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A REMARKABLE EELLIPTIC CURVE

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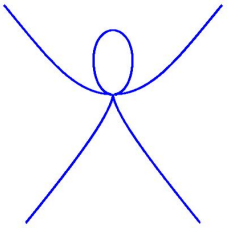
To the Memory of Wolfgang Ebeling

Abstract. We describe a system of plane algebraic curves defined over $\mathbb{Z}$, attached naturally to the exponential function. One of these is a remarkable curve of degree 6 that has genus equal to 1, and looks like



As this sextic curve has rational points, it is an elliptic curve and can be transformed over $\mathbb{Q}$ into the curve **1584.j1** of the *L-functions and modular forms database*, LMFDB for short. One is left to wonder what the number 11, appearing in the factorisation $1584 = 2^4 \cdot 3^2 \cdot 11$, has to do with the exponential function.

1. The exponential function e^x is the first object we encounter in elementary analysis that transcends the algebraic geometrical world of algebraic functions. Being its own derivative and having no zeros or poles, it seems to lack clear geometrical features. A little later we learn to extend e^x into the complex domain and to study the function e^z, $z = x + iy$. We discover that it has an imaginary period $2\pi i$, and is related, via Euler's relation

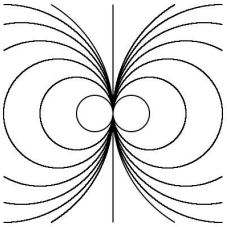
$$e^{x+iy} = e^x \left(\cos y + i \sin y\right),$$

to the functions sin and cos, the circle, trigonometry and cyclotomy.

2. In the real domain, the function e^x stretches between the limit value 0 for $x = -\infty$ and $+\infty$ for $x = +\infty$, thus showing that the point $z = \infty$ is an *essential singularity*. To study this special point, it is convenient to perform the inversion $z \mapsto 1/z$ and study $e^{1/z}$ in a neigbourhood of $z = 0$. Decomposing $z = x + iy$ in real and imaginary parts, the inversion takes the form

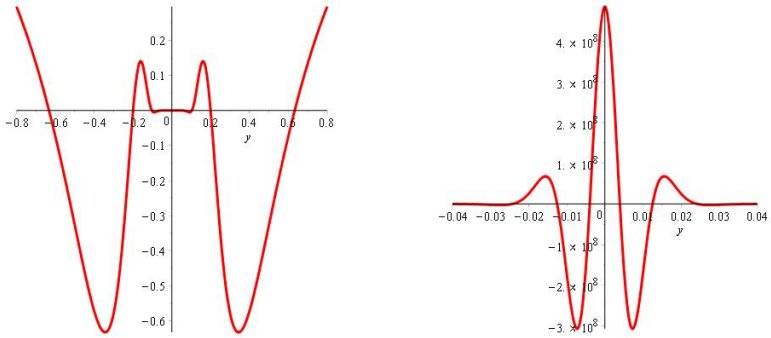
$$(x, y) \mapsto (x/(x^2 + y^2), -y/(x^2 + y^2)),$$

The level sets of $|e^z| = e^x$ are the vertical lines $x =$ constant and these are mapped to the system of circles tangent to the line $x = 0$ and with center on the line $y = 0$, the level sets of $|e^{1/z}| = e^{x/(x^2+y^2)}$.



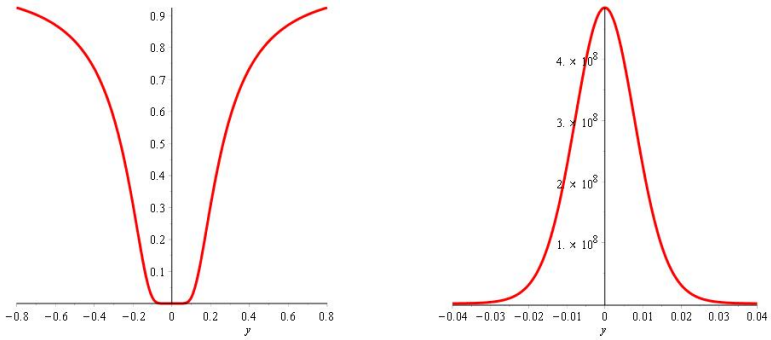
If we approach the origin $z = 0$ from the right halfplane $x = \mathrm{Re}(z) > 0$, the function $e^{1/z}$ is unbounded; coming from the left halfplane $\mathrm{Re}(z) < 0$, the function approches 0. The restriction to the imaginary axis $x = 0$ is $e^{1/(iy)}$ and has $\cos(1/y)$ as real part and exhibits the well-known wild oscillatory behaviour near 0.

3. To get a clearer view of what is happening, one can plot, for small $x =$ constant, the graph of the function $\mathrm{Re}(e^{1/(x+iy)}) = e^{x/(x^2+y^2)} \cos \frac{y}{x^2+y^2}$. For y large this function tends rapidly to its limiting value 1, so we look at a small interval around $y = 0$. For $x < 0$ one observes an oscillatory behaviour, whose amplitude suddenly drops to very small values, forming a *valley* around the origin. For $x > 0$ we have complementary behaviour: the function is small well away from the origin, but develops oscillations that suddenly blow up when we approach the origin.



Plot of $\mathrm{Re}(e^{1/(x+iy)})$ for $x = -0.05$ (left) and $x = 0.05$ (right).
(Note the difference in scaling on the axes.)

The graphs of $|e^{1/(x+iy)})| = e^{x/(x^2+y^2)}$ for $x =$ constant suggest the presence of two inflectional points that may serve to demarcate the location of the regions where the function is very small.



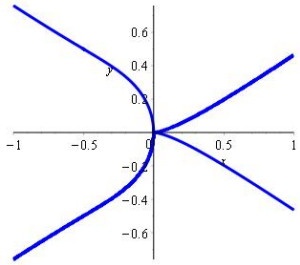
Plot of $|(e^{1/(x+iy)}|$ for $x = -0.05$ (left) and $x = 0.05$ (right).
(Note the difference in scaling on the axes.)

These pictures illustrate the striking *difference in scale* for positive and corresponding negative values of x: the width of the valley for $x = -c < 0$ seems to be much bigger than the width of the corresponding peak for $x = c > 0$.

4. It is easy to calculate the location of these inflectional points by computing the second derivative of $e^{x/(x^2+y^2)}$ with respect to y. Its vanishing is determined by the polynomial factor $xF_2(x, y)$, where

$$F_2(x, y) := (x^2 - 3y^2)(x^2 + y^2) - 2xy^2,$$

which defines an irreducible quartic curve C_2 with a surprising singularity of type D_5 at the origin, consisting of a smooth branch and a transverse cuspidal branch.



The rational curve C_2 defined by $F_2 = 0$.

Hence we are dealing with a rational curve which can be parametrised by the pencil of lines through 0, which intersect the curve C_2 in a unique further point; the substitution $y = mx$ leads to

$$x = \frac{2m^2}{(1+m^2)(1-3m^2)}, \quad y = m \cdot x.$$

In the half-plane $x > 0$, C_2 exhibits it cuspidal branch, which is intersected by the line $x = c > 0$ in two points, which span an interval of length

$$R_c = \sqrt{2}c^{3/2}(1 + \frac{c}{2} + \ldots),$$

whereas in the half-plane $x < 0$ the curve C_2 is approximated near 0 by a parabola, and now the two points of intersection of C_2 with the line $x = -c$ (same positive c as before) span an interval of length $L_c = 2\sqrt{\frac{2}{3}}c^{1/2}(1 - \frac{c}{2} + \ldots)$. The ratio between these intervals is

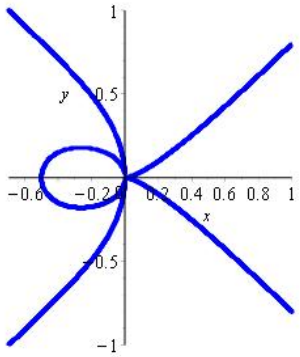
$$R_c/L_c \sim \frac{1}{2} \cdot \sqrt{3} \cdot c.$$

As a consequence, the central peak is indeed much narrower for small $x = c > 0$ than the corresponding central valley for $x = -c < 0$, explaining the feature observed in **3.**

5. Surprised by the beauty of the curve C_2, one asks: why not take a look at higher derivatives of $e^{x/(x^2+y^2)}$? The third derivative factors out the irreducible polynomial

$$F_3(x,y) = 6(x^2-y^2)(x^2+y^2)^2 + 3x^5 - 6x^3y^2 - 9xy^4 - 2x^2y^2.$$

The curve defined by $F_3 = 0$ is a sextic C_3, with a fourfold point at the origin, where we find two smooth and a cuspidal branch. One of the smooth branches closes up at the point $(x, y) = (-1/2, 0)$ to a loop lying in the left half-plane.



The curve C_3 defined by $F_3 = 0$.

In the affine (x, y)-plane, the curve has the origin as unique singular point. But we will look also at the projective closure of this curve, lying in the projective closure of the complexification $\mathbb{C}^2$ of $\mathbb{C}$ considered as $\mathbb{R}$-vector space. As is visible from the equation, it passes doubly through the two circular points $(\pm i : 1 : 0)$ on the line at infinity common to all circles in $\mathbb{R}^2$.

The *genus* of an irreducible plane curve C is defined to be the topological genus of the Riemann surface, obtained as the normalisation $\widetilde{C}$ of C. It is by far the most important invariant of a curve. We refer to [1] and in particular [11] for more information. The genus can be computed via the *Riemann-Clebsch formula* ([1], p.830) in terms of its degree and types of singularities as follows:

$$\text{genus}(C) = \frac{(d-1)(d-2)}{2} - \sum_p \delta(C, p),$$

where $\delta(C, p)$ is the so-called *δ-invariant* of the singularity (C, p) at $p \in C$. For an ordinary double point one has $\delta = 1$, so each node decreases the genus by one. The invariant $\delta(C, p)$ is also called the *virtual number of double points* of the singularity, as by a small generic perturbation of the normalisation map the singularity will split in $\delta(C, p)$ double points. If a singularity consists of several branches through p, the δ-invariant can be computed using

$$\delta(C \cup D, p) = \delta(C, p) + \delta(D, p) + (C \cdot D)_p,$$

which is obvious from the interpretations in terms of virtual double points. Using this formula inductively, one finds for example $\delta = m(m-1)/2$ for the singularity consisting of m smooth branches passing through a point, with distinct tangents, and indeed, if we shift these m lines a little bit, we obtain generically $m(m-1)/2$ intersections points. We now apply these ideas to determine the genus of our curve C_3. As we have two smooth branches A and B and one cuspidal branch C at the origin, and each pair of branches have mutual intersection multiplicity $= 2$, we obtain for the δ-invariant of C_3 at 0:

$$\begin{aligned} \delta(C_3, 0) &= \delta(A, 0) + \delta(B, 0) + \delta(C, 0) + (A \cdot B) + (A \cdot C) + (B \cdot C) \\ &= 0 + 0 + 1 + 2 + 2 + 2 = 7. \end{aligned}$$

Together with the double points of C_3 at the circular points, we find from the Riemann-Clebsch formula:

$$\text{genus}(C_3) = \frac{(6-1)(6-2)}{2} - 7 - 1 - 1 = 1,$$

so C_3 is a curve of genus 1. As we have the rational point $(-1/2, 0)$ lying on C_3, the curve is in fact an *elliptic curve defined over* $\mathbb{Q}$. We note the presence of the further rational points $(-1/6, \pm 1/6)$ on C_3.

6. The pencil of lines through the origin intersect the curve C_3 in two further points, representing C_3 as double cover of the projective line of directions at 0. Setting $y = mx$ in $F_3(x, y)$, we obtain

$$F_3(x, mx) = -x^4(Ax^2 + Bx + C),$$

where

$$A = 6(m-1)(m+1)(m^2+1)^2, \quad B = 3(3m^2-1)(m^2+1), \quad C = 2m^2.$$

The x-coordinate of the two further intersection points of C_3 with the line $y = mx$ are the roots of $Ax^2 + Bx + C$. So the ramification points of the double cover are determined by the discriminant

$$D = B^2 - 4AC = 3(m^2+1)^2(11m^4 - 2m^2 + 3)$$

of the quadratic term. This shows that the curve C_3 is a double cover ramified over the four roots of the quartic

$$11m^4 - 2m^2 + 3.$$

For a general quartic $am^4+4bm^3+6cm^2+4dm+e$ one can compute the j-invariant of its four roots using the classical Cayley invariants (see [12], p.189 and p.345):

$$S := ae-4bd+3c^2, \quad T := ace+2bcd-ad^2-b^2e-c^3$$

as

$$j := 1728\frac{S^3}{S^3-27T^2}$$

Setting in $a=11, b=0, c=-1/3, d=0, e=3$ we obtain

$$j = \frac{62500}{33} = \frac{2^2\cdot 5^6}{3\cdot 11},$$

which determines the elliptic curve as a complex curve. However, the above quartic has only complex roots, and it is not directly clear how to find the familiar Weierstrass normal form from here.

7. It is much more convenient to first perform the inversion $z\mapsto 1/z$, or

$$(x,y)\mapsto (x/(x^2+y^2), -y/(x^2+y^2))$$

on our curve, so in a way 'undoing' the initial inversion to go from e^z to $e^{1/z}$. The corresponding substitution into F_3 leads to:

$$F_3\left(\frac{x}{x^2+y^2}, -\frac{y}{x^2+y^2}\right) = \frac{G_3(x,y)}{(x^2+y^2)^4},$$

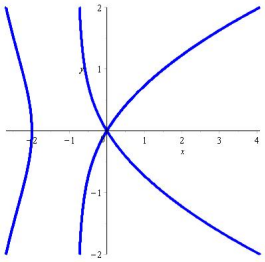
where

$$G_3 := -2x^2y^2+3x^3-9xy^2+6x^2-6y^2.$$

As the inversion is an involutory transformation, we also have

$$G_3\left(\frac{x}{x^2+y^2}, -\frac{y}{x^2+y^2}\right) = \frac{F_3(x,y)}{(x^2+y^2)^4}.$$

Indeed, from the standpoint of projective geometry, we are performing a *Cremona transformation* based at 0 and the two circular points. Any conic through these three points (i.e. any linear combination of xz, yz, x^2+y^2) intersects C_3 generally in $2\cdot 6-4-2-2=4$ further points, so this transforms our sextic C_3 into a *quartic curve* Q, defined by the vanishing of the above polynomial G_3.



The quartic curve Q defined by $G_3=0$.

The (projective closure of) the curve Q has $(0:0:1)$ and $(0:1:0)$ as singular points, which are both ordinary double points, so the genus is indeed $3-1-1=1$. We note that the equation for Q is of a very special form:

$$0 = -(2x^2+9x+6)y^2+3x^2(x+2), \quad y^2 = 3x^2\frac{x+2}{2x^2+9x+6}.$$

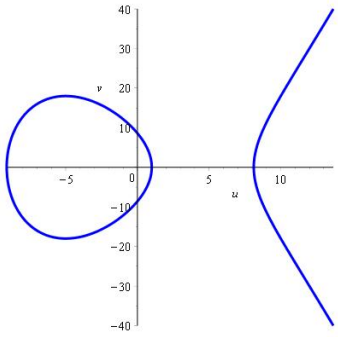
The introduction of the variable $y' := y(2x^2+9x+6)/x$ defines a birational transformation of the plane and converts this equation into the form

$$y'^2 = 3(x+2)(2x^2+9x+6) = 6x^3+39x^2+72x+36.$$

Completing the cube and scaling $u = 6x+13$, $v = 6y'$ brings the equation in *minimal Weierstrass form*

$$v^2 = u^3 - 75u + 74,$$

defining a standard elliptic curve E in the (u, v)-plane.



The elliptic curve E defined by $v^2 = u^3 - 75u + 74$.

Combining the transformations, we can express x and y in u and v:

$$x = \frac{u-13}{6}, \quad y = \left(\frac{v}{2}\right)\frac{u-13}{u^2+u-74},$$

and indeed

$$G_3(\frac{u-13}{6}, \left(\frac{v}{2}\right)\frac{u-13}{u^2+u-74}) = \frac{(u-13)^2(u^3-75u+74-v^2)}{72(u^2+u-74)},$$

checking that E is mapped to Q. The composition

$$(u, v) \mapsto (x, y) \mapsto (x/(x^2+y^2), -y/(x^2+y^2))$$

is the sought for birational map from E to the sextic C_3.

7. There are powerful software tools to handle algebraic curves and in particular elliptic curves.

MAPLE [9] has a nice package to deal with algebraic curves, their singularities, Puiseux expansions, their genera, etc. It has a built-in routine `Weierstrassform` that can be applied to find the Weierstrass normal form for the elliptic curve defined by the sextic C_3. However, the algorithm produces a rather complicated transformation, whose geometric origin is obscure. The above simple algebra-geometric approach leads to a much simpler transformation.

PARI/GP [10] is a powerful software, pitched toward the computation of arithmetical invariants of (elliptic) curves defined over number fields. It can count points, handle modular forms and L-functions and much more. Convenient is the command `ellidentify`, which tells us that our curve E has the Cremona label **1584f1** [4], so the conductor of our curve is $1584 = 2^4 \cdot 3^2 \cdot 11$. The conductor can be considered as a refinement of the discriminant, which for the cubic $u^3 - 75u - 74$ is $-2^6 \cdot 3^7 \cdot 11$. It is made up of the same prime factors, but the exponents occuring in the conductor are usually (much) lower. For the precise definition we refer to the book of [13]. Elliptic curves are commonly ordered according to their conductor, which is most basic measure of their arithmetic complexity. The primes p dividing the conductor are exactly the primes for which the reduction modulo p of the elliptic curve is singular. So we learn that our curve E remains smooth if we reduce modulo p for all $p \neq 2, 3, 11$. The number of points of our curve modulo such a p is of the form $1 + p - a_p$, where a_p is the Fourier-coefficient of a modular form of weigth two for the modular group $\Gamma_0(1584)$.

The LMFDB (L-functions and Modular Forms Data Base) [8] represents a huge extension of the earlier tables by J. Cremona [4]; it assigns to our curve the label **1584.j1** and presents further data attached to the curve in an attractive form. We learn that the Mordell-Weil group of the curve is $\mathbb{Z} \oplus \mathbb{Z}/2$. The torsion point $(u, v) = (1, 0)$ of E corresponds to $(-2, 0)$ on Q and $(-1/2, 0)$ on C_3. The generator of infinite order $(u, v) = (-5, 18)$ is mapped to $(-3, 3)$ on Q and to $(-1/6, 1/6)$ of C_3. Furthermore, the curve E posseses 11 (!) points with integral coordinates, which map to special rational points on Q and C_3.

The curve $ny^2 = f(x)$ $(n \in \mathbb{Z})$ is called a *quadratic twist* of the curve $y^2 = f(x)$. The substitution $y \mapsto \sqrt{n}y$ shows that these two curves are isomorphic over $\mathbb{C}$, but not over $\mathbb{Q}$ if n is not a square. There exists a twist that makes the twisted curve have conductor as low as possible, called the *minimal twist*. From LMFDB we also learn that our curve has the curve **264.c1** with equation $y^2 = x^3 + x^2 - 8x$ as minimal quadratic twist ($264 = 2^3 \cdot 3 \cdot 11$ and that this curve has Mordell-Weil group $\mathbb{Z}/2$.

For general background information on algebraic curves, among the many books available, my favourites are [1] and [2]. For the arithmetic theory of elliptic curves [13] is indispensable.

8. Why stop at the third derivative? Write $\phi := \exp(x/(x^2 + y^2))$ and $\partial := \partial/\partial y$, so

$$\partial\phi = h\phi, \quad h := \partial\Big(\frac{x}{x^2 + y^2}\Big).$$

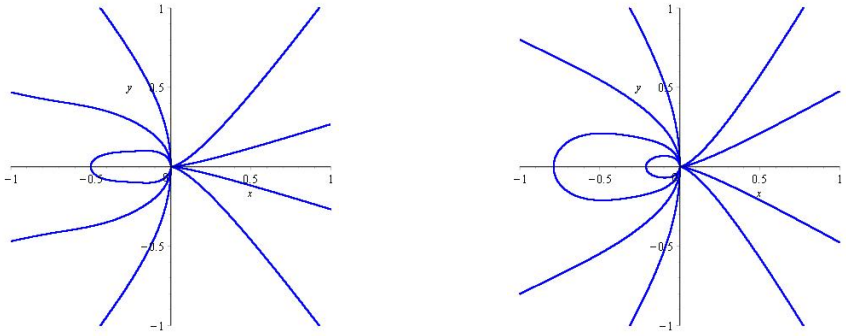
For the n-th derivative one then has

$$\partial^n\phi = h_n\phi, \quad h_{n+1} = \partial h_n + h \cdot h_n,$$

so that

$$h_n = \frac{f_n}{(x^2 + y^2)^{2n}},$$

where $f_n \in \mathbb{Z}[x, y]$ is a polynomial of degree $3n - 1$. For n odd f_n factorises as $f_n = xyF_n$, and for n even as $f_n = xF_n$, where for small n the polynomial $F_n \in \mathbb{Z}[x, y]$ appears to be irreducible (of degree $3n - 3$ (n odd) resp. $3n - 2$ (n even)). (Note $f_0 = 1$, $f_1 = -2xy$, so there are no curves C_0 or C_1.)



The curve C_4 of genus 4 and C_5 of genus 7.

The curve C_n defined by $F_n = 0$ has a unique singularity in $\mathbb{C}^2$, lying at the origin, where it has $(n - 1)$ smooth local branches with common vertical tangent and $\lfloor \frac{n}{2} \rfloor$ cuspidal branches with common horizontal tangent. Furthermore it has ordinary $(n - 1)$-fold points at the circular points. Assuming irreducibility of F_n for all n, the genus of C_n can be computed with help of

the δ-invariants of the singularities and the Riemann-Clebsch formula. We find:

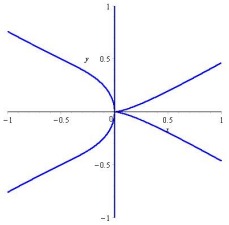
$$\text{genus}(C_n) = \begin{cases} (n-2)(3n-4)/4 & \text{if } n \text{ is even} \\ (3n^2+1)/4 & \text{if } n \text{ is odd} \end{cases}$$

In the real domain, the curves C_n have an insect-like appearance. The $(n-1)$ smooth branches are all in the halfspace $x \le 0$; $\lfloor \frac{n-1}{2} \rfloor$ close up to 'heads', whereas the remaining branches form pairs of 'arms'; the cuspidal branches are in the halfspace $x \ge 0$ and form the 'legs'.

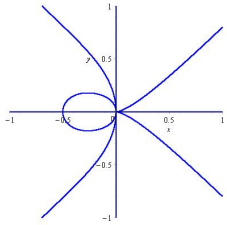
9. Of course, we do not need to restrict to y derivatives only. It is simple to show that

$$P_{a,b} := (x^2+y^2)^{2(a+b)} \left(\frac{\partial^{a+b}}{\partial x^a y^b} e^{x/(x^2+y^2)} \right) e^{-x/(x^2+y^2)}$$

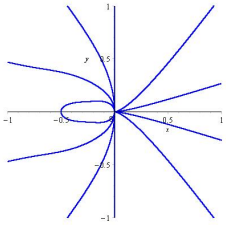
is a polynomial of degree $3(a+b)-1$. For b an odd number, $P_{a,b}$ is divisible by y; for $a=0$ there is an additional factor x. Clearing out these factors, we obtain a polynomial $F_{a,b}$ that turns out to define an irreducible curve $C_{a,b}$ and $C_n = C_{0,n}$ for small values of a and b.



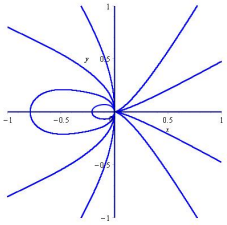
$g(C_{02}) = 0,$



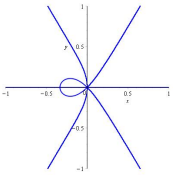
$g(C_{03}) = 1,$



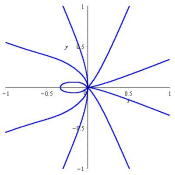
$g(C_{04}) = 4,$



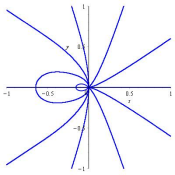
$g(C_{05}) = 7.$



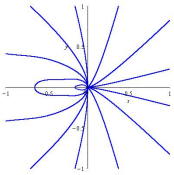
$g(C_{11}) = 0,$



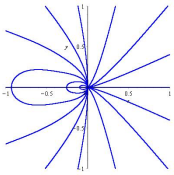
$g(C_{12}) = 3,$



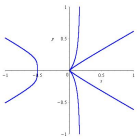
$g(C_{13}) = 6,$



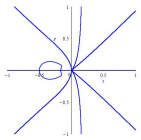
$g(C_{14}) = 13,$



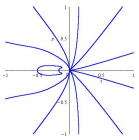
$g(C_{15}) = 18.$



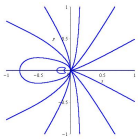
$g(C_{20}) = 0,$



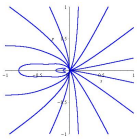
$g(C_{21}) = 3,$



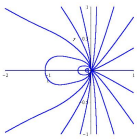
$g(C_{22}) = 10,$



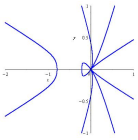
$g(C_{23}) = 15,$



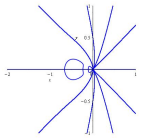
$g(C_{24}) = 26,$



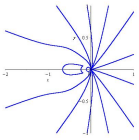
$g(C_{25}) = 33.$



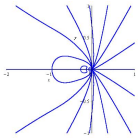
$g(C_{30}) = 2,$



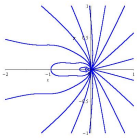
$g(C_{31}) = 7,$



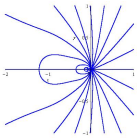
$g(C_{32}) = 18,$



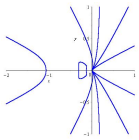
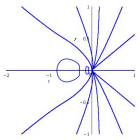
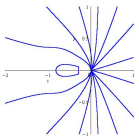
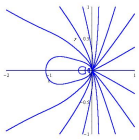
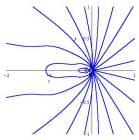
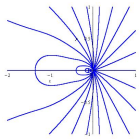
$g(C_{33}) = 25,$



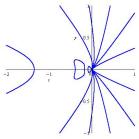
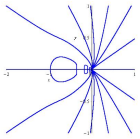
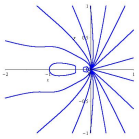
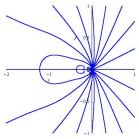
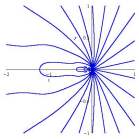
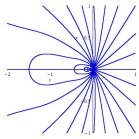
$g(C_{34}) = 40,$



$g(C_{35}) = 49.$

$g(C_{40}) = 7,$ $g(C_{41}) = 14,$ $g(C_{42}) = 29,$ $g(C_{43}) = 38,$ $g(C_{44}) = 57,$ $g(C_{45}) = 68.$

$g(C_{50}) = 13,$ $g(C_{51}) = 22,$ $g(C_{52}) = 41,$ $g(C_{53}) = 52,$ $g(C_{54}) = 75,$ $g(C_{55}) = 88.$

The curve $C_{a,b}$ has the origin as only singular point in the affine plane and furthermore ordinary $a+b$-fold singularities at the two circle points. The genus is basically a quadratic function of a and b, to be more precise we seems to have

$$\text{genus}(C_{a,b}) = 9a^2/8 + 3ab + 3b^2/4 - 13a/4 - 5b/2 + \begin{cases} 2 & \text{if } a \equiv 0 \bmod 2, \\ 13/8 & \text{if } a \equiv 0 \bmod 2,\ b \equiv 0 \bmod 4, \\ 9/8 & \text{if } a \equiv 0 \bmod 2,\ b \equiv 2 \bmod 4, \end{cases}$$

when b is even and

$$\text{genus}(C_{a,b}) = a^2 + 3ab + 3b^2/4 - 7a/2 - 3b + \begin{cases} 7/4 & \text{if } a \equiv 1 \bmod 2, \\ 13/4 & \text{if } a = 0, \\ 9/4 & \text{if } a \equiv 0 \bmod 2,\ a > 0, \end{cases}$$

when b is odd.

10. It is somewhat of a mystery to me how the curve C_3, with its rich specific arithmetic geometry, like its Galois representation of conductor $1584 = 2^4 3^2 11$, can arise without any specific choices from the exponential function, one of the most basic functions one can imagine. But mathematics apparently is like that. There is a philosophy about the non-existent field $\mathbb{F}_1$ with one element, which is about the idea that there should exist a deeper base than $\mathbb{Z}$ over which some sort of algebraic geometry can be done and over which one can deal with the zeros of Riemanns zeta function $\zeta(s)$. The most basic functions like the exponential function and the gamma function certainly have to belong to that elusive 'geometry and analysis over $\mathbb{F}_1$', [3].

Of course, there are many other algebraic geometric objects one can 'naturally' attach to the exponential function. For example, we can look at the Taylor polynomial

$$f_n(x) := 1 + x + \frac{1}{2!}x^2 + \ldots + \frac{1}{n!}x^n$$

of degree n of e^x. The location of the roots of $f_n(nx)$ and their convergence to the *Szegő curve* defined by $\{z \in \mathbb{C} \mid |ze^{1-z}| = 1\}$ is a well-known topic in classical analysis [14]. One can look at the sequence of hyperelliptic curves $y^2 = f_n(x)$, which for $n = 3, 4$ lead to two further 'natural' elliptic curves. But, in a sense I find hard to make precise, these curves look more artificial to me.

Periods in the sense of Kontsevich and Zagier [7] are integrals of rational functions over domains defined by algebraic inequalities, all with coefficients in $\mathbb{Q}$. They include numbers like $\sqrt{2}, \log 2$ and π. Such numbers are attached to motives in ordinary algebraic geometry [6]. Similarly, Kontsevich and Zagier [7] introduced the notion of an *exponential period*, defined as appropriate integral of the exponential of a rational function, like in the theory of oscillatory integrals. There is a corresponding theory of *exponential motives* [5]. Certainly in the context

of that theory, the function $e^{1/z}$ belongs to the very simplest class of examples. It appears that the curves $C_{a,b}$ are natural *algebraic geometric sattelites* of the exponential motive defined by $e^{1/z}$, and as such has some obvious generalisations. One may look at a similar system of curves for e^{1/z^2}. Here one also finds an elliptic curve, **576.c3**, with Mordel-Weil group $\mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$. In general, if $g, S \in K := \mathbb{Q}(x_1, x_2, \ldots, x_n)$ are two rational functions, $\partial \in Der(K)$ a derivation of K, and $\phi := g.e^S$, then repeated differentiation leads to a sequence

$$g_1, g_2, g_3, \ldots, g_n, \ldots \in \mathbb{Q}(x_1, x_2, \ldots, x_n)$$

defined recursively by

$$g_1 := g, g_2 = \partial(g_1) + g_1 \cdot \partial S, \ldots, \ g_{n+1} := \partial g_n + g_n \cdot \partial S, \ldots$$

and which appear as prefactors of $\partial^n \phi = g_n \phi$. The classical motives attached to the g_n are in a similar way algebraic-geometrical satellites of the exponential motive with exponential periods $\int e^S \omega$, $\omega := g \cdot dx_1 dx_2 \ldots dx_n$, [7], [5].

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References

[1] E. Brieskorn, H. Knörrer, *Plane algebraic curves*, Modern Birkhäuser Classics (2012). DOI: 10.1007/978-3-0348-0493-6
[2] H. Clemens, *A Scrapbook on Complex Curve Theory*, The University Series in Mathematics, Plenum Press (New York-London) (1980). DOI: 10.1007/978-1-4684-7000-0
[3] A. Connes, C. Consani, *Schemes over $\mathbb{F}_1$ and zeta functions*, Compositio Math. **146** (2010), p.1383-1415. DOI: 10.1112/S0010437X09004692
[4] J. Cremona, *The elliptic curve database for conductors to* 130000, ANTS-VII proceedings, Lecture Notes in Computer Science, vol. **4076**, (2006), p. 11-29. DOI: 10.1007/11792086_2
[5] J. Fresan, P. Jossen, *Exponential Motives*, in preparation.
[6] A. Huber, S. Müller-Stach, *Periods and Nori-Motives*, Ergebnisse der Mathematik 3.Folge, Band 65, Springer Verlag (2017). DOI: 10.1007/978-3-319-50926-6_1
[7] M. Kontsevich, D. Zagier, *Periods*, In: Engquist, B., Schmid, W. (eds) Mathematics Unlimited - 2001 and Beyond. Springer, Berlin, Heidelberg, (2001), p.771-808. DOI: 10.1007/978-3-642-56478-9_39
[8] LMFDB Collaboration, *The L-functions and modular forms database*, lmfdb.org
[9] Maple 2015. Maplesoft, a division of Waterloo Maple Inc., Waterloo, Ontario.
[10] C. Batut, K. Belabas, D. Bernardi, H. Cohen, M. Olivier, *User's Guide to PARI-GP*, Laboratoire A2X, Université Bordeaux I, France, (1998).
[11] P. Popescu-Pampu, *What is the genus?* Springer Lecture Notes in Mathematics **2162**, (2016). DOI: 10.1007/978-3-319-42312-8
[12] G. Salmon, *Higher Algebra* (1885), Reprint Chelsea Publishing Co. New York (1964).
[13] J. H. Silverman, *The Arithmetic of Elliptic Curves*, Graduate Texts in Mathematics, Band 106), Second edition (2009), Springer, Berlin, Heidelberg. DOI: 10.1007/978-0-387-09494-6
[14] G. Szegő, *Über eine Eigenschaft der Exponentialreihe*, Sitzungsber. Berl. Math. Ges. 23 (1924), p. 50-64.

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