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IRREDUCIBILITY CRITERION FOR QUASI-ORDINARY POLYNOMIALS

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Abstract. Using the notion of approximate roots and that of generalized Newton sets, we give a local criterion for a quasi ordinary polynomial to be irreducible. Such a criterion is useful in the study of singularities of quasi-ordinary hypersurfaces. It generalizes the criterion given by S.S. Abhyankar for algebraic plane curves.

Introduction

Let $\mathbf{K}$ be an algebraically closed field of characteristic zero, and let $\mathbf{R} = \mathbf{K}[[x_1, \ldots, x_e]] = \mathbf{K}[[\underline{x}]]$ be the ring of formal power series in $x_1, \ldots, x_e$ over $\mathbf{K}$. Let $F = y^n + a_1(\underline{x})y^{n-1} + \ldots + a_n(\underline{x})$ be a nonzero polynomial of $\mathbf{R}[y]$, and suppose that F is irreducible in $\mathbf{R}[y]$. Suppose that $e = 1$ and let g be a nonzero polynomial of $\mathbf{R}[y]$, then define the intersection multiplicity of F with g, denoted $\mathrm{int}(F, g)$, to be the x-order of the y resultant of F and g. The set of $\mathrm{int}(F, g), g \in \mathbf{R}[y]$, defines a semigroup, denoted $\Gamma(F)$. It is well known that a set of generators of $\Gamma(F)$ can be computed from polynomials having maximal contact with F (see [1]), namely, there exist $g_1, \ldots, g_h$ such that $n, \mathrm{int}(F, g_1), \ldots, \mathrm{int}(F, g_h)$ generate $\Gamma(F)$ and for all $1 \le k \le h$, the Newton-Puiseux expansion of g_k coincides with that of F up to a characteristic exponent of F. In [1], Abhyankar introduced a special set of polynomials called the approximate roots of F. These polynomials have the advantage that they can be calculated from the equation of F by using the Tschirnhausen transform. Suppose that $e \ge 2$ and that the y-discriminant of f, denoted by $D_y(F)$, is of the form $x_1^{N_1} \ldots . x_e^{N_e} . u(x_1, \ldots, x_e)$, where $N_1, \ldots, N_e \in \mathbf{N}$ and u is a unit in $\mathbf{R}$ (such a polynomial is called quasi-ordinary polynomial). By the Abhyankar-Jung Theorem (see [2]), the roots of $F(\underline{x}, y) = 0$ are all in $\mathbf{K}[[x_1^{\frac{1}{n}}, \ldots, x_e^{\frac{1}{n}}]]$, in particular there exists a power series $y(t_1, \ldots, t_e) = \sum_{p \in \mathbf{N}^e} c_p t_1^{p_1} \ldots . t_e^{p_e} \in \mathbf{K}[[t_1, \ldots, t_e]]$ such that $F(t_1^n, \ldots, t_e^n, y(t_1, \ldots, t_e)) = 0$ and the other roots of $F(t_1^n, \ldots, t_e^n, y) = 0$ are the conjugates of $y(t_1, \ldots, t_e)$ with respect to the nth roots of unity in $\mathbf{K}$. Given a polynomial g of $\mathbf{R}[y]$, we define the order of g to be the leading exponent with respect to the lexicographical order of the smallest homogeneous component of $g(t_1^n, \ldots, t_e^n, y(t_1, \ldots, t_e))$. The set of orders of polynomials of $\mathbf{R}[y]$ defines a semigroup, denoted $\Gamma(F)$. It turns out that, as in the curve case, there exists a set of approximate roots of F whose orders generate $\Gamma(F)$ (see [6], [8]). Furthermore,

(*) these approximate roots of F are quasi-ordinary and irreducible

In Section 4. we introduce the notion of generalized Newton set of a polynomial with respect to a set of polynomials and a set of elements of $\mathbb{N}^e$, and we define the notion of the straightness of such a set. It turns out that

(**) F is straight with respect to its set of approximate roots and the set of generators of its semigroup.

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The main result of the paper is that the two properties above, together with some numerical conditions, characterize irreducible quasi-ordinary polynomials (see Theorem 5.1.).

Note that if $e = 1$, then any nonzero element of $\mathbf{K}[[x]][y]$ is quasi-ordinary, in particular our results generalize those of Abhyankar given in [3].
The paper is organized as follows: in Section 1 we discuss the main properties of an irreducible quasi-ordinary polynomial F. In Section 2 we introduce the notion of approximate roots of a polynomial in one variable over a commutative ring with unity. By [6], the orders of the approximate roots together with the canonical basis of $(n\mathbf{Z})^e$ give a set of generators of the semigroup of F. We recall this property in Section 3. Sections 4 and 5 are devoted to the irreducibility criterion: in Section 4 we introduce the notion of generalized Newton polygon, and we define the notion of straightness of a polynomial with respect to a set of polynomials, then we use these notions in Section 5 in order to decide if a given quasi-ordinary polynomial is irreducible. We finally end the paper with some examples in Section 6.

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1. The semigroup of a quasi-ordinary polynomial

Let $\mathbf{K}$ be an algebraically closed field of characteristic zero, and let $\mathbf{R} = \mathbf{K}[[x_1, \ldots, x_e]]$ (denoted by $\mathbf{K}[[\underline{x}]]$) be the ring of formal power series in $x_1, \ldots, x_e$ over $\mathbf{K}$. Let $F = y^n + a_1(\underline{x})y^{n-1} + \ldots + a_n(\underline{x})$ be a nonzero polynomial of $\mathbf{R}[y]$ and assume, after a possible change of variables, that $a_1(\underline{x}) = 0$. Suppose that the discriminant of F is of the form $x_1^{N_1} \cdot \ldots \cdot x_e^{N_e} . u(x_1, \ldots, x_e)$, where $N_1, \ldots, N_e \in \mathbf{N}$ and $u(\underline{x})$ is a unit in $\mathbf{R}$. We call F a quasi-ordinary polynomial. It follows from the Abhyankar-Jung Theorem (see [2]) that there exists a formal power series $y(\underline{t}) = y(t_1, \ldots, t_e) \in \mathbf{K}[[t_1, \ldots, t_e]]$ (denoted by $\mathbf{K}[[\underline{t}]]$) such that $F(t_1^n, \ldots, t_e^n, y(\underline{t})) = 0$. Furthermore, if F is an irreducible polynomial, then we have:

$$F(t_1^n, \ldots, t_e^n, y) = \prod_{i=1}^{n} (y - y(w_1^i t_1, \ldots, w_e^i t_e))$$

where $(w_1^i, \ldots, w_e^i)_{1 \leq i \leq n}$ are distinct elements of $(U_n)^e$, U_n being the group of nth roots of unity in $\mathbf{K}$.
Suppose that F is irreducible and let $y(\underline{t})$ be as above. Write $y(\underline{t}) = \sum_p c_p \underline{t}^p$ and define the support of $y(\underline{t})$, denoted $\text{Supp}(y(\underline{t}))$, to be the set $\{p | c_p \neq 0\}$. Obviously the support of $y(w_1 t_1, \ldots, w_e t_e)$ does not depend on $w_1, \ldots, w_e \in U_n$. We denote it by $\text{Supp}(F)$ and we call it the support of F. Given $a, b \in \mathbb{N}^e$, we say that $a \leq b$ (resp. $a < b$) if $a \leq b$ coordinate-wise (resp. $a \leq b$ coordinate-wise and $a \neq b$). By [9], there exists a finite sequence of elements in $\text{Supp}(F)$, denoted $m_1, \ldots, m_h$, such that

i) $m_1 < m_2 < \ldots < m_h$. .
ii) If $p \in \text{Supp}(F)$, then $p \in (n\mathbb{Z})^e + \sum_{p \in m_i + \mathbb{N}^e} m_i \mathbb{Z}$.
iii) $m_i \notin (n\mathbb{Z})^e + \sum_{j<i} m_j \mathbb{Z}$ for all $i = 1, \ldots, h$.

The set of elements of this sequence is called the set of characteristic exponents of F, or the $\underline{m}$-sequence associated with F.

Let $glex$ be the well-ordering on $\mathbb{N}^e$ defined as follows: $\underline{\alpha} <_{glex} \underline{\beta}$ if and only if $|\alpha| = \sum_{i=1}^e \alpha_i < |\beta| = \sum_{i=1}^e \beta_i$ or $|\alpha| = |\beta|$ and $\alpha <_{lex} \beta$ (where lex denotes the lexicographical order).

Definition 1.1. Let $u = \sum_p c_p \underline{t}^p$ in $\mathbf{K}[[\underline{t}]]$ be a nonzero formal power series. We denote by $\text{In}(u)$ the initial form of u: if $u = u_d + u_{d+1} + \ldots$ denotes the decomposition of u into a sum of homogeneous components, then $\mathrm{In}(u) = u_d$. We set $O_{\underline{t}}(u) = d$ and we call it the $\underline{t}$-order of u. We denote by $\exp_{glex}(u)$ the greatest exponent of u with respect to glex. We denote by $\mathrm{inco}_{glex}(u)$ the coefficient $c_{\exp_{glex}(u)}$, and we call it the initial coefficient of u. We finally set $\mathrm{M}_{glex}(u) = \mathrm{inco}_{glex}(u)\underline{t}^{exp_{glex}(u)}$, and we call it the initial monomial of u.

Remark 1.2. Let $u(\underline{t}) \in \mathbf{K}[[\underline{t}]]$ be a nonzero formal power series, and let $\mathrm{In}(u)$ be the initial form of u. Let $\prec$ be a well-ordering on $\mathbb{N}^e$ and define the leading exponent of u to be the leading exponent of $\mathrm{In}(u)$ with respect to $\prec$. If $\prec$ is not the lexicographical order, then we get a different notion of leading exponent (resp. initial coefficient, resp. initial monomial) of u. Note that if $\mathrm{In}(u)$ is a monomial, then these notions do not depend on the choice of $\prec$.

Denote by $\mathrm{Root}(f)$ the set of n roots of $F(t_1^n, \ldots, t_e^n, y) = 0$ introduced above and let $y(\underline{t})$ be an element of this set. We have the following:

Lemma 1.3. (See [9], paragraph 5.) $\mathrm{In}(y(\underline{t}) - z(\underline{t}))$ is a monomial for all $z(\underline{t}) \in \mathrm{Root}(f) - \{y(\underline{t})\}$. Furthermore, $\{\exp_{glex}(y(\underline{t}) - z(\underline{t})) | z(\underline{t}) \in \mathrm{Root}(f) - \{y(\underline{t})\}\} = \{m_1, \ldots, m_h\}$.

Let g be a nonzero element of $\mathbf{R}[y]$. The order of g with respect to F, denoted $O_{glex}(F, g)$, is defined to be $\exp_{glex}(g(t_1^n, \ldots, t_e^n, y(\underline{t}))$. Note that it does not depend on the choice of the root $y(\underline{t})$ of $F(t_1^n, \ldots, t_e^n, y) = 0$. The set $\{O_{glex}(F, g) | g \in \mathbf{R}[y], g \notin (F)\}$ defines a subsemigroup of $\mathbf{Z}^e$. We call it the semigroup associated with F and we denote it by $\Gamma(F)$ (see [6], [8],[10], and [11] for the several definitions of the semigroup of F).

Let $\underline{m}_0 = (m_0^1, \ldots, m_0^e)$ be the canonical basis of $(n\mathbb{Z})^e$. Let I_e be the unit $e \times e$ matrix, and let $D_1 = n^e$ and for all $1 \leq i \leq h$, let D_{i+1} be the gcd of the (e, e) minors of the matrix $(nI_e, {m_1}^T, \ldots, {m_i}^T)$ (where T denotes the transpose of a matrix). Since $m_i \notin (n\mathbf{Z})^e + \sum_{j<i} m_j\mathbf{Z}$ for all $1 \leq i \leq h$, then $D_{i+1} < D_i$. We call $(D_1, \ldots, D_{h+1})$ the $\underline{D}$-sequence associated with F, and we denote it by $\mathrm{GCDM}(m_0^1, \ldots, m_0^e, m_1, \ldots, m_h)$. We define the sequence $(e_i)_{1 \leq i \leq h}$ to be $e_i = \dfrac{D_i}{D_{i+1}}$ for all $1 \leq i \leq h$, and we call it the $\underline{e}$-sequence associated with F.

Let $F_0 = \mathbf{K}((\underline{x}))$ and let $\mathbf{F}_k = \mathbf{F}_{k-1}(x_1^{\frac{m_k^1}{n}} \ldots . x_e^{\frac{m_k^e}{n}})$ for all $k = 1, \ldots, h$. In particular we have:

$$\mathbf{F}_0 \subseteq \mathbf{F}_1 \subseteq \mathbf{F}_2 \subseteq \ldots \subseteq \mathbf{F}_h = \mathbf{F}_0(x_1^{\frac{m_1^1}{n}} \ldots . x_e^{\frac{m_1^e}{n}}, \ldots, x_1^{\frac{m_h^1}{n}} \ldots . x_e^{\frac{m_h^e}{n}})$$

Proposition 1.4. With the notations above, we have the following:

i) If $y(\underline{x})$ is a root of $F(\underline{x}, y) = 0$, then $F_h = \mathbf{K}((y(\underline{x})))$.

ii) For all $k = 1, \ldots, h$, $\mathbf{F}_k$ is an algebraic extension of degree e_k of $\mathbf{F}_{k-1}$.

iii) For all $k = 1, \ldots, h$, $\mathbf{F}_k$ is an algebraic extension of degree $e_k.e_{k-1}.\ldots.e_1$ of $\mathbf{F}_0$.

iv) $n = \deg_y(F) = e_1.\ldots.e_h = \dfrac{D_1}{D_{h+1}} = \dfrac{n^e}{D_{h+1}}$. In particular $D_{h+1} = n^{e-1}$.

Proof. . ii), iii), and iv) are obvious. For a proof of i) see [9], Paragraph 5. □

Remark 1.5. (see [9]) Conversely, let $N \in \mathbb{N}^*$ and let $Y(\underline{t}) = \sum_p c_p\underline{t}^p \in \mathbf{K}[[\underline{t}]]$, and suppose that there exists a finite sequence of elements $m'_1, \ldots, m'_{h'}$ of $\mathrm{Supp}(Y(\underline{t}))$ such that the following holds:

i) $m'_1 < m'_2 < \ldots < m'_{h'}$.
ii) If $p \in \mathrm{Supp}(Y(\underline{t}))$, then $p \in (N\mathbb{Z})^e + \sum_{p \in m'_i + \mathbb{N}^e} m'_i\mathbb{Z}$.
iii) $m_i \notin (N\mathbb{Z})^e + \sum_{j<i} m'_j\mathbb{Z}$ for all $i = 1, \ldots, h'$.

Let $\bar{F}(\underline{x},y)$ be the minimal polynomial of $Y(\underline{x}^{\frac{1}{N}})=\sum_p c_p\underline{x}^{\frac{p}{N}}$. If $\deg_y(\bar{F})=N$, then $\mathbf{F}_0(Y(\underline{x}^{\frac{1}{N}}))=\mathbf{F}_0(x_1^{\frac{{m'}^1_1}{N}}\ldots\ldots x_e^{\frac{{m'}^e_1}{N}},\ldots,x_1^{\frac{{m'}^1_{h'}}{N}}\ldots\ldots x_e^{\frac{{m'}^e_{h'}}{N}})$. In particular, for all $Z(\underline{t})\in\mathrm{Root}(\bar{F})$, $\mathrm{In}(Y(\underline{t})-Z(\underline{t}))=\tilde{a'}.\underline{t}^{m'_k}$, where $\tilde{a'}\in\mathbf{K}^*$ and $1\leq k\leq h'$. This implies that $D_y(\bar{F})=a.\underline{x}^{\alpha}(1+u(\underline{x}))$, where $a\in\mathbf{K}^*$ and $u(\underline{0})=0$, i.e. $\bar{F}$ is a quasi-ordinary polynomial.

The result of Proposition 1.4. has also the following interpretation: let $M_0=(n\mathbf{Z})^e$ and let $M_i=(n\mathbf{Z})^e+\sum_{j=1}^i m_j\mathbf{Z}$ for all $1\leq i\leq h$. Then $M_0\subseteq M_1\subseteq\cdots\subseteq M_h\subseteq\mathbb{Z}^e$. In particular, since M_0 and $\mathbb{Z}^e$ are free abelian groups of rank e, then for all $1\leq i\leq h$, M_i is a free abelian group of rank e. Furthermore, e_i is the index of the lattice M_{i-1} in M_i.
Let $1\leq i\leq h$ and let $v_1,\cdots,v_e$ be a basis of M_i, and recall that D_{i+1} is the determinant of the matrix $(v_1^T,\cdots,v_e^T)$. We have the following:

Proposition 1.6. Let v be a nonzero element of $\mathbb{Z}^e$ and let $\tilde{D}$ be the gcd of the (e,e) minors of the matrix $(v_1^T,\ldots,v_e^T,v^T)$. Then $\tilde{D}$ is also the gcd of the (e,e) minors of the matrix $(nI_e,m_1^T,\cdots,m_i^T,v^T)$. With these notations, we have the following:

i) $v\in M_i$ if and only if $\tilde{D}=D_{i+1}$.

ii) $\dfrac{D_{i+1}}{\tilde{D}}.v\in M_i$ and if $D_{i+1}>\tilde{D}$ then for all $1\leq k<\dfrac{D_{i+1}}{\tilde{D}}$, $k.v\notin M_i$.

In particular, since $m_{i+1}\notin M_i$, then $D_{i+2}>D_{i+1}$, $e_{i+1}m_{i+1}\in M_i$, and $km_{i+1}\notin M_i$ for all $1\leq k<e_{i+1}$.

Proof. . i) For all $k=1,\ldots,e$, let $\tilde{D}_k$ be the determinant of the matrix $(v_1^T,\ldots,v_{k-1}^T,v^T,v_{k+1}^T,\ldots,v_e^T)$. If $\tilde{D}=D_{i+1}$ then D_{i+1} divides $\tilde{D}_k$. In particular the Cramer system $\lambda_1v_1+\ldots+\lambda_ev_e=v$ has the unique solution $\lambda_k=\dfrac{\tilde{D}_k}{D_{i+1}}\in\mathbb{Z}$. Conversely, if $v\in M_i$, then there exist unique integers $\lambda_1,\ldots,\lambda_e$ such that $v=\lambda_1v_1+\ldots+\lambda_ev_e$, but $(\lambda_1,\ldots,\lambda_e)$ is the unique solution to the (e,e) system $a_1v_1+\ldots+a_ev_e=v$, in particular $\lambda_k=\dfrac{\tilde{D}_k}{D_{i+1}}$ for all $k=1,\ldots,e$. This proves that $\tilde{D}=D_{i+1}$.

ii) Let the notations be as in i) and let $1\leq k\leq\dfrac{D_{i+1}}{\tilde{D}}$. Let $\bar{D}$ be the gcd of the (e,e) minors of the matrix $[v_1^T,\cdots,v_e^T,(k.v)^T]$. Clearly $\bar{D}=\gcd(k\tilde{D}_1,\cdots,k\tilde{D}_e,D_{i+1})$. If $k=\dfrac{D_{i+1}}{\tilde{D}}$, then $\bar{D}=\gcd(D_{i+1}\dfrac{\tilde{D}_1}{\tilde{D}},\cdots,D_{i+1}\dfrac{\tilde{D}_e}{\tilde{D}},D_{i+1})=D_{i+1}$, which implies by i) that $k.v\in M_i$. Suppose that $D_{i+1}>\tilde{D}$ and that $1\leq k<\dfrac{D_{i+1}}{\tilde{D}}$. If $k.v\in M_i$, then $\bar{D}=D_{i+1}$, which implies that D_{i+1} divides $\gcd(k\tilde{D}_1,\cdots,k\tilde{D}_e,kD_{i+1})=k.\tilde{D}$. This is a contradiction because $k.\tilde{D}<D_{i+1}$.

$\square$

The following result will be used later in the paper:

Corollary 1.7. Let the notations be as in Remark 1.5., i.e. $N\in\mathbb{N}^*$, $Y(\underline{t})=\sum_p c_p\underline{t}^p\in\mathbf{K}[[\underline{t}]]$, and there exists a finite sequence of elements $m'_1,\ldots,m'_{h'}$ of $\mathrm{Supp}(Y(\underline{t}))$ such that the following holds:
i) $m'_1<m'_2<\ldots<m'_{h'}$.
ii) If $p\in\mathrm{Supp}(Y(\underline{t}))$ then $p\in(N\mathbb{Z})^e+\sum_{p\in m'_i+\mathbb{N}^e}m'_i\mathbb{Z}$.
iii) $m'_i\notin(N\mathbb{Z})^e+\sum_{j<i}m'_j\mathbb{Z}$ for all $i=1,\ldots,h'$.

Let $F(\underline{x},y)$ be the minimal polynomial of $Y(\underline{x}^{\frac{1}{N}})=\sum_p c_p \underline{x}^{\frac{p}{N}}$ over $\mathbf{K}((\underline{x}))$ and suppose that $\deg_y F=N$. Let $m\in\mathbb{N}^e, m'_{h'}<_{glex} m$, and let $\bar{Y}(\underline{t})=Y(\underline{t})+c_m\underline{t}^m, c_m\in\mathbf{K}^*$. Let finally $\bar{F}(\underline{x},y)$ be the minimal polynomial of $\bar{Y}(\underline{x}^{\frac{1}{N}})$ over $\mathbf{K}((\underline{x}))$. We have the following:

1) $\deg_y(\bar{F})\geq N$ and $\deg_y(\bar{F})=N$ if and only if $m\in M_{h'}=(N\mathbb{Z})^e+\sum_{i=1}^{h'} m'_i\mathbb{Z}$.

2) If $m\in m'_{h'}+\mathbb{N}^e$, then $\bar{F}$ is quasi-ordinary.

Proof. . 1) Let $(D_1=N^e,\ldots,D_{h'+1}=N^{e-1})$ be the $\underline{D}$-sequence associated with F. We have $\deg_y\bar{F}\geq N.[\mathbf{F}_0(\bar{Y}(\underline{x}^{\frac{1}{N}})),\mathbf{F}_{h'}]\geq N$, and $m\in M_{h'}$ if and only if $\mathbf{F}_{h'}=\mathbf{F}_0(\bar{Y}(\underline{x}^{\frac{1}{N}}))$, and this holds if and only if $\deg_y(\bar{F})=N$.

2) If $m\in M_{h'}$ (resp. $m\notin M_{h'}$), then $\bar{Y}(\underline{x})$ and $m'_1,\ldots,m'_{h'}$ (resp. $\bar{Y}(\underline{x})$ and $m'_1,\ldots,m'_{h'},m'_{h'+1}=m$) satisfy the conditions of Remark 1.5., and $\bar{F}$ is quasi-ordinary. □

Let $d_i=\dfrac{D_i}{D_{h+1}}$ for all $1\leq i\leq h+1$. In particular $d_1=n$ and $d_{h+1}=1$. The sequence $(d_1,d_2,\ldots,d_{h+1})$ is called the gcd-sequence of F or the $\underline{d}$-sequence associated with F. Let $(r_0^1,\cdots,r_0^e)=(m_0^1,\cdots,m_0^e)$ be the canonical basis of $(n\mathbb{Z})^e$ and define the sequence $(r_k)_{1\leq k\leq h}$ by $r_1=m_1$ and:

$$r_{k+1}=e_k r_k+m_{k+1}-m_k$$

for all $1\leq k\leq h-1$. We call $(r_0^1,\cdots,r_0^e,r_1,\cdots,r_h)$ the $\underline{r}$-sequence associated with F. Note that each of the sequences $(m_k)_{1\leq k\leq h}$ and $(r_k)_{1\leq k\leq h}$ determines the other. More precisely $m_1=r_1$ and $r_k d_k=m_1d_1+\sum_{j=2}^k(m_j-m_{j-1})d_j$ (resp. $m_k=r_k-\sum_{j=1}^{k-1}(e_j-1)r_j$) for all $2\leq k\leq h$. In particular $M_k=(n\mathbb{Z})^e+\sum_{j=1}^k m_j\mathbb{Z}=(n\mathbb{Z})^e+\sum_{j=1}^k r_j\mathbb{Z}$ for all $k=1,\ldots,h$. It also follows that $\mathrm{GCDM}(r_0^1,\cdots,r_0^e,r_1,\cdots,r_h)=\mathrm{GCDM}(m_0^1,\cdots,m_0^e,m_1,\cdots,m_h)$, in particular, the results of Proposition 1.6. hold if we replace $(m_1,\cdots,m_h)$ by $(r_1,\cdots,r_h)$.

Corollary 1.8. (see also [6], Lemma 3.3.) Let $(r_0^1,\cdots,r_0^e,r_1,\cdots,r_h)$ be the $\underline{r}$-sequence associated with F. For all $1\leq k\leq h-1$, we have:

i) $r_kd_k<r_{k+1}d_{k+1}$.

ii) $e_kr_k\in M_{k-1}$.

iii) For all $1\leq i<e_k, ir_k\notin M_{k-1}$.

Proof. . This results from Proposition 1.6. and the equalities above. □

Let $\phi(\underline{t})=(t_1^p,\ldots,t_e^p,Y(\underline{t}))$ and $\psi(\underline{t})=(t_1^q,\ldots,t_e^q,Z(\underline{t}))$ be two nonzero elements of $\mathbf{K}[[\underline{t}]]^{e+1}$. We define the contact between ϕ and ψ, denoted $\mathrm{c}_{glex}(\phi,\psi)$, to be the element $\dfrac{1}{pq}\exp_{glex}(Y(t_1^q,\ldots,t_e^q)-Z(t_1^p,\ldots,t_e^p))$.

We define the contact between F and ϕ, denoted $\mathrm{c}_{glex}(F,\phi)$, to be the maximal element of

$$\{\mathrm{c}_{glex}(\phi,(t_1^n,\ldots,t_e^n,y(\underline{t})))|y(\underline{t})\in\mathrm{Root}(f)\}.$$

Let $g=y^m+b_1(\underline{x})y^{m-1}+\ldots+b_m(\underline{x})$ be a nonzero polynomial of $\mathbf{R}[y]$. Suppose that g is an irreducible quasi-ordinary polynomial and let $\psi(\underline{t})=(t_1^m,\ldots,t_e^m,Z(\underline{t}))$ be a root of $g(t_1^m,\ldots,t_e^m,y)=0$. We define the contact between F and g, denoted $\mathrm{c}_{glex}(F,g)$, to be the contact between F and ψ. Note that this definition does not depend on the choice of the root $Z(\underline{t})$ of g, and that if $F.g$ is a quasi-ordinary polynomial, then $\mathrm{In}(F(\psi(\underline{t}))=M_{glex}(F(\psi(\underline{t})))$. In this case, the contact $c_{glex}(F,g)$ coincides with the notion of contact introduced in [4] and [12]. The following Proposition generalizes a well known result for plane curves. It calculates the order $O_{glex}(F,g)$ in terms of the contact $\mathrm{c}_{glex}(F,g)$ and the characteristic sequences of F. When $F.g$ is quasi-ordinary, this result has been proved in [12], Proposition 2.14 and Proposition 5.9.

Proposition 1.9. Let $g = y^m + b_1(\underline{x})y^{m-1} + \ldots + b_m(\underline{x})$ be an irreducible quasi-ordinary polynomial of $\mathbf{R}[y]$ and suppose that $m \leq n$. If $c = c_{glex}(F, g)$ then we have the following:

i) If $nc <_{glex} m_1$, then $O_{glex}(F, g) = nmc$.

ii) Otherwise, let $1 \leq q \leq h-1$ be the smallest integer such that $m_q \leq_{glex} nc <_{glex} m_{q+1}$, then $O_{glex}(F, g) = (r_q d_q + (nc - m_q)d_{q+1}) \cdot \dfrac{m}{n}$. In particular $O_{glex}(F, g) <_{glex} r_{q+1}h_{q+1} \cdot \dfrac{m}{n}$.

Proof. . The proof is technical. It uses the same arguments as in the case of plane curves (see also [12], Proposition 5.9.). We shall consequently omit the details.

□

2. G-adic expansions

Let $\mathbf{S}$ be a commutative ring with unity and let $\mathbf{S}[y]$ be the ring of polynomials in y with coefficients in $\mathbf{S}$. Let $f = y^n + a_1 y^{n-1} + \ldots + a_n$ be a monic polynomial of $\mathbf{S}[y]$ of degree $n > 0$ in y. Let $d \in \mathbf{N}$ and suppose that d divides n. Let g be a monic polynomial of $\mathbf{S}[y]$ of degree $\dfrac{n}{d}$ in y. There exist unique polynomials $a_1(y), \ldots, a_d(y) \in \mathbf{S}[y]$ such that:

$$f = g^d + \sum_{i=1}^{d} a_i(y).g^{d-i}$$

and for all $1 \leq i \leq d$, $\deg_y(a_i(y)) < \dfrac{n}{d} = \deg_y g$ (where $\deg_y$ denotes the y-degree). The equation above is called the g-adic expansion of f. Assume that d is a unit in $\mathbf{S}$. The Tschirnhausen transform of f with respect to g, denoted $\tau_f(g)$, is defined to be $\tau_f(g) = g + d^{-1}a_1$. Note that $\tau_f(g) = g$ if and only if $a_1 = 0$. By [1], $\tau_f(g) = g$ if and only if $\deg_y(f - g^d) < n - \dfrac{n}{d}$. If one of these equivalent conditions is satisfied, then the polynomial g is called a d-th approximate root of f. By [1], there exists a unique d-th approximate root of f. We denote it by $\mathrm{App}_d(f)$.

Let $n = d_1 > d_2 > \cdots > d_h$ be a sequence of integers such that d_{i+1} divides d_i for all $1 \leq i \leq h-1$, and set $e_i = \dfrac{d_i}{d_{i+1}}$, $1 \leq i \leq h-1$ and $e_h = +\infty$. For all $1 \leq i \leq h$, let g_i be a monic polynomial of $\mathbf{S}[y]$ of degree $\dfrac{n}{d_i}$ in y. Set $\underline{G} = (g_1, \ldots, g_h)$ and let $B = \{\underline{\theta} = (\theta_1, \ldots, \theta_h) \in \mathbb{N}^h, 0 \leq \theta_i < e_i$ for all $1 \leq i \leq h\}$. Then f can be uniquely written as $f = \sum_{\underline{\theta} \in B} a_{\underline{\theta}}.\underline{g}^{\underline{\theta}}$ where $\underline{g}^{\underline{\theta}} = g_1^{\theta_1}.\ldots.g_h^{\theta_h}$ and $a_{\underline{\theta}} \in \mathbf{S}$ for all $\underline{\theta} \in B$. We call this expansion the $\underline{G}$-adic expansion of f.

3. Generators of the semigroup of F

Let the notations be as in Sections 1. and 2., in particular $F = y^n + a_2(\underline{x})y^{n-2} + \ldots + a_n(\underline{x})$ is an irreducible quasi-ordinary polynomial of $\mathbf{R}[y] = \mathbf{K}[[\underline{x}]][y]$. We have the following:

Theorem 3.1. (see [6], [8]) Let the notations be as above, and let $d_1 = n, \ldots, d_h, d_{h+1} = 1$ be the gcd-sequence of F. The d_k-th approximate root $\mathrm{App}_{d_k}(F)$ is an irreducible quasi-ordinary polynomial for all $k = 1, \ldots, h$. Furthermore, $c_{glex}(F, \mathrm{App}_{d_k}(F)) = \dfrac{m_k}{n}$ and $O_{glex}(F, \mathrm{App}_{d_k}(F)) = r_k$.

Let $\underline{G} = (g_1, \ldots, g_h, g_{h+1})$ be the d_k-th approximate roots of F, $1 \leq k \leq h+1$, and recall that $g_1 = y$, $g_{h+1} = F$. Let $B(\underline{G}) = \{\underline{\theta} = (\theta_1, \ldots, \theta_h, \theta_{h+1}) \in \mathbf{N}^{h+1} | \theta_{h+1} < +\infty$ and $0 \leq \theta_k < e_k$ for all $1 \leq k \leq h\}$.

Lemma 3.2. (see [8], (2.3)) Given two elements $\underline{\theta}^1, \underline{\theta}^2 \in B(\underline{G})$ and two elements $\underline{\gamma}^1, \underline{\gamma}^2 \in \mathbf{N}^e$, if $\theta^1_{h+1} = \theta^2_{h+1}$ and $\underline{\theta}^1 \neq \underline{\theta}^2$, then $\sum_{i=1}^e \gamma^1_i r^i_0 + \sum_{k=1}^h \theta^1_k r_k \neq \sum_{i=1}^e \gamma^2_i r^i_0 + \sum_{k=1}^h \theta^2_k r_k$.

Let $\bar{F}(\underline{x}, y)$ be a monic polynomial of $\mathbf{R}[y]$ and let

$$\bar{F} = \sum_{\underline{\theta} \in B(\underline{G})} c_{\underline{\theta}}(\underline{x}) g_1^{\theta_1} \cdot \ldots \cdot g_h^{\theta_h} \cdot g_{h+1}^{\theta_{h+1}}$$

be the $\underline{G}$-adic expansion of $\bar{F}$. Let $\mathrm{Supp}_{\underline{G}}(\bar{F}) = \{\underline{\theta} \in B(\underline{G}) | c_\theta \neq 0\}$ and let $B'(\underline{G}) = \{\underline{\theta} \in \mathrm{Supp}_{\underline{G}}(\bar{F}) | \theta_{h+1} = 0\}$. Clearly F divides $\bar{F}$ if and only if $B'(\underline{G}) = \emptyset$. Otherwise, by Lemma 3.2., there is a unique $\underline{\theta}_0 \in \mathrm{Supp}_{\underline{G}}(\bar{F})$ such that $O_{glex}(F, \bar{F}) = O_{glex}(F, M(c_{\underline{\theta}_0}(\underline{x})) g_1^{\theta_0^1} \cdot \ldots \cdot g_h^{\theta_0^h}) = O_{glex}(F, M(c_{\underline{\theta}_0}(\underline{x}))) + \sum_{i=1}^h \theta^i_0 r_i$. We set $M_{\underline{G}}(\bar{F}) = M_{glex}(c_{\underline{\theta}_0}(\underline{x})) g_1^{\theta_0^1} \cdot \ldots \cdot g_h^{\theta_0^h}$ and we call it the $\underline{G}$-initial monomial of $\bar{F}$. This leds to the following proposition:

Proposition 3.3. (see also [6], [8])With the notations above, $r^1_0, \ldots, r^e_0, r_1, \ldots, r_h$ generate $\Gamma(F)$.

Lemma 3.4. (see also [6], Prop. 2.3. or [11], Lemmas 7.4. and 7.5.) Let $\bar{F}$ be a non zero polynomial of $\mathbf{R}[y]$. If $\deg_y(\bar{F}) < \dfrac{n}{d_k}$ for some $1 \leq k \leq h$, then $O_{glex}(F, \bar{F}) \in < r^1_0, \ldots, r^e_0, r_1, \ldots, r_{k-1} >$. More precisely, there are unique $\theta^1_0, \cdots, \theta^e_0, \theta_1, \cdots, \theta_{k-1} \in \mathbb{N}$ such that $O_{glex}(F, \bar{F}) = \sum_{i=1}^e \theta^i_0 r^i_0 + \sum_{j=1}^{k-1} \theta_j r_j$ where $0 \leq \theta_j < e_j$ for all $1 \leq j \leq k-1$.

Proof. . Let the notations be as above, and let

$$\bar{F} = \sum_{\underline{\theta} \in B(\underline{G})} c_\theta(\underline{x}) g_1^{\theta_1} \cdot \ldots \cdot g_h^{\theta_h} \cdot g_{h+1}^{\theta_{h+1}}$$

be the $\underline{G}$-adic expansion of $\bar{F}$. Since $\deg_y(\bar{F}) < \dfrac{n}{d_k}$, then for all $\underline{\theta} \in \mathrm{Supp}_{\underline{G}}(\bar{F})$, $\theta_k = \cdots = \theta_h = 0$. This implies the result. $\square$

4. Generalized Newton sets

Let $n \in \mathbb{N}, n > 1$ and let $\underline{r}_0 = (r^1_0, \ldots, r^e_0)$ be the canonical basis of $(n\mathbb{Z})^e$. Let $r_1 < \ldots < r_h$ be a sequence of elements of $\mathbb{N}^e$. Set $D_1 = n^e$ and for all $1 \leq k \leq h$, let D_{k+1} be the GCD of the $e \times e$ minors of the $e \times (e+k)$ matrix $(nI_e, (r_1)^T, \ldots, (r_k)^T)$. Suppose that n^{e-1} divides D_k for all $1 \leq k \leq h+1$ and that $D_{h+1} = n^{e-1}$, and also that $D_1 > D_2 > \ldots > D_{h+1}$, in such a way that if we set $d_1 = n$ and $d_k = \dfrac{D_k}{n^{e-1}}$ for all $2 \leq k \leq h+1$, then $d_1 = n > d_2 > \ldots > d_{h+1} = 1$.

For all $1 \leq k \leq h+1$, let g_k be a monic polynomial of degree $\dfrac{n}{d_k}$ in y and set $\underline{G} = (g_1, \ldots, g_h, g_{h+1})$, $\underline{r} = (r_1, \ldots, r_h)$. Let H be a nonzero polynomial of $\mathbf{R}[y]$, and let

$$H = \sum_{\underline{\theta} \in B(\underline{G})} c_{\underline{\theta}}(\underline{x}) g_1^{\theta_1} \cdot \ldots \cdot g_h^{\theta_h} g_{h+1}^{\theta_{h+1}}$$

where $B(\underline{G}) = \{\underline{\theta} = (\theta_1, \ldots, \theta_h, \theta_{h+1}) | \theta_{h+1} < +\infty$ and $0 \leq \theta_i < \dfrac{d_i}{d_{i+1}} \forall 1 \leq i \leq h\}$, be the $\underline{G}$-adic expansion of H. Let $\mathrm{Supp}_{\underline{G}}(H) = \{\underline{\theta} \in B(\underline{G}) | c_{\underline{\theta}} \neq 0\}$ and let $B'(\underline{G}) = \{\underline{\theta} \in \mathrm{Supp}_{\underline{G}}(H) | \theta_{h+1} = 0\}$. Suppose that $B'(\underline{G}) \neq \emptyset$. Given $\underline{\theta} \in B'(\underline{G})$, if $\underline{\gamma}_\theta = \exp_{glex}(c_{\underline{\theta}}(\underline{x}))$, we shall associate with the monomial $c_{\underline{\theta}}(\underline{x}) g_1^{\theta_1} \cdot \ldots \cdot g_h^{\theta_h}$ the e-tuple

$$< (\underline{\gamma}_{\underline{\theta}}, \underline{\theta}), (\underline{r}_0, \underline{r}) >= \sum_{i=1}^{e} \gamma_{\theta_i} r_0^i + \sum_{j=1}^{h} \theta_j r_j.$$

We set $N_{\underline{G}}(H) = \{< (\underline{\gamma}_{\underline{\theta}}, \underline{\theta}), (\underline{r}_0, \underline{r}) >, \underline{\theta} \in B'(\underline{G})\}$, and we call it the $\underline{G}$-Newton set of H. By Lemma 3.2., there is a unique $\underline{\theta}^0 \in B'(\underline{G})$ such that if $\underline{\gamma}_{\theta^0} = \exp_{glex}(c_{\underline{\theta}^0}(\underline{x}))$, then:

$$< (\underline{\gamma}_{\theta^0}, \underline{\theta}^0), (\underline{r}_0, \underline{r}) >= \min_{glex}(N_{\underline{G}}(H))$$

where $\min_{glex}$ means the minimal element with respect to the well-ordering $glex$. We set $\mathrm{fO}(\underline{r}, \underline{G}, H) = < (\underline{\gamma}_{\theta^0}, \underline{\theta}^0), (\underline{r}_0, \underline{r}) >$ and we call it the formal order of H with respect to $(\underline{r}, \underline{G})$. We also set $M_{\underline{G}}(H) = M_{glex}(c_{\underline{\theta}^0}(\underline{x})).g_1^{\theta_1^0}.\ldots.g_h^{\theta_h^0}$ and we call it the initial monomial of H with respect to $(\underline{r}, \underline{G})$. If $B'(\underline{G}) = \emptyset$, then we set $\mathrm{fO}(\underline{r}, \underline{G}, H) = (+\infty, \ldots, +\infty)$. Note that this holds if and only if g_{h+1} divides H.

Let $f = y^n + a_1(\underline{x})y^{n-1} + \ldots + a_n(\underline{x})$ be a quasi-ordinary polynomial of $\mathbf{R}[y]$ and let $d \in \mathbb{N}, d > 1$ be a divisor of n. Let g be a monic polynomial of $\mathbf{R}[y]$ of degree $\dfrac{n}{d}$ in y and let $f = g^d + \alpha_1(\underline{x}, y)g^{d-1} + \ldots + \alpha_d(\underline{x}, y)$ be the g-adic expansion of f. We associate with f the set of points:

$$\{(\mathrm{fO}(\underline{r}, \underline{G}, \alpha_k), (d-k)\mathrm{fO}(\underline{r}, \underline{G}, g)), k = 0, \ldots, d\} \subseteq \mathbb{N}^e \times \mathbb{N}^e$$

We denote this set by $\mathrm{GNS}(f, \underline{r}, \underline{G}, g)$ and we call it the generalized Newton set of f with respect to $(\underline{r}, \underline{G}, g)$ (note that, since $\alpha_0 = 1$, then $\mathrm{fO}(\underline{r}, \underline{G}, \alpha_0) = \underline{0} \in \mathbb{N}^e$).

Definition 4.1. We say that f is straight with respect to $(\underline{r}, \underline{G}, g)$ if the following holds:

i) $\mathrm{fO}(\underline{r}, \underline{G}, \alpha_d) = d.\mathrm{fO}(\underline{r}, \underline{G}, g)$ and $\mathrm{fO}(\underline{r}, \underline{G}, \alpha_d) << (\underline{\gamma}_{\underline{\theta}}, \underline{\theta}), (\underline{r}_0, \underline{r}) >$ for all $\underline{\theta} \in N_{\underline{G}}(\alpha_d - M_{\underline{G}}(\alpha_d))$.

ii) For all $1 \leq k \leq d-1$, and for all $\underline{\theta} \in N_{\underline{G}}(\alpha_k)$, $k.\mathrm{fO}(\underline{r}, \underline{G}, g) \leq< (\underline{\gamma}_{\underline{\theta}}, \underline{\theta}), (\underline{r}_0, \underline{r}) >$.

We say that f is strictly straight with respect to $(\underline{r}, \underline{G}, g)$ if the inequality in ii) is a strict inequality.

Example 4.2. i) Let $f = (y^2 - x^3)^2 - x^5y + x^{10} \in \mathbf{K}[[x]][y]$, and let $r_0 = 4, r_1 = 6, r_2 = 13$, $\underline{G} = (g_1 = y, g_2 = y^2 - x^3, g_3 = f)$, $\underline{r} = (r_1, r_2)$: $f = g_2^2 - x^5 g_1$ is the g_2-expansion of f. Furthermore, $\mathrm{fO}(\underline{r}, \underline{G}, g_2) = r_2 = 13$, $\mathrm{fO}(\underline{r}, \underline{G}, x^5 g_1 + x^{10}) = 5r_0 + r_1 = 26 < 10r_0 = 40$. In particular, $\mathrm{GNS}(f, \underline{r}, \underline{G}, g_2) = \{(0, 26), (26, 0)\}$, and f is strictly straight with respect to $(\underline{r}, \underline{G}, g_2)$. Note that f is irreducible, and that $\Gamma(f) =< 4, 6, 13 >$.

ii) Let f be as in i), and let $r_0 = 4, r_1 = 10, r_2 = 13$. If $\underline{G} = (g_1 = y, g_2 = y^2 - x^3, g_3 = f)$ and $\underline{r} = (10, 13)$, then $\mathrm{GNS}(f, \underline{r}, \underline{G}, g_2) = \{(0, 26), (30 = 5r_0 + r_1, 0)\}$, in particular, f is not straight with respect to $(\underline{r}, \underline{G}, g_2)$.

5. **The criterion**

Let $f = y^n + a_1(x)y^{n-1} + \ldots + a_n(x)$ be a nonzero quasi-ordinary polynomial of $\mathbf{R}[y]$ and assume, after possibly a change of variables, that $a_1(\underline{x}) = 0$. Let $\underline{r}_0 = (r_0^1, \ldots, r_0^e)$ be the canonical basis of $(n\mathbb{Z})^e$ and let $D_1 = n^e$, $d_1 = n$. Let $g_1 = y$ be the d_1-th approximate root of f and set $r_1 = \exp_{glex}(a_n(\underline{x}))$. Let D_2 be the gcd of the (e, e) minors of the $e \times (e+1)$ matrix $(nI_e, {r_1}^T)$. We set $d_2 = \dfrac{D_2}{n^{e-1}}$, $g_2 = \mathrm{App}_{d_2}(f)$, and $e_2 = \dfrac{d_1}{d_2} = \dfrac{n}{d_2}$. Similarly we shall construct $r_k, g_k, d_{k+1}, e_k,\ k \geq 2$ as follows: given $(r_1, \dots, r_{k-1})$ and $(d_1, \dots, d_k)$, let g_k be the d_k-th approximate root of f, and let

$$f = g_k^{d_k} + \beta_2^k g_k^{d_k-2} + \ldots + \beta_{d_k}^k$$

be the g_k-adic expansion of f. We set $r_k = \mathrm{fO}(\underline{r}^k, \underline{G}^k, \beta_{d_k}^k)$, where $(\frac{r_0^1}{d_k}, \dots, \frac{r_0^e}{d_k})$ denotes the canonical basis of $(\frac{n}{d_k}\mathbb{Z})^e$, $\underline{r}^k = (\frac{r_1}{d_k}, \dots, \frac{r_{k-1}}{d_k})$ and $\underline{G}^k = (g_1, \dots, g_{k-1})$. We also set $D_{k+1} =$ the gcd of the (e,e) minors of the matrix $[nI_e, r_1^T, \dots, r_k^T]$, $d_{k+1} = \dfrac{D_{k+1}}{n^{e-1}}$, and $e_k = \dfrac{d_k}{d_{k+1}}$. With these notations we have the following:

Theorem 5.1. The quasi-ordinary polynomial f is irreducible if and only if the following holds:
i) There is an integer h such that $d_{h+1} = 1$.
ii) $g_1, \cdots, g_h$ are irreducible quasi-ordinary polynomials.
iii) For all $1 \leq k \leq h-1, r_k d_k < r_{k+1} d_{k+1}$.
iv) For all $2 \leq k \leq h+1$, g_k is strictly straight with respect to $(\underline{r}^k, \underline{G}^k, g_{k-1})$.

We shall first prove the following results:

Lemma 5.2. Let $c \in \mathbf{K}^*$. The quasi-ordinary polynomial $F = y^N - cx_1^{\alpha_1} \ldots . x_e^{\alpha_e}$ is irreducible in $\mathbf{R}[y]$ if and only if $\gcd(N, \alpha_1, \dots, \alpha_e) = 1$, or equivalently if and only if the gcd of the (e,e) minors of the matrix $(NI_e, (\alpha_1, \dots, \alpha_e)^T)$ is N^{e-1}.

Proof. . Let $\tilde{c}$ be an N-th root of c in $\mathbf{K}$ and let $d = \gcd(n, \alpha_1, \dots, \alpha_e)$. If $d > 1$, then $F = \prod_{w^d=1}(y^{\frac{N}{d}} - w\tilde{c}x_1^{\frac{\alpha_1}{d}} \ldots . x_e^{\frac{\alpha_e}{d}})$, which is a contradiction. Conversely, let $Y = \tilde{c}x_1^{\frac{\alpha_1}{N}} \ldots . x_e^{\frac{\alpha_e}{N}} \in \mathbf{K}((x_1^{\frac{1}{N}}, \dots, x_e^{\frac{1}{N}}))$. Then F is the minimal polynomial of Y over $\mathbf{K}((\underline{x}))$. In particular it is irreducible. □

Proposition 5.3. Let $F = y^N + b_2(\underline{x})y^{N-2} + \ldots + b_N(\underline{x})$ be an irreducible quasi-ordinary polynomial of degree N in y, and let $(m'_k)_{1 \leq k \leq h'}$ be the set of characteristic exponents of F. Let also $(d'_k)_{1 \leq k \leq h'+1}$ (resp. $(r'_k)_{1 \leq k \leq h'}$) be the $\underline{d}$-sequence (resp. the $\underline{r}$-sequence) of F. Let F' be a quasi-ordinary polynomial of $\mathbf{R}[y]$ and assume that F' is monic of degree N in y. If $r'_{h'} d'_{h'} <_{glex} \mathrm{O}_{glex}(F, F')$, then F' is irreducible in $\mathbf{R}[y]$.

Proof. . Assume that F' is not irreducible and let $\tilde{F'}$ be an irreducible component of F' in $\mathbf{R}[y]$. Let $C = c_{glex}(F, \tilde{F'})$ be the contact of F with $\tilde{F'}$. If $m'_{h'} <_{glex} NC$, then $\deg_y(\tilde{F'}) \geq N = \deg_y(F')$ (see Corollary 1.7.), which is a contradiction. Finally $NC \leq_{glex} m'_{h'}$, in particular, by Proposition 1.9., $O_{glex}(F, \tilde{F'}) \leq_{glex} r'_{h'} d'_{h'} \dfrac{\deg_y(\tilde{F'})}{N}$. Since this is true for all irreducible components of F', then $O_{glex}(F,' F) \leq_{glex} r'_{h'} d'_{h'} \dfrac{\deg_y(F')}{N} = r'_{h'} d'_{h'}$, which contradicts the hypothesis. □

Proof of Theorem 5.1. Suppose first that f is irreducible. Condition i) follows from the results of Section 1, condition ii) follows from Theorem 3.1., and condition iii) is nothing but Corollary 1.8.,i). Now for all $1 \leq k \leq h+1$, g_k is an irreducible quasi-ordinary polynomial and $g_1, \dots, g_{k-1}$ are the approximate roots of g_k. In particular, to prove iv), it suffices to prove that $f = g_{h+1}$ is strictly straight with respect to $(\underline{r}, \underline{G}, g_h)$. Let

$$f = g_h^{d_h} + \beta_2^h g_h^{d_h-2} + \ldots + \beta_{d_h}^h$$

be the g_h-adic expansion of f and let $\Gamma^{h-1}(f)$ be the semigroup generated by $r_1^0,\ldots,r_e^0,r_1,\ldots,r_{h-1}$. We have the following:

- For all $2\le i\le h-1$, $O_{glex}(\beta_i^h,f)\in\Gamma^{h-1}(f)$ (by Lemma 3.4.).
- For all $0<a<d_h=e_h, a.r_h\notin\Gamma^{h-1}(f)$ (by Corollary 1.8.).

It follows that for all $2\le i\le h-1, O_{glex}(\beta_i^h,f)\neq i.r_h$ and for all $2\le i\neq j\le d_h-1$, $O_{glex}(\beta_i^h,f)+(d_h-i)r_h\neq O_{glex}(\beta_j^h,f)+(d_h-j)r_h$. Since $O_{glex}(g_h^{d_h},f)=r_hd_h$, then $O_{glex}(\beta_{d_h}^h,f)=r_hd_h$ and $i.r_h<O_{glex}(\beta_i^h,f)$ for all $2\le i\le d_h-1$. The other assertions follow by a similar argument.

Conversely suppose that f satisfies the conditions i), ii), iii), and iv). We shall prove by induction on h that f is irreducible. Suppose that $h=1$, then $f=y^n+a_2(\underline{x})y^{n-2}+\ldots+a_n(\underline{x})$, $\underline{G}=(y,f)$, and $\underline{r}=r_1=\exp_{glex}(a_n(x))$. Now condition iv) implies that $i.\exp_{glex}(a_n(\underline{x}))<\exp_{glex}(a_i(\underline{x}))$ for all $2\le i\le n-1$. Furthermore, $D_2=n^{e-1}$ by condition i). In particular $F=y^n+M_{glex}(a_n(\underline{x}))$ is irreducible by Lemma 5.2. Since $r_1d_1<O_{glex}(F,f)=O_{glex}(f-F,f)$, then f is irreducible by Proposition 5.3.

Let $h>1$ and assume that g_k is an irreducible quasi-ordinary polynomial for all $1\le k\le h$. Let $m_0^1=r_0^1,\cdots,m_0^e=r_0^e,m_1=r_1$ and for all $2\le i\le h$, let:

$$m_i=r_i-\sum_{k=1}^{i-1}(e_k-1)r_k$$

Let $f=g_h^{d_h}+\beta_2^hg_h^{d_h-2}+\ldots+\beta_{d_h}^h$ be the g_h-adic expansion of f and let $Y(\underline{t})=\sum_pY_p\underline{t}^p$ be a root of $g_h(t_1^{\frac{n}{d_h}},\ldots,t_e^{\frac{n}{d_h}},y)=0$. Since the quasi-ordinary polynomial g_h is irreducible, then the $\underline{m}$-sequence associated with g_h is $(\frac{m_0^1}{d_h},\ldots,\frac{m_0^e}{d_h},\frac{m_1}{d_h},\cdots,\frac{m_{h-1}}{d_h})$. In particular,

$$\mathrm{GCDM}(\frac{m_0^1}{d_h},\ldots,\frac{m_0^e}{d_h},\frac{m_1}{d_h},\cdots,\frac{m_{h-1}}{d_h})=((\frac{n}{d_h})^e,\frac{d_2}{d_h}(\frac{n}{d_h})^{e-1},\cdots,\frac{d_{h-1}}{d_h}(\frac{n}{d_h})^{e-1},(\frac{n}{d_h})^{e-1}).$$

Note that, by Corollary 1.7., since $\deg_yg_h<n$, then $Y_{\frac{m_h}{d_h}}=0$.

Let λ be an indeterminate and let

$$y(\underline{t},\lambda)=\sum_pY_p\underline{t}^{d_h\cdot p}+\lambda\underline{t}^{m_h}=Y(\underline{t}^{d_h})+\lambda\underline{t}^{m_h}.$$

Let $F(\underline{x},y,\lambda)$ be the minimal polynomial of $y(\underline{x}^{\frac{1}{n}},\lambda)$ over $\mathbf{K}(\lambda)((\underline{x}))$. Conditions i) and iii) imply that the polynomial F is an irreducible quasi-ordinary polynomial of $\mathbf{K}(\lambda)[[\underline{x}]][y]$, of degree n in y. Furthermore, the $\underline{m}$-sequence (resp. the $\underline{r}$-sequence) associated with F is $(m_0^1,\ldots,m_0^e,m_1,\cdots,m_h)$ (resp. $(r_0^1,\ldots,r_0^e,r_1,\cdots,r_h)$), and

$$\mathrm{GCDM}(m_0^1,\ldots,m_0^e,m_1,\cdots,m_{h-1},m_h)=(n^e,d_2n^{e-1},\cdots,d_{h-1}n^{e-1},d_hn^{e-1},n^{e-1}).$$

Now an easy calculation shows that $c_{glex}(F,g_h)=\dfrac{m_h}{n}$, hence $O_{glex}(F,g_h)=r_h$. Furthermore, if we denote by $Y_1(\underline{t})=Y(\underline{t}),Y_2(\underline{t}),\cdots,Y_{\frac{n}{d_h}}(\underline{t})$ the set of roots of $g_h(t_1^{\frac{n}{d_h}},\cdots,t_e^{\frac{n}{d_h}},y)=0$, then we have:

$$M_{glex}(y(\underline{t},\lambda)-Y_k(t_1^{d_h},\cdots,t_e^{d_h}))=\begin{cases}\lambda t^{m_h} & \text{if } k=1\\ a_kt^{d_h\exp_{glex}(Y_1-Y_k)}, a_k\neq 0 & \text{if } k>1.\end{cases}$$

In particular, $\exp_{glex}(g_h(t_1^n,\cdots,t_e^n,y(\underline{t},\lambda)) = m_h+d_h\exp_{glex}(D_y(g_h)) = m_h+\sum_{k=1}^{h-1}(e_k-1)r_k = r_h$, finally, if $a = a_2\cdots a_{\frac{n}{d_h}}$, then:

$$g_h(t_1^n,\cdots,t_e^n,y(\underline{t},\lambda)) = a.\lambda t^{r_h}.u(\underline{t},\lambda)$$

where $u(\underline{t},\lambda)$ is a unit in $\mathbf{K}(\lambda)[[\underline{t}]]$. Let $M_{\underline{G}^h}(\beta^h_{d_h}) = c.\underline{x}^{\underline{\theta}_0}.g_1^{\theta_1}.\ldots.g_{h-1}^{\theta_{h-1}}$, where $\underline{G}^h = (g_1,\ldots,g_h)$ and $c\in\mathbf{K}^*$. We have:

$$O_{glex}(M_{\underline{G}^h}(\beta^h_{d_h}),F) = \sum_{i=1}^{e}\theta_0^i r_0^i + \sum_{k=1}^{h-1}\theta_k r_k$$

which is r_hd_h by condition iv). By the same condition, the following hold:

- $\beta_{d_h}(t_1^n,\cdots,t_e^n,Y(\underline{t},\lambda)) = \bar{c}\underline{t}^{r_hd_h}(1+\bar{u}(\underline{t},\lambda))$, where $\bar{u}(\underline{0},\lambda)=0$ and $\bar{c}\neq 0$.
- $r_hd_h < \exp_{glex}(\beta_i g_i^{d_h-i}(t_1^n,\cdots,t_e^n,Y(\underline{t},\lambda)))$.

In particular $f(t_1^n,\cdots,t_e^n,y(\underline{t},\lambda)) = (\bar{c}+\lambda)t^{r_hd_h}.u_1(\underline{t},\lambda)$, where $u_1(\underline{t},\lambda)$ is a unit in $\mathbf{K}(\lambda)[[\underline{t}]]$. Finally $r_hd_h < \mathrm{O}_{glex}(F(\underline{x},y,-\bar{c}),f)$, which implies by Proposition 5.3. that f is irreducible.

6. Examples

Example 1: Let $f = y^8-2x_1x_2y^4+x_1^2x_2^2-x_1^3x_2^2\in\mathbf{K}[[x_1,x_2]][y]$. Then we have:

- $D_1 = n^2 = 8^2 = 64, d_1 = n = 8,\ r_0^1 = (8,0), r_0^2 = (0,8),\ g_1 = \mathrm{App}_{d_1}(f) = y$, and $r_1 = O(f,g_1) = (2,2)$.
- D_2 is the gcd of the 2×2 minors of the matrix $(8\cdot I_e,(2,2)^T)$, then $D_2 = 16 = 8.2$, in particular $d_2 = 2$. Since $f = (y^4-x_1x_2)^2-x_1^3x_2^2$, then $g_2=\mathrm{App}_{d_2}(f) = y^4-x_1x_2$. Let $\underline{r}^2 = (\dfrac{r_0^1}{d_2},\dfrac{r_0^2}{d_2},\dfrac{r_1}{d_2}) = ((4,0),(0,4),(1,1))$ and $\underline{G}^2 = (g_1)$, then $r_2 = \mathrm{fO}(\underline{r}^2,\underline{G}^2,x_1^3x_2^2) = 3(4,0)+2(0,4) = (12,8)$.
- D_3 is the gcd of the 2×2 minors of the matrix $(8I_2,(2,2)^T,(12,8)^T)$, then $D_3 = 8$, in particular $d_3 = 1$.
- Now $\mathrm{GNP}(g_2,\underline{r}^2,\underline{G}^2) = \{((0,0),4.(1,1)),((4,4),(0,0))\}$ and $\mathrm{GNP}(f,\underline{r}^3 = (r_0^1,r_0^2,r_1,r_2),\underline{G}^3 = (g_1,g_2)) = \{((0,0),2.(12,8)),((24,16),(0,0))\}$, then the strict straightness condition is verified. Since $g_1 = y$ is irreducible, then so is g_2, but g_2 is quasi-ordinary and $r_1d_1 < r_2d_2$, then f is irreducible. Note that $m_2 = r_2-(\frac{d_1}{d_2}-1)r_1 = (12,8)-3(2,2) = (6,2))$ is the second characteristic exponent of f.

Example 2: Let $f = y^8-2x_1x_2y^4+x_1^2x_2^2-x_1^4x_2^2-x_1^5x_2^3\in\mathbf{K}[[x_1,x_2]][y]$. Then we have:

- $D_1 = n^2 = 8^2 = 64, d_1 = n = 8,\ r_0^1 = (8,0), r_0^2 = (0,8),\ g_1 = \mathrm{App}_{d_1}(f) = y$, and $r_1 = (2,2)$.
- D_2 is the gcd of the 2×2 minors of the matrix $(8I_2,(2,2)^T)$, then $D_2 = 16 = 8.2$, in particular $d_2 = 2$. Since $f = (y^4-x_1x_2)^2-x_1^4x_2^2-x_1^5x_2^3$, then $g_2=\mathrm{App}_{d_2}(f) = y^4-x_1x_2$. Let $\underline{r}^2 = (\dfrac{r_0^1}{d_2},\dfrac{r_0^2}{d_2},\dfrac{r_1}{d_2}) = ((4,0),(0,4),(1,1))$ and $\underline{G}^2 = (g_1)$, then $r_2 = \mathrm{fO}(\underline{r}^2,\underline{G}^2,x_1^4x_2^2) = 4(4,0)+2(0,4) = (16,8)$.
- D_3 is the gcd of the 2×2 minors of the matrix $(8\cdot I_2,(2,2)^T,(16,8)^T)$, then $D_3 = 16$, in particular $d_3 = d_2 = 2$. In particular f is not irreducible. Note that in this example the strict straightness condition is verified for f and g_2.

Example 3: Let $f = y^8-2x_1x_2y^4+x_1^3x_2^2-x_1y^5\in\mathbf{K}[[x_1,x_2]][y]$. Then we have:

- $D_1 = n^2 = 8^2 = 64, d_1 = n = 8,\ r_0^1 = (8,0), r_0^2 = (0,8),\ g_1 = \mathrm{App}_{d_1}(f) = y$, and $r_1 = (3,2)$.
- D_2 is the gcd of the 2×2 minors of the matrix $(8I_2,(3,2)^T)$, then $D_2 = 8$, in particular $d_2 = 1$.
- $\mathrm{GNP}(f,\underline{r}^2 = (r_0^1,r_0^2,r_1),\underline{G}^2 = (g_1)) = \{((0,0),8.(3,2)),((8,0),5.(3,2)),((8,0)+(0,8),4.(3,2)),(3.(8,0)+2.(0,8),(0,0))\} = \{((0,0),(24,16)),((8,0),(15,10)),((8,8),(12,8)),((24,16),(0,0))\}$. Here the strict straightness is not verified, then f is not irreducible.

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