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Fronts of Whitney umbrella – a differential geometric approach via blowing up

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Abstract

We investigate the differential geometric ingredients for Whitney umbrella, which is known as the only stable singularity of surface to 3-dimensional Euclidean space. We obtain several criteria of the singularity types of fronts of Whitney umbrella in terms of differential geometric language we discuss.

Contents

1 Introduction

H. Whitney [28] has found Whitney umbrella (also known as the cross-cap) as singularities which are not avoidable by small perturbation. This is very important singularity type, since it is the only singularity of a map of surface to 3-dimensional Euclidean space which is stable under small deformations. This singularity is fundamental in the context of differential topology but it does not seems that Whitney umbrella is a subject of differential geometry,

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at least before C. Gutierrez and J. Sotomayor 's paper [10]. Motivated by Darboux's classification ([5]), they aimed to determine the configuration of lines of curvature near Whitney umbrella, and complete it in [9]. J. W. Bruce and J. M. West [3] investigated functions on Whitney umbrella using singularity theory. In [8], we show that a unified treatment for differential geometric properties for regular and singular maps $g : (\boldsymbol{R}^2, \mathbf{0}) \to (\boldsymbol{R}^3, \mathbf{0})$ and show that Whitney umbrella is not a bad singularity ([8, proposition 4.2]) from the view point of investigating singularities of distance squared functions. In other words, Whitney's umbrella is as good as (or as bad as) Darbouxian umbilics (i.e., both are characterised by the condition that $\operatorname{rank} R(g,0) = 4$ in the notation of [8]). Several authors continue to investigate the configuration of the solution curves of particular binary differential equations (i.e. lines of curvature, asymptotic and characteristic curves) in [20, 24]. In [19], the classification of parabolic lines of Whitney umbrella is used to investigate the projections of smooth surface in $\boldsymbol{R}^4$ to 3-spaces.

When we consider parallel surfaces of a regular surface, we are not able to avoid singularities. These singularities are often called front and this subject is investigated by M. Kokubu, W. Rossman, K. Saji, M. Umehara and K. Yamada [16]. They mean by a front a map $g : (\boldsymbol{R}^2, \mathbf{0}) \to \boldsymbol{R}^3$ such that there exists a well-defined normal $\mathbf{n} : (\boldsymbol{R}^2, \mathbf{0}) \to S^2 \subset \boldsymbol{R}^3$ so that $(g, \mathbf{n}) : (\boldsymbol{R}^2, \mathbf{0}) \to (\boldsymbol{R}^3, \mathbf{0}) \times S^2$ is an immersion. Cuspidal edges and swallowtails are typical singularity types of fronts. In [16], criteria for the these singularities are given. Furthermore, criteria for the cuspidal lips and the cuspidal beaks are given in [15], criterion for the cuspidal butterfly is given in [14], and criteria for the D_4-singularities are given in [23]. Whitney umbrella is not a front in their sense, since any unit normals defined at regular points near the singular point cannot extend continuously to the singular point.

Physically, the wave propagation is described by Huygens's principle: every point to which a luminous disturbance reached becomes a source of a spherical wave, and the sum of these secondary waves determines the form of the wave front at any subsequent time. We remark that this does not require the notion of unit normal vectors. Mathematically, a wave front is the envelope of the spherical waves, and this requires us to investigate the singularities of the members of the family of functions:

$$\Phi : (\boldsymbol{R}^2, \mathbf{0}) \times \boldsymbol{R}^3 \to \boldsymbol{R}, \qquad (u,v) \times (x,y,z) \mapsto -\frac{1}{2}(\|(x,y,z) - g(u,v)\|^2 - {t_0}^2) \tag{1.1}$$

where t_0 is a constant. The family Φ is an unfolding of the distance squared function $\varphi(u,v) = \Phi(u,v,x_0,y_0,z_0)$ where (x_0,y_0,z_0) is a point in $\boldsymbol{R}^3$, and the discriminant set $\mathcal{D}(\Phi)$ of Φ is a (wave) front of g at distance $|t_0|$ where

$$\mathcal{D}(\Phi) = \{(x,y,z) \in \boldsymbol{R}^3 \ ; \ \Phi = \Phi_u = \Phi_v = 0 \text{ for some } (u,v) \in (\boldsymbol{R}^2, \mathbf{0})\}.$$

For regular surfaces, we investigate the distance squared unfolding Φ and show several criteria for singularity types of parallel surfaces in terms of differential geometric language (principal curvatures, ridge points, sub-parabolic points, etc.) in [7]. In this paper, we investigate singularities of the distance squared unfolding for Whitney's umbrella, and show similar criteria for versality (Theorems 3.7). To investigate the singularities of the distance squared unfolding for Whitney's umbrella, we need several differential geometric languages of Whitney umbrella.

We also investigate the focal sets (caustics) of Whitney umbrella, since the bifurcation set $\mathcal{B}(\Phi)$ of Φ represents the focal set where

$$\mathcal{B}(\Phi) = \{(x, y, z) \in \boldsymbol{R}^3 \ ; \ \Phi_u = \Phi_v = \Phi_{uu}\Phi_{vv} - {\Phi_{uv}}^2 = 0 \text{ for some } (u, v) \in (\boldsymbol{R}^2, \mathbf{0})\}.$$

In Section 2, we introduce some differential geometric ingredients (principal curvatures, ridge, sub-parabolic points, etc.) for Whitney umbrella. For regular surfaces in Euclidean 3-space, several authors investigate ridge points and sub-parabolic points; see for example [2], [4], [18], [21], and [22]. The ridge points were first studied in details by I. Porteous [21] in terms of singularities of distance squared functions. The ridge line is the locus of points where one principal curvature has an extremal value along lines of the same principal curvature. The sub-parabolic points were studied in details by J. W. Bruce and T. C. Wilkinson [4] in terms of singularities of folding maps. The sub-parabolic line is the locus of points where one principal curvature has an extremal value along lines of the other principal curvature. Recently, in the case of the hyperbolic space, the analogous notion to the ridge point of hypersurfaces is introduced in [13], and the analogous notion to the sub-parabolic point of smooth surfaces is introduced in [12]. We develop the differential geometric ingredients over Whitney umbrella, which seem to be missing pieces of knowledge of the people who work on singularity theory and differential geometry. Since Whitney umbrella is a singularity of rank one, the tangent planes of nearby point degenerate to the tangent line at the singularity, and the normal lines are developed to the normal plane at the singular point. This means that we have a chance to have a bounded normal curvature in one direction at singular point.

In Section 2.1, we first show that for Whitney umbrella there is a well-defined unit normal via the double oriented blowing-up ([11, example (a) in p. 221]):

$$\tilde{\pi} : \boldsymbol{R} \times S^1 \to \boldsymbol{R}^2, \qquad (r, \theta) \mapsto (r\cos\theta, r\sin\theta). \tag{1.2}$$

Then we are able to talk about principal curvatures and principal directions via $\tilde{\pi}$, and we discuss their asymptotic behaviours in Section 2.3.

Let $\mathcal{M}$ denote the quotient space of $\boldsymbol{R} \times S^1$ with identification $(r, \theta) \sim (-r, \theta + \pi)$. Then we obtain a natural map

$$\pi : \mathcal{M} \to \boldsymbol{R}^2, \qquad [(r, \theta)] \mapsto (r\cos\theta, r\sin\theta), \tag{1.3}$$

which we usually call a blow up. We remark that $\mathcal{M}$ is topologically a Möbius strip. It is a natural problem to ask configurations of the parabolic line, ridge lines, sub-parabolic lines etc. on $\mathcal{M}$ near the exceptional set $X = \pi^{-1}(0, 0)$. We show

(1) The parabolic line intersects with X in at most two points (Proposition 2.3),

(2) Along an arc which reaches the singularity of Whitney umbrella, one principal curvature κ_1 is bounded if the arc is not tangent to the double point locus, and the other principal curvature κ_2 tends to infinity (Lemma 2.2),

(3) The ridge line with respect to κ_1 intersects with X in at most four points (Lemma 2.6, see Lemma 2.7 also) in generic context,

(4) The ridge line with respect to κ_2 intersects with X at two points (Proposition 2.10),

(5) The sub-parabolic line with respect to κ_1 intersects with X in at most three points (Lemma 2.11), and

(6) A constant principal curvature (CPC) line intersects with X in at most four points (Proposition 2.16).

In Section 3, we investigate singularities of the distance squared unfolding Φ defined by (1.1). We define the focal conic in the normal plane as a counterpart of focal points, and discuss versality of the unfolding Φ of φ, which is one of the fundamental notion in singularity theory. As a consequence, we are able to determine singularity types of caustics and fronts of Whitney umbrella in Section 4. We summarise our results as follows:

(1) If (x_0, y_0, z_0) is on the focal conic, then φ is at least A_2-singularity.

(2) If $(x_0, y_0, z_0) \neq g(0,0)$ does not correspond to the ridge over Whitney umbrella, then φ has an A_2-singularity and Φ is an $\mathcal{R}^+$-versal unfolding (and a $\mathcal{K}$-versal unfolding). The caustic is nonsingular at (x_0, y_0, z_0), and the front has the cuspidal edge at (x_0, y_0, z_0).

(3) If $(x_0, y_0, z_0) \neq g(0,0)$ corresponds to the first-order ridge over Whitney umbrella, then φ has an A_3-singularity and Φ is $\mathcal{R}^+$-versal. Thus the caustic has the cuspidal edge at (x_0, y_0, z_0). Moreover if (x_0, y_0, z_0) does not correspond to the sub-parabolic point over Whitney umbrella, then Φ is a $\mathcal{K}$-versal unfolding. We thus conclude that the front has the swallowtail at (x_0, y_0, z_0)

(4) If $(x_0, y_0, z_0) \neq g(0,0)$ corresponds to the first-order ridge and the sub-parabolic over Whitney umbrella, and the CPC line has definite (resp. indefinite) Morse singularity on X, then the front is the cuspidal lips (resp. cuspidal beaks) at (x_0, y_0, z_0).

(5) If $(x_0, y_0, z_0) \neq g(0,0)$ corresponds to the second-order ridge and does not correspond to the sub-parabolic point over Whitney umbrella, then the front is the cuspidal butterfly at (x_0, y_0, y_0).

See Theorem 3.7 and Theorem 4.3 for a precise statement. We remark that there are no D_4-singularities (or worse) for distance squared function φ at Whitney umbrella.

2 Differential geometry for Whitney umbrella

We consider a smooth map $g : U \to \boldsymbol{R}^3$ given by $g(u, v) = (g_1(u, v), g_2(u, v), g_3(u, v))$ which defines a surface in $\boldsymbol{R}^3$, where $U \subset \boldsymbol{R}^2$ is an open subset. The map g possibly has singularities. The map $g : (\boldsymbol{R}^2, \mathbf{0}) \to (\boldsymbol{R}^3, \mathbf{0})$ has a *Whiney umbrella* singularity at $(0, 0)$ if it is $\mathcal{A}$-equivalent to the map germ:

$$(\boldsymbol{R}^2, \mathbf{0}) \to (\boldsymbol{R}^3, \mathbf{0}), \quad (u, v) \mapsto (u, uv, v^2).$$

We remark that some authors distinguish between Whitney umbrellas and cross-caps as follows: the Whitney umbrella is the zero-set of the function $x^2 z - y^2 = 0$; the cross-cap is the image of the map that is $\mathcal{A}$-equivalent to $(u, v) \mapsto (u, uv, v^2)$ (see, for example, [3] and [24]). But the authors prefer to use the word "Whitney umbrella" for map germs with respect for H. Whitney's work.

Away from singularities a unit normal vector is defined by $\mathbf{n} = (g_u \times g_v)/\|g_u \times g_v\|$, and the first and second fundamental forms for g are given by

$$\mathrm{I} = E\,du^2 + 2F\,du\,dv + G\,dv^2, \quad \mathrm{II} = L\,du^2 + 2M\,du\,dv + N\,dv^2,$$

respectively, where

$$E = \langle g_u, g_u\rangle, \quad F = \langle g_u, g_v\rangle, \quad G = \langle g_v, g_v\rangle, \quad L = \langle g_{uu}, \mathbf{n}\rangle, \quad M = \langle g_{uv}, \mathbf{n}\rangle, \quad N = \langle g_{vv}, \mathbf{n}\rangle.$$

The principal curvatures κ_1 and κ_2 are the roots of the equation

$$\begin{vmatrix} L - \kappa_i E & M - \kappa_i F \\ M - \kappa_i F & N - \kappa_i G \end{vmatrix} = 0.$$

If a non-zero vector $\mathbf{v}_i = (\xi_i, \eta_i)$ $(i = 1, 2)$ is the principal vector with principal curvature κ_i, then

$$\begin{pmatrix} L - \kappa_i E & M - \kappa_i F \\ M - \kappa_i F & N - \kappa_i G \end{pmatrix} \begin{pmatrix} \xi_i \\ \eta_i \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.1}$$

We can choose (ξ_i, η_i) so that the tangent vector $\xi_i g_u + \eta_i g_v$ is of unit length.

We investigate the asymptotic behaviour of these ingredients near Whitney umbrella.

2.1 The unit normal vectors

Now we suppose that g has a rank one singularity at $(0, 0)$. Take the image of dg_0 to be the x-axis. Then we may write g as

$$g(u, v) = \left(u,\ \frac{1}{2}(a_{02}u^2 + 2a_{11}uv + a_{02}v^2) + O(u, v)^3,\ \frac{1}{2}(b_{20}u^2 + 2b_{11}uv + b_{02}v^2) + O(u, v)^3\right).$$

We consider the unit normal vector $\tilde{\mathbf{n}} = \mathbf{n} \circ \tilde{\pi}$ in the coordinates (r, θ), where $\tilde{\pi}$ is as in (1.2). By a straightforward calculation we show that the unit normal vector $\tilde{\mathbf{n}}$ is expressed as follows:

$$\tilde{\mathbf{n}}(r, \theta) = \frac{(0 + O(r),\ -b_{11}\cos\theta - b_{02}\sin\theta + O(r),\ a_{11}\cos\theta + a_{02}\sin\theta + O(r))}{\sqrt{({a_{11}}^2 + {b_{11}}^2)\cos^2\theta + 2(a_{11}a_{02} + b_{11}b_{02})\cos\theta\sin\theta + ({a_{02}}^2 + {b_{02}}^2)\sin^2\theta}}.$$

If the singular point of g is a Whitney umbrella, then

$$\begin{vmatrix} a_{11} & a_{02} \\ b_{11} & b_{02} \end{vmatrix} \neq 0.$$

We thus conclude the unit normal vector $\tilde{\mathbf{n}}$ is well-defined on $\{r = 0\}$, since

$$(a_{11}a_{02} + b_{11}b_{02})^2 - ({a_{11}}^2 + {b_{11}}^2)({a_{02}}^2 + {b_{02}}^2) = -(a_{11}b_{02} - a_{02}b_{11})^2.$$

2.2 Normal form of Whitney umbrella

For a regular surface, we can take the z-axis as the normal line, and, after suitable rotation if necessary, we can express the surface in the Monge normal form:

$$(u,v) \mapsto \big(u,\ v,\ \frac{1}{2}(k_1 u^2 + k_2 v^2) + O(u,v)^3\big)$$

For Whitney umbrella we can perform similar computations and obtain the following normal form theorem of Whitney umbrella.

Proposition 2.1. *Let $g : (\boldsymbol{R}^2, \mathbf{0}) \to (\boldsymbol{R}^3, \mathbf{0})$ be a smooth map with a Whitney umbrella at $(0,0)$. Then there are a rotation $T : \boldsymbol{R}^3 \to \boldsymbol{R}^3$ and a diffeomorphism $\phi : (\boldsymbol{R}^2, \mathbf{0}) \to (\boldsymbol{R}^2, \mathbf{0})$ so that*

$$T \circ g \circ \phi(u,v) = \left(u,\ uv + B(v) + O(u,v)^{k+1},\ \sum_{j=2}^{k} A_j(u,v) + O(u,v)^{k+1}\right) \quad (k \geq 3),$$

where

$$B(v) = \sum_{i=3}^{k} \frac{b_i}{i!} v^i, \quad \text{and} \quad A_j(u,v) = \sum_{i=0}^{j} \frac{a_{i,j-i}}{i!(j-i)!} u^i v^{j-i} \quad \text{with} \quad a_{02} \neq 0.$$

The result was first proved in [27], but we repeat the proof for completeness.

Proof. Take the image of dg_0 to be the x-axis. Then we may write g as

$$g(u,v) = \Big(u,\ \sum_{i+j=2} \frac{b^*_{ij}}{i!j!} u^i v^j + O(u,v)^3,\ \sum_{i+j=2} \frac{a^*_{ij}}{i!j!} u^i v^j + O(u,v)^3\Big) \quad \text{with} \quad \begin{vmatrix} b^*_{11} & b^*_{02} \\ a^*_{11} & a^*_{02} \end{vmatrix} \neq 0.$$

Take θ so that $(\cos\theta, \sin\theta) = (a^*_{02}, b^*_{02})/\sqrt{{a^*_{02}}^2 + {b^*_{02}}^2}$, and set a rotation of $\boldsymbol{R}^3$

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

and a change of coordinates

$$\psi(u,v) = \left(u,\ \frac{1}{\begin{vmatrix} b^*_{11} & b^*_{02} \\ a^*_{11} & a^*_{02} \end{vmatrix}} \left(-\frac{1}{2} \begin{vmatrix} b^*_{20} & b^*_{02} \\ a^*_{20} & a^*_{02} \end{vmatrix} u + \sqrt{{a^*_{02}}^2 + {b^*_{02}}^2}\, v\right)\right).$$

Then we have

$$T \circ g \circ \psi(u,v) = \left(u, uv + \sum_{i+j=3} \frac{\beta_{ij}}{i!j!} u^i v^j + O(u,v)^4,\ \sum_{i+j=2} \frac{\alpha_{ij}}{i!j!} u^i v^j + O(u,v)^3\right)$$

for some constants α_{ij} and β_{ij}. Setting $B_k = \sum_{i+j=k} b_{ij} u^i v^j/(i!j!)$ $(k \geq 3)$ and replacing v by $v + \sum_{i+j=k-1} c_{ij} u^i v^j/(i!j!)$, we have

$$uv + B_k + O(u,v)^{k+1} = uv + \sum_{i+j=k} \frac{ic_{i-1,j} + b_{ij}}{i!j!} u^i v^j + O(u,v)^{k+1}.$$

For a suitable choice of c_{ij} $(i+j=k-1)$, we can reduce this to $b_{(0,k)} v^k/k! + O(u,v)^{k+1}$. Hence, we obtain the result. □

2.3 Principal curvatures and principal directions

Coefficients of the first and second fundamental forms. Throughout the rest of the paper, we suppose that g is given in the normal form of Whitney umbrella:

$$g(u,v) = \left(u,\ uv + B(v) + O(u,v)^5,\ \sum_{j=2}^{4} A_j(u,v) + O(u,v)^5\right), \tag{2.2}$$

where

$$B(v) = \sum_{i=3}^{4} \frac{b_i}{i!} v^i, \quad \text{and} \quad A_j(u,v) = \sum_{i=0}^{j} \frac{a_{i,j-i}}{i!(j-i)!} u^i v^{j-i} \quad \text{with} \quad a_{02} \neq 0.$$

Then we have

$$g_u = \left(1,\ v + O(u,v)^4,\ \sum_{j=2}^{4} (A_j)_u + O(u,v)^4\right),$$
$$g_v = \left(0,\ u + B_v + O(u,v)^4,\ \sum_{j=2}^{4} (A_j)_v + O(u,v)^4\right)$$

and thus have

$$\begin{aligned} E &= 1 + v^2 + (A_{2u})^2 + 2A_{3u}A_{2u} + O(u,v)^4, \\ F &= uv + A_{2u}A_{2v} + A_{3u}A_{2v} + A_{3v}A_{2u} + \frac{1}{2}b_3 v^3 + O(u,v)^4, \\ G &= u^2 + (A_{2v})^2 + 2A_{3v}A_{2v} + b_3 uv^2 + O(u,v)^4. \end{aligned} \tag{2.3}$$

Since

$$g_u \times g_v = \left(\sum_{j=2}^{3} \left(v\,(A_j)_v - u\,(A_j)_u\right) - \frac{b_3}{2} v^2 A_{2u} + O(u,v)^4,\right.$$
$$\left. -\sum_{j=2}^{4} (A_j)_v + O(u,v)^4,\ u + \frac{1}{2}b_3 v^2 + \frac{1}{6} b_4 v^3 + O(u,v)^4\right),$$

we have

$$\|g_u \times g_v\|^2 = \lambda_2 + \lambda_3 + \lambda_4 + O(u,v)^5,$$

where

$$\begin{aligned} &\lambda_2 = u^2 + A_{2v}^2, \qquad \lambda_3 = 2A_{3v}A_{2v} + b_3uv^2, \\ &\lambda_4 = 2A_{4v}A_{2v} + A_{3v}^2 + (uA_{2u} - vA_{2v})^2 + \frac{1}{3}b_4uv^3 + \frac{1}{4}{b_3}^2v^4. \end{aligned} \tag{2.4}$$

It follows that

$$\tilde{\mathbf{n}}(0,\theta) = \frac{1}{\mathcal{A}}(0,\ -a_{11}\cos\theta - a_{02}\sin\theta,\ \cos\theta), \tag{2.5}$$

where $\mathcal{A} = \mathcal{A}(\theta) = \sqrt{\cos^2\theta + (a_{11}\cos\theta + a_{02}\sin\theta)^2}$. Since $a_{02} \neq 0$, $\tilde{\mathbf{n}}(0,\theta)$ defines an isomorphism from real projective line $P^1(\boldsymbol{R})$

$$P^1(\boldsymbol{R}) \to P^1(\boldsymbol{R}), \qquad \theta \mapsto \tilde{\mathbf{n}}(0,\theta).$$

We set

$$l = \langle g_{uu}, g_u \times g_v \rangle, \qquad m = \langle g_{uv}, g_u \times g_v \rangle, \qquad n = \langle g_{vv}, g_u \times g_v \rangle.$$

Since

$$\begin{aligned} g_{uu} &= \left(0,\ 0,\ \sum_{j=2}^{4} (A_j)_{uu} + O(u,v)^3\right), \\ g_{uv} &= \left(0,\ 1 + O(u,v)^3,\ \sum_{j=2}^{4} (A_j)_{uv} + O(u,v)^3\right), \\ g_{vv} &= \left(0,\ B_{vv} + O(u,v)^3, \sum_{j=2}^{4} (A_j)_{vv} + O(u,v)^3\right), \end{aligned}$$

l, m, and n are expressed as follows:

$$l = l_1 + l_2 + l_3 + O(u,v)^4, \quad m = m_1 + m_2 + m_3 + O(u,v)^4, \quad n = n_1 + n_2 + n_3 + O(u,v)^4,$$

where

$$\begin{aligned} &l_1 = a_{20}u, \quad l_2 = uA_{3uu} + \frac{1}{2}a_{20}b_3v^2, \quad l_3 = uA_{4uu} + \frac{1}{2}b_3A_{3uu}v^2 + \frac{1}{6}a_{20}b_4v^3 \\ &m_1 = -a_{02}v, \quad m_2 = uA_{3uv} - A_{3v} + \frac{1}{2}a_{11}b_3v^2, \\ &m_3 = uA_{4uv} - A_{4uv} + \frac{1}{2}b_3A_{3uv} + \frac{1}{6}a_{11}b_4v^3 \\ &n_1 = a_{02}u, \quad n_2 = uA_{3vv} + \frac{1}{2}b_3(a_{02}v^2 - 2vA_{2v}), \\ &n_3 = uA_{4vv} + \frac{1}{2}b_3v(vA_{3vv} - 2A_{3v}) + \frac{1}{6}b_4v^2(a_{02}v - 3A_{2v}). \end{aligned} \tag{2.6}$$

By using the Taylor series for $L(ru, rv)$, $M(ru, rv)$, and $N(ru, rv)$ in r we obtain

$$
\begin{aligned}
L &= \frac{a_{20}u}{\sqrt{\lambda_2}} + \frac{2l_2\lambda_2 - l_1\lambda_3}{2\lambda_2^{3/2}}r + \frac{8l_3{\lambda_2}^2 - 4l_2\lambda_2\lambda_3 - 4l_1\lambda_2\lambda_4 + l_1\lambda_3}{8\lambda_2^{5/2}}r^2 + O(r^3),\\
M &= \frac{-a_{02}v}{\sqrt{\lambda_2}} + \frac{2m_2\lambda_2 - m_1\lambda_3}{2\lambda_2^{3/2}}r + \frac{8m_3{\lambda_2}^2 - 4m_2\lambda_2\lambda_3 - 4m_1\lambda_2\lambda_4 + m_1\lambda_3}{8\lambda_2^{5/2}}r^2 + O(r^3),\\
N &= \frac{a_{02}u}{\sqrt{\lambda_2}} + \frac{2n_2\lambda_2 - n_1\lambda_3}{2\lambda_2^{3/2}}r + \frac{8n_3{\lambda_2}^2 - 4n_2\lambda_2\lambda_3 - 4n_1\lambda_2\lambda_4 + n_1\lambda_3}{8\lambda_2^{5/2}}r^2 + O(r^3).
\end{aligned}
\tag{2.7}
$$

Principal curvatures.

Lemma 2.2. *The principal curvatures $\tilde{\kappa}_i = \kappa_i \circ \tilde{\pi}$ are expressed as follows:*

$$
\begin{aligned}
\tilde{\kappa}_1(r,\theta) &= k_{10}(\theta) + k_{11}(\theta)r + k_{12}(\theta)r^2 + O(r^3),\\
\tilde{\kappa}_2(r,\theta) &= \frac{1}{r^2}\left[k_{20}(\theta) + k_{21}(\theta)r + O(r^2)\right],
\end{aligned}
\tag{2.8}
$$

where

$$
k_{10} = \frac{A_2^*\sec\theta}{\mathcal{A}}, \tag{2.9}
$$

$$
k_{11} = \frac{1}{\mathcal{A}^3}\begin{pmatrix} 6a_{02}\tilde{A}_3\tilde{A}_{2v}\tan\theta + 2\tilde{A}_{3u}(a_{11}\tilde{A}_{2v} + \cos\theta)\cos\theta \\ -\tilde{A}_{3v}\tilde{A}_{2v}(a_{20}\cos^2\theta + a_{02}\sin^2\theta)\sec\theta \end{pmatrix} + \frac{b_3A_2^*\tilde{A}_{2v}^2\tan^2\theta}{2\mathcal{A}^3}, \tag{2.10}
$$

$$
\begin{aligned}
k_{12} = \frac{1}{2a_{02}\mathcal{A}^5}\Big[&24a_{02}\mathcal{A}^4\tilde{A}_4\sec\theta - 12a_{02}\mathcal{A}^2\tilde{A}_3\tilde{A}_{3v}\tilde{A}_{2v}\sec\theta\\
&- 2a_{02}\mathcal{A}^2\tilde{A}_{4v}(\tilde{A}_{2u}\tilde{A}_{2v} + \mathcal{A}^2\tan\theta + \cos\theta\sin\theta)\\
&- \tilde{A}_{3v}^2\begin{pmatrix} 2\tilde{A}_{2v}^2({a_{11}}^2 - a_{20}a_{02}) + 4a_{11}\tilde{A}_{2v}\cos\theta \\ +(a_{20}a_{02} + 2)\cos^2\theta - a_{02}\sin^2\theta \end{pmatrix}\cos\theta\\
&- 8a_{02}\mathcal{A}^4A_2^*\tilde{A}_2^2\sec^3\theta - 8a_{02}\mathcal{A}^4A_2^*\sin\theta\tan\theta - a_{02}\mathcal{A}^2{A_2^*}^3\sec\theta\Big],
\end{aligned}
\tag{2.11}
$$

$$
k_{20} = \frac{a_{02}\cos\theta}{\mathcal{A}^3}, \tag{2.12}
$$

$$
\begin{aligned}
k_{21} = &\frac{1}{\mathcal{A}^5}\left[(-3a_{02}\tilde{A}_{3v}\tilde{A}_{2v} + \mathcal{A}^2\tilde{A}_{3vv})\cos\theta\right]\\
&+ \frac{1}{2\mathcal{A}^5}\left[b_3(-2\mathcal{A}^2\tilde{A}_{2v} + a_{02}\mathcal{A}^2\sin\theta - 3a_{02}\cos^2\theta\sin\theta)\sin\theta\right].
\end{aligned}
\tag{2.13}
$$

Here $A_2^* = a_{20}\cos^2\theta - a_{02}\sin^2\theta$, $\tilde{A}_{2v} = A_{2v}\circ\tilde{\pi}\mid_{r=1}$, $\tilde{A}_{3v} = A_{3v}\circ\tilde{\pi}\mid_{r=1}$, and so on.

Proof. The principal curvatures κ_i are the roots of the equation

$$
(EG - F^2)k^2 - (EN - 2FM + GL)k + (LN - M^2) = 0.
$$

From (2.3), (2.4), (2.6), and (2.7), it follows that

$$
(EG - F^2)\circ\tilde{\pi} = a_2r^2 + a_3r^3 + a_4r^4 + O(r^5),
$$

$$
\begin{aligned}
-(EN-2FM+GL)\circ\tilde{\pi} &= b_0+b_1r+b_2r^2+O(r^3),\\
(LN-M^2)\circ\tilde{\pi} &= c_0+c_1r+c_2r^2+O(r^3),
\end{aligned}
$$

where

$$
a_2=\mathcal{A}^2,\quad b_0=-\frac{a_{02}\cos\theta}{\mathcal{A}},\quad c_0=\frac{a_{02}A_2^*}{\mathcal{A}^2},
$$

and the coefficients a_3, a_4, b_1, b_2, c_1, and c_2 are the trigonometric polynomials in the coefficients appearing in the terms of degree four or less in the normal form of Whitney umbrella. Therefore, we obtain

$$
\begin{aligned}
\tilde{\kappa}_1 &= -\frac{c_0}{b_0}+\frac{b_1c_0-b_0c_1}{{b_0}^2}r+\frac{{b_1}^2c_0-b_0b_2c_0+a_2{c_0}^2-b_0b_1c_1+{b_0}^2c_2}{{b_0}^3}r^2+O(r^3),\\
\tilde{\kappa}_2 &= \frac{1}{r^2}\Big(-\frac{b_0}{a_2}+\frac{a_3b_0-a_2b_1}{{a_2}^3}r+O(r^2)\Big).
\end{aligned}
$$

We thus obtain (2.9)–(2.13) by a straightforward calculation. □

Gaussian curvature. Since the Gaussian curvature K is the product of the principal curvatures, it does not depend on the choice of the unit normal vector. From (2.9) and (2.12), the Gaussian curvature is expressed as follows:

$$
\tilde{K}(r,\theta)=K\circ\pi(r,\theta)=\frac{1}{r^2}\left(\frac{a_{02}(a_{20}\cos^2\theta-a_{02}\sin^2\theta)}{{\mathcal{A}(\theta)}^4}+O(r)\right),
$$

where π is as in (1.3). By this expression, we say that a point $(0,\theta_0)$ on the Möbius strip $\mathcal{M}$ is *elliptic*, *hyperbolic*, or *parabolic point over Whitney umbrella* if $r^2\tilde{K}(0,\theta_0)$ is positive, negative, or zero, respectively. We often omit the phrase "over Whitney umbrella" if no confusion is possible from the context.

We immediately have the following proposition.

Proposition 2.3. (1) *There is no parabolic point over Whitney umbrella if and only if* $a_{20}a_{02}<0$.

(2) *There is one parabolic point over Whitney umbrella if and only if* $a_{20}=0$.

(3) *There are two parabolic points over Whitney umbrella if and only if* $a_{20}a_{02}>0$.

Furthermore, in terms of the parabolic line in a domain, Whitney umbrella is classified into three types. We say that Whitney umbrella is *hyperbolic*, *elliptic*, or *parabolic* if the parabolic line has an A_1^+-singularity (an isolated point), A_1^--singularity (a pair of smooth curves intersecting transversally), or A_2-singularity (a cusp), respectively. In the case of the hyperbolic Whitney umbrella all non-singular points near the Whitney umbrella singularity are hyperbolic. In the case of the elliptic Whitney umbrella the parabolic line divides the surface into hyperbolic and elliptic regions (see [27] for details).

From (2.6), the parabolic line in the Möbius strip $\mathcal{M}$ is expressed by the equation

$$
a_{02}A_2^*+\big[(a_{20}\tilde{A}_{3vv}+a_{02}\tilde{A}_{3uu})\cos^2\theta+2a_{02}(\tilde{A}_{3uv}\cos\theta-\tilde{A}_{3v})\sin\theta-a_{11}b_3A_2^*\sin\theta\big]r+O(r^2)=0.
$$

If $a_{20}a_{02} < 0$, then the parabolic line dose not meet with the exceptional set $X = \pi^{-1}(0,0)$ on $\mathcal{M}$, in which case the surface is the hyperbolic Whitney umbrella. If $a_{20}a_{02} > 0$, then the parabolic line meets with X at two parabolic point over Whitney umbrella, in which case the surface is the elliptic Whitney umbrella. If $a_{20} = 0$, then the parabolic line meets with X at one parabolic point $(r,\theta) = (0,0)$, which is the sub-parabolic point over Whitney umbrella (See Section 2.4). Calculating the tangent vector of the parabolic line at $(0,0)$ on $\mathcal{M}$, we show that the parabolic line meets tangentially with X at $(0,0)$ if and only if $a_{30} \neq 0$. In this case, we have the parabolic Whitney umbrella (This classification according to the coefficients of the normal form of Whitney umbrella is also obtained in [20]). Moreover, the parabolic point $(0,0)$ is the singular point of the parabolic line if and only if $a_{30} = 0$, equivalently this point is the ridge point over Whitney umbrella (See Section 2.4). In this case, this point is of Morse type if and only if $3{a_{21}}^2 + 2a_{40}a_{02} \neq 0$.

Principal directions.

Lemma 2.4. *The unit principal vectors $\tilde{\mathbf{v}}_i$ in the coordinates (r,θ) are expressed as follows:*

$$\tilde{\mathbf{v}}_1 = (\sec\theta + O(r))\frac{\partial}{\partial r} + \left(\frac{-2\tilde{A}_{3v} + b_3\tilde{A}_{2v}\sin\theta\tan\theta}{2a_{02}} + O(r)\right)\frac{\partial}{\partial\theta},$$
$$\tilde{\mathbf{v}}_2 = \frac{1}{r^2}\left[\left(\frac{\sin\theta}{\mathcal{A}}r + O(r^2)\right)\frac{\partial}{\partial r} + \left(\frac{\cos\theta}{\mathcal{A}} + O(r)\right)\frac{\partial}{\partial\theta}\right].$$

Proof. From the equation (2.1), one of the vectors along the principal vectors $\mathbf{v}_i$ in the coordinates (u,v) are given by

$$\xi_i\frac{\partial}{\partial u} + \eta_i\frac{\partial}{\partial v} = (N - \kappa_i G)\frac{\partial}{\partial u} + (-M + \kappa_i F)\frac{\partial}{\partial v}. \tag{2.14}$$

Since

$$\frac{\partial}{\partial u} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta} \quad \text{and} \quad \frac{\partial}{\partial v} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta},$$

the vector (2.14) can be lifted by π and we obtain

$$\tilde{\xi}_i\frac{\partial}{\partial r} + \tilde{\eta}_i\frac{\partial}{\partial\theta} = \Big[(N\circ\tilde{\pi} - \tilde{\kappa}_i G\circ\tilde{\pi})\cos\theta + (\tilde{\kappa}_i F\circ\tilde{\pi} - M\circ\tilde{\pi})\sin\theta\Big]\frac{\partial}{\partial r}$$
$$+\frac{1}{r}\Big[(\tilde{\kappa}_i F\circ\tilde{\pi} - M\circ\tilde{\pi})\cos\theta + (\tilde{\kappa}_i G\circ\tilde{\pi} - N\circ\tilde{\pi})\sin\theta\Big]\frac{\partial}{\partial\theta}.$$

From (2.3), (2.4), (2.6), and (2.7), we have

$$F\circ\tilde{\pi} = F_2 r^2 + F_3 r^3 + O(r^4), \qquad G\circ\tilde{\pi} = G_2 r^2 + G_3 r^3 O(r^4),$$
$$M\circ\tilde{\pi} = M_0 + M_1 r + M_2 r^2 + O(r^3), \qquad N\circ\tilde{\pi} = N_0 + N_1 r + N_2 r^2 + O(r^3),$$

where

$$F_2 = \cos\theta\sin\theta + \tilde{A}_{2u}\tilde{A}_{2v}, \quad F_3 = \tilde{A}_{3u}\tilde{A}_{2v} + \tilde{A}_{3v}\tilde{A}_{2u} + \frac{1}{2}b_3\sin^3\theta,$$
$$G_2 = \cos^2\theta + \tilde{A}_{2v}^2, \quad G_3 = 2\tilde{A}_{3v}\tilde{A}_{2v} + b_3\cos\theta\sin^2\theta,$$
$$M_0 = -\frac{a_{02}\sin\theta}{\mathcal{A}}, \quad N_0 = \frac{a_{02}\cos\theta}{\mathcal{A}},$$

and coefficients L_1, M_1, N_1, L_2, M_2, and N_2 are the trigonometric polynomials in the coefficients appearing in the terms of degree four or less in the normal form of Whitney umbrella. It follows that $\tilde{\xi}_i$ and $\tilde{\eta}_i$ are expressed as follows:

$$\begin{aligned}
\tilde{\xi}_1 &= N_0\cos\theta - M_0\sin\theta + (N_1\cos\theta - M_1\sin\theta)r + O(r^2),\\
\tilde{\eta}_1 &= -M_1\cos\theta - N_1\sin\theta + [(F_2k_{10} - M_2)\cos\theta + (G_2k_{10} - N_2)\sin\theta]\,r + O(r^2),\\
\tilde{\xi}_2 &= \big(-G_2k_{20} + N_0\big)\cos\theta + \big(F_2k_{20} - M_0\big)\sin\theta\\
&\quad + [(-G_3k_{20} - G_2k_{21} + N_1)\cos\theta + (F_3k_{20} + F_2k_{21} - M_1)\sin\theta]\,r + O(r^2),\\
\tilde{\eta}_2 &= \frac{1}{r}\,[k_{20}(F_2\cos\theta + G_2\sin\theta)\\
&\quad + ((F_3k_{20} + F_2k_{21} - M_1)\cos\theta + (G_3k_{20} + G_2k_{21} - N_1)\sin\theta)r + O(r^2)\big].
\end{aligned}$$

After a long calculation, it follows that $\tilde{\xi}_i$ and $\tilde{\eta}_i$ are expressed as follows:

$$\begin{aligned}
\tilde{\xi}_1 &= \tilde{\xi}_{10} + \tilde{\xi}_{11}r + O(r^2), & \tilde{\eta}_1 &= \tilde{\eta}_{10} + \tilde{\eta}_{11}r + O(r^2),\\
\tilde{\xi}_2 &= \tilde{\xi}_{20} + \tilde{\xi}_{21}r + O(r^2), & \tilde{\eta}_2 &= \frac{1}{r}\left[\tilde{\eta}_{20} + \tilde{\eta}_{21}r + O(r^2)\right],
\end{aligned}$$

where

$$\begin{aligned}
\tilde{\xi}_{10} &= \frac{a_{02}}{\mathcal{A}}, \quad \tilde{\eta}_{10} = \frac{1}{2\mathcal{A}}\big(-2\tilde{A}_{3v}\cos\theta + b_3\tilde{A}_{2v}\sin^2\theta\big),\\
\tilde{\xi}_{11} &= \frac{1}{2\mathcal{A}^3}\Big[2\big(\tilde{A}_{3v}(a_{02}\tilde{A}_{2v}\cos\theta - a_{11}\tilde{A}_{2v}\sin\theta - \cos\theta\sin\theta)\\
&\quad - a_{02}\tilde{A}_{3uv}\tilde{A}_{2v} + \tilde{A}_{3vv}(a_{11}\tilde{A}_{2v} + \cos\theta)\big)\\
&\quad - b_3\tilde{A}_{2v}(a_{11}\tilde{A}_{2v} + \mathcal{A}^2\cos^2\theta + \cos\theta)\sin\theta\Big],\\
\tilde{\eta}_{11} &= \frac{1}{12\mathcal{A}^3}\Big[-24\mathcal{A}^2\tilde{A}_{4v}\cos\theta + 12\tilde{A}_{3v}^2\tilde{A}_{2v}\cos\theta + 24\tilde{A}_2\tilde{A}_{2v}A_2^*\cos\theta\\
&\quad + 12a_{02}\tilde{A}_{2u}^2(3a_{11}\tilde{A}_{2v}\cos\theta + {a_{02}}^2\sin^2\theta)\cos\theta\sin\theta - 24a_{11}a_{02}\tilde{A}_{2v}^3\sin^3\theta\\
&\quad + 12{a_{02}}^3\tilde{A}_{2u}^2\cos\theta\sin^3\theta + 12a_{20}{a_{11}}^3\tilde{A}_{2u}\cos^5\theta + 60{a_{11}}^2{a_{02}}^2\tilde{A}_{2v}\cos\theta\sin^4\theta\\
&\quad + 24\mathcal{A}^2A_2^*\cos\theta + 12a_{20}{a_{11}}^4\cos^5\theta - {a_{02}}^5\sin^5\theta\tan\theta + 4b_4\mathcal{A}^2\tilde{A}_{2v}\sin^3\theta\\
&\quad -3{b_3}^2\tilde{A}_{2v}\cos\theta\sin^4\theta - 6b_3\tilde{A}_{3v}(\tilde{A}_{2v}^2 - \cos^2\theta)\sin^2\theta\Big],\\
\tilde{\xi}_{20} &= \frac{2a_{02}}{\mathcal{A}^3}(\tilde{A}_2\tilde{A}_{2v} + \cos^2\theta\sin\theta)\sin\theta, \quad \tilde{\eta}_{20} = \tilde{\xi}_{20}\cot\theta, \quad \tilde{\eta}_{21} = \tilde{\xi}_{21}\cot\theta,\\
\tilde{\xi}_{21} &= \frac{1}{\mathcal{A}^5}\Big[3a_{11}a_{02}\mathcal{A}^2\tilde{A}_3 + {a_{02}}^2\mathcal{A}^2\tilde{A}_{3u}\sin\theta\\
&\quad + \tilde{A}_{3v}(2a_{02}\tilde{A}_2\tilde{A}_{2v}^2 - a_{11}\mathcal{A}^2 - \mathcal{A}^2 - 8a_{11}a_{02}\cos\theta\sin\theta)\cos\theta\\
&\quad - a_{02}\tilde{A}_{3uv}(2\tilde{A}_2\tilde{A}_{2v}^2 + a_{11}\tilde{A}_{2v}^2\cos\theta\sin\theta + 3a_{02}\cos^2\theta\sin^2\theta)\\
&\quad + \tilde{A}_{3vv}(2\tilde{A}_2\tilde{A}_{2v} + a_{11}\tilde{A}_{2u}\tilde{A}_{2v}^2 + a_{11}\tilde{A}_{2v}^2\sin\theta + 2\mathcal{A}^2\sin\theta - 3{a_{02}}^2\sin^3\theta)\Big]\cos\theta\sin\theta\\
&\quad - \frac{b_3}{2\mathcal{A}^5}\Big[2\tilde{A}_2\tilde{A}_{2v}(a_{11}\tilde{A}_{2v}^2 + \tilde{A}_{2v}\cos\theta - 2a_{11}\cos^2\theta) + 3\tilde{A}_{2u}\tilde{A}_{2v}^2\cos^2\theta
\end{aligned}$$

$$+\, 5a_{11}\tilde{A}_{2v}^2 \cos^2\theta \sin\theta + 3\tilde{A}_{2v}\cos^3\theta\sin\theta + a_{02}\cos^3\theta\sin^2\theta\Big]\sin^2\theta.$$

The unit principal vectors $\tilde{\mathbf{v}}_i$ are given by

$$\tilde{\mathbf{v}}_i = \frac{1}{\sqrt{\tilde{E}\tilde{\xi}_i^2 + 2\tilde{F}\tilde{\xi}_i\tilde{\eta}_i + \tilde{G}\tilde{\eta}_i^2}} \left(\tilde{\xi}_i \frac{\partial}{\partial r} + \tilde{\eta}_i \frac{\partial}{\partial \theta}\right), \tag{2.15}$$

where $\tilde{E}$, $\tilde{F}$, and $\tilde{G}$ are the coefficients of the first fundamental form of g in the coordinates (r,θ). We calculate that

$$\begin{aligned}
\tilde{E} &= \cos^2\theta + 4\left(\cos^2\theta\sin^2\theta + \tilde{A}_2^2\right) r^2 + O(r^3),\\
\tilde{F} &= -r\cos\theta\sin\theta\\
&\qquad + 2\left[(\cos^2\theta - \sin^2\theta)\cos\theta\sin\theta + \tilde{A}_2\big(\tilde{A}_{2v}\cos\theta - \tilde{A}_{2u}\sin\theta\big)\right] r^3 + O(r^4),\\
\tilde{G} &= r^2\sin^2\theta + \left[(\cos^2\theta - \sin^2\theta)^2 + \big(\tilde{A}_{2v}\cos\theta - \tilde{A}_{2u}\sin\theta\big)^2\right] r^4 + O(r^5).
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
\tilde{E}\tilde{\xi}_1^2 + 2\tilde{F}\tilde{\xi}_1\tilde{\eta}_1 + \tilde{G}\tilde{\eta}_1^2 &= \frac{{a_{02}}^2\cos^2\theta}{\mathcal{A}^2} + O(r),\\
\tilde{E}\tilde{\xi}_2^2 + 2\tilde{F}\tilde{\xi}_2\tilde{\eta}_2 + \tilde{G}\tilde{\eta}_2^2 &= \frac{4{a_{02}}^2(\tilde{A}_2\tilde{A}_{2v} + \cos^2\theta\sin\theta)^2}{\mathcal{A}^4} r^2 + O(r^3).
\end{aligned}$$

This completes proof together with (2.15). □

Remark 2.5. The (unit) principal vector $\tilde{\mathbf{v}}_1$ is extendible on $\{(r,\theta)\,;\, r\neq 0 \text{ or } \cos\theta\neq 0\}$ and thus the principal field defined by $\mathbf{v}_1$ is extendible on the Möbius strip $\mathcal{M}$ except on the set $\{(r,\theta)\,;\, r=0,\, \cos\theta=0\}$. The principal curvature vector $r^2\tilde{\mathbf{v}}_2$ is extendible over $\boldsymbol{R}\times S^1$ even though $\tilde{\mathbf{v}}_2$ is not. So the principal field defined by $\mathbf{v}_i$ is extendible over $\mathcal{M}$.

2.4 Ridge points and sub-parabolic points over Whitney umbrella

Ridge points. By the computation in the previous subsection, we can express $\tilde{\mathbf{v}}_i\tilde{\kappa}_i$ as follows:

$$\tilde{\mathbf{v}}_1\tilde{\kappa}_1(r,\theta) = R_{110}(\theta) + R_{111}(\theta)r + \cdots, \qquad \tilde{\mathbf{v}}_2\tilde{\kappa}_2(r,\theta) = \frac{1}{r^4}(R_{210}(\theta) + R_{211}(\theta)r + \cdots,$$

where $\tilde{\mathbf{v}}_i\tilde{\kappa}_i$ denotes the directional derivative of the principal curvature $\tilde{\kappa}_i$ in the principal vector $\tilde{\mathbf{v}}_i$. We say that a point (r_0,θ_0) is a *ridge point relative to the principal vector* $\tilde{\mathbf{v}}_1$ (resp. $\tilde{\mathbf{v}}_2$) if $\tilde{\mathbf{v}}_1\tilde{\kappa}_1(r_0,\theta_0)=0$ (resp. $r^4\tilde{\mathbf{v}}_2\tilde{\kappa}_2(r_0,\theta_0)=0$). If the ridge point (r_0,θ_0) is over Whitney umbrella (that is, $r_0=0$) this is equivalent that $R_{110}(\theta_0)=0$ (resp. $R_{210}(\theta_0)=0$). It is possible that $R_{i10}(\theta)$ has multiple roots. We say that $(0,\theta_0)$ is a *multiple ridge point relative to* $\tilde{\mathbf{v}}_i$ if θ_0 is a multiple root of $R_{i10}(\theta)$. We say that a point $(0,\theta_0)$ is a *n-th order ridge point relative to* $\tilde{\mathbf{v}}_1$ *(resp.* $\tilde{\mathbf{v}}_2$*) over Whitney umbrella* if $R_{im0}(\theta_0)=0$ $(1\leqq m\leqq n)$ and $R_{i,n+1,0}(\theta_0)\neq 0$, where

$$\tilde{\mathbf{v}}_1^{(m)}\tilde{\kappa}_1(r,\theta) = R_{1m0}(\theta) + R_{1m1}(\theta)r + \cdots, \quad \tilde{\mathbf{v}}_2^{(m)}\tilde{\kappa}_2(r,\theta) = \frac{1}{r^{2+2m}}(R_{2m0}(\theta) + R_{2m1}(\theta)r + \cdots.$$

Here, $\tilde{\mathbf{v}}_i^{(n)}\tilde{\kappa}_i$ denotes the n-th time directional derivative of $\tilde{\kappa}_i$ in the direction $\tilde{\mathbf{v}}_i$. The ridge line relative to $\tilde{\mathbf{v}}_i$ near the exceptional set $X = \pi^{-1}(0,0)$ is expressed by the equation:

$$R_{i10}(\theta) + R_{i11}(\theta)r + \cdots = 0.$$

In terms of the normal form of Whitney umbrella, we have $\tilde{\kappa}_1(0,\theta)$ tends to infinity as θ approaches $\pm\pi/2$, by (2.9), and, after some calculations, we obtain

$$\tilde{\mathbf{v}}_1\tilde{\kappa}_1(r,\theta) = \frac{\Gamma_3(\theta)\sec^3\theta}{\mathcal{A}(\theta)} + O(r), \tag{2.16}$$

$$\tilde{\mathbf{v}}_2\tilde{\kappa}_1(r,\theta) = \frac{1}{r^2}\left(-\frac{a_{02}\Gamma_3^*(\theta)\sec\theta}{\mathcal{A}(\theta)^4} + O(r)\right), \tag{2.17}$$

$$\tilde{\mathbf{v}}_2\tilde{\kappa}_2(r,\theta) = \frac{1}{r^4}\left(-\frac{3{a_{02}}^2(a_{11}\cos\theta + a_{02}\sin\theta)\cos\theta}{\mathcal{A}(\theta)^6} + O(r)\right), \tag{2.18}$$

$$\tilde{\mathbf{v}}_1\tilde{\kappa}_2(r,\theta) = \frac{1}{r^3}\left(-\frac{2a_{02}}{\mathcal{A}(\theta)^3} + O(r)\right), \tag{2.19}$$

where

$$\Gamma_3(\theta) = 6\tilde{A}_3\cos\theta - b_3\tilde{A}_{2v}\sin^3\theta, \quad \Gamma_3^*(\theta) = 2\tilde{A}_2\tilde{A}_{2v} + 2\cos^2\theta\sin\theta.$$

Lemma 2.6. *A point $(0,\theta_0)$ with $\cos\theta_0 \neq 0$ is a ridge point relative to $\tilde{\mathbf{v}}_1$ if and only if $\Gamma_3(\theta_0) = 0$. Moreover, the point $(0,\theta_0)$ is a first order ridge point relative to $\tilde{\mathbf{v}}_1$ if and only if $\Gamma_4(\theta_0) \neq 0$, where*

$$\begin{aligned}\Gamma_4(\theta) = {} & 24a_{02}\tilde{A}_4\cos^2\theta - 12\tilde{A}_{3v}^2\cos^2\theta - 12a_{02}A_2^*\tilde{A}_2^2 - 12a_{02}A_2^*\cos^2\theta\sin^2\theta \\ & - a_{02}b_4\tilde{A}_{2v}\cos\theta\sin^4\theta + 12{b_3}^2\tilde{A}_{3v}\tilde{A}_{2v}\cos\theta\sin^2\theta - 3b_3\tilde{A}_{2v}^2\sin^4\theta.\end{aligned}$$

Proof. The expansion (2.16) implies the first assertion. Since $\cos\theta_0 \neq 0$, the condition $\Gamma_3(\theta_0) = 0$ is equivalent to

$$\begin{aligned}a_{30} = & -\big[\cos\theta_0(3a_{21}\cos^2\theta_0\sin\theta_0 + 3a_{12}\cos\theta_0\sin^2\theta_0 + a_{03}\sin^3\theta_0) \\ & - b_3\sin^3\theta_0(a_{11}\cos\theta_0 + a_{02}\sin\theta_0)\big]\sec^4\theta_0.\end{aligned}$$

Using the above relation, we can reduce $\tilde{\mathbf{v}}_1^2\tilde{\kappa}_1(0,\theta_0)$ to

$$\tilde{\mathbf{v}}_1^2\kappa_1(0,\theta_0) = \frac{\Gamma_4(\theta_0)\sec^5\theta_0}{a_{02}\mathcal{A}(\theta_0)^3},$$

and the proof is complete. □

Lemma 2.7. *If $b_3 \neq 0$, then there are at most four ridge points relative to $\tilde{\mathbf{v}}_1$ over Whitney umbrella.*

Proof. The ridge points relative to $\tilde{\mathbf{v}}_1$ over Whitney umbrella are given by $\Gamma_3(\theta) = 0$; equivalently,

$$a_{30} + 3a_{21}\tan\theta + 3a_{12}\tan^2\theta + (a_{30} - a_{11}b_3)\tan^3\theta - a_{02}b_3\tan^4\theta = 0.$$

This implies assertions. □

Remark 2.8. When $b_3 = a_{03} = a_{12} = a_{21} = a_{30} = 0$, the multiplicity of ridge points is not defined but the order is defined. In fact, we have $\Gamma_4(\theta) \not\equiv 0$.

Proposition 2.9. *Suppose that a point $(0, \theta_0)$ is a ridge point relative to $\tilde{\mathbf{v}}_1$ over Whitney umbrella, and that the ridge line relative to $\tilde{\mathbf{v}}_1$ is non-singular at $(0, \theta_0)$. Then the ridge line is tangent to X at $(0, \theta_0)$ if and only if $(0, \theta_0)$ is the multiple ridge point relative to $\tilde{\mathbf{v}}_1$.*

Proof. It follows form (2.16) that if and only if $\Gamma_3'(\theta_0) = 0$, the ridge line relative to $\tilde{\mathbf{v}}_1$ is tangent to X at $(0, \theta_0)$. Hence, we have proved the proposition. □

From (2.18), we have the following proposition.

Proposition 2.10. *There are two simple ridge points relative to $\tilde{\mathbf{v}}_2$ over Whitney umbrella. That is, the ridge line relative to $\tilde{\mathbf{v}}_2$ is not tangent to X.*

Sub-parabolic points. By the computation in the previous subsection, we can express $\tilde{\mathbf{v}}_i\tilde{\kappa}_j$ $(i \neq j)$ as follows:

$$\tilde{\mathbf{v}}_1\tilde{\kappa}_2(r, \theta) = \frac{1}{r^3}(P_{10}(\theta) + P_{11}(\theta)r + \cdots), \quad \tilde{\mathbf{v}}_2\tilde{\kappa}_1(r, \theta) = \frac{1}{r^2}(P_{20}(\theta) + P_{21}(\theta)r + \cdots).$$

We say that a point (r_0, θ_0) is a *sub-parabolic point relative to the principal vector* $\tilde{\mathbf{v}}_1$ (resp. $\tilde{\mathbf{v}}_2$) if $r^3\tilde{\mathbf{v}}_1\tilde{\kappa}_2(r_0, \theta_0) = 0$ (resp. $r^2\tilde{\mathbf{v}}_2\tilde{\kappa}_1(r_0, \theta_0) = 0$). When the sub-parabolic point is over Whitney umbrella (that is, $r_0 = 0$), we obtain $P_{i0}(\theta_0) = 0$. A point $(0, \theta_0)$ is said to be a *multiple sub-parabolic point relative to $\mathbf{v}_i$ over Whitney umbrella* if θ_0 is a multiple root of $P_{i0}(\theta) = 0$. The sub-parabolic line relative to $\mathbf{v}_i$ near X is expressed by the equation:

$$P_{i0}(\theta) + P_{i1}(\theta)r + \cdots = 0.$$

From (2.17) we have the following lemma.

Lemma 2.11. *A point $(0, \theta_0)$ is a sub-parabolic point relative to $\tilde{\mathbf{v}}_2$ if and only if $\Gamma_3^*(\theta_0) = 0$ holds.*

From (2.19), we have the following proposition.

Proposition 2.12. *There is no sub-parabolic point relative to $\tilde{\mathbf{v}}_1$ over Whitney umbrella.*

Lemma 2.13. *There is at least one and at most three sub-parabolic points relative to $\tilde{\mathbf{v}}_2$ over Whitney umbrella.*

Proof. The sub-parabolic points relative to $\tilde{\mathbf{v}}_2$ over Whitney umbrella are given by $\Gamma_3^*(\theta) = 0$; equivalently,

$$a_{20}a_{11} + (2 + a_{20}a_{02} + 2{a_{11}}^2)\tan\theta + 3a_{11}a_{02}\tan^2\theta + {a_{02}}^2\tan^3\theta = 0.$$

Since $a_{02} \neq 0$, the equation has at least one and at most three roots and we have completed the proof of the Lemma. $\square$

It follows from (2.17) that we obtain the following proposition.

Proposition 2.14. *Suppose that a point $(0, \theta_0)$ is a sub-parabolic point relative to $\tilde{\mathbf{v}}_2$ over Whitney umbrella, and that the sub-parabolic line relative to $\tilde{\mathbf{v}}_2$ is non-singular at $(0, \theta_0)$. Then the sub-parabolic line is tangent to X at $(0, \theta_0)$ if and only if $(0, \theta_0)$ is the multiple sub-parabolic point relative to $\tilde{\mathbf{v}}_2$.*

Example 2.15. We set $(a_{20}, a_{11}, a_{02}) = (-3, 0, 1)$ in the normal form of Whitney umbrella, then we have

$$\Gamma_3^*(\theta) = -\cos^2\theta\sin\theta + \sin^3\theta.$$

The roots of $\Gamma_3^*(\theta) = 0$ are $\theta = \pm\pi/4$ and $\theta = 0$. Hence, we have the distinct three simple sub-parabolic points $(0, \pm\pi/4)$ and $(0, 0)$ relative to $\tilde{\mathbf{v}}_2$ over Whitney umbrella.

(1) We take $(a_{30}, a_{21}, a_{12}, a_{03}, b_3) = (-1, 0, 10/9, 0, 1)$. Then we have

$$\Gamma_3(\theta) = -\cos^4\theta + \frac{10}{3}\cos^2\theta\sin^2\theta - \sin^4\theta.$$

The roots of $\Gamma_3(\theta) = 0$ are $\theta = \pm\pi/3$ and $\theta = \pm\pi/6$. Hence, we have four distinct simple ridge points $(0, \pm\pi/3)$ and $(0, \pm\pi/6)$ relative to $\tilde{\mathbf{v}}_1$ over Whitney umbrella.

(2) We take $(a_{30}, a_{21}, a_{12}, a_{03}, b_3) = (-3, 0, 4/3, 0, 1)$. Then we have

$$\Gamma_3(\theta) = -3\cos^4\theta + 4\cos^2\theta\sin^2\theta - \sin^4\theta.$$

The roots of $\Gamma_3(\theta) = 0$ are $\theta = \pm\pi/3$ and $\theta = \pm\pi/4$. Hence, we have four distinct simple ridge points $(0, \pm\pi/3)$ and $(0, \pm\pi/4)$ relative to $\tilde{\mathbf{v}}_1$ over Whitney umbrella. Remark that the points $(0, \pm\pi/4)$ are ridge relative to $\tilde{\mathbf{v}}_1$ and sub-parabolic relative to $\tilde{\mathbf{v}}_2$.

2.5 Constant principal curvature lines

We set Σ_k^1 (resp. Σ_k^2) $(k \geq 0)$ as the image of

$$\{(r, \theta) \in \boldsymbol{R} \times S^1\,;\, \tilde{\kappa}_1(r, \theta) = \pm k\} \quad (\text{resp. } \{(r, \theta) \in \boldsymbol{R} \times S^1\,;\, \tilde{\kappa}_2(r, \theta) = \pm k\})$$

by the double covering $\boldsymbol{R} \times S^1 \to \mathcal{M}$. We call $\Sigma_k = \Sigma_k^1 \cup \Sigma_k^2$ by the *constant principal curvature* (*CPC*) *line with a constant value of* k. Remark that $X \cap \Sigma_k^2 = \emptyset$. Remark also that Σ_0 is nothing but the parabolic line, which we already described in Proposition 2.3. Remember that $\mathcal{M}$ is non-orientable and the induced image of $\tilde{\mathbf{n}}$, by $\tilde{\pi} : \boldsymbol{R} \times S^1 \to \boldsymbol{R}^2$, covers "all possible unit normals". We remark that Σ_k $(k > 0)$ is the singular set of the front of g at distance $1/k$.

Proposition 2.16. *A CPC line Σ_k $(k \neq 0)$ intersects with X in at most four points.*

Proof. The number of the intersection points of Σ_k and X equals the number of roots of the equation $|\tilde{\kappa}_1(0, \theta)| = k$. From (2.9), we obtain the equation

$$\left| \frac{(a_{20} \cos^2 \theta - a_{02} \sin^2 \theta) \sec \theta}{\sqrt{\cos^2 \theta + (a_{11} \cos \theta + a_{02} \sin \theta)^2}} \right| = k.$$

Squaring both sides and setting the equation to 0, we get

$$\left[{a_{20}}^2 - k^2({a_{11}}^2 + 1)\right] \cos^4 \theta - 2a_{02}a_{11}k^2 \cos^3 \theta \sin \theta - a_{02}(2a_{20} - a_{02}k) \cos^2 \theta \sin^2 \theta + {a_{02}}^2 \sin^4 \theta = 0.$$

Dividing both sides by $\cos^4 \theta$, we obtain

$${a_{20}}^2 - k^2({a_{11}}^2 + 1) - 2a_{02}a_{11}k^2 \tan \theta - a_{02}(2a_{20} - a_{02}k^2) \tan^2 \theta + {a_{02}}^2 \tan^4 \theta = 0.$$

Since $a_{02} \neq 0$, this equation is quartic in $\tan \theta$ and we have thus completed the proof. □

Setting $C(r, \theta) = \tilde{\kappa}_1(r, \theta)^2 - k^2$ $(k \neq 0)$ and by (2.8), we have

$$C(r, \theta) = -k^2 + k_{10}(\theta)^2 + 2k_{10}(\theta)k_{11}(\theta)r + \left(2k_{10}(\theta)k_{12}(\theta) + k_{11}(\theta)^2\right)r^2 + \cdots . \qquad (2.20)$$

From (2.16) and (2.18), the principal vectors $\tilde{\mathbf{v}}_i$ can be written in

$$\begin{aligned}
\tilde{\mathbf{v}}_1(r, \theta) &= (x_{10}(\theta) + x_{11}(\theta)r + \cdots) \frac{\partial}{\partial r} + (y_{10}(\theta) + y_{11}(\theta)r + \cdots) \frac{\partial}{\partial \theta}, \\
\tilde{\mathbf{v}}_2(r, \theta) &= \frac{1}{r^2} \Big[\left(x_{21}(\theta)r + x_{22}(\theta)r^2 + \cdots\right) \frac{\partial}{\partial r} + (y_{20}(\theta) + y_{21}(\theta)r + \cdots) \frac{\partial}{\partial \theta} \Big].
\end{aligned}$$

Note that $x_{10}(\theta) \neq 0$ and $y_{20}(\theta) \neq 0$. Therefore, the directional derivatives of $\tilde{\kappa}_1(r, \theta)$ by $\tilde{\mathbf{v}}_i$ are expressed as follows:

$$\begin{aligned}
\tilde{\mathbf{v}}_1\tilde{\kappa}_1(r, \theta) = {} & x_{10}(\theta)k_{11}(\theta) + y_{10}(\theta)k'_{10}(\theta) \\
& + (2x_{10}(\theta)k_{12}(\theta) + x_{11}(\theta)k_{11}(\theta) + y_{10}(\theta)k'_{11}(\theta) + y_{11}(\theta)k'_{10}(\theta))r + \cdots , \\
\tilde{\mathbf{v}}_2\tilde{\kappa}_1(r, \theta) = {} & \frac{1}{r^2} \left[y_{20}(\theta)k'_{10}(\theta) + (x_{21}(\theta)k_{11}(\theta) + y_{20}(\theta)k'_{11}(\theta) + y_{21}(\theta)k'_{10}(\theta))r + \cdots \right].
\end{aligned} \qquad (2.21)$$

The following lemma provides the criterion for the singularity of the CPC line intersecting with X in terms of the configurations of the ridge line and the sub-parabolic line.

Lemma 2.17. *Suppose that a point $(0, \theta_0)$ is not parabolic and that the CPC line Σ_k meets X at $(0, \theta_0)$. Then the CPC line Σ_k is singular at $(0, \theta_0)$ if and only if $(0, \theta_0)$ is the ridge point relative to $\tilde{\mathbf{v}}_1$ and the sub-parabolic point relative to $\tilde{\mathbf{v}}_2$. In this case, the singularity is of Morse type if and only if the ridge line relative to $\tilde{\mathbf{v}}_1$ and the sub-parabolic line relative to $\tilde{\mathbf{v}}_2$ intersect transversely at $(0, \theta_0)$.*

Proof. Let us use expansions (2.20) and (2.21). Now we have $k_{10}(\theta_0) \neq 0$. The CPC line Σ_k is singular at $(0, \theta_0)$ if and only if $C_r(0, \theta_0) = C_\theta(0, \theta_0) = 0$. By computation, we have

$$C_r(0, \theta_0) = 2k_{10}(\theta_0)k_{11}(\theta_0) \quad \text{and} \quad C_\theta(0, \theta_0) = 2k_{10}(\theta_0)k'_{10}(\theta_0).$$

It follows that Σ_k has singularity at $(0,\theta_0)$ if and only if $k'_{10}(\theta_0) = k_{11}(\theta_0) = 0$. From (2.21), we deduce that $(0,\theta_0)$ is the ridge point relative to $\tilde{\mathbf{v}}_1$ and the sub-parabolic point relative to $\tilde{\mathbf{v}}_2$ if and only if $k'_{10}(\theta_0) = k_{11}(\theta_0) = 0$. This completes the proof of the first assertion.

We show the second assertion. Assume that $(0,\theta_0)$ is a singularity of Σ_k. Then $(0,\theta_0)$ is a Morse singularity if and only if

$$\begin{vmatrix} C_{rr}(0,\theta_0) & C_{r\theta}(0,\theta_0) \\ C_{r\theta}(0,\theta_0) & C_{\theta\theta}(0,\theta_0) \end{vmatrix} \neq 0.$$

This is equivalent to

$$2k''_{10}(\theta_0)k_{12}(\theta) - {k'_{11}(\theta_0)}^2 \neq 0.$$

It follows from (2.21) that the ridge line relative to $\tilde{\mathbf{v}}_1$ and the sub-parabolic line relative to $\tilde{\mathbf{v}}_2$ intersect transversely at $(0,\theta_0)$ if and only if

$$\begin{vmatrix} 2x_{10}(\theta_0)k_{12}(\theta_0) + y_{10}(\theta_0)k'_{11}(\theta_0) & x_{10}(\theta_0)k'_{11}(\theta_0) + y_{10}(\theta_0)k''_{10}(\theta) \\ y_{20}(\theta_0)k'_{11}(\theta_0) & y_{20}(\theta_0)k''_{10}(\theta_0) \end{vmatrix} \neq 0;$$

equivalently,

$$x_{10}(\theta_0)y_{20}(\theta_0)(2k''_{10}(\theta_0)k_{12}(\theta) - {k'_{11}(\theta_0)}^2) \neq 0.$$

We thus have completed of the proof of the second assertion. □

Lemma 2.18. *Suppose that a point $(0,\theta_0)$ is not parabolic and that the CPC line Σ_k meets X at $(0,\theta_0)$. Then the CPC line Σ_k is tangent to X at $(0,\theta_0)$ if and only if $(0,\theta_0)$ is a sub-parabolic point relative to $\tilde{\mathbf{v}}_2$ which is not a ridge point relative to $\tilde{\mathbf{v}}_1$.*

Proof. Let us consider expansions (2.20) and (2.21). From (2.20), the equation of the tangent line of Σ_k at $(0,\theta_0)$ is then

$$k_{11}(\theta_0)r + k'_{10}(\theta_0)(\theta - \theta_0) = 0.$$

It follows that the CPC line Σ_k is tangent to X if and only if $k'_{10}(\theta_0) = 0$ and $k_{11}(\theta_0) \neq 0$. From (2.21), we show that the point $(0,\theta_0)$ is a sub-parabolic point relative to $\tilde{\mathbf{v}}_2$ which is not a ridge point relative to $\tilde{\mathbf{v}}_1$ if and only if $k'_{10}(\theta_0) = 0$ and $k_{11}(\theta_0) \neq 0$. □

Lemma 2.19. *Assume that the CPC line Σ_k and the ridge line relative to $\tilde{\mathbf{v}}_1$ meet at a point $(0,\theta_0)$ which is not parabolic over Whitney umbrella. Then*

(1) *These two curves intersect transversely at the point $(0,\theta_0)$ if and only if the point $(0,\theta_0)$ is a first order ridge point relative to $\tilde{\mathbf{v}}_1$.*

(2) *These two curves are tangent at the point $(0,\theta_0)$ if and only if the point $(0,\theta_0)$ is a second or higher order ridge point relative to $\tilde{\mathbf{v}}_1$.*

Proof. Let us consider expansions (2.20) and (2.21). Remark that $k_{10}(\theta_0) \neq 0$. The ridge line relative to $\tilde{\mathbf{v}}_1$ passes through $(0,\theta_0)$, that is, $(0,\theta_0)$ is a ridge point relative to $\tilde{\mathbf{v}}_1$ over Whitney umbrella if and only if

$$x_{10}(\theta_0)k_{11}(\theta_0) + y_{10}(\theta_0)k'_{10}(\theta_0) = 0.$$

Since $x_{10} \neq 0$, this is equivalent to

$$k_{11}(\theta_0) = -\frac{y_{10}(\theta_0)k'_{10}(\theta_0)}{x_{10}(\theta_0)}. \tag{2.22}$$

The CPC line Σ_k and the ridge line relative to $\tilde{\mathbf{v}}_1$ intersect transversely at $(0, \theta_0)$ if and only if the determinant

$$\begin{vmatrix} k_{11}(\theta_0) & 2x_{10}(\theta_0)k_{12}(\theta_0) + x_{11}(\theta_0)k_{11}(\theta_0) + y_{10}(\theta_0)k'_{11}(\theta_0) + y_{11}(\theta_0)k'_{10}(\theta_0) \\ k'_{10}(\theta_0) & x_{10}(\theta_0)k'_{11}(\theta_0) + x'_{10}(\theta_0)k_{11}(\theta_0) + y_{10}(\theta_0)k''_{10}(\theta_0) + y'_{10}(\theta_0)k'_{10}(\theta_0) \end{vmatrix} \tag{2.23}$$

is not zero. Otherwise, these two curves are tangent at $(0, \theta_0)$ if and only if the determinant (2.23) is zero. By using (2.22), the determinant (2.23) is expanded as

$$\begin{aligned} &-\frac{1}{x_{10}(\theta_0)^2}k'_{10}(\theta_0)\big[2x_{10}(\theta_0)^3k_{12}(\theta_0) + 2x_{10}(\theta_0)^2y_{10}(\theta_0)k'_{11}(\theta_0) \\ &\quad + x_{10}(\theta_0)^2y_{11}(\theta_0)k'_{10}(\theta_0) + x_{10}(\theta_0)y_{10}(\theta_0)^2k''_{10}(\theta_0) - x_{10}(\theta_0)x_{11}(\theta_0)y_{10}(\theta_0)k'_{10}(\theta_0) \\ &\qquad + x_{10}(\theta_0)y_{10}(\theta_0)^2k'_{10}(\theta_0) - y_{10}(\theta_0)^2x'_{10}(\theta_0)k'_{10}(\theta_0)\big], \end{aligned}$$

and we obtain

$$\begin{aligned} \tilde{\mathbf{v}}_1^2\tilde{\kappa}_1(0, \theta_0) = \frac{1}{x_{10}(\theta_0)}\Big[&2x_{10}(\theta_0)^3k_{12}(\theta_0) + 2x_{10}(\theta_0)^2y_{10}(\theta_0)k'_{11}(\theta_0) + x_{10}(\theta_0)^2y_{11}(\theta_0)k'_{10}(\theta_0) \\ &+ x_{10}(\theta_0)y_{10}(\theta_0)^2k''_{10}(\theta_0) - x_{10}(\theta_0)x_{11}(\theta_0)y_{10}(\theta_0)k'_{10}(\theta_0) \\ &+ x_{10}(\theta_0)y_{10}(\theta_0)^2k'_{10}(\theta_0) - y_{10}(\theta_0)^2x'_{10}(\theta_0)k'_{10}(\theta_0)\Big]. \end{aligned}$$

Hence, we conclude that the determinant (2.23) is zero (resp. non-zero) if and only if $(0, \theta_0)$ is a first order (resp. second or higher order) ridge point relative to $\tilde{\mathbf{v}}_1$, and we have completed the proof. □

3 Singularities of the distance squared unfolding

We assume that $g : (\boldsymbol{R}^2, \mathbf{0}) \to (\boldsymbol{R}^3, \mathbf{0})$ is given by (2.2). In this section, we investigate the singularities of the members of the family of the distance squared function:

$$\Phi : (\boldsymbol{R}^2, \mathbf{0}) \times \boldsymbol{R}^3 \to \boldsymbol{R}, \qquad (u, v) \times (x, y, z) \mapsto -\frac{1}{2}(\|(x, y, z) - g(u, v)\|^2 - {t_0}^2) \tag{3.1}$$

where t_0 is a constant. We set $\varphi(u, v) = \Phi(u, v, x_0, y_0, z_0)$ where (x_0, y_0, z_0) is a point in $\boldsymbol{R}^3$, and take t_0 so that $\varphi(0, 0) = 0$, that is, ${t_0}^2 = {x_0}^2 + {y_0}^2 + {z_0}^2$.

Now we recall the definition of the normal plane. When the map $g : (\boldsymbol{R}^2, \mathbf{0}) \to (\boldsymbol{R}^3, \mathbf{0})$ has Whitney umbrella at $(0, 0)$, the image of $dg_{(0,0)}$ is a line in $\boldsymbol{R}^3$. We call the plane perpendicular to this line the *normal plane at Whitney umbrella.*

Proposition 3.1. *The following conditions are equivalent:*

(1) *The function φ has at least an A_1-singularity at* $(0, 0)$*;*

(2) *The point* (x_0, y_0, z_0) *is on the normal plane at Whitney umbrella;*

(3) *There exist* $\rho_0 \in \boldsymbol{R}$ *and* θ_0 *with* $\theta_0 \in [-\pi/2, \pi/2]$ *such that* $(x_0, y_0, z_0) = \rho_0 \tilde{\mathbf{n}}(0, \theta_0)$.

Proof. The unfolding Φ can be expressed as follows:

$$\Phi(u, v, x, y, z) = c_{00}(x, y, z) + xu + \sum_{i+j=2}^{4} \frac{1}{i!j!} c_{ij}(x, y, z) u^i v^j + O(u, v)^5. \tag{3.2}$$

We have

$$c_{00} = \frac{1}{2}({t_0}^2 - x^2 - y^2 - z^2), \quad c_{20} = a_{20}z - 1, \quad c_{11} = y + a_{11}z, \quad c_{02} = a_{02}z,$$
$$c_{30} = a_{30}z, \quad c_{21} = a_{21}z, \quad c_{12} = a_{12}z, \quad c_{03} = a_{03}z + b_3 y,$$
$$c_{40} = -3{a_{20}}^2 + a_{40}z, \quad c_{31} = -3a_{20}a_{11} + a_{31}z, \quad c_{22} = -2 - 2{a_{11}}^2 - a_{20}a_{02} + a_{22}z,$$
$$c_{13} = -a_{11}a_{02} + a_{13}z, \quad c_{04} = -3{a_{02}}^2 + b_4 y + a_{04}z.$$

Then φ can be written in the form

$$\varphi(u, v) = x_0 u + \sum_{i+j=2}^{4} \frac{1}{i!j!} c^0_{ij} u^i v^j + O(u, v)^5, \tag{3.3}$$

where $c^0_{ij} = c_{ij}(x_0, y_0, z_0)$. It follows that φ has at least an A_1-singularity at $(0, 0)$ if and only if $x_0 = 0$. Directly from the definition of the normal form we obtain that the image of $dg_{(0,0)}$ is the x-axis. Hence, the normal plane is the yz-plane. Thus (1) and (2) are equivalent.

Next, suppose (2) holds. Since $a_{02} \neq 0$,

$$\tilde{\mathbf{n}}(0, \theta) = \frac{(0, -a_{11}\cos\theta - a_{02}\sin\theta, \cos\theta)}{\sqrt{\cos^2\theta + (a_{11}\cos\theta + a_{02}\sin\theta)^2}}$$

expands in all direction in the yz-plane. Hence, there exist $\rho_0 \in \boldsymbol{R}$ and θ_0 with $\theta_0 \in [-\pi/2, \pi/2]$ such that $(x_0, y_0, z_0) = \rho_0 \tilde{\mathbf{n}}(0, \theta_0)$.

Finally, suppose (3) holds. Then we have $x_0 = 0$. Thus (3) implies (1). □

3.1 Focal conics

Proposition 3.2. *The points* (x_0, y_0, z_0) *at which* φ *has at least* A_2*-singularity (valid for the rest of the paper "an* A_k*-singularity") at* $(0, 0)$ *form a conic in the normal plane.*

Proof. Assume that φ has at least an A_1-singularity at $(0, 0)$, that is, $x_0 = 0$. Then the determinant of the Hessian of φ at $(0, 0)$ is given by

$$\begin{vmatrix} \varphi_{uu}(0,0) & \varphi_{uv}(0,0) \\ \varphi_{uv}(0,0) & \varphi_{vv}(0,0) \end{vmatrix} = \begin{vmatrix} c^0_{20} & c^0_{11} \\ c^0_{11} & c^0_{02} \end{vmatrix} = -{y_0}^2 - 2a_{11}y_0 z_0 + (a_{20}a_{02} - {a_{11}}^2){z_0}^2 - a_{02}z_0.$$

Therefore, the locus of the equation

$$-y^2 - 2a_{11}yz + (a_{20}a_{02} - {a_{11}}^2)z^2 - a_{02}z = 0$$

is the set of the points at which φ is an A_k-singularity at $(0, 0)$. Thus we complete the proof. □

Lemma 3.3. *The function* φ *has an* A_k*-singularity at* $(0,0)$ *if and only if* $(x_0, y_0, z_0) = (0,0,0)$ *or*

$$(x_0, y_0, z_0) = \frac{1}{\tilde{\kappa}_1(0,\theta_0)}\tilde{\mathbf{n}}(0,\theta_0) \quad \text{with} \quad \tilde{\kappa}_1(0,\theta_0) \neq 0, \quad \text{where} \quad \theta_0 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

Proof. Suppose that φ is at least an A_1-singularity at $(0,0)$. By Proposition 3.1, we have $(x_0, y_0, z_0) = \rho_0 \tilde{\mathbf{n}}(0,\theta_0)$ where $\rho_0 \in \boldsymbol{R}$ and $\theta_0 \in [-\pi/2, \pi/2]$. Substituting this into the equation $-{y_0}^2 - 2a_{11}y_0z_0 + (a_{20}a_{02} - {a_{11}}^2){z_0}^2 - a_{02}z_0 = 0$, we obtain

$$\rho_0 = 0, \qquad \text{or} \qquad \rho_0 = \frac{\sqrt{\cos^2\theta_0 + (a_{11}\cos\theta_0 + a_{02}\sin\theta_0)^2}}{(a_{20}\cos^2\theta_0 - a_{02}\sin^2\theta_0)\sec\theta_0}.$$

When $\rho_0 = 0$, the point (x_0, y_0, z_0) coincides with $(0,0,0)$. In the later case, ρ_0 coincides with the principal radius $1/\tilde{\kappa}_1(0,\theta_0)$. □

For this reason, we call the set of the points (x, y, z) at which φ has an A_k-singularity at $(0,0)$ the *focal conic of Whitney umbrella*. Focal conics are classified into three types as shown in Figure 1. The following proposition provides a classification of focal conics.

Proposition 3.4. (1) *The focal conic is an ellipse if and only if* $a_{20}a_{02} < 0$.

(2) *The focal conic is a hyperbola if and only if* $a_{20}a_{02} > 0$*, in which case its asymptotes are parallel to* $y + (a_{11} \pm \sqrt{a_{20}a_{02}})z = 0$.

(3) *The focal conic is a parabola if and only if* $a_{20} = 0$*, in which case its axis of symmetry is parallel to* $y + a_{11}z = 0$.

Proof. Remark that the focal conic is the zero locus of

$$FC(y,z) = -y^2 - 2a_{11}yz + (a_{20}a_{02} - {a_{11}}^2)z^2 - a_{02}z.$$

Firstly, we assume that $a_{20} \neq 0$. Replacing y and z by $y - a_{11}/(2a_{20})$ and $z + 1/(2a_{20})$, respectively. Then the equation $FC(y,z) = 0$ has the form

$$-\frac{a_{02}}{4a_{20}} - (y + a_{11}z)^2 + a_{20}a_{02}z^2 = 0.$$

This form implies the assertion (1) and (2).

Next, we assume that $a_{20} = 0$. Then the equation $FC(y,z) = 0$ reduces to

$$-(y + a_{11}z)^2 - a_{02}z = 0.$$

This implies the assertion (3). □

The following proposition provides properties of the focal conic. It is easy to verify this proposition and we omit its proof.

Proposition 3.5. *Let* g *be given in the normal form of Whitney umbrella and let* C_k *denote the circle centred at the origin with radius* $1/k$ *in the normal plane.*

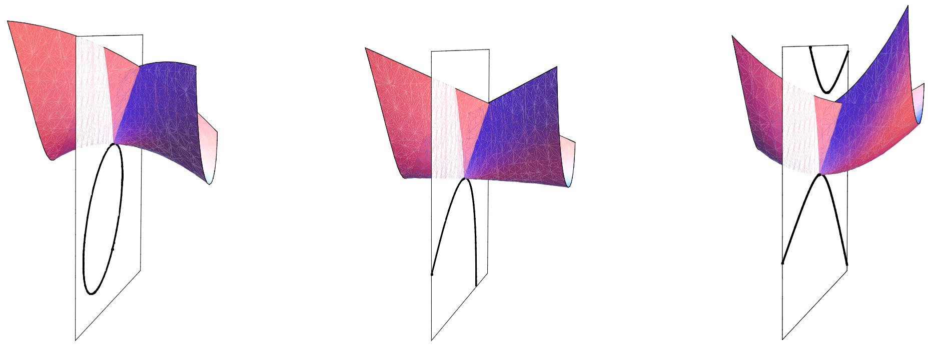


Figure. 1: The three types of focal conics: focal ellipse left, focal parabola center, focal hyperbola right.

(1) *The y-axis is tangent to the focal conic at the origin.*

(2) *The circle C_k is a subset of the front at distance $1/k$.*

(3) *For $k > 0$ we have $\#\Sigma_k \cap X = \#C_k \cap \{(y,z) \ ; \ FC(y,z) = 0\}$ where X is the exceptional set $X = \pi^{-1}(0,0)$ on the Möbius strip $\mathcal{M}$. Since $\#C_k \cap \{(y,z) \ ; \ FC(y,z) = 0\}$ is at most four, $\#\Sigma_k \cap X$ is also at most four (cf. Proposition 2.16). Moreover, we have*

$$C_k \cap \{(y,z) \ ; \ FC(y,z) = 0\} = \bigcup_{\theta : \tilde{\kappa}_1(0,\theta) = k} \boldsymbol{R}\tilde{n}(0,\theta) \cap \{(y,z) \ ; \ FC(y,z) = 0\} \setminus \{(0,0)\}$$
$$= \bigcup_{\theta : \tilde{\kappa}_1(0,\theta) = k} \frac{1}{k}\tilde{\mathbf{n}}(0,\theta).$$

If $\tilde{\mathbf{n}}(0,\theta)$ is parallel to the axis of symmetry of the focal parabola or the asymptotes of the focal hyperbola, then $\boldsymbol{R}\tilde{n}(0,\theta) \cap \{(y,z) \ ; \ FC(y,z) = 0\} \setminus \{(0,0)\} = \emptyset$ and $\tilde{\kappa}_1(0,\theta) = 0$, that is, $(0,\theta)$ is a parabolic point over Whitney umbrella.

(4) *The circle C_k is tangent to the focal conic at $\tilde{\mathbf{n}}(0,\theta)/k$ if and only if $(0,\theta)$ a is sub-parabolic point relative to $\tilde{\mathbf{v}}_2$ over Whitney umbrella. For any focal conic, there exists at least one and at most three values of k such that C_k is tangent to the focal conic. This implies that the number of the sub-parabolic points relative to $\tilde{\mathbf{v}}_2$ over Whitney umbrella is at least one and at most three (cf. Lemma 2.13).*

(5) *The origin of the normal plane is the vertex of the focal conic if and only if $a_{11} = 0$. In this case, we have $\tilde{\kappa}_1(0,-\theta) = \tilde{\kappa}_1(0,\theta)$, and $(r,\theta) = (0,0)$ is a sub-parabolic point relative to $\tilde{\mathbf{v}}_2$.*

(6) *When the focal conic is a parabola, $(r,\theta) = (0,0)$ is parabolic (in fact, $\tilde{\kappa}_1(0,0) = 0$) and sub-parabolic relative to $\tilde{\mathbf{v}}_2$ over Whitney umbrella (i.e., $\Gamma_3^*(0) = 0$).*

We obtain the following corollary by Propositions 2.3 and 3.4.

Corollary 3.6. (1) *There is no parabolic point over Whitney umbrella, that is, the parabolic line dose not meet with X, if and only if the focal conic is an ellipse, in which case we have the hyperbolic Whitney umbrella.*

(2) *There is one parabolic point over Whitney umbrella, that is, the parabolic line meets with X at one point, if and only if the focal conic is a parabola. In this case the intersection point of the parabolic line and X is a sub-parabolic point relative to $\tilde{\mathbf{v}}_2$. Furthermore, the parabolic line is tangent to X at the point if and only if the point is not a ridge point relative to $\tilde{\mathbf{v}}_1$. In this case, we have the parabolic Whitney umbrella.*

(3) *There are two parabolic points, that is, the parabolic line meets with X at two points, if and only if the focal conic is a hyperbola, in which case we have the elliptic Whitney umbrella.*

3.2 Versality of distance squared unfolding

We do not repeat here the definition of versal unfolding, which is fundamental in singularity theory. Please refer to [1] for elegant explanation, [17] for elementary introduction, and [25] for carefully prepared survey. The notation in [25] becomes the standard in singularity theory.

Theorem 3.7. *Suppose that $g : (\boldsymbol{R}^2, \mathbf{0}) \to (\boldsymbol{R}^3, \mathbf{0})$ is given in the normal form of Whitney umbrella. Assume that $\Phi : (\boldsymbol{R}^2, \mathbf{0}) \times \boldsymbol{R}^3 \to \boldsymbol{R}$ is the distance squared function defined by (3.1) and that $\varphi(u, v) = \Phi(u, v, x_0, y_0, z_0)$ where (x_0, y_0, z_0) is a point in $\boldsymbol{R}^3$.*

(1) *Suppose that $(x_0, y_0, z_0) = \tilde{\mathbf{n}}(0, \theta_0)/\tilde{\kappa}_1(0, \theta_0) \neq (0, 0, 0)$ with $\tilde{\kappa}_1(0, \theta_0) \neq 0$ and ${t_0}^2 = {x_0}^2 + {y_0}^2 + {z_0}^2$, where $\theta_0 \in (-\pi/2, \pi/2)$.*

   (a) *The function $\varphi(u, v)$ has an A_2-singularity at $(0, 0)$ if and only if $(0, \theta_0)$ is not a ridge point relative to $\tilde{\mathbf{v}}_1$ over Whitney umbrella. In this case, Φ is $\mathcal{R}^+$ and $\mathcal{K}$-versal unfolding of φ.*

   (b) *The function φ has an A_3-singularity at $(0, 0)$ if and only if $(0, \theta_0)$ is a first order ridge point relative to $\tilde{\mathbf{v}}_1$ over Whitney umbrella. In this case, Φ is an $\mathcal{R}^+$-versal unfolding of φ. Moreover, Φ is a $\mathcal{K}$-versal unfolding of φ if and only if $(0, \theta_0)$ is not a sub-parabolic point relative to $\tilde{\mathbf{v}}_2$ over Whitney umbrella.*

(2) *Suppose that $(x_0, y_0, z_0) = (0, 0, 0)$ and $t_0 = 0$. Then φ has an A_3-singularity at $(0, 0)$. In this case, Φ is neither an $\mathcal{R}^+$-versal nor a $\mathcal{K}$-versal unfolding of φ.*

Proof. Let us use expansions of Φ and φ as in (3.2) and (3.3), respectively. We remark that the coefficient a_{02} appearing in the normal form of Whitney umbrella is not zero.

(1) We first prove the condition for the point $(0, 0)$ to be an A_2 or A_3-singularity of φ. Lemma 3.3 now shows that φ has an A_k-singularity at $(0, 0)$. By (2.5) and (2.9), we have

$$(x_0, y_0, z_0) = \left(0,\ -\frac{(a_{11}\cos\theta_0 + a_{02}\sin\theta_0)\cos\theta_0}{a_{20}\cos^2\theta - a_{02}\sin^2\theta_0},\ \frac{\cos^2\theta_0}{a_{20}\cos^2\theta_0 - a_{02}\sin^2\theta_0}\right) \neq (0, 0, 0).$$

Simple calculations show that

$$\begin{pmatrix} c_{20}^0 & c_{11}^0 \\ c_{11}^0 & c_{02}^0 \end{pmatrix} = \frac{a_{02}}{a_{20}\cos^2\theta_0 - a_{02}\sin^2\theta_0} \begin{pmatrix} \sin^2\theta_0 & -\cos\theta_0\sin\theta_0 \\ -\cos\theta_0\sin\theta_0 & \cos^2\theta_0 \end{pmatrix}.$$

Taking s and (ξ,η) so that

$$s\begin{pmatrix} \eta^2 & -\xi\eta \\ -\xi\eta & \xi^2 \end{pmatrix} = \begin{pmatrix} c^0_{20} & c^0_{11} \\ c^0_{11} & c^0_{02} \end{pmatrix},$$

we obtain

$$s = \frac{a_{02}}{a_{20}\cos^2\theta_0 - a_{02}\sin^2\theta_0} \quad \text{and} \quad (\xi,\eta) = (\cos\theta_0, \sin\theta_0).$$

Setting $c = 2s(c^0_{21}\xi^2 + 2c^0_{12}\xi\eta + c^0_{03}\eta^2)/\xi^4$ and replacing v by $v + (\eta/\xi)u - cu^2$, we have

$$\Phi = c_{00}(x,y,z) + xu + \sum_{i+j=2}^{4} \frac{1}{i!j!}\hat{c}_{ij}(x,y,z)u^i v^j + O(u,v)^5,$$

where

$$\begin{aligned}
\hat{c}_{20} &= \frac{1}{\xi^2}(c_{20}\xi^2 + 2c_{11}\xi\eta + c_{02}\eta^2), \quad \hat{c}_{11} = \frac{1}{\xi}(c_{11}\xi + c_{02}\eta), \quad \hat{c}_{02} = c_{02},\\
\hat{c}_{30} &= \frac{1}{s\xi^5}\Big(s\xi^2(c_{30}\xi^3 + 3c_{21}\xi^2\eta + 3c_{12}\xi\eta^2 + c_{03}\eta^3) - 3(c_{11}\xi + c_{02}\eta)(c^0_{21}\xi^2 + 2c^0_{12}\xi\eta + c^0_{02}\eta^2)\Big)\\
\hat{c}_{21} &= \frac{1}{s\xi^4}\left[s\xi^2(c_{21}\xi^2 + 2c_{12}\xi\eta + c_{02}\eta^2) - c_{02}(c^0_{21}\xi^2 + 2c^0_{12}\xi\eta + c^0_{02}\eta^2)\right],\\
\hat{c}_{12} &= \frac{1}{\xi}(c_{21}\xi + c_{03}\eta), \quad \hat{c}_{03} = c_{03},\\
\hat{c}_{40} &= \frac{1}{\xi^4}(c^0_{40}\xi^4 + 4c^0_{31}\xi^3\eta + 6c^0_{22}\xi^2\eta^2 + 4c^0_{13}\xi^3\eta + c^0_{04}\eta^4)\\
&\quad - \frac{3}{s\xi^6}(c^0_{21}\xi^2 + 2c^0_{12}\xi\eta + c^0_{03}\eta^2)\left[2(c_{21}\xi^2 + 2c_{12}\xi\eta + c_{02}\eta^2) - (c^0_{21}\xi^2 + 2c^0_{12}\xi\eta + c^0_{03}\eta^2)\right].
\end{aligned}$$

Therefore, φ is expressed as

$$\varphi = \frac{1}{2}\hat{c}^0_{02}v^2 + \frac{1}{6}(\hat{c}^0_{30}u^3 + 3\hat{c}^0_{12}uv^2 + \hat{c}^0_{03}v^3) + \sum_{i+j=4}\frac{1}{i!j!}\hat{c}^0_{ij}u^i v^j + O(u,v)^5,$$

where $\hat{c}^0_{ij} = \hat{c}_{ij}(x_0,y_0,z_0)$. By this form, φ does not have D_4 or worse singularities. This form also shows that φ has an A_2 or A_3 singularity for (x_0,y_0,z_0) at $(0,0)$ if and only if $\hat{c}^0_{30} \neq 0$, or $\hat{c}^0_{30} = 0$ and $\hat{c}^0_{40} \neq 0$, respectively. After some computations, we extract that

$$\hat{c}^0_{30} = \frac{\Gamma_3(\theta_0)\sec^2\theta_0}{a_{20}\cos^2\theta_0 - a_{02}\sin^2\theta_0} \quad \text{and} \quad \hat{c}^0_{40} = \frac{\Gamma_4(\theta_0)\sec^4\theta_0}{a_{02}(a_{20}\cos^2\theta_0 - a_{02}\sin^2\theta_0)}.$$

Therefore, from Lemma 2.6 it follows that φ has an A_2 or A_3-singularity at $(0,0)$ if and only if $(0,\theta_0)$ is not a ridge point relative to $\tilde{\mathbf{v}}_1$ or a first order ridge point relative to $\tilde{\mathbf{v}}_1$, respectively.

We now turn to the versality of Φ. Firstly, we prove Case (a). Suppose that $(0,0)$ is an A_2-singularity. We remark that A_2-singularity is 3-determined. To show the $\mathcal{R}^+$-versality and the $\mathcal{K}$-versality of Φ, it is enough to verify that, respectively,

$$\mathcal{E}_2 = \langle\varphi_u,\varphi_v\rangle_{\mathcal{E}_2} + \langle\Phi_x|_{\boldsymbol{R}^2\times p_0}, \Phi_y|_{\boldsymbol{R}^2\times p_0}, \Phi_z|_{\boldsymbol{R}^2\times p_0}\rangle_{\boldsymbol{R}} + \langle 1\rangle_{\boldsymbol{R}} + \langle u,v\rangle^4_{\mathcal{E}_2}, \quad \text{and} \tag{3.4}$$

$$\mathcal{E}_2 = \langle \varphi_u, \varphi_v, \varphi \rangle_{\mathcal{E}_2} + \langle \Phi_x|_{\boldsymbol{R}^2 \times p_0}, \Phi_y|_{\boldsymbol{R}^2 \times p_0}, \Phi_z|_{\boldsymbol{R}^2 \times p_0} \rangle_{\boldsymbol{R}} + \langle u, v \rangle^4_{\mathcal{E}_2}, \tag{3.5}$$

where $p_0 = (x_0, y_0, z_0)$. The coefficients of $u^i v^j$ of functions appearing in (3.4) and (3.5) are given by the following table:

	1	u	v	u^2	uv	v^2	u^3	u^2v	uv^2	v^3
Φ_x	0	$\boxed{1}$	0	0	0	0	0	0	0	0
Φ_y	$-y_0$	0	0	α_{20}	1	0	α_{30}	α_{21}	α_{12}	α_{03}
Φ_z	$\boxed{-z_0}$	0	0	β_{20}	β_{11}	β_{02}	β_{30}	β_{21}	β_{12}	β_{03}
φ_u	0	0	0	$\boxed{\frac{1}{2}\hat{c}^0_{30}}$	0	$\frac{1}{2}\hat{c}^0_{12}$	$\frac{1}{6}\hat{c}^0_{40}$	$\frac{1}{2}\hat{c}^0_{31}$	$\frac{1}{2}\hat{c}^0_{22}$	$\frac{1}{6}\hat{c}^0_{13}$
φ_v	0	0	$\boxed{\hat{c}^0_{02}}$	0	$\hat{c}^0_{12}$	$\frac{1}{2}\hat{c}^0_{03}$	$\frac{1}{6}\hat{c}^0_{31}$	$\frac{1}{2}\hat{c}^0_{22}$	$\frac{1}{2}\hat{c}^0_{13}$	$\frac{1}{6}\hat{c}^0_{04}$
φ	0	0	0	0	0	$\frac{1}{2}\hat{c}^0_{02}$	$\frac{1}{6}\hat{c}^0_{30}$	0	$\frac{1}{2}\hat{c}^0_{12}$	$\frac{1}{6}\hat{c}^0_{03}$
$u\varphi_u$	0	0	0	0	0	0	$\boxed{\frac{1}{2}\hat{c}^0_{30}}$	0	$\frac{1}{2}\hat{c}^0_{12}$	0
$v\varphi_u$	0	0	0	0	0	0	0	$\frac{1}{2}\hat{c}^0_{30}$	0	$\frac{1}{2}\hat{c}^0_{12}$
$u\varphi_v$	0	0	0	0	$\boxed{\hat{c}^0_{02}}$	0	0	$\hat{c}^0_{12}$	$\frac{1}{2}\hat{c}^0_{03}$	0
$v\varphi_v$	0	0	0	0	0	$\boxed{\hat{c}^0_{02}}$	0	0	$\hat{c}^0_{12}$	$\frac{1}{2}\hat{c}^0_{03}$
$u^2\varphi_v$	0	0	0	0	0	0	0	$\boxed{\hat{c}^0_{02}}$	0	0
$uv\varphi_v$	0	0	0	0	0	0	0	0	$\boxed{\hat{c}^0_{02}}$	0
$v^2\varphi_v$	0	0	0	0	0	0	0	0	0	$\boxed{\hat{c}^0_{02}}$

Here,

$$\alpha_{ij} = \frac{\partial \hat{c}_{ij}}{\partial y}(x_0, y_0, z_0), \quad \beta_{ij} = \frac{\partial \hat{c}_{ij}}{\partial z}(x_0, y_0, z_0),$$

and boxed elements are non-zero. Hence, Gauss's elimination method using boxed elements as pivots leads to that the matrix presented by this table is full rank. Thus the equality (3.5) holds. The case of (3.4) is similar.

Next, we consider Case (b). Suppose that $(0,0)$ is an A_3-singularity. We remark that A_3-singularity is 4-determined. To show the $\mathcal{R}^+$-versality and the $\mathcal{K}$-versality of Φ, it is enough to verify that

$$\mathcal{E}_2 = \langle \varphi_u, \varphi_v \rangle_{\mathcal{E}_2} + \langle \Phi_x|_{\boldsymbol{R}^2 \times p_0}, \Phi_y|_{\boldsymbol{R}^2 \times p_0}, \Phi_z|_{\boldsymbol{R}^2 \times p_0} \rangle_{\boldsymbol{R}} + \langle 1 \rangle_{\boldsymbol{R}} + \langle u, v \rangle^5, \quad \text{and} \tag{3.6}$$

$$\mathcal{E}_2 = \langle \varphi_u, \varphi_v, \varphi \rangle_{\mathcal{E}_2} + \langle \Phi_x|_{\boldsymbol{R}^2 \times p_0}, \Phi_y|_{\boldsymbol{R}^2 \times p_0}, \Phi_z|_{\boldsymbol{R}^2 \times p_0} \rangle_{\boldsymbol{R}} + \langle u, v \rangle^5, \tag{3.7}$$

respectively. The coefficients of $u^i v^j$ of functions appearing in (3.6) and (3.7) are given by the following table:

	1	u	v	u^2	uv	v^2	u^3	u^2v	uv^2	v^3	u^4
Φ_x	0	$\boxed{1}$	0	0	0	0	0	0	0	0	0
Φ_y	$-y_0$	0	0	α_{20}	1	0	α_{30}	α_{21}	α_{12}	α_{03}	α_{40}
Φ_z	$-z_0$	0	0	β_{20}	β_{11}	β_{02}	β_{30}	β_{21}	β_{12}	β_{03}	β_{40}
φ_u	0	0	0	0	0	$\frac{1}{2}\hat{c}^0_{12}$	$\boxed{\frac{1}{6}\hat{c}^0_{40}}$	$\frac{1}{2}\hat{c}^0_{31}$	$\frac{1}{2}\hat{c}^0_{22}$	$\frac{1}{6}\hat{c}^0_{13}$	$\frac{1}{24}\hat{c}^0_{50}$
φ_v	0	0	$\boxed{\hat{c}^0_{02}}$	0	$\hat{c}^0_{12}$	$\frac{1}{2}\hat{c}^0_{03}$	$\frac{1}{6}\hat{c}^0_{31}$	$\frac{1}{2}\hat{c}^0_{22}$	$\frac{1}{2}\hat{c}^0_{13}$	$\frac{1}{6}\hat{c}^0_{04}$	$\frac{1}{24}\hat{c}^0_{41}$
φ	0	0	0	0	0	$\frac{1}{2}\hat{c}^0_{02}$	0	0	$\frac{1}{2}\hat{c}^0_{12}$	$\frac{1}{6}\hat{c}^0_{03}$	$\frac{1}{24}\hat{c}^0_{40}$
$u\varphi_u$	0	0	0	0	0	0	0	0	$\frac{1}{2}\hat{c}^0_{12}$	0	$\boxed{\frac{1}{6}\hat{c}^0_{40}}$
$v\varphi_u$	0	0	0	0	0	0	0	0	0	$\frac{1}{2}\hat{c}^0_{12}$	0
$u\varphi_v$	0	0	0	0	$\boxed{\hat{c}^0_{02}}$	0	0	$\hat{c}^0_{12}$	$\frac{1}{2}\hat{c}^0_{03}$	0	$\frac{1}{6}\hat{c}^0_{31}$
$v\varphi_v$	0	0	0	0	0	$\boxed{\hat{c}^0_{02}}$	0	0	$\hat{c}^0_{12}$	$\frac{1}{2}\hat{c}^0_{03}$	0
$u^2\varphi_v$	0	0	0	0	0	0	0	$\boxed{\hat{c}^0_{02}}$	0	0	0
$uv\varphi_v$	0	0	0	0	0	0	0	0	$\boxed{\hat{c}^0_{02}}$	0	0
$v^2\varphi_v$	0	0	0	0	0	0	0	0	0	$\boxed{\hat{c}^0_{02}}$	0

	u^iv^j $(i+j\leq 3)$	u^4	u^3v	u^2v^2	uv^3	v^4
$u^3\varphi_u$	0	0	$\boxed{\hat{c}^0_{02}}$	0	0	0
$u^2v\varphi_u$	0	0	0	$\boxed{\hat{c}^0_{02}}$	0	0
$uv^2\varphi_u$	0	0	0	0	$\boxed{\hat{c}^0_{02}}$	0
$v^3\varphi_u$	0	0	0	0	0	$\boxed{\hat{c}^0_{02}}$

The equality (3.6) holds if and only if the matrix presented by this table except the first column is full rank. This requires that α_{20} or β_{20} is non-zero. Similarly, (3.7) holds if and only if

$$\begin{vmatrix} y_0 & \alpha_{20} \\ z_0 & \beta_{20} \end{vmatrix} \neq 0. \tag{3.8}$$

Some calculations show that

$$\alpha_{20} = 2\tan\theta_0 \quad \text{and} \quad \beta_{20} = a_{20} + 2a_{11}\tan\theta_0 + a_{02}\tan^2\theta_0.$$

Now we assume that $(\alpha_{20}, \beta_{20}) = (0, 0)$. Then we have $\theta_0 = 0$ and $a_{20} = 0$. Hence, $\tilde{\kappa}_1(0, \theta_0) = 0$. Since this opposes the assumption $\tilde{\kappa}_1(0, \theta_0) \neq 0$, the equality (3.6) holds.

We now turn to (3.7). A Calculation shows that

$$\begin{vmatrix} y_0 & \alpha_{20} \\ z_0 & \beta_{20} \end{vmatrix} = \frac{2\Gamma^*_2(\theta_0)\sec\theta_0}{A^*_2(\theta_0)}.$$

By Lemma 2.11, it follows that (3.8) holds if and only if $(0, \theta_0)$ is not a sub-parabolic point relative to $\tilde{\mathbf{v}}_2$ over Whitney umbrella.

(2) Now we can reduce φ to

$$\begin{aligned}\varphi = & -\frac{1}{2}u^2 - \frac{1}{8}{a_{20}}^2u^4 - \frac{1}{2}a_{20}a_{11}u^3v \\ & -\frac{1}{4}(2 + 2{a_{11}}^2 + a_{20}a_{02})u^2v^2 - \frac{1}{2}a_{11}a_{02}uv^3 - \frac{1}{8}{a_{02}}^2v^4 + \cdots .\end{aligned}$$

It follows that φ has an A_3-singularity at $(0,0)$. Next, we show that Φ is not $\mathcal{R}^+$-versal. Since A_3-singularity is 4-determined, Φ is an $\mathcal{R}^+$-versal unfolding of φ if and only if the following equality holds.

$$\mathcal{E}_2 = \langle \varphi_u, \varphi_v \rangle_{\mathcal{E}_2} + \langle \Phi_x|_{\boldsymbol{R}^2\times\{\mathbf{0}\}}, \Phi_y|_{\boldsymbol{R}^2\times\{\mathbf{0}\}}, \Phi_z|_{\boldsymbol{R}^2\times\{\mathbf{0}\}} \rangle_{\boldsymbol{R}} + \langle 1 \rangle_{\boldsymbol{R}} + \langle u, v \rangle^5.$$

Since

$$\begin{aligned}&\Phi_x = u + \cdots , \quad \Phi_y = uv + \cdots , \quad \Phi_z = \frac{1}{2}(a_{20}u^2 + 2a_{11}uv + a_{02}v^2) + \cdots , \\ &\varphi_u = -u + \cdots , \\ &\varphi_v = -\frac{1}{2}\left[a_{20}a_{11}u^3 + (2 + a_{20}a_{02} + 2{a_{11}}^2)u^2v + a_{11}a_{02}uv^2 + {a_{02}}^2v^3\right] + \cdots ,\end{aligned}$$

the sum of two ideals $\langle \varphi_u, \varphi_v \rangle_{\mathcal{E}_2} + \langle \Phi_x|_{\boldsymbol{R}^2\times\{\mathbf{0}\}}, \Phi_y|_{\boldsymbol{R}^2\times\{\mathbf{0}\}}, \Phi_z|_{\boldsymbol{R}^2\times\{\mathbf{0}\}} \rangle_{\boldsymbol{R}}$ does not contain v, and Φ is not an $\mathcal{R}^+$-versal unfolding of φ. In a similar way, we can prove that Φ is not $\mathcal{K}$-versal. □

4 Singularities of caustics and fronts of Whitney umbrella

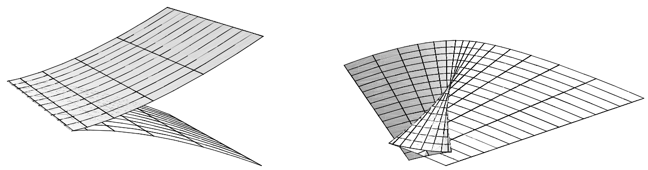
If a smooth function germ $f : (\boldsymbol{R}^2, \mathbf{0}) \to (\boldsymbol{R}, 0)$ is right equivalent to A_2-singularity, then the discriminant set of a $\mathcal{K}$-versal unfolding $F : (\boldsymbol{R}^2 \times \boldsymbol{R}^3, \mathbf{0}) \to (\boldsymbol{R}, 0)$ of f is locally diffeomorphic to the discriminant set of the following unfolding:

$$G(u, v, x, y, z) = u^3 \pm v^2 + x + yu.$$

The singularity of the discriminant set of G is the cuspidal edge. Here, the *cuspidal edge* is the image of a map germ $\mathcal{A}$-equivalent to $(u, v) \mapsto (u, v^2, v^3)$ at the origin. The picture of the cuspidal edge is shown in Figure 2 (i). Similarly, if a smooth function f is right equivalent to A_3-singularity, then the discriminant (resp. bifurcation) set of a $\mathcal{K}$-versal (resp. $\mathcal{R}^+$) unfolding F is locally diffeomorphic to the discriminant (resp. bifurcation) set of the following unfolding:

$$G(u, v, x, y, z) = u^4 \pm v^2 + x + yu + zu^2 \quad (\text{resp. } \hat{G}(u, v, x, y, z) = u^4 \pm v^2 + xu^2 + yu).$$

The singularity of the discriminant set of G (resp. bifurcation set of $\hat{G}$) is the swallowtail (resp. cuspidal edge). Here, the *swallowtail* is the image of a map germ $\mathcal{A}$-equivalent to $(u, v) \mapsto (u, 3v^4 + uv^2, 4v^3 + 2uv)$ at the origin. The picture of the swallowtail is shown in Figure 2 (ii). Therefore, Theorem 3.7 leads to the following.



(i) (ii)

Figure. 2: (i) Cuspidal edge; (ii) swallowtail.

Theorem 4.1. *Let $g : (\boldsymbol{R}^2, \mathbf{0}) \to (\boldsymbol{R}^3, \mathbf{0})$ be given in the normal form of Whitney umbrella. Suppose that $(0, \theta_0)$ is not a parabolic point over Whitney umbrella where $\theta_0 \in (-\pi/2, \pi/2)$.*

(1) *Suppose that $(0, \theta_0)$ is not a ridge point relative to $\tilde{\mathbf{v}}_1$ over Whitney umbrella. Then the front of g at distance $1/|\tilde{\kappa}_1(0, \theta_0)|$ is locally diffeomorphic to a cuspidal edge at $\tilde{\mathbf{n}}(0, \theta_0)/\tilde{\kappa}_1(0, \theta_0)$.*

(2) *Suppose that $(0, \theta_0)$ is a first order ridge point relative to $\tilde{\mathbf{v}}_1$ over Whitney umbrella. Then the caustic of g is locally diffeomorphic to a cuspidal edge at $\tilde{\mathbf{n}}(0, \theta_0)/\tilde{\kappa}_1(0, \theta_0)$. Additionally, if $(0, \theta_0)$ is not a sub-parabolic point relative to $\tilde{\mathbf{v}}_2$ over Whitney umbrella, then the front of g at distance $1/|\tilde{\kappa}_1(0, \theta_0)|$ is locally diffeomorphic to a swallowtail at $\tilde{\mathbf{n}}(0, \theta_0)/\tilde{\kappa}_1(0, \theta_0)$.*

Theorem 4.1 (1) and Proposition 2.16 imply that the front of g has at most four cuspidal edge singularities on C_k. Similarly, Theorem 4.1 (2) and Lemma 2.7 imply that the front of g has at most four swallowtail singularities on C_k. If for example g is given in the normal form of Whitney umbrella determined by coefficients

$$(a_{20}, a_{11}, a_{02}, a_{30}, a_{21}, a_{12}, a_{03}, b_3) = (3, 0, 1, -7, 0, 8/3, 0, 1),$$

then the front of g at distance $1/\sqrt{2}$ has four swallowtail singularities on $C_{\sqrt{2}}$.

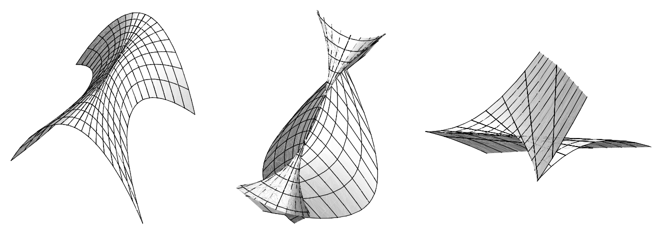
In Theorem 4.3 below, we give the criteria for the cuspidal lips, the cuspidal beaks and the cuspidal butterfly of fronts of Whitney umbrella. To prove Theorem 4.3, we use the criteria for these singularities of parallel surfaces of regular surfaces, which shown in the authors' previous work [7]. We present these criteria as the following:

Theorem 4.2 ([[7], theorem 5·3])**.** *Suppose that $g : U \to \boldsymbol{R}^3$ is a smooth map which defines a regular surface in $\boldsymbol{R}^3$ and that g^t denotes the parallel surface of g at distance t. Assume that $\kappa_i(p) \neq 0$.*

(1) *Assume that $g(p)$ is a first order ridge point relative to the principal vector $\mathbf{v}_i$ and a sub-parabolic point relative to the other principal vector, and* $\det(\text{Hess } \kappa_i(p)) > 0$ *(resp. < 0), where* Hess κ_i *denotes the Hessian matrix of κ_i. Then the parallel surface g^t at distance $t = 1/\kappa_i(p)$ is locally diffeomorphic to a cuspidal lips (resp. cuspidal beaks) at $g^t(p)$.*

(2) *Assume that $g(p)$ is a second order ridge point relative to the principal vector $\mathbf{v}_i$ and not a sub-parabolic point relative to the other principal vector. Then the parallel surface g^t at distance $t = 1/\kappa_i(p)$ is locally diffeomorphic to a cuspidal butterfly at $g^t(p)$.*

Here, the *cuspidal lips* is the image of a map germ $\mathcal{A}$-equivalent to $(u,v) \mapsto (3u^4 + 2u^2v^2, u^3 + uv^2, v)$ at the origin, the *cuspidal beaks* is the image of a map germ $\mathcal{A}$-equivalent to $(u,v) \mapsto (3u^4 - 2u^2v^2, u^3 - uv^2, v)$ at the origin, and the *cuspidal butterfly* is the image of a map germ $\mathcal{A}$-equivalent to $(u,v) \mapsto (4u^5 + u^2v, 5u^4 + 2uv, v)$ at the origin. The pictures of these singularities are shown in Figure 3.



(i) (ii) (iii)

Figure. 3: (i) Cuspidal lips; (ii) cuspidal beaks; (iii) cuspidal butterfly.

Theorem 4.3. *Let $g : (\boldsymbol{R}^2, \mathbf{0}) \to (\boldsymbol{R}^3, \mathbf{0})$ be given in the normal of Whitney umbrella. Suppose that $(0, \theta_0)$ is not a parabolic point over Whitney umbrella, where $\theta_0 \in (-\pi/2, \pi/2)$.*

(1) *Assume that $(0, \theta_0)$ is a first order ridge point relative to $\tilde{\mathbf{v}}_1$ and sub-parabolic point relative to $\tilde{\mathbf{v}}_2$ over Whitney umbrella, and that* $\det(\mathrm{Hess}\,\tilde{\kappa}_1(0,\theta_0)) > 0$ *(resp.* < 0*). Then the front of g at distance $1/|\tilde{\kappa}_1(0,\theta_0)|$ is locally diffeomorphic to a cuspidal lips (resp. cuspidal beaks) at $\tilde{\mathbf{n}}(0,\theta_0)/\tilde{\kappa}_1(0,\theta_0)$.*

(2) *Assume that $(0, \theta_0)$ is a second order ridge point relative to $\tilde{\mathbf{v}}_1$ and not a sub-parabolic point relative to $\tilde{\mathbf{v}}_2$ over Whitney umbrella. Then the front of g at distance $1/|\tilde{\kappa}_1(0,\theta_0)|$ is locally diffeomorphic to a cuspidal butterfly at $\tilde{\mathbf{n}}(0,\theta_0)/\tilde{\kappa}_1(0,\theta_0)$.*

Proof. We shall prove the assertion (1). The proof of the assertion (2) is similar and we omit the detail. We set $\tilde{g} = g \circ \tilde{\pi} : \boldsymbol{R} \times S^1 \to \boldsymbol{R}^3$ and $\tilde{g}^t(r,\theta) = \tilde{g}(r,\theta) + t\tilde{\mathbf{n}}(r,\theta)$ $(t \neq 0)$. Then $\tilde{g}^t$ is the parallel surface of $\tilde{g}$ at distance t, whose image is the front of g at distance $|t|$. The principal radii of $\tilde{g}^t$ are given by

$$\frac{1}{\tilde{\kappa}_i^t} = \frac{1}{\tilde{\kappa}_i} - t. \tag{4.1}$$

We consider two parallel surfaces of $\tilde{g}$: one is at distance $t_0 = 1/\tilde{\kappa}_1(0,\theta_0)$ by $\tilde{g}^{t_0}$, and the other is at distance $t_1 \neq t_0$ by $\tilde{g}^{t_1}$. We remark that $\tilde{g}^{t_0}$ is the parallel surface of $\tilde{g}^{t_1}$ at distance $1/\tilde{\kappa}_1(0,\theta_0) - t_1 = 1/\tilde{\kappa}_1^{t_1}(0,\theta_0)$. Now suppose that $\tilde{g}^{t_1}(0,\theta_0)$ is a first order ridge point relative to $\tilde{\mathbf{v}}_1^{t_1}$ and a sub-parabolic point relative to $\tilde{\mathbf{v}}_2^{t_1}$, that is,

$$\tilde{\mathbf{v}}_1^{t_1}\tilde{\kappa}_1^{t_1}(0,\theta_0) = 0, \quad (\tilde{\mathbf{v}}_1^{t_1})^2\tilde{\kappa}_1^{t_1}(0,\theta_0) \neq 0, \quad \text{and} \quad \tilde{\mathbf{v}}_2^{t_1}\tilde{\kappa}_1^{t_1}(0,\theta_0) = 0, \tag{4.2}$$

and that $\det(\mathrm{Hess}\,\tilde{\kappa}_1^{t_1}(0,\theta_0)) > 0$ (resp. < 0). Then it follows from Theorem 4.2 that $\tilde{g}^{t_0}$ is locally diffeomorphic to a cuspidal lips (resp. cuspidal beaks) at $\tilde{g}^{t_0}(0,\theta_0)$.

Since the lines of curvature on all parallel surfaces correspond to one another (cf. [6] p. 121, see also [26] p. 159), it follows that $\tilde{\mathbf{v}}_1^{t_1}$ (resp. $\tilde{\mathbf{v}}_2^{t_1}$) is parallel to $\tilde{\mathbf{v}}_1$ (resp. $r^2\tilde{\mathbf{v}}_2$) at $(0,\theta_0)$. This and (4.1) show that the condition (4.2) holds if and only if

$$\tilde{\mathbf{v}}_1\tilde{\kappa}_1(0,\theta_0)=0,\quad \tilde{\mathbf{v}}_1^2\tilde{\kappa}_1(0,\theta_0)\neq 0,\quad \text{and}\quad r^2\tilde{\mathbf{v}}_2\tilde{\kappa}_1(0,\theta_0)=0.$$

These conditions are equivalent to that $(0,\theta_0)$ is a first order ridge point relative to $\tilde{\mathbf{v}}_1$ and a sub-parabolic point relative to $\tilde{\mathbf{v}}_2$ over Whitney umbrella. Moreover, it follows from (4.1) that the sign of $\det(\text{Hess } \tilde{\kappa}_1^{t_1}(0,\theta_0))$ is the same as the sign of $\det(\text{Hess } \tilde{\kappa}_1(0,\theta_0))$. Thus we have completed the proof. □

We obtain criteria for the cuspidal edge, the swallowtail, the cuspidal lips, the cuspidal beaks, and the cuspidal butterfly of fronts of Whitney umbrella, and the criterion for the cuspidal edge of caustics of Whitney umbrella except at singular point of Whitney umbrella (Table 1). Finding a normal form of the caustic there is an open problem.

Table 1: Criteria for singularities of caustics and fronts of Whitney umbrella.

		caustic	front
no ridges	—	non singular	cuspidal edge
1-ridges	not sub-parabolic	cuspidal edge	swallowtail
	sub-parabolic	—	cuspidal lips or beaks if CPC is Morse
2-ridges	not sub-parabolic	—	cuspidal butterfly

Example 4.4. Let g be given in the normal form of Whitney umbrella determined by coefficients

$$(a_{20},a_{11},a_{02},a_{30},a_{21},a_{12},a_{03},b_3)=(0,0,1,0,1,-1,0,0).$$

Then we obtain

$$\begin{aligned}
\tilde{\mathbf{n}}(0,\theta)&=(0,-\sin\theta,\cos\theta),\\
\tilde{\kappa}_1(0,\theta)&=-\sin\theta\tan\theta,\\
\Gamma_3(\theta)&=3\cos^3\theta\sin\theta-3\cos^2\theta\sin^2\theta,\\
\Gamma_3^*(\theta)&=2\cos^2\theta\sin\theta+\sin^3\theta,\\
\Gamma_4(\theta)&=-3\cos^6\theta+3\cos^5\theta\sin\theta-3\cos^4\theta\sin^2\theta+12\cos^2\theta\sin^4\theta+3\sin^6\theta.
\end{aligned}$$

We set $k=1/\sqrt{2}$. The CPC line Σ_k and the exceptional set $X=\pi^{-1}(0,0)$ meet at two points $(r,\theta)=(0,\pm\pi/4)$. Therefore, the front of g at distance $\sqrt{2}$ has two singular points on C_k at

$$\frac{\tilde{\mathbf{n}}(0,\pi/4)}{\tilde{\kappa}_1(0,\pi/4)}=(0,1,-1)\quad\text{and}\quad\frac{\tilde{\mathbf{n}}(0,-\pi/4)}{\tilde{\kappa}_1(0,-\pi/4)}=(0,-1,1).$$

Conditions Γ_3, Γ_3^*, and Γ_4 are shown in Table 2. From Table 2, it follows that $(0,\pi/4)$ is the first order ridge point relative to $\tilde{\mathbf{v}}_1$ and not sub-parabolic point relative to $\tilde{\mathbf{v}}_2$ over Whitney umbrella, and $(0,-\pi/4)$ is neither the ridge point relative to $\tilde{\mathbf{v}}_1$ nor sub-parabolic

Table 2: Conditions for points to be the ridge or sub-parabolic point.

	$\Gamma_3(\theta)$	$\Gamma_3^*(\theta)$	$\Gamma_4(\theta)$
$\theta = \pi/4$	0	$3/2$	$3/2$
$\theta = -\pi/4$	$-3/2$	$3/4$	$-$

point relative to $\tilde{\mathbf{v}}_2$ over Whitney umbrella. Hence, the front of g at distance $\sqrt{2}$ is locally diffeomorphic to the swallowtail at $(0, 1, -1)$. Moreover, this front is locally diffeomorphic to the cuspidal edge at $(0, -1, -1)$. The picture of this front is shown in Figure 4. The thick circle in Figure 4 is C_k.

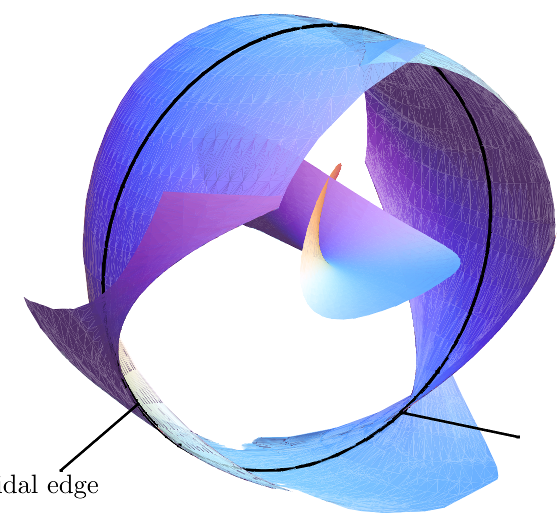


Figure. 4: The front of Whitney umbrella g as in Example 4.4 at distance $\sqrt{2}$.

References

[1] V. I. Arnol'd, S. M. Guseĭn-Zade and A. N. Varchenko. *Singularities of differentiable maps. Vol. I.* Monogr. Math. 82, (Birkhäuser Boston, Inc., 1986.)

[2] J. W. Bruce, P. J. Giblin and F. Tari. Families of surface: focal sets, ridges and umbilics. *Math. Proc. Cambridge Philos. Soc.* **125** (1999), 243–268. DOI: 10.1017/S0305004198003004

[3] J. W. Bruce and J. M. West. Functions on cross-caps. *Math. Proc. Cambridge Philos. Soc.* **123** (1998), 19–39. DOI: 10.1017/S0305004197002132

[4] J. W. Bruce and T. C. Wilkinson. Folding maps and focal sets. In *Singularity theory and its applications, Part I* (Coventry, 1988/1989), 63–72. Lecture Notes in Math. **1462**, (Springer, 1991.)

[5] G. Darboux. *Leçons sur la Théorie générale des Surfaces, IV.* (Gauthier-Villars, 1896).

[6] A. R. Forsyth. *Lectures on the Differential Geometry of Curves and Surfaces.* (Cambridge Univ. Press, 1920).

[7] T. Fukui and M. Hasegawa. Singularities of parallel surfaces. to appear in Tohoku Mathematical Journal.

[8] T. Fukui and J. J. Nuño Ballesteros. Isolated roundings and flattennings of submanifolds in Euclidian spaces. *Tohoku Math. J.* **57** (2005), 469–503. DOI: 10.2748/tmj/1140727069

[9] R. Garcia, C. Gutierrez and J. Sotomayor. Lines of Principal curvature around umbilics and Whitney umbrellas. *Tohoku Math. J.* **52** (2000), 163–172. DOI: 10.2748/tmj/1178224605

[10] C. Gutierrez and J. Sotomayor. Lines of principal curvature for mappings with Whitney umbrella singularities. *Tohoku Math. J.* **38** (1986), 551–559. DOI: 10.2748/tmj/1178228407

[11] H. Hironaka. Stratification and flatness. In *Real and complex singularities* (Oslo, 1976), 199–265. (Sijthoff and Noordhoff, 1977.)

[12] S. Izumiya, M. Takahashi and F. Tari. Folding maps on spacelike and timelike surfaces and duality. *Osaka J. Math.* **47** (2010), 839–862.

[13] S. Izumiya, D. Pei and M. Takahashi. Singularities of evolutes of hypersurfaces in hyperbolic space. *Proc. Edinb. Math. Soc.* **47** (2004), 131–153. DOI: 10.1017/S0013091503000312

[14] S. Izumiya and K. Saji. The mandala of Legendrian dualities for pseudo-spheres in Lorentz-Minkowski space and "flat" spacelike surfaces. *J. Singul.* **2** (2010), 92–127. DOI: 10.5427/jsing.2010.2g

[15] S. Izumiya, K. Saji and M. Takahashi. Horospherical flat surfaces in hyperbolic 3-space. *J. Math. Soc. Japan.* **62** (2010), 789–849. DOI: 10.2969/jmsj/06230789

[16] M. Kokubu, W. Rossman, K. Saji, M. Umehara and K. Yamada. Singularities of flat fronts in hyperbolic 3-space. *Pacific J. Math.* **221** (2005), 303–351. DOI: 10.2140/pjm.2005.221.303

[17] Y. C. Lu. *Singularity theory and an introduction to catastrophe theory.* (Springer-Verlag, 1976.) DOI: 10.1007/978-1-4612-9909-7

[18] R. J. Morris. The sub-parabolic lines of a surface. In *The mathematics of surfaces, VI* (Uxbridge, 1994), 79–102. Inst. Math. Appl. Conf. Ser. New Ser. **58**, (Oxford Univ. Press, 1996.)

[19] J. J. Nuño Ballesteros and F. Tari. Surfaces in $\boldsymbol{R}^4$ and their projections to 3-spaces. *Proc. Roy. Soc. Edinburgh Sect. A* **137** (2007), 1313–1328.

[20] J. M. Oliver. On pairs of foliations of a parabolic cross-cap. *Qual. Theory Dyn. Syst.* **10** (2011), 139–166. DOI: 10.1007/s12346-011-0042-0

[21] I. R. Porteous. The normal singularities of surfaces in $\mathbb{R}^3$. In *Singularities, Part 2* (Arcata, 1981), 379–393. Proc. Sympos. Pure Math. **40**, (Amer. Math. Soc., 1983.)

[22] I. R. Porteous. *Geometric differentiation for the intelligence of curves and surfaces (Second edition).* (Cambridge University Press, 2001.)

[23] K. Saji. Criteria for D_4 singularities of wave fronts. *Tohoku Math. J.* **63** (2011), 137–147. DOI: 10.2748/tmj/1303219939

[24] F. Tari. Pairs of geometric foliations on a cross-cap. *Tohoku Math. J.* **59** (2007), 233-258. DOI: 10.2748/tmj/1182180735

[25] C. T. C. Wall. Finite Determinacy of Smooth Map-Germs. *Bull. London Math. Soc.* **13** (1981), 481–539. DOI: 10.1112/blms/13.6.481

[26] C. E. Weatherburn. *Differential geometry of three dimensions.* (Cambridge University Press, 1955.)

[27] J. M. West. The differential geometry of the cross-cap. Ph.D. thesis, The University of Liverpool, 1995.

[28] H. Whitney. The general type of singularity of a set of $2n-1$ smooth functions of n variables. *Duke Math. J.* **10** (1943), 161-172. DOI: 10.1215/S0012-7094-43-01016-6

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