Journal *of* Singularities
Volume 30 (2026), 216-241

Special volume in honor of the life and mathematics of Wolfgang Ebeling
DOI: 10.5427/jsing.2026.30i

A LOCAL ZARISKI-VAN KAMPEN THEOREM

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Abstract. The term"Zariski-van Kampen theorem" refers to results which allow to compute homology (or homotopy) groups of smooth quasi-projective varieties, comparing with a general hyperplane section. Here we concentrate on the local analogue and prove a corresponding theorem not only for homology but also for homotopy groups. This paves at the same time the way to complete the treatment of the global case.

0. Introduction

A classical tool for studying the topology of projective varieties is the Lefschetz theorem on hyperplane sections: Suppose that $X \subset \mathbb{P}^N(\mathbb{C})$ is a smooth subvariety of pure dimension n and L a hyperplane in $\mathbb{P}^N$ (which is general enough but here this is not necessary). Then $H_k(X, X \cap L; \mathbb{Z}) = 0$ for $k < n$, i.e. $H_k(X \cap L; \mathbb{Z}) \to H_k(X; \mathbb{Z})$ is bijective for $k < n-1$ and surjective for $k = n-1$. See [L].

Later R. Bott [B] as well as Andreotti and Frankel [AF1] gave a proof using Morse theory, which made it possible to pass from homology to homotopy groups: $(X, X \cap L)$ is $(n-1)$-connected.

A corresponding local theorem has been proved by the author ([Hm], Satz 1.2).

If L is generic the (global) Lefschetz theorem above can be extended to smooth quasi-projective varieties (Lefschetz-Zariski theorem), as shown by Lê D. T. and the author [HL2]. The name of Zariski comes from the fact that O. Zariski [Z] studied already the special case of a quasi-projective variety of the form $X = \mathbb{P}_n \setminus H$, H being a hypersurface. A complete proof in this case was given in [HL1]. Also, the authors showed there that the theorem in question can be deduced from a corresponding local analogue. They realized that we get an even better result in the local case when replacing the hyperplane through the origin by a nearby hyperplane (strong local Lefschetz-Zariski theorem).

The transition to the more general case of a smooth quasi-projective variety mentioned above as well as the local analogue was accomplished in [HL2], [HL3].

Finally, the smoothness condition was dropped in [HL4], the dimension being replaced by the rectified homotopical depth.

Goresky and MacPherson extended classical Morse theory to singular spaces in order to have a systematic tool, in particular for proving Lefschetz theorems [GM].

Turning back to the original Lefschetz theorem, note that Lefschetz also studied the kernel of the epimorphism $H_{n-1}(X \cap L; \mathbb{Z}) \to H_{n-1}(X; \mathbb{Z})$, and obtained the so-called second Lefschetz theorem, see [L], [W], [AF2]. Similar questions can be asked for the other situations above, too. The case of the fundamental group of the complement of a plane projective curve was treated by van Kampen [vK], so we speak of Zariski-van Kampen theorems instead of Lefschetz-Zariski theorems in the case of quasi-projective varieties or its local analogue.

In this paper we concentrate on a strong local Zariski-van Kampen theorem which is analogous to the strong local Lefschetz-Zariski theorem mentioned before, which will be recalled in section

2020 *Mathematics Subject Classification.* 32S05, 32S50.
Key words and phrases. Zariski-van Kampen, variation, open book decomposition, nearby fibre.

1. First we will work in the framework of homology, the global case being already accomplished in [Ch2].

In the latter case it is traditional to start from a variation mapping, see [Ch2], we prefer to start with a straightforward procedure which appears to be technically simpler.

But after all we will establish the connection with the very geometric concept of variation.

To be more precise:

Let U be an open neighbourhood of 0 in $\mathbb{C}^N$, Z and $\bar{X}$ closed complex analytic subsets of U, $0 \in Z \subset \bar{X}$, $X = \bar{X} \setminus Z$ smooth and purely n-dimensional. Choose a Whitney stratification of U which is compatible with $\bar{X}$ and Z. Let $f : U \to \mathbb{C}$ be a holomorphic function, $f(0) = 0$. Suppose $0 < \epsilon \ll 1$, $f^{-1}(0) \setminus \{0\}$ smooth and transverse to $\partial \mathrm{B}_\epsilon$ with the induced stratification, where $\mathrm{B}_\epsilon := \{z \in \mathbb{C}^N \mid \|z\| \leq \epsilon\}$. Choose $0 < \alpha \ll \epsilon$. Put $X_\epsilon := X \cap \mathrm{B}_\epsilon$, $F_t := f^{-1}(t) \cap X_\epsilon$ and $F := F_\alpha$. Note that F is a manifold with boundary called ∂F.

Now (X_ϵ, F) is $(n-1)$-connected, by the strong local Lefschetz theorem (see Cor. 1.6a below), so $H_k(F) \overset{\simeq}{\to} H_k(X_\epsilon), k \leq n-2$.

Notice that $f : X_\epsilon \cap \{|f| = \alpha\} \to \{t \in \mathbf{C} \mid |t| = \alpha\}$ defines a fibre bundle. Let $h : F \to F$ be a monodromy, i.e. $h = \phi_1$, where $\phi_t, t \in [0,1]$, is a family of homeomorphisms $F \to F_{e^{2\pi i t}\alpha}$ with $\phi_0 = \mathrm{id}$. Moreover, suppose that it is a geometric monodromy, i.e. ϕ_t is compatible with a trivialization of $f|\partial X_\epsilon \cap \{|f| \leq \alpha\} \to \{|t| \leq \alpha\}$ - in particular, $h|\partial F = \mathrm{id}$.

Let $\mathrm{var}_{n-1}(f) : H_{n-1}(F, \partial F) \to H_{n-1}(F)$ be the corresponding variation, defined by $\mathrm{var}_{n-1}(f)([c]) := [h \circ c - c]$. Then we will prove the following homological strong local Zariski-van Kampen theorem:

Theorem 0.1: $H_{n-1}(F) / \operatorname{Im} \mathrm{var}_{n-1}(f) \simeq H_{n-1}(X_\epsilon)$.

In special cases (e.g. $Z = \{0\}$) this is well-known.

More important is that we will prove a corresponding homotopical version if $n \geq 2$: As in the case of homology we do not start from the notion of a variation which is geometrically defined but use it finally.

So let $\cdot \in \partial F$ be a base point, then let $\mathrm{var}_{n-1}(f) : \pi_{n-1}(F, \partial F, \cdot) \to \pi_{n-1}(F, \cdot)$ be defined by $\mathrm{var}_{n-1}(f)([\beta]) := [\beta^- \perp h \circ \beta]$, see beginning of section 4. Then we will prove the following theorem, inspired by a conjecture of D. Chéniot and C. Eyral in the global case [CE]:

Theorem 0.2: Let $\cdot \in \partial F$.

a) $\pi_{n-1}(F, \cdot) / \operatorname{Im} \mathrm{var}_{n-1}(f) \simeq \pi_{n-1}(X_\epsilon, \cdot)$ if $n \geq 3$.

b) $\pi_1(F, \cdot)/U \simeq \pi_1(X_\epsilon, \cdot)$ if $n = 2$, U being the subgroup generated by $\operatorname{Im} \mathrm{var}_1(f)$.

The proofs of these theorems will be given in Section 6, after Corollary 6.3 resp. Theorem 6.6.

This paper is dedicated to the memory of my friend Wolfgang Ebeling. We had the same thesis advisor, Egbert Brieskorn, which joined us together, we met often at conferences on singularity theory and had fruitful conversations. I have appreciated his friendly behaviour very much!

Furthermore I would like to thank Denis Chéniot very much for his critical remarks.

1. On the theorem of Lefschetz-Zariski

Here we recall the theorem of Lefschetz-Zariski and prove a complement which is due to Chéniot [Ch1] in the global case. We start from a semi-local statement which allows to treat the local situation, too, and will turn out to be useful elsewhere.

We do not restrict to (sc. integral) homology but include homotopy groups as well, where we encounter difficulties in context with excision. However when we restrict to the question whether pairs of spaces are n-connected the investigation is easier:

Lemma 1.1: Let (A, B) be a pair of CW complexes, and let A_1, A_2 be subcomplexes of some CW complex.
a) (A, B) is n-connected if and only if A has the homotopy type of a space obtained from B by attaching cells of dimension $> n$.
b) If $(A_1, A_1 \cap A_2)$ is n-connected the pair $(A_1 \cup A_2, A_2)$ is n-connected.
c) If (A, B) is n-connected the pair $(I^m, \partial I^m) \times (A, B)$ is $(m+n)$-connected, where $I = [0, 1]$.

Proof: a) see [H] Prop. 4.16.

b) This follows using the Blakers-Massey homotopy excision theorem. But it is simpler to use a).

c) Use a). In fact it is sufficient to look at the case $m = 1$. □

Now let U be open in $\mathbb{C}^N$, $Z \subset \bar{X}$ closed complex-analytic subsets of U, $X = \bar{X} \setminus Z$ smooth of pure dimension n, $r > 0$ general, $\mathrm{B}_r := \{z \in \mathbb{C}^N \mid \|z\| \leq r\} \subset U$, $X_r := X \cap \mathrm{B}_r$.

Theorem 1.2: a) $(X_r, \partial X_r)$ is $(n-1)$-connected.
b) If $N = n, \bar{X} = U, Z$ of codimension c, the pair $(\mathrm{B}_r \setminus Z, \partial \mathrm{B}_r \setminus Z)$ is $(n+c-2)$-connected.

Proof: Induction on n. Induction step: Choose a Whitney stratification of $(\bar{X}, Z)$ and apply stratified Morse theory, see [GM], to $z \mapsto \phi(z) := -\|z\|^2$. (More precisely: If ϕ happens not to be a Morse function replace ϕ by a suitable small perturbation.)

a) Let z_0 be a critical point. If $z_0 \in X$ we apply usual Morse theory: the index is $\geq n$, so the local Morse data is $(n-1)$-connected. If $z_0 \in Z$ this point is contained in a stratum S of dimension i. Then the tangential Morse data is $(i-1)$-connected. Moreover, let j be the index of the critical point z_0 of $\phi|S$, so $j \geq i$, we may take $I^{2i-j} \times (I^j, \partial I^j)$ as tangential Morse data. The complex link is of the form

$$\mathcal{L} = L \cap X \cap \{\|z - z_0\| \leq \rho\},$$

where L is a general affine subspace of $\mathbb{C}^N$ of codimension $i+1$ passing nearby z_0, and $0 < \rho \ll r$. So $(\mathcal{L}, \partial\mathcal{L})$ is $(n-i-2)$-connected, by induction hypothesis. Hence the normal Morse data is $(n-i-1)$-connected, because it is homotopy equivalent to $(I, \partial I) \times (\mathcal{L}, \partial\mathcal{L})$ ([GM] II 2.6, Cor. 1), using Lemma 1.1c). So the Morse data at z_0 is $(n-1)$-connected, using Lemma 1.1c) again. This implies our statement.

b) Again let z_0 be a critical point. If $z_0 \notin Z$ we must have $z_0 = 0$ and the index is $2n$, hence the local Morse data is $(2n-1)$-connected, so $(n+c-2)$-connected. If $z_0 \in Z$, in a stratum S of dimension i, $j \geq i$ being the index of the critical point z_0 of $\phi|S$, the tangential Morse datum is again $(i-1)$-connected: we may take $I^{2i-j} \times (I^j, \partial I^j)$. The complex link is of the form

$$\mathcal{L} = L \cap \{\|z - z_0\| \leq \rho\} \setminus Z,$$

where L is a general affine subspace of $\mathbb{C}^N$ of codimension $i+1$ passing nearby z_0, and $0 < \rho \ll r$. So $(\mathcal{L}, \partial\mathcal{L})$ is $(n+c-i-3)$-connected, by induction hypothesis.

Hence the normal Morse data is $(n+c-i-2)$-connected, which implies, by Lemma 1.1c), that the Morse data at z_0 is $(n+c-2)$-connected, which was to be proved. □

Corollary 1.3: Let $Z \subset \bar{X}$ be complex analytic subsets of $\mathbb{P}_N, X = \bar{X} \setminus Z$ smooth of pure dimension n, L a general hyperplane in $\mathbb{P}_N$.
a) Lefschetz-Zariski theorem: $(X, X \cap L)$ is $(n-1)$-connected.
b) Chéniot's theorem ([Ch1]): If $N = n, \bar{X} = \mathbb{P}_n$, the pair $(\mathbb{P}_n \setminus Z, L \setminus Z)$ is $(n+c-2)$-connected, where Z is of codimension c.

Proof: a) Identify $\mathbb{P}_N \setminus L$ with $\mathbb{C}^N$ and let $r \gg 0$. The preceding theorem implies that $(X, X\setminus \mathrm{B}_r^\circ)$ is $(n-1)$-connected. Now $X \cap L$ is a deformation retract of $X \setminus \mathrm{B}_r^\circ$, because L is general, so we get the desired conclusion.

b) This is a straightforward analogue. □

Finally return to the data for Theorem 1.2. Let $\ell : \mathbb{C}^N \to \mathbb{C}$ be linear, $H_t = \{\ell = t\}$. Assume that $H_0, \bar{X}$ and $\partial \mathrm{B}_r$ intersect transversally in the stratified sense. Let $0 < \alpha \ll r$.

First:

Lemma 1.4: Suppose that $(X_r \cap H_\alpha, \partial X_r \cap H_\alpha)$ is k-connected. Then

$$(\partial(\{|\ell| \leq \alpha\} \cap X_r), H_\alpha \cap X_r)$$

is $(k+1)$-connected.

Proof: By Lemma 1.1c), $(I, \partial I) \times (X_r \cap H_\alpha, \partial X_r \cap H_\alpha)$ is $(k+1)$-connected. This implies that the homeomorphic pair

$$(X_r \cap \{|\ell| = \alpha, \operatorname{Im} \ell \geq 0\}, \partial(X_r \cap \{|\ell| = \alpha, \operatorname{Im} \ell \geq 0\}))$$

is $(k+1)$-connected. Then

$$(\partial(X_r \cap \{|\ell| \leq \alpha\}), (X_r \cap \{|\ell| = \alpha, \operatorname{Im} \ell \leq 0\}) \cup (\partial X_r \cap \{|\ell| \leq \alpha\}))$$

is $(k+1)$-connected, too, by Lemma 1.1b). And $X_r \cap H_\alpha$ is a deformation retract of

$$(X_r \cap \{|\ell| = \alpha, \operatorname{Im} \ell \leq 0\}) \cup (\partial X_r \cap \{|\ell| \leq \alpha\})).$$

□

Theorem 1.5: a) $(\partial(\{|\ell| \leq \alpha\} \cap X_r), H_\alpha \cap X_r)$ is $(n-1)$-connected.
b) If $N = n$, $\bar{X} = U$, and $Z \subset U$ has codimension c,

$$(\partial(\{|\ell| \leq \alpha\} \cap B_r) \setminus Z, H_\alpha \cap B_r \setminus Z)$$

is $(n+c-2)$-connected.

Proof: This follows from Theorem 1.2 and Lemma 1.4. □

Corollary 1.6: Suppose that $0 < \alpha \ll \epsilon \ll 1$ and that H_0 is transversal to $\bar{X} \setminus \{0\}$ near 0 in the stratified sense, $0 \in Z$.
a) (Strong local Lefschetz-Zariski theorem, [HL3] Main Theorem II.1.4:) $(X_\epsilon, H_\alpha \cap X_\epsilon)$ is $(n-1)$-connected.
b) If $N = n$, $\bar{X} = U$, and $Z \subset U$ has codimension c, the pair
$(B_\epsilon \setminus Z, H_\alpha \cap B_\epsilon \setminus Z)$ is $(n+c-2)$-connected.

In fact, $\partial(\{|\ell| \leq \alpha\} \cap X_\epsilon)$ is a deformation retract of X_ϵ.

Note that a) is a special case of the strong local Lefschetz-Zariski theorem proved in [HL4]. The essential point is again that

$$(X_\epsilon \cap H_\alpha, \partial(X_\epsilon \cap H_\alpha))$$

is $(n-2)$-connected.

2. Auxiliary lemmas

Lemma 2.1: Let (A, B, C) be a triple of CW complexes. Suppose that C is a deformation retract of A.
a) For $\cdot \in C, k \geq 1$, the mapping $\partial : \pi_{k+1}(A, B, \cdot) \to \pi_k(B, C, \cdot)$ is bijective.
More precisely, it is the composition of the bijective mappings

$$\pi_{k+1}(A, B, \star) \to \ker(\pi_k(B, \star) \to \pi_k(A, \star)) \to \pi_k(B, C, \star)$$

b) If (B, C) is k-connected, $k \geq 0$, the pair (A, B) is $(k+1)$-connected.

Proof: a) $\pi_k(A, C, \cdot) = 0$ for all $k \geq 1$. For $k \geq 2$ we get our statement by the exact homotopy sequence of the triple (A, B, C), for $k = 1$ we get at least surjectivity.
In order prove injectivity in the case $k = 1$, note that $\partial = \phi \circ \partial'$ with $\partial' : \pi_{k+1}(A, B, \cdot) \to \pi_k(B, \cdot)$, $\phi : \pi_k(B, \cdot) \to \pi_k(B, C, \cdot)$.
Injectivity for $k = 1$: Suppose that $c_1, c_2 \in \pi_2(A, B, \cdot), \partial(c_1) = \partial(c_2)$. We have that $\pi_1(B, C, \cdot)$ maps bijectively onto the set

$$U \backslash \pi_1(B, \cdot) := \{Ug \mid g \in \pi_1(B, \cdot)\}$$

where $U = \psi(\pi_1(C, \cdot)$ and $\psi : \pi_1(C, \cdot) \to \pi_1(B, \cdot)$, see [H] 4.1, Exercise 5. Here, note that we may suppose that B is connected, hence A and C are connected, too. Since $a_1 = \partial'(c_1)$ and $a_2 = \partial'(c_2)$ are mapped to the same element of $\pi_1(B, C, \cdot)$ we get that $Ua_1 = Ua_2$, so $a_1 a_2^{-1} \in U$, hence $a_1 a_2^{-1}$ is mapped to the distinguished element of $\pi_1(B, C, \cdot)$. The same holds for $c_1 c_2^{-1}$, so $c_1 c_2^{-1} = 1$, hence $c_1 = c_2$.
In particular, $\partial' : \pi_{k+1}(A, B, \cdot) \to \pi_k(B, \cdot)$ is injective, and because of the exact sequence

$$\pi_{k+1}(A, B, \cdot) \to \pi_k(B, \cdot) \to \pi_k(A, \cdot)$$

the image of ∂' is $\ker(\pi_k(B, \cdot) \to \pi_k(A, \cdot))$. The rest is clear.
b) Note that (A, B) is obviously always 0-connected: Since C is a deformation retract of A, every path component of A meets C, hence B, too.
Also, if (B, C) is 0-connected, $\cdot \in B$, we have $\pi_1(A, B, \cdot) = 0$: Let α be a representative of this set, i.e. a path in A from some point $\star$ of B to $\cdot$. Since (B, C) is 0-connected we can find paths β, γ in B from $\star$ resp. $\cdot$ to points of C. Then $\beta^- \star \alpha \star \gamma$ is homotopic to a path δ in C because C is a deformation retract of A. So α is homotopic to $\beta \star \delta \star \gamma^-$ which is a path in B, hence α represents the trivial element.
The rest follows from a). □

Then, in particular, $\pi_k(B, C, \cdot)$ is an abelian group if $k \geq 2$.

In the case $k = 1$, $\pi_1(B, C, \cdot)$ gets the induced structure of a group.

Now we turn to a special case. Let (F, F') be a pair of CW complexes.

Lemma 2.2: Lemma 2.1 can be applied in the following case: $A = I \times F$, $B = (\partial I \times F) \cup (I \times F')$, i.e. $(A, B) = (I, \partial I) \times (F, F')$, $C := (\{1\} \times F) \cup (I \times F')$.

Proof: C is a weak deformation retract of A because $\{1\} \times F$ is a deformation retract of both C and A. □

Later on we will have the situation that we have a mapping from a set to a group, in this case we will modify the notion of an epimorphism:

Definition: Let $\phi : M \to G$ be a mapping where M is a set and G is a group. Then ϕ is said to generate the target group if $\phi(M)$ generates the group G.

If M is a group too, and ϕ a group homomorphism, this notion coincides with the one of surjectivity, of course. And if $\pi : M' \to M$ is surjective, $\psi : G \to G'$ is an epimorphism of groups and if ϕ generates the target group the same holds for $\psi \circ \phi \circ \pi$.

Now we have:

Lemma 2.3: Let (F, F') be a pair of CW complexes which is $(n-1)$-connected, $\cdot \in F'$, $\star := (0, \cdot)$, and

$$B := (\partial I \times F) \cup (I \times F')\,,\; C := (\{1\} \times F) \cup (I \times F').$$

Then the mapping

$$\pi_n(F, F', \cdot) \to \pi_n(B, C, \star)$$

induced by $x \mapsto (0, x)$
a) is bijective if $n \geq 3$,
b) is surjective if $n = 2$,
c) generates the target group if $n = 1$.

Proof: a),b): We have

$$B = (\{0\} \times F) \cup C\,,\; (\{0\} \times F) \cap C = \{0\} \times F'.$$

Our assumption implies that

$$(\{0\} \times F, \{0\} \times F') \,\text{and}\, (C, \{0\} \times F')$$

are both $(n-1)$-connected: note that $(C, I \times F')$, hence $(C, \{0\} \times F')$, is $(n-1)$-connected because the same holds for $(\{1\} \times F, \{1\} \times F')$.
By Blakers-Massey:

$$\pi_n(\{0\} \times F, \{0\} \times F', \star) \to \pi_n(B, C, \star)$$

is bijective if $n \leq 2n-3$, i.e. $n \geq 3$, and surjective if $n \leq 2n-2$, i.e. $n \geq 2$. This yields our assertion.
c) By Lemma 2.2, we have bijections

$$\pi_2(A, B, \star) \to H \to \pi_1(B, C, \star)$$

with $H := \ker(\pi_1(B, \star) \to \pi_1(A, \star))$.
Now let us take an element of $\pi_1(B, C, \star)$, it comes from an element $[\gamma] \in H \subset \pi_1(B, \star)$. We may assume that γ is a composition $\gamma_1 \star \ldots \star \gamma_s$ of curves γ_i with initial and end point in $\{0\} \times F'$ which lie in C if i is odd resp. in $\{0\} \times F$ if i is even: We have $B = C \cup (\{0\} \times F)$, where B is a CW complex written as the union of two subcomplexes. We may assume that $\gamma(0)$ is a vertex of B. By cellular approximation, see [H] Theorem 4.8, we may assume that the image of γ is contained in the 1-skeleton of B. This image is compact, so we may pass from the 1-skeleton to a union of finitely many cells of dimension ≤ 1, see [H] Prop. A.1, so to a finite graph. After subdivision we can pass from the graph to a simplicial complex of dimension ≤ 1, so by simplicial approximation, see [Sp] Theorem 3.4.8, combined with [Sp] Lemma 3.4.2, we may achieve indeed that γ can be decomposed into paths in C or $\{0\} \times F$, respectively. (The reader is invited to find a more direct argument.)
We may assume that F is connected. Then we may join the junction point p_i between γ_i and γ_{i+1} with $\star$ by paths α_i resp. β_i such that
α_i is a path from $\star$ to p_i in $\{0\} \times F$,
β_i is a path from $\star$ to p_i in C.
Here, let α_0 and β_0 be the constant paths at $\star$.
Then $[\gamma]$ can be written as a product of elements of $\pi_1(B, \star)$ represented by closed paths of the following form:
a) $\alpha_{i-1} \star \gamma_i \star \alpha_i^-$, i even,
b) $\beta_{i-1} \star \gamma_i \star \beta_i^-$, i odd,

c) $\alpha_i \star \beta_i^-$, i even,
d) $\beta_i \star \alpha_i^-$, i odd.
We may achieve that these are elements of H:
In order to see this, choose first β_1 such that the path in b) represents an element of H: otherwise, look at the image in $\pi_1(A,\star)$, and modify β_1 using the isomorphism $\pi_1(C,\star) \simeq \pi_1(A,\star)$. Then choose α_1 such that the path in d) represents an element of H, using $\pi_1(\{0\} \times F) \simeq \pi_1(A,\star)$, afterwards α_2 such that the path in a) represents such an element, using $\pi_1(\{0\} \times F) \simeq \pi_1(A,\star)$ again, then β_2 such that the path in c) represents such an element, using $\pi_1(C,\star) \simeq \pi_1(A,\star)$, and so on. Note that we may take $\alpha_s = \beta_s =$ constant path because we started with $[\gamma] \in H$.

Let us look at these cases:
a) This path lies in $\{0\} \times F$ and represents an element which comes already from $\pi_1(F, F', \cdot)$.
b) This path lies in C and represents the trivial element of $\pi_1(B, C, \star)$ because

$$\pi_1(C,\star) \to \pi_1(B,\star) \to \pi_1(B,C,\star)$$

is exact.
c) The inverse of $[\alpha_i \star \beta_i^-]$ is represented by $\beta_i \star \alpha_i^-$ which reduces to the case considered in d).
d) This path represents in $\pi_1(B,C,\star)$ the same element as α_i^-, so this element comes from $\pi_1(F, F', \cdot)$.
Altogether we have written the given element of H as a product of elements which stem from $\pi_1(B,C)$, up to inversion. $\square$

3. Monodromy

It is not the aim to present something new in this subsection; the purpose is to prepare the discussion of the notion of geometric monodromy in a general context, see Section 5.

a) Monodromy and associated bundle

Let $f : E \to S^1$ be a (topological) fibre bundle with "typical" fibre $F := f^{-1}(1)$. Recall that a homeomorphism $h : F \to F$ is called a *monodromy* for E if and only if there is a continuous family of homeomorphisms $\phi_t : F \to f^{-1}(e^{2\pi i t}), t \in I$, such that $\phi_0 = \mathrm{id}, \phi_1 = h$.

Lemma 3.1: Let $f : E \to S^1$ be a fibre bundle with typical fibre F.
Then there is a corresponding monodromy.

Proof: Let $q : I \to S^1 : \tau \mapsto e^{2\pi i \tau}$. Then $q^*E \to I$ is a trivial fibre bundle, because I is contractible. Hence there is a trivialization of q^*E, so a corresponding mapping

$$\phi : I \times F \to q^*E \to E.$$

We may assume that $\phi(0,x) = x$ for $x \in F$, otherwise let $\phi_0(x) := \phi(0,x)$ and replace $\phi(\tau,x)$ by $\phi(\tau, \phi_0^{-1}(x))$. Then $h(x) := \phi(1,x)$ defines a monodromy. $\square$

Now let F be a Hausdorff space, $h : F \to F$ a homeomorphism. Then we can construct an associated fibre bundle $f_h : E_h \to S^1$:

$$E_h := (I \times F)/\sim,$$

where $(1,x) \sim (0,h(x))$, and $f_h([\tau,x]) := e^{2\pi i\tau}$. We may identify F with $f_h^{-1}(1)$, using $x \mapsto [0,x]$. Note that h is a corresponding monodromy: use $\phi_\tau(x) := [\tau, x]$.

How does this fibre bundle depend on h? Instead of h we consider two such homeomorphisms h_0, h_1, keep the notation $[\tau,x]$ in the case E_{h_1} and write $[[\tau,x]]$ instead of $[\tau,x]$ in the case of E_{h_0}.

Lemma 3.2: Let $h_0, h_1 : F \to F$ be homeomorphisms where F is a locally finite CW complex. Then the following conditions are equivalent:

(i) There is a homeomorphism $\phi : E_{h_1} \to E_{h_0}$ with $\phi|F = \mathrm{id}$ such that the diagram

$$\begin{array}{ccc} E_{h_1} & \xrightarrow{\phi} & E_{h_0} \\ f_{h_1} \searrow & & \downarrow f_{h_0} \\ & & S^1 \end{array}$$

is commutative,

(ii) h_0 and h_1 are isotopic, i.e. there is a continuous mapping $H : I \times F \to F$ such that $h_t := H(t, .), t \in I$, is a homeomorphism and h_0, h_1 are the given mappings.

Proof: Note that the assumption on F implies that F is a Hausdorff space which is locally compact and locally connected.
We may replace the condition (ii) by a different one:
(II) there is a continuous family of homeomorphisms

$$K_t : F \to F, t \in I$$

such that $K_0 = \mathrm{id}$ and $K_1 = h_0^{-1} \circ h_1$. Here continuity is meant with respect to the compact-open topology.
(ii) $\Rightarrow$ (II): Put $K_t := h_0^{-1} \circ h_t$. And $t \mapsto K_t$ îs continuous if and only if $(t, x) \mapsto K_t(x)$ is continuous, see [H] Prop. A.14.
(II) $\Rightarrow$ (ii): Put $h_t := h_0 \circ K_t$.
(i) $\Rightarrow$ (II): Note that ϕ is of the form

$$[t, x] \mapsto [[t, K_t(x)]]$$

so $K_0 = \mathrm{id}$. Then

$$[1, x] \mapsto [[1, K_1(x)]] = [[0, h_0(K_1(x))]]$$

on the other hand

$$[1, x] = [0, h_1(x)] \mapsto [[0, h_1(x)]]$$

so $h_0 \circ K_1 = h_1$, $K_1 = h_0^{-1} \circ h_1$.
(II) $\Rightarrow$ (i): Put $\phi([t, x]) := [[t, K_t(x)]]$. This is well-defined, ϕ is continuous, and $\phi|F = \mathrm{id}$. Furthermore, ϕ is bijective: $\phi^{-1}([[t, x]]) = [t, K_t^{-1}(x)]$. And $t \mapsto K_t^{-1}$ is continuous, since F is Hausdorff, locally compact and locally connected, hence the group of homeomorphisms $F \to F$ with the compact-open topology is a topological group by [Ar] Theorem 4. $\square$

And we have a different characterization of the notion of monodromy:

Lemma 3.3: If $h : F \to F$ is a homeomorphism, $f : E \to S^1$ a fibre bundle, F a locally finite CW complex, then there is an isomorphism $E_h \simeq E$ with $[0, x] \mapsto x, x \in F$, if and only if h is a monodromy for E.

Proof: "$\Rightarrow$": Let us start from an isomorphism $\phi : E_h \to E$ with $[0, x] \mapsto x, x \in F$. Then it induces an isomorphism $I \times F \simeq q^*E_h \to q^*E$ which is of the form $(t, x) \mapsto (t, \phi([t, x]))$ with $\phi([0, x]) = x$. Then $\phi([1, x]) = \phi([0, h(x)]) = h(x)$, so h is a monodromy.
"$\Leftarrow$": Let (ϕ_t) be chosen according to the definition of a monodromy. Then we get a mapping $I \times F \to E : (t, x) \mapsto \phi_t(x)$ which induces a bijective continuous mapping $E_h \to E$ with $[0, x] \mapsto x, x \in F$. As in the proof of Lemma 3.2 we get that the latter is an isomorphism. $\square$

Lemma 3.4: Let E be a fibre bundle over S^1, h_0 a monodromy, $h_1 : F \to F$ a homeomorphism, F a locally finite CW complex. Then the following conditions are equivalent:
a) h_1 is a monodromy, too,

b) there is an isomorphism $E \simeq E_{h_1}$ with $x \mapsto [0, x], x \in F$,
c) h_0, h_1 are isotopic.

Proof: Because of Lemma 3.3, we may replace the hypothesis that E is a fibre bundle over S^1 and h_0 a monodromy by the hypothesis that h_0 is a homeomorphism and $E = E_{h_0}$. Then a) $\Leftrightarrow$ b) follows from Lemma 3.3, whereas b) $\Leftrightarrow$ c) follows from Lemma 3.2. $\square$

b) Homotopy monodromy

We may modify the context: instead of a fibre bundle look at a (Hurewicz) fibration, cf. [Sp] 2.2. This is a weaker notion, cf. [Sp] Cor. 2.7.14. Then we can weaken the notion of monodromy and speak of a homotopy monodromy instead: Let $f : E \to S^1$ be a fibration and F a fibre. Then call $h : F \to F$ a *homotopy monodromy* if and only if there is a continuous family of mappings $\phi_\tau : F \to f^{-1}(e^{2\pi i\tau})$ such that $\phi_0 = \mathrm{id}$ and $\phi_1 = h$.

Lemma 3.5: Let $f : E \to S^1$ be a fibration and $F := f^{-1}(1)$. Then there is a homotopy monodromy.

Proof: This follows from the homotopy lifting property used to define fibrations. In the case of a fibre bundle, F being a locally finite CW complex, one can use Lemma 3.1, of course. $\square$

And similar to Lemma 3.4a)$\Leftrightarrow$c) we have:

Lemma 3.6: Let h_0 be a homotopy monodromy, $h_1 : F \to F$ continuous. Then the following conditions are equivalent:
a) h_1 is a homotopy monodromy,
b) h_0 and h_1 are homotopic.

Proof: a) $\Rightarrow$ b): This follows from the homotopy lifting property as well, see [Sp] Cor. 2.8.11. More explicitly, one could also argue as follows:
Since h_0, h_1 are homotopy monodromies, we have, for $t = 0, 1$, mappings $\psi_t : I \times F \to E$ with $\{\tau\} \times F \to f^{-1}(e^{2\pi i\tau})$ such that $(0, x) \mapsto x$ and $(1, x) \mapsto h_t(x)$. Now look at the commutative diagram

$$\begin{array}{ccc} (I \times \{0,1\} \cup \{0\} \times I) \times F & \to & E \\ \downarrow & \nearrow & \downarrow \\ I \times I \times F & \to & S^1 \end{array}$$

where the upper horizontal is given by $(\tau, t, x) \mapsto \psi_t(\tau)$ on $I \times \{0,1\} \times F$ and $(0, t, x) \mapsto x$ on $\{0\} \times I \times F$. By homotopy lifting one obtains the diagonal, as loc.cit., or easier, if F is a CW complex: by the homotopy lifting with respect to the pair $(I \times F, \{0,1\} \times F)$, and restricting a homotopy between h_0 and h_1 to $\{1\} \times I \times F$.
b) $\Rightarrow$ a): By definition, we have a mapping $\psi_0 : I \times F \to E$ with $\{\tau\} \times F \to f^{-1}(e^{2\pi i\tau})$ such that $(0, x) \mapsto x$ and $(1, x) \mapsto h_0(x)$. Now look at the commutative diagram

$$\begin{array}{ccc} (I \times \{0\} \cup \{0,1\} \times I) \times F & \to & E \\ \downarrow & \nearrow & \downarrow \\ I \times I \times F & \to & S^1 \end{array}$$

where the upper horizontal is defined by $(\tau, 0, x) \mapsto \psi_0(\tau, x)$, $(0, t, x) \mapsto x$, and $(1, t, x) \mapsto h_t(x)$.

Then apply homotopy lifting again. Restricting to $I \times \{1\} \times F$ we see that h_1 is a homotopy monodromy too. $\square$

c) Wang sequence

Now we restrict to fibre bundles.

As we just have seen, homotopy monodromies are homotopic, hence the homological monodromy $h_* : H_k(F) \to H_k(F)$ does not depend on the choice of a homotopy monodromy h. The latter appears in the Wang sequence, using it we obtain an alternative proof of the independence of the homological monodromy:

Theorem 3.7: Let $f : E \to S^1$ be a fibre bundle, $F = f^{-1}(1)$.
a) For every k there is a canonical isomorphism

$$w_k : H_k(F) \to H_{k+1}(E, F)$$

which will be defined in the proof below.
b) If h is a homotopy monodromy the mapping

$$\partial \circ w_k : H_k(F) \to H_k(F)$$

coincides with $id - h_*$.
c) We have an exact sequence (Wang sequence)

$$\ldots \to H_{k+1}(E) \to H_k(F) \stackrel{h_*-id}{\to} H_k(F) \to H_k(E) \to \ldots$$

The Wang sequence is well-known, see e.g. [Sp] Cor. 8.5.6.

From b) we see again that $h_* : H_k(F) \to H_k(F)$ does not depend on the choice of the homotopy monodromy h, as promised.

Proof: Put $E_- := f^{-1}(S^1 \cap \{\operatorname{Im} z \leq 0\})$, analogously E_+, and $F' := f^{-1}(-1)$.
a) w_k is defined as the composition of the following mappings:

$$\begin{array}{c} H_k(F) \\ \downarrow \\ H_k(F \cup F', F') \\ \uparrow\simeq \\ H_{k+1}(E_+, F \cup F') \\ \downarrow \\ H_{k+1}(E, E_-) \\ \uparrow\simeq \\ H_{k+1}(E, F) \end{array}$$

Most of the mappings are induced by inclusion, except for the second one which is the boundary operator of the long exact homology sequence of the triple $(E_+, F \cup F', F')$.

All mappings are isomorphisms: the first and the third one by excision, the second one because F' is a deformation retract of E_+, the last one because F is a deformation retract of E_-.

b) We expand the diagram in a) to a commutative diagram, where $J = [0, 2]$:

$$\begin{array}{ccc}
H_k(F) & & \\
\downarrow & \searrow & \\
H_k(\{0\} \times F) & \to & H_k(F) \\
\downarrow & & \downarrow \\
H_k(\{0,1\} \times F, \{1\} \times F) & \to & H_k(F \cup F', F') \\
\uparrow & & \uparrow \\
H_{k+1}(I \times F, \{0,1\} \times F) & \to & H_{k+1}(E_+, F \cup F') \\
\downarrow\simeq & & \downarrow\simeq \\
H_{k+1}(J \times F, (\{0\} \cup [1,2]) \times F) & \to & H_{k+1}(E, E_-) \\
\uparrow & & \uparrow \\
H_{k+1}(J \times F, \{0,2\} \times F) & \to & H_{k+1}(E, F) \\
\downarrow & & \downarrow \\
H_k(\{0,2\} \times F) & \to & H_k(F)
\end{array}$$

In the left vertical, most of the mappings are induced by inclusion, except for the first one which is induced by $x \mapsto (0, x)$, the third one which is the boundary operator of the exact homology sequence of the triple

$$(I \times F, \{0,1\} \times F, \{1\} \times F)$$

and the last one which is a boundary operator.

All vertical mappings are isomorphisms except the lowest ones: note that $\{1\} \times F$ and $\{0,2\} \times F$ are deformation retracts of $I \times F$ resp. $(\{0\} \cup [1,2]) \times F$, respectively.

All horizontal mappings are induced by the mapping $(t, x) \mapsto \phi_t(x)$, where ϕ_t is chosen as in the definition of a homotopy monodromy h, with $J = [0, 2]$ instead of I, hence ϕ_{2t} instead of the mapping ϕ_t before.

If we walk from $H_k(F)$ in the upper left corner first to the right and then down we get $\partial \circ w_k$, if we walk down first and finally to the right we obtain $id - h_*$, as we see from the following diagram:

$$\begin{array}{ccc}
[c] & \in & H_k(F) \\
\downarrow & & \downarrow \\
[0 \times c] & \in & H_k(\{0\} \times F) \\
\downarrow & & \downarrow \\
-[\partial\varepsilon \times c] & \in & H_k(\{0,1\} \times F, \{1\} \times F) \\
\uparrow & & \uparrow\simeq \\
-[\varepsilon \times c] & \in & H_{k+1}(I \times F, \{0,1\} \times F) \\
\downarrow & & \downarrow \\
-[\varepsilon' \times c] & \in & H_{k+1}(J \times F, (\{0\} \cup [1,2]) \times F) \\
\uparrow & & \uparrow \\
-[\varepsilon' \times c] & \in & H_{k+1}(J \times F, \{0,2\} \times F) \\
\downarrow & & \downarrow \\
[0 \times c - 2 \times c] & \in & H_k(\{0,2\} \times F) \\
\downarrow & & \downarrow \\
[c - h \circ c] & \in & H_k(F)
\end{array}$$

Here ε and ε' represent the canonical generators of $H_1(I, \partial I)$ resp. $H_1(J, \partial J)$.

c) Note that the Wang sequence follows from the long exact homology sequence of the pair (E, F), using a) resp. b). □

4. Variation

Let (F, F') be a pair of topological spaces. Let $h : F \to F$, $h|F' = \text{id}$, $k \geq 0$.

Then we can define the homological variation associated to h (it is not necessary to suppose that h is a homeomorphism):

Definition (cf. [La]): Use singular homology (with integral coefficients). Then let

$$\mathrm{var}_k^{[h]} : H_k(F, F') \to H_k(F)$$

be defined by

$$\mathrm{var}_k^{[h]}([c]) := [h \circ c - c]$$

if $[c]$ is the class of a singular k-chain in $H_k(F, F')$.

There is a homotopical analogue, see [CE]: Let $k \geq 1$, $\cdot$ a base point in F'. Then let $\mathrm{var}_k^{[h]} : \pi_k(F, F', \cdot) \to \pi_k(F, \cdot)$ be defined as follows:

Let $\alpha : I^k \to F$ define a relative k-cell for $(F, F', \cdot)$. Then $\mathrm{var}_k^{[h]}([\alpha])$ is represented by the k-cell $\alpha \perp h \circ \alpha$ for $(F, \cdot)$ defined by:

$(t_1, \ldots, t_k)$ maps to $\alpha(t_1, \ldots, t_{k-1}, 1-2t_k)$ if $0 \leq t_k \leq \frac{1}{2}$, and maps to $(h \circ \alpha)(t_1, \ldots, t_{k-1}, 2t_k - 1)$ if $\frac{1}{2} \leq t_k \leq 1$.

This is well-defined because $h|F' = \mathrm{id}$, and it is a homomorphism if $k \geq 2$, cf. [CE].

For example, in the case $h = \mathrm{id}$ we have $\mathrm{var}_k^{[h]} = 0$ and - if $k \geq 1$ - $\mathrm{var}_k^{[h]} = 0$.

And we have:

Lemma 4.1: Assume that $h_0, h_1 : F \to F$ with $h_0|F' = h_1|F' = \mathrm{id}$ are homotopic relative to F', i.e. such that there is there is a homotopy between h_0, h_1 which is constant on F'. Then:
a) $\mathrm{var}_k^{[h_0]} = \mathrm{var}_k^{[h_1]}$,
b) $\mathrm{var}_k^{[h_0]} = \mathrm{var}_k^{[h_1]}$ if $k \geq 1$.

Proof: a) see [La].
b) Let $h_s, s \in I$, define a homotopy between h_0 and h_1 which is constant on F'. Let α be a relative k-cell of $(F, F', \cdot)$. Then $\alpha \perp h_s \circ \alpha$ defines a homotopy between $\alpha \perp h_0 \circ \alpha$ and $\alpha \perp h_1 \circ \alpha$. □

We may weaken the hypothesis that h_0, h_1 are homotopic relative to F': Given h_0, h_1 as above, then call h_0, h_1 *strongly homotopic* if there is a homotopy between h_0 and h_1, given by $(h_t)_{t \in I}$, such that $t \mapsto h_t|F'$ is homotopic to $t \mapsto id$ relative to $\partial I \times F'$. This means that there is a continuous family of mappings

$$H_{s,t} : F' \to F', s, t \in I$$

such that $H_{0,t} = h_t|F'$ and $H_{s,t} = \mathrm{id}$ if $t = 0, t = 1$ or $s = 1$.

Lemma 4.2: Lemma 4.1 still holds if we merely suppose that h_0, h_1 are strongly homotopic.

Proof: a) Let $j \in \{0, 1\}$. Put $\tilde{F} := F \cup (I \times F')$, where F' is identified with $\{0\} \times F'$, and $\tilde{h}_j$ the extension of h_j by the identity. This means that we can look at

$$\tilde{h}_j : (\tilde{F}, \tilde{F}') \to (\tilde{F}, \tilde{F}')$$

instead of $h_j : (F, F') \to (F, F')$, where $\tilde{F}' := \{1\} \times F'$. We could modify this definition by taking $\check{F}' := [0,1] \times F'$ instead of $\tilde{F}'$ and write $\check{h}_j$ instead of $\tilde{h}_j$ in this case.

The advantage of passing to $\tilde{h}_j$ is that there is a homotopy between $\tilde{h}_0$ and $\tilde{h}_1$ which is constant on $\tilde{F}'$:

Let h_τ and $H_{s,\tau}$ be chosen as in the definition of "strongly homotopic". Then define $\tilde{h}_\tau : \tilde{F} \to \tilde{F}$ as follows:
$\tilde{h}_\tau(x) := h_\tau(x)$ if $x \in F$,
$\tilde{h}_\tau(s,x) := H_{s,\tau}(x)$ if $s \in [0,1], x \in F'$.
Note that we have identified F' with $\{0\} \times F'$, and indeed $\tilde{h}_\tau(0,x) = H_{0,\tau}(0,x) = h_\tau(x)$ if $x \in F'$, so $\tilde{h}_\tau$ is well-defined, $\tilde{h}_\tau|\tilde{F}' = \mathrm{id}$.

Now we have a commutative diagram, with $h = h_0$ or $h = h_1$:

$$\begin{array}{ccc} H_k(F,F') & \overset{\mathrm{var}_k^{[h]}}{\to} & H_k(F) \\ \downarrow & & \downarrow \\ H_k(\tilde{F},\check{F}') & \overset{\mathrm{var}_k^{[\tilde{h}]}}{\to} & H_k(\tilde{F}) \\ \uparrow & & \uparrow \\ H_k(\tilde{F},\tilde{F}') & \overset{\mathrm{var}_k^{[\tilde{h}]}}{\to} & H_k(\tilde{F}) \end{array}$$

The vertical arrows are isomorphisms.

Furthermore, note that there is a homotopy between $\tilde{h}_0$ and $\tilde{h}_1$ which is constant on $\tilde{F}'$, namely $\tilde{h}_\tau, \tau \in [0,1]$. By Lemma 4.1, we conclude that $\mathrm{var}_k^{\tilde{h}_0} = \mathrm{var}_k^{\tilde{h}_1}$, hence $\mathrm{var}_k^{[h_0]} = \mathrm{var}_k^{[h_1]}$.

b) We argue similarly in the case of homotopy groups but we must be aware of the base points: Let $\cdot \in F' \subset \check{F}'$ and $\star := (\cdot, 1) \in \tilde{F}' \subset \check{F}'$. There is a canonical path from $\cdot$ to $\star$ in $\check{F}'$ which induces base change isomorphisms $\pi_k(\check{F}, \cdot) \to \pi_k(\check{F}, \star)$ and $\pi_k(\tilde{F}, \check{F}', \cdot) \to \pi_k(\tilde{F}, \check{F}', \star)$ such that the following diagram is commutative for $h = h_0$ or $h = h_1$:

$$\begin{array}{ccc} \pi_k(\tilde{F},\check{F}',\cdot) & \to & \pi_k(\tilde{F},\cdot) \\ \downarrow & & \downarrow \\ \pi_k(\tilde{F},\check{F}',\star) & \to & \pi_k(\tilde{F},\star) \end{array}$$

Altogether we obtain a commutative diagram for $h = h_0$ or $h = h_1$, the verticals being isomorphisms:

$$\begin{array}{ccc} \pi_k(F,F',\cdot) & \overset{\mathrm{var}_k^{[h]}}{\to} & \pi_k(F,\cdot) \\ \downarrow & & \downarrow \\ \pi_k(\tilde{F},\check{F}',\cdot) & \overset{\mathrm{var}_k^{[\tilde{h}]}}{\to} & \pi_k(\tilde{F},\cdot) \\ \downarrow & & \downarrow \\ \pi_k(\tilde{F},\check{F}',\star) & \overset{\mathrm{var}_k^{[\tilde{h}]}}{\to} & \pi_k(\tilde{F},\star) \\ \uparrow & & \uparrow \\ \pi_k(\tilde{F},\tilde{F}',\star) & \overset{\mathrm{var}_k^{[\tilde{h}]}}{\to} & \pi_k(\tilde{F},\star) \end{array}$$

Again, by Lemma 4.1, $\mathrm{var}_k^{\tilde{h}_0} = \mathrm{var}_k^{\tilde{h}_1}$, hence $\mathrm{var}_k^{h_0} = \mathrm{var}_k^{h_1}$. □

We will apply this in the next section to a particular geometric situation.

5. Geometric monodromy for generalized open book decompositions

a) Generalized open book decompositions

It is well known that the concept of open book decompositions (see [Qu]) is useful in singularity theory (see e.g. [St]).

Here we introduce the notion of a generalized open book decomposition in the following version, motivated by the application in the proof of Theorem 0.1 and 0.2:

Definition: Let X be a Hausdorff space. A *generalized open book decomposition* of X is given by a mapping

$$g : X \to \mathbb{D}^2 := \{t \in \mathbb{C} \mid |t| \leq 1\}$$

such that, with $X^1 := g^{-1}(S^1)$ and $X^2 := \overline{g^{-1}((\mathbb{D}^2)^\circ)}$, hence $X = X^1 \cup X^2$, the following holds:
$g|X^1 : X^1 \to S^1$ is a (locally trivial) fibre bundle and
$g|X^2 : X^2 \to \mathbb{D}^2$ is a trivial fibre bundle.
Let $F := g^{-1}(1)$, $F' := g^{-1}(1) \cap X^2$. Here we suppose that (F, F') is a pair of locally finite CW complexes. Then X, together with such a decomposition, is called a *generalized open book*.

Note that X^2 plays the role of a neighbourhood of the "binding" $g^{-1}(0)$ in X on which we have a distance function from the binding, given by $|g|$. And we renounce to fix a trivialization of $g|X^2$!

We suppose from now on that (F, F') always denotes a pair of locally finite CW complexes, as in the case of an open book above and in the classical case where F is a compact differentiable manifold with boundary F'. One can also pass to the stratified case, see [G]. The word "generalized" is used because we do not suppose here that we work with manifolds.

Remark: We may introduce a more general notion: Let us call $g : X \to B$ a compound fibre bundle with basis (B, B') if the condition above holds with B, B' instead of $\mathbb{D}^2, S^1$ and "locally trivial" instead of "trivial" . So a generalized open book decomposition is a *compound fibre bundle* with base $(\mathbb{D}^2, S^1)$.

Suppose now that X is a generalized open book. Then there are continuous mappings $\chi : \mathbb{D}^2 \times F' \to X^2$ and $\psi : I \times F \to X^1$ such that $\chi(z, \cdot)$ maps $\{z\} \times F'$ homeomorphically onto $g^{-1}(z) \cap X^2$ and ψ maps $\{\tau\} \times F$ homeomorphically onto $g^{-1}(e^{2\pi i\tau})$.

But is not clear whether the following condition (C) is automatically satisfied:
χ and ψ can be chosen such that $\chi(e^{2\pi i\tau}, x) = \psi(\tau, x)$ if $(\tau, x) \in I \times F'$. (C)

A special case is a *usual open book decomposition*: X compact differentiable manifold with boundary and corners, X^1, X^2 differentiable manifolds with common boundary $X^1 \cap X^2$, g differentiable, $g|X^1 : X^1 \to S^1$ and $g|X^2 : X^2 \to \mathbb{D}^2$ submersions. One can modify X then such that X becomes a differentiable manifold. This case is considered in [St].

Recall that proper submersions of differentiable manifolds yield fibre bundles.

We can consider a more general situation where X is a union of strata of some stratified differentiable manifold, the precise hypotheses being as follows: Let M be a differentiable manifold, endowed with a Whitney stratification, $g^* : M \to \mathbb{C}$ a stratified submersion. Suppose that $A' \subset A$ are closed subsets of M which are unions of strata, $\alpha > 1$, $g^*|A' : A' \to \{|z| < \alpha\}$ proper, $A \cap \{|g^*| = 1\}$ compact. Then $X = X^1 \cup X^2$ with $X^1 := A \cap \{|g^*| = 1\}$, $X^2 := A' \cap \{|g^*| \leq 1\}$ and $g := g^*|X$ is a generalized open book: apply Thom's first isotopy lemma.

More generally, let B be a closed subset of A which is a union of strata, then we may take off B in the definition of X, X^1 and X^2 and still get a generalized open book: we call it a *stratified open book*.

On the other hand there is the following construction: Let (F, F') be a pair of locally finite CW complexes, $h : F \to F$ a homeomorphism such that $h|F'$ is the identity. Let $f_h : E_h \to S^1$ be defined as in section 3 (after Lemma 3.1), we write g_h instead of f_h now. Let $X^1 := E_h$, $X^2 := \mathbb{D}^2 \times F'$ and $X = X^1 \cup X^2$, where $[\tau, x] \in X^1$, with $x \in F'$, is identified with $(e^{2\pi i\tau}, x) \in X^2$, and let $X_h := X^1 \cup X^2$. Put $g_h : X_h \to \mathbb{D}^2$: $[\tau, x] \mapsto e^{2\pi i\tau}$ if $[\tau, x] \in X^1$, $g_h|X^2 \to \mathbb{D}^2$ being the projection onto the first factor. Then $g_h : X_h \to \mathbb{D}^2$ defines a generalized open book; identify F with $g_h^{-1}(1)$.

Lemma 5.1: Condition (C) holds in the case of
a) a usual open book,
b) a stratified open book,
c) the generalized open book X_h.

Proof: a) We can find χ by the theorem of Ehresmann. The standard vector field on S^1 induces a vector field on $S^1 \times F'$, hence on $X^1 \cap X^2$. It can be suitably extended to X^1. By integration we obtain ψ.
b) By hypothesis, g^* is a stratified submersion with respect to a Whitney stratification. Then choose corresponding control data which are compatible with g^*, see [M] Prop. 7.1.
Then
$$g^*|A' : A' \to \{|z| < \alpha\}$$
is a proper controlled submersion as in [M] §9. Let F' be the fibre over 1. By Thom's first isotopy lemma, see [M] Cor. 10.2, we have a stratified homeomorphism $G' : \{|t| < \alpha'\} \times F' \to A'$ such that $G'(1, x) = x$ and $g(G'(t, x)) = t$, which is compatible with the pre-stratification structures, cf. [M] Cor. 10.3.
Differentiating $\tau \mapsto e^{i\pi\tau}z$ we obtain a controlled vector field, cf. [M] §9, on $S^1 \times F'$, hence on $\{|g| = 1\} \cap A'$, which is compatible with g. We can extend the latter to a controlled vector field on $\{|g| = 1\} \cap A$ which is compatible with g, proceeding by induction as in the proof of [M] Prop. 9.1. We can pass to $\rho^*(\{|g| = 1\} \cap A$, where $\rho : I \to S^1 : t \mapsto e^{2\pi it}$. Then, by Thom's first isotopy lemma again, we get a stratified homeomorphism $I \times F \to \rho^*(\{|g| = 1\} \cap A$ of the form $(\tau, x) = (\tau, H'(\tau, x))$ where $f(H'(\tau, x)) = e^{i\pi\tau}, H'(0, x) = x$. We obtain G, H by restriction of G', H' to the complement of B, respectively.
c) Here, take $\chi :=$ id and $\psi(\tau, x) := [\tau, x]$ in order to obtain condition (C). □

Similar to the case a) or b), a possible strategy in general is to start with a trivialization $\mathbb{D}^2 \times F' \to X^2$ (it was supposed to exist) - this can be used to define χ. Let
$$q : [0, 1] \to S^1 : \tau \mapsto e^{2\pi i\tau}.$$
Then we get a trivialization $[0, 1] \times F' \to q^*(X^1 \cap X^2)$ of $q^*(X^1 \cap X^2) \to [0, 1]$, too. We extend it - if possible (this is a question for topologists) - to a trivialization of $q^*X^1 \to [0, 1]$ which gives ψ.

Lemma 5.2: Let $h_0, h_1 : F \to F$ be homeomorphisms such that $h_0|F' = h_1|F' =$ id. Then the following conditions are equivalent:
a) There is a homeomorphism $\phi : X_{h_1} \to X_{h_0}$ such that $\phi|F =$ id and that the diagram
$$\begin{tikzcd} X_{h_1} \arrow[r, "\phi"] \arrow[d, "g_{h_1}"'] & X_{h_0} \arrow[dl, "g_{h_0}"] \\ \mathbb{D}^2 & \end{tikzcd}$$
is commutative,
b) h_0 and h_1 are *strongly isotopic* (cf. the notion "strongly homotopic" used in Lemma 4.2), i.e. there are continuous families of homeomorphisms:
$(h_t : F \to F)_{t \in I}$, where h_0, h_1 are already given, and $(H_{s,t} : F' \to F')_{s,t \in I}$ such that

$H_{0,t} = h_t|F', H_{s,0} = H_{s,1} = H_{1,t} = \text{id}$ for all s, t.

Proof: We consider a third condition, too:
c) There are continuous families of homeomorphisms
$K_t : F \to F, t \in I$, such that $K_0 = \text{id}$ and $K_1 = h_0^{-1} \circ h_1$,
$L_z : F' \to F', z \in \mathbb{D}^2$, such that $K_t|F' = L_{e^{2\pi it}}$.

a) $\Rightarrow$ c): First, ϕ must map $X^1_{h_1}$ onto $X^1_{h_0}$ and $X^2_{h_1}$ onto $X^2_{h_0}$, then $\phi|F = \text{id}$.
From the proof of Lemma 3.2 we know already that $\phi|X^1_{h_1}$ must be of the form $[\tau, x] \mapsto [[\tau, K_\tau(x)]]$ with $K_0 = \text{id}$, $K_1 = h_0^{-1} \circ h_1$, so K_τ has the desired property.

Furthermore, $\phi|X^2_{h_1} \to X^2_{h_0}$ must be of the form $(z, x) \mapsto (z, L_z(x))$ where $L_{e^{2\pi it}}(x) = K_t(x)$ for $x \in F', t \in I$, i.e. L_z has the desired property.

c) $\Rightarrow$ a): Put $\phi([t, x]) := ([[t, K_t(x)]]$ and $\phi(z, x) := (z, L_z(x)), x \in F'$. This is well-defined and gives a continuous mapping $X_{h_1} \to X_{h_0}$ which is bijective, the inverse mapping being given using K_t^{-1} resp. L_z^{-1}. Similarly as in the proof of Lemma 3.2 we see that the inverse mapping is continuous, too.

b) $\Rightarrow$ c): Put $K_t := h_0^{-1} \circ h_t$, so $K_t|F' = h_t$, $L_{\rho+(1-\rho)e^{2\pi it}} := H_{\rho,t}$. The latter is well-defined because of the hypothesis on $H_{\rho,t}$.

c) $\Rightarrow$ b): Put $h_t := h_0 \circ K_t$, hence $K_t|F' = h_t$, and $H_{\rho,t} := L_{\rho+(1-\rho)e^{2\pi it}}$. □

b) Geometric monodromy

Definition: If $g : X \to \mathbb{D}^2$ defines an arbitrary generalized open book, $F := g^{-1}(1)$, $F' := F \cap X^2$, then call a homeomorphism $h : F \to F$ with $h|F' = \text{id}$ a *geometric monodromy* if one of the equivalent conditions of the following lemma holds:

Lemma 5.3: Let $h : F \to F$ with $h|F' = \text{id}$ and X a generalized open book with fibre $F := g^{-1}(1)$. The following conditions are equivalent:

a) h is a monodromy of $g|X^1$, and there is a homeomorphism $\chi : \mathbb{D}^2 \times F' \to X^2$ which is compatible with the projections to $\mathbb{D}^2$ such that $\chi(e^{2\pi it}, x) = \phi_t(x), x \in F'$, ϕ_t being chosen as in the definition of a monodromy,

b) there is a homeomorphism $\phi : X_h \to X$ which is compatible with the projections.

Proof: a) $\Rightarrow$ b): Define $\phi([t, x]) := \phi_t(x), x \in F$, $\phi(z, x) := \chi(z, x)$, $z \in \mathbb{D}^2, x \in F'$. Then ϕ is continuous and bijective, similarly as in the proof of Lemma 3.3 we get that ϕ is an isomorphism.

b) $\Rightarrow$ a): We may suppose that $\phi([0, x]) = x, x \in F$. Then $\phi_t(x) := \phi([t, x]), \chi(z, x) := \phi(z, x)$.
□

Lemma 5.4: Let X be a generalized open book.
a) If condition (C) holds there is a geometric monodromy.
b) Suppose that h_0 is a geometric monodromy and $h_1 : F \to F$ a homeomorphism such that $h_1|F' = \text{id}$. Then h_1 is a geometric monodromy, too, if and only if h_0 and h_1 are strongly isotopic.

Proof: a) Let χ and ψ be chosen as in condition (C) and put $h(x) := \psi(1, x)$. Then we have a homeomorphism $\phi : X_h \to X$: $\phi([\tau, x]) := \psi(\tau, x)$, $\phi|X^2_h := \chi$.

b) By the definition of a geometric monodromy, we can replace X by X_{h_0}. So h_1 is a geometric monodromy, too if and only if there is a homeomorphism $X_{h_0} \to X_{h_1}$ which is compatible with the projections. Now apply Lemma 5.2. □

c) Geometric homotopy monodromy

We may weaken the notion of geometric monodromy to the one of a geometric homotopy monodromy. This has the advantage that we need no longer assumption (C):

A continuous mapping $h : F \to F$ with $h|F' = \mathrm{id}$ is called a *geometric homotopy monodromy* if there is a continuous mapping $\phi : I \times F \to X^1$ with $\phi(0, x) = x, \phi(1, x) = h(x), g(\phi(\tau, x)) = e^{2\pi i\tau}$ and a continuous mapping $\chi : \mathbb{D}^2 \times F' \to X^2$ with $g(\chi(z, x)) = z$, $\chi(e^{2\pi i\tau}, x) = \phi(\tau, x)$.

Lemma 5.5: There is a geometric homotopy monodromy. In the case $F' = \emptyset$ this corresponds to Lemma 3.5.

Proof: Apply the homotopy lifting property for the pair (F, F'), see [H] 4.48: First, look at the commutative diagram

$$\begin{array}{ccc} \{1\} \times F' & \to & X^2 \\ \downarrow & \nearrow & \downarrow g \\ \mathbb{D}^2 \times F' & \to & \mathbb{D}^2 \end{array}$$

The lower horizontal is obvious, the upper one given by $(1, x) \mapsto x$. By homotopy lifting we get a diagonal $\chi^* : \mathbb{D}^2 \times F' \to X^2$.
Then we pass to the commutative diagram

$$\begin{array}{ccc} I \times F' \cup \{0\} \times F & \to & X^1 \\ \downarrow & \nearrow & \downarrow g \\ I \times F & \to & S^1 \end{array}$$

The lower horizontal is given by $(\tau, x) \mapsto e^{2\pi i\tau}$, the upper one by $(\tau, x) \mapsto \chi^*(e^{2\pi i\tau}, x)$ on $I \times F'$ and by $(0, x) \mapsto x$ on $\{0\} \times F$. By the homotopy lifting property for the pair (F, F') we obtain a diagonal $\phi : I \times F \to X^1$. Restriction to $\{1\} \times F$ gives a geometric homotopy monodromy. □

Note that, in connection with geometric homotopy monodromy, instead of χ we may use $\xi : I^2 \times F' \to X^2$ and ask that

$$g(\xi(\rho, \tau, x)) = \rho + (1 - \rho)e^{2\pi i\tau}, \quad \xi(0, \tau, x) = \phi(\tau, x), \quad \xi(\rho, 0, x) = \xi(\rho, 1, x) = \xi(1, \tau, x) = x.$$

Lemma 5.6: Suppose that h_0 is a geometric homotopy monodromy. Then $h_1 : F \to F$ with $h_1|F' = \mathrm{id}$ is a geometric homotopy monodromy as well if and only if h_0 and h_1 are strongly homotopic.
In the case $F' = \emptyset$ this corresponds to Lemma 3.6.

Proof:
"⇒": As we have seen, geometric homotopy monodromies lead to mappings $\phi : I \times F \to X^1$ and $\xi : I^2 \times F' \to X^2$. So in our case we have for $t = 0, 1$ mappings $\phi_t : I \times F \to X^1$ and $\xi_t : I^2 \times F' \to X^2$.
First we have a commutative diagram

$$\begin{array}{ccc} (I \times \{0, 1\} \times I \cup I^2 \times \{0, 1\} \cup \{1\} \times I^2) \times F' & \to & X^2 \\ \downarrow & \nearrow & \downarrow g \\ I^3 \times F' & \to & \mathbb{D}^2 \end{array}$$

The lower horizontal map is $(\rho, \tau, t, x) \mapsto \rho + (1 - \rho)e^{2\pi i\tau}$, the upper one by $(\rho, \tau, t, x) \mapsto x$ on $(I \times \{0, 1\} \times I \cup \{1\} \times I^2) \times F'$, whereas on $I^2 \times \{t\} \times F'$ by $(\rho, \tau, t, x) \mapsto \xi_t(\rho, \tau, x)$, $t = 0, 1$.
By homotopy lifting with respect to the pair $(I^2 \times F', \partial I^2 \times F')$, we can find the diagonal mapping called ξ^*.

Now look at the commutative diagram

$$\begin{array}{ccc} I^2 \times F' \cup (I \times \{0,1\} \cup \{0\} \times I) \times F & \to & X^1 \\ \downarrow & \nearrow & \downarrow g \\ I^2 \times F & \to & S^1 \end{array}$$

Here, the lower horizontal arrow is defined by $(\tau, t, x) \mapsto e^{2\pi i \tau}$, the upper one by $(\tau, t, x) \mapsto \xi^*(0, \tau, t, x)$ on $I^2 \times F'$, on $I \times \{t\} \times F$, $t = 0, 1$, by $(\tau, t, x) \mapsto \psi_t(\tau, x)$, and on $\{0\} \times I \times F$ by $(\tau, t, x) \mapsto x$. By the homotopy lifting property with respect to $(I \times F, (\partial I \times F) \cup (I \times F'))$ we can find the diagonal arrow.

Then, passing to $\tau = 1$, we obtain a strong homotopy between h_0 and h_1.

"$\Leftarrow$": Look at the commutative diagram

$$\begin{array}{ccc} (I \times \{0,1\} \times I \cup I^2 \times \{0\} \cup \{0,1\} \times I^2) \times F' & \to & X^2 \\ \downarrow & \nearrow & \downarrow \\ I^3 \times F' & \to & \mathbb{D}^2 \end{array}$$

The lower horizontal map is given as before, the upper one is defined by $(\rho, \tau, t, x) \mapsto x$ on $(I \times \{0\} \times I \cup \{1\} \times I^2) \times F'$, whereas $(\rho, \tau, t, x) \mapsto \xi_0(\rho, \tau, x)$ on $I^2 \times \{0\} \times F'$ and $\mapsto \phi_{\rho,t}(x)$ on $I \times \{1\} \times I \times F'$. We can find a diagonal arrow ξ^+, by homotopy lifting with respect to the pair $(I^2 \times F', (\partial I^2) \times F')$ again.

Finally we look at the commutative diagram

$$\begin{array}{ccc} (\{0,1\} \times I \cup I \times \{0\}) \times F \cup I^2 \times F' & \to & X^1 \\ \downarrow & \nearrow & \downarrow \\ I^2 \times F & \to & S^1 \end{array}$$

The upper arrow is defined on $I^2 \times F'$ by $(\tau, t, x) \mapsto \xi^+(0, \tau, t, x)$, on $\{0\} \times I \times F$ by $(0, t, x) \mapsto x$, on $\{1\} \times I \times F$ by the mapping h_t attached to a strong homotopy, on $I \times \{0\} \times F$ using ψ_0. The lower horizontal mapping is defined again by $(\tau, t, x) \mapsto e^{2\pi i \tau}$.

We can find a diagonal arrow, using homotopy lifting with respect to $(I \times F, (\partial I) \times F) \cup (I \times F'))$. Altogether we have found mappings defined on $I^3 \times F$ resp. $I^2 \times F$ wich show for $t = 1$ that h_1 is indeed a geometric homotopy monodromy too. $\square$

Remark: If $h_0, h_1 : F \to F$ are given, $h_0|F' = h_1|F' = \mathrm{id}$, h_0 being a homeomorphism, we see that h_0 and h_1 are strongly homotopic if and only if h_1 is a geometric homotopy monodromy of X_{h_0}. This is because h_0 is a geometric monodromy, hence a geometric homotopy monodromy, of X_{h_0}.

6. Variation of a generalized open book decomposition

a) Definitions

We assume that we have a generalized open book as before. From Lemma 4.2 and 5.6 we deduce:

Corollary 6.1: The mappings $\mathrm{var}_k^{[h]}$ and $\mathrm{var}_k^{[h]}$, h being a geometric homotopy monodromy, do not depend on the particular choice of h. So we write $\mathrm{var}_k(g)$ resp. $\mathrm{var}_k(g)$ instead.
Note that $\mathrm{var}_k(g)$ is defined for $k \geq 1$ and a homomorphism if $k \geq 2$.

If condition (C) holds we may use a geometric monodromy h instead and apply Lemma 5.4 instead of Lemma 5.6.

For the following note that F is a deformation retract of $(X_1 \cap \{\operatorname{Im} g \le 0\}) \cup X_2$.

b) Homology

Now we concentrate first upon the case of homology groups. Here we get a generalization of Theorem 3.7 where we had $F' = \emptyset$:

Theorem 6.2: a) There is a canonical isomorphism $w_k : H_k(F, F') \to H_{k+1}(X, F)$ which will be defined in the proof below.
b) The mapping $\partial \circ w_k : H_k(F, F') \to H_k(F)$ coincides with $-\operatorname{var}_k(g)$.
c) There is an exact sequence

$$\ldots \to H_{k+1}(X) \to H_k(F, F') \overset{\operatorname{var}_k(g)}{\to} H_k(F) \to H_k(X) \to \ldots .$$

Proof: a) w_k is defined as the composition of the following isomorphisms, with

$$X_1^+ := X_1 \cap \{\operatorname{Im} g \ge 0\} \text{ etc.:}$$

$$\begin{array}{c} H_k(F, F') \\ \downarrow \\ H_k(\{g = \pm 1\} \cup X_{12}^+, \{g = -1\} \cup X_{12}^+) \\ \uparrow \\ H_{k+1}(X_1^+, \{g = \pm 1\} \cup X_{12}^+) \\ \downarrow \\ H_{k+1}(X, X_1^- \cup X_2) \\ \uparrow \\ H_{k+1}(X, F) \end{array}$$

All mappings are induced by inclusion, except for the second one which is the boundary operator of the long exact homology sequence of the triple

$$(X_1^+, \{g = \pm 1\} \cup X_{12}^+, \{g = -1\} \cup X_{12}^+).$$

All arrows are isomorphisms: the second and the last arrow are isomorphisms because $\{g = -1\} \cup X_{12}^+$ and F are deformation retracts of X_1^+ resp. $X_1^- \cup X_2$, the first and the third one by excision.

b) If we compose with $\partial : H_{k+1}(X, F) \to H_k(F)$ we obtain (essentially) the right hand side of the following commutative diagram:

$$\begin{array}{ccc} H_k(F, F') & & \\ \downarrow & & \\ H_k(\{0\} \times F, \{0\} \times F') & \searrow & \\ \downarrow & & \\ H_k((\partial I \times F) \cup (I \times F'), (\{1\} \times F) \cup (I \times F')) & \to & H_k(\{g = \pm 1\} \cup X_{12}^+, \{g = -1\} \cup X_{12}^+) \\ \uparrow & & \uparrow \\ H_{k+1}(I \times F, (\partial I \times F) \cup (I \times F')) & \to & H_{k+1}(X_1^+, \{g = \pm 1\} \cup X_{12}^+) \\ \downarrow & & \downarrow \\ H_{k+1}(J \times F, (\{0\} \cup [1,2]) \times F) \cup (J \times F')) & \to & H_{k+1}(X, X_1^- \cup X_2) \\ \uparrow & & \uparrow \\ H_{k+1}(J \times F, (\partial J \times F) \cup (J \times F')) & \to & H_{k+1}(X, F \cup X_2) \\ \downarrow & & \downarrow \\ H_k(\partial J \times F) \cup (J \times F')) & \to & H_k(F \cup X_2) \\ & & \uparrow\simeq \\ & & H_k(F) \end{array}$$

The horizontal arrows are induced by $(t, x) \mapsto \phi(t, x) = \phi_t(x)$, where ϕ_t is chosen according to the definition of a geometric homotopy monodromy but using J instead of I.

Now let us look at the left vertical arrows, without the last one: The first one is induced by $x \mapsto (0,x)$, the third is the boundary operator of the long exact homology sequence to the triple $(I \times F, (\partial I \times F) \cup (I \times F'), (\{1\} \times F) \cup (I \times F')$, the other arrows are induced by the inclusion. The third arrow is bijective because $(\{1\} \times F) \cup (I \times F')$ is a deformation retract of $I \times F$. The other arrows are obviously bijective, too.

If we walk from the upper left corner to the right, then down and compose finally with the inverse of $H_k(F) \to H_k(F \cup X_2)$ we obtain $\partial \circ w_k$.

Now we walk down on the left:

$$
\begin{array}{ccc}
[c] & \in & H_k(F, F') \\
\downarrow & & \downarrow \\
[0 \times c] & \in & H_k(\{0\} \times F, \{0\} \times F') \\
\downarrow & & \downarrow \\
-[\partial\varepsilon \times c] & \in & H_k((\partial I \times F) \cup (I \times F'), (\{1\} \times F) \cup (I \times F')) \\
\uparrow & & \uparrow \\
-[\varepsilon \times c] & \in & H_{k+1}(I \times F, (\partial I \times F) \cup (I \times F')) \\
\downarrow & & \downarrow \\
-[\varepsilon \times c] & \in & H_{k+1}(J \times F, (\{0\} \cup [1,2]) \times F) \cup (J \times F')) \\
\uparrow & & \uparrow \\
-[\varepsilon' \times c] & \in & H_{k+1}(J \times F, (\partial J \times F) \cup (J \times F')) \\
\downarrow & & \downarrow \\
-[\partial\varepsilon' \times c - \varepsilon' \times \partial c] & \in & H_k((\partial J \times F) \cup (J \times F'))
\end{array}
$$

Again, ε and ε' represent the canonical generators go $H_1(I, \partial I)$ resp. $H_1(J, \partial J)$. If we go to the right we must apply the homomorphism ϕ_* induced by ϕ.

Now we have a commutative diagram:

$$
\begin{array}{ccc}
H_k(\partial J \times F) & \to & H_k(F) \\
\downarrow & & \downarrow \\
H_k((\partial J \times F) \cup (J \times F')) & \to & H_k(F \cup X_2),
\end{array}
$$

where the horizontal mappings yield $-[\partial\varepsilon' \times c] \mapsto [c - h_*(c)]$. But in the lower line we can replace $[\partial\varepsilon' \times c]$ by $[\partial\varepsilon' \times c + \varepsilon' \times \partial c]$ because $\phi_*([\varepsilon' \times \partial c]) = 0$, i.e. $\phi_*(\varepsilon' \times \partial c)$ is a boundary:

First, ∂c is a $(k-1)$-cycle of ∂F.

By definition of a geometric homotopy monodromy we have a continuous mapping $\chi : \mathbb{D}^2 \times F' \to \mathbb{D}^2$ such that the restrictions of the mappings ϕ and $\chi \circ (\rho \times id)$ to $[0,2] \times F'$ coincide, where $\rho : [0,2] \to S^1$: $t \mapsto e^{it}$.

So we have $\phi_*(\varepsilon' \times \partial c) = \chi_*(\rho_*(\varepsilon') \times \partial c)$.

Now $\rho_*(\varepsilon')$ is a 1-cycle of $\mathbb{D}^2$, hence a boundary, because $\mathbb{D}^2$ is contractible. So there is a 2-chain κ of $\mathbb{D}^2$ such that $\rho_*(\varepsilon') = \partial\kappa$. Then $\phi_*(\rho_*(\varepsilon') \times \partial c) = \chi_*((\partial\kappa) \times \partial c)$.
But $\partial(\kappa \times \partial c) = (\partial\kappa) \times \partial c + \kappa \times \partial\partial c$; hence $\chi_*(\partial\kappa \times \partial c) = \partial\chi_*(\kappa \times \partial c)$ is a boundary. This proves our statement. (Acknowledgment: It was C. Eyral who proposed such a type of reasoning to me.)

c) This follows from a), b) and the exact homology sequence for the pair (F, F'). $\square$

Remark: From a),b) we obtain a new description of $\mathrm{var}_k(g)$.

Corollary 6.3: Suppose that $H_k(X, F) = 0$ for $k \leq n-1$. Then $H_k(F) \to H_k(X)$ is bijective for $k \leq n-2$, surjective for $k = n-1$, which means that $\mathrm{var}_k(g) = 0$ for $k \leq n-2$, and $\mathrm{Im}\,\mathrm{var}_{n-1}(g)$ is the kernel of $H_{n-1}(F) \to H_{n-1}(X)$.

Proof of Theorem 0.1: Note that X_ϵ admits the deformation retract $\partial(X_\epsilon \cap \{|f| \leq \alpha\})$ which is a stratified open book.

In fact, let M be a suitable open neighborhood of $\partial(\bar{X}_\epsilon \cap \{|f| \leq \alpha\})$ in $\mathbb{C}^N$. Choose a stratification of M such that $A := \bar{X}_\epsilon \cap M, B := Z \cap A, A' := \partial \bar{X}_\epsilon$ and the intersections of these sets are unions of strata. Then we may achieve the conditions for a stratified open book for $\partial(X_\epsilon \cap \{|f| \leq \alpha\})$, with $X^1 := X_\epsilon \cap \{|f| = \alpha\}$, $X^2 := \partial X_\epsilon \cap \{|f| \leq \alpha\}$, $g := \frac{f}{\alpha}$, and apply Theorem 6.1. Of course, $\partial(X_\epsilon \cap \{|f| \leq \alpha\})$ is a deformation retract of X_ϵ. $\square$

c) Homotopy groups

This time there is also a statement about homotopy groups: We use base points $\cdot$ in F' and $* := (0, \cdot)$.

Theorem 6.4: a) Let $k \geq 1$. Then there is a canonical map $W_k : \pi_k(F, F', \cdot) \to \pi_{k+1}(X, F, \cdot)$ which is a homomorphism if $k \geq 2$.

b) For $k \geq 1, \partial \circ W_k = \kappa \circ \mathrm{var}_k(g)$, κ being the inversion mapping $c \mapsto c^{-1}$.

Proof: a) We define $W_k : \pi_k(F, F', \cdot) \to \pi_{k+1}(X, F, \cdot)$, $k \geq 1$, similarly as in the proof of Theorem 6.2, by composition:

$$\begin{array}{c}
\pi_k(F, F', \cdot) \\
\downarrow \\
\pi_k(\{g = \pm 1\} \cup X_{12}^+, \{g = -1\} \cup X_{12}^+, \cdot) \\
\uparrow \\
\pi_{k+1}(X_1^+, \{g = \pm 1\} \cup X_{12}^+, \cdot) \\
\downarrow \\
\pi_{k+1}(X, X_1^- \cup X_2, \cdot) \\
\uparrow \\
\pi_{k+1}(X, F, \cdot)
\end{array}$$

All mappings are induced by the inclusion, except for the second one which is a boundary operator. And note that the second and the last arrow are bijective: The second one because $\{g = -1\} \cup X_{12}^+$ is a deformation retract of X_1^+, see Lemma 2.2, and the last one because F is a deformation retract of $X_1^- \cup X_2$. So W_k is well-defined, and is a homomorphism if $k \geq 2$.

b) The two upper arrows of the preceding diagram form the right hand side of the following commutative diagram:

$$\begin{array}{ccc}
\pi_k(F, F' \cdot) & & \\
\downarrow & & \\
\pi_k(\{0\} \times F, \{0\} \times F', *) & \searrow & \\
\downarrow & & \\
\pi_k((\partial I \times F) \cup (I \times F'), (I \times F') \cup (\{1\} \times F), *) & \to & \pi_k(\{g = \pm 1\} \cup X_{12}^+, \{g = -1\} \cup X_{12}^+, \cdot) \\
\uparrow & & \uparrow \\
\pi_{k+1}(I \times F, (\partial I \times F) \cup (I \times F'), *) & \to & \pi_{k+1}(X_1^+, \{g = \pm 1\} \cup X_{12}^+, \cdot)
\end{array}$$

On the left, the first vertical arrow is induced by $x \mapsto (0, x)$, the second one by inclusion, the third one is a boundary operator; the horizontal arrows are induced by $(t, x) \mapsto \phi_t(x)$. In the definition of W_k we can therefore replace the first two arrows by the mapping obtained by first going down and finally to the right.

If we continue with the last arrows of the diagram used for a), we end up with the left vertical of the following commutative diagram:

$$\begin{array}{ccc}
\pi_k(F, F', \cdot) & & \\
\downarrow & & \\
\pi_k(\{0\} \times F, \{0\} \times F', *) & & \\
\downarrow & & \\
\pi_k((\partial I) \times F) \cup (I \times F'), (I \times F') \cup (\{1\} \times F), *) & & \\
\uparrow & & \\
\pi_{k+1}(I \times F, (\partial I \times F) \cup (I \times F'), *) & \rightarrow & \pi_k((\partial I) \times F) \cup (I \times F'), *) \\
\downarrow & & \downarrow \\
\pi_{k+1}(X_1^+, \{g = \pm 1\} \cup X_{12}^+, \cdot) & \rightarrow & \pi_k(\{g = \pm 1\} \cup X_{12}^+, \cdot) \\
\downarrow & & \downarrow \\
\pi_{k+1}(X, X_1^- \cup X_2, \cdot) & \rightarrow & \pi_k(X_1^- \cup X_2, \cdot) \\
\uparrow & & \uparrow \\
\pi_{k+1}(X, F \cup X_2, \cdot) & \rightarrow & \pi_k(F \cup X_2, \cdot) \\
\uparrow & & \uparrow \\
\pi_{k+1}(X, F, \cdot) & \rightarrow & \pi_k(F, \cdot)
\end{array}$$

If we walk down on the left and finally to the right we obtain $\partial \circ W_k$, if we walk down on the right instead we obtain $\kappa \circ \mathrm{var}_k(g)$; we see this as follows:

Let us compute the image of $[\beta] \in \pi_k(F, F' \cdot)$.

(i) The image in $\pi_k(\{0\} \times F, \{0\} \times F', *)$ as well as in $\pi_k((\partial I \times F) \cup (I \times F'), (I \times F') \cup (\{1\} \times F), *)$ is represented by $(t_1, \ldots, t_k) \mapsto (0, \beta(t_1, \ldots, t_k))$.

(ii) The inverse image in $\pi_{k+1}(I \times F, (\partial I \times F) \cup (I \times F'), *)$ is represented by $(t_1, \ldots, t_{k+1})$ maps to

$(4t_k(1 - t_{k+1}), \beta(t_1, \ldots, t_{k-1}, 1)), 0 \le t_k \le \frac{1}{4}$,
$(1 - t_{k+1}, \beta(t_1, \ldots, t_{k-1}, 4t_k(t_{k+1} - 1) + 2 - t_{k+1})), \frac{1}{4} \le t_k \le \frac{1}{2}$,
$((4t_{k+1} - 4)t_k + 3 - 3t_{k+1}, \beta(t_1, \ldots, t_{k-1}, t_{k+1})), \frac{1}{2} \le t_k \le \frac{3}{4}$,
$(0, \beta(t_1, \ldots, t_{k-1}, (4 - 4t_{k+1})t_k + 4t_{k+1} - 3)), \frac{3}{4} \le t_k \le 1$.

In fact, the image of this element in $\pi_k((\partial I \times F) \cup (I \times F'), *)$, hence in

$$\pi_k((\partial I \times F) \cup (I \times F'), (I \times F') \cup (\{1\} \times F), *),$$

is represented by
$(t_1, \ldots, t_k) \mapsto \gamma(t_1, \ldots, t_k) :=$
$(4t_k, \beta(t_1, \ldots, t_{k-1}, 1)), 0 \le t_k \le \frac{1}{4}$,
$(1, \beta(t_1, \ldots, t_{k-1}, 2 - 4t_k)), \frac{1}{4} \le t_k \le \frac{1}{2}$,
$(3 - 4t_k, \beta(t_1, \ldots, t_{k-1}, 0)), \frac{1}{2} \le t_k \le \frac{3}{4}$,
$(0, \beta(t_1, \ldots, t_{k-1}, -3 + 4t_k)), \frac{3}{4} \le t_k \le 1$.

And this k-cell is homotopic to $(t_1, \ldots, t_k) \mapsto (0, \beta(t_1, \ldots, t_k))$, using the homotopy

$$(t_1, \ldots, t_k, s) \mapsto \gamma(t_1, \ldots, t_{k-1}, -\frac{3}{4}t_k s + t_k + \frac{3}{4}s).$$

(iii) As we have seen, we end up with an element in $\pi_k((\partial I \times F) \cup (I \times F'), *)$ which we have described explicitly by γ. The image in $\pi_k(\{g = \pm 1\} \cup X_{12}^+, \cdot)$ is represented by $(t_1, \ldots, t_k)$ maps to

$\phi(4t_k, \beta(t_1, \ldots, t_{k-1}, 1)), 0 \le t_k \le \frac{1}{4}$,
$\phi(1, \beta(t_1, \ldots, t_{k-1}, 2 - 4t_k)), \frac{1}{4} \le t_k \le \frac{1}{2}$,
$\phi(3 - 4t_k, \beta(t_1, \ldots, t_{k-1}, 0)), \frac{1}{2} \le t_k \le \frac{3}{4}$,
$\phi(0, \beta(t_1, \ldots, t_{k-1}, -3 + 4t_k)), \frac{3}{4} \le t_k \le 1$.

(iv) The image of this element in $\pi_k(X_1^- \cup X_2, \cdot)$ coincides with the image of the element of $\pi_k(F \cup X_2, \cdot)$ which is represented by $(t_1, \ldots, t_k)$ maps to

$\phi(8t_k, \beta(t_1, \ldots, t_{k-1}, 1)), 0 \le t_k \le \frac{1}{4}$,

$\phi(2, \beta(t_1, \ldots, t_{k-1}, 2-4t_k)), \frac{1}{4} \leq t_k \leq \frac{1}{2}$,
$\phi(6-8t_k, \beta(t_1, \ldots, t_{k-1}, 0)), \frac{1}{2} \leq t_k \leq \frac{3}{4}$,
$\phi(0, \beta(t_1, \ldots, t_{k-1}, -3+4t_k)), \frac{3}{4} \leq t_k \leq 1$.
Here use the homotopy $(t_1, \ldots, t_k, s) :=$
$\phi(4t_k(1+s), \beta(t_1, \ldots, t_{k-1}, 1)), 0 \leq t_k \leq \frac{1}{4}$,
$\phi(1+s, \beta(t_1, \ldots, t_{k-1}, 2-4t_k)), \frac{1}{4} \leq t_k \leq \frac{1}{2}$,
$\phi((3-4t_k)(1+s), \beta(t_1, \ldots, t_{k-1}, 0)), \frac{1}{2} \leq t_k \leq \frac{3}{4}$,
$\phi(0, \beta(t_1, \ldots, t_{k-1}, -3+4t_k)), \frac{3}{4} \leq t_k \leq 1$.
(v) It is also the image of the element of $\pi_k(F, \cdot)$ represented by $(t_1, \ldots, t_k)$ maps to
$h(\beta(t_1, \ldots, t_{k-1}, 1)), 0 \leq t_k \leq \frac{1}{4}$,
$h(\beta(t_1, \ldots, t_{k-1}, 2-4t_k), \frac{1}{4} \leq t_k \leq \frac{1}{2}$,
$\beta(t_1, \ldots, t_{k-1}, 0), \frac{1}{2} \leq t_k \leq \frac{3}{4}$,
$\beta(t_1, \ldots, t_{k-1}, -3+4t_k)), \frac{3}{4} \leq t_k \leq 1$.
Here, recall that $\phi(t, x) = \chi(e^{\pi i t}, x), x \in F'$, and use the homotopy $(t_1, \ldots, t_k, s)$ maps to
$\chi(s+(1-s)e^{8\pi i t_k}, \beta(t_1, \ldots, t_{k-1}, 1)), 0 \leq t_k \leq \frac{1}{4}$,
$\beta(t_1, \ldots, t_{k-1}, 2-4t_k), \frac{1}{4} \leq t_k \leq \frac{1}{2}$,
$\chi(s+(1-s)e^{-8\pi i t_k}), \beta(t_1, \ldots, t_{k-1}, 0)), \frac{1}{2} \leq t_k \leq \frac{3}{4}$,
$h(\beta(t_1, \ldots, t_{k-1}, -3+4t_k)), \frac{3}{4} \leq t_k \leq 1$.
(vi) Finally, we have a different representative of the element of $\pi_k(F, \cdot)$ just obtained: $(t_1, \ldots, t_k)$ maps to
$h(\beta(t_1, \ldots, t_{k-1}, 1-2t_k)), 0 \leq t_k \leq \frac{1}{2}$,
$\beta(t_1, \ldots, t_{k-1}, 2t_k-1)), \frac{1}{2} \leq t_k \leq 1$,
i.e. $(h \circ \beta) \perp \beta$ which represents $\kappa(\mathrm{var}_k(g)([\beta]))$, by the following lemma.

Here, use the homotopy $(t_1, \ldots, t_k, s)$ maps to
$h(\beta(t_1, \ldots, t_{k-1}, 1-2st_k)), 0 \leq t_k \leq \frac{1}{4}$,
$h(\beta(t_1, \ldots, t_{k-1}, (1-2t_k)(2-s)), \frac{1}{4} \leq t_k \leq \frac{1}{2}$,
$\beta(t_1, \ldots, t_{k-1}, s(2t_k-1)), \frac{1}{2} \leq t_k \leq \frac{3}{4}$,
$\beta(t_1, \ldots, t_{k-1}, 4t_k - 2st_k - 3 + 2s), \frac{3}{4} \leq t_k \leq 1$.
Altogether b) is proved. $\square$

It remains to show the following auxiliary lemma:

Lemma 6.5: Let (X, Y) be a pair of topological spaces, $\cdot \in Y$, $a, b \in \pi_k(X, Y, \cdot), k \geq 1$, represented by α resp. β, $\alpha|\{0\} \times I^{k-1} = \beta|\{0\} \times I^{k-1}$.
a) $[\alpha \perp \beta] = a^{-1}b$ if $Y = \{\cdot\}$,
b) $[\alpha \perp \alpha] = 1$,
c) $[\beta \perp \alpha] = [\alpha \perp \beta]^{-1}$.

Proof:
a) This is obvious if $k = 1$. For $k \geq 2$ note that we may use the coordinate t_k instead of t_1 in order to get the (abelian) group structure on $\pi_k(X, \cdot)$:

Let $\gamma_j : I^k \to X, j = 1, 2$, represent elements of $\pi_k(X, \cdot)$,
$\gamma_3(t_1, \ldots, t_k) := \gamma_1(t_1, \ldots, t_{k-1}, 2t_k)$ for $t_k \leq \frac{1}{2}$,
$\gamma_3(t_1, \ldots, t_k) := \gamma_2(t_1, \ldots, t_{k-1}, 2t_k - 1)$ for $t_k \geq \frac{1}{2}$.
We must show that $[\gamma_1] + [\gamma_2] = [\gamma_3]$.

Let $\gamma'_j(t_1, \ldots, t_k) := \gamma_j(t_k, t_2, \ldots, t_{k-1}, t_1)$. Then, by definition, $[\gamma'_1] + [\gamma'_2] = [\gamma'_3]$, and $[\gamma_j] = -[\gamma'_j]$ (here it is sufficient to look at the case $X = I^k/\partial I^k$), so indeed $[\gamma_1] + [\gamma_2] = [\gamma_3]$.
b) $\alpha \perp \alpha$ is defined by $(\ldots, t_k)$ maps to

$\alpha(\dots, 1-2t_k)$ if $0 \le t_k \le \frac{1}{2}$,
$\alpha(\dots, 2t_k - 1)$ if $\frac{1}{2} \le t_k \le 1$.
Here $\dots$ stands for $t_1, \dots, t_{k-1}$.

Now use the homotopy to a constant map: $(\dots, t_k, s)$ maps to
$\alpha(\dots, 1-2(1-s)t_k), t_k \le \frac{1}{2}$
$\alpha(\dots, -2(1-s)(1-t_k)+1), t_k \ge \frac{1}{2}$.

c) Using the proof of a), we see that $(a \perp b)(b \perp a)$ is represented by the mapping: $(t_1, \dots, t_k)$ maps to
$\alpha(\dots, 1-4t_k), 0 \le t_k \le \frac{1}{4}$,
$\beta(\dots, 4t_k - 1), \frac{1}{4} \le t_k \le \frac{1}{2}$,
$\beta(\dots, 3-4t_k), \frac{1}{2} \le t_k \le \frac{3}{4}$,
$\alpha(\dots, 4t_k - 3), \frac{3}{4} \le 1$.

Now this mapping is homotopic to the mapping $(\dots, t_k)$ maps to
$\alpha(\dots, 1-2t_k), 0 \le t_k \le \frac{1}{2}$,
$\alpha(\dots, 2t_k - 1), \frac{1}{2} \le t_k \le 1$,

hence by b) to a constant map: use the homotopy $(t_1, \dots, t_k, s)$ maps to
$\alpha(\dots, 1 - \frac{4t_k}{1+s})$, $0 \le t_k \le \frac{1}{4}(1+s)$,
$\beta(\dots, \frac{4t_k}{1+s} - 1)$, $\frac{1}{4}(1+s) \le t_k \le \frac{1}{2}$,
$\beta(\dots, \frac{4-4t_k}{1+s} - 1)$, $\frac{1}{2} \le t_k \le \frac{3-s}{4}$,
$\alpha(\dots, \frac{4t_k - 4}{1+s} + 1)$, $\frac{3-s}{4} \le t_k \le 1$.

Remark: Theorem 6.4 yields an alternative definition of $\mathrm{var}_k(g)$. It shows also that $\mathrm{var}_k(g)$ is a homomorphism if $k \ge 2$. And note that, contrary to the case of homology, we cannot consider the case $F' = \emptyset$ because we need a base point in F'.

Theorem 6.6: Assume that $n \ge 2$ and that (F, F') is $(n-2)$-connected.

a) (X, F) is $(n-1)$-connected, and $W_{n-1} : \pi_{n-1}(F, F', \cdot) \to \pi_{n-1}(X, F, \cdot)$ is bijective for $n \ge 4$, surjective if $n = 3$ and generates the target group if $n = 2$.

b) $\pi_{n-1}(F, \cdot) \to \pi_{n-1}(X, \cdot)$ is surjective, the kernel being the image of $\mathrm{var}_{n-1}(g)$ if $n \ge 3$ resp. generated by the image of $\mathrm{var}_{n-1}(g)$ if $n = 2$.

Proof: a) Look at the left column of the last diagram in the proof of Theorem 6.4. The composition of the first three arrows is $\pi_k(F, F', \cdot) \to \pi_{k+1}((I, \partial I) \times (F, F', \star)$, and this mapping is bijective if $k < n-1$, surjective if $k = n-1 \ge 2$, and generates the target group if $k = n-1 = 1$ - use Lemma 2.3. And the arrow

$$\pi_{k+1}(X_1^+, \{g = \pm 1\} \cup X_{12}^+, \cdot) \to \pi_{k+1}(X, X_1^- \cup X_2, \cdot) \to \pi_k(X_1^- \cup X_2, \cdot)$$

is bijective if $k < n-1$ and surjective if $k = n-1$. This follows from the Blakers-Massey theorem.

b) This follows from a) and Theorem 6.4. ☐

Proof of Theorem 0.2: This follows from Theorem 6.6, proceeding as in the proof of Theorem 0.1 above. ☐

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