary proof for it. This fibration is called the "key bundle".)

## 1.5. Computation of the groups $\mathrm{Cob}\,\Sigma^{1,0}(n)$

In the case of fold maps $\tau = \{\Sigma^0, \Sigma^{1,0}\}$ where $\Sigma^0$ denotes the germ of maximal rank and $\Sigma^{1,0}$ denotes that of a Whitney umbrella $(R^2,0) \longrightarrow (R^3,0)$ (multiplied by the germ of identity $(R^{n-2},0) \longrightarrow (R^{n-2},0)$).

Hence here $\eta = \Sigma^{1,0}$ and $\tau' = \Sigma^0$. Note that a $\tau'$–map is nothing else but an immersion (cooriented and of codimension 1). Hence $X_{\tau'} = \Gamma S^1$. Now the key bundle looks as follows:

$$(**) \qquad X\Sigma^{1,0} \xrightarrow{\Gamma S^1} \Gamma T\widetilde{\xi}_{\Sigma^{1,0}}\,.$$

It is not hard to see (see also [RSz]) that the bundle $\widetilde{\xi}_{\Sigma^{1,0}}$ is $2\varepsilon^1 \oplus \gamma^1$, and so $T\widetilde{\xi}_{\Sigma^{1,0}} = S^2RP^\infty$.

Now the bundle $(**)$ gives the following exact sequence of homotopy groups:

$$\pi_{n+1}(\Gamma S^1) \longrightarrow \pi_{n+1}(X\Sigma^{1,0}) \longrightarrow \pi_{n+1}(\Gamma S^2RP^\infty) \xrightarrow{\partial} \pi_n(\Gamma S^1), \quad \text{i.e.,}$$

$$\pi^s(n) \longrightarrow \mathrm{Cob}\,\Sigma^{1,0}(n) \longrightarrow \pi^s_{n-1}(RP^\infty) \xrightarrow{\partial} \pi^s(n-1) \longrightarrow$$

**Lemma 3.** *The boundary map $\partial$ coincides with the map $8_*$ and hence with the Kahn–Priddy homomorphism $\lambda_*$.*

*Proof of Theorem A.* is immediate from Theorem 1, Theorem 2 and Lemma 3. □

*Proof of Lemma 3.* The boundary map $\partial\colon \pi^s_{n+1}(S^2RP^\infty) \approx \pi_{n+1}(X\Sigma^{1,0},\Gamma S^1) \longrightarrow \pi_n(\Gamma S^1)$ can be interpreted geometrically as follows:

The source group $\pi_{n+1}(X\Sigma^{1,0},\Gamma S^1)$ is isomorphic to the cobordism group of fold maps

$$f\colon (M^n,\partial M^n) \longrightarrow (D^{n+1},S^n) \text{ such that } f^{-1}(S^n)=\partial M^n,$$

and the map $\partial f = f\Big|_{\partial M^n}$ is an immersion of $\partial M^n$ into $S^n$. (Here $M^n$ is an oriented compact smooth $n$–dimensional manifold with boundary $\partial M^n$.)

Let $[f]$ denote the (relative) cobordism class of $f$, and let $[\partial f]$ be that of the immersion $\partial f\colon \partial M^n \longrightarrow S^n$. Then $\partial[f]=[\partial f]$.

Now let $V$ denote the set of singular points of $f$. This is a submanifold of $M^n$ of codimension 2. The restriction of $f$ to $V$ is an immersion, its image we denote by $\widetilde{V}$ $(=f(V))$. Let $\widetilde{T}$ be the (immersed) tubular neighbourhood of $\widetilde{V}$. More precisely there exist a $D^3$–bundle $T' \longrightarrow V$ over $V$, a submersion $F$ of $T'$ into $D^n$, $(F(T')=\widetilde{T})$ and $F$ extends the immersion $f \mid V\colon V \longrightarrow D^n$. The bundle $T' \longrightarrow V$ has the form $2\varepsilon^1 \oplus \zeta^1$, where $\zeta^1$ is a line bundle.

Let $T$ be the tubular neighbourhood of $V$ in $M$ such that $f(T)\subset \widetilde{T}$. The map $f\big|_T\colon T \longrightarrow \widetilde{T}$ can be decomposed into a map $\widehat{f}\colon T \longrightarrow T'$ and the submersion $F\colon T' \longrightarrow \widetilde{T}$, where $\widehat{f}$ maps each fiber $D^2$ of the bundle $T \longrightarrow V$ into a fiber $D^3$ of $T' \longrightarrow V$ as a Whitney umbrella and $\widehat{f}^{-1}(\partial T')=\partial T$. On the boundary of each fiber $D^3$ we obtain a "curved figure 8" as image of $\widehat{f}$. The manifold with boundary $M\setminus T$ will be denoted by $W$. Note that its boundary is $\partial W = \partial_1 W \sqcup \partial_2 W$, where $\partial_1 W = \partial M$, and $\partial_2 W = \partial T$.

The image of $\partial_2 W$ at $f$ is the union of the ($F$–images of the) above mentioned curved figures 8. This is a codimension 2 framed immersed submanifold in $D^{n+1}$, we will denote it by $\widetilde{V}$. (The first framing is the $F$–image of the inside normal vector of $\partial T'$ in $T'$. The second framing is the normal vector of the curved figure 8 in $S^2=\partial D^3$.)

It remained to show the following two claims.

**Claim a).** *$\widetilde{V}$ with the given 2–framing is framed cobordant to the immersion $\partial f\colon \partial M \looparrowright S^n$ (compared with the framed embedding $S^n \subset D^{n+1}$).*

**Claim b).** *$\widetilde{V}$ is obtained from the immersion $f\big|V\colon V^{n-2} \looparrowright D^{n+1}$ by the figure 8 construction.*

We have to make some remarks in order to clarify the above statements a) and b).

To a): The framed immersion $\partial f\colon \partial M \looparrowright S^n$ and its composition with $i: S^n \subset D^{n+1} \subset R^{n+1}$ (with the added second framing, the inside normal vectors of $S^n$ in $D^{n+1}$) represent the same element in $\pi^s(n-1)$. Indeed, the composition with $i$ corresponds to applying the suspension homomorphism in homotopy groups of spheres. But the cobordism group of framed *immersions* is isomorphic to the corresponding *stable* homotopy group of spheres, so the suspension homomorphism gives the identity map of these groups.

To b): The figure 8 construction was defined for a codimension one immersed submanifold in a Euclidean space. Here we apply it to the codimension 3 immersed submanifold $\widetilde{V}^{n-2}$ in $D^{n+1}$. But $\widetilde{V}^{n-2}$ has two linearly independent normal vector fields in $D^{n+1}$ as was described above. Identify $D^{n+1}$ with $R^{n+1}$ and apply the so-called multicompression theorem by Rourke–Sanderson [RS], thus one can make the two normal vectors parallel to the last two coordinate axes in $R^{n+1}$, we can project the immersion to $R^{n-1}$ and then we have a codimension 1–immersion, so claim b) makes sense. (This needs some more clarification since the multicompression theorem deals with embeddings. See below.)

*Proof of Claim a).* It can be supposed that the center $c$ of $D^{n+1}$ does not belong to $\partial\widetilde{T} \cup f(M)$. Let us omit from $D^{n+1}$ a small ball centered around $c$ and still disjoint from $\partial\widetilde{T} \cup f(M)$. In the remaining manifold $S^n \times I$ the direction of $I$ will be called vertical. Take the product with an $R^q$ for big enough $q$, so that the immersions $f\big|_W \colon W^n \looparrowright S^n \times I$ and $\partial T \looparrowright S^n \times I$ become embeddings after small perturbations.

Now $W$ is embedded in $S^n \times I \times R^q$, it is framed with $q+1$ normal vectors ($q$ are parallel to the coordinate axes of $R^q$).

One can suppose that the two boundary components of $W$ are embedded as follows

1) $\partial_1 W = \partial M$ is embedded into $S^n \times \{0\} \times R^q$.

2) $\partial_2 W = \partial T$ is embedded into the interior part $\operatorname{int}(S^n \times I) \times R^q$.

Both have $(q+2)$-framings.

Now applying the multicompression theorem we make by an isotopy the first framing vector (the one coming from the normal vector of $\partial\widetilde{T}$ in $\widetilde{T}$) vertical, i.e. parallel to the direction of $I$ in $S^n \times I \times R^q$, while the $q$ last framing vectors (coming from $R^q$) we keep parallel to themselves. The other boundary component $\partial_1 W \subset S^n \times \{0\} \times R^q$ is kept fixed.

We arrive at such an embedding of $W$ in $S^n \times I \times R^q$ for which the outward normal vector along $\partial_2 W$ in $W$ is vertical (i.e. parallel to the direction of $I$). Now by a vertical shift we can deform $\partial_2 W$ into $S^n \times \{1\} \times R^q$.

This deformation can be extended to $W$. Projecting into $S^n \times I$ this new position of $W$ in $S^n \times I \times R^q$ we obtain an immersion cobordism between the immersions of the two boundary components.

On the first component $(\partial_1 W)$ we obtain $\partial[f]$. On the second component we obtain the same framed cobordism class as was that of $\widetilde{V}$ in $\partial\widetilde{T} \subset D^{n+1}$ (the union of curved figures 8). Claim a) is proved. $\square$

*Proof of Claim b).* Deform the immersed manifold $\widetilde{V}^{n-1}$ (formed by the union of curved figures 8) as follows. Contract each curved figure 8 by an isotopy along the corresponding sphere $S^2$ into a small neighbourhood of its double point obtaining an almost flat (very small) figure 8. As we have noticed the normal bundle of $f(V)$ in $D^{n+1}$ has the form $2\varepsilon^1 \oplus \zeta^1 = \varepsilon^1_1 \oplus \varepsilon^1_2 \oplus \zeta^1$. The first trivial normal line bundle $\varepsilon^1_1$ can be identified with the direction of the double line of the umbrella, the second one $\varepsilon^1_2$ can be the symmetry axes of the figures 8 (both of the curved and the flattened ones). $\zeta^1$ is the direction orthogonal to the symmetry axes.

Now considering the maps of $V$ and $\widetilde{V}$ in $R^{n+1}$ (instead of $D^{n+1}$) applying again the multicompression theorem we make the two trivial normal directions $\varepsilon^1_2$ and $\varepsilon^1_1$ parallel to the last two coordinate axes and then project $V$ to $R^{n-1}$. In this way we obtain a new immersion $g \colon V^{n-2} \looparrowright R^{n-1}$ and $\varepsilon^1_2$ (the symmetry axes of the figures 8) will be parallel to the normal vector of $R^{n-1}$ in $R^n$. Now the (flattened) figures 8 (of $\widetilde{V}$) are placed exactly as by the original figure 8 construction applied to $g$.

It remained to note that the described deformations do not change the cobordism class of a framed immersion. Claim b) is proved. $\square$

Thus Theorem A is also proved. $\square$

**Remark.** The stable homotopy groups of $RP^\infty$ were computed by Liulevicius [Liu] in dimensions not greater than 9.

Below in the first line we show his result, in the second one the stable homotopy groups of spheres. These two lines by Theorem A give the groups $\operatorname{Cob}\Sigma^{1,0}(n)$ for $n \le 10$ given in the third line. (Here for example $(Z_2)^3$ stands for $Z_2 \oplus Z_2 \oplus Z_2$.)

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $\pi_n^s(RP^\infty)$ | $Z_2$ | $Z_2$ | $Z_8$ | $Z_2$ | 0 | $Z_2$ | $Z_{16}\oplus Z_2$ | $(Z_2)^3$ | $(Z_2)^4$ | ? |
| $\pi^s(n)$ | $Z_2$ | $Z_2$ | $Z_{24}$ | 0 | 0 | $Z_2$ | $Z_{240}$ | $(Z_2)^2$ | $(Z_2)^3$ | $Z_6$ |
| $\operatorname{Cob}\Sigma^{1,0}(n)$ | 0 | 0 | $Z_3$ | 0 | $Z_2$ | 0 | $Z_{15}$ | $Z_2$ | $Z_2$ | $Z_6$ |

**Remark.** It follows from part b) of Theorem A that all the odd torsion part of $\operatorname{Cob}\Sigma^{1,0}(n)$ can be represented by immersions. In particular, in dimensions $n=3$ and $n=7$ all the elements can be represented by immersions of the sphere $S^3$ and $S^7$, respectively, since the $J$-homomorphism is onto in these dimensions.

## Part 2. Cusp maps

### 2.1. Formulation of the results

Here we consider cusp maps, i.e. maps having at most cusp singularities. (In the previous terms these are $\tau$–maps for $\tau=\{\Sigma^0,\Sigma^{1,0},\Sigma^{1,1}\}$.) The cobordism group of cusp maps of oriented $n$–dimensional manifolds in $R^{n+1}$ will be denoted by $\operatorname{Cob}\Sigma^{1,1}(n)$. We shall compute these groups modulo their 2–primary and 3–primary parts. Let $C_{\{2,3\}}$ be the minimal class of groups containing all 2–primary and 3–primary groups.

**Theorem B.** $$\operatorname{Cob}\Sigma^{1,1}(n) \underset{\mathcal{C}_{\{2,3\}}}{\approx} \pi^s(n)\oplus\pi^s(n-4)$$

*where* $\underset{\mathcal{C}_{\{2,3\}}}{\approx}$ *means isomorphism modulo the class* $\mathcal{C}_{\{2,3\}}$*, and* $\pi^s(m)$ *denotes the* $m^{th}$ *stable homotopy group of spheres.*

### 2.2. Preliminaries on Morin maps

Morin maps are those of types $\Sigma^{1,0},\Sigma^{1,1,0},\dots,\Sigma^{1_r,0},\dots$, $r=1,2,\dots$ (See [AGV].)

For $\eta=\Sigma^{1_r,0}$ the universal normal bundle $\widetilde{\xi}_\eta$ will be denoted by $\widetilde{\xi}_r$. It was established in [RSz] and [R] that the structure group of $\widetilde{\xi}_r$ is $Z_2$ and the bundle $\widetilde{\xi}_r$ is associated to a representation $\lambda_2\colon Z_2\longrightarrow O(2r+1)$ with the property that $\lambda_2(Z_2)\subset SO(2r+1)$ precisely when $r$ is even. It follows that $\widetilde{\xi}_r$ is the direct sum $i\cdot\gamma^1\oplus j\cdot\varepsilon^1$, where $i+j=2r+1$, and $i\equiv r \bmod 2$. Here $\varepsilon^1$, $\gamma^1$ are the trivial and the universal line bundles respectively. Hence the Thom space $T\widetilde{\xi}_r$ is $S^j(RP^\infty/RP^{i-1})$. It is easy to see that for any odd $p$ the reduced mod $p$ cohomology $\overline{H}^*(T\widetilde{\xi}_r;Z_p)$ vanishes if $r$ is odd, and the natural inclusion $S^{2r+1}\subset T\widetilde{\xi}_r$ (as a "fiber") induces isomorphism of the cohomology groups with $Z_p$–coefficients for $r$ even. Consequently by Serre's generalization of the Whitehead theorem [S2] – the inclusion $\Gamma S^{2r+1}\subset\Gamma T\widetilde{\xi}_r$ (recall $\Gamma=\Omega^\infty S^\infty$) induces isomorphism of the odd torsion parts of the homotopy groups for $r$ even, while for $r$ odd $\pi_*(\Gamma T\widetilde{\xi}_r)$ are finite 2–primary groups.

### 2.3. Computation of the cusp cobordism groups

In the case of cusps $r=2$ and we have that the inclusion $\Gamma S^5\subset\Gamma T\widetilde{\xi}_2$ is a mod $\mathcal{C}_2$ homotopy equivalence ($\mathcal{C}_2$ is the class of 2–primary groups).

Let us consider the following pull-back diagram defining the space $X^{\mathrm{fr}}\Sigma^{1,1}$

$$\begin{array}{ccc} X^{\mathrm{fr}}\Sigma^{1,1} & \longrightarrow & X\Sigma^{1,1} \\ \Big\downarrow X\Sigma^{1,0} & & \Big\downarrow X\Sigma^{1,0} \\ \Gamma S^5 & \longrightarrow & \Gamma T\widetilde{\xi}_2 \end{array}$$

(Note that $X^{\mathrm{fr}}\Sigma^{1,1}$ is the classifying space of those cusp maps for which the normal bundle of the $\Sigma^{1,1}$–stratum in the target is trivialized. Equivalently these are the cusp maps for which the kernel of the differential is trivialized over the cusp-stratum.)

The horizontal maps of the diagram induce isomorphisms of the odd-torsion parts of the homotopy groups. Now we show that the homotopy exact sequence of the left-hand side fibration "almost has a splitting".

**Definition.** Let $p\colon E \xrightarrow{F} B$ be a fibration and let $t$ be a natural number. We say that this fibration has an algebraic $t$–splitting if for each $i$ there is homomorphism $S_i\colon \pi_i(B) \longrightarrow \pi_i(E)$ such that the composition of $S_i$ with the map $p_*$ induced by $p$ is a multiplication by $t$. We say that the fibration $p$ has a geometric $t$–splitting if it has an algebraic one such that all $S_i$ are induced by a map $s\colon B \longrightarrow E$ (the same map $s$ for each $i$).

**Lemma 4.** *The fibration $X^{\mathrm{fr}}\Sigma^{1,1} \longrightarrow \Gamma S^5$ has a 6–splitting.*

**Remark.** We shall prove this only in algebraic sense, since we will need only that. For the existence of geometric splitting we give only a hint.

*Proof of Theorem B.* is immediate from Lemma 4. Indeed,

$$\mathrm{Cob}\,\Sigma^{1,1}(n) \approx \pi_{n+1}(X\Sigma^{1,1}) \underset{\mathcal{C}_2}{\approx} \pi_{n+1}(X^{\mathrm{fr}}\Sigma^{1,1}).$$

Now the homotopy exact sequence of the fibration $p^{\mathrm{fr}}\colon X^{\mathrm{fr}}\Sigma^{1,1} \xrightarrow{X\Sigma^{1,0}} \Gamma S^5$ has a 6–splitting, hence modulo the class $\mathcal{C}_{\{2,3\}}$ we have

$$\pi_{n+1}(X^{\mathrm{fr}}\Sigma^{1,1}) \underset{\mathcal{C}_{\{2,3\}}}{\approx} \pi^s_{n+1}(S^5) \oplus \pi_{n+1}(X\Sigma^{1,0}) \underset{\mathcal{C}_2}{\approx} \pi^s(n-4) \oplus \pi^s(n).$$

(In the last mod $\mathcal{C}_2$ isomorphism we used Theorem A.)

Theorem B is proved except Lemma 4. □

*Proof of Lemma 4.* will follow from the following two claims.

**Claim 1.** *If there is a map of an oriented 4–dimensional manifold into $R^5$ with $t$ cusp points (algebraically counting them), then the fibration $p^{\mathrm{fr}}$ has a $t$–splitting (algebraically).*

**Claim 2.** *There is a cusp-map $f\colon M^4 \longrightarrow R^5$ with $t$–cusp points (counting algebraically).*

*Proof of Claim 1.* Let $f\colon M^4 \longrightarrow R^5$ be a cusp-map with $t$ cusp-points. Let $x$ be an element in $\pi_m(\Gamma S^5) \approx \pi^s(m-5)$. It can be represented by a framed, immersed $(m-5)$–dimensional manifold $A^{m-5}$ in $R^m$, let us denote its immersion by $\alpha$.

Take the product $A^{m-5} \times M^4$ and its map into the direct product $A^{m-5} \times D^5$ by $\mathrm{id}_A \times f$. Now the target $A^{m-5} \times D^5$ can be mapped by a submersion $F$ into $R^m$ onto the immersed tubular neighbourhood of $\alpha(A)$ using the framing to map the $D^5$–fibers). The composition $A \times M^4 \longrightarrow R^m$ is clearly a cusp map and its cusp-singularity stratum represents the element $t \cdot x$ in $\pi^s(m-5)$.

Claim 1 is proved (at least its algebraic version. The geometric one follows from the fact that we use the same element $[f\colon M^4 \longrightarrow D^5]$ for any element $x \in \pi^s(i)$ and for any dimension $i$ to construct the element $S_i(x)$. The classifying space $\Gamma S^5$ can be obtained as the limit of target spaces of codimension 5 framed immersions.) □

*Proof of Claim 2.* is a compilation of the following two theorems.

**Theorem** ([Sz1], [Sz2], [L])**.** *Given a generic immersion $g\colon M \looparrowright Q \times R^1$ and a natural number $r$, let us denote by $\Delta_{r+1}(g)$ the manifold of (at least) $(r+1)$–tuple points in $M^n$. Let $f\colon M \longrightarrow Q$ be the composition of $g$ with the projection $Q \times R^1 \longrightarrow Q$. Let us denote by $\Sigma^{1_r}(f)$ the closure* *of the set of $\Sigma^{1_r}$ singular points of $f$. Then the manifolds $\Delta_{r+1}(g)$ and $\Sigma^{1_r}(f)$ are cobordant. If $M$ and $Q$ are oriented and* $\dim Q - \dim M$ *is odd, then these manifolds are oriented and they are oriented-cobordant.*

**Theorem 5** (Eccles–Mitchell [EM])**.** *There is an oriented closed 4–dimensional manifold $M^4$ and an immersion $g\colon M^4 \looparrowright R^6$ with (algebraically) 2 triple points.*

□

Theorem B is proved. □

## Part 3. Higher Morin maps

Most of the previous arguments can be applied in the computation of cobordism groups of Morin maps having at most $\Sigma^{1_r}$ singular points for any $r$ (the codimensions of the considered maps are still equal to one, and the maps are cooriented). The only problem is that we need a generalization of the Theorem of Eccles–Mitchell.

Below we shall give a weak form of such a generalization. This will allow us to compute the groups $\operatorname{Cob}\Sigma^{1_r}(n)$ modulo the $p$–primary part for $p \le r+1$.

Notation: Let $\mathcal{C}\{p \le 2r+1\}$ denote the minimal class of groups containing all $p$–primary groups for any prime $p \le 2r+1$. The main result of this Part 3 is the following.

**Theorem C.** *Let us denote by* $\operatorname{Cob}\Sigma^{1_i}(n)$ *the cobordism group of $\Sigma^{1_i}$–map of oriented $n$–manifolds in $R^{n+1}$ (i.e. $\tau$–maps for $\tau = \{\Sigma^0, \Sigma^{1,0}, \dots, \Sigma^{1,1,\dots,1},\ i$ digits $1\}$). Then for any $r$*

$$\operatorname{Cob}\Sigma^{1_{2r+1}}(n) \underset{\mathcal{C}_2}{\approx} \operatorname{Cob}\Sigma^{1_{2r}}(n) \underset{\mathcal{C}\{p\le 2r+1\}}{\approx} \bigoplus_{i=0}^{r} \pi^s(n-4i).$$

The proof is very similar to that given in Parts 1 and 2. It goes by induction on $r$. First we give a weak analogue of the Theorem of Eccles and Mitchell.

**Lemma 1.** a) *For any natural number $k$ there is a positive integer $t(k)$ such that for any immersion of an oriented, closed, smooth $4k$–dimensional manifold in $R^{4k+2}$ the algebraic number of $(2k+1)$–tuple points is divisible by $t(k)$, and there is a case when this number is precisely $t(k)$.*

b) *The number $t(k)$ coincides with the order of the cokernel of the stable Hurewicz homomorphism*

$$\pi^s_{4k+2}(CP^\infty) \longrightarrow H_{4k+2}(CP^\infty).$$

Before proving Lemma 1 it will be useful to recall a result on the cokernel of the stable Hurewicz map.

**Theorem** (Arlettaz [A])**.** *Let $X$ be a $(b-1)$–connected space and let $\varrho_j$ be the exponent of the stable homotopy group of spheres $\pi^s(j)$. Let $h_m\colon \pi^s_m(X) \longrightarrow H_m(X)$ be the stable Hurewicz homomorphism. Then $(\varrho_1 \dots \varrho_{m-b-1})(\operatorname{coker} h_m) = 0$.*

Next we recall a theorem of Serre on the prime divisors of the numbers $\varrho_j$.

**Theorem** (Serre [S1])**.** $\pi^s(i) \otimes Z_p = 0$ *if $i < 2p-3$ and $\pi^s(2p-3) \otimes Z_p = Z_p$.*

Hence $\varrho_j$ is not divisible by a prime $p$ if $p > \frac{j+3}{2}$, in other words, $\varrho_j$ may have a prime $p$ as a divisor only if $p \le \frac{j+3}{2}$.

Applying Arlettaz' theorem to $X = CP^\infty$, $b = 2$, $m = 4r+2$ we obtain that $t(r)$ has no prime divisor greater than $2r+1$.

*Proof of Theorem C.* should be clear now, since it is completely analogous to that of Theorem B.

First we consider the "key bundle"

$$X\Sigma^{1_{2r+1}} \longrightarrow \Gamma T\widetilde{\xi}_{2r+1} \ \text{ with fiber } \ X\Sigma^{1_{2r}}.$$

Remember that $H^*(\Gamma T\widetilde{\xi}_{2r+1}; Z_p) = 0$ for any odd $p$, so – by the mod $\mathcal{C}$ Whitehead theorem [S2] we obtain the first (mod $\mathcal{C}_2$) isomorphism in the Theorem

$$\operatorname{Cob}\Sigma^{1_{2r+1}}(n) \underset{\mathcal{C}_2}{\approx} \operatorname{Cob}\Sigma^{1_{2r}}(n).$$

In order to prove the second (mod $\mathcal{C}\{p \leq 2r+1\}$) isomorphism recall that modulo the class $\mathcal{C}_2$ the key bundle

$$X\Sigma^{1_{2r}} \longrightarrow \Gamma T\widetilde{\xi}_{2r} \quad \text{with fibre } \ X\Sigma^{1_{2r-1}}$$

can be replaced by the bundle

$$X^{\mathrm{fr}}\Sigma^{1_{2r}} \longrightarrow \Gamma S^{4r+1} \ \text{ with the same fibre.}$$

The later bundle has a(n algebraic) $t(r)$–splitting.

By Lemma 1 and the theorems of Arletaz and Serre $t(r)$ has no prime divisor greater than $2r+1$, hence by induction on $r$ we obtain the second isomorphism in Theorem C. □

*Proof of Lemma 1.* By Herbert's theorem the algebraic number of the $(2k+1)$–tuple points of an immersion $f\colon M^{4k} \looparrowright R^{4k+2}$ is $\langle \overline{p}_1{}^k, [M^{4k}]\rangle$. The immersion $f$represents an element $[f]$ of the corresponding cobordism group of immersions of oriented $4k$–manifolds in $R^{4k+2}$. This cobordism group is isomorphic to the group $\pi^s_{4k+2}(MSO(2))$, the element of the later group corresponding to $[f]$ will be denoted by $[\alpha_f]$. Here $\alpha_f$ is the Pontrjagin–Thom map $S^{q+4k+2} \longrightarrow S^q MSO(2)$, for $q$ big enough.

Let us consider the following composition of maps

$$\pi^s_{4k+2}(MSO(2)) \underset{①}{\longrightarrow} H_{4k+2+q}\big(S^q MSO(2)\big) \underset{②}{\overset{\approx}{\longrightarrow}} H_{4k}(BSO(2)) \underset{③}{\overset{\approx}{\longrightarrow}} Z.$$

Here ① is the stable Hurewicz homomorphism, ② is the Thom isomorphism in the homologies

$$x \longrightarrow S^q U_2 \cap x$$

where $U_2$ is the Thom class of $MSO(2)$ and $S^qU_2$ its $q^{\text{th}}$ suspension, and also the Thom class of $S^q MSO(2)$.

③ is the evaluation on the class $p_1{}^k$

$$y \longrightarrow \langle y, p_1{}^k\rangle.$$

Since the maps ② and ③ are isomorphisms, the cokernel of this composition is the same as the cokernel of ①, i.e. of the stable Hurewicz homomorphism.

On the other hand, we show that the image of this composition map is $t(k)Z$, and that will prove part b) of Lemma 1. (Part a) follows then as well, since the rational stable Hurewicz homomorphism

$$\pi^s_m(X) \otimes Q \longrightarrow H_m(X; Q) \ \text{ is an isomorphism.)}$$

**Claim.** *The composition of the maps ①, ②, ③ has image $t(k) \cdot Z$.*

**Proof** It is enough to show that the image of $[\alpha_f] \in \pi^s_{4k+2}(MSO(2))$ is $\langle \overline{p}_1{}^k, [M^{4k}]\rangle$.

$[\alpha_f]$ goes by the map ① to $(\alpha_f)_*[S^{q+4k+2}]$, that is mapped by ② to $(\alpha_f)_*[S^{q+4k+2}] \cap S^qU_2$.

Let $\nu$ be the normal bundle of $f$, let us denote by $T\nu$ its Thom space, let $pr\colon S^{q+4k+2} \longrightarrow S^qT\nu$ be the Pontrjagin–Thom map, $\beta_f\colon S^qT\nu \longrightarrow S^q\,MSO(2)$ the fiberwise map of Thom spaces that on the base spaces is the map $\nu_f\colon M \longrightarrow BSO(2)$ inducing the normal bundle $\nu$.

Now $(\alpha_f)_*[S^{q+4k+2}] = (\beta_f)_* \circ pr_*[S^{q+4k+2}] = (\beta_f)_*[S^qT\nu]$. Here $[S^qT\nu]$ is the fundamental homology class of $S^qT\nu$. Therefore

$$\begin{aligned}\left\langle {p_1}^k, (\alpha_f)_*[S^{q+4k+2}] \cap S^qU_2 \right\rangle &= \left\langle {p_1}^k, (\beta_f)_*[S^qT\nu] \cap S^qU_2 \right\rangle \\ &= \left\langle {p_1}^k, (\nu_f)_*[M] \right\rangle = \left\langle {\nu_f}^* {p_1}^k, [M] \right\rangle = \left\langle {\overline{p}_1}^k, [M] \right\rangle.\end{aligned}$$

$\square$

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