A POLYNOMIAL GENERALIZATION OF THE EULER CHARACTERISTIC FOR ALGEBRAIC SETS.

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WITH AN APPENDIX BY J. V. RENNEMO

ABSTRACT. We present a method to compute the Euler characteristic of an algebraic subset of $\mathbb{C}^n$. This method relies on classical tools such as Gröbner basis and primary decomposition. The existence of this method allows us to define a new invariant for such varieties. This invariant is related to the problem of counting rational points over finite fields. In an appendix, Jørgen Vold Rennemo proves the relation between this invariant and the Chern-Schwartz-MacPherson class of the variety.

1. INTRODUCTION

One of the main invariants of a topological space is its Euler characteristic. It was initially defined for cell complexes, but several extensions have been defined to more general classes of spaces. In the setting of complex algebraic varieties, the natural extension is the Euler characteristic with compact support. In [8] Szafraniec gives a method to compute the Euler characteristic of a complex algebraic set by using methods from the real geometry. Aluffi gave in [1] another method based on the computation of Chern-Schwartz-MacPherson classes. In this paper, we present another method, that only makes use of the basic properties of the Euler characteristic, and classical results on algebraic sets. This way of computing the Euler characteristic gives naturally a stronger invariant, which we define.

The method works as follows. Consider $V \subseteq \mathbb{C}^n$ an irreducible algebraic set of dimension d and degree g. Take a generic linear projection $\pi : \mathbb{C}^n \to \mathbb{C}^d$. If we consider π restricted to V, it is a $g : 1$ branched cover. The branching locus Δ and its preimage $\pi \mid_V^{-1} (\Delta)$ can be computed. From the additivity and the multiplicativity for covers of the Euler characteristic, we have the following formula:

$$\chi(V) = g \cdot \chi(\mathbb{C}^d) - g \cdot \chi(\Delta) + \chi(\pi \mid_V^{-1} (\Delta)).$$

So the computation of $\chi(V)$ is reduced to the computation of the Euler characteristic of algebraic sets of lower dimension, allowing us to use a recursion process.

In the previous method, we make use of the fact that $\chi(\mathbb{C}^d) = 1$. If instead of making this substitution, we keep track of $\chi(\mathbb{C}^d)$ as a formal symbol, we obtain a stronger invariant $F(V)$. This invariant is defined as a polynomial in $\mathbb{Z}[L]$, which turns out to coincide with the counting polynomial defined by Plesken in [6], and has some interesting properties: the dimension, degree, and Euler characteristic of an algebraic set can be computed from this polynomial. It also gives information on the number of points on some varieties over finite fields. This relation with finite varieties could be used to compute this invariant by counting points.

In Sections 2 and 3 we show the preliminary results that prove the correctness of the method to compute the Euler characteristic, and describe the algorithm. Section 4 is devoted to the generalization of this method to the new invariant, which is also defined and some of its properties

are shown. This invariant is defined after both the variety and the used projection, although we conjecture that, assuming general position, every projection would give the same result, making the invariant independent of the projection. After the first version of this paper was made public, Jørgen Vold Rennemo contacted the author proposing a way to prove the main conjecture. His proof consists on showing that this invariant is directly related to the Chern-Schwartz-MacPherson class of the variety. This proof is included in the appendix.

The extension of this invariant to projective varieties is discussed in Section 5. In Section 6 we include some examples. In particular, we show that in the case of hyperplane arrangements this invariant coincides with the characteristic polynomial. Finally, the relationship of the invariant with the number of points over finite fields is shown in Section 7. As an annex, Section 8 includes an implementation of the algorithms in Sage together with some timings.

2. Theoretical justification

Let $V = V(I) \subseteq \mathbb{C}^n$ be the algebraic set determined by a radical ideal I. Without loss of generality, we can assume that it is in general position (in the sense that we will precise later). By computing the associated primes of I we obtain the decomposition in irreducible components $V = V_1 \cup \cdots \cup V_c$. The Euler characteristic $\chi(V)$ can be expressed as $\chi(V_1) - \chi((V_1) \cap (V_2 \cup \cdots \cup V_c)) + \chi(V_2 \cup \cdots \cup V_c)$. The variety $(V_1) \cap (V_2 \cup \cdots \cup V_c)$ is an algebraic set of lower dimension. So, by a double induction argument (over the dimension and over the number of irreducible components), we may reduce the problem of computing $\chi(V)$ to the case where V is either zero-dimensional or irreducible.

If V is zero-dimensional, it consists of a number of isolated points, and its Euler characteristic equals the number of points. This number of points can be computed as the degree of the homogenization of the radical of I (which can be computed via the Hilbert polynomial, see [3, Chapter 5] for example).

For the case of an irreducible variety $V = V(I) \subseteq \mathbb{C}^n$ being $I \trianglelefteq \mathbb{C}[x_1, \ldots, x_n]$ a radical ideal of Krull dimension d, we will distinguish the homogeneous case from the non homogeneous.

If I is a homogeneous ideal, the variety V has a conic structure (it is formed by a union of lines that go through the origin). It means that V is contractible and hence its Euler characteristic is 1.

For the non homogeneous case, consider the projection

$$\begin{array}{rcl} \pi : \mathbb{C}^n & \to & \mathbb{C}^d \\ (x_1, \ldots, x_n) & \mapsto & (x_1, \ldots x_d) \end{array}$$

We may assume (applying a generic linear change of coordinates if necessary) that the following condition is satisfied:

Definition 1. Consider $I_h \trianglelefteq \mathbb{C}[x_0, x_1, \ldots, x_n]$ the homogenization of I. We will say that I is in **general position** if $\sqrt{I_h + (x_0, x_1, \ldots, x_d)} \supseteq (x_0, x_1, \ldots, x_n)$.

Theorem 2. *The previous condition is satisfied by any ideal I after a generic linear change of variables. Moreover, when this condition is satisfied the map π restricted to V is surjective.*

Proof. If we consider the projectivization $\bar{V} \subseteq \mathbb{CP}^n$, the projection π consists of taking as a center the $n - d - 1$ dimensional subspace $S = \{[x_0 : x_1 : \cdots : x_n] \mid x_0 = x_1 = \cdots = x_d = 0\}$. Since the dimension of $\bar{V}$ is d, the intersection $\bar{V} \cap S$ is generically empty. We may hence assume that after a generic linear change of variables, $S \cap \bar{V} = \emptyset$. This intersection is given precisely by the ideal $\sqrt{I_h + (x_0, x_1, \ldots, x_d)}$, which is homogeneous. The condition of $\bar{V} \cap S$ being empty is equivalent to the ideal I being in general position.

In this situation the preimage by $\pi \mid_V$ of a point $[1 : x_1 : \cdots : x_d]$ is given by the intersection of the subspace $\{[y_0 : y_1 : \cdots : y_n] \mid y_1 = y_0 x_1, \ldots, y_d = y_0 x_d\}$ with $\bar{V}$. By the genericity assumption, this intersection does not have points in the infinity. By dimension arguments, this intersection cannot be empty, and must be contained in the affine part of $\bar{V}$. We have then proved that π restricted to V is surjective.

$\square$

The intersection of a generic linear subspace of dimension $n-d$ with $\bar{V}$ is a union of g distinct points, g being the degree of I_h. This degree can be computed through the Hilbert polynomial. Since I is in general position, all of the intersections of $\bar{V}$ with the fibers of π will happen in the affine part. This means that π restricted to V is a branched cover of degree g. We will see now that the branching locus of this cover is contained in a subvariety of $\mathbb{C}^d$ that can be computed.

Assume $I = (f_1, \ldots, f_s)$ is in general position. Consider the matrix

$$M := \begin{pmatrix} \frac{\partial f_1}{\partial x_{d+1}} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_s}{\partial x_{d+1}} & \cdots & \frac{\partial f_s}{\partial x_n} \end{pmatrix}$$

and the ideal J generated by its $(n-d) \times (n-d)$ minors.

Theorem 3. *The branching locus of $\pi \mid_V$ is contained in the zero locus of the elimination ideal $(I + J) \cap \mathbb{C}[x_1, \ldots, x_d]$.*

Proof. Consider a point $p = (x_1, \ldots, x_n) \in V$. If the linear space $\pi^{-1}(\pi(p))$ intersects V at p transversely, then there is no ramification at p, since it means that at a neighborhood of p the map $\pi \mid_V$ is a diffeomorphism. This condition of transversality can be expressed as follows: the complex normal space of V in p and the complex normal space of $\pi^{-1}(\pi(p))$ generate the complex tangent space of $\mathbb{C}^n$ in p. We can identify in a natural way the complex tangent space of $\mathbb{C}^n$ in p with the vector space $\mathbb{C}^n$ itself. The complex normal space of V in p is generated by the rows of the matrix

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1}(p) & \cdots & \frac{\partial f_1}{\partial x_n}(p) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_s}{\partial x_1}(p) & \cdots & \frac{\partial f_s}{\partial x_n}(p) \end{pmatrix}.$$

The complex normal space of $\pi^{-1}(\pi(p))$ is generated by the first d vectors of the canonical basis. A Gaussian elimination argument tells us that these two spaces generate the whole space if and only if the matrix M has rank $(n-d)$. So the set of points of V where $\pi \mid_V$ ramifies is contained in the set S of zeros of $I + J$.

The zero set of the elimination ideal $\mathbb{C}[x_1, \ldots x_d] \cap (I + J)$ is the Zariski closure of $\pi(S)$.

$\square$

Since $V \setminus \pi^{-1}(\pi(S))$ is a cover over $\mathbb{C}^d \setminus \pi(S)$ of degree g, we have that

$$\chi(V) = \chi(V \setminus \pi^{-1}(\pi(S))) + \chi(V \cap \pi^{-1}(\pi(S))) = g \cdot \chi(\mathbb{C}^d \setminus \pi(S)) + \chi(V \cap \pi^{-1}(\pi(S))) =$$

$$= g \cdot (\chi(\mathbb{C}^d) - \chi(\pi(S))) + \chi(V \cap \pi^{-1}(\pi(S))) = g - g \cdot \chi(\pi(S)) + \chi(V \cap \pi^{-1}(\pi(S))).$$

Both $\pi(S)$ and $V \cap \pi^{-1}(\pi(S))$ are varieties of dimension smaller than V, so, by induction hypothesis, we can compute their Euler characteristic in the same form.

3. Description of the algorithm

Now we will describe, step by step, an algorithm to compute the Euler characteristic of the zero set of an ideal $I = (f_1, \ldots, f_s)$.

Algorithm 1. (Compute the Euler characteristic of the algebraic set defined by the ideal I):

(1) Check if I is homogeneous. If it is, return 1.
(2) Compute the associated primes $(I_1, \ldots, I_m)$ of I (see [3, Chapter 4]).
(3) If there is more than one associated prime, we have that

$$\chi(V(I)) = \chi(V(I_1)) + \chi(V(I_2 \cap \cdots \cap I_m)) - \chi(V(I_1 + (I_2 \cap \cdots \cap I_m))).$$

by recursion, each summand can be computed with this algorithm. The following parts of this algorithm consider only the irreducible case, since we have already computed the associated primes, we will assume that I_1 is prime.
(4) Compute the dimension d and the degree g of $V(I)$. If d is zero, return g.
(5) Check that I is in general position. This can be done by computing a Gröbner basis of $\sqrt{I_h + (x_0, x_1, \ldots, x_d)}$ (where I_h is the homogenization of I) and using it to check that $x_{d+1}, \ldots, x_n$ are in it. If it is not in general position, apply a generic linear change of variables and start again the algorithm.
(6) Construct the ideal J generated by the $(n-d) \times (n-d)$ minors of the matrix

$$M := \begin{pmatrix} \frac{\partial f_1}{\partial x_{d+1}} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_s}{\partial x_{d+1}} & \cdots & \frac{\partial f_s}{\partial x_n} \end{pmatrix}.$$

(7) Compute the elimination ideal $K = (I+J) \cap \mathbb{C}[x_1, \cdots, x_d]$ (this can be done by computing a Gröbner basis w.r.t. an elimination ordering).
(8) Compute by recursion $\chi(V(K))$ and $\chi(V(I+K))$. Return the number $g - g \cdot \chi(V(K)) + \chi(V(I+K))$.

4. A finer invariant

The previous method essentially consists in decomposing our variety V in pieces, each of which is compared to $\mathbb{C}^i$ through linear maps that are unbranched covers. At the end of the day, it gives us a linear combination (with integer coefficients), of the Euler characteristic of $\mathbb{C}^i$.

Now we will show that we can actually keep the information in this linear combination, defining a slightly different invariant. This information will be kept in a polynomial $F_\pi(V) \in \mathbb{Z}[L]$, where L^i will play the role of $\chi(\mathbb{C}^i)$

We follow the same method as before but with two differences:

- If the ideal I is homogeneous, we don't end returning a 1. Instead, we continue the algorithm, taking as I_h the ideal generated by I inside $\mathbb{C}[x_0, \ldots, x_n]$.
- In the final step, we return $g \cdot L^d - g \cdot F_\pi(V(K)) + F_\pi(V(I+K)))$ instead of $g - g \cdot \chi(V(K)) + \chi(V(I+K))$.

So the algorithm results like this:

Algorithm 2. (Compute the polynomial $F_\pi(V)$ associated to an algebraic set $V(I)$ in general position).

(1) Compute the associated primes $(I_1, \ldots, I_m)$ of I.
(2) If there is more than one associated prime, we have that

$$F_\pi(V(I)) = F_\pi(V(I_1)) + F_\pi(V(I_2 \cap \cdots \cap I_m)) - F_\pi(V(I_1 + (I_2 \cap \cdots \cap I_m))).$$

by recursion, each summand can be computed with this algorithm. The following parts of this algorithm consider only the irreducible case, since we have already computed the associated primes, we will assume that I_1 is prime.

(3) Compute the dimension d and the degree g of $V(I)$. If d is zero, return g.
(4) Construct the ideal J generated by the $(n-d) \times (n-d)$ minors of the matrix

$$
M := \begin{pmatrix} \frac{\partial f_1}{\partial x_{d+1}} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_s}{\partial x_{d+1}} & \cdots & \frac{\partial f_s}{\partial x_n} \end{pmatrix}.
$$

(5) Compute the elimination ideal $K = (I + J) \cap \mathbb{C}[x_1, \cdots, x_d]$.
(6) Compute by recursion $F_\pi(V(K))$ and $F_\pi(V(I + K))$. Return the polynomial $gL^d - g \cdot F_\pi(V(K)) + F_\pi(V(I + K))$.

Note that both algorithms 1 and 2 can run differently if we apply a linear change of coordinates to I (which would change the projection π). The topological properties of the Euler characteristic tells us that the final result of the algorithm 1 will coincide with the Euler characteristic regardless of this linear change of coordinates. But in the case of $F_\pi(V)$ we cannot ensure such a result. Nevertheless, for two sufficiently generic projections, algorithm 2 will follow the same exact steps, so we can define $F(V)$ as the polynomial obtained by the algorithm 2 for generic projections.

More precisely, there must exist a Zariski open set $T \subseteq GL(n, \mathbb{C})$ such that, the polynomial $F_\pi(\sigma(V(I)))$ is the same for every linear change of coordinates $\sigma \in T$.

Definition 4. Given an ideal $I \trianglelefteq \mathbb{C}[x_1, \ldots, x_n]$, we define the polynomial $F(V(I))$ as the polynomial $F_\pi(\sigma(V(I))$ for any $\sigma \in T$.

We will say that I or $V(I)$ are in **generic position**, or that we are in **generic coordinates** if $F_\pi(V(I)) = F(V(I))$.

Since so far we have no algorithmic criterion to determine if a projection is generic enough or not, the generic case can be computed by introducing the parameters of the projection, and computing the Gröbner basis with those parameters. Experimental evidence suggested the following conjecture:

Conjecture 1. *If an ideal is in general position, it is also in generic position.*

Remark 5. Note that this conjecture is equivalent to saying that, assuming general position, the polynomial $F(V)$ is an invariant of V. That is: it does not depend on the projection.

Shortly after the first version of this paper was made public, Jørgen Vold Rennemo proved that the polynomial $F(V)$ contains essentially the same information as the Chern-Schwartz-MacPherson class. This result proves the conjecture, since the CSM class is an invariant of the variety. His proof is included in the appendix at the end of this document.

Some partial results in this direction are easy to show.

Lemma 6. *If V in in general position, the leading term of $F_\pi(V)$ coincides with the leading term of $F(V)$.*

Proof. It is immediate to check that the degree of $F_\pi(V)$ coincides with the dimension of V, and that the leading coefficient of $F_\pi(V)$ coincide with the degree of V, regardless of the projection used to compute it. $\square$

Remark 7. The value of $F_\pi(V)$ at $L = 1$ equals $\chi(V)$, independently of the choices of projections made for its computation, as long as we are in general position.

These two results actually show that $F(V)$ is independent of the projection for the case of curves (since in this case it is a degree 1 polynomial whose leading term and value at 1 are fixed).

We will now show that the invariant F_π behaves well with respect to the product of varieties:

Proposition 8. *Let $I_1 \trianglelefteq \mathbb{C}[x_1, \ldots, x_n]$ and $I_2 \trianglelefteq \mathbb{C}[y_1, \ldots, y_m]$ be two ideals on polynomial rings with separated variables, and let $V_1 \subseteq \mathbb{C}^n$ and $V_2 \subseteq \mathbb{C}^m$ be their corresponding algebraic sets of dimensions d_1 and d_2 respectively. Consider the ideal*

$$I := I_1 + I_2 \trianglelefteq \mathbb{C}[x_1 \ldots, x_{d_1}, y_1, \ldots, y_{d_2}, x_{d_1+1}, \ldots, x_n, y_{d_2+1}, \ldots, y_m].$$

Its corresponding algebraic set is $V = V_1 \times V_2 \subseteq \mathbb{C}^n \times \mathbb{C}^m = \mathbb{C}^{n+m}$. Denote by $\pi_1 : V_1 \to \mathbb{C}^{d_1}$ the map defined by the projection on the first coordinates. Analogously, denote $\pi_2 : V_2 \to \mathbb{C}^{d_2}$ and $\pi : V \to \mathbb{C}^{d_1+d_2}$. Then $F_\pi(V) = F_{\pi_1}(V_1) \cdot F_{\pi_2}(V_2)$.

Proof. Without loss of generality, we can assume that we are in the irreducible case. We will work on induction over the dimension. If V_1 or V_2 are zero dimensional, the result is immediate.

Consider g_1, g_2 the degrees of V_1 and V_2, and g the degree of V. It is easy to check that $g = g_1 \cdot g_2$. Is is also immediate to check that, if $V_2 = \mathbb{C}^m$, the statement holds (that is, $F_\pi(V_1 \times \mathbb{C}^m) = F_\pi(V_1) \cdot L^m$). Consider also Δ_1, Δ_2 and Δ the branching loci of π_1, π_2 and π respectively.

Now we will show that $\Delta = (\Delta_1 \times \mathbb{C}^{d_2}) \cup (\mathbb{C}^{d_1} \times \Delta_2)$. Let

$$p = (x_1, \ldots, x_{d_1}, y_1, \ldots, y_{d_2}) \in \mathbb{C}^{d_1} \times \mathbb{C}^{d_2}.$$

The set of points in V that project on p is the product of the set of points in V_1 that project in $(x_1 \ldots, x_{d_1})$ and the set of points in V_2 that project in $(y_1, \ldots, y_{d_2})$. This set has less than $g_1 \cdot g_2$ points if and only if $(x_1, \ldots, x_{d_1}) \in \Delta_1$ or $(y_1, \ldots, y_{d_1}) \in \Delta_2$. It is immediate also that $(\Delta_1 \times \mathbb{C}^{d_2}) \cap (\mathbb{C}^{d_1} \times \Delta_2) = \Delta_1 \times \Delta_2$. By induction hypothesis, we have that

$$F_\pi(\Delta) = L^{d_2} \cdot F_{\pi_1}(\Delta_1) + L^{d_1} \cdot F_{\pi_2}(\Delta_2) - F_{\pi_1}(\Delta_1) \cdot F_{\pi_2}(\Delta_2).$$

Reasoning analogously, we can conclude that

$$F_\pi(\pi^{-1}(\Delta)) = F_{\pi_1}(\pi_1^{-1}(\Delta_1)) \cdot F_{\pi_2}(V_2) + F_{\pi_2}(\pi_2^{-1}(\Delta_2)) \cdot F_{\pi_1}(V_1) - F_{\pi_1}(\pi_1^{-1}(\Delta_1)) \cdot F_{\pi_2}(\pi_2^{-1}(\Delta_2)).$$

So, summarizing, we have that

$$\begin{array}{rcl} F_\pi(V) & = & g_1 g_2 (L^{d_1+d_2} - F_\pi(\Delta)) + F_\pi(\pi^{-1}(\Delta)) \\ F_{\pi_1}(V_1) & = & g_1 (L^{d_1} - F_{\pi_1}(\Delta_1)) + F_{\pi_1}(\pi_1^{-1}(\Delta_1)) \\ F_{\pi_2}(V_2) & = & g_2 (L^{d_2} - F_{\pi_2}(\Delta_2)) + F_{\pi_2}(\pi_2^{-1}(\Delta_2)). \end{array}$$

Using all the previous formulas one can easily check that $F_\pi(V) = F_{\pi_1}(V_1) \cdot F_{\pi_2}(V_2)$. □

Since conjecture 1 is true, the same result is true for $F(V)$. In fact, a weaker condition is enough: if the product of two generic projections is generic, then the invariant F is multiplicative. This could be useful, for example, to give a criterion to check if an algebraic set can be the product of two nontrivial algebraic sets. If $F(V)$ is irreducible in $\mathbb{Z}[L]$, then V couldn't be a product.

5. The projective case

To compute the Euler characteristic of the projective variety $\bar{V}$ defined by a homogeneous ideal $I_h \trianglelefteq \mathbb{C}[x_0, \ldots, x_n]$ we can also use algorithm 1. In order to do so, we will consider the hyperplane H "at infinity" given by the equation $x_0 = 0$. This allows us to decompose $\bar{V}$ as its affine part $V := \bar{V} \setminus H$ and its part at infinity $\bar{V}^\infty := \bar{V} \cap H$. It is clear that $\chi(\bar{V}) = \chi(V) + \chi(\bar{V}^\infty)$.

The affine part V is an affine variety defined by the ideal obtained by substituting $x_0 = 1$ in the generators of I_h, whose Euler characteristic can be computed as seen before.

The part at infinity $\bar{V}^{\infty}$ is a projective variety embedded in a projective space of less dimension. The homogeneous ideal that defines it is obtained by substituting $X_0 = 0$ in the generators of I_h. Its Euler characteristic can be computed by recursion. If we are in the case of $\mathbb{CP}^1$, $\bar{V}$ will consist on a finite number of points, which can be computed as the degree of $\sqrt{I_h}$.

This same idea of decomposing a projective variety as a disjoint union of affine pieces can be used to extend the definition of F_π to projective varieties. The invariant of a projective variety is defined as the sum of the invariant of its affine pieces.

Now we will show a different way to compute the Euler characteristic of a projective variety using the polynomial $F(V)$.

Theorem 9. *Let $I = \mathbb{C}[x_0, \ldots, x_n] \cdot (f_1, \ldots, f_s)$ be a homogeneous ideal in generic position. Assume that the generators $f_1, \ldots, f_s$ are homogeneous. Denote by $I_0 := (I + \mathbb{C}[x_0, \ldots, x_n] \cdot (x_0)) \cap \mathbb{C}[x_1, \ldots, x_n]$, and $I_1 := (I + \mathbb{C}[x_0, \ldots, x_n] \cdot (x_0 - 1)) \cap \mathbb{C}[x_1, \ldots, x_n]$. That is, the ideals that represent the intersection of $V(I)$ with the hyperplanes $\{x_0 = 0\}$ and $\{x_0 = 1\}$ respectively, seeing the two hyperplanes as ambient spaces. Then the following formula holds:*

$$F(V(I)) = (L - 1) \cdot F(V(I_1)) + F(V(I_0))$$

Proof. By induction on the dimension of $V(I)$. If the dimension is zero, $V(I)$ must consist only on the origin, since I is homogeneous. In this case, $V(I_1)$ is empty, and $V(I_0)$ is also the origin. We have that

$$1 = F(V(I)) = (L - 1) \cdot 0 + 1 = (L - 1) \cdot F(V(I_1)) + F(V(I_0)).$$

If the dimension d of $V(I)$ is positive, consider the ideals J, K and $H = I + K$ as before. Construct also H_0, H_1, K_0 and K_1 in the same way as I_0 and I_1. Note that, since we are in generic position, the ideals H_0' and K_0' needed to compute $F(V(I_0))$ are precisely H_0 and K_0 (that is, specializing $x_0 = 0$ and then computing the minors of the matrix M and the elimination ideal is the same as computing the minors of the matrix and the elimination and then specializing). The same happens with J_1 and K_1.

By induction hypothesis, we have that

$$F(V(H)) = (L - 1) \cdot F(V(H_1)) + F(V(H_0))$$

and

$$F(V(K)) = (L - 1) \cdot F(V(K_1)) + F(V(K_0)).$$

Now we have that

$$F(V(I)) \;=\; g \cdot L^d - g \cdot F(V(K)) + F(V(H)) \;=$$

$$g \cdot L^d - g \cdot ((L - 1) \cdot F(V(K_1)) + F(V(K_0))) + (L - 1) \cdot F(V(H_1)) + F(V(H_0)) \;=$$

$$(L - 1) \cdot (g \cdot L^{d-1} - g \cdot F(V(K_1)) + F(V(H_1))) + g \cdot L^{d-1} - g \cdot F(V(K_0)) + F(V(H_0)) =$$

$$(L - 1) \cdot F(V(I_1)) + F(V(I_0)),$$

and this proves the result. $\square$

This theorem allows us to relate the invariant F of the affine algebraic set defined by a homogeneous ideal, and the invariant F of the projective variety defined by the same ideal as follows:

Corollary 10. *Let I be a homogeneous ideal in $\mathbb{C}[x_1, \ldots, x_n]$ in generic position. Let V be the algebraic set defined by I, and V' the projective variety defined by the same ideal. Then $F(V) = (L - 1) \cdot F(V') + 1$.*

Proof. Applying the previous result recursively, we have that $F(V(I))$ equals

$$(L - 1) \cdot F(V(I_1)) + (L - 1) \cdot F(V((I_0)_1)) + \cdots$$

$$+ (L - 1) \cdot F(V((\cdots (I_0) \cdots)_0)_1)) + F(V((\cdots (I_0) \cdots)_0)).$$

Note that, if we compute $F(V')$ through the sum of the affine pieces, we obtain precisely $F(V(I_1)) + F(V((I_0)_1)) + \cdots + F(V((\cdots (I_0) \cdots)_0)_1))$. Since $V((\cdots (I_0) \cdots)_0))$ consists only of the origin, we have the result.

This last result can be interpreted as the fact that V is the complex cone over V'. That is, $V \setminus \{0\}$ is the product $V' \times \mathbb{C}^*$.

$\square$

6. Examples

Both the polynomial $F(V)$ and the Hilbert polynomial P_I have the same degree, and the leading term is determined by the degree of V. This would point in the direction of considering that they contain the same information. The following example shows this is not the case:

Example 11. Consider the conics $C_1, C_2 \in \mathbb{C}^2$ given by $C_1 := V(x^2 + y^2 - 1)$ and $C_2 := V(x^2 + y^2)$. If we compute the Hilbert polynomial of the corresponding homogeneous ideals in $\mathbb{C}[x, y, z]$ we get that $P_{(x^2 + y^2 + z^2)} = P_{(x^2 + y^2)} = 2 \cdot t + 1$.

Let's now compute the polynomial $V(C_1)$ using the canonical projection $\pi : \mathbb{C}^2 \to \mathbb{C}$ to the first component. This projection is a $2 : 1$ cover of $\mathbb{C}$ branched along the points ± 1. Over each of these points, there is only one preimage. So finally we have that $F(C_1) = 2(L - 2) + 1 + 1 = 2L - 2$.

On the other hand, the curve C_2 also projects $2 : 1$ over $\mathbb{C}$, but now there is only one branching point ($x = 0$). So the result is $F(C_2) = 2(L - 1) + 1 = 2L - 1$.

That is, this example shows that the polynomial $F(V)$ contains information that is not contained in the Hilbert polynomial.

An important example of algebraic sets is the case of hyperplane arrangements. We will now recall some related notions (see [5, Chapter II]).

Definition 12. Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{C}^n$. Its **intersection lattice** $L(\mathcal{A})$ is the set of all its intersections ordered by reverse inclusion, with the convention that the intersection of the empty set is $\mathbb{C}^n$ itself.

The Möbius function is the only function $\mu : L \to \mathbb{Z}$ satisfying that:

$$\mu(\mathbb{C}^n) = 1$$
$$\sum_{Y \leq X} \mu(Y) = 0 \quad \forall X \in L \setminus \{\mathbb{C}^n\}.$$

Definition 13. The **characteristic polynomial** of $\mathcal{A}$ is defined as

$$\chi(\mathcal{A}, L) := \sum_{X \in L(\mathcal{A})} \mu(X) \cdot L^{dim(X)}.$$

Theorem 14 (Deletion-Restriction). *Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{C}^n$, and H a hyperplane of $\mathcal{A}$. Let $\mathcal{A}'$ be the arrangement resulting from eliminating H from $\mathcal{A}$, and let $\mathcal{A}''$ be the hyperplane arrangement inside H induced by the intersection with $\mathcal{A}'$. Then the following formula holds:*

$$\chi(\mathcal{A}, L) = \chi(\mathcal{A}', L) - \chi(\mathcal{A}'', L).$$

Now we will see how this characteristic polynomial relates to the polynomial $F(V)$.

Theorem 15. *Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{C}^n$. The following holds:*

$$F(\mathcal{A}) = L^n - \chi(\mathcal{A}, L).$$

Proof. By induction on the number of hyperplanes. The case of only one hyperplane is immediate.

If there are more than one hyperplane, take one hyperplane H of $\mathcal{A}$. Since $\mathcal{A} = \mathcal{A}' \cup H$, we have, by additivity, that

$$F(\mathcal{A}) = F(\mathcal{A}') + F(H) - F(\mathcal{A}' \cap H) = F(\mathcal{A}') + L^{n-1} - F(\mathcal{A}'').$$

Both $\mathcal{A}'$ and $\mathcal{A}''$ are hyperplane arrangements with less hyperplanes than $\mathcal{A}$, so, by induction hypothesis the following formulas hold:

$$F(\mathcal{A}') = L^n - \chi(\mathcal{A}', L)$$
$$F(\mathcal{A}'') = L^{n-1} - \chi(\mathcal{A}'', L).$$

Substituting these formulas in the previous one, and using the deletion-restriction theorem we get the result.   □

This result tells us that the characteristic polynomial of $\mathcal{A}$ can be thought of as the polynomial F of its complement. If we consider a projective arrangement $\mathcal{A}$ and its affine cone $\hat{\mathcal{A}}$, we can use Corollary 10 to relate $\chi(\hat{\mathcal{A}}, L)$ and $F(\mathcal{A})$. The result in the appendix then relates $F(\mathcal{A})$ with the Chern-Schwartz-MacPherson class of its complement. This way we obtain a new proof of the formula in [2, Theorem 3.1].

7. Counting points over finite fields

In this section we will see how the polynomial $F(V)$ can be related to the number of points of the variety considered over a finite field. Let's illustrate this fact with an example.

Example 16. Consider the conic given by the equation

$$x_1^2 + x_2^2 - 1$$

in the affine plane over the field of 5 elements $\mathbb{F}_5$.

The set of rational points is the following:

$$(0, 1), (0, 4), (1, 0), (4, 0)$$

If we project it to the x_1 axis, we see that over the point (0), we have two preimages, as expected by the degree. Over the points (1) and (4), we have just one point of the curve, since the cover ramifies there. But over the points (2) and (3) we have no points of the curve. The reason for this is that the equations $2^2 + x_2^2 - 1$ and $3^2 + x_2^2 - 1$ have no roots over $\mathbb{F}_5$. However, they do have all their solutions over a quadratic extension $\mathbb{F}_{25}$ of degree 2. In particular, if we look at the points of $\mathbb{F}_5 \times \mathbb{F}_{25}$ that satisfy the equation, we obtain the set

$$\{(0, 1), (0, 4), (1, 0), (2, a + 2), (2, 4a + 3), (3, a + 2), (3, 4a + 3), (4, 0)\},$$

where a is an element of $\mathbb{F}_{25}$ that has minimal polynomial $x^2 + 3$ over $\mathbb{F}_5$.

It might seem strange to consider the set of points in $\mathbb{F}_5 \times \mathbb{F}_{25}$ that satisfy a given equation. But this set is in fact an algebraic set. Indeed, it can be expressed as the set of points of the affine plane over $\mathbb{F}_{25}$ that satisfy the equations

$$x_1^2 + x_2^2 - 1$$
$$x_1^5 - x_1$$

Recall that the polynomial $F(V)$ for such a conic was $f = 2L - 2$. In this case, we have obtained 8 points in total, which is precisely the value of f for $L = 5$. A quick look at the points shows why this happens: there are two points over each value of $\mathbb{F}_5$, with the exception of the two branching points, where there is only one.

Note that both algorithms 1 and 2 can be run over finite fields in the same way as they run over the rationals. With a small exception, though: to ensure the existence of a change of coordinates that puts the ideal in general position, we might need to work in a finite field extension. Once done that, both algorithms would mimic the steps given by the algorithm run over $\mathbb{Q}$, except if some leading coefficient becomes zero. But that would happen only for a finite number of prime numbers p.

Notation 17. We can then define the polynomial $F_p(V)$ as the result of running algorithm 2 in a field of characteristic p.

As we have seen before, $F_p(V) = F(V)$ for almost every prime number p.

Notation 18. Given a polynomial
$$f = a_0 + a_1 L + a_2 L^2 + \cdots + a_n L^n \in \mathbb{Z}[L]$$
and a list of numbers $(d_1, \ldots, d_s)$ with $s \geq n$, we will denote by $f(d_1, \ldots, d_s)$ the number
$$f(d_1, \ldots, d_s) = a_0 + a_1 d_1 + a_2 d_1 d_2 + \cdots + a_n d_1 d_2 \cdots d_n$$

Another difference between the way the algorithms would run over $\mathbb{Q}$ and over finite fields lies in the primary decomposition. But then again, this will only happen for a finite number of primes.

Theorem 19. *Given an ideal $I \trianglelefteq \mathbb{F}_p[x_1, \ldots, x_n]$, and the corresponding $f := F_p(V)$ of degree s, there exists a list of numbers $(d_1 \leq \cdots \leq d_n)$ such that the number of points in*
$$\left(\mathbb{F}_{p^{d_1}} \times \cdots \times \mathbb{F}_{p^{d_n}} \right) \cap V(I)$$
equals the number $f(p^{d_1}, \ldots, p^{d_n})$.

Moreover, for any such list $(d_1 \leq \cdots \leq d_n)$ and any d'_i multiple of d_i there exists another list $(d_1 \leq \cdots \leq d_{i-1} \leq d'_i \leq d'_{i+1} \leq \cdots \leq d'_n)$ satisfying the same property.

Proof. By induction over the dimension. If I is zero dimensional, $V(I)$ consists of $F_p(V) = deg(I)$ distinct points, whose coordinates lie in a sufficiently big extension $\mathbb{F}_{p^{d_1}}$. Any sequence starting with d_1 would satisfy the theorem.

If I has dimension $d > 0$ and degree g, we have that $F_p(V(I)) = g(L^d - F_p(V(K))) + F_p(V(I + K))$. By induction, we can assume that both K and $I + K$ satisfy the result. Let $(d_1^1 \leq \cdots \leq d_d^1)$ and $(d_1^2 \leq \cdots \leq d_n^2)$ be the corresponding sequences. Take $d_1 = lcm(d_1^1, d_1^2)$. There exist two sequences $(d_1 \leq d_2^{1'} \leq \cdots \leq d_d^{1'})$ and $(d_1 \leq d_2^{2'} \leq \cdots \leq d_n^{2'})$ that are valid for K and $I + K$ respectively, and coinciding in the first term. Repeating this reasoning we can obtain two sequences $(d_1 \leq \cdots \leq d_d)$ and $(d_1 \leq \cdots \leq d_d \leq d_{d+1} \leq \cdots \leq d_n)$ that are valid for K and $I + K$ respectively.

Note that, since both $F_p(V(K))$ and $F_p(V(I + K))$ are of dimension at most $d - 1$, the terms $d_d, \ldots, d_n$ can be changed arbitrarily and still the sequence would be valid for K and $I + K$.

Now take any point $q := (q_1, \ldots, q_d) \in (\mathbb{F}_{p^{d_1}} \times \cdots \times \mathbb{F}_{p^{d_d}}) \setminus V(K)$. Taking an appropriate field $\mathbb{F}_{p^{n_q}}$, we can ensure that there are exactly g points in $\{(x_1, \ldots, x_n) \in (\mathbb{F}_{p^{n_q}})^n \mid x_1 = q_1, \ldots, x_d = q_d\} \cap V$. We can do the same for every point q, and take a common field extension $\mathbb{F}_{p^s}$ of all the different $\mathbb{F}_{p^{n_q}}$. This way, we have that there are exactly g points of $V \cap (\mathbb{F}_{p^{d_1}} \times \cdots \times \mathbb{F}_{p^{d_d}} \times \mathbb{F}_{p^s} \times \cdots \times \mathbb{F}_{p^s})$ over each point of $(\mathbb{F}_{p^{d_1}} \times \cdots \times \mathbb{F}_{p^{d_d}}) \setminus V(K)$.

By definition, we have that

$$F_p(V) = g(L^d - F_p(V(K))) + F_p(V(I+K)).$$

Now if we restrict ourselves to the points in $\mathbb{F}_{p^{d_1}} \times \cdots \times \mathbb{F}_{p^{d_d}} \times \mathbb{F}_{p^s} \times \cdots \times \mathbb{F}_{p^s}$, we have that

$$\#V = g \cdot (p^{d_1} p^{d_2} \cdots p^{d_d} - \#V(K)) + \#V(I+K).$$

Making use of the induction hypothesis, and the fact that $F_p(V(K))$ and $F_p(V(I+K))$ are of dimension less than d, the result follows easily.

$\square$

Corollary 20. *Let V be an algebraic set in $\mathbb{C}^n$ defined by an ideal $I \trianglelefteq \mathbb{K}[x_1, \ldots, x_n]$, where $\mathbb{K}$ is an algebraic extension of $\mathbb{Q}$. Then for almost every prime p there exists a list of positive integers $(d_{1,p}, \ldots, d_{n,p})$ such that the number of points in*

$$\left(\mathbb{F}_{p^{d_{1,p}}} \times \cdots \times \mathbb{F}_{p^{d_{n,p}}} \right)$$

that satisfy the equations of I equals the number $F(V(I))(p^{d_{1,p}}, \ldots, p^{d_{n,p}})$.

This corollary provides a different way to compute the polynomial $F(V(I))$ by counting points over finite fields. If we know the value of $F(V(I))(S)$ for a sufficient number of such sequences S, recovering the coefficients of the polynomial is a simple linear algebra problem.

8. Code and timings

Here we show an implementation in Sage ([7]) of the two algorithms. This same code can be found at http://www.sagenb.org/home/pub/3961.

8.1. **Implementation of Algorithm 1.**

```
def Euler_characteristic(I):
    R=I.ring()
    if I.is_one():
        return 0
    if R.ngens()==1:
        return sum([j[0].degree() for j in I.gen().factor()])
    J1=I.radical()
    if J1.is_homogeneous():
        return 1
    primdec=J1.associated_primes()
    J1=primdec[0]
    m=len(primdec)
    if m>1:
        J2=R.ideal(1)
        for j in [1..m-1]:
            J2=J2.intersection(primdec[j])
        return Euler_characteristic(J1)+Euler_characteristic(J2)-Euler_characteristic(J1+J2)
    P=J1.homogenize().hilbert_polynomial()
    if P.is_zero():
        deg=0
    else:
        deg=P.leading_coefficient()*P.degree().factorial()
    if deg==1:
        return 1
```

```
    dim=J1.dimension()
    n=R.ngens()
    vars1=R.gens()[0:n-dim]
    vars2=R.gens()[n-dim:n]
    varpiv=vars1[-1]
    IH=J1.homogenize()
    S=IH.ring()
    JH=IH+S.ideal(S.gens()[n-dim:])
    if JH.dimension()>0:
        det=0
        while det==0:
            MH=random_matrix(R.base_ring(),n)
            det=MH.determinant()
        L=list(MH*vector(list(R.gens())))
        return Euler_characteristic(R.hom(L)(J1))
    if dim==0:
        return deg
    M=matrix([[f.derivative(v) for v in vars1] for f in J1.gens()])
    J=R.ideal(M.minors(n-dim))
    K=(J+J1).elimination_ideal(vars1)
    S=PolynomialRing(R.base_ring(),vars2)
    H=R.hom([S(0) for j in vars1]+[S(j) for j in vars2])
    C=deg-deg*Euler_characteristic(H(K))+Euler_characteristic(K+J1)
    return C
```

This algorithm may be very slow (since it involves several Gröbner basis computations), but in several interesting cases, it gives a useful answer in reasonable time. Let's show here some examples.

In the case of curves and surfaces, the result is often reasonably fast; but it may vary a lot if a random change of variables has to be applied. Here we show a few examples. These tests have been run on a Dual-Core AMD Opteron 8220.

Three examples of plane curves:

```
sage: R.<x,y>=QQ[]
sage: time Euler_characteristic(R.ideal(x^5+1))
5
Time: CPU 0.16 s, Wall: 0.17 s
sage: time Euler_characteristic(R.ideal(y^4+x^3-1))
-5
Time: CPU 0.19 s, Wall: 0.20 s
time Euler_characteristic(R.ideal(x^2+y^2-5*x^2*y^4+x*y-1))
-8
Time: CPU 17.82 s, Wall: 17.82 s
```

A curve and a surface in $\mathbb{C}^3$:

```
S.<x,y,z>=QQ[]
time Euler_characteristic(S.ideal(x^5+y^2+2*x*y+1,3*x-5*y*x+y^2+1))
10
Time: CPU 0.17 s, Wall: 0.18 s
time Euler_characteristic(S.ideal(x^5+y^2+2*x*y+1))
-3
```

```
Time: CPU 0.49 s, Wall: 0.49 s
```

8.2. **Implementation of Algorithm 2.**

```
@parallel(7)
@cached_function
def FV(I,var='L'):
    FS=PolynomialRing(ZZ,var)
    L=FS.gen()
    R=I.ring()
    if R.ngens()==0:
        return 0
    if I.is_zero():
        return L^R.ngens()
    if I.is_one():
        return 0
    if R.ngens()==1:
        return FS(sum([j[0].degree() for j in I.gen().factor()]))
    J1=I.radical()
    if J1.is_homogeneous():
        S1=PolynomialRing(R.base_ring(),R.gens()[0:-1])
        H1=R.hom(list(S1.gens())+[S1(1)])
        H2=R.hom(list(S1.gens())+[S1(0)])
        resulp=FV([H1(J1),H2(J1)])
        d=dict([[a[0][0][0],a[1]] for a in resulp])
        [i1,i2]=[d[H1(J1)],d[H2(J1)]]
        return (L-1)*(i1)+i2
    primdec=J1.associated_primes()
    J1=primdec[0]
    m=len(primdec)
    if m>1:
        J2=R.ideal(1)
        for j in [1..m-1]:
            J2=J2.intersection(primdec[j])
            resulp=FV([J1,J2,J1+J2])
            d=dict([[a[0][0][0],a[1]] for a in resulp])
            [i1,i2,i3]=[d[J1],d[J2],d[J1+J2]]
        return i1+i2-i3
    P=J1.homogenize().hilbert_polynomial()
    if P.is_zero():
        deg=0
    else:
        deg=P.leading_coefficient()*P.degree().factorial()
    dim=J1.dimension()
    if deg==1:
        return FS(L^dim)
    n=R.ngens()
    vars1=R.gens()[0:n-dim]
    vars2=R.gens()[n-dim:n]
    varpiv=vars1[-1]
```

```
    IH=J1.homogenize()
    S=IH.ring()
    JH=IH+S.ideal(S.gens()[n-dim:])
    if JH.dimension()>0:
        det=0
        while det==0:
            MH=random_matrix(R.base_ring(),n)
            det=MH.determinant()
        L=list(MH*vector(list(R.gens())))
        return FV(R.hom(L)(J1))
    if dim==0:
        return FS(deg)
    M=matrix([[f.derivative(v) for v in vars1] for f in J1.gens()])
    J=R.ideal(M.minors(n-dim))
    K=(J+J1).elimination_ideal(vars1)
    S=PolynomialRing(R.base_ring(),vars2)
    H=R.hom([S(0) for j in vars1]+[S(j) for j in vars2])
    d=dict([[a[0][0][0],a[1]] for a in FV([H(K),K+J1])])
    [i1,i2]=[d[H(K)],d[K+J1]]
    C=deg*FS(L^dim-i1)+i2
    return C
```

This implementation makes use of the Sage framework for parallel computations, allowing to use several processor cores at the same time to compute the intermediate steps. It also caches the already computed results in case they would be needed later.

Note that the implementation assumes that the ideal is in generic position. Since Conjecture 1 is true, it is enough to check that it is in general position (recall that, to do so, one has to compute the ideal $\sqrt{I_h + (x_0, \dots, x_d)}$ and see if it contains all the monomials x_i).

There is another way to check if the ideal is in generic position without using Conjecture 1. Considering a ring that contains the parameters of the possible linear transformations, and write the matrix $(a_{i,j})$ whose entries are these parameters. This matrix represents a generic linear change of coordinates, so we can apply it to our ideal and run the computation over this ring with parameters. The Gröbner basis computations in this ring would mimic the ones in the original one, for generic values of the parameters. Unluckily, the methods to compute the primary decomposition do not work over rings with parameters. One way to proceed is to compute the primary decomposition over the original ring without parameters. Then compute the discriminant with parameters, and then choose a value for the parameters in the open part of the Gröbner cover (see [4] for a definition and algorithm).

Again, this method can be very slow, but in some cases it is fast enough to be useful. Here we present some of those examples:

The complex 2-sphere:

```
sage: R.<x,y,z>=QQ[]
sage: time FV(R.ideal(x^2+y^2+z^2-1))
2*L^2 - 2*L + 2
Time: CPU 0.07 s, Wall: 0.32 s
```

Another surface:

```
sage: time FV(R.ideal(x^3+y^3+z^3-1))
3*L^2 - 6*L + 12
Time: CPU 0.07 s, Wall: 0.29 s
```

The intersection of the two:

```
sage: time FV(R.ideal(x^3+y^3+z^3-1,x^2+y^2+z^2-1))
6*L - 15
Time: CPU 0.08 s, Wall: 43.01 s
```

and their union (the timing is done after cleaning the cache of the function):

```
sage: time FV(R.ideal((x^3+y^3+z^3-1)*(x^2+y^2+z^2-1)))
5*L^2 - 14*L + 29
Time: CPU 0.08 s, Wall: 43.02 s
```

notice the additivity of the polynomial.

The Whitney umbrella:

```
sage: time FV(R.ideal(x*y^2-z^2))
3*L^2 - 4*L + 2
Time: CPU 0.13 s, Wall: 3.44 s
```

The 3-sphere:

```
sage: S.<x,y,z,t>=QQ[]
sage: time FV(S.ideal(x^2+y^2+z^2+t^2-1))
2*L^3 - 2*L^2 + 2*L - 2
Time: CPU 0.09 s, Wall: 0.43 s
```

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APPENDIX: THE POLYNOMIAL INVARIANT AS A CHERN-SCHWARTZ-MACPHERSON CLASS

Jørgen VOLD RENNEMO

ABSTRACT: After a change of variables, the polynomial $F(V(I))$ defined in the paper equals the Chern-Schwartz-MacPherson class of $V(I)$ considered as a subset of $\mathbb{P}^n$.

Let X be a reduced and irreducible variety. The Chern-Schwartz-MacPherson class (abbreviated CSM class) is a construction which to any constructible set $Y \subseteq X$ assigns a class $c_{SM}(Y) \in H_*(X, \mathbb{Z})$. For our purposes, the essential properties of the CSM class are the following.

(1) If Y is complete and nonsingular and $i : Y \to X$ denotes the embedding, then $c_{\mathrm{SM}}(Y) = i_*(c_\bullet(T_Y) \cap [Y])$.
(2) If Y is a disjoint union of constructible sets Y_1, Y_2, then $c_{\mathrm{SM}}(Y) = c_{\mathrm{SM}}(Y_1) + c_{\mathrm{SM}}(Y_2)$.
(3) Let $f : X \to Z$ be a proper morphism of reduced and irreducible varieties, and assume that restricted to Y the morphism f is set-theoretically g-to-1. Then the relation $f_*(c_{\mathrm{SM}}(Y)) = g \cdot c_{\mathrm{SM}}(f(Y))$ holds.

For the construction and general properties of CSM classes, see [Fu, Ex 19.1.7] or [Mac]. What we call $c_{\mathrm{SM}}(Y)$ corresponds in the notation of these references to $c_*(1_Y)$, where $1_Y : X \to \mathbb{Z}$ is the characteristic function of Y.

Identify the homology group $H_*(\mathbb{P}^n, \mathbb{Z})$ with $\mathbb{Z}[Q]/(Q^{n+1})$, letting Q^d correspond to the class of a linear d-dimensional subspace $\mathbb{P}^d \subseteq \mathbb{P}^n$. Note that this is strictly an isomorphism of abelian groups and is not compatible with any of the usual multiplicative structures on the groups. We can now state the result.

Proposition. *Let I be an ideal as in Definition 4 of the paper, and consider $V(I)$ as a locally closed subset of $\mathbb{P}^n$ via the standard compactification $\mathbb{C}^n \subset \mathbb{P}^n$. After the change of variables $L \to 1 + Q$, the equation $F(V(I)) = c_{SM}(V(I))$ holds.*

Proof. Let $V = V(I)$. As $F(V)$ is defined by Algorithm 2 of the paper, it suffices to show that after the variable change $L \to (1 + Q)$ and replacing F_π with c_{SM} everywhere, this algorithm gives a computation of $c_{SM}(V)$. The only steps that need justification in this setting are Steps 2, 3 and 6. The second listed property of the CSM class implies that the inclusion-exclusion principle holds for c_{SM}, and so the relation in Step 2 is valid. The CSM class of a point is the ordinary class of the point, showing the validity of Step 3.

It remains to justify Step 6, in other words we must verify that

$$c_{\mathrm{SM}}(V) = g(1 + Q)^d - g\, c_{\mathrm{SM}}(V(K)) + c_{\mathrm{SM}}(V(I + K)).$$

In what follows we shall consider algebraic sets which can be considered as subsets of more than one variety. In order to compare the CSM classes with respect to the different inclusions, we introduce the following notation: If Y is an algebraic subset of X, denote by $c_{SM}^X(Y)$ the CSM class of Y considered as a subset of X, i.e. so that $c_{\mathrm{SM}}^X(Y)$ is an element of $H_*(X)$.

Let $\mathbb{C}^d \subset \mathbb{P}^n$ be a linear space, and compute $c_{\mathrm{SM}}^{\mathbb{P}^n}(\mathbb{C}^d)$ by

$$c_{\mathrm{SM}}^{\mathbb{P}^n}(\mathbb{C}^d) = c_{\mathrm{SM}}^{\mathbb{P}^n}(\mathbb{P}^d) - c_{\mathrm{SM}}^{\mathbb{P}^n}(\mathbb{P}^{d-1}) = (1 + Q)^d,$$

where the last step follows from the first property of the CSM class.

Now, let $\pi : V \to \mathbb{C}^d$ be the restriction of a generic projection $\pi' : \mathbb{C}^n \to \mathbb{C}^d$, and let $U \subseteq \mathbb{C}^d$ be the complement of $V(K)$ as defined in the algorithm. Theorem 3 of the paper shows that over U the map π is g-to-1. In view of the computation of $c_{\mathrm{SM}}^{\mathbb{P}^n}(\mathbb{C}^d)$, the justification of Step 6 now amounts to showing the relation

$$c_{\mathrm{SM}}^{\mathbb{P}^n}(V) = g(c_{\mathrm{SM}}^{\mathbb{P}^n}(\mathbb{C}^d) - c_{\mathrm{SM}}^{\mathbb{P}^n}(\mathbb{C}^d \setminus U)) + c_{\mathrm{SM}}^{\mathbb{P}^n}(\pi^{-1}(\mathbb{C}^d \setminus U)),$$

which is equivalent to

$$(8.1) \qquad\qquad c_{\mathrm{SM}}^{\mathbb{P}^n}(V) = g\, c_{\mathrm{SM}}^{\mathbb{P}^n}(U) + c_{\mathrm{SM}}^{\mathbb{P}^n}(\pi^{-1}(\mathbb{C}^d \setminus U)).$$

Let X be the blow up of $\mathbb{P}^n$ along the $\mathbb{P}^{n-d-1}$ that is the centre of the projection $\pi' : \mathbb{C}^n \to \mathbb{C}^d$, and let $p : X \to \mathbb{P}^n$ be the blowup morphism. Since π' is generic, for dimension reasons we may assume that the centre of the projection is disjoint from the closure $\overline{V}$ of V in $\mathbb{P}^n$. Let $\pi'' : X \to \mathbb{P}^d$ be the unique extension of π' to X. We thus have the following diagram.

$$V \hookrightarrow \mathbb{C}^n \hookrightarrow X \xrightarrow{\ p\ } \mathbb{P}^n$$

The left hand triangle and the square are commutative, and the right hand triangle is not. However, when we restrict to the subgroup of $H_*(X, \mathbb{Z})$ generated by cycles with support disjoint from the exceptional divisor of p, the pushforward maps on homology satisfy $i_* \circ \pi''_* = p_*$.

We now have

$$c_{\mathrm{SM}}^{\mathbb{P}^n}(V) = p_*(c_{\mathrm{SM}}^X(V)) = i_* \pi''_*(c_{\mathrm{SM}}^X(V)).$$

The last equality follows from the fact that $c_{\mathrm{SM}}^X(V)$ is the pushforward from $\overline{V}$ to X of the class $c_{\mathrm{SM}}^{\overline{V}}(V)$, and hence it is represented by a cycle that is disjoint from the exceptional divisor in X.

We now compute

$$i_* \pi''_*(c_{\mathrm{SM}}^X(V)) = i_* \pi''_*(c_{\mathrm{SM}}^X(\pi^{-1}(U))) + i_* \pi''_*(c_{\mathrm{SM}}^X(\pi^{-1}(\mathbb{C}^d \setminus U)))$$

$$= i_*(g\, c_{\mathrm{SM}}^{\mathbb{P}^d}(U)) + p_*(c_{\mathrm{SM}}^X(\pi^{-1}(\mathbb{C}^d \setminus U)))$$

$$= g\, c_{\mathrm{SM}}^{\mathbb{P}^n}(U) + c_{\mathrm{SM}}^{\mathbb{P}^n}(\pi^{-1}(\mathbb{C}^d \setminus U)),$$

which proves (8.1). $\square$

I thank the referee for pointing out an important mistake in the submitted version of this appendix.

References

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