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A NEW CODING THEORY, FOR NORMAL SURFACES, AND ADE SINGULARITIES, I

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Dedicated to Wolfgang Ebeling, a gentle spirit and a deep mathematician, in memoriam.

Abstract. In this article we extend the theory of the binary codes (the strict code $\mathcal{K}$ and the extended code $\mathcal{K}'$), associated to a projective nodal surface, to a coding theory for normal surfaces, with special consideration of the surfaces with ADE (Rational Double Points) singularities.

We define a new theory of generalized labeled codes, establish in the geometric case basic restrictions for the weights of these codes, and some basic inequalities. A crucial method that we establish is the extension of the concept of 'code shortening' to the case of generalized codes: this is the algebraic counterpart of the geometric notion of a partial smoothing of the singular points, and leads to the concept of ancestors, which we illustrate through several examples.

1. Introduction

The interplay between coding theory and lattice theory is a central thread in mathematics, as witnessed for instance by Wolfgang Ebeling's works [Eb94], [Eb87], among many other ones (as [Con-Slo88]).

In a sense, codes can be viewed as a finite approximation to lattices, which are free abelian groups endowed with a $\mathbb{Z}$-valued symmetric bilinear form. In geometry, codes may be used to describe the saturation of a given sublattice $\mathcal{L}$ inside a lattice Λ, which usually for us is the integral second cohomology group $H^2(S,\mathbb{Z})$ of an algebraic surface (or of an oriented compact 4-manifold).

The advantage of coding theory over a finite field is of being essentially linear algebra, and as such it has stunning connections with the classification of configurations of sets of points in finite projective spaces.

For geometric applications, however, one feels the need to enlarge the realm of coding theory, replacing codes over finite fields by more general embeddings of a finite abelian group inside a direct sum of labeled abelian groups. The underlying geometric idea is to view the first homology group H_1 of the smooth part of a normal complete algebraic surface as the quotient of a direct sum of local homology groups, labeled by a class of surface singularities. For ADE singularities, the class is just a holomorphic isomorphism class, while, in the most general case, we have an equivalence class including the relations generated by equisingular deformation.

Cumbersome as it may look at the onset, the key feature, as in [Cat22], is to relate partial smoothing of singularities to an operation in coding theory, called shortening. Whereas, for codes given by a vector space $\mathcal{K}$ of functions defined on a set Σ, shortening is a very simple concept, amounting to taking the functions with support contained in a subset $\mathcal{N} \subset \Sigma$, in our general theory the concept is more subtle, due to its direct geometric meaning related to partial smoothings, which may produce several singular points out of a complicated singular point.

But the main idea is that, once a singular surface is unobstructed (this means that we may independently achieve all the local deformations of the singularities) one can define the concept

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of ancestors, which are roughly speaking the singular surfaces with a maximal set of singularities, and from which one can determine all other classes of singular surfaces (see Definition 23).

We can then view the stratification of the space of singular surfaces of a fixed degree d in $\mathbb{P}^3 := \mathbb{P}^3_{\mathbb{C}}$ with ADE singularities, especially for $d \leq 4$, as the collection of partial smoothings of a finite set of ancestors (the case $d = 3$ is illustrated in the final section).

To make the above discussion more concrete (otherwise it would be hardly understandable), I want now to illustrate the above idea describing the main motivation for the present work: the success of binary coding theory for the study of nodal surfaces, and in particular the following theorem, proven in [Cat22], which determined the irreducible components of the space of nodal quartic surfaces in $\mathbb{P}^3$ (the complete classification, in spite of a lot of classical work, [Rohn86], [Roh87], [Jes16], was still open).

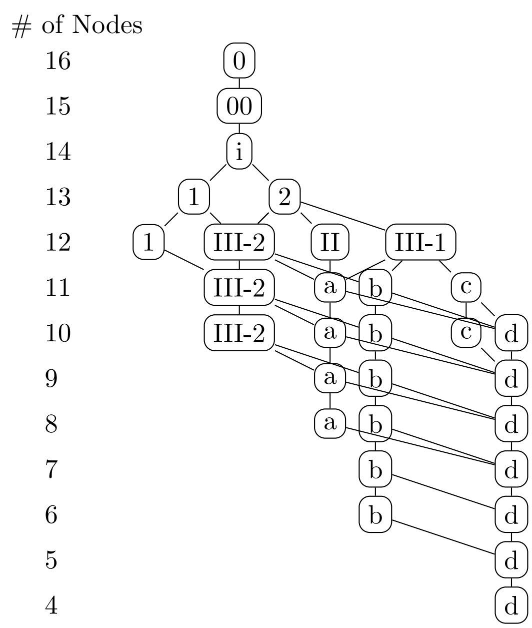
Theorem 1. *(I) The subset of the Nodal Severi variety $\mathcal{F}(4, \nu)$ corresponding to the nodal quartic surfaces with a fixed number ν of nodes (it must be $\nu \leq 16$) and fixed binary code $\mathcal{K}'$ is smooth and connected, hence irreducible.*

(II) The possible extended codes $\mathcal{K}'$ appearing are exactly all the shortenings of the extended Kummer code $\mathcal{K}'_{Kum}$, and are listed as: cases (0), (00), (i), (1), (2) , (II),(III-1), (III-2), (a), (b), (c), (d); hence we have 11 nontrivial codes in their minimal code-embedding.

(III) For $\nu \in \{0, 1, 2, 3, 4, 5, 14, 15, 16\}$, $\mathcal{F}(4, \nu)$ has exactly one irreducible component, and exactly two irreducible components for $\nu = 6, 7, 13$, exactly three irreducible components for $\nu = 8, 9$, exactly four irreducible components for $\nu = 11, 12$, exactly five irreducible components for $\nu = 10$.

(III) A component of $\mathcal{F}(4, \nu)$ is in the closure of another component of $\mathcal{F}(4, \nu')$ if and only if $\nu \geq \nu'$ and the second code is a shortening of the first. Hence the stratification of $\mathcal{F}(4, \nu)$ is made of 34 strata, where each stratum is in the boundary of another according to the above Genealogy tree = Kummer nontrivial genealogy for nodal quartic surfaces.

Kummer genealogy tree of nodal quartic surfaces.



The above main theorem for quartics, Theorem 1, is based, as already said, on binary coding theory, but the main new idea, beyond the use of the Torelli theorem for K3 surfaces (see [PS-Shaf71]) and Nikulin's theorems on primitive embeddings of lattices in the K3 lattice [Nik79], is really the relation between unobstructedness and shortenings (Theorem 21).

Nikulin' s theory of primitive embeddings of lattices offers criteria for existence and unicity of such embeddings of lattices, and these results are very useful for the wider class of nodal polarized K3 surfaces.

If we fix a 'potential' K3 code $\mathcal{K}'$, and $\mathcal{L}$ is the lattice generated by the (-2)-curves corresponding to the nodes, $\mathcal{K}'$ determines a lattice $\mathcal{L}'$ such that $\mathcal{L}'/\mathcal{L} \cong \mathcal{K}'$.

If the code is geometrically realized by a nodal surface X, then $\mathcal{L}'$ shall be the saturation $\mathcal{L}^{sat} := \Lambda \cap \mathbb{Q}\mathcal{L}$ of $\mathcal{L}$ inside the cohomology Λ of the desingularization of X, and shall have a primitive embedding into Λ. It turns out, using the results of Nikulin on integral quadratic forms, that there is at most one primitive embedding of $\mathcal{L}'$ inside Λ, and one can then decide whether such a primitive embedding exists. If it exists, we conclude that the potential code is effectively realized by a nodal surface X.

In this way the theorem for quartics extends also to all nodal K3 surfaces as follows (this is an abridged version, see [Cat22] for more details).

Theorem 2. *The connected components of the Nodal Severi variety $\mathcal{F}_{K3}(d,\nu)$ of nodal K3' s of degree d (d is any even number) and with ν nodes are in bijection with the isomorphism classes of their extended codes $\mathcal{K}'$ (equivalently, of the pair of codes $\mathcal{K} \subset \mathcal{K}'' \subset \mathbb{F}_2^\nu$).*

Apart from some sporadic cases, which are however obtained via the projection from a node of a nodal K3 surface, we have several main stream cases, that is, we get all the codes $\mathcal{K}''$ as shortenings from five fixed ancestors:

- *i) If $d \equiv 0 \pmod 8$ of the K8 code (generated by the strict Kummer code $\mathcal{K}_{Kum}$ and by the characteristic function of an affine plane π).*
- *ii) If $d \equiv 2 \pmod 8$ of the code generated by the linear functions on $\mathbb{F}_2^4$ and by the quadratic function $x_1y_1 + x_2y_2 + 1$.*
- *iii) If $d \equiv 4 \pmod 8$ and the weight $t = 2$ does not occur, of the extended Kummer code $\mathcal{K}''_{Kum}$.*
- *iv) If $d \equiv 4 \pmod 8$ and the weight $t = 2$ occurs, we get all the shortenings of the special code $\mathcal{K}'$ occurring for $\nu = 14$, $k' = 4$.*
- *v) If $d \equiv 6 \pmod 8$ we get all the shortenings $\mathcal{K}'$ of the two codes occurring for $\nu = 15$: the first $\mathcal{K}'$ is the strict Kummer code, in the second $\mathcal{K}'$ there is a weight 4 vector.*

The above theorems, yielding five genealogy trees, suggest the possibility (dream?) of understanding the equisingular components of the space of polarized singular K3 surfaces, whose singularities are Rational Double Points = ADE Singularities, in terms of suitable ancestors, which should replace the role taken by the 16-nodal Kummer quartic surfaces for the case of nodal quartics. This should work for quartic surfaces with ADE singularities without the need to consider sporadic cases. In this direction, we plan in the sequel to this paper to see whether it is possible to combine the methods of this article with other methods and results ([Ura87], [Ura90] and [GPP24]), where some ADE configurations on quartic surfaces and on unpolarized K3 surfaces were investigated.

An ancestor is a component which is closed, hence with a maximal set of singularities, and each ancestor determines a genealogy tree of components which have the ancestor in their closure.

Since in the case of nodal surfaces the incidence relation between components (the former being contained in the closure of the latter) is determined by the property that the associated binary codes are the latter a shortening of the former, our purpose in this article is to describe a generalized coding theory achieving this purpose for surfaces with ADE singularities. And our investigation is directed towards all surfaces with ADE singularities, also those of higher degree.

Our first and main goal in this article will then be to introduce a new coding theory for normal surfaces with singularities of a fixed type, describing in greater generality the concepts we use.

We shall restrict later on to the case of codes associated to surfaces with ADE singularities, because these singularities are amenable to an easier treatment: here the geometric ideas used to describe a more restricted class of shortenings, which we call geometrically driven shortenings, lead easily to a complete description and classification.

We hope that this first general algebraic step may be found interesting and useful also for other applications than the geometric ones which motivated us. [1]

We illustrate now the contents of the paper and some of our main results.

We begin by recalling some general results for a normal complex space with isolated singularities, implying that under suitable assumptions the first homology group of the smooth part is a quotient of the direct sum of the local homology groups at the singular points, see Theorem 3 and Proposition 4.

Then we focus on the local fundamental groups and local homology groups of ADE-singularities, which are going to be crucial for our results.

We describe then, as a warm up, the general picture of a normal crossing configuration of rational curves on a complex surface, and the concept of geometric shortening as the procedure of deleting some of the rational curves from the configuration. We illustrate the effect on the local first homology groups, exhibiting a few examples.

These examples are to be seen as a key grasp for understanding the general abstract definitions which are given later, of labeled generalized codes, and their shortenings.

We then concentrate on the case where the codes are finite, in this case we can define the concept of the weight and refined weight of a vector, the latter amounts to the number of times where some coordinates are equal and have the same labeling.

In the subsequent section we define a special class of shortenings, which we call geometrically driven, and show that for unobstructed surfaces these are the algebraic counterpart to a partial smoothing of some of the singular points (Theorem 21). We also establish, in Theorem 22, the unobstructedness of K3 surfaces with ADE singularities.

In section 5 we restrict to ADE-codes, and, using the primary decomposition, we prove several results (Propositions 28, 29, 32, 33) yielding restrictions for the weights of the primary components, namely for the weights of vectors of prime order, the most important cases being the orders $N = 2$, $N = 3$, $N = 5$, and, in general, the almost simple case.

The case $N = 3$ reminds us of the work of Barth and Rams [Barth98], [Barth00], [Ba-Ra05], [Ba-Ra07], who considered the case of surfaces with A_2 singularities, cusps in their terminology, which lead to codes $\mathcal{K}$ over the field $\mathbb{F}_3 = \mathbb{Z}/3$, which they investigated for low degree surfaces.

Later on, we extend to our general setting, using the binary component of the code, the inequality observed by Beauville in [Bea80].

We proceed then with the important step of establishing in this general context the theory of the extended code $\mathcal{K}'$, which keeps track of the polarization of the projective surface Y, and whose underlying group is the dual of the first homology of the complement, inside the smooth locus, of a smooth hyperplane section H.

The extended code, as already illustrated, plays the key role in the theory, since it allows to compute the saturation, inside the cohomology $\Lambda := H^2(S, \mathbb{Z})$ of the minimal resolution S of Y, of the sublattice $\mathcal{L}$ generated by the exceptional curves and by a smooth hyperplane section (Theorem 38).

We provide then some restrictions for the weights also for vectors of the extended code $\mathcal{K}'$.

In the last section we conclude showing some examples, describing for instance the surfaces which are the ancestors for the cubic surfaces in $\mathbb{P}^3$ with ADE singularities, their codes, and then indicating some ancestors for non-polarized K3 surfaces; we hope, in the sequel to this article, to find more substantial applications.

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[1] It was pointed out by Sascha Kurz that codes whose alphabet is a ring have been studied, see [BHOS98], [SAS17]; but our notion is quite more general: forgetting much of the information about $\mathcal{K}$, there remains a code

$$\mathrm{trace}(\mathcal{K}) \subset (G_1 \times G_2 \times \cdots \times G_n),$$

where the G_i's are finite abelian groups.

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2. Local to global homology of Normal compact complex spaces X with isolated singularities

Let X be a normal compact complex space of dimension n with isolated singularities, let

$$s := s(X) := |\operatorname{Sing}(X)|$$

be the number of its singular points, and denote further by Σ the singular set of X

$$\Sigma := \operatorname{Sing}(X) = \{P_1, \dots, P_s\}.$$

The topological structure of the smooth part $X^* := X \setminus \operatorname{Sing}(X)$ is an important invariant of X, which is not changed by equisingular deformations (see [Wahl98]); in particular the fundamental group $\pi_1(X^*)$ and its Abelianization, the first homology group $H_1(X^*, \mathbb{Z})$, are two such basic invariants.

Let S be a minimal resolution of singularities of X, and let H be a smooth divisor in X not passing through the singular points of X (in the case where X is projective, H shall be chosen to be a very ample divisor).

Recall the following result of [Cat84], pages 489-490, which shall be applied in two important cases:

- the case where S is a minimal resolution of singularities of X, and D is the reduced exceptional divisor E of $p : S \to X$;
- ditto with D the union of E with the inverse image of H.

Theorem 3. *Let S be a smooth compact complex manifold of complex dimension n, and let $D = D_1 \cup \dots \cup D_\nu$ be a reduced divisor, where each D_i is irreducible.*

Then the surjection $H_1(S \setminus D, \mathbb{Z}) \twoheadrightarrow H_1(S, \mathbb{Z})$ has kernel equal to the cokernel of

$$\rho : H^{2n-2}(S, \mathbb{Z}) \to H^{2n-2}(D, \mathbb{Z}) = \oplus_1^\nu \mathbb{Z}[D_i].$$

For $n = 2$, $\rho(L) = \sum_i (L \cdot D_i)[D_i]$.

Given the singular point P_i, let U_i be a neighbourhood of P_i obtained by intersecting X with a small Euclidean ball with centre P_i, set $U_i^* := U_i \setminus \{P_i\}$ and consider the local fundamental group

$$\pi_{1,loc}(X_{P_i}) := \pi_1(U_i^*) =: \Gamma_i,$$

whose abelianization is called the local homology group

$$H_1(P_i) := H_{1,loc}(X_{P_i}, \mathbb{Z}) := H_1(U_i^*, \mathbb{Z}) = (\Gamma_i)^{ab}.$$

We focus on the natural local to global homomorphism

$$gl : [\oplus_{P_i \in \operatorname{Sing}(X)} H_1(P_i) = H_{1,loc}(X_{p_i}, \mathbb{Z})] \to H_1(X^*, \mathbb{Z}) =: H_1. \tag{1}$$

This homomorphism gl will be particularly interesting in the case where the local homology groups are finite and also $H_1(X^*, \mathbb{Z})$ is finite.

The following proposition is proven in [Cat22].

Proposition 4. *(i) Let X be a normal compact complex space of dimension $n \geq 2$ with isolated singularities, with a normal crossing divisor resolution of singularities S having $H_1(S, \mathbb{Z}) = 0$.*

Then the local to global homomorphism gl (see (1)) is surjective, and its image $H_1 := H_1(X^*, \mathbb{Z})$ is isomorphic to coker(ρ)*, hence fits into an exact sequence*

$$H^{2n-2}(S, \mathbb{Z}) \to H^{2n-2}(D, \mathbb{Z}) = \oplus_1^\nu \mathbb{Z}[D_i] \to H_1 \to 0.$$

(ii) The same is true if X admits a smoothing (a flat deformation over the disk in $\mathbb{C}$ with general fibre a smooth manifold Y), such that the general fibre Y is simply connected, and the Milnor fibres are 1-connected. This happens for instance if X is a complete intersection in $\mathbb{P}^N$.

In this latter case we have the stronger result that $\pi_1(X^*)$ is normally generated by a quotient of the free product $\Gamma_1 * \Gamma_2 * \cdots * \Gamma_\nu$ of the local fundamental groups at the singular points.

Proof. Concerning (i), we apply Theorem 3 by slightly changing the notation. Namely, for each $P \in \mathrm{Sing}(X)$, we let E_P be the exceptional divisor, so that D is the disjoint union of the divisors E_P.

Then, as in [Cat84] loc. cit. $H_1(X^*, \mathbb{Z}) = H_1(S \setminus D, \mathbb{Z}) = H_1(S \setminus V, \mathbb{Z})$, where V is a tubular neighbourhood of D.

Since the kernel of the surjection $\pi_1(S \setminus D) \to \pi_1(S)$ is normally generated by small geometrical loops γ_i around the (irreducible) divisors D_i, it follows that $H_1(S \setminus D, \mathbb{Z})$ is generated by the homology classes $[\gamma_i]$ of these loops. These classes $[\gamma_i]$ come from $H_1(V_P \setminus E_P, \mathbb{Z}) = H_{1,loc}(X_P)$.

In turn, by Lefschetz duality, $H_1(S \setminus V, \mathbb{Z}) \cong H^{2n-1}(S \setminus V, \partial V, \mathbb{Z})$ which by excision equals $H^{2n-1}(S, \bar{V}, \mathbb{Z}) \cong H^{2n-1}(S, D, \mathbb{Z})$ which equals then the cokernel of

$$\rho : H^{2n-2}(S, \mathbb{Z}) \twoheadrightarrow H^{2n-2}(D, \mathbb{Z}).$$

For the proof of (ii), observe that X deforms to a smooth manifold Y, with $\pi_1(Y) = 1$; this is a consequence of Lefschetz' theorem, if X is a complete intersection of dimension $n \geq 2$.

Define F to be the union of the Milnor fibres F_i, together with a tree connecting the base point to each Milnor fibre with a respective segment (in the case where X is a complete intersection that $\pi_1(F) = 1$ follows by Hamm's extension [Hamm71] of Milnor's theorem [Mil68] stating that F_i is homotopically equivalent to a bouquet of spheres).

The first van Kampen 's theorem says that $1 = \pi_1(Y)$ is the quotient of the free product $\pi_1(Y^*) * \pi_1(F) = \pi_1(Y^*)$ by the subgroup normally generated by $\pi_1(F^*)$, where $F^* := F \cap Y^*$. Now, by construction, $\pi_1(F^*)$ is the free product $\Gamma_1 * \Gamma_2 * \cdots * \Gamma_\nu$, hence the desired assertion follows.

When we abelianize, we obtain a surjection of $\Gamma_1 * \Gamma_2 * \cdots * \Gamma_\nu$ onto $H_1(Y^*, \mathbb{Z})$, which therefore factors through a surjection

$$\Gamma_1^{ab} \times \Gamma_2^{ab} \times \cdots \times \Gamma_\nu^{ab} \twoheadrightarrow H_1(Y^*, \mathbb{Z}).$$

$\square$

2.1. **The case of normal surfaces.** Let us consider first the simplest possible case, where X is a nodal surface, that is, all singular points of X are **nodes**, hypersurface singularities of multiplicity two with nondegenerate Hessian (hence locally biholomorphic to the singularity $y_1 y_2 - y_3^2 = 0$ inside $\mathbb{C}^3$).

A node is the quotient $\mathbb{C}^2/(\pm 1)$, since, if u_1, u_2 are local coordinates on $\mathbb{C}^2$, the quotient is embedded by $y_i := u_i^2,\ i = 1, 2$ and by $y_3 := u_1 u_2$.

In particular, the local fundamental group is then $\mathbb{Z}/2$.

Example 5. *If X is a nodal surface in $\mathbb{P}^3$, with $\Sigma = \mathrm{Sing}(X) = \{P_1, \ldots, P_s\}$, then $H_1(X^*, \mathbb{Z}) \cong (\mathbb{Z}/2)^k$ for some $k \leq s$, since there is a natural surjection $\oplus_1^s (\mathbb{Z}/2) \twoheadrightarrow H_1(X^*, \mathbb{Z})$.*

In order to justify our forthcoming definitions of generalized codes let us look at the example of the binary codes associated to nodal surfaces.

Example 6. *Let X be a nodal surface in $\mathbb{P}^3$: then its binary code $\mathcal{K} \subset (\mathbb{Z}/2)^s$ is defined as the image of the injective map dual to the surjection*

$$\oplus_1^s (\mathbb{Z}/2) \twoheadrightarrow H_1(X^*, \mathbb{Z}).$$

Example 7. *Take more generally X to be a normal surface whose singularities are rational double points (ADE singularities), and whose minimal resolution S is simply connected. Since*

X deforms differentiably to $Y = S$, see for instance [Cat09], saying that S is simply connected is equivalent to saying that X has a simply connected smoothing.

In this case again gl is surjective.

Moreover, since S is simply connected, then

$$\Lambda := H^2(S, \mathbb{Z})$$

is a unimodular lattice, that is, a free abelian group endowed (by Poincaré duality) with a unimodular bilinear form.

Set $\mathcal{L} := \oplus_1^\nu \mathbb{Z} D_i$: since the intersection matrix $(D_i \cdot D_j)$ is negative definite, then $\mathcal{L} \hookrightarrow \Lambda$.

2.2. **Local homology groups of ADE-singularities, also called Rational Double Points and Kleinian Singularities.** Following for instance the notation of [Cat08], pages 71-74, we consider the Kleinian singularities

$$X_x = \mathbb{C}^2/G, \ G < SU(2, \mathbb{C}).$$

Let $G' < \mathbb{P}SU(2, \mathbb{C}) \cong SO(3)$ be the image of G, and let $\hat{G}$ be the full inverse image of G' through the exact sequence

$$1 \to \{\pm 1\} \to SU(2, \mathbb{C}) \to \mathbb{P}SU(2, \mathbb{C}) \to 1,$$

so that

$$1 \to \{\pm 1\} \to \hat{G} \to G' \to 1.$$

Clearly, $\pi_{1,x}(P) \cong G$, while $H_1(x) \cong (G)^{ab}$, according to the following table [2] ($H_1(x) \cong (G')^{ab}$ except for the case of A_n with n odd).

- For a singularity of type A_n, $G = H_1(x) = \mathbb{Z}/(n+1)\mathbb{Z}$.
- For a singularity of type D_{2m+1}, G is the binary dihedral group, G' is the dihedral group, but with a shift of index,
$$G = \hat{D}_{2m-1}, \ G' = D_{2m-1}, \ H_1(x) = \mathbb{Z}/4\mathbb{Z}.$$
- For a singularity of type D_{2m},
$$G = \hat{D}_{2m-2}, \ G' = D_{2m-2}, \ H_1(x) = (\mathbb{Z}/2\mathbb{Z})^2$$
- For a singularity of type E_6
$$G = \hat{\mathfrak{A}}_4, \ G' = \mathfrak{A}_4, \ H_1(x) = \mathbb{Z}/3\mathbb{Z}.$$
- For a singularity of type E_7
$$G = \hat{\mathfrak{S}}_4, \ G' = \mathfrak{S}_4, \ H_1(x) = \mathbb{Z}/2\mathbb{Z}.$$
- For a singularity of type E_8
$$G = \hat{\mathfrak{A}}_5, \ G' = \mathfrak{A}_5, \ H_1(x) = 0.$$

2.3. **Local homology of normal surface singularities and groups of normal crossing curve configurations.** Let $x \in X$ be (the germ of) a normal surface singularity, and $p : S \to X$ a relatively minimal (global) normal crossing resolution of singularities, so that $p^{-1}(x)$ is a connected normal crossing divisor [3]

$$D = \cup_i D_i$$

such that the intersection matrix $(D_i \cdot D_j)$ is negative definite (this implies that $K_s \cdot D_i \geq 0$, as noted in [Cat06], from which we borrow the following facts).

Let $T = T(D)$ be a good tubular neighbourhood of D, so that $(T \setminus D) \cong X \setminus \{x\}$.

The local fundamental group $\pi_1(X \setminus \{x\}) = \pi_1(T \setminus D)$ can be computed as a group with generators and relations, see [Cat06] pages 140 and following.

[2] about which we shall provide more details in the ensuing subsections.

[3] When dealing with a singularity of type D_n, we shall switch notation to $D = \cup_i C_i$.

In view of the surjection $\pi_1(T \setminus D) \twoheadrightarrow \pi_1(D)$ if we assume that the local first homology group $H_1(x) := H_1(X \setminus \{x\}, \mathbb{Z})$ is finite, it follows that each D_i is a smooth rational curve, and we have a graph of rational curves with zero first homology (a tree).

Under these two assumptions, we have a presentation, where we associate a generator γ_i to each component D_i, a generator γ_{ij} for each point $D_i \cap D_j$, and we set $-m_i := D_i^2$:

$$\pi_1(T \setminus D) = \langle \gamma_i, \gamma_{ij} \,|\, \gamma_{ij} = \gamma_j, \gamma_i^{m_i} = \prod_j \gamma_{ij}, [\gamma_i, \gamma_{ij}] = 1 \rangle.$$

We have

$$H_1(x) = \pi_1(T \setminus D)^{ab} = (\oplus \mathbb{Z}\gamma_i)/(\oplus \mathbb{Z}\gamma_j)^\vee,$$

in other words, $H_1(x)$ is the cokernel of the intersection matrix.

Hence it is a finite abelian group if $(D_i \cdot D_j)$ is negative definite.

The Rational Double Points, also called ADE singularities, yield a case where we have a graph of smooth rational curves with $m_i = 2, \forall i$.

The above calculation yields not only the local homology group $H_1(x)$, but also a set of generators γ_i.

i) A_n corresponds to a linear tree of n rational curves, and here we have as generator γ_1, and the relations amount to $\gamma_j = j\gamma_1$, and to $2\gamma_n = \gamma_{n-1} \Leftrightarrow (n+1)\gamma_1 = 0$.

ii) D_n corresponds to a linear tree of $(n-1)$ rational curves, plus γ_n intersecting γ_{n-2} (this means, by abuse of notation, that the curve C_n corresponding to γ_n intersects C_{n-2}): here the relations amount to $\gamma_j = j\gamma_1$, for $j \leq n-2$, and to

$$2\gamma_n = 2\gamma_{n-1} = \gamma_{n-2} = (n-2)\gamma_1, (n-1)\gamma_1 = \gamma_n + \gamma_{n-1},$$
$$\Leftrightarrow \gamma_{n-1} = \gamma_1 + \gamma_n, 2\gamma_1 = 0, 2\gamma_n = (n-2)\gamma_1,$$

hence we get $H_1(x) = \mathbb{Z}/4$ for n odd, with generator γ_n, and $H_1(x) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$ for n even, with generators γ_1, γ_n.

iii) For E_6 we get a linear tree of five rational curves, corresponding to $\gamma_1, \gamma_2, \gamma_3, \gamma_5, \gamma_6$, and γ_4 intersecting γ_3.

We get $H_1(x) = \mathbb{Z}/3$ with generator $\gamma_1 = \gamma_5$, and moreover $\gamma_2 = \gamma_6 = 2\gamma_1$, $\gamma_3 = \gamma_4 = 0$.

iv) For E_7 we get a linear tree of six rational curves, corresponding to $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6$, and γ_7 intersecting γ_4.

We get $H_1(x) = \mathbb{Z}/2$ with generator $\gamma_1 = \gamma_7 = \gamma_3$, and moreover

$$\gamma_2 = \gamma_4 = \gamma_5 = \gamma_6 = 0.$$

v) For E_8 we get a linear tree of seven rational curves, corresponding to $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7$, and with γ_8 intersecting γ_5. We get $H_1(x) = 0$.

Looking at the p-primary components, we see that for $p > 3$ we get only a contribution from A_n, with $p|(n+1)$ (hence $n \geq 4$), for $p = 3$ only A_{3k-1} and E_6, for $p = 2$: A_{2m+1} , D_n, E_7.

2.4. **Elementary geometric shortenings.** If we have as above a global normal crossing divisor $D = \cup D_i$ with tubular neighbourhood $T = T(D)$, we consider the operation of deleting one curve D_i from D, obtaining $D' := D(i) := \cup_{j \neq i} D_j$, a union of connected normal crossing divisors.

Observe now that $T \setminus D'$ contains a union $T(i)$ of tubular neighbourhoods of the components of $D' = D(i)$.

Assume now that all the curves D_i are smooth rational.

Clearly $\pi_1(T \setminus D)$ surjects onto $\pi_1(T \setminus D')$ and the kernel is normally generated by γ_i, hence the same holds true for their abelianizations, hence $H_1(x) := H_1(T \setminus D)$ surjects onto $H_1'(x) := H_1(T \setminus D')$.

On the other hand, also $H_1(T(i) \setminus D(i))$ surjects onto $H_1'(x) := H_1(T \setminus D')$ by the Mayer-Vietoris exact sequence, and because we are adding a disk bundle over the punctured curve

$D_i \setminus (\cup_{j \neq i} D_j)$, whose first homology is generated by the elements γ_{ij}, which are in the image of $H_1(T(i) \setminus D(i))$.

Therefore $H_1'(x)$, a quotient of $H_1(x)$, is also a quotient of $H_1(T(i) \setminus D(i)) =: \oplus_y H_1'(x,y)$, where y runs through the connected components of $T(i) \setminus D(i)$ (the cumbersome notation is clear in the case where D is the exceptional normal crossing divisor in the resolution of a singular point x of a normal surface, and the points y are the singular points of a partial smoothing of the singularity x).

Example 8. Consider the case of A_n-singularities,

$$A_n := \{(u,v,z) \mid z^2 - u^2 + v^{n+1} = 0\}.$$

The semiuniversal deformation of the singularity consists of the surfaces with equation

$$z^2 - u^2 + P(v) = 0, \; P = v^{n+1} + a_{n-1} v^{n-1} + \cdots + a_0,$$

whose singular set is the set $\{z = u = P(v) = P_v(v) = 0\}$, hence consists of the points $(0, \eta, 0)$, where η is a multiple root of $P(v)$.

The simplest (the elementary) deformations are of the form $P(v) = v^{m+1}(v-t)^{n-m}$.

Then the A_n-singularity deforms to an A_m plus an A_{n-m-1}-singularity. Taking the minimal resolution, the outcome is to take the A_n-diagram and delete the curve with index $m+1$.

In terms of fundamental groups, we take the quotient of $H_1(x) \cong \mathbb{Z}/(n+1)$ by the relation $\gamma_{m+1} = 1$: since $\gamma_{m+1} = \gamma_1^{m+1} = \gamma_n^{n-m}$, and $\gamma_n = \gamma_1^n$, we get

$$H_1'(x) \cong \mathbb{Z}/(m+1, n-m) = \mathbb{Z}/d,$$

where $d := \gcd(m+1, n-m)$.

Whereas the groups $H_1'(x,y)$ are just $\mathbb{Z}/(m+1)$ and $\mathbb{Z}/(n-m)$, whose direct sum $\mathbb{Z}/(m+1) \oplus \mathbb{Z}/(n-m)$ obviously maps onto $\mathbb{Z}/d$, with kernel $\cong \mathbb{Z}/LCM(n-m, m+1)$.

The above example will illuminate later on the concept of geometrically driven elementary (code) shortenings [4].

In the case of D_n an elementary geometric shortening leads either to D_m plus A_{n-m-1}, or to A_{n-1}, or to $A_1 \cup A_1 \cup A_{n-3}$.

The case of E_6, E_7, E_8 is also fun, E_6 can go by deleting the middle curve to $A_2 \cup A_2 \cup A_1$, correspondingly $(\mathbb{Z}/3)^2 \oplus (\mathbb{Z}/2)$ maps onto $0 = H_1'(x)$.

3. Generalized codes and their shortenings

3.1. **Duality for abelian groups.** Let A be an abelian group.

Then we define the **Weakly dual abelian group**[5] as:

$$A^* := \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z} \oplus \mathbb{Q}). \tag{2}$$

If A is finitely generated, then we define its **dual** as

$$A^\vee := \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z} \oplus \mathbb{Z}), \tag{3}$$

observing that, if A is finite, then, using the notation of [God58], page 6 [6]

$$A^\vee = \hat{A} := \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z}). \tag{4}$$

Observe that we have also a notion of duality for homomorphisms, and that, if A is finitely generated, then

$$A \hookrightarrow (A^\vee)^\vee,$$

[4] For precision, we distinguish between geometrically driven shortenings, which are shortenings of a code, and geometric shortenings, which are shortenings of a configurations of curves

[5] The Pontrjagin dual is defined instead as $\mathrm{Hom}(A, \mathbb{R}/\mathbb{Z})$, for any locally compact abelian group A, and biduality holds.

[6] loc. cit. contains the incorrect claim that every cyclic group embeds in $\mathbb{Q}/\mathbb{Z}$.

and indeed **biduality** holds:

$$(A^\vee)^\vee \cong A.$$

3.2. **Generalized Codes.** We start with some quite general definitions.

Definition 9. Let $\mathcal{C}$ be a finite **set of labeled finitely generated abelian groups**, that is, $\mathcal{C}$ a finite set, given together with a map to the category of finitely generated abelian groups: to each $j \in \mathcal{C}$ is associated a finitely generated abelian group V_j (and possibly $V_j \cong V_i$ for $i \neq j$).

Then a **$\mathcal{C}$-labeled-generalized code** consists of

(1) a finite set Σ,

(2) a map $h : \Sigma \to \mathcal{C}$ which, for shorthand notation, shall be indicated as the association to $x \in \Sigma$ of an abelian group

$$H_1(x) := V_{h(x)} \in \mathcal{C},$$

(3) a surjection $\mathsf{k} : \oplus_{x\in\Sigma} H_1(x) \twoheadrightarrow H_1$ [7]

hence (equivalently, since $\mathcal{C}$ consists of finitely generated abelian groups) also

(3') an injective homomorphism

$$\mathsf{k}^\vee : H_1^\vee \hookrightarrow \oplus_{x\in\Sigma} H_1(x)^\vee,$$

and we set then

$$\mathcal{K} := \mathrm{im}(\mathsf{k}^\vee).$$

(4) Two such codes (Σ, h, k), $(\Sigma_2, h_2, \mathsf{k}_2)$ are equivalent if there is a bijection $\phi : \Sigma \to \Sigma_2$ such that $h = h_2 \circ \phi$, and carrying $\ker(\mathsf{k})$ to $\ker(\mathsf{k}_2)$ (hence inducing an isomorphism of the cokernels $\mathrm{coker}(\mathsf{k}) \cong \mathrm{coker}(\mathsf{k}_2)$, and, dualizing, carrying $\mathcal{K}$ to $\mathcal{K}_2$).

Remark 10. *(i) When defining the equivalence of codes, it is convenient, as in the case of vector codes, not to allow automorphisms of $H_1(x)$.*

In our case, a very good reason is that if we take an A_n tree of smooth rational curves $E = E_1 \cup \dots \cup E_n$ contained in a smooth complex surface, and with $E_i^2 = -2$, then $\mathcal{T}(E)$ being a tubular neighbourhood of E, then $\pi_1(\mathcal{T}(E) \setminus E) = \mathbb{Z}/(n+1)$, yet the graph produces two standard generators, γ_1 and γ_n, the inverse of each other. In fact, as we saw, the (more general) algorithm of Lemma 5 of [Cat06] shows that $\gamma_m = \gamma_1^m$, and that $1 = \gamma_1^{n+1}$. But, in the case where $(n+1)$ is prime, then each γ_m is a generator.

(ii) In the case of ADE singularities one may allow automorphisms which come from an automorphism of the exceptional tree, for instance for A_n, to allow exchanging γ_1 with γ_n. This might be useful later on for classification, but allowing it right away might make the general theory more complicated.

Definition 11. A **naive generalized code** is obtained in the special case where $V_i \cong V_j$ implies $i = j \in \mathcal{C}$.

We have a **finite generalized code** (labeled or unlabeled) if each V_j is a finite abelian group.

Remark 12. *One can associate a naive generalized code to a labeled generalized code, by a sort of forgetful map.*

It suffices, for each maximal set of mutually isomorphic abelian groups V_j, to choose an isomorphism of each V_j with a fixed group V_j'.

---

[7] This datum includes therefore also the specification of the codomain, a finitely generated abelian group H_1, depending on k.

3.3. **Finite generalized Codes and their primary decomposition.** In this case all the abelian groups occurring in

$$\mathsf{k} : \oplus_{x\in\Sigma} H_1(x) \twoheadrightarrow H_1$$

are finite and we can take their primary decomposition, for instance we write

$$H_1 = \oplus_{p=\mathrm{prime}} H_{1,p}.$$

Proposition 13. *A finite generalized code is a direct sum of its p-primary components,*

$$\mathcal{K} = \oplus_p \mathcal{K}_p,$$

$$\mathcal{K}_p := \mathrm{im}(\mathsf{k}_p^\vee),\ \mathsf{k}_p^\vee : H_{1,p}^\vee \hookrightarrow\ \oplus_{x\in\Sigma} H_{1,p}(x)^\vee.$$

Two finite generalized codes are equivalent if and only if their p-primary components are equivalent.

Proof. This follows since the primary decomposition is unique, indeed every homomorphism $\psi : A \to B$, where A is p-primary, and the p-primary component of B is 0, is equal to 0. □

3.4. **Weights of finite generalized Codes.** We now define the code vectors, and their weights, as follows.

Definition 14. Define a **code vector** as an element

$$v \in \mathcal{K} \cong H_1^\vee \hookrightarrow\ \oplus_{x\in\Sigma} H_1(x)^\vee.$$

Since $v : H_1 \to \mathbb{Q}/\mathbb{Z}$, $v(H_1)$ is a finite subgroup of $\mathbb{Q}/\mathbb{Z}$, hence of the cyclic group $(\frac{1}{M}\mathbb{Z})/\mathbb{Z} \cong \mathbb{Z}/M\mathbb{Z}$, where $M \in \mathbb{N}$ is the exponent of the group H_1.

We define the **order** of v as the positive integer N such that $\mathrm{im}(v) = (\frac{1}{N}\mathbb{Z})/\mathbb{Z}$.

For each x we get the composition

$$v_x : H_1(x) \to H_1 \to \Big(\frac{1}{N}\mathbb{Z}\Big)/\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z},$$

and, letting, for $j \in \mathcal{C}$, the corresponding abelian group be V_j, we define

- the **refined weight** as the number of times, for $j \in \mathcal{C}$, that a given nonzero linear form $\xi : V_j \to \mathbb{Q}/\mathbb{Z}$ occurs:

$$w(v; j, \xi) := |\{x \,|\, h(x) = j,\ v_x = \xi\}|.$$

- The **label weight** as

$$w(v; j) := |\{x \,|\, h(x) = j,\ v_x \neq 0\}|.$$

- The **Hamming weight** $w(v)$ as

$$w(v) := \sum_{j,\xi\neq 0} w(v; j, \xi) = \sum_j w(v; j). \tag{5}$$

Observe that, if $V_j = \mathbb{Z}/N_j\mathbb{Z}$, then ξ is determined by $\xi(1) \in \mathbb{Z}/N\mathbb{Z}$.

3.5. **Labeled generalized code shortenings.** Given a labeled generalized code

$$\mathsf{k} : V := \oplus_{x\in\Sigma} H_1(x) \twoheadrightarrow H_1,$$

we define shortenings via elementary shortenings.

Definition 15. (1) An elementary shortening associated to $z \in \Sigma$ is obtained replacing $H_1(z)$ by a $\mathcal{C}$-labeled generalized code

$$\mathsf{k}'' : V'' := \oplus_{y \mapsto z} H_1'(z,y) \twoheadrightarrow H_1'(z),$$

where $H_1'(z)$ is a quotient of $H_1(z)$, and replacing H_1 by the quotient H_1' of H_1 by the image of $M_j := \ker(H_1(z) \twoheadrightarrow H_1'(z))$ inside H_1.

Clearly we have

$$k' : V' := (\oplus_{y \mapsto z} H_1'(z,y)) \bigoplus (\oplus_{x \in \Sigma, x \neq z} H_1(x)) \twoheadrightarrow H_1'(z) \bigoplus (\oplus_{x \in \Sigma, x \neq z} H_1(x)) \twoheadrightarrow H_1'.$$

(2) We say that we have a full elementary shortening if $H_1'(z) = 0$.

(3) Every shortening is defined as a composition of elementary shortenings.

(4) We say that an elementary shortening is driven if, for each $j \in \mathcal{C}$, k'' belongs to a list prescribed a priori (as in the case of geometrically driven shortenings).

Remark 16. *(i) Observe that, dually,*

$$\mathcal{K}' = (H_1')^\vee = (H_1)^\vee \cap (V')^\vee = \mathcal{K} \cap (V')^\vee.$$

(ii) Using a forgetful map we can also define the shortening of a naive generalized code.

3.6. **Length of generalized codes?** The definition of length can be given in several ways, the most natural one being to define the length as the cardinality of Σ (alternatively, one can sum, over $x \in \Sigma$, the number of summands for a canonical decomposition of $H_1(x)$ (a decomposition with cyclic summands, be it the Frobenius decomposition, or the primary decomposition).

This naive notion has the disadvantage, as we easily infer, that the length does not decrease by a shortening, a fact which looks like a contradiction in words.

For this reason we shall define in the next section the notion of rank, for the geometric codes associated to normal surfaces: these do indeed decrease via a shortening.

4. Plumbing of curves and geometrically driven shortenings of codes of normal surfaces, ADE generalized codes

Definition 17. (i) ADE generalized codes are the labeled generalized codes where $\mathcal{C}$ is the class of all ADE surface singularities. Here to each singular point $x \in X$ we associate the local homology group $H_1(x)$ (given together with a set of generators corresponding to the exceptional divisors of the minimal normal crossing resolution).

(ii) A more general class is the class of the codes of normal surfaces with rational singularities, where the labeling is given by the configuration of exceptional curves in a minimal normal crossing resolution (the curves are rational, and the self intersection numbers are part of the datum of the configuration). In fact, the isomorphism classes of ADE singularities are determined by their configurations (for them, the self intersection numbers are $= -2$).

(iii) elementary geometrically driven shortenings correspond to deleting an irreducible curve $D_i \leq D$ from the normal crossing divisor D.

(iv) geometrically driven shortenings correspond to deleting a certain number of the irreducible curves D_i; then, contracting the remaining curves, one gets a normal surface with a finite number of singularities, which are ADE singularities if we started from an ADE singularity.

Definition 18. Given a code of a normal surface, where the labeling is given by the configuration of exceptional curves in a minimal normal crossing resolution, the **rank** of the code is given by the number of exceptional curves. [8]

[8]For an ADE singularity the index equals this number, e.g. n for A_n, 8 for E_8.

Remark 19. *Given the dual graph of a connected normal crossing configuration D of smooth rational curves D_i, and self intersection numbers $D_i^2 = -m_i \in \mathbb{Z}$, we can construct, by plumbing, a smooth (non-compact) complex surface T containing the configuration. If the intersection matrix $D_i \cdot D_j$ is negative definite, by Grauert's theorem* [Grau62] *we can contract D to a point x, obtaining a normal (non-compact) complex surface Y with a point x, with* $\mathrm{Sing}(Y) \subset \{x\}$.

x is singular if and only if $\pi_1(T \setminus D) \neq 1$, as shown by Mumford [Mum61].

The isomorphism class of the singularity may vary, but for us the labeling is given by the configuration D (including the multiplicities), which is an invariant for equisingular deformation. One can of course also consider more general configurations.

Geometrically driven shortenings replace an ADE singularity by several other ones, yielding an ADE code with quotient $H_1'(x)$ which is a quotient of the local homology $H_1(x)$.

Remark 20. *As shown in the particular case of example 8, a geometrically driven shortening corresponds to a partial smoothing of an ADE singularity, as shown by Burns-Wahl* [B-W74], *and as we shall discuss in the next subsection.*

4.1. **Code shortenings and unobstructedness.** The condition of unobstructedness means that all the local deformations of the singular points of X can be simultaneously achieved by a global deformation of X.

More precisely, unobstructedness means that the deformations of X have a submersion onto the space of local deformations of the singularities, so that one can obtain independent smoothings of all the singular points. New results in this direction have been obtained by Dimca and Kloostermann [Dim17] [Kloos17].

In fact, (see for instance [Wahl76], [Cat88], [Sern06]) for a normal surface Y in $\mathbb{P} := \mathbb{P}^3$, we have (by definition) the exact sequence

$$0 \to N_Y^* \to \Omega^1_{\mathbb{P}} \otimes \mathcal{O}_Y \to \Omega^1_Y \to 0,$$

where the conormal sheaf $N_Y^* \cong \mathcal{O}_Y(-d)$ s generated by the differential of F, where F is the polynomial equation (of degree d) of Y.

Dualizing with respect to $\mathcal{O}_Y$, we get an exact sequence

$$0 \to \Theta_Y := \mathcal{H}om(\Omega^1_Y, \mathcal{O}_Y) \to \Theta_{\mathbb{P}} \otimes \mathcal{O}_Y \to N_Y \cong \mathcal{O}_Y(d) \to \mathcal{E}xt^1(\Omega^1_Y, \mathcal{O}_Y) \to 0,$$

which can be split into two short exact sequences, the second one being:

$$0 \to N_Y' \to N_Y \cong \mathcal{O}_Y(d) \to \mathcal{E}xt^1(\Omega^1_Y, \mathcal{O}_Y) \to 0.$$

The sheaf $\mathcal{E}xt^1(\Omega^1_Y, \mathcal{O}_Y)$ is, for a normal surface, concentrated on the singular points, and the stalk at each of these points is equal to the quotient Tjurina algebra

$$\mathcal{O}_{\mathbb{P}}/(F, \partial F/\partial x_i),$$

where the x_i are local (or projective) coordinates.

Unobstructedness means, when we pass to the long exact cohomology sequence, in view of the surjection $H^0(\mathcal{O}_{\mathbb{P}}(d)) \to H^0(\mathcal{O}_Y(d))$, exactness of

$$0 \to H^0(N_Y') \to H^0(\mathcal{O}_Y(d)) \to H^0(\mathcal{E}xt^1(\Omega^1_Y, \mathcal{O}_Y)) \to 0.$$

The main result is that then the projective space $\mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}}(d)))$, which parametrizes surfaces of degree d, at the point corresponding to Y, has a submersion onto the manifold $\mathcal{L}ocDef(Y)$ of local deformations of the singularities of Y (see [Tju70]). Hence in the neighbourhood of Y we can find all possible local deformations of the singularities of Y.

We have the following criterion relating shortenings and partial smoothings:

Theorem 21. *Let Y be an unobstructed surface of degree d in $\mathbb{P}^3$ with ADE singularities: then all geometrically driven shortenings of the code $\mathcal{K} := H_1^\vee$ are realized by some normal surface of* *degree d, which is a partial smoothing of Y. And, conversely, any partial smoothing of Y yields a geometrically driven shortening of the code $\mathcal{K} := H_1^\vee$.*

Proof. Let $\Sigma := \mathrm{Sing}(Y)$; it suffices to show the statement for a partial smoothing of one singular point x, since we can independently achieve all the possible local deformations of the singularities.

There is a local deformation Y' of Y which is equisingular at the singular points different from x and achieves any local deformation of the singularity x.

By the Brieskorn-Tyurina theorem [Bries68] [Tju70], there is a smooth family with connected base whose fibres contain the respective resolutions $\tilde{Y}$ and $\tilde{Y}'$, and Y^* is diffeomorphic to the complement of the union of all the exceptional curves E_i, while $(Y')^*$ is diffeomorphic to the complement of the union of some configurations of exceptional curves E'_j, which includes the configurations of the curves E_i which are exceptional and do not lie over x.

The local deformation amounts to a deformation of the tubular neighbourhood $T(D_x)$ of the exceptional divisor D_x, and as proven by Burns and Wahl [B-W74] Theorem 2.14 and Remark 2.15, both the deformations of $T(D_x)$ and the deformations of the germ (Y, x) are smooth of the same dimension, and there is a map between them which is the quotient by the action of the Weyl group of the singularity. The deformations of the tubular neighbourhood $T(D_x)$ amount to destroying some of the exceptional curves in D_x, and this leads to a geometrically driven shortening of the code $H_1(x)$.

In fact, $H_1(Y^*, \mathbb{Z})$ surjects onto $H_1((Y')^*, \mathbb{Z})$ with kernel generated by the γ_i 's which are geometric loops around the irreducible curves E_i which do not remain holomorphic submanifolds after the deformation. Then $H_1(x) \to H_1(Y^*, \mathbb{Z})$, and $H_1(x)$ surjects onto $\oplus_y H_1(y)$, where the y's are the singularities of Y' to which x deforms.

The previous argument shows also the converse assertion. □

The next result follows essentially from [B-W74] and might be known to specialists; we give a simple and short proof but without giving full references for some well known facts.

Theorem 22. *If Y is a singular projective K3 surface, that is, Y is a surface with ADE singularities, and it has a trivial canonical divisor, then Y is projectively unobstructed, that is, one can independently smoothen all the exceptional curves while preserving the ample divisor class H.*

Proof. Indeed, as already observed, by [B-W74] Theorem 2.14 and Remark 2.15, it suffices to show that the deformations of the smooth K3 surface S and the normal crossing divisor $D = E + H = D_1 + \cdots + D_m$, where E consists of the ADE configurations, has a submersion onto the local deformations of the exceptional components of D.

Now, infinitesimal deformations of the pair (S, D), that is, preserving the components of D, are governed by the cohomology group

$$H^1(\Theta_S(-\log D_1, \dots, -\log D_m)) \cong H^1(\Omega^1_S(\log D_1, \dots, \log D_m)(K_S))^\vee,$$

where the isomorphism is provided by Serre duality. These map to infinitesimal deformations of S, governed by $H^1(\Theta_S)$.

Use now the triviality of K_S to infer that

$$\Omega^1_S(\log D_1, \dots, \log D_m)(K_S) \cong \Omega^1_S(\log D_1, \dots, \log D_m),$$

and the long exact cohomology sequence associated to the residue sequence

$$0 \to \Omega^1_S \to \Omega^1_S(\log D_1, \dots, \log D_m) \to \oplus_1^m \mathcal{O}_{D_i} \to 0.$$

Since the intersection matrix $(D_i \cdot D_j)$ is non-degenerate, the Chern classes of the curves D_i are linearly independent. And since, as observed in [Cat84], the coboundary map

$$\oplus_1^m H^0(\mathcal{O}_{D_i}) \to H^1(\Omega^1_S)$$

is given by the Chern classes of the divisors D_i, it follows that this coboundary map is injective.

Since $H^0(\Omega^1_S) = 0 = H^2(\Omega^1_S)$, and the D_i are rational curves for $i < m$, we have an exact sequence

$$0 \to \oplus_1^m H^0(\mathcal{O}_{D_i}) \to H^1(\Omega^1_S) \to H^1(\Omega^1_S(\log D_1, \dots, \log D_m)) \to H^1(\mathcal{O}_{D_m}) \to 0.$$

Dualizing it, one sees that, since S is unobstructed, that is, $Def(S)$ is an open set in $H^1(\Theta_S)$ (which is the dual of $H^1(\Omega^1_S)$), the deformations of S map onto the dual of $(\oplus_1^m H^0(\mathcal{O}_{D_i}))$, so that one can independently smoothen all the exceptional curves $D_1, \dots, D_{m-1}$ while keeping the ample divisor class D_m.

□

We end this section with a formal definiton of the notion of ancestor, which is a key concept.

Definition 23. 1) Given a finite partially ordered set, an **ancestor** is a maximal element.

2) Given a stratified space, with connected strata $\mathcal{S}_j$ which are locally closed (and with closure a union of strata), one says that $\mathcal{S}_1$ is bigger or equal to $\mathcal{S}_2$ if $\mathcal{S}_1$ is in the closure of $\mathcal{S}_2$. The **ancestors** are the maximal elements, that is, the closed strata.

The notion of ancestor depends on the choice of the stratified space: it could be

- the space of nodal surfaces of degree d in $\mathbb{P}^3$, with strata given by equisingular deformation;
- the space of surfaces of degree d in $\mathbb{P}^3$ with ADE singularities (same type of stratification);
- all surfaces of degree d in $\mathbb{P}^3$, with strata given by topologically equisingular deformation.

The only ancestors in the last case are the planes counted with multiplicity equal to the degree d.

5. ADE-Codes for surfaces X with RDP=ADE singularities

In the case of nodal surfaces X we get the usual binary code.

If $\Sigma = \mathrm{Sing}(X) = \{P_1, \dots, P_s\}$, then the **strict code** $\mathcal{K} \subset (\mathbb{Z}/2)^\Sigma$ associated to X is the image of the injective linear map dual to the surjection $(\mathbb{Z}/2)^\Sigma \twoheadrightarrow H_1(X^*, \mathbb{Z})$.

One of the key difficulties of the theory, also in the nodal case, is concerned with the determination of the possible sets of weights that a normal surface of degree d in $\mathbb{P}^3$ can have. The easiest restrictions come from applying the Riemann-Roch theorem.

We attempt to find similar restrictions for generalized ADE codes, recall here our notation $H_1 := H_1(X^*, \mathbb{Z})$.

5.1. Restrictions for the weights.

Given a vector $v \in H_1^\vee$ of order (exactly) N, it determines a surjection, which is the composition

$$\oplus_{x \in \Sigma} H_1(x) \twoheadrightarrow H_1 \twoheadrightarrow \mathbb{Z}/N.$$

Now, v determines a normal finite Galois covering $f \colon Y \to X$ with Galois group $\mathbb{Z}/N$, and branched only on a set $\Sigma(v)$ of singular points.

Remark 24. *(1) Let the image of $H_1(x) \to \mathbb{Z}/N$ be isomorphic to $\mathbb{Z}/N_x$, that is, equal to $(s_x\mathbb{Z})/(N\mathbb{Z})$, where $s_x N_x = N$. Then*

$$N = \mathrm{lcm}\{N_x \,|\, x \in \Sigma\} \quad \text{that is, } 1 = \gcd\{s_x \,|\, x \in \Sigma\}.$$

(2) For the case of ADE singularities either $N_x \in \{2, 3, 4\}$ or it can be $N_x > 4$ only in the case where x is of type A_{n_x} with $n_x \geq 4$, and then N_x is a divisor of $n_x + 1$.

We assume now that X has only ADE singularities, and let $\Sigma(v)$ be the subset of singular points of X for which $H_1(x) \to \mathbb{Z}/N$ is nontrivial, that is, $\Sigma(v)$ is the set of branching points for $f : Y \to X$.

Clearly the E_8 points are not in $\Sigma(v)$. For each x, let N_x be the integer such that the image of $H_1(x) \to \mathbb{Z}/N$ is isomorphic to $\mathbb{Z}/N_x$.

Then the singular set of Y is contained in the inverse image of $operatornameSing(Y) = \Sigma$, and inverse image of x consists of N/N_x points which are (possibly) singular points, corresponding to the surjection $H_1(x) \to \mathbb{Z}/N_x$.

Set for simplicity of notation $r := N_x$. We shall now determine which type of ADE singularity are the points lying over x.

Observe preliminarily that, if $H_1(x)$ is cyclic, then the integer r determines a unique type of singularity, whereas for $H_1(x) = (\mathbb{Z}/2)^2$, we have three possibilities (and this only occurs for x of type D_{2m}).

i) For A_n, we have then $(n+1) = rs$ and we get r singular points of type A_{s-1}, since the local covering is $\mathbb{C}^2/\mu_s \to \mathbb{C}^2/\mu_{n+1}$, μ_s being the cyclic group of s-th roots of 1.

ii) For D_{2m+1}, $H_1(x)$ is cyclic of order 4.

Consider first the case with $r = 4$: here we have a surjection $\hat{D}_{2m-1} \to \mathbb{Z}/4$ whose kernel is the cyclic group μ_{2m-1}. In fact $\hat{D}_n$ is generated by $\sigma(u,v) = (iv, iu)$ and by $\rho(u,v) = (\zeta_n u, \zeta_n^{-1} v)$, and moreover ρ is in the commutator subgroup for n odd, because $\sigma\rho\sigma^{-1} = \rho^{-1} \Rightarrow [\sigma,\rho] = \rho^{-2}$.

The conclusion is that, over x, lie $N/4$ singularities of type A_{2m-2}.

iii) For D_{2m+1}, $r = 2$, we have a surjection $\hat{D}_{2m-1} \to \mathbb{Z}/2$ and the kernel is $\cong \mathbb{Z}/2 \times \mu_{2m-1}$, hence we find $N/2$ singularities of type A_{4m-3}.

iv) For D_{2m+2}, $r = 2$, we have a surjection $\hat{D}_{2m} \to \mathbb{Z}/2$, factoring through the abelianization $H_1(x) \cong (\mathbb{Z}/2)^2$. $\hat{D}_{2m}$ is generated by σ as above, and $\rho(u,v) := (\zeta_{4m}^{-1} u, \zeta_{4m} v)$. In the abelianization $2[\sigma] = 2[\rho] = 0$. We have three cases, according to the choice of an index 2 subgroup as the kernel.

v) Taking as respective kernels: $\langle \rho \rangle \cong \mu_{4m}$, $\langle \sigma, \rho^2 \rangle \cong \hat{D}_m$, $\langle \sigma\rho, \rho^2 \rangle \cong \hat{D}_m$, we have accordingly $N/2$ singularities of respective types $A_{4m-1}, D_{m+2}, D_{m+2}$.

vi) For E_7, the double covering corresponds to the inclusion $\mathfrak{A}_4 < \mathfrak{S}_4$, and we get $N/2$ singularities of type E_6.

vii) For E_6, $H_1(x) = \mathbb{Z}/3$, hence $r = 3$, which corresponds to the surjection $\mathfrak{A}_4 \to \mathfrak{A}_3$, with kernel $(\mathbb{Z}/2)^2$, hence we get $N/2$ singularities of type D_4.

Remark 25. *i) In* [Cat87] *Theorem 2.1, page 89 , table 2 page 90, it is shown that* $A_{2k+1}/b \cong D_{k+3}$, *where* $A_{2k+1} = \{z^2 + x^2 + y^{2k+2} = 0\}$, *and* $b(x,y,z) := (x, -y, -z)$.

While, for $a(x,y,z) := (x,-y,z)$, $A_{2k+1}/a \cong A_k$, *but this quotient is ramified in codimension 1, while the quotient for* $e(x,y,z) := (-x,-y,-z)$, A_{2k+1}/e, *is not an ADE singularity (see Theorem 2.5, page 92 of* [Cat87]*).*

ii) It is fun to make a picture of the triple covering in the case E_6 *for the minimal resolution. Algebraically, we get equations from* $z^2 = x^3 + y^4$ *and* $w^3 = x/y$.

iii) Taking the full covering associated to $H_1(x)$ *for the case* D_{2m}, *we get* A_{2m-3} *singularities, see also table 3, page 93 of* [Cat87].

Remark 26. *In view of the primary decomposition (Proposition 13), in order to obtain restrictions, it suffices to consider the case where we have a vector* v *whose order is a prime power* $N = p^k$. *And recall that, for* $p > 3$, *we get only a contribution from* A_n, *with* $p|(n+1)$, *for* $p = 3$ *only from* A_{3k-1} *and* E_6, *for* $p = 2$ *from* A_{2m+1} , D_n, E_7.

5.2. **Restriction on the weights for vectors** v **of order** $N = 2$**.** Let S be, unlike elsewhere, the minimal resolution of the singular points of X, $x \in \mathcal{N} \subset \Sigma$, contributing to the vector v.

Then the double covering $f : Y \to X$ induces a double covering $F : Z \to S$, determined by a line bundle L on S such that

$$2L \equiv \sum_{x \in \mathcal{N}} B_x,$$

where B_x is a reduced exceptional divisor, such that $B_x \neq 0 \; for \; x \in \mathcal{N}$.

We need first to calculate the self intersection numbers B_x^2. For this purpose we need to observe that, if $H_1(x)$ is cyclic, and $x \in \mathcal{N}$, then B_x consists of the curves for which γ_i is an odd multiple of the generator.

i) A_n appears in $\mathcal{N}$ only if $n = 2m+1$ is odd, in this case $H_1(x) \cong \mathbb{Z}/(2m+2)$ and B_x consists of the odd labelled curves, these are $m+1$ disjoint (-2)-curves, hence
$$B_x^2 = -2(m+1).$$

ii) For E_7 we get, using our previous calculations, $B_x = D_1 + D_3 + D_7$, hence
$$B_x^2 = -6.$$

iii) For D_n, n odd, $B_x = C_n + C_{n-2}$, hence
$$B_x^2 = -4.$$

iv) For D_{2m}, we have generators γ_1, γ_n, and three options for the surjection to $\mathbb{Z}/2$. But the two options: $\gamma_{2m} \mapsto 0$, respectively $\gamma_{2m-2} \mapsto 0$, are related by a symmetry.

In the first case $B_x = C_1 + C_3 + \cdots + C_{2m-3} + C_{2m-1}$, hence
$$B_x^2 = -2m.$$

If instead $\gamma_1 \mapsto 0$, then $B_x = C_{2m} + C_{2m-1}$, hence
$$B_x^2 = -4.$$

Definition 27. (i) In the case of a singularity of type D_{2m}, the previous computations suggest to distinguish the possible surjections $H_1(x) \to \mathbb{Z}/2$ as being of subtype $(-)$ if $\gamma_1 \mapsto 0$, and of subtype $(+)$ otherwise. Because, for subtype $(-)$, $B_x^2 = -4$, while, for subtype $(+)$, $B_x^2 = -2m$.

(ii) For convenience of notation we shall consider singularities of type D_{2m+1} as being of subtype $(-)$, since then $B_x^2 = -4$.

Proposition 28. *Let v be a code vector of order $N = 2$. Define $\mathcal{N}$ as the subset of the singular points $x \in \Sigma$ such that $H_1(x)$ surjects onto $\mathbb{Z}/2$. Let $t(A_{2m+1})$ be the A_{2m+1}-weight of v, that is, the number of points in $\mathcal{N}$ which are of type A_{2m+1}. Let similarly $t(E_7)$ be the number of points in $\mathcal{N}$ which are of type E_7, $t(D_n, -)$ the number of points in $\mathcal{N}$ which are of type D_n and of subtype $(-)$ for v, $t(D_{2m}, +)$ the number of points in $\mathcal{N}$ which are of type D_{2m}, and of subtype $(+)$ for v.*

Then
$$(**) \quad \sum_m ((m+1)\; t(A_{2m+1}) + m\; t(D_{2m}, +)) + \sum_n 2\; t(D_n, -) + 3\; t(E_7)$$
is divisible by 4, and indeed by 8 if K_S is divisible by 2.

Proof. Observe that $F_*(\mathcal{O}_Z) = \mathcal{O}_S \oplus \mathcal{O}_S(-L)$, and by Riemann-Roch
$$\chi(-L) = \chi(\mathcal{O}_S) + \frac{1}{2}L(L + K_S) = \chi(\mathcal{O}_S) + \frac{1}{2}L^2 \Rightarrow 2|L^2.$$
In fact $K_S L = 0$, in view of $2L \equiv \sum_{x \in \mathcal{N}} B_x$, and since the divisors B_x are exceptional and K_S is the pull back of K_X. The same formula implies
$$\sum_{x \in \mathcal{N}} B_x^2 \equiv 0 \pmod 8,$$
equivalent to $(**)$.

To get divisibility of $(**)$ by 8 when K_S is 2-divisible, we need that L^2 be divisible by 4, that is, that
$$\chi(\mathcal{O}_Z) = 2\chi(\mathcal{O}_S) + \frac{1}{2}L^2$$

be divisible by 2.

This is true, as in Proposition 2.11 of [Cat81], because: if $Y \to X$ is the corresponding 2-1 covering of normal surfaces, then K_Y is divisible by 2, being the pull back of K_X, hence there exists a Cartier divisor G such that $K_Y \equiv 2G$.

Then $\chi(\mathcal{O}_Y(G))$ is an even integer since it equals $2h^0(\mathcal{O}_Y(G)) + h^1(\mathcal{O}_Y(G))$, where the last vector space $H^1(\mathcal{O}_Y(G))$ carries a bilinear alternating non-degenerate form, hence has even dimension. Now, $\chi(\mathcal{O}_Z) = \chi(\mathcal{O}_Y)$ and $\chi(\mathcal{O}_Y(G)) = \chi(\mathcal{O}_Y) - \frac{1}{2}G^2$ hence we are done if G^2 is divisible by 4.

We conclude because $G = f^*(M)$, hence $G^2 = 2M^2$ and M^2 is even because the intersection form is even.

$\square$

5.3. **Restriction on the weights for vectors v of order $N = 3$ or a larger number.** Let S be now the minimal resolution of the singularities $x \in \mathcal{N} \subset \Sigma$ of X contributing to the vector v.

Then the triple covering $f : Y \to X$ induces a singular triple covering $F : Z \to S$, determined by a line bundle L on S such that

$$3L \equiv \sum_{x \in \mathcal{N}} B_x,$$

where B_x is an exceptional divisor, $B_x \neq 0$ *for* $x \in \mathcal{N}$; moreover the multiplicities of the components D_i in B_x are either 1 or -1 according to the image of γ_i in $\mathbb{Z}/3$ being 1 or 2.

i) A_n appears in $\mathcal{N}$ only if $n + 1 = 3s$, in this case $H_1(x) \cong \mathbb{Z}/(3s)$ and B_x consists of the curves with label congruent to 1 modulo 3, minus the curves with label congruent to 2 modulo 3.

The associated reduced divisors form s A_2 strings, hence

$$B_x^2 = -6s.$$

ii) For E_6 we get, using our previous calculations, $B_x = D_1 - D_2 + D_5 - D_6$, hence

$$B_x^2 = -12.$$

Proposition 29. *Let v be a code vector of order $N = 3$. Define $\mathcal{N}$ as the subset of the singular points $x \in \Sigma$ such that $H_1(x)$ surjects onto $\mathbb{Z}/3$. Let $t(A_{3s-1})$ be the A_{3s-1}-weight of v, that is, the number of points in $\mathcal{N}$ which are of type A_{3s-1}. Let similarly $t(E_6)$ be the number of points in $\mathcal{N}$ which are of type E_6,*

Then

$$(***) \;\; \sum_s s\ t(A_{3s-1}) + 2t(E_6) \equiv \sum_s s\ t(A_{3s-1}) - t(E_6)$$

is divisible by 3.

Proof. Observe that $F_*(\mathcal{O}_Z) = \mathcal{O}_S \oplus \mathcal{O}_S(-L) \oplus \mathcal{O}_S(-2L)$.

And, as in the proof of Proposition 28, by Riemann-Roch

$$\chi(-L) = \chi(\mathcal{O}_S) + \frac{1}{2}L(L + K_S) = \chi(\mathcal{O}_S) + \frac{1}{2}L^2 \Rightarrow 2|L^2.$$

Since $3L \equiv \sum_{x \in \mathcal{N}} B_x$, we get

$$9L^2 = (\sum_{x \in \mathcal{N}} B_x)^2 = -6[2(t(E_6) + \sum_s s\ t(A_{3s-1})].$$

$\square$

Corollary 30. *Assume that the singular points of Y are just A_2 singularities, cusps in the terminology of* [Ba-Ra05], [Ba-Ra07].

Then we have a vector code $\mathcal{K}$ over $\mathbb{F}_3$, and the weight of a code vector is always divisible by 3.

Barth and Rams proved that for sextic surfaces in $\mathbb{P}^3$ with cusps, the minimal weight is 18, and that the weights $18, 24, 27$ can occur.

Barth showed that, for quartics, there cannot be 9 cusps, but there is a quartic with 8 cusps ([Barth98], [Barth00]).

Definition 31. A code vector v of order N is said to be **almost simple** if $w(v; j, \xi) = 0$ for V_j non-cyclic, and, for V_j cyclic, for $\xi(1) \neq 1, N-1$.

Proposition 32. *Let v be an almost simple code vector of order $N > 4$. Denote $\mathcal{N}$ the subset of the singular points $x \in \Sigma$ such that $H_1(x)$ surjects onto $\mathbb{Z}/N$: these are A_n strings, where $n+1 = Ns$.*

Let $t(A_{Ns-1})$ be the A_{Ns-1}-weight of v, that is, the number of points in $\mathcal{N}$ which are of type A_{Ns-1} (under our assumption $t(A_{Ns-1}) = w(v; A_{Ns-1}, 1) + w(v; A_{Ns-1}, N-1)$).

Then

$$(****) \ \sum_s s \ t(A_{Ns-1})$$

is divisible by N.

Proof. Again

$$F_*(\mathcal{O}_Z) = \mathcal{O}_S \oplus \mathcal{O}_S(-L) \oplus \mathcal{O}_S(-2L) \oplus \cdots \oplus \mathcal{O}_S(-(N-1)L).$$

And, as in the proof of Proposition 28, by Riemann-Roch $2 | L^2$.

Now, for $x \in \mathcal{N}$, B_x consists of s strings of A_{N-1} singularities, with multiplicities $1, 2 \ldots, N-1$. We can write such a string as $Z_{N-1} + Z_{N-2} + \cdots + Z_1$, where the Z_j's are the fundamental cycles of the string where the first $N-1-j$ curves are deleted. Then by induction

$$(Z_{N-1} + Z_{N-2} + \cdots + Z_1)^2 = -2 - 2(N-2) + (Z_{N-2} + \cdots + Z_1)^2 =$$
$$-2(N-1) + (Z_{N-2} + \cdots + Z_1)^2 = -N(N-1).$$

It follows that for each string we get a contribution to B_x^2 equal to $-N(N-1)$.

Since $NL \equiv \sum_{x \in \mathcal{N}} B_x$, we get

$$N^2 L^2 = (\sum_{x \in \mathcal{N}} B_x)^2 = -N(N-1) \ \sum_s s \ \ t(A_{Ns-1}),$$

and $(****)$ follows.

□

In general, for $N \geq 4$, one has more complicated formulae depending on the refined weights. We treat for instance the case $N = 5$.

Proposition 33. *For vectors v of order $N = 5$, we get that*

$$5 \mid [\sum_s w(v; A_{5s-1}, 1) + w(v; A_{5s-1}, 4) - w(v; A_{5s-1}, 2) - w(v; A_{5s-1}, 3)].$$

Proof. The points $x \in \mathcal{N}$ with $v_x(1) = 1, 4$, give the same contribution as the one calculated in Proposition 32, namely -20 while those with $v_x(1) = 2, 3$, since the multiplicities of the curves in B_x are $(2, 4, 1, 3)$ or these in reverse order, give a contribution equal to $-2(30)+2(8+4+3) = -30$; dividing by 5 we get coefficients $+1$, respectively -1, modulo 5.

□

6. The binary code $\mathcal{K}[2]$ of surfaces in $\mathbb{P}^3$ with ADE singularities, and the B-inequality

For Y a normal surface in $\mathbb{P}^3$, with ADE singularities, let $\tilde{Y}$ be the minimal resolution of the singular points.

We shall denote by $\mathcal{E}$ the set of irreducible exceptional curves E_i in $\tilde{Y}$: these are all smooth rational curves with selfintersection -2.

Definition 34. Given a generalized ADE code

$$\mathcal{K} := \mathrm{im}(\mathsf{k}^\vee),\ \mathsf{k}^\vee : H_1^\vee \hookrightarrow\ \oplus_{x\in\Sigma} H_1(x)^\vee,$$

we define $\mathcal{K}[2]$ to be the set of code vectors of order $N = 2$.

$\mathcal{K}[2]$ is a $\mathbb{Z}/2 =: \mathbb{F}_2$ vector space, and indeed we have

$$\mathcal{K}[2] = \mathrm{Hom}(\mathbb{Z}/2, H_1^\vee) \cong \mathrm{Hom}(H_1, \mathbb{Z}/2) = \mathrm{Hom}(H_1(Y^*, \mathbb{Z}), \mathbb{Z}/2) = H^1(Y^*, \mathbb{Z}/2).$$

Corollary 35. *Let Y be an ADE surface in $\mathbb{P}^3$, and let $\tilde{Y}$ be the minimal resolution of its singularities: then, setting $\nu := |\mathcal{E}|$, the binary code of Y, $\mathcal{K}[2] \subset (\mathbb{Z}/2)^\nu$, equals to the kernel of the map*

$$\oplus_1^\nu (\mathbb{Z}/2)[E_i] \to H^2(\tilde{Y}, \mathbb{Z}/2).$$

The vectors $v \in \mathcal{K}[2]$ correspond to subsets $\mathcal{E}_v$ of $\mathcal{E}$ such that $\sum_{i\in\mathcal{E}_v} E_i$ is linearly equivalent to $2L$, for some divisor class L on $\tilde{Y}$.

The cardinality t of such sets $\mathcal{E}_v$, i.e., the weights of the code vectors in $\mathcal{K}[2]$, are divisible by 4, and indeed by 8 if the degree $d := deg(Y)$ is even.

Finally, the dimension $k[2]$ of $\mathcal{K}[2]$ satisfies

$$(B)\ k[2] \geq \sum_x \delta_x - \Big[\frac{1}{2}\ b_2(\tilde{Y})\Big].$$

Here, $[[a]]$ denoting the smallest integer bigger than a, $\delta_x := [[\frac{n}{2}]]$ for A_n-singularities, $\delta_x := [[\frac{n+1}{2}]]$ for D_n-singularities, $\delta_x := 3, 4, 4$ for the respective singularities E_6, E_7, E_8.

Proof. Since Y has ADE singularities, by the theorem of Brieskorn and Tjurina [Bries68] [Tju70], $\tilde{Y}$ is diffeomorphic to a smooth surface of degree $d = deg(Y)$ in $\mathbb{P}^3$, which is simply connected, by Lefschetz' theorem. In particular, $H^2(\tilde{Y}, \mathbb{Z})$ is a free $\mathbb{Z}$-module, $H^2(\tilde{Y}, \mathbb{Z}/2) = H^2(\tilde{Y}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}/2$, and the second assertion follows immediately from the first.

By theorem 3, applied to $X := \tilde{Y}$ with $D := E_1 \cup \cdots \cup E_\nu$, we get that

$$H_1(Y^*, \mathbb{Z}) \cong \mathrm{coker}[H^2(\tilde{Y}, \mathbb{Z}) \to \oplus_1^\nu \mathbb{Z}[E_i]].$$

By the exact sequence

$$H^2(\tilde{Y}, \mathbb{Z}) \to \oplus_1^\nu \mathbb{Z}[E_i] \to H_1(Y^*, \mathbb{Z}) \to 0,$$

we tensor with $\mathbb{Z}/2$ and obtain an exact sequence

$$H^2(\tilde{Y}, \mathbb{Z}/2) \to \oplus_1^\nu \mathbb{Z}/2[E_i] \to H_1(Y^*, \mathbb{Z}/2) \to 0,$$

and taking the dual $\mathbb{Z}/2$-vector spaces, we obtain the exact sequence

$$0 \to \mathcal{K}[2] \hookrightarrow \oplus_1^\nu (\mathbb{Z}/2)[E_i] \to H^2(\tilde{Y}, \mathbb{Z}/2).$$

The statement about $|\mathcal{E}_v|$ was proven in [Cat81], Proposition 2.11, and reproven here in Proposition 28. The final assertion follows since the dimension of an isotropic subspace for the intersection form, which is nondegenerate, is at most $1/2$ of the dimension $b_2(\tilde{Y})$ of $H^2(\tilde{Y}, \mathbb{Z}/2)$.

Then we observe that for each singularity x we have δ_x equal to the maximal number of disjoint (-2)-curves, which generate an isotropic subspace.

∎

Remark 36. *i) The B-inequality is named after Beauville, see* [Bea80].
ii) If $\tilde{Y}$ *is a K3 surface,* $b_2(\tilde{Y}) = 22$*, hence* $\mathcal{K}[2]$ *has dimension at least* $\sum_x \delta_x - 11$.
iii) The generalized (B)-inequality was observed and used effectively in [Pet21].

7. The extended code $\mathcal{K}'$ and sublattice saturation.

In this section we extend the notion of the extended code $\mathcal{K}'$ from the case of nodal surfaces to the case of normal surfaces, in particular, surfaces with ADE singularities.

Recall that, for Y a nodal surface in $\mathbb{P}^3$ of even degree d, if H is a smooth hyperplane section of Y, then the **extended code** $\mathcal{K}'$ is defined as the kernel

$$\mathcal{K}' := ker[\oplus_1^\nu (\mathbb{Z}/2)[E_i] \oplus (\mathbb{Z}/2)[H] \to H^2(S, \mathbb{Z}/2)].$$

More generally, if Y is a normal projective surface, and $S = \tilde{Y}$ is the minimal normal crossing resolution of singularities, we set $D := E_1 \cup \dots \cup E_\nu \cup H$, where the E_i's are the exceptional curves, and H corresponds to the inverse image of a smooth hyperplane section, disjoint from $\Sigma := \mathrm{Sing}(Y)$.

We apply now Theorem 3 to the case where S is a smooth compact surface with $H_1(S, \mathbb{Z}) = 0$, $D = D_1 \cup \dots \cup D_{\nu+1}$ is a normal crossing divisor, with D_i is irreducible, hence $H_1(S \setminus D, \mathbb{Z})$ is the cokernel of

$$\rho : H^2(S, \mathbb{Z}) \to H^2(D, \mathbb{Z}) = \oplus_1^{\nu+1} \mathbb{Z}[D_i],$$

and $\rho(L) = \sum_i (L \cdot D_i)[D_i]$.

Definition 37. The underlying group of the extended code $\mathcal{K}'$ is defined as the dual of $H_1(Y^* \setminus H, \mathbb{Z})$.

The definition of $\mathcal{K}'$ as a generalized labeled code is through the surjection (6) of the next Theorem 38.

Theorem 38. *Let Y be a normal surface in $\mathbb{P}^3$, let S be a minimal normal crossing resolution of Y, and let S' be the complement in S of the union of the exceptional curves E_i with a smooth hyperplane section H (not passing through any node).*

(i) Then the first homology group $H_1(S', \mathbb{Z})$ is a quotient

$$\oplus_{x \in \Sigma} H_1(x) \oplus (\mathbb{Z}/d)[H] \twoheadrightarrow H_1(S', \mathbb{Z}),$$

where $d = H^2$ is the degree of Y;

(ii) moreover, its extended code $\mathcal{K}'$, which has underlying group equal to the dual of $H_1(S', \mathbb{Z}) = H_1(Y^* \setminus H, \mathbb{Z})$ determines the saturation of the lattice $\mathcal{L}$ generated by the curves E_i and by the hyperplane section H inside the lattice $\Lambda := H^2(S, \mathbb{Z})$.

More precisely, the saturation $\mathcal{L}^{sat}$ is such that $\mathcal{L}^{sat}/\mathcal{L} \cong (\mathcal{K}')^\vee$, and we have a surjection

$$\oplus_{x \in \Sigma} H_1(x) \oplus (\mathbb{Z}/d')[H] \to (\mathcal{K}')^\vee, \tag{6}$$

where $d' := \gcd(d, m)$, $m := \mathrm{lcm}(\{m_x, x \in \Sigma\}$, and m_x is the exponent of the group $H_1(x)$.

The same result holds for normal surfaces Y whose minimal resolution $S = \tilde{Y}$ has $H_1(S, \mathbb{Z}) = 0$ and is such that the hyperplane class H yields an indivisible element in $\mathrm{Pic}(S)$.

Proof. We have the exact sequence

$$\rho : \Lambda := H^2(S, \mathbb{Z}) \to H^2(D, \mathbb{Z}) = \oplus_1^\nu \mathbb{Z}[E_i] \oplus \mathbb{Z}H =: \mathcal{L} \to H_1(S \setminus D, \mathbb{Z}) \to 0,$$

and $\mathcal{L}$ embeds in Λ because the intersection product on $\mathcal{L}$ is non-degenerate.

Then $\mathcal{L}/\rho(\Lambda)$ is a quotient of $\mathcal{L}/\rho(\mathcal{L}) = \oplus_{x \in \Sigma} H_1(x) \oplus (\mathbb{Z}/d)[H]$ and (i) follows.

We defined the group underlying the code $\mathcal{K}'$ as the dual of

$$H_1(Y^* \setminus H, \mathbb{Z}) \cong \mathrm{coker}(H^2(S, \mathbb{Z}) \to \oplus_1^\nu \mathbb{Z}[E_i] \oplus \mathbb{Z}H), \tag{7}$$

the cokernel of the dual map of the embedding inside the lattice $\Lambda := (H^2(S, \mathbb{Z}) = H^2(\tilde{Y}, \mathbb{Z})$ of the lattice $\mathcal{L}$ (generated by the curves E_i and by the hyperplane section H).

In view of the exact sequence

$$0 \to \mathcal{L} \to \Lambda \to F \oplus T \to 0,$$

where T is a torsion group and F a free abelian group, dualizing we get (in view of the unimodularity of Λ)

$$0 \to F^\vee \to \Lambda \to \mathcal{L}^\vee \to Ext^1(T, \mathbb{Z}) \to 0,$$

showing that $T \cong H_1(Y^* \setminus H, \mathbb{Z})$.

Consider now the saturation $\mathcal{L}' := \mathcal{L}^{sat}$ of $\mathcal{L}$ inside Λ; $\mathcal{L}'$ is primitively embedded.

Since $\mathcal{L} \subset \mathcal{L}'$, and $\mathcal{L}'/\mathcal{L} \cong T$, it follows that $\mathcal{L}' = \mathcal{L}^{sat}$ is generated by $\mathcal{L}$ and by classes $\lambda \in \Lambda$ of the form

$$\lambda = \sum_{x \in \Sigma} \big(\frac{1}{m_x} \sum_i a_i E_{i,x}\big) + \frac{1}{d} bH, \;\; a_i, b \in \mathbb{Z},$$

where m_x is the exponent of the group $H_1(x)$.

Since $m := \operatorname{lcm}\{m_x, x \in \Sigma\}$, then, since $m\lambda \in \Lambda$, we obtain that $\frac{mb}{d} H \in \Lambda$.

But H is indivisible (cf. for instance [Cat81], page 464), hence $d := H^2$ divides mb; if d is relatively prime to m, this implies that d divides b, hence we may assume $b = 0$. Else, setting $d' = \gcd(m, d)$, $d = d'c, m = d'c'$, then $b = cb'$; hence

$$\lambda = \sum_{x \in \Sigma} (\frac{1}{m_x} \sum_i a_i E_{i,x}) + \frac{1}{d'} b' H, a_i, b' \in \mathbb{Z}.$$

The only facts that we have used are that $\operatorname{Pic}(\tilde{Y})$ is a free abelian group, because $H_1(S) = 0$, and that the class of H is indivisible.

□

7.1. **Weight restrictions for the extended code.** If we have now an extended code

$$\mathcal{K}' \hookrightarrow \oplus_{x \in \Sigma} H_1(x)^\vee \oplus (\mathbb{Z}/d')[H]$$

and a vector v which is not in $\mathcal{K}$, hence with 'second' coordinate $m \in \mathbb{Z}/d'$, we can again consider the primary decomposition, and obtain restrictions for the labeled weight using the same method as for $\mathcal{K}$, namely taking the cyclic covering branched on H and on a subset $\mathcal{N}$ of the singular set Σ.

We shall work out more systematically these restrictions in the sequel to this first part, but let us give easy extensions of the previous Propositions 28 and 29.

Proposition 39. *Let v be a code vector in $\mathcal{K}' \setminus \mathcal{K}$ of order $N = 2$. Define $\mathcal{N}$ as the subset of the singular points $x \in \Sigma$ such that $H_1(x)$ surjects onto $\mathbb{Z}/2$. Let $t(A_{2m+1})$ be the A_{2m+1}-weight of v, that is, the number of points in $\mathcal{N}$ which are of type A_{2m+1}. Let similarly $t(E_7)$ be the number of points in $\mathcal{N}$ which are of type E_7, $t(D_n, -)$ the number of points in $\mathcal{N}$ which are of type D_n, and of subtype $(-)$ for v if n is even, $t(D_{2m}, +)$ the number of points in $\mathcal{N}$ which are of type D_{2m}, and of subtype $(+)$ for v.*

Then

$$(**) \;\; \sum_m ((m+1)\; t(A_{2m+1}) + m\; t(D_{2m}, +)) + \sum_n 2t(D_n, -) + 3t(E_7) - \frac{d}{2} - K_S \cdot H$$

is divisible by 4.

Proof. Observe again that $F_*(\mathcal{O}_Z) = \mathcal{O}_S \oplus \mathcal{O}_S(-L)$, and by Riemann-Roch

$$\chi(-L) = \chi(\mathcal{O}_S) + \frac{1}{2} L(L + K_S) = \chi(\mathcal{O}_S) + \frac{1}{2}(L^2 + LK_S).$$

We have $K_S L = \frac{1}{2} K_S H$, in view of $2L \equiv \sum_{x \in \mathcal{N}} B_x + H$, and since the B_x are exceptional.

Hence 8 divides

$$4L^2 + 2K_S H = \sum_{x\in\mathcal{N}} B_x^2 + H^2 + 2K_S H,$$

and this condition is equivalent to $(**)$.

$\square$

Proposition 40. *Let v be a code vector in $\mathcal{K}' \setminus \mathcal{K}$ of order $N = 3$, and let $v(H) =: \epsilon = \pm 1$.*

Define $\mathcal{N}$ as the subset of the singular points $x \in \Sigma$ such that $H_1(x)$ surjects onto $\mathbb{Z}/3$. Let $t(A_{3s-1})$ be the A_{3s-1}-weight of v, that is, the number of points in $\mathcal{N}$ which are of type A_{3s-1}. Let similarly $t(E_6)$ be the number of points in $\mathcal{N}$ which are of type E_6,

Then d is divisible by 3 and

$$(***)\ \sum_s s\ t(A_{3s-1}) - t(E_6) + \frac{d}{3} + \epsilon K_S H$$

is divisible by 3.

Proof. Observe that $F_*(\mathcal{O}_Z) = \mathcal{O}_S \oplus \mathcal{O}_S(-L) \oplus \mathcal{O}_S(-L)$.

And, as in the proof of Proposition 28, by Riemann-Roch

$$\chi(-L) = \chi(\mathcal{O}_S) + \frac{1}{2}L(L+K_S).$$

Since $3L \equiv \sum_{x\in\mathcal{N}} B_x + \epsilon H$, we get

$$L(L+K_S) = L^2 + \frac{1}{3}\epsilon K_S H,$$

and

$$9L^2 = (\sum_{x\in\mathcal{N}} B_x)^2 + H^2, \quad 9L(L+K_S) = (\sum_{x\in\mathcal{N}} B_x)^2 + H^2 + 3\epsilon K_S H,$$

hence 18 divides

$$= -6[2t(E_6) + \sum_s s\ t(A_{3s-1})] + d + 3\epsilon K_S H.$$

The previous formulae show that d is divisible by 3, and that $d + 3\epsilon K_S H$ is an even number, hence 3 divides

$$= -2[2t(E_6) + \sum_s s\ t(A_{3s-1})] + \frac{d}{3} + \epsilon K_S H,$$

and this is equivalent to

$$(***)\ \sum_s s\ t(A_{3s-1}) - t(E_6) + \frac{d}{3} + \epsilon K_S H$$

being divisible by 3.

$\square$

8. Examples: the ancestors for surfaces of degree ≤ 3 with ADE singularities and some K3 ancestors

If Y is a singular normal surface in $\mathbb{P}^3$ of degree d, then, for $d = 2$, Y must be the quadric cone.

Moreover, for all d, Y has ADE singularities [Art66] if the singular points are rational and have multiplicity 2.

Even if almost everything is known for cubic surfaces ($d = 3$), we want to describe the ancestors from our point of view (see especially Table 9.1 of [Dol12], and also Section 8 of [Cat21]).

Observe before hand that the minimal resolution S of Y has $b_2(S) = 7$, hence the number of exceptional curves is smaller or equal to 6, and if equality holds, we surely have an ancestor (since this number, which we called rank, is maximal).

i) For nodal surfaces the ancestor is the 4-nodal Cayley cubic

$$\mathcal{C}_3 := \left\{ x \;\Big|\; \sum_0^3 \frac{x_0x_1x_2x_3}{x_j} = 0 \right\}.$$

It is also an ADE ancestor because $\mathcal{C}_3$ is the only cubic with at least four singular points.

ii) The next examples are ancestors because the rank, i.e. the number of exceptional curves, will be equal to 6.

iii) Another ancestor is the cubic with 3 A_2 singularities (3 cusps)

$$\mathcal{Y}_3^{cu} := \{x \mid x_0^3 - x_1x_2x_3 = 0\}.$$

iv) A third ancestor is the cubic with an E_6-singularity,

$$\mathcal{Y}_3^{E_6} := \{x \mid x_0x_3^2 - g(x_1, x_2, x_3) = 0\},$$

where, at the point $(1, 0, 0, 0)$, the plane cubic $g = 0$ has the line $x_3 = 0$ as an inflectionary tangent.

v) The fourth ancestor is the surface S obtained blowing up 6 infinitely near points in the plane such that the first 3 do not lie on a line but the 6 do lie on a conic: then the anticanonical system of S maps to a cubic $Y^{5,1}$ which has an A_5 singularity and a disjoint A_1 singularity.

Because, if E'_i is the total transform of the i-th point, then $H := 3L - \sum_i E'_i$ has $H^2 = 3$ and contracts each $E_i, \forall i \geq 2$, and also contracts the proper transform of the conic $Q \in |2L - \sum_i E'_i|$.

We can now describe the associated codes, even without using the restrictions for the weights of $\mathcal{K}'$.

But we observe that Propositions 29 and 40 imply that for vectors in $\mathcal{K}$ (of order 3) the difference between the number of A_2-singularities and the number of E_6-singularities appearing is divisible by 3. Whereas, for vectors in $\mathcal{K}' \setminus \mathcal{K}$ the same number is congruent to 0 or 2 modulo 3.

In general the sublattice $\mathcal{L} = \mathcal{L}_0 \oplus \mathbb{Z}H$, where $\mathcal{L}_0 := \oplus_i \mathbb{Z}E_i$, is contained in the lattice $\Lambda = H^2(S, \mathbb{Z})$, and we have the inclusions

$$\mathcal{L} = \mathcal{L}_0 \oplus \mathbb{Z}H \subset \mathcal{L}^{sat} \subset \Lambda = \Lambda^\vee \subset (\mathcal{L}^{sat})^\vee \subset \mathcal{L}_0^\vee \oplus \mathbb{Z}H^\vee.$$

We have seen that $(\mathcal{K}')^\vee \cong \mathcal{L}^{sat}/\mathcal{L}$, and that $\mathcal{K}^\vee = \operatorname{coker}(\Lambda \to \mathcal{L}_0^\vee)$.

i) For the Cayley cubic

$$\mathcal{C}_3 := \left\{ x \;\Big|\; \sum_0^3 \frac{x_0x_1x_2x_3}{x_j} = 0 \right\},$$

the code $\mathcal{K} = \mathbb{Z}/2(\sum_1^4 e_i) \subset (\mathbb{Z}/2)^4$(well known, see also [Cat22]), and it equals $\mathcal{K}'$ since $d = 3$ is odd.

ii) In the next examples, since the lattices $\mathcal{L}, \Lambda$, have the same rank, $\mathcal{L}^{sat} = \Lambda$, hence $(\mathcal{K}')^\vee = \Lambda/\mathcal{L}$. Since Λ is unimodular, and with the same rank as $\mathcal{L}$, Λ corresponds to an isotropic subspace of the discriminant lattice

$$\Delta := (\mathcal{L}')^\vee/\mathcal{L}' = ((\mathcal{L}'_0)^\vee/\mathcal{L}'_0) \oplus ((\mathbb{Z}/3)H).$$

iii) For the cubic with 3 cusps (A_2 singularities)

$$\mathcal{Y}_3^{cu} := \{x \mid x_0^3 - x_1x_2x_3 = 0\},$$

$\Delta \cong (\mathbb{Z}/3)^3 \oplus \mathbb{Z}/3$, hence $\mathcal{K}' \cong (\mathbb{Z}/3)^2$, while $\mathcal{K} \cong \mathbb{Z}/3$. The last assertion because Y is a quotient $Y = \mathbb{P}^2/(\mathbb{Z}/3)$, with action having as fixed points exactly the 3 coordinate points: hence $\pi_1(Y^*) = \mathbb{Z}/3$.

$\Lambda/\mathcal{L}$ is generated by $\frac{1}{3}(E_1 - E_2 - E_3 + E_4 + H)$ and by $\frac{1}{3}(E_1 - E_2 - E_5 + E_6 + H)$.

iv) For the cubic with an E_6-singularity,

$$\mathcal{Y}_3^{E_6} := \{x \mid x_0 x_3^2 - g(x_1, x_2, x_3) = 0\},$$

$\Delta \cong \mathbb{Z}/3 \oplus \mathbb{Z}/3$, and $\mathcal{K}' \cong \mathbb{Z}/3$, $\mathcal{K} = 0$. $\Lambda/\mathcal{L}$ is generated by $\frac{1}{3}(E_1 - E_2 - E_5 + E_6 + H)$.

v) For the cubic with an A_5 singularity and a disjoint A_1 singularity, $\Delta \cong (\mathbb{Z}/6 \oplus \mathbb{Z}/2) \oplus \mathbb{Z}/3$, hence $\mathcal{K}' \cong \mathbb{Z}/6$ and $\mathcal{K} \cong \mathbb{Z}/2$. Here the generator of $\Lambda/\mathcal{L}$ can be directly found using the representation of S as a blow up of the plane:

$$\frac{1}{6}\big[- 2H + 3E_6 - \sum_1^5 iE_i \big].$$

Once we have the ancestors, we can find all the other codes by geometrically driven shortenings, in view of the fact that cubic surfaces with ADE singularities are unobstructed.

We shall next show a (very partial) list of ancestors for (non-polarized) K3 surfaces, based on the following Proposition.

Proposition 41. *Let $Y = T/G$, where G is a finite group acting on a 2-dimensional complex torus with a strictly positive but finite number of fixpoints, and such that Y has only ADE singularities, so that Y has trivial canonical sheaf ω_Y.*

Then Y is an ancestor.

Moreover, let $T = \mathbb{C}^2/\Lambda$, and let Γ be the group of affine transformations such that $Y = \mathbb{C}^2/\Gamma$, so that we have an exact sequence

$$1 \to \Lambda \to \Gamma \to G \to 1.$$

Then the code group $\mathcal{K}$ is isomorphic to Γ^{ab}, and we have an exact sequence

$$\Lambda_G \to \Gamma^{ab} \to G^{ab} \to 1,$$

where Λ_G is the group of G-covariants, the quotient of Λ by the subgroup generated by $\{\lambda - g\lambda \mid \lambda \in \Lambda\}$.

If G is cyclic, then $\Lambda_G \hookrightarrow \Gamma^{ab}$.

Moreover, if $\Gamma = \Lambda \rtimes G$, then $\mathcal{K} \cong G \oplus \Lambda_G$.

Proof. If Y were a partial smoothing of another singular K3 surface X with ADE singularities, then since $\pi_1(X^*) \twoheadrightarrow \pi_1(Y^*)$, then also $\pi_1(X^*)$ would be infinite and by [Camp04] also X would be a Torus quotient, and the configuration of exceptional curves of Y would be a geometric shortening of another torus quotient.

Inspecting the list contained in Theorem 3.6 of [GPP24], based on the results of [Fuj88], we see that this cannot occur for the cases (5)-(10), because the number of exceptional curves is 19.

For the cases (1)-(4), one sees that the configuration of exceptional curves of Y is not a geometric shortening of another one in the list.

We have $\pi_1(Y^*) = \Gamma$, since $\mathbb{C}^2$ is the orbifold universal covering of Y^*, hence $\mathcal{K}$ is isomorphic to Γ^{ab}.

Now, let $\lambda_1, \dots, \lambda_4$ be a system of generators of Λ, $g_1, \dots, g_r$ be a system of generators of G, and $\gamma_1, \dots, \gamma_r$ be lifts of these generators to Γ.

Then Γ^{ab} has the above as generators, and we must add the relations

$$[\gamma_j, \lambda_i] = 1 = [\gamma_h, \gamma_j], \forall i = 1, 2, 3, 4,\ j, h, = 1, \dots, r.$$

Then the image of Λ inside Γ^{ab} factors through Λ_G; and if G is cyclic we have an exact sequence.

In the case of a semidirect product, G is a subgroup, hence we may take $\gamma_j = g_j$.

$\square$

Example 42. For K3 surfaces, two ancestors are well known ([Nik79], [Barth98]):

1) the Kummer surfaces $X = T/\pm 1$, with 16 A_1 singularities, (16 nodes), which are quartic surfaces in $\mathbb{P}^3$ if T is a principally polarized 2-dimensional torus: here $G = \{\pm 1\}$, and since $\Gamma = \Lambda \rtimes G$ we have $\mathcal{K} \cong (\mathbb{Z}/2)^5$;

2) the K3 surfaces $Y = (F \times F)/\mathbb{Z}/3$ with action generated by (ζ, ζ^2). Here F is the Fermat cubic, ζ is a primitive 3rd root of unity, and Y has 9 cusps. Here $\Lambda = (\mathbb{Z} \oplus \zeta\mathbb{Z})^2$, the covariants $(\mathbb{Z} \oplus \zeta\mathbb{Z})_G \cong \mathbb{Z}/3$, again we have a semidirect product, hence $\mathcal{K} \cong (\mathbb{Z}/3)^3$.

Barth [Barth00] proved that the latter may only yield surfaces of degree $d \equiv 0, 2 \pmod 6$.

3) Theorem 3.6 of [GPP24] contains a list of the exceptional configurations for 10 quotients of a 2-dimensional complex torus, the first two being the above two examples 1) and 2). They are all ancestors, by Proposition 41.

In the cases (5)-(10) the number of exceptional curves is 19, in the cases (3) and (4) the number of exceptional curves for Y is 18, but the configuration of exceptional curves of Y is not a direct geometric shortening of another one in the list with 19 exceptional curves.

4) Needless to say, an interesting question concerns the degrees of the possible polarizations on the above surfaces in the cases (3)-(10).

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