

THE FUNDAMENTAL GROUP OF TRIGONAL LOCI OF STRATA OF ABELIAN DIFFERENTIALS

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dedicated to the memory of Wolfgang Ebeling

ABSTRACT. We investigate locally closed subspaces of projectivized strata of abelian differentials which classify trigonal curves with canonical divisor a multiple of a trigonal divisor. We describe their orbifold structure using linear systems on Segre-Hirzebruch surfaces and obtain results for their orbifold fundamental groups.

Most notable among these orbifolds is the connected component $\mathbf{PH}_4^{ev}(6)$, the projectivisation of the space $\mathcal{H}_4^{ev}(6)$ of abelian differentials on non-hyperelliptic genus 4 curves with a single zero of multiplicity 6 providing an even spin structure. Its orbifold fundamental group is identified with the quotient of the Artin group of type E_8 by its maximal central subgroup.¹

1. INTRODUCTION

Wolfgang Ebeling was the advisor of my thesis and following that, he provided constant support and inspiration during the preparation of my habilitation thesis. His way to study and to teach singularity theory had an essential impact on the choice of its topic which aimed at a better understanding of fundamental groups of discriminant complements in versal unfoldings of hypersurface singularities.

These groups have since been a recurring object in the study of various spaces of curves and in the present paper we want to use our approach to the study of moduli spaces $\mathcal{H}_g$ of abelian differentials, particularly – regarding the $\mathcal{H}_g$ as moduli stacks – to the study of their orbifold structure. The $\mathcal{H}_g$ classify pairs (C, φ) consisting of a complex algebraic curve C of genus g and a non-zero section $\varphi \in H^0(C, \omega_C)$ of the canonical bundle ω_C over C .

The associated projectivized moduli spaces $\mathbf{PH}_g$ classify pairs (C, D) , where D is an effective divisor of degree $2g - 2$ on C given as the zero divisor of a differential φ .

The two spaces decompose naturally into strata $\mathcal{H}_g(k_1, \dots, k_r)$ and $\mathbf{PH}_g(k_1, \dots, k_r)$ according to the multiplicities of the zeroes of φ and those of the points of D , respectively. It is the celebrated achievement of Kontsevich and Zorich [KoZ] to find and characterize the connected components of these strata.

Since then it is an intriguing problem how they relate to the moduli space of curves $\mathcal{M}_g$ with Kontsevich and Zorich putting a special emphasis on the following two question which they conjectured to have positive answers:

Are the strata orbifold quotients of contractible spaces?

Do their fundamental groups relate nicely with mapping class groups?

Addressing the second question Calderon Salter [CaS] successfully described the image of the monodromy induced by the forgetful map $\mathcal{H}_g(k_1, \dots, k_r) \rightarrow \mathcal{M}_g$ using techniques primarily from geometric group theory.

¹This result was previously obtained by Giannini [Gia], and we give an independent proof using an alternative argument.

A more algebraic geometric approach is taken by Looijenga and Mondello [LoM]. They describe the orbifold fundamental groups for strata of genus 3 and settle the first question in the affirmative for most of these strata, but they express their doubt, whether their orbifold fundamental groups can be shown to be commensurable with mapping class groups.

We propose the study of a new kind of loci defined in analogy with the purely hyperelliptic strata $\mathcal{H}^{hyp}(2g-2)$ and $\mathcal{H}^{hyp}(g-1, g-1)$, which are exceptional components of $\mathcal{H}_g(2g-2)$ and $\mathcal{H}_g(g-1, g-1)$ respectively. Recall from [KoZ, Rem. 3]:

Points of $\mathcal{H}^{hyp}(2g-2)$ (respectively of $\mathcal{H}^{hyp}(g-1, g-1)$) are Abelian differentials on hyperelliptic curves of genus g which have a single zero of multiplicity $2g-2$ invariant under the hyperelliptic involution (respectively a pair of zeroes of orders $g-1$ symmetric to each other with respect to the hyperelliptic involution).

Pairs (C, D) in $\mathbf{PH}^{hyp}(2g-2)$ and in $\mathbf{PH}^{hyp}(g-1, g-1)$ can thus be characterized by the property that D is a multiple of a divisor in the g_2^1 of C . In the first case D has support in a single point, it has support in a pair of distinct points in the second case.

For our generalization we increase the gonality and consider trigonal curves with canonical divisors a multiple of a divisor in the trigonal linear system. The precise definition reads as follows:

Definition 1.1. A pair $(C, D) \in \mathbf{PH}_g$ is called *strictly trigonal* if

- i) C is strictly trigonal, i.e. not hyperelliptic with a trigonal linear system g_3^1 , see 2,
- ii) D is an integral multiple of a divisor L in a trigonal linear system g_3^1 of C .

The subspace $\mathbf{PH}_g^{tri}$ of $\mathbf{PH}_g$ of strictly trigonal pairs is the moduli space of these pairs.

We will see below that these spaces are non-empty only if $g = 3k + 1$ for some $k \geq 1$ and only intersect strata of types $(6k)$, $(4k, 2k)$ or $(2k, 2k, 2k)$ with spin structure of the parity of g . The number of zeroes is 1, 2 and 3, respectively, for the differentials.

In these cases we can describe the orbifold structure in very concrete terms using discriminant complements of suitable linear systems on Hirzebruch surfaces (later on, we will be more precise and explain all ingredients):

Theorem 1.2. *Loci of strictly trigonal abelian differentials in $\mathbf{PH}_g$ of genus $g = 3k + 1$, $k \geq 1$ can be identified in the orbifold sense as*

$$\begin{aligned} \mathbf{PH}_{3k+1}^{tri}(6k) &\cong (\mathcal{L}_1^k - \mathcal{D}_1)/\mathbb{C}^* \\ \mathbf{PH}_{3k+1}^{tri}(4k, 2k) &\cong \mathcal{L}_2^k - \mathcal{D}_2 \\ \mathbf{PH}_{3k+1}^{tri}(2k, 2k, 2k) &\cong (\mathcal{L}_3^k - \mathcal{D}_3 - \mathcal{D}')/\mathbb{C} \rtimes (\mathbb{C}^*)^2 \end{aligned}$$

where $\mathcal{L}_i^k$ are linear subsystems of $|3\sigma_0|$ on the Hirzebruch surface $\mathbf{F}_{k+1}$, the $\mathcal{D}_i$ are the respective discriminant divisors corresponding to singular curves, and $\mathcal{D}'$ is the divisor corresponding to abelian differentials with less than 3 zeroes.

The same kind of description was obtained in [BoL] for the locus of trigonal curves of maximal Maroni invariant and its orbifold fundamental group – in fact equal to that of $\mathbf{PH}_{3k+1}^{tri}$ – was determined using unfoldings of isolated plane curve singularities.

Similarly here, for types $(6k)$ and $(2k, 2k, 2k)$ we are going to describe the topology of the discriminant complements in the theorem with the help of discriminant complements in unfoldings of isolated plane curve singularities. Their fundamental groups, studied in [L10] and [L18] under the name of *discriminant knot groups*, see 6.1, can be used to express the orbifold fundamental groups in the present setting:

Corollary 1.3. *The orbifold fundamental group of $\mathbf{PH}_{3k+1}^{tri}(6k)$ fits into a short exact sequence*

$$1 \longrightarrow \mathbb{Z} \longrightarrow \pi^K(y^3 + x^{3k+2}) \longrightarrow \pi_1^{orb} \mathbf{PH}_{3k+1}^{tri}(6k) \longrightarrow 1$$

as quotient by a central subgroup of the discriminant knot group π^K of the singular plane curve germ $y^3 + x^{3k+2}$.

Corollary 1.4. *The orbifold fundamental group of $\mathbf{PH}_{3k+1}^{tri}(2k, 2k, 2k)$ fits into a short exact sequence*

$$1 \rightarrow \mathbb{Z}^2 \rightarrow \pi^K(y^3 + x^{3k+3}) \rtimes \pi^K(y^3 + x^2) \rightarrow \pi_1^{orb} \mathbf{PH}_{3k+1}^{tri}(2k, 2k, 2k) \rightarrow 1$$

as quotient of a semi-direct product of discriminant knot groups π^K of singular plane curve germs $y^3 + x^{3k+3}$ and $y^3 + x^2$.

Another aspect we want to emphasize arises in the special case $k = 1, g = 4$ of our results. From the basic observation that all curves of genus 4 are trigonal we can sharpen the theorem to obtain an alternative proof of [Gia, Thm.1] (this is not possible for $k > 1$ by a simple dimension count).

Corollary 1.5. *The projectivization $\mathbf{PH}_4^{ev}(6)$ of the space $\mathcal{H}_4^{ev}(6)$ of abelian differentials of even spin structure is isomorphic to a quotient*

$$\mathbf{PH}_4^{ev}(6) \cong (\mathbb{C}^8 - \mathcal{D}_{E_8})/\mathbb{C}^*$$

of the unfolding of the simple E_8 singularity and the orbifold fundamental group is isomorphic to a quotient of the Artin group of type E_8 by its centre:

$$\pi_1^{orb} \mathbf{PH}_4^{ev}(6) \cong Ar(E_8)/Centre$$

This result is very similar to results of [LoM]. Indeed it should be interesting to look at $\mathbf{PH}_4^{ev}(6)$ from the point of view of anti-canonical divisors on a degree one delPezzo surface.

Conspicuously we make no claim in case of type $(4k, 2k)$ where obviously $\pi_1^{orb} = \pi_1$. However, the discriminant complements $\mathcal{L}_2^k - \mathcal{D}$ is not induced from a versal unfolding of plane curve germs and an identification with a discriminant knot of a singular plane curve germs is not possible.

Let us sketch though an idea how to get around that obstacle: As finitely presented groups, the groups above should be studied as instances of *secondary braid groups* 2Br associated to positive braid words, as proposed by the author and investigated together with Baader [BaL]. Indeed, there is the following identification with secondary braid groups associated to positive words in the standard generators σ_1, σ_2 of the braid group Br_3 :

$$\pi^K(y^3 + x^{3k+2}) \cong {}^2Br((\sigma_1\sigma_2)^{3k+2})$$

$$\pi^K(y^3 + x^{3k+3}) \cong {}^2Br((\sigma_1\sigma_2)^{3k+3})$$

$$\pi^K(y^3) \cong Br_3 \cong {}^2Br((\sigma_1\sigma_2)^2)$$

These positive words define braids which close to the links in S^3 of the respective plane curve singularities. Thanks to quasi-homogeneity the links at infinity for the singular affine curves are the same. Moreover it is true that the links at infinity of the affine smooth curves in the vector spaces V_1^k, V_2^k, V_3^k , see (1), corresponding to $\mathcal{L}_1^k, \mathcal{L}_2^k$ and $\mathcal{L}_3^k$ are equal to the closure of the respective braids

$$(\sigma_1\sigma_2)^{3k+2}, \quad \sigma_1(\sigma_2\sigma_1)^{3k+2} = (\sigma_1\sigma_2)^{3k+2}\sigma_1 \quad \text{and} \quad (\sigma_1\sigma_2)^{3k+3}$$

So by interpolation and after some encouraging first investigations we are confident to propose the following conjecture:

Conjecture 1.6. *The orbifold fundamental group of $\mathbf{PH}_{3k+1}^{tri}(4k, 2k)$ is given by an isomorphism*

$$\pi_1^{orb} \mathbf{PH}_{3k+1}^{tri}(4k, 2k) \cong {}^2Br(\sigma_1(\sigma_2\sigma_1)^{3k+2}),$$

where the secondary braid group has a finite presentation by

$$\left\langle t_1, \dots, t_{6k+3} \left| \begin{array}{ll} t_i t_j = t_j t_i & \text{if } |i - j| > 2 \\ t_i t_j t_i = t_j t_i t_j & \text{if } |i - j| \leq 2 \\ t_i t_{i+1} t_{i+2} t_i = t_{i+1} t_{i+2} t_i t_{i+1} & \end{array} \right. \right\rangle.$$

While this has to be deferred to a future paper, we are going to proof our stated results pursuing the following steps. First we are going to review and use canonical geometry of trigonal curves. Then we reduce the canonical global quotient construction of our loci to a quotient constructions on a vector space of polynomials.

The two subsequent section provide a set of transformations and employ them to accomplish the proof of the theorem. In the final section we give the proofs of the corollaries.

2. CANONICAL GEOMETRY

There is a well-known approach to the study of non-hyperelliptic trigonal curves and their moduli. We refer to [StV] which provides all the information necessary for the following set-up.

Let C be a non-hyperelliptic curve of genus $g \geq 4$, then it is canonically embedded into projective space by its canonical linear system.

$$\phi_{|\omega_C|} : C \rightarrow \mathbf{P}^{g-1} := \mathbf{PH}^0(C, \omega_C)^\vee.$$

Every canonical divisor of C is thus identified with a hyperplane section of the image $C_{\text{can}} := \phi(C)$.

If moreover C is trigonal, i.e. C has a base point free g_3^1 , then the canonical image of C is contained in a rational scroll S_C projectively equivalent to some

$$S_{mn} = \left\{ (z_0 : \dots : z_{g-1}) \mid \text{rk} \begin{pmatrix} z_0 & \dots & z_{n-1} & z_{n+1} & \dots & z_{n+m} \\ z_1 & \dots & z_n & z_{n+2} & \dots & z_{n+m+1} \end{pmatrix} < 2 \right\},$$

where $g = m + n + 2$ and $m \leq n$. The scroll S_C for the curve C is cut out by the quadrics containing C_{can} such that the divisors of the g_3^1 on C and the lines on S_C correspond bijectively.

The g_3^1 on C is unique except for the case of $g = 4$ and $n = m = 1$ when there are two g_3^1 on C and two corresponding rulings of S_C by lines but that case will not play a role in this article.

The difference $e := n - m$ is called the *Maroni invariant* and known to take all integral values subject to

$$0 \leq n - m \leq \left\lfloor \frac{g+2}{3} \right\rfloor, \quad n - m \equiv_2 g.$$

The Maroni invariant determines the smooth model of S_{mn} to be the rational ruled Segre-Hirzebruch surface

$$\mathbf{F}_{n-m} := \mathbf{P} \mathcal{O}(n) \oplus \mathcal{O}(m)$$

The zero section of $\mathcal{O}(n)$ provides a moveable curve σ_0 on $\mathbf{F}_{n-m}$, the zero section of $\mathcal{O}(m)$ gives a curve E , which has self-intersection number $-e \leq 0$ and which is rigid for $e > 0$. Together with the class of a fibre L of the ruling, either section class generates the Picard group of $\mathbf{F}_e$.

The surface S_{nm} is then the image for a map associated with the complete linear system

$$| \sigma_0 + mL | = | E + nL |$$

which defines an embedding except for the case $g = 4, e = 2$ when $m = 0$ and the exceptional section E of $\mathbf{F}_2$ is contracted. In that case $S_{2,0}$ is the quadric cone in $\mathbf{P}^3$. The image of E is in any case given by a parametrization

$$E : \{ (0 : \dots : 0 : a^m : a^{m-1}b : \dots : ab^{m-1} : b^m) \mid (a : b) \in \mathbf{P}^1 \},$$

a twisted rational curve for $m > 0$ and the point $(0 : \dots : 0 : 1)$ if $m = 0$.

We are now sufficiently prepared to prove the following result, that being part of a strictly trigonal pair, see Def.1.1, puts additional conditions on the trigonal curve.

Proposition 2.1. *Suppose (C, D) is a strictly trigonal pair in $\mathbf{PH}_g$, then*

- i) *there is $k > 0$ such that $g = 3k + 1$,*
- ii) *S_C is projectively equivalent to $S_{2k, k-1}$ (thus C has maximal Maroni invariant).*
- iii) *D is cut out by a hyperplane H with $H \cap S_{2k, k-1}$ projectively equivalent to*

$$\begin{cases} 2kL & \text{if } k = 1 \\ 2kL + E & \text{if } k > 1 \end{cases}$$

as a divisor, where L is a line of the scroll.

Proof: Since D is an integer multiple of a divisor $D' \in g_3^1(C)$, this is also true for their degrees, so 3 divides $2g - 2$ and thus divides $g - 1$. Hence $g = 3k + 1$ and $D = 2kD'$. Since no strictly trigonal curves for $g \leq 2$ exist, $k > 0$ and so part i) is proved.

A canonical image of a trigonal C lies on a rational scroll S_{nm} with $e = n - m \leq k + 1$ of the same parity as g and $k + 1$. Via $\mathbf{F}_e \rightarrow S_{nm}$ we consider C as a curve on $\mathbf{F}_e$, its g_3^1 as the restriction of the linear system $|L|$ and any hyperplane section of S_{nm} as an effective divisor H_S on $\mathbf{F}_e$. Then

$$C \in |3\sigma_0 + (2m - n + 2)L|, \quad H_S \in |\sigma_0 + mL| = |E + nL|.$$

Accordingly the hyperplane section D of C is given as $H_S \cap C$ on $\mathbf{F}_e$ and the defining property of strictly trigonal pairs poses a strong condition on H_S :

$$(1) \quad H_S \cap C = D = 2kD' = 2kL \cap C \quad \text{as divisors of degree } 2g - 2 = 6k$$

$$(2) \quad H_S \cap C = D = D' = L' \cap C \quad \text{as sets}$$

for a single fibre L' of the ruling. The class of H_S forces this divisor to have the following irreducible decomposition

$$H_S = \sigma_H + aL$$

with some smooth section σ_H to the ruling of $\mathbf{F}_e$ and $a \geq 0$. Indeed, σ_H must be disjoint from C which implies $a = n = 2k$ and thus the remaining claims. Otherwise $\sigma_H \cap C = \sigma_H \cap L$ were a single point contributing with multiplicity at least 3 to both $H_S \cap C$ and $L \cap C$ which is impossible since all three divisors are smooth and σ_H intersects L transversally. $\square$

Let us note, that on $S_{2k, k-1} \subset \mathbf{P}^{g-1}$ the hyperplane H_0 given by $z_0 = 0$ cuts out the effective divisor

$$H_S = H_0 \cap S_{2k, k-1} = E + 2kL_0$$

where L_0 is a line of the ruling and there is a map $\mathbb{C}^2 \rightarrow S_{2k, k-1}$ which is an isomorphism onto the complement of $E \cup L_0$:

$$\begin{aligned} \mathbb{C}^2 &\longrightarrow S_{2k, k-1} \\ x, y &\mapsto (1 : x : \dots : x^n : x : xy : \dots : x^m y) \end{aligned}$$

It induces an isomorphism

$$|3\sigma_0 + (2m - n + 2)L| \cong \mathbf{P}V^k$$

where $V^k \subset \mathbb{C}[x, y]$ is the vector space of polynomials $f = sy^3 + r(x)y^2 + p(x)y + q(x)$ spanned by monomials

$$x^i y^j \quad \text{of weighted degree } i + (k+1)j \leq 3k+3$$

that is, $s, r(x), p(x), q(x) \in \mathbb{C}[x]$ are polynomials of degrees at most $0, k+1, 2k+2, 3k+3$, respectively.

In the end we want to discard the $f \in V^k$ in a discriminant divisor corresponding to singular curves, hence we define

Definition 2.2. An element $f \in V^k$ is called *regular* if it has the following equivalent properties:

- i) $C_f \in |3\sigma_0 + (2m - n + 2)L|$ corresponding to $f = 0$ is a smooth curve on $\mathbf{F}_{k+1}$,
- ii) $s \neq 0$, $f = 0$ defines a smooth curve in $\mathbb{C}^2$ and C_f has no singularity on L_0 .

3. ORBIFOLD STRUCTURE

Let us now turn to the moduli spaces of strictly trigonal pairs. The orbifold structure for the open part $\mathbf{PH}_g^{\text{non-hyp}}$ of $\mathbf{PH}_g$ corresponding to non-hyperelliptic curves is given by the identification with a global quotient

$$\mathbf{PH}_g^{\text{non-hyp}} \cong \left\{ (C, H) \subset \mathbf{P}^{g-1} \mid \begin{array}{l} C \text{ a smooth canonical curve} \\ H \text{ a hypersurface} \end{array} \right\} / \mathbf{PGL}_g$$

Then $\mathbf{PH}_g^{\text{tri}}$ is non-empty only if $g = 3k + 1$ and it is closed in $\mathbf{PH}_g^{\text{non-hyp}}$ corresponding to pairs (C, H) such that

C meets the following equivalent conditions:

- i) it is trigonal of Maroni invariant $k + 1$.
- ii) it is trigonal with S_C projectively equivalent to $S_{2k, k-1}$.

H meets the following equivalent conditions:

- iii) it intersects C in a single divisor of its g_3^1 .
- iv) it intersects the ruled surface S_C in its section of negative self-intersection and a single divisor of its ruling.

The aim of the next steps is to successively reduced the dimension of the spaces involved in the global quotient. For the moment, we consider the whole strictly trigonal locus at once:

$$\mathbf{PH}_g^{\text{tri}} = \mathbf{PH}_g^{\text{tri}}(6k) \dot{\cup} \mathbf{PH}_g^{\text{tri}}(4k, 2k) \dot{\cup} \mathbf{PH}_g^{\text{tri}}(2k, 2k, 2k).$$

Proposition 3.1. *The moduli space $\mathbf{PH}_g^{\text{tri}}$, $g = 3k + 1$, with its orbifold structure can be represented as a global quotient*

$$\{C \mid C \text{ canonical curve on } S_{2k, k-1}\} / \text{Stab}_{\mathbf{PGL}}(S_{2k, k-1}, H_0)$$

Proof: The group of projective equivalences $\mathbf{PGL}_{3k+1}$ is transitive on all pairs (S, H) in $\mathbf{P}^{3k}$ such that S is projectively equivalent to $S_{2k, k-1}$ and such that H has property *iv)* above with respect to S .

In fact, the map $(C, H) \mapsto (S_C, H)$ is $\mathbf{PGL}_{3k+1}$ -equivariant and induces an isomorphism in the orbifold sense of the two quotients. $\square$

Here we see the first instance of how to pass from a quotient description by the action of a group G on a space to that of a smaller group on a smaller space:

Take the fibre of a G -equivariant map to a G -homogenous space and the stabilizer subgroup.

This kind of passing between quotient description is an elementary kind of Morita equivalence.

Proposition 3.2. *The moduli space $\mathbf{PH}_g^{\text{tri}}$, $g = 3k + 1$, with its orbifold structure can be represented as a global quotient*

$$\{f \in V^k \mid f = 0 \text{ non-singular on } \mathbf{F}_{k+1}\} / G \times \mathbb{C}^*$$

where G is the group of transformations induced by automorphisms of $\mathbf{F}_{k+1}$ preserving H_0 , i.e.

$$x, y \mapsto ax + a_0, by + b_{k+1}x^{k+1} + \dots + b_0$$

with $a, b \in \mathbb{C}^*$, $a_0, b_0, \dots, b_{k+1} \in \mathbb{C}$, and the factor $\mathbb{C}^*$ acts by scalar multiplication on V^k .

Proof: We use [StV] which describes orbits of curves on $\mathbf{F}_{k+1}$ by the action on a fixed dense open $\mathbb{C}^2 \subset \mathbf{F}_{k+1}$ by a group $\tilde{G}$ of birational transformations

$$(x, y) \mapsto \left(\frac{ax + a_0}{a'x + a'_0}, \frac{by + b_0 + b_1x + \cdots + b_{k+1}x^{k+1}}{(a'x + a'_0)^{k+1}} \right)$$

with $\begin{pmatrix} a & a_0 \\ a' & a'_0 \end{pmatrix}$ invertible, $b \in \mathbb{C}^*$ and $b_0, \dots, b_{k+1} \in \mathbb{C}$.

Their result [StV, Prop.1.2] implies that the projective equivalence classes of trigonal canonical curves of positive Maroni invariant $k+1$ correspond bijectively to $\tilde{G}$ -orbits of smooth curves on $\mathbf{F}_{k+1}$ given by some $f \in V^k$.

Now we identify the open $\mathbb{C}^2 \subset \mathbf{F}_{k+1}$ with the complement of $H_0 \cap S_{2k, k-1}$. Then the stabilizer of $(S_{2k, k-1}, H_0)$ is identified with the subgroup of $\tilde{G}$ corresponding to biregular transformations. They are obtained for $a' = 0$ which implies $a'_0 \neq 0$. We may normalize to $a'_0 = 1$ and get our claim. $\square$

Let us state again, that the locus $\mathbf{PH}_g^{tri}$, $g = 3k+1$, which we just have given as a global quotient, decomposes into the three loci,

$$\mathbf{PH}_g^{tri}(6k), \quad \mathbf{PH}_g^{tri}(4k, 2k), \quad \mathbf{PH}_g^{tri}(2k, 2k, 2k)$$

corresponding to curves which intersect L_0 in 1, 2 and 3 points, respectively, which is equal to the number of zeros of $sy^3 + r_{k+1}y^2 + p_{2k+2}y + q_{3k+3}$.

4. TRANSFORMATION STEPS

In this section we obtain a few preliminary results on transformations.

Lemma 4.1. *If $f \in V_k$ is regular, then for $r_0 \in \mathbb{C}$ the transformation*

$$x, y \mapsto x, y - \frac{1}{3s} (r(x) - r_0 x^{k+1})$$

maps f to f' with $r'(x) = r_0 x^{k+1}$.

If moreover $\deg_x(r(x) - r_0 x^{k+1}) < k$ then

$$p'_{2k+1} = p_{2k+1}, \quad p'_{2k+2} = p_{2k+2}, \quad q'_{3k+2} = q_{3k+2}, \quad q'_{3k+3} = q_{3k+3}.$$

Proof: The transformation is well defined, since regularity of the curve implies $s \neq 0$. It is then simply the Tschirnhaus transformation for the polynomial in y transforming the coefficients which are polynomials in x .

If the given degree bound holds, it is easy to check that the polynomials r, p, q are only affected in degrees less than $k, 2k+1, 3k+2$, respectively. $\square$

The following three results are proved by similar elementary arguments:

Lemma 4.2. *If $f \in V_k$ with $q_{3k+3} = p_{2k+2} = 0$ and $sr_{k+1}q_{3k+2} \neq 0$ then the transformation*

$$x, y \mapsto x, y - \frac{1}{2} \frac{p_{2k+1}}{r_{k+1}} x^k$$

maps f to f' with $q'_{3k+3} = p'_{2k+2} = 0$, $s'r'_{k+1}q'_{3k+2} \neq 0$ and

$$p'_{2k+1} = 0.$$

Lemma 4.3. *If $f \in V_k$ with $q_{3k+3} = p_{2k+2} = p_{2k+1} = 0$ and $sr_{k+1}q_{3k+2} \neq 0$ then the transformation*

$$x, y \mapsto x - \frac{1}{k} \frac{r_k}{r_{k+1}}, y$$

maps f to f' with $q'_{3k+3} = p'_{2k+2} = p'_{2k+1} = 0$, $s'r'_{k+1}q'_{3k+2} \neq 0$ and

$$r'_k = 0.$$

Lemma 4.4. *If $f \in V_k$ with $q_{3k+3} = p_{2k+2} = 0$ and $sq_{3k+2} \neq 0$ then the transformation*

$$x, y \mapsto x - \frac{1}{3k+2} \frac{q_{3k+1}}{q_{3k+2}}, y$$

maps f to f' with $q'_{3k+3} = p'_{2k+2} = 0$, $s'q'_{3k+2} \neq 0$ and

$$q'_{3k+1} = 0, \quad p'_{2k+1} = p_{2k+1}.$$

Also the next result is similar, though it needs some more care to show its validity.

Lemma 4.5. *If $f \in V_k$ with $r(x) = 0$ and $\{f = 0\}$ non-singular intersecting the line L_0 at infinity in two points, then the transformation*

$$x, y \mapsto x, y - \frac{3q_{3k+3}}{2p_{2k+2}} x^{k+1}$$

maps f to f' with $q'_{3k+3} = 0 = p'_{2k+2}$ and

$$r'(x) = r'_{k+1} x^{k+1}, \quad r'_{k+1} \neq 0$$

Proof: The curve intersects the line L_0 in the roots of

$$sy^3 + p_{2k+2}y + q_{3k+3}$$

thus $p_{2k+2} \neq 0$, for otherwise the number of roots is either one or three according to q_{3k+3} equal to zero or not. Hence the transformation is well defined.

The vanishing discriminant implies $27q_{3k+3}^2 s = -4p_{2k+2}^3$, thus $q_{3k+3} \neq 0$ and one can check that the polynomial

$$-4p_{2k+2}^3 y^3 + 27q_{3k+3}^2 p_{2k+2} y + 27q_{3k+3}^3$$

has a simple root at $3q_{3k+3}/p_{2k+2}$ and a double root at $-3q_{3k+3}/2p_{2k+2}$. Hence the given transformation yields a polynomial f' with $r'_{k+1} = 9sq_{3k+3}/2p_{2k+2} \neq 0$ and a double root at 0, so finally $p'_{2k+2} = 0 = q'_{3k+3}$. $\square$

We further need a non-vanishing result to be exploited later.

Lemma 4.6. *Suppose $f \in V_k$ restricted to the line L_0 at infinity has a double or triple zero at $y = 0$ and $\{f = 0\}$ is non singular on $\mathbf{F}_{k+1}$, then*

$$q_{3k+2} \neq 0$$

for the coefficient of x^{3k+2} in the monomial expansion of f .

Proof: At $y = 0$ on the line L_0 , the linear local expansion of f in $x_0 = 1/x$ and y is

$$f(x_0, y) = q_{3k+3} + p_{2k+2}y + q_{3k+2}x_0 + h.o.t.$$

The first two coefficients are zero, since $x = \infty, y = 0$ is a zero of f , and at least doubly so along $x = \infty$. In order to have non-vanishing gradient at $x = \infty, y = 0$ the last coefficient q_{3k+2} must be non-vanishing. $\square$

A final prerequisite concerns an alternative choice of basis for the $(\mathbb{C}^*)^3$ subgroup of transformations.

Lemma 4.7. *The actions of $(\mathbb{C}^*)^3$ on V^k given by*

$$x^i y^j \mapsto a^i b^j c x^i y^j, \quad x^i y^j \mapsto \lambda^{6k+4-2i-2kj-j} \mu^{3k+3-i-kj-j} \varrho^{9k+6-3i-3kj-2j} x^i y^j,$$

respectively, are equal under a group automorphism.

Proof: The group automorphism is given by

$$a \mapsto \lambda^{-2i} \mu^{-i} \varrho^{-3i} b \mapsto \lambda^{-2kj-j} \mu^{-kj-j} \varrho^{-3kj-2j} c \mapsto \lambda^{6k+4} \mu^{3k+3} \varrho^{9k+6}.$$

It is obviously a group homomorphism, it has an inverse, since the corresponding 3 by 3 matrix is invertible over the integers. Finally it translates the first action to the second one. $\square$

Proposition 4.8. *If $f \in V^k$ and $\{f = 0\}$ non-singular intersecting the line L_0 at infinity in one point, then in the G -orbit of f intersects V_6 in the orbit of some*

$$f' \in V^k \quad \text{with} \quad s = q_{3k+2} = 1, r_0 = \cdots = r_{k+1} = p_{2k+2} = q_{3k+1} = q_{3k+3} = 0$$

by the residual action of $\mathbb{C}^*$ via

$$p_i \mapsto t^{6k+4-3i} p_i, \quad q_i \mapsto t^{9k+6-3i} q_i$$

Proof: Since $\{f = 0\}$ is non-singular, by Lemma 4.1 with $r_0 = 0$ any f of the claim transforms to a polynomial f' with $r'(x) = 0$. By assumption there is only one zero on L_0 for f , thus $y = 0$ is a triple zero on L_0 for f' , $p'_{2k+2} = 0$ and $q'_{3k+3} = 0$. Lemma 4.6 then implies $q'_{3k+2} \neq 0$. By Lemma 4.4 we may then additionally impose f' to have $q'_{3k+1} = 0$.

The stabilizer group of the locus of such f' is the abelian group $(\mathbb{C}^*)^3$ of Lemma 4.7. The action by λ, μ, ϱ on the coefficients of f' is

$$s \mapsto \lambda s, \quad p_i \mapsto \lambda^{4k+3-2i} \mu^{2k+2-i} \varrho^{6k+4-3i} p_i, \quad q_i \mapsto \lambda^{6k+4-2i} \mu^{3k+3-i} \varrho^{9k+6-3i} q_i$$

With $\lambda = 1/s$ and $\mu = 1/q_{3k+2}$ we arrive at f' in the claimed set with the residual action of $t = \varrho$ via $p_i \mapsto \varrho^{6k+4-3i} p_i, q_i \mapsto \varrho^{9k+6-3i} q_i$ as given in the claim. $\square$

Proposition 4.9. *If $f \in V^k$ and $\{f = 0\}$ non-singular intersecting the line L_0 at infinity in two points, then in the G -orbit of f there is a unique $f' \in V^k$ with*

$$s' = r'_{k+1} = q'_{3k+2} = 1, \quad r'_0 = \cdots = r'_k = 0, \quad p'_{2k+1} = p'_{2k+2} = q'_{3k+3} = 0.$$

Proof: Similar to the last proof, a succession of transformations according to Lemma 4.1 with $r_0 = 0$, Lemma 4.5, Lemma 4.2, Lemma 4.3 and Lemma 4.1 again but with $r_0 = r_{k+1}$ is possible thanks to Lemma 4.6. It yields f' in the orbit of f with

$$s' r'_{k+1} q'_{3k+2} \neq 0 \quad \text{and} \quad r_0 = \cdots = r_k = 0, \quad p_{2k+1} = p_{2k+2} = 0.$$

The stabilizer group of the locus of such f' is again the abelian group $(\mathbb{C}^*)^3$ of Lemma 4.7. The action by λ, μ, ϱ on non-vanishing coefficients of f' is

$$s \mapsto \lambda s, \quad r_{k+1} \mapsto \varrho^{-1} r_{k+1}, \quad q_{3k+2} \mapsto \mu q_{3k+2}$$

With $\lambda = 1/s$, $\mu = 1/q_{3k+2}$ and $\varrho = r_{k+1}$ we arrive at f' in the claimed set with no residual action. $\square$

Proposition 4.10. *If $f \in V^k$ and $\{f = 0\}$ non-singular intersecting the line L_0 at infinity in three points, then the G -orbit of f intersects $V_{2,2,2}$ in the orbit of some*

$$f' \in V^k \quad \text{with} \quad s = 1, r_0 = \cdots = r_{k+1} = 0$$

by the residual action of $(a, a_0, t) \in (\mathbb{C}^* \rtimes \mathbb{C}) \times \mathbb{C}^*$ via

$$p(x) \mapsto t^2 p(ax + a_0), \quad q(x) \mapsto t^3 q(ax + a_0).$$

Proof: By a transformation according to Lemma 4.1 with $r_0 = 0$ the polynomial f' is obtained. The stabilizing subgroup can be identified with transformations induced by

$$x \mapsto ax + a_0$$

and the action of a scalars $\lambda, t \in \mathbb{C}^*$ via

$$s \mapsto \lambda s, \quad p_i \mapsto \lambda t^2 p_i, \quad q_i \mapsto \lambda t^3 q_i.$$

With $\lambda = 1/s$ we arrive at the form of f' of the claim and the residual action there. $\square$

Let us note, that the three propositions make statements about the slices of V^k given by the affine subspaces

$$(1) \quad \begin{aligned} V_1^k &= \{f \in V^k \mid s = q_{3k+2} = 1, r = 0, p_{2k+2} = 0, q_{3k+3} = 0\} \\ V_2^k &= \{f \in V^k \mid s = r_{k+1} = q_{3k+2} = 1, r_0, \dots, r_k, p_{2k+1}, p_{2k+2}, q_{3k+3} = 0\} \\ V_3^k &= \{f \in V^k \mid s = 1, r_0 = \dots = r_{k+1} = 0\} \end{aligned}$$

5. PROOF OF THE MAIN THEOREM

Next we apply the results of the previous section to a proof of our main theorem.

Proof of Theorem 1.2: Let us start to address the first claim about the global quotient for $\mathbf{PH}_{3k+1}^{tri}(6k)$.

At the end of section 2 we established the identification with the global quotient

$$\{f \in V^k \text{ regular} \mid C_f \text{ on } \mathbf{F}_{k+1} \text{ intersects } L_0 \text{ in one point}\} / G \times \mathbb{C}^*.$$

In particular, the hypotheses of Prop.4.8 are met by all the f involved.

The transformations used in the proof of Prop.4.8 can be performed simultaneously on all elements f , since they depend algebraically on the coefficients of f .

Hence we get the global quotient description of $\mathbf{PH}_{3k+1}^{tri}(6k)$ by

$$\{f \in V^k \text{ regular} \mid s = q_{3k+2} = 1, r(x) = 0, p_{2k+2} = q_{3k+1} = q_{3k+3} = 0\} / \mathbb{C}^*$$

Define the linear subsystem $\mathcal{L}_1^k$ of $|3\sigma_0|$ by

$$\mathcal{L}_1^k = \{[f] \in \mathbf{PV}^k \mid s = q_{3k+2}, r(x) = 0, p_{2k+2} = q_{3k+1} = q_{3k+3} = 0\}.$$

Then the affine subspace

$$\{f \in V^k \mid s = q_{3k+2} = 1, r(x) = 0, p_{2k+2} = q_{3k+1} = q_{3k+3} = 0\}$$

maps bijectively to the complement of the hyperplane $s = 0$ in $\mathcal{L}_1^k$. Since this hyperplane is contained in the discriminant $\mathcal{D}_1$ of $\mathcal{L}_1^k$, that map induces an isomorphism

$$\{f \in V^k \text{ regular} \mid s = q_{3k+2} = 1, r(x) = 0, p_{2k+2} = q_{3k+1} = q_{3k+3} = 0\} \cong \mathcal{L}_1^k - \mathcal{D}_1.$$

We transfer the $\mathbb{C}^*$ -action on the left hand side through this isomorphism to the right hand side to get the claimed global quotient for $\mathbf{PH}_{3k+1}^{tri}(6k)$:

$$(\mathcal{L}_1^k - \mathcal{D}) / \mathbb{C}^*.$$

The analogous strategy yields $\mathbf{PH}_{3k+1}^{tri}(4k, 2k) \cong (\mathcal{L}_2^k - \mathcal{D})$ in the second case with

$$\mathcal{L}_2^k = \{[f] \in \mathbf{PV}^k \mid s = q_{3k+2} = r_{k+1}, r_0 = 0, \dots, r_k = 0, p_{2k+1} = p_{2k+2} = q_{3k+3} = 0\}$$

and with $\mathcal{L}_3^k = \{[f] \in \mathbf{PV}^k \mid r(x) = 0\}$ yields

$$\mathbf{PH}_{3k+1}^{tri} \cong (\mathcal{L}_3^k - \mathcal{D}_3) / \mathbb{C} \rtimes (\mathbb{C}^*)^2.$$

To get the third case, it suffices then to additionally discard the divisor $\mathcal{D}'$ of pairs belonging to $\mathbf{PH}_{3k+1}^{tri}(4k, 2k) \cup \mathbf{PH}_{3k+1}^{tri}(6k)$. $\square$

6. ORBIFOLD FUNDAMENTAL GROUPS AND GENUS 4

In this final section we address the proofs of the corollaries from the introduction concerning the orbifold fundamental group of trigonal loci and the special situation of genus 4 curves. We present here the two main ingredients before going into the proof.

First, for any plane curve singularity $f = f(x, y)$ with an isolated singular point at the origin the local fundamental group of the complement $U_f - \mathcal{D}$ of the discriminant in any versal unfolding is well-defined up to isomorphism and thus we define (see also [L18]):

Definition 6.1. Given a plane curve singularity $f = f(x, y)$ with an isolated singular point at the origin, then

$$\pi^K(f) := \pi_1^{loc}(U_f - \mathcal{D})$$

is called the *discriminant knot group* of f .

Second, the orbifold fundamental group for an orbifold given as a global quotient of a manifold X by a topological group G is by definition the topological fundamental group of the quotient

$$(EG \times X)/G$$

where G acts freely on the contractible space EG and diagonally on the product.

Then there is a long exact homotopy sequence associated to the diagonal free action

$$\pi_2((EG \times X)/G) \rightarrow \pi_1 G \rightarrow \pi_1(EG \times X) \rightarrow \pi_1((EG \times X)/G) \rightarrow \pi_0 G$$

in which we may identify $\pi_1(EG \times X) = \pi_1 X$ and $\pi_1((EG \times X)/G) = \pi_1^{orb}(X/G)$.

Proof of Corollary 1.3: The moduli space $\mathbf{PH}_{3k+1}^{tri}(6k)$ was shown to be representable as a quotient of the complement $U_1^k := V_1^k - \mathcal{D}$ of the discriminant in V_1^k by the group $\mathbb{C}^*$. Hence we get from the long exact homotopy sequence above

$$\mathbb{Z} \rightarrow \pi_1 U_1^k \rightarrow \pi_1^{orb}(U_1^k/\mathbb{C}^*) \rightarrow 1$$

The map from $\mathbb{Z} = \pi_1 \mathbb{C}^*$ is injective, since so is the map to $H_1 U_1^k$, which is essentially multiplication by the non-torsion homology class represented by any free orbit of $S^1 \subset \mathbb{C}^*$.

The affine space V_1^k is an unfolding of the singularity given by the polynomial $y^3 + x^{3k+2}$, namely the unfolding over the vector space spanned by the monomials $x^i y^j$ of weighted degree $3i + (3k+2)j < 9k+6$. By the arguments given in [L22, section 3.2] the fundamental group of the discriminant complement U_1^k is naturally isomorphic to that of the discriminant complement of the universal unfolding of $y^3 + x^{3k+2}$. Thus by definition, it is the discriminant knot group $\pi^K(y^3 + x^{3k+2})$.

Hence we get the short exact sequence of the claim. $\square$

Remark 6.2. A finite presentation for $\pi^K(y^3 + x^{3k+2})$ was given in [L10] and with the argument in [L22] it can also be given with generators t_i $i = 1, \dots, 6k+2$ and relations

- i) of braid type : $t_i t_j t_i = t_j t_i t_j$ for all i, j with $|i - j| \leq 2$
- ii) of commutation type : $t_i t_j = t_j t_i$ for all i, j with $|i - j| > 2$
- iii) of triangle type: $t_i t_{i+1} t_{i+2} t_i = t_{i+1} t_{i+2} t_i t_{i+1}$ for all $i < 6k+1$

The central element which has to be factored can be expressed as

$$(t_1 \dots t_{6k+2})^{9k+6}$$

Proof of Corollary 1.4: The moduli space $\mathbf{PH}_{3k+1}^{tri}(2k, 2k, 2k)$ was shown to be representable as a quotient of the complement $U_3^k := V_3^k - \mathcal{D} - \mathcal{D}'$ of $\mathcal{D} \cup \mathcal{D}'$ in V_3^k by the group $\mathbb{C} \rtimes (\mathbb{C}^*)^2$. Hence we get from the long exact homotopy sequence above

$$\mathbb{Z}^2 \rightarrow \pi_1 U_3^k \rightarrow \pi_1^{orb} U_3^k / \mathbb{C}^* \rightarrow 1$$

The map from $\mathbb{Z}^2 = \pi_1(\mathbb{C}^*)^2$ is injective, since so is the map to $H_1 U_3^k$: Indeed, linking with the divisors $\mathcal{D}$ and $\mathcal{D}'$ gives a map $H_1 U_3^k \rightarrow \mathbb{Z}^2$ such that the composition

$$\mathbb{Z}^2 = \pi_1(\mathbb{C}^*)^2 \rightarrow H_1 U_3^k \rightarrow \mathbb{Z}^2$$

is injective.

Moreover, we need to identify

$$\pi_1 U_3^k \cong \pi^K(y^3 + x^{3k+3}) \rtimes \pi^K(y^3 + x^2)$$

This is possible since the map

$$\begin{aligned} U_3^k &\rightarrow \{Y^3 + pY + q \mid 4p^3 \neq -27q^2\} \\ f &\mapsto Y^3 + p_{2k+2}Y + q_{3k+3} \end{aligned}$$

is close enough to a fibre bundle. In fact, by the results in [L18], each point in the base has a topological disc neighbourhood, such that the preimage has fundamental group $\pi^K(y^3 + x^{3k+3})$, the discriminant knot group of the fibre over $Y^3 + 1$.

The base space is a versal unfolding of the polynomial Y^3 which defines a simple singularity. In particular, it has fundamental group the discriminant knot group $\pi^K(y^3 + x^2)$. Since its universal covering space is contractible the second homotopy group is trivial.

By the above, there is a short exact sequence coming from a longer exact homotopy sequence:

$$1 \rightarrow \pi^K(y^3 + x^{3k+3}) \rightarrow \pi_1 U_3^k \rightarrow \pi^K(y^3 + x^2) \rightarrow 1$$

There exists a section for the topological spaces given by the map

$$Y^3 + pY + q \mapsto y^3 + px^{2k+2}y + qx^{3k+3} + 1$$

accordingly the group $\pi_1 U_3^k$ above in the middle is a semi-direct product of the other two as claimed. $\square$

Remark 6.3. A finite presentation for the two factors is again known and is in fact of the form as given in Rem.6.2, but how the quotient acts on the normal factor needs a more careful analysis which is deferred to a future investigation.

For the proof of the final corollary 1.5 we note that $\mathbb{C}^8 - \mathcal{D}_{E_8}$ is $\mathbb{C}^*$ -equivariantly equal to $V_1^1 - \mathcal{D}$ since

$$V_1^1 = y^3 + x^5 + \text{span}\{1, x, x^2, x^3, y, yx, yx^2, yx^3\}$$

In particular, both have fundamental group $\pi^K(y^3 + x^5) = \text{Ar}(E_8)$ and the generator of $\pi_1(\mathbb{C}^*)$ maps to the generator of the center. By the proof of the theorem $\mathbf{PH}_4^{tri}(6) \cong (V_1^1 - \mathcal{D})/\mathbb{C}^*$, so the corollary follows as soon as the following lemma is proven.

Lemma 6.4.

$$\mathcal{H}_4^{ev}(6) = \mathcal{H}_4^{tri}(6)$$

Proof: Let us first say something about even and odd which was already mentioned in the introduction: For pairs (C, D) in the strictly trigonal loci the canonical divisors $D = 2kL|_C$ come with a theta-characteristic $kL|_C$ of parity equal to that of

$$h^0(kL|_C) = h^0(\mathcal{O}_C(kL)) = k+1 \cong_2 g \pmod{2}.$$

Indeed $h^0(kL) = k+1 + h^1(\mathcal{O}_{\mathbf{F}}(kL - C))$ from the long exact cohomology sequence of

$$0 \rightarrow \mathcal{O}_{\mathbf{F}}(kL - C) \rightarrow \mathcal{O}_{\mathbf{F}}(kL) \rightarrow \mathcal{O}_C(kL) \rightarrow 0$$

and $h^1(\mathcal{O}_{\mathbf{F}}(kL - C))$ vanishes by the long exact cohomology sequence of

$$0 \rightarrow \mathcal{O}_{\mathbf{F}}(-B) \rightarrow \mathcal{O}_{\mathbf{F}} \rightarrow \mathcal{O}_B \rightarrow 0$$

where $B = 2\sigma_0 + E + L \in |C - kL|$ is effective and connected.

In the case at hand $g = 4$ therefore every pair C, D on the right determines an even theta-characteristic and thus also belongs to the left hand side. Conversely every pair C, D in the set on the left has a curve C with an effective theta characteristic. Hence C belongs to the thetanull divisor of genus 4 curves. C is not hyperelliptic, since every hyperelliptic curve with a 6-uple canonical divisor is in the disjoint stratum $\mathcal{H}_4^{hyp}(6)$. Therefore C maps canonically to a trigonal curve on the quadric cone and is a trigonal curve with a unique g_3^1 .

Consider now the effective canonical divisor D : By assumption it is the double $D = 2D'$ of an effective divisor D' which is an even theta-characteristic. Therefore D' has degree 3 and being effective and even implies that D' belongs to the unique g_3^1 of C .

We may thus conclude, that (C, D) belongs to the right hand side as well. $\square$

Similar claims are not true for larger k simply for dimension reasons: The dimension of $\mathcal{H}(2g - 2)$ is $2g - 1 = 6k + 1$ and so is the dimension of its image in the moduli space. On the other hand, the dimension of the locus of trigonal curves with maximal Maroni invariant in genus $g = 3k + 1$ is $5k + 3$ and those with a total ramification point form a divisor in it of dimension $5k + 2$.

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