ON A NEWTON FILTRATION FOR FUNCTIONS ON A CURVE SINGULARITY

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ABSTRACT. In a previous paper, there was defined a multi-index filtration on the ring of functions on a hypersurface singularity corresponding to its Newton diagram generalizing (for a curve singularity) the divisorial one. Its Poincaré series was computed for plane curve singularities non-degenerate with respect to their Newton diagrams. Here we use another technique to compute the Poincaré series for plane curve singularities without the assumption that they are non-degenerate with respect to their Newton diagrams. We show that the Poincaré series only depends on the Newton diagram and not on the defining equation.

INTRODUCTION

In [2, 3], there were defined two multi-index filtrations on the ring $\mathcal{O}_{\mathbb{C}^n,0}$ of germs of holomorphic functions in n variables associated to a Newton diagram Γ in $\mathbb{R}^n$ and to a germ of an analytic function $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ with this Newton diagram. We assumed that the function f was non-degenerate with respect to its Newton diagram Γ. These filtrations are essentially filtrations on the ring $\mathcal{O}_{V,0} = \mathcal{O}_{\mathbb{C}^n,0}/(f)$ of germs of functions on the hypersurface singularity $V = \{f = 0\}$. They correspond to the quasihomogeneous valuations on the ring $\mathcal{O}_{\mathbb{C}^n,0}$ defined by the facets of the diagram Γ. These facets correspond to some components of the exceptional divisor of a toric resolution of the germ f constructed from the diagram Γ. Such a component defines the corresponding divisorial valuation on the ring $\mathcal{O}_{\mathbb{C}^n,0}$. For $n \geqslant 3$ (and for a Γ-non-degenerate f) these valuations induce divisorial valuations on the ring $\mathcal{O}_{V,0}$ and define the corresponding multi-index filtration on it. The filtration defined in [2] was expected to be a certain "simplification" of the divisorial one. This appeared not to be the case. For example, a general formula for the Poincaré series of this filtration is not known even for the number of variables $n = 2$. For Newton diagrams of special type, A. Lemahieu identified this filtration with a so called embedded filtration on $\mathcal{O}_{V,0}$ [7]. In [7], a formula for the Poincaré series of the embedded filtration for a hypersurface singularity was given. H. Hamm studied the embedded filtration and the corresponding Poincaré series for complete intersection singularities [6].

In [3], there was given an "algebraic" definition of the divisorial valuation corresponding to a Newton diagram (for $n \geqslant 3$) somewhat similar to the definition in [2]. Roughly speaking, the difference consists in using the ring $\mathcal{O}_{\mathbb{C}^n,0}[x_1^{-1}, \dots, x_n^{-1}]$ instead of $\mathcal{O}_{\mathbb{C}^n,0}$. For $n = 2$, this definition does not give, in general, a valuation, but an order function (see the definition below). For a Γ-non-degenerate $f \in \mathcal{O}_{\mathbb{C}^2,0}$, this order function was described as a "generalized divisorial valuation" defined by the divisorial valuations corresponding to all the points of intersection of the resolution (normalization) $\widetilde{V}$ of the curve V with the corresponding component of the exceptional divisor. This permitted to apply the technique elaborated in [1] and to compute the corresponding Poincaré series. (This technique has no analogue which could be applied to

degenerate f, or to the case $n > 2$, or to the filtration defined in [2].) In this case the Poincaré series depends only on the Newton diagram Γ and does not depend on the function f with $\Gamma_f = \Gamma$.

The definitions in [2] and [3] make also sense for functions f degenerate with respect to their Newton diagrams. Here we compute the Poincaré series of the filtration introduced in [3] for $n = 2$ directly from the definition without the assumption that f is non-degenerate with respect to the Newton diagram. We show that the answer is the same as in [3, Corollary 1] for non-degenerate f. Thus, for $n = 2$, the Poincaré series of this filtration depends only on the Newton diagram Γ. One can speculate that the same holds for $n \geqslant 3$ and for the Poincaré series of the filtration defined in [2].

We hope that some elements of the technique used here can be applied to the case $n \geqslant 3$ and/or to the filtration defined in [2] as well.

One motivation to study (multi-variable) Poincaré series of filtrations comes from the fact that they are sometimes related or even coincide with appropriate monodromy zeta functions or with Alexander polynomials (see e.g. [1]). We show that the obtained formula for the Poincaré series has a relation to the (multi-variable) Alexander polynomial of a collection of functions.

1. FILTRATIONS ASSOCIATED TO NEWTON DIAGRAMS

Let $(V, 0)$ be a germ of a complex analytic variety and let $\mathcal{O}_{V,0}$ be the ring of germs of holomorphic functions on $(V, 0)$. A map $v : \mathcal{O}_{V,0} \to \mathbb{Z}_{\geqslant 0} \cup \{+\infty\}$ is an *order function* on $\mathcal{O}_{V,0}$ if $v(\lambda g) = v(g)$ for a non-zero $\lambda \in \mathbb{C}$ and $v(g_1 + g_2) \geqslant \min\{v(g_1), v(g_2)\}$. (If, moreover, $v(g_1 g_2) = v(g_1) + v(g_2)$, the map v is a *valuation* on $\mathcal{O}_{V,0}$.) A collection $\{v_1, v_2, \ldots, v_r\}$ of order functions on $\mathcal{O}_{V,0}$ defines a multi-index filtration on $\mathcal{O}_{V,0}$:

$$(1) \qquad J(\underline{v}) := \{g \in \mathcal{O}_{V,0} : \underline{v}(g) \geqslant \underline{v}\}$$

for $\underline{v} = (v_1, \ldots, v_r) \in \mathbb{Z}_{\geqslant 0}^r$, $\underline{v}(g) = (v_1(g), \ldots, v_r(g))$, $\underline{v}' = (v_1', \ldots, v_r') \geqslant \underline{v}'' = (v_1'', \ldots, v_r'')$ iff $v_i' \geqslant v_i''$ for all $i = 1, \ldots, r$. (It is convenient to assume that the equation (1) defines the subspaces $J(\underline{v}) \subset \mathcal{O}_{V,0}$ for all $\underline{v} \in \mathbb{Z}^r$.) The *Poincaré series* $P_{\{v_i\}}(\underline{t})$ ($\underline{t} = (t_1, \ldots, t_r)$) of the filtration (1) can be defined as

$$(2) \qquad P_{\{v_i\}}(\underline{t}) := \frac{\left(\sum_{\underline{v} \in \mathbb{Z}^r} \dim(J(\underline{v})/J(\underline{v}+\underline{1}))\underline{t}^{\underline{v}}\right) \prod_{i=1}^r (t_i - 1)}{(t_1 t_2 \cdots t_r - 1)},$$

where $\underline{1} = (1, \ldots, 1) \in \mathbb{Z}^r$, $\underline{t}^{\underline{v}} = t_1^{v_1} \cdots t_r^{v_r}$ (see e.g. [1]; it is defined when the dimensions of all the factor spaces $J(\underline{v})/J(\underline{v}+\underline{1})$ are finite). In [1] it was explained that the Poincaré series (2) is equal to the integral with respect to the Euler characteristic

$$(3) \qquad P_{\{v_i\}}(\underline{t}) = \int_{\mathbb{P}\mathcal{O}_{V,0}} \underline{t}^{\underline{v}(g)} d\chi$$

over the projectivization $\mathbb{P}\mathcal{O}_{V,0}$ of the space $\mathcal{O}_{V,0}$. (In the integral, $t_i^{+\infty}$ has to be assumed to be equal to zero.)

Let $f \in \mathcal{O}_{\mathbb{C}^n,0}$ be a function germ with the Newton diagram $\Gamma = \Gamma_f \subset \mathbb{R}^n$, $V := \{f = 0\}$. Let γ_i, $i = 1, \ldots, r$, be (all) the facets of the diagram Γ and let $\ell_i(\bar{k}) = c_i$ be the reduced equation of the hyperplane containing the facet γ_i. One has $\ell_i(\bar{k}) = \sum_{j=1}^n \ell_{ij} k_j$ ($\bar{k} = (k_1, \ldots, k_n)$), where ℓ_{ij} are positive integers, $\gcd(\ell_{i1}, \ldots, \ell_{in}) = 1$.

For $g \in \mathcal{O}_{\mathbb{C}^n,0}[x_1^{-1}, \ldots, x_n^{-1}]$, $g = \sum_{\bar{k} \in \mathbb{Z}^n} a_{\bar{k}} \bar{x}^{\bar{k}}$ ($\bar{x} = (x_1, \ldots, x_n)$), let $\operatorname{supp} g := \{\bar{k} \in \mathbb{Z}^n : a_{\bar{k}} \neq 0\}$ and let

$$u_i(g) := \min_{\bar{k}: a_{\bar{k}} \neq 0} \ell_i(\bar{k}).$$

One can see that u_i is a valuation on $\mathcal{O}_{\mathbb{C}^n,0} \subset \mathcal{O}_{\mathbb{C}^n,0}[x_1^{-1}, \ldots, x_n^{-1}]$. For a Newton diagram Λ in $\mathbb{R}^n$, let

$$u_i(\Lambda) := \min_{\bar{k} \in \Lambda} \ell_i(\bar{k}).$$

(It is also equal to $u_i(g)$ for any germ g with the Newton diagram Λ.) Let $g_{\gamma_i}(\bar{x}) := \sum\limits_{\bar{k}:\ell_i(\bar{k})=u_i(g)} a_{\bar{k}}\bar{x}^{\bar{k}}$.

The following two collections of order functions on $\mathcal{O}_{\mathbb{C}^n,0}$ corresponding to the pair (Γ, f) were defined in [2] and [3] respectively:

$$(4) \qquad\qquad v_i'(g) \quad := \quad \sup_{h \in \mathcal{O}_{\mathbb{C}^n,0}} u_i(g + hf),$$

$$(5) \qquad\qquad v_i''(g) \quad := \quad \sup_{h \in \mathcal{O}_{\mathbb{C}^n,0}[x_1^{-1}, \ldots, x_n^{-1}]} u_i(g + hf).$$

(v_i' and v_i'' are, in general, not valuations, at least when $n = 2$ or when f is degenerate with respect to its Newton diagram Γ.) They can be considered as order functions on the ring $\mathcal{O}_{V,0} = \mathcal{O}_{\mathbb{C}^n,0}/(f)$ as well. (These order functions and moreover the corresponding Poincaré series are, in general, different.)

Assume that the function f is non-degenerate with respect to its Newton diagram Γ and let $p : (X, D) \to (\mathbb{C}^n, 0)$ be a toric resolution of f corresponding to the Newton diagram Γ. The facets $\gamma_1, \ldots, \gamma_r$ of Γ correspond to some components (say, $E_1, \ldots, E_r$) of the exceptional divisor D. Let $\widetilde{V}$ be the strict transform of the hypersurface singularity V (it is a smooth complex manifold) and let $\mathcal{E}_i := \widetilde{V} \cap E_i$, $i = 1, \ldots, r$.

For $n \geqslant 3$, the set $\mathcal{E}_i$ is an irreducible component of the exceptional divisor $\mathcal{D} = D \cap \widetilde{V}$ of the resolution $p_{|\widetilde{V}} : (\widetilde{V}, \mathcal{D}) \to (V, 0)$. The divisorial valuation $v_{\mathcal{E}_i}$ on $\mathcal{O}_{V,0}$ defined by this component coincides with v_i'': see [3]. For $n = 2$, the set $\mathcal{E}_i$ is, in general, reducible (if the integer length s_i of the facet (edge) γ_i is greater than 1). Let $\mathcal{E}_i = \bigcup\limits_{j=1}^{s_i} \mathcal{E}_i^{(j)}$ be the decomposition into irreducible components ($\mathcal{E}_i^{(j)}$ are points on the curve $\widetilde{V}$). One can show that in this case $v_i''(g) = \min\limits_j v_{\mathcal{E}_i^{(j)}}(g)$, where $v_{\mathcal{E}_i^{(j)}}$ are the corresponding divisorial valuations on $\mathcal{O}_{V,0}$. This order function v_i'' can be regarded as a generalized divisorial valuation.

2. THE POINCARÉ SERIES

Let Γ be a Newton diagram in $\mathbb{R}^2$ with the facets (edges) $\gamma_1, \ldots, \gamma_r$ and let f be a function germ $(\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ with the Newton diagram Γ. One can see that $f = x^a y^b \prod\limits_{i=1}^{r} f_i$, where f_i is such that $f_{\gamma_i} = \lambda_i \bar{x}^{\bar{k}_i} (f_i)_{\gamma_i}$ for certain $\lambda_i \in \mathbb{C}^*$ and $\bar{k}_i \in \mathbb{Z}_{\geqslant 0}^2$. The Newton diagram Γ_i of the germ f_i consists of one segment congruent (by a shift; in particular, parallel) to the facet γ_i with the vertices on the coordinate lines in $\mathbb{R}^2$.

Let $\underline{M}_i := \underline{u}(\Gamma_i)$, i.e. $\underline{M}_i = (M_{i1}, \ldots, M_{ir})$, where $M_{ij} := u_j(\Gamma_i)$. (One can see that $\underline{M}_i = s_i \underline{m}_i$ in the notations of [3].)

Theorem 1. *One has*

$$(6) \qquad\qquad P_{\{v_i''\}}(\underline{t}) = \frac{\prod\limits_{i=1}^{r}(1 - \underline{t}^{\underline{M}_i})}{(1 - \underline{t}^{\underline{u}(x)})(1 - \underline{t}^{\underline{u}(y)})}.$$

Corollary 1. *For the number of variables $n = 2$, the Poincaré series $P_{\{v_i''\}}(\underline{t})$ depends only on the Newton diagram Γ and does not depend on f with $\Gamma_f = \Gamma$.*

For the proof of Theorem 1 we need some auxiliary statements. We first introduce some notations.

For a Newton diagram Λ in $\mathbb{R}^2$, let Σ_Λ be the corresponding Newton polygon: $\Sigma_\Lambda := \bigcup_{\bar{q} \in \Lambda} \left(\bar{q} + \mathbb{R}^2_{\geqslant 0} \right)$. Let $\mathcal{O}^\Lambda$ be the set of functions $g \in \mathcal{O}_{\mathbb{C}^2,0}$ with the Newton diagram $\Gamma_g = \Lambda$. For $\underline{v} \in \mathbb{Z}^r_{\geqslant 0}$, let $\mathcal{O}^\Lambda_{\underline{v}} := \{ g \in \mathcal{O}^\Lambda : \underline{v}''(g) = \underline{v} \}$. The set $\mathcal{O}_{\mathbb{C}^2,0} \setminus \{0\}$ is the disjoint union of the sets $\mathcal{O}^\Lambda$ over all diagrams Λ. According to (3) one has

(7) $$ P_{\{v''_i\}}(\underline{t}) = \sum_\Lambda \int_{\mathbb{P}\mathcal{O}^\Lambda} \underline{t}^{\underline{v}''(g)} d\chi \,. $$

For $g \in \mathcal{O}^\Lambda$, one has $v''_i(g) \geqslant u_i(\Lambda)$. We shall first show that the integrals in (7) can be restricted only to functions $g \in \mathbb{P}\mathcal{O}^\Lambda$ with $\underline{v}''(g) = \underline{u}(\Lambda)$.

Proposition 1. *For a Newton diagram Λ in $\mathbb{R}^2$, let $\underline{v} \in \mathbb{Z}^r_{\geqslant 0}$ be such that $\underline{v} > \underline{u}(\Lambda)$, i.e. $v_i \geqslant u_i(\Lambda)$ for all $i = 1, \ldots, r$ and $v_i > u_i(\Lambda)$ for some i. Then the set $\mathbb{P}\mathcal{O}^\Lambda_{\underline{v}}$ has Euler characteristic equal to zero.*

Remark. The direct analogue of this proposition does not hold for the filtration defined by the order functions $\{v'_i\}$. As an example one can take $f(x, y) = y^5 + xy^2 + x^2y + x^5$ whose Newton diagram Γ has the set of vertices $\{(0,5), (1,2), (2,1), (5,0)\}$. One has $\ell_1(\bar{k}) = 3k_x + k_y$, $\ell_2(\bar{k}) = k_x + k_y$, $\ell_3(\bar{k}) = k_x + 3k_y$. Let Λ be the Newton diagram with the set of vertices $\{(0,5), (1,2)\}$. One has $\underline{u}(\Lambda) = (5,3,7)$. Let us take $\underline{v} = (7,3,7)$. One can see that for the order functions $\{v'_i\}$ the set $\mathcal{O}^\Lambda_{\underline{v}}$ consists of the germs $g(x,y) = \sum a_{ij}x^iy^j$ from $\mathcal{O}^\Lambda$ with $a_{05} = a_{12} \neq 0$ and $a_{06} = a_{13} = 0$. This gives $\chi(\mathbb{P}\mathcal{O}^\Lambda_{\underline{v}}) = 1$. For the order functions $\{v''_i\}$ the set $\mathcal{O}^\Lambda_{\underline{v}}$ consists of the germs with $a_{05} = a_{12} \neq 0$, $a_{06} = a_{13} = 0$ and $a_{07} - a_{14} + a_{21} \neq 0$. This gives $\chi(\mathbb{P}\mathcal{O}^\Lambda_{\underline{v}}) = 0$ in accordance with Proposition 1.

For the proof of Proposition 1 we need two lemmas.

Let $\mathcal{O}^\Lambda_{\underline{v}}$ be non-empty. For $g \in \mathcal{O}^\Lambda$ with $\underline{v}''(g) = \underline{v}$ and for $i = 1, \ldots, r$, one can find $h(= h_i) \in \mathcal{O}_{\mathbb{C}^2,0}[x^{-1}, y^{-1}]$ such that the Newton diagram of $g + hf$ lies in the (closed) half-plane $H_i := \{ \bar{k} : \ell_i(\bar{k}) \geqslant v_i \}$, but there are no h for which the Newton diagram of $g + hf$ lies in the open half-plane $\{ \bar{k} : \ell_i(\bar{k}) > v_i \}$. Let Λ^* be the union of the compact edges of the (infinite) polygon $\Sigma^*_\Lambda := \left(\bigcap_{i=1}^r H_i \right) \cap \Sigma_\Lambda$, where Σ_Λ is the Newton polygon corresponding to Λ. (Λ^* is not, in general, a Newton diagram since it may have non-integral vertices. Nevertheless we shall use the name "diagram" for it.)

Lemma 1. *In the situation described above, there exists an index i ($1 \leqslant i \leqslant r$) such that Λ^* has an edge δ_i parallel to γ_i and (strictly) longer than γ_i.*

Proof. We shall prove that there exists an edge of the diagram Λ^* which is (strictly) longer than the edge of Λ parallel to it. This implies that this edge is parallel to a certain edge γ_i of the diagram Γ and is longer than it. Since we assumed $\mathcal{O}^\Lambda_{\underline{v}}$ being non-empty, all edges of Λ^* are parallel to edges of Λ. Let $a_0 < a_1 < \ldots < a_\sigma$ be the k_x-coordinates of all the vertices of Λ. Let $b_0 \leqslant b_1 \leqslant \ldots \leqslant b_\sigma$ be the k_x-coordinates of the corresponding vertices of Λ^*, i.e. $[b_{i-1}, b_i]$ is the projection of the segment of Λ^* parallel to the segment of Λ projected to $[a_{i-1}, a_i]$ (b_{i-1} and b_i may coincide). One can see that $a_0 = b_0$ and $a_\sigma \leqslant b_\sigma$. Then either $b_i = a_i$ for all $i = 0, 1, \ldots, \sigma$ or $[a_{i-1}, a_i] \subsetneq [b_{i-1}, b_i]$ for some $i \in \{0, 1, \ldots, \sigma\}$. But the first case cannot happen since $\Lambda^* \neq \Lambda$. $\qquad \square$

We shall also use the following generalized version of the division with remainder for Laurent polynomials.

Lemma 2. *Let $p(z)$ and $q(z)$ be Laurent polynomials in z. Assume that $\operatorname{supp} q$ has length s, i.e. $q(z) = \sum\limits_{i=0}^{s} b_i z^{d+i}$ with $b_0 \neq 0$, $b_s \neq 0$, and let d' be an integer. Then the polynomial $p(z)$ has a unique representation of the form $p(z) = q(z)a(z) + r(z)$ with $r(z) = \sum\limits_{i=0}^{s-1} c_i z^{d'+i}$.*

Proof of Proposition 1. Let i be as in Lemma 1. Let the integer length of γ_i be equal to s_i. Then the segment δ_i contains at least s_i integer points. Let $Q_1, \ldots, Q_{s_i}$ be s_i consecutive integer points on the segment δ_i. Let $g \in \mathcal{O}^\Lambda$ be such that $\underline{v}''(g) = \underline{v}$ ($> \underline{u}(\Lambda)$) and let $\widetilde{g} = g + hf$ be a Laurent polynomial such that $\operatorname{supp} \widetilde{g} \subset H_i = \{\ell_i(\bar{k}) \geqslant v_i\}$. Lemma 2 implies that $\widetilde{g}_{\gamma_i}(x, y) = f_{\gamma_i}(x, y)p_i(x, y) + r_i(x, y)$, where $\operatorname{supp} r_i \subset \{Q_1, \ldots, Q_{s_i}\}$ and $r_i \neq 0$ (otherwise $v_i(g) > v_i = u_i(\Lambda^*)$). (Let us recall that $\operatorname{supp} f_{\gamma_i}$ consists of $s_i + 1$ consecutive points on the line containing γ_i.) Moreover the polynomial r_i depends only on g and does not depend on the choice of $\widetilde{g}$ (i.e. on the choice of h).

One can see that all functions g' of the form $g + (\lambda - 1)r_i$ with $\lambda \neq 0$ lie in $\mathcal{O}^\Lambda$ and satisfy the condition $v_i''(g') = v_i''(g)$. Thus the set $\mathbb{P}\mathcal{O}_{\underline{v}}^\Lambda$ is fibred by $\mathbb{C}^*$-families and therefore its Euler characteristic is equal to zero. $\square$

Proposition 1 implies that

$$(8) \qquad P_{\{v_i''\}}(\underline{t}) = \sum_\Lambda \int_{\mathbb{P}\mathcal{O}_{\underline{u}(\Lambda)}^\Lambda} \underline{t}^{\underline{v}''(g)} d\chi \,.$$

Proposition 2. *Suppose that a Newton diagram Λ contains an edge δ not congruent to any edge of Γ, i.e. either not parallel to all the edges γ_i, or parallel to one of them, but of another length. Then $\chi(\mathbb{P}\mathcal{O}_{\underline{u}(\Lambda)}^\Lambda) = 0$.*

Proof. Assume first that the edge δ is either not parallel to all the edges γ_i, or it is parallel to γ_i, but is shorter than γ_i. Let $\bar{q} = (q_x, q_y)$ and $\bar{q}' = (q_x', q_y')$, $q_x > q_x'$, be the vertices of the edge δ and let Λ' be the set of points $\bar{k}$ in Λ with $k_x \geqslant q_x$. A function germ $g \in \mathcal{O}_{\underline{u}(\Lambda)}^\Lambda$ can be represented as $g_1 + g_2$, where $\operatorname{supp} g_1 \subset \Lambda'$, $\operatorname{supp} g_2 \subset \Sigma_\Lambda \setminus \Lambda'$. (Note that $g_1 \neq 0$ and $g_2 \neq 0$.) One can see that all the functions of the form $g_1 + \lambda g_2$ with $\lambda \neq 0$ lie in $\mathcal{O}_{\underline{u}(\Lambda)}^\Lambda$. Thus the set $\mathbb{P}\mathcal{O}_{\underline{u}(\Lambda)}^\Lambda$ is fibred by $\mathbb{C}^*$-families and therefore its Euler characteristic is equal to zero.

Now assume that the edge δ of the diagram Λ is parallel to γ_i and is longer than it. Let $\bar{q} = (q_x, q_y)$ and $\bar{q}' = (q_x', q_y')$, $q_x > q_x'$, (respectively $\bar{q}_0 = (q_{0x}, q_{0y})$ and $\bar{q}_0' = (q_{0x}', q_{0y}')$, $q_{0x} > q_{0x}'$) be the vertices of the edge δ (respectively of the edge γ_i) and let Λ' be defined as above: $\Lambda' = \{\bar{k} \in \Lambda : k_x \geqslant q_x\}$. Let $g(\bar{x}) = \sum a_{\bar{k}} \bar{x}^{\bar{k}} \in \mathcal{O}_{\underline{u}(\Lambda)}^\Lambda$, $f(\bar{x}) = \sum c_{\bar{k}} \bar{x}^{\bar{k}}$ ($\bar{x} = (x, y)$) and let

$$g_1(\bar{x}) = g_{\Lambda'}(\bar{x}) - a_{\bar{q}}\bar{x}^{\bar{q}} + (a_{\bar{q}}/c_{\bar{q}_0})f_{\gamma_i}(\bar{x}) \cdot \bar{x}^{\bar{q} - \bar{q}_0},$$

where $g_{\Lambda'}(\bar{x}) = \sum\limits_{\bar{k} \in \Lambda'} a_{\bar{k}}\bar{x}^{\bar{k}}$, $g_2 = g - g_1$. One has $\operatorname{supp} g_1 \subset \Lambda' \cup (\bar{q}, \bar{q}')$, $\operatorname{supp} g_2 \subset \Sigma_\Lambda \setminus \Lambda'$, $f_{\gamma_i} \nmid (g_2)_{\gamma_i}$ (in $\mathcal{O}_{\mathbb{C}^2,0}[x^{-1}, y^{-1}]$), where $(\bar{q}, \bar{q}')$ denotes the open line segment connecting the two points. All the functions of the form $g_1 + \lambda g_2$ with $\lambda \neq 0$ lie in $\mathcal{O}_{\underline{u}(\Lambda)}^\Lambda$. Thus the set $\mathbb{P}\mathcal{O}_{\underline{u}(\Lambda)}^\Lambda$ is again fibred by $\mathbb{C}^*$-families and therefore its Euler characteristic is equal to zero. $\square$

Proposition 3. *Let the Newton diagram Λ consist (only) of segments congruent to γ_i for $i \in I \subset \{1, \ldots, r\}$. Then $\chi(\mathbb{P}\mathcal{O}_{\underline{u}(\Lambda)}^\Lambda) = (-1)^{\#I}$.*

Proof. For $I = \emptyset$, the statement is obvious. Let $I \neq \emptyset$. Let $\mathbb{P}\mathcal{O}_i^\Lambda$ be the set of functions $g \in \mathbb{P}\mathcal{O}_{\underline{u}(\Lambda)}^\Lambda$ with $f_{\gamma_i}|g_{\gamma_i}$ (in $\mathcal{O}_{\mathbb{C}^2,0}[x^{-1}, y^{-1}]$). One has

$$\mathbb{P}\mathcal{O}_{\underline{u}(\Lambda)}^\Lambda = \mathbb{P}\mathcal{O}^\Lambda \setminus \bigcup_{i \in I} \mathbb{P}\mathcal{O}_i^\Lambda .$$

Therefore

$$(9) \qquad \chi(\mathbb{P}\mathcal{O}_{\underline{u}(\Lambda)}^\Lambda) = \chi(\mathbb{P}\mathcal{O}^\Lambda) + \sum_{I' \subset I, I' \neq \emptyset} (-1)^{\#I'} \chi\left(\bigcap_{i \in I'} \mathbb{P}\mathcal{O}_i^\Lambda\right) .$$

Let $\bar{q}_i$, $i = 0, 1, \ldots, \#I$, be the vertices of the diagram Λ. The set $\mathbb{P}\mathcal{O}^\Lambda$ consists of functions $g(\bar{x}) = \sum a_{\bar{k}} \bar{x}^{\bar{k}}$ with $a_{\bar{q}_i} \neq 0$ for $i = 0, 1, \ldots, \#I$ and $a_{\bar{k}} = 0$ for $\bar{k} \notin \Sigma^\Lambda$. Its Euler characteristic is equal to zero. Assume that $I' \subsetneq I$, $I' \neq \emptyset$. Let $[\bar{q}, \bar{q}']$ be an edge of Λ congruent to γ_i, $i \in I \setminus I'$ ($\bar{q} = (q_x, q_y)$, $\bar{q}' = (q'_x, q'_y)$, $q_x > q'_y$). Let $\Lambda' = \{\bar{k} \in \Lambda : k_x \geqslant q_x\}$. For $g \in \bigcap_{i \in I'} \mathbb{P}\mathcal{O}_i^\Lambda$, let $g_1(\bar{x}) = g_{\Lambda'}(\bar{x})$, $g_2 = g - g_1$. All the functions of the form $g_1 + \lambda g_2$ with $\lambda \neq 0$ belong to $\bigcap_{i \in I'} \mathbb{P}\mathcal{O}_i^\Lambda$. Thus $\bigcap_{i \in I'} \mathbb{P}\mathcal{O}_i^\Lambda$ is fibred by $\mathbb{C}^*$-families and therefore $\chi(\bigcap_{i \in I'} \mathbb{P}\mathcal{O}_i^\Lambda) = 0$.

Let $f_I(\bar{x}) := \prod_{i \in I} f_i(\bar{x})$. The intersection $\bigcap_{i \in I} \mathbb{P}\mathcal{O}_i^\Lambda$ (the Euler characteristic of which corresponds to $I' = I$ in (9)) consists of the functions $g \in \mathbb{P}\mathcal{O}_{\mathbb{C}^2,0}$ such that $g_\Lambda(\bar{x}) = \lambda \bar{x}^{\bar{a}} (f_I)_{\Gamma_{f_I}}$ where Γ_{f_I} is the Newton diagram of f_I, $\lambda \neq 0$ and $\bar{x}^{\bar{a}}$ is a certain monomial. Therefore

$$\chi\left(\bigcap_{i \in I} \mathbb{P}\mathcal{O}_i^\Lambda\right) = 1.$$

$\square$

Proof of Theorem 1. Propositions 1 and 2 imply that

$$(10) \qquad P_{\{v_i''\}}(\underline{t}) = \sum_\Lambda \int_{\mathbb{P}\mathcal{O}_{\underline{u}(\Lambda)}^\Lambda} \underline{t}^{\underline{v}''(g)} d\chi$$

where the sum runs over all diagrams Λ consisting only of edges congruent to some of the edges γ_i of the diagram Γ. Let the edges of Λ be congruent to the edges γ_i with $i \in I = I(\Lambda)$. Proposition 3 implies that the summand in (10) corresponding to such a diagram Λ is equal to $(-1)^{\#I}\underline{t}^{\underline{u}(\Lambda)}$. All the diagrams of this sort are obtained from the diagrams $\Gamma_I = \Gamma_{f_I}$ by shifts by non-negative integral vectors $\bar{k}$, i.e. $\Lambda = \bar{k} + \Gamma_I$. One has $\underline{u}(\Lambda) = \underline{\ell}(\bar{k}) + \sum_{i \in I} \underline{M}_i$. Therefore

$$
\begin{aligned}
P_{\{v_i''\}}(\underline{t}) &= \sum_{\bar{k} \in \mathbb{Z}_{\geqslant 0}^2} \sum_{I \subset \{1, \ldots, r\}} (-1)^{\#I} \underline{t}^{\underline{\ell}(\bar{k}) + \sum_{i \in I} \underline{M}_i} \\
&= \left(\sum_{\bar{k} \in \mathbb{Z}_{\geqslant 0}^2} \underline{t}^{\underline{\ell}(\bar{k})}\right) \cdot \left(\sum_{I \subset \{1, \ldots, r\}} (-1)^{\#I} \underline{t}^{\sum_{i \in I} \underline{M}_i}\right) \\
&= \frac{\prod_{i=1}^r (1 - \underline{t}^{\underline{M}_i})}{(1 - \underline{t}^{\underline{u}(x)})(1 - \underline{t}^{\underline{u}(y)})}.
\end{aligned}
$$

$\square$

3. RELATION WITH AN ALEXANDER POLYNOMIAL

One can see that the equation (6) gives the Poincaré series $P_{\{v_i''\}}(\underline{t})$ as a finite product/ratio of "cyclotomic" binomials of the form $(1 - \underline{t}^{\underline{M}})$ with $\underline{M} \in \mathbb{Z}_{>0}^r$. This looks similar to the usual A'Campo type expressions for monodromy zeta functions or for Alexander polynomials of algebraic links [4]. Here we shall describe a relation between the Poincaré series (6) and a certain Alexander polynomial.

A notion of the multi-variable Alexander polynomial for a finite collection of germs of functions on $(\mathbb{C}^n, 0)$ was defined in [8]: see Proposition 2.6.2 therein. (In [8] it is called the (multi-variable) zeta function.) In a somewhat more precise form this definition can be found in [5]. (The definition in [5] gives the one for a collection of functions if one considers the corresponding principal ideals.)

As above, let Γ be a Newton diagram in $\mathbb{R}^2$ with the edges $\gamma_1, \ldots, \gamma_r$ of integer lengths $s_1, \ldots, s_r$ and let $p : (X, D) \to (\mathbb{C}^2, 0)$ be a toric modification of $(\mathbb{C}^2, 0)$ corresponding to the diagram Γ. For $i = 1, \ldots, r$, let $\widetilde{C}_i$ be a germ of a smooth curve on X transversal to the component E_i of the exceptional divisor D. Let $C_i = p(\widetilde{C}_i)$ be the image of $\widetilde{C}_i$ in $(\mathbb{C}^2, 0)$ (a *curvette* corresponding to the component E_i) and let $L_i = C_i \cap S_\varepsilon^3$ be the corresponding knot in the 3 sphere $S_\varepsilon^3 = S_\varepsilon^3(0)$ for $\varepsilon > 0$ small enough. The curve C_i can be defined by an equation $g_i = 0$ where g_i is a function germ $(\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ with the Newton diagram consisting of one segment parallel to γ_i, with (integer) length 1 and with the vertices on the coordinate lines.

Let $\Delta_{\underline{g}}(\underline{t})$ and $\Delta_{\underline{g}^{\underline{s}}}(\underline{t})$ be the Alexander polynomials of the collections of functions $\underline{g} = (g_1, \ldots, g_r)$ and $\underline{g}^{\underline{s}} = (g_1^{s_1}, \ldots, g_r^{s_r})$ respectively. The polynomial $\Delta_{\underline{g}}(\underline{t})$ is the classical Alexander polynomial $\Delta^L(\underline{t})$ of the link $L = \bigcup L_i$ (see e.g. [4]). A one-variable analogue of $\Delta_{\underline{g}^{\underline{s}}}(\underline{t})$ is considered in [4, I.5] as the Alexander polynomial of the multilink $L(\underline{s}) := \bigcup s_i L_i$. One has

$$\Delta_{\underline{g}}(\underline{t}) = \frac{\prod_{i=1}^r (1 - \underline{t}^{\underline{m}_i})}{(1 - \underline{t}^{\underline{u}(x)})(1 - \underline{t}^{\underline{u}(y)})},$$

$$\Delta_{\underline{g}^{\underline{s}}}(\underline{t}) = \frac{\prod_{i=1}^r (1 - \underline{t}^{\underline{s}\,\underline{m}_i})}{(1 - \underline{t}^{\underline{u}(x)})(1 - \underline{t}^{\underline{u}(y)})},$$

where $\underline{s}\,\underline{m}_i = (s_1 m_{1i}, s_2 m_{2i}, \ldots, s_r m_{ri})$. The main result of [1] says that $\Delta_{\underline{g}}(\underline{t}) = \Delta^L(\underline{t})$ coincides with the Poincaré series of the filtration corresponding to the Newton diagram of the function $\prod_{i=1}^r g_i$.

Let the *reduced Poincaré series* of the filtration defined by $\{v_i''\}$ be

$$\widetilde{P}_{\{v_i''\}}(\underline{t}) := P_{\{v_i''\}}(\underline{t})/P_{\{u_i\}}(\underline{t}), \quad \text{where} \quad P_{\{u_i\}}(\underline{t}) = \frac{1}{(1 - \underline{t}^{\underline{u}(x)})(1 - \underline{t}^{\underline{u}(y)})}$$

is the Poincaré series of the filtration defined by the quasihomogeneous valuations $\{u_i\}$ on $\mathcal{O}_{\mathbb{C}^2,0}$. One has

(11) $$\widetilde{P}_{\{v_i''\}}(\underline{t}) = \prod_{i=1}^r (1 - \underline{t}^{s_i \underline{m}_i}).$$

Let

$$\widetilde{\Delta}_{\underline{g}^{\underline{s}}}(\underline{t}) := \Delta_{\underline{g}^{\underline{s}}}(\underline{t})/\Delta_{\underline{g}^{\underline{s}}}^x(\underline{t}) \cdot \Delta_{\underline{g}^{\underline{s}}}^y(\underline{t})$$

where $\Delta_{\underline{g}^{\underline{s}}}^x(\underline{t})$ and $\Delta_{\underline{g}^{\underline{s}}}^y(\underline{t})$ are the Alexander polynomials of the sets of functions $\underline{g}^{\underline{s}} = \{g_1^{s_1}, \ldots, g_r^{s_r}\}$ restricted to the coordinate axes $\mathbb{C}_x$ and $\mathbb{C}_y$ respectively. One can regard $\widetilde{\Delta}_{\underline{g}^{\underline{s}}}(\underline{t})$ as the Alexander

polynomial of the set of functions $\underline{g}^{\underline{s}}$ restricted to the complex torus $(\mathbb{C}^*)^2 \subset \mathbb{C}^2$. One has

$$(12) \qquad \widetilde{\Delta}_{\underline{g}^{\underline{s}}}(\underline{t}) = \prod_{i=1}^{r}(1 - \underline{t}^{\underline{s}\,\underline{m}_i}).$$

One can see that a relation between (11) and (12) can be described in the following way. Consider products of r ordered cyclotomic binomials in r variables. Such a product

$$\prod_{i=1}^{r}(1 - \underline{t}^{\underline{N}_i}), \quad \underline{N}_i = (N_{i1}, \dots, N_{ir}),$$

can be described by the corresponding $r \times r$-matrix $N := (N_{ij})$. The transposition of the matrix induces an involution on the set of products of this sort. One can see that this involution maps the product (11) for the Poincaré series to the product (12) for the Alexander polynomial.

References

[1] A. Campillo, F. Delgado, S. M. Gusein-Zade: The Alexander polynomial of a plane curve singularity and integrals with respect to the Euler characteristic. International Journal of Mathematics **14**, no.1, 47–54 (2003). DOI: 10.1142/S0129167X03001703

[2] W. Ebeling, S. M. Gusein-Zade: Multi-variable Poincaré series associated with Newton diagrams. Journal of Singularities **1**, 60–68 (2010). DOI: 10.5427/jsing.2010.1d

[3] W. Ebeling, S. M. Gusein-Zade: On divisorial filtrations associated with Newton diagrams. Journal of Singularities **3**, 1–7 (2011). DOI: 10.5427/jsing.2011.3a

[4] D. Eisenbud, W. Neumann: Three-dimensional link theory and invariants of plane curve singularities. Annals of Mathematics Studies, 110. Princeton University Press, Princeton, NJ, 1985.

[5] S. M. Gusein-Zade, F. Delgado, A. Campillo: Alexander polynomials and Poincaré series of sets of ideals. Funktsional. Anal. i Prilozhen. **45**, no.4, 40–48 (2011) (Engl. translation in Funct. Anal. Appl. **45**, no.4, 271–277 (2011)). DOI: 10.1007/s10688-011-0029-2

[6] H. Hamm: On the Newton filtration for functions on complete intersections. arXiv: 1103.0654. arχiv: 1103.0654v2

[7] A. Lemahieu: Poincaré series of embedded filtrations. Math. Res. Lett. **18**, no.5, 815–825 (2011).

[8] C. Sabbah: Modules d'Alexander et $\mathcal{D}$-modules. Duke Math. J. **60**, no.3, 729–814 (1990). DOI: 10.1215/S0012-7094-90-06030-2

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