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DARBOUX INTEGRABILITY VIA SINGULARITIES OF INVARIANT CURVES AT INFINITY

HANS-CHRISTIAN VON BOTHMER

To the memory of Wolfgang Ebeling, for coffee, cookies, and singularities.

Abstract. We extend the computation of the invariant $\eta(\omega, C, a)$ defined in [4] to special points on the line at infinity and show that, as in the affine case, its value is determined purely by the geometry of the integral curve C. By incorporating points at infinity, the invariant η yields effective geometric criteria that certify Darboux integrability in cases not covered by affine data alone. As an application we construct six new codimension-11 components of the degree-3 center variety.

Introduction

In 1878 Darboux showed that first integrals of planar polynomial vector fields can be obtained by studying their algebraic invariant curves and the associated cofactors [13]. In particular, if the cofactors of several invariant curves satisfy a nontrivial linear relation, then a rational first integral exists and can be written explicitly as a Darboux product. Darboux also observed—and later results made precise—that the existence of sufficiently many invariant curves forces such a dependence automatically. The number of invariant curves required can in fact be reduced considerably by taking into account the multiplicities of the invariant curves and the singularities *of the differential form* [12].

In [4] Darboux theory was developed in a different direction by taking into account the singularities *of the integral curves* themselves. This makes the computation entirely independent of the differential form and hence applies uniformly to all differential forms admitting a fixed configuration of integral curves. While [4] treated only affine singularities, the present note extends the approach to the line at infinity and, in Theorem 4.15, treats both affine and infinite singularities on an equal footing.

A central application of Darboux theory is to the Poincaré center problem, which seeks criteria ensuring that a singularity of a planar polynomial vector field is a center rather than a focus [22]. The existence of a rational first integral forces trajectories to be closed, so Darboux integrability provides a direct mechanism for producing centers [13]. In the quadratic case every center arises this way, yielding a complete classical description [15]. For cubic systems, many centers are explained by Darboux integrability and substantial classifications are known, but a full classification of Darboux-integrable cubic vector fields remains open.

Lacking a complete classification, a complementary approach is to construct families of Darboux-integrable degree-3 centers, with the hope of eventually capturing all of them and guiding a fuller classification through examples. The first systematic list of such examples was compiled by Żołądek in [31]. Numerical evidence over finite fields [8, 25] seems to suggest that we are still far from a complete list: in codimension 11, for instance, we found only five components of the center variety in the literature [3], [26], whereas characteristic-p data seem to suggest there are at least 97.

To assess the usefulness of our refinement of Darboux's method, we exhibit six new codimension-11 components of the degree-3 center variety and recover two previously known ones. More precisely, we work with submaximal sextics C (degree-6 curves with simple singularities of total Milnor number 18) for which the line at infinity is bitangent to C. These curves enjoy several favorable numerical coincidences: the expected counts predict

(1) a unique degree-3 polynomial 1-form ω (up to scale) having C as an integral curve;
(2) a single zero of ω outside $C \cup L_\infty$; and
(3) an expected codimension 11 for the family of normalized differential forms obtained by varying C in its equisingular family.

With a careful choice of the singularity types and component structure of C, our extension of Darboux's method to infinity guarantees the existence of an integrating factor and hence a center at the extra zero of ω.

Across all our examples, Darboux integrability cannot be certified from affine data alone—singularities at infinity provide the missing constraints. This shows that extending η to singularities at infinity can substantially expand the range of accessible examples.

The paper is organized into the following sections:

1. We fix notation and recall Darboux's method.
2. We review the *inverse problem* (finding all differential forms having a given curve as an integral curve). The general solution is due to [11]. We recall a version that incorporates the singularities of C following [4].
3. We recall the definition of our invariant η from [4] and how, at affine points, η can be computed purely from the geometry of C.
4. We extend the definition of η to the line at infinity and show how to compute it there (Proposition 4.9). The main result of this paper, our extension of Darboux integrability, is stated in Theorem 4.15.
5. We obtain an upper bound for the number of zeros of ω outside C in terms of the singularities of C.
6. We summarize what is known about the Poincaré center problem in degree 3 over finite fields.
7. We give a blueprint for constructing codimension-11 components from submaximal sextics and use characteristic-p information to select promising baskets of singularities. We then construct six new components and show how two previously known ones are recovered by the same method.

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1. Preliminaries

In this article, we describe a plane autonomous system by a differential form $\omega = P\,\mathrm{dx} + Q\,\mathrm{dy}$ where $P, Q \in \mathbb{C}[x, y]$ are polynomials of degree at most d.

Definition 1.1. Let ω be a differential form, and let $F, \mu \in \mathbb{C}[[x, y]]$ be power series. If

$$dF = \mu\omega$$

then F is called a *first integral* and μ an *integrating factor* of ω

For a given ω it can often be very difficult to decide, whether a first integral exists. Darboux realized in 1878 that the existence of algebraic integral curves can help to answer this question:

Definition 1.2. Let $C \in \mathbb{C}[x, y]$ be a polynomial and $\{C = 0\}$ the plane algebraic curve defined by C. The zero locus $\{C = 0\}$ is called an *algebraic integral curve* of a differential form ω if and only if

$$dC \wedge \omega|_C = 0 \iff dC \wedge \omega = C \cdot K_C$$

for some polynomial 2-form $K_C = k_C \,\mathrm{dx} \wedge \mathrm{dy}$. In this situation the 2-form K_C is called the *cofactor* of C and k_C its coefficient polynomial.

In a slight abuse of notation, we also denote the algebraic curve $\{C = 0\}$ by the same letter C.

Theorem 1.3 (Darboux 1878)**.** *Let ω be a differential form, $C_1, \dots, C_r$ algebraic integral curves of ω and $K_1, \dots, K_r$ their cofactors.*

- *If $\sum \alpha_i K_i = -d\omega$ for appropriate $\alpha_i \in \mathbb{C}$ then $\mu = \prod C_i^{\alpha_i}$ is a rational integrating factor of ω.*
- *If $\sum \alpha_i K_i = 0$ for appropriate $\alpha_i \in \mathbb{C}$ then $F = \prod C_i^{\alpha_i}$ is a first integral of ω.*

Proof. For the first claim we calculate:

$$\begin{aligned} d(\mu\omega) &= d\mu \wedge \omega + \mu d\omega \\ &= \mu\big(\frac{d\mu}{\mu} \wedge \omega + d\omega\big) \\ &= \mu(d\log\mu \wedge \omega + d\omega) \\ &= \mu\big(\sum \alpha_i d\log C_i \wedge \omega + d\omega\big) \\ &= \mu\big(\sum \alpha_i \frac{dC_i}{C_i} \wedge \omega + d\omega\big) \\ &= \mu\big(\sum \alpha_i K_i + d\omega\big) \\ &= 0 \end{aligned}$$

consequently there exists an F with $dF = \mu\omega$.

For the second claim we observe that

$$dF = \mu\omega \iff dF \wedge \omega = 0.$$

We now compute

$$\begin{aligned} dF \wedge \omega &= F\Big(\frac{dF}{F} \wedge \omega\Big) \\ &= F(d\log F \wedge \omega) \\ &= F\sum \alpha_i d\log C_i \wedge \omega \\ &= F\sum \alpha_i K_i \\ &= 0 \end{aligned}$$

□

Sometimes the following trivial observation is useful:

Lemma 1.4. *Let C, D be plane algebraic curves without common components and ω a differential form. Then C and D are integral curves of ω if and only if $C \cup D$ is an integral curve of ω.*

Furthermore if K_C, K_D and $K_{C\cup D}$ are the respective cofactors, we have

$$K_C + K_D = K_{C\cup D}.$$

Proof. Locally, outside the intersection points of C and D, the integral curve condition holds for both curves individually if and only it holds for the union. Since the integral curve condition is a closed condition this equivalence must also hold globally.

For the cofactors we have

$$\begin{aligned} CDK_{C\cup D} &= d(CD) \wedge \omega \\ &= \big((dC)D + CdD\big) \wedge \omega \\ &= CK_C D + CDK_D \\ &= CD(K_C + K_D) \end{aligned}$$

□

2. The inverse problem

For a given configuration of integral curves $\{C_1, \dots, C_n\}$ one can consider the so called *inverse problem* of finding all differential forms ω that have this curve configuration as integral curve. The inverse problem was solved in [11]. Here we recall a version of this solution that takes into account the singularities of the curve configuration from [4].

Definition 2.1. Let $C_1, \dots, C_r \in \mathbb{C}[x, y, z]$ be homogeneous polynomials. Then

$$\begin{pmatrix} C_{1x} & C_{1y} & C_1 & & \\ \vdots & \vdots & & \ddots & \\ C_{rx} & C_{ry} & & & C_r \end{pmatrix}$$

is called the *Darboux Matrix* of the configuration $C_1, \dots, C_r$.

Proposition 2.2. *Let $C_1, \dots, C_r$ be a configuration of plane curves, and write $K_i = k_i \,\mathrm{dx} \wedge \mathrm{dy}$. A differential form $\omega = P\,\mathrm{dx} + Q\,\mathrm{dy}$ has integral curves C_i with cofactors K_i if and only if*

$$M \cdot (Q, -P, -k_1, \dots, -k_r)^t = 0,$$

where M is the Darboux matrix of $C_1, \dots, C_r$.

Proof. The definition of integral curve gives, using $K_i = k_i \,\mathrm{dx} \wedge \mathrm{dy}$,

$$\begin{aligned} 0 &= dC_i \wedge \omega - C_i \cdot K_i \\ &= (C_{ix}\,\mathrm{dx} + C_{iy}\,\mathrm{dy}) \wedge (P\,\mathrm{dx} + Q\,\mathrm{dy}) - (C_i k_i)\,\mathrm{dx} \wedge \mathrm{dy} \\ &= (C_{ix} Q - C_{iy} P - C_i k_i)\,\mathrm{dx} \wedge \mathrm{dy}\,. \end{aligned}$$

Writing these equations in matrix form gives the claimed identity. □

Remark 2.3. We can consider P, Q, C_i and k_i either as affine or as homogeneous polynomials in Proposition 2.2. Notice that the condition of the proposition is compatible with homogenization and dehomogenization. Observing that

$$d\omega = d(P\,\mathrm{dx} + Q\,\mathrm{dy}) = (Q_x - P_y)\,\mathrm{dx} \wedge \mathrm{dy}$$

we see that also the conditions of Darboux' Theorem can be written as conditions between P, Q, C_i and k_i, and make sense for affine and projective polynomials. These conditions are again compatible with homogenization and dehomogenization.

Notice also that we only homogenize the coefficients of dx, dy and dx ∧ dy. This is therefore slightly different from pulling back differential forms from $\mathbb{C}^2$ to $\mathbb{P}^2$ where one would for example pull back dx to

$$d\left(\frac{x}{z}\right) = \frac{z\,\mathrm{dx} - x\,\mathrm{dz}}{z^2}.$$

We now restrict to the case of one integral curve. For the inverse problem this is not a restriction, since, by Lemma 1.4, a reduced (possibly reducible) curve $C = C_1 \cup \cdots \cup C_r$ is an integral curve of a differential form if and only if all its irreducible factors are.

Definition 2.4. Let $C \in \mathbb{C}[x, y, z]$ be a homogeneous polynomial of degree e with no multiple factors, and let

$$M_C := (C_x, C_y, C)$$

be its Darboux matrix. We define

$$\mathcal{V}_C(d) := (\ker M_C)_d$$

to be the space of degree-d differential forms that admit C as an integral curve.

The vector $H_C := (C_y, -C_x, 0)^t$ is always in the kernel of M_C and represents the Hamiltonian vector field associated to C. Let

$$\mathcal{V}_C^H(d) := (H_C)_d$$

be the *vector space of trivial differential forms*. Its elements are of the form FH_C, where $F \in \mathbb{C}[x, y, z]$ is homogeneous of degree $d - e + 1$.

We now turn to singularity theory

Definition 2.5. Let $C \in \mathbb{C}[x, y, z]$ be a homogeneous polynomial with no multiple factors and no components at infinity, and let P be a point on the curve defined by C. We denote the Tjurina number of C at P by $t(P)$.

If P lies on the line at infinity $L_\infty = \{z = 0\}$, choose local coordinates at P such that L_∞ is given by $z = 0$ and $P = (0, 0)$. We then define the *modified Tjurina number* of C at P by

$$t_z(P) := \dim \frac{\mathbb{C}[[y, z]]}{(C, C_y, zC_z)}.$$

Lemma 2.6. *Let C be a plane curve intersecting the line at infinity $L_\infty = \{z = 0\}$ at a point P such that $C \cup L_\infty$ has a quasi-homogeneous singularity at P. Then*

$$t_z(P) = t(P) + i - 1,$$

where i is the intersection number of C with L_∞ at P.

Proof. The proof is the same as that of [4, Prop. 3.5], which works verbatim under the present hypothesis. □

Example 2.7. The so-called *simple singularities* are listed in Table 1. The simple singularities are all quasi-homogeneous and therefore the Tjurina number t is equal to the Milnor number m. Furthermore, for simple singularities, the maximal number c of independent conditions imposed on a polynomial by the requirement that it has such a singularity at some point is equal to the Tjurina number.

name	local equation	$m = t = c$
node (A_1)	$x^2 - y^2$	1
cusp (A_2)	$x^2 - y^3$	2
tacnode (A_3)	$x^2 - y^4$	3
A_n	$x^2 - y^{n+1}$	n
triple point (D_4)	$x^3 - y^3$	4
D_n	$y(x^2 - y^{n-2})$	n
E_6	$x^3 - y^4$	6
E_7	$x(x^2 - y^3)$	7
E_8	$x^3 - y^5$	8

TABLE 1. The simple singularities

Theorem 2.8. *Let $C \in \mathbb{C}[x, y, z]$ be a homogeneous polynomial of degree e with no multiple factors and no components at infinity, let $M_C = (C_x, C_y, C)$ be the Darboux matrix, and let $X \subset \mathbb{P}^2$ be the finite scheme defined by the vanishing of M_C. We denote by $\deg X$ the length of X. Then*

$$\dim \mathcal{V}_C(d) \geq \binom{d-e+3}{2} + \binom{d+1}{2} - (e-1)^2 + \deg X =: \delta$$

where the first summand is 0 if $e > d+1$. We call δ the expected dimension of differential forms with integral curve C.

Let C_{sing} be the set of singular points of C outside of the line at infinity, and let C_∞ be the set of points of C that lie on the line at infinity. Then

$$\deg X = \sum_{P \in C_{\text{sing}}} t(P) + \sum_{P \in C_\infty} t_z(P),$$

where $t(P)$, $t_z(P)$ denote the Tjurina number and modified Tjurina number of the curve C at P.

Proof. [4, Prop. 4.4]. □

For our later applications , it is convenient to introduce the following global invariants.

Definition 2.9. Let C be a reduced plane curve with isolated singularities. We define the *total Milnor number* and *total Tjurina number* of C by

$$\mu(C) := \sum_{P \in \text{Sing}(C)} \mu(P), \qquad \tau(C) := \sum_{P \in \text{Sing}(C)} t(P),$$

where $\mu(P)$ and $t(P)$ denote the local Milnor number and local Tjurina number of C at P.

Example 2.10 (plane sextics)**.** Let C be a plane sextic curve. We want to find degree 3 differential forms ω having C as integral curve. Since $6 > 3 + 1$ the expected number of such differential forms is

$$\delta = \binom{3+1}{2} - (6-1)^2 + \deg X = \deg X - 19$$

So we need at least $\deg X = 20$ to ensure the existence of ω.

The maximum total Milnor number of a plane sextic with simple singularities is 19; see [19]. Such sextics are called *maximal*. To obtain $\deg X = 20$ we need in addition that C has at least one intersection of multiplicity 2 with the line at infinity. Unfortunately in this situation all zeros of ω lie either on C or on line at infinity, see Example 5.7.

If C is a submaximal sextic, i.e. one with total Milnor number 18 we need two intersections of multiplicity 2 (or one of multiplicity 3) to obtain $\delta > 0$. In this situation ω has generically one zero outside of C and the line at infinity. The examples of this paper are all of this type.

Since the intersection multiplicity of C with the line at infinity is 6, the contribution to $\deg X$ from infinity is at most 5. Hence the smallest possible total Milnor number of C with $\delta > 0$ is 15. Plane sextics with simple singularities have been classified; see [27, 29]. These classifications contain hundreds of cases, providing a rich source of examples for interesting cubic differential forms.

3. An analytic invariant

In [4] we introduced an analytic invariant η attached to a triple $(\omega, \{C_1, \dots, C_r\}, a)$ where ω is a differential form in the plane, $C_1, \dots, C_r$ are algebraic integral curves of ω, and a is any point in the plane:

Definition 3.1. Let ω be a differential form with algebraic integral curves $C_1, \dots, C_r$, cofactors $K_1, \dots, K_r$, and let a be a zero of ω. Then we define

$$\eta := \eta(\omega, \{C_1, \dots, C_r\}, a) := \big(K_1(a) : \cdots : K_r(a) : d\omega(a)\big).$$

Here $K_1(a), \dots, K_r(a), d\omega(a)$ are elements of the one-dimensional vector space of 2-forms at a, so the right-hand side is well defined as a projective ratio, i.e.

$$(u_1 : \cdots : u_{n+1}) = (\lambda u_1 : \cdots : \lambda u_{n+1}) \quad \text{for } \lambda \neq 0.$$

The degenerate case $(0 : \cdots : 0)$ is also allowed.

While η is defined everywhere, it carries the most information if a is a zero of ω. The main property of η is, that it depends in many situations only on the singularities of the curve configuration, but not on ω. To make this precise we need the following:

Definition 3.2. Let C be a algebraic curve and $a \in C$ a point. We say that C has a quasi-homogeneous singularity at a if there exists local analytic coordinates x_a, y_a around a and a weighted grading $\deg_{\mathrm{a}}$ on $\mathbb{C}[x_a, y_a]$ such that the local equation C_a of C at a is quasi-homogeneous with respect to $\deg_{\mathrm{a}}$.

Example 3.3. The curve $C = x^2 - y^3$ has a quasi-homogeneous singularity at $a = (0, 0)$, since for $x_a = x$ and $y_a = y$ the equation local equation $C_a = C$ is quasi-homogeneous with respect to the grading given by $\deg_{\mathrm{a}} x_a = 3$ and $\deg_{\mathrm{a}} y_a = 2$. Indeed all monomials of C_a have degree 6 in this grading

Definition 3.4. Let $C = C_1 \cup \cdots \cup C_r$ be a plane curve. A point $P \in C$ is called *η-geometric* if

(1) C has a quasi-homogeneous singularity at P.
(2) P is *not* a point where exactly two distinct curves of $\{C_1, \dots, C_r\}$ intersect transversally.

Theorem 3.5. *Let $C = C_1 \cup \cdots \cup C_r$ be an algebraic integral curve for a differential form ω, and let a be an η–geometric point of C not lying on the line at infinity. Then either*

$$\eta = (\deg_{\mathrm{a}} C_1 : \cdots : \deg_{\mathrm{a}} C_r : \deg_{\mathrm{a}} x_a + \deg_{\mathrm{a}} y_a),$$

or $\eta = (0 : \cdots : 0)$.

Proof. The case in which a lies on a single component C_i is covered by [4, Prop. 5.3]. In particular, the case of an A_1 singularity on a single component is included in this proposition. If a lies on exactly two components, [4, Prop. 5.7] applies under the additional hypothesis

$$\deg_{\mathrm{a}} C \;>\; \deg_{\mathrm{a}} x_a + \deg_{\mathrm{a}} y_a.$$

Type	Equation	η, if not (0:0)
node (A_1)	$x^2 - y^2$	1:1
cusp (A_2)	$x^2 - y^3$	6:5
tacnode (A_3)	$(x - y^2)(x + y^2)$	2:2:3
A_n, n even	$x^2 - y^{n+1}$	(2n+2):(n+3)
A_n, n odd	$(x - y^{\frac{n+1}{2}})(x + y^{\frac{n+1}{2}})$	(n+1):(n+1):(n+3)
triple point (D_4)	$y(x - y)(x + y)$	1:1:1:2
D_n, n even	$y(x^2 - y^{\frac{n}{2}-1})(x^2 + y^{\frac{n}{2}-1})$	2:(n-2):(n-2):n
D_n, n odd	$y(x^2 - y^{n-2})$	2:(2n-4):n
E_6	$x^3 - y^4$	12:7
E_7	$x(x^2 - y^3)$	9:5
E_8	$x^3 - y^5$	15:8

TABLE 2. η for simple singularities

This extra hypothesis fails only at an A_1 singularity, that is, when the two components meet transversally at a. However, by our assumption that a is η-geometric, this case does not occur. The general case, where a lies on at least three components, follows by a straightforward adaptation of the argument in [4, Prop. 5.7]. □

Example 3.6. For simple singularities we obtain the values in Table 2 when all analytic branches lie on distinct components. If several branches lie on the same component, we add the corresponding entries (i.e. sum the branch weights in that coordinate). For example, three smooth components meeting in a triple point yield $\eta = (1\!:\!1\!:\!1\!:\!2)$, whereas a single component with a triple point yields $\eta = (3\!:\!2)$.

Example 3.7. We now demonstrate how η can be used in a concrete example to prove Darboux integrability using only the arrangement of singularities of an integral curve:

Let C be the dual curve of a smooth cubic E. Since E has 9 distinct inflection points, C is a sextic with 9 cusps and hence total Tjurina number 18. Since not all cusps lie on a line, some line through two cusps contains no third cusp, and we may assume that this line is the line at infinity. We are therefore in the situation of Example 2.10 and obtain a degree 3 differential form ω which has C as integral curve with some cofactor K.

At each of the 7 cusps $a_1, \dots, a_7$ that do not lie at infinity we have $\eta = (6 : 5)$ or $(0 : 0)$. This implies that $5K(a_i) - 6d\omega(a_i) = 0$ for $i = 1, \dots, 7$. Now either $5K - 6d\omega$ vanishes identically or its zero locus is a conic in the plane passing through these 7 distinct cusps. In the second case, the intersection multiplicity of the conic with the sextic at each a_i is at least 2, so the total intersection number is at least $2 \cdot 7 = 14$. Since the sextic is irreducible, Bézout's theorem gives total intersection number $2 \cdot 6 = 12$, a contradiction. It follows that $d\omega = \frac{5}{6}K$. But this is exactly the condition for the existence of a Darboux integrating factor.

Notice the following: If we start with an elliptic curve E defined over the real numbers (or even the rationals), we obtain $C, \omega, d\omega$ and K over the same field, but the coordinates of the cusps are in general defined only over $\mathbb{C}$. Nevertheless the argument above still implies $d\omega = \frac{5}{6}K$ and hence ω still has a Darboux integrating factor. In other words, even singularities which are not defined over the real numbers, do help to proving the existence a Darboux integrating factor for a real differential form.

Theorem 3.5 describes η at quasi-homogeneous singularities. We now complement this by showing that non-quasi-homogeneous singularities contribute only the degenerate value $(0 : \dots : 0)$.

To extend the picture beyond the quasi-homogeneous case, we first study how the cofactor reflects the local geometry of the curve.

Lemma 3.8. *Let C be an algebraic integral curve of a differential form ω, and let K_C be its cofactor. Let $a \in C$ be a singular point of C. If $K_C(a) \neq 0$, then C is quasi-homogeneous at a.*

Proof. Choose local coordinates around a and write

$$\omega = P \,\mathrm{dx} + Q \,\mathrm{dy} \qquad \text{and} \qquad K_C = k_C \,\mathrm{dx} \wedge \mathrm{dy}\,.$$

Since C is an integral curve of ω, we have locally

$$C_x Q - C_y P - C k_C = 0.$$

Because $K_C(a) \neq 0$, we have $k_C(a) \neq 0$, so k_C is a unit in the local ring at a. Hence

$$C = k_C^{-1}(C_x Q - C_y P) \in (C_x, C_y).$$

By Saito's criterion [23], this implies that C is quasi-homogeneous at a. □

To treat the remaining case, we analyze the lowest-degree terms of the integral curve equation with respect to a suitable weighted grading induced by the Newton polygon.

Notation 3.9. Let C be a reduced germ of plane curve that is neither smooth nor a node. After a suitable change of coordinates, we may assume that the germ is taken at $(0,0)$ and that $C \in \mathbb{C}[[x,y]]$. In this situation the Newton polygon of C contains at least one compact edge. We use the following notation:

(1) Δ denotes the steepest compact edge of the Newton polygon of C;
(2) C_Δ denotes the sum of the terms of C whose exponents lie on Δ;
(3) $\deg x = a$ and $\deg y = b$ denote the grading on $\mathbb{C}[[x,y]]$, where $-\frac{a}{b}$ is the slope of Δ, written with coprime positive integers a and b.

Note that C_Δ is quasi-homogeneous with respect to this grading.

We first show that one can choose local coordinates in such a way that the initial form along the steepest Newton edge avoids certain degenerate cases.

Lemma 3.10. *Let C be a reduced curve germ such that C is neither smooth nor a node. Then one can choose local coordinates x, y such that none of the following holds:*

(1) C_Δ is a monomial;
(2) $C_\Delta = \lambda(x - cy^a)^k$ for some nonzero $\lambda, c \in \mathbb{C}$;
(3) $C_\Delta = \lambda(y - cx^b)^k$ for some nonzero $\lambda, c \in \mathbb{C}$.

Proof. Let $C' \in \mathbb{C}[[x,y]]$ be a branch of C. Since C' is irreducible, its Newton polygon has only one compact edge Δ' with slope $-\frac{a'}{b'}$. After possibly switching x and y, we may assume that $a' \leq b'$. Let

$$y = \sum_i \alpha_i x^{i/n}$$

be a Puiseux expansion of C'. See, for example, [10, Chapter 2]. Then the smallest exponent with nonzero coefficient α_i is $\frac{i}{n} = \frac{b'}{a'} \geq 1$.

We now subtract the integral part of the Puiseux expansion by the invertible formal change of coordinates

$$x \mapsto x, \qquad y \mapsto y + \sum_{i/n \in \mathbb{N}} \alpha_i x^{i/n}.$$

In the new coordinates, either no terms remain, in which case C' is smooth, or the first nonzero exponent is of the form $\frac{b'}{a'}$, where $a' \neq 1$. Since $\frac{b'}{a'} \geq 1$, it follows that $b' \neq 1$.

We now show that C has the claimed properties in these coordinates. Since C is neither smooth nor a node, case 1 cannot occur. It remains to exclude cases 2 and 3.

To this end, let Δ be the steepest edge of the Newton polygon of C, and let $-\frac{a}{b}$ be its slope. We distinguish the following cases:

(1) C' is smooth. Then it contributes a factor y to C_Δ.
(2) C' is not smooth. In this case we have $\frac{b}{a} \leq \frac{b'}{a'}$, since C' is a branch of C.
  (a) $\frac{b}{a} < \frac{b'}{a'}$. Then the initial term of C' with respect to the grading defined by Δ is $y^{a'}$, so C' contributes a factor $y^{a'}$ to C_Δ.
  (b) $\frac{b}{a} = \frac{b'}{a'}$. Since both pairs (a, b) and (a', b') are coprime, we have $a = a'$ and $b = b'$. Hence $a \neq 1$ and $b \neq 1$ by the choice of coordinates above.

In all cases, it follows that cases 2 and 3 do not occur. □

With this choice of coordinates, we can now compare the lowest-degree terms in the integral curve equation and deduce that the vanishing of the cofactor forces the vanishing of $d\omega$ at a singular point.

Proposition 3.11. *Let ω be a 1-form on the plane with reduced integral curve C and cofactor K. If C is singular at a point a and $K(a) = 0$, then $d\omega(a) = 0$.*

Proof. Suppose, for contradiction, that $d\omega(a) \neq 0$. We choose coordinates as in Lemma 3.10 and the grading $\deg x = a$ and $\deg y = b$ as in Notation 3.9. We also set $\deg \mathrm{dx} = a$ and $\deg \mathrm{dy} = b$, and write

$$C = C_m + C_{m+1} + \dots,$$
$$\omega = \omega_n + \omega_{n+1} + \dots,$$
$$K = K_\ell + K_{\ell+1} + \dots$$

as sums of quasi-homogeneous components with respect to this grading, where the dots denote terms of higher degree. In particular, by assumption,

$$\deg K = \deg(k\ \mathrm{dx} \wedge \mathrm{dy}) > a + b.$$

We now consider the integral curve condition

$$dC \wedge \omega = KC$$

and compare the lowest-degree terms on both sides. There are two possibilities:

(1) $dC_m \wedge \omega_n = K_\ell C_m$ and $n = \ell$;
(2) $dC_m \wedge \omega_n = 0$ and $n < \ell$.

Now $d\omega(a)$ depends only on the coefficients of $x\,\mathrm{dy}$ and $y\,\mathrm{dx}$, each of which has degree $a + b$. Hence, in the first case, if $n = \ell > a + b$, then $d\omega(a) = 0$. Thus it remains to consider the second case.

Although C is reduced, C_m need not be reduced. Let g be the gcd of $(C_m)_x$ and $(C_m)_y$. Then

$$\omega_n = \frac{h}{g}\big((C_m)_x\ \mathrm{dy} - (C_m)_y\ \mathrm{dx}\big),$$

so

$$n = \deg \omega_n \geq \deg C_m - \deg g = m - \deg g.$$

For ω_n to contribute to $d\omega(a)$, we must have

$$m - \deg g \leq n \leq a + b.$$

Since C_m is quasi-homogeneous, we may write

$$C_m = x^u y^v \prod_{i=1}^{k} (x^b - c_i y^a).$$

By our choice of coordinates, C_m cannot be a monomial. Hence $k \geq 1$.

Now $m - \deg g$ is the degree of the reduced part of C_m, and therefore

$$m - \deg g \geq ab,$$

with equality if and only if $k = 1$, $u = 0$, and $v = 0$.

Thus

$$ab \leq m - \deg g \leq a + b.$$

Since

$$ab \leq a + b \iff (a-1)(b-1) \leq 1,$$

the only possibilities in positive integers are $a = 1$, $b = 1$, or $a = b = 2$. Since a and b are coprime, the case $(a, b) = (2, 2)$ is excluded. Hence $a = 1$ or $b = 1$.

Again, by our choice of coordinates, this is impossible. Therefore $d\omega(a) = 0$. □

Conversely, the vanishing of $d\omega$ also forces the vanishing of the cofactor.

Corollary 3.12. *Let ω be a 1-form on the plane with reduced integral curve C and cofactor K. If C is singular at a point a and $d\omega(a) = 0$, then $K(a) = 0$.*

Proof. If the singularity of C at a is quasi-homogeneous, then Theorem 3.5 implies that $K(a) = 0$, since already $d\omega(a) = 0$. If the singularity is non-quasi-homogeneous, then $K(a) = 0$ by contraposition from Lemma 3.8. □

It remains to pass from the vanishing of the cofactor of the total curve C to the vanishing of the cofactors of its individual components.

Theorem 3.13. *Let $C = C_1 \cup \cdots \cup C_r$ be an algebraic integral curve of a differential form ω, and let a be a point of C at which C has a non-quasi-homogeneous singularity. Assume that a is a zero of ω. Then*

$$\eta = (0 : \cdots : 0).$$

Proof. Let K_C be the cofactor of C. Since the singularity of C at a is not quasi-homogeneous, Lemma 3.8 implies that $K_C(a) = 0$. Hence Proposition 3.11 yields

$$d\omega(a) = 0.$$

Now let K_i be the cofactor of C_i. By Corollary 3.12, we have $K_i(a) = 0$ for every singular component C_i.

Assume for the moment that at least one component of C is singular at a. Without loss of generality, we may assume that this component is C_1. Let C_i be any other component of C, and consider

$$C' = C_1 \cup C_i.$$

Let K' be the cofactor of C'. Since C' is singular at a and $d\omega(a) = 0$, Corollary 3.12 implies that $K'(a) = 0$. Since also

$$K'(a) = K_1(a) + K_i(a)$$

by the additivity of cofactors, we obtain $K_i(a) = 0$. Hence $\eta = (0 : \cdots : 0)$ whenever at least one component of C is singular at a.

This leaves the case where all components C_i are smooth at a. If there are two components C_i and C_j that intersect nontransversally at a, then $C_i \cup C_j$ is quasi-homogeneous but not a node. Hence Theorem 3.5 applied to $C_i \cup C_j$ implies that

$$(K_i(a) : K_j(a) : d\omega(a)) = (0 : 0 : 0),$$

since already $d\omega(a) = 0$. In particular.

$$K_i(a) = K_j(a) = 0.$$

The only remaining case is that all components of C are smooth at a and intersect transversally there. But then the singularity of C at a is quasi-homogeneous, contradicting our assumption. Therefore $\eta = (0 : \cdots : 0)$. □

4. Computing the invariant at infinity

There are two ways to define η for a point a at infinity: One is to use Definition 3.1 locally at a chart around a with respect to coordinates in that chart. The other is to consider $d\omega, C_i$ and K_i in Definition 3.1 as homogeneous polynomials and evaluate at the projective coordinates of a. Since $d\omega$ and the K_i are of the same degree $\deg \omega - 1$ and their ratio is well defined. While the first definition is easier to predict from the singularities of C, it is the second definition that fits best with our application to Darboux integrability.

Unfortunately the values one obtains from these two definition are not equal. In this section we show how one can obtain one from the other.

We start with some notation: Consider the projective plane $\mathbb{P}^2$ with coordinates $(X : Y : Z)$, the chart $U_{Z\neq 0}$ with coordinates $x := \frac{X}{Z}$ and $y := \frac{Y}{Z}$, and the chart $U_{X\neq 0}$ with coordinates $y := \frac{Y}{X}$ and $z := \frac{Z}{X}$. The transformation map φ from $U_{X\neq 0}$ to $U_{Z\neq 0}$, defined on the intersection, is then given by $\varphi(y, z) = \left(\frac{1}{z}, \frac{y}{z}\right)$. We call $U_{X\neq 0}$ the *chart at infinity*.

Definition 4.1. Let F be a polynomial differential i-form, $i = 0, 1, 2$ on $U_{Z\neq 0}$ with coefficients of degree d. Then we denote its homogenization by F^h i.e

$$F^h := F\left(\frac{X}{Z}, \frac{Y}{Z}\right) \cdot \begin{cases} Z^d & \text{if } F \text{ is a polynomial} \\ Z^{d+2} & \text{if } F \text{ is a 1-form} \\ Z^{d+3} & \text{if } F \text{ is a 2-form.} \end{cases}$$

Dehomogenizing this with respect to X we obtain a polynomial differential i-form on the chart $U_{X\neq 0}$ at infinity we denote this by

$$F' := F^h(1 : y : z).$$

Notice that this implies

$$F' = \varphi^*(F) \cdot \begin{cases} z^d & \text{if } F \text{ is a polynomial} \\ z^{d+2} & \text{if } F \text{ is a 1-form} \\ z^{d+3} & \text{if } F \text{ is a 2-form.} \end{cases}$$

where φ^* is the pull-back map associated to φ.

Example 4.2. We have

$$(\mathrm{dx})' = \varphi^*(\mathrm{dx})z^2 = d\left(\frac{1}{z}\right)z^2 = -\frac{1}{z^2}\,\mathrm{dz} \cdot z^2 = -\,\mathrm{dz}$$

similarly

$$(\mathrm{dy})' = \varphi^*(\mathrm{dy})z^2 = d\left(\frac{y}{z}\right)z^2 = \frac{z\,\mathrm{dy} - y\,\mathrm{dz}}{z^2}\cdot z^2 = z\,\mathrm{dy} - y\,\mathrm{dz}$$

We can now formally define the extention of our invariant to infinity:

Definition 4.3. Let ω be a differential form with algebraic integral curve C and cofactor K_C and $a \in \mathbb{P}^2$ a projective point. Then

$$\eta := \eta(\omega, C, a) := \big((K_C)^h(a) : (d\omega)^h(a)\big).$$

Notice that for $a = (x : y : 1)$ a projective point *not* at infinity, this definition agrees with our previous one.

Remark 4.4. η is well defined. Indeed, since C is an integral curve with cofactor K, we have formula

$$dC \wedge \omega = CK$$

and hence $\deg K = \deg \omega - 1 = \deg d\omega$.

Remark 4.5. Let ω be a differential form with algebraic integral curve C and cofactor K_C and $a = (1 : y : z)$ a point in the chart at infinity. On the one hand, we have by definition

$$\eta(\omega, C, a) = \big((K_C)'(a) : (d\omega)'(a)\big),$$

while computing in the chart at infinity, would give

$$\eta(\omega', C', a) = \big(K_{C'}(a) : d(\omega')(a)\big),$$

assuming that C' is an algebraic integral curve of ω' with some cofactor $K_{C'}$ (which we prove below). Unfortunately these values differ, since taking differentials does not commute with homogenization.

Let us look in more detail at the geometry in the chart at infinity:

Proposition 4.6. *Let ω be a differential form. Then the line at infinity defined by $z = 0$ is an algebraic integral curve of ω' with cofactor*

$$K_z := \frac{\mathrm{dz} \wedge \omega'}{z}.$$

In particular this quotient is a polynomial 2-form.

Proof. Let $\omega = P\,\mathrm{dx} + Q\,\mathrm{dy}$. Then

$$\begin{aligned}\mathrm{dz} \wedge \omega' &= \mathrm{dz} \wedge \big(P'(\mathrm{dx})' + Q'(\mathrm{dy})'\big)\\ &= \mathrm{dz} \wedge \big(-P'\,\mathrm{dz} + Q'(z\,\mathrm{dy} - y\,\mathrm{dz})\big)\\ &= Q'z\,\mathrm{dz} \wedge \mathrm{dy}\end{aligned}$$

So indeed $\omega' \wedge \mathrm{dz}$ is divisible by z. We now check, that z is an integral curve of ω' with cofactor K_z

$$\mathrm{dz} \wedge \omega' = z\frac{\mathrm{dz} \wedge \omega'}{z} = zK_z$$

□

Proposition 4.7. *Let ω be a differential form with algebraic integral curve C and cofactor K_C, then C' is an algebraic integral curve of ω'.*

If we denote the cofactor of C' by $K_{C'}$, we have the following formulas:

$$\begin{aligned}(K_C)' &= K_{C'} - (\deg C)K_z\\ (d\omega)' &= d(\omega') - (\deg \omega + 2)K_z.\end{aligned}$$

Proof. Since C is an integral curve of ω with cofactor C and φ^* is a ring homomorphism with have

$$\varphi^*(dC) \wedge \varphi^*(\omega) = \varphi^*(C) \cdot \varphi^*(K_C). \qquad (*)$$

Multiplying both sides with $z^{\deg C + \deg \omega + 3}$ we obtain

$$(dC)' \wedge \omega' = zC' \cdot (K_C)'.$$

Since taking differentials commutes with pullback, we have

$$\begin{aligned} d(C') &= d\big(z^{\deg C}\,\varphi^*(C)\big) \\ &= (\deg C) z^{\deg C - 1}\,\varphi^*(C)\,\mathrm{dz} + z^{\deg C}\varphi^*(dC) \\ &= \frac{(\deg C) C'\,\mathrm{dz}}{z} + \frac{(dC)'}{z} \end{aligned}$$

and hence

$$\begin{aligned} d(C') \wedge \omega' &= \frac{(\deg C) C'\,\mathrm{dz}}{z} \wedge \omega' + \frac{(dC)'}{z} \wedge \omega' \\ &= (\deg C) C' \frac{\mathrm{dz} \wedge \omega'}{z} + \frac{zC' \cdot (K_C)'}{z} \qquad \text{(using } (*)) \\ &= (\deg C) C' \cdot K_z + C' \cdot (K_C)' \\ &= C' \cdot \big((\deg C) K_x + (K_C)'\big). \end{aligned}$$

If follows that C' is an integral curve of ω' with cofactor

$$K_{C'} = (\deg C) K_z + (K_C)'.$$

Solving for $(K_C)'$ gives the first formula.

We also have

$$\begin{aligned} d(\omega') &= d\big(z^{\deg \omega + 2}\,\varphi^*(\omega)\big) \\ &= (\deg \omega + 2) z^{\deg \omega + 1} dz \wedge \varphi^*(\omega) + z^{\deg \omega + 2}\varphi^*(d\omega) \\ &= (\deg \omega + 2) \frac{\mathrm{dz} \wedge \omega'}{z} + (d\omega)' \\ &= (\deg \omega + 2) K_z + (d\omega)'. \end{aligned}$$

which gives the second formula. □

With this data we define a second invariant at a point at infinity, which can be computed geometrically in the chart at infinity using the results of Section 3.

Definition 4.8. Let ω be a differential form with algebraic integral curve C and and $a \in U_{X \neq 0}$ a point in the chart at infinity. Then we define

$$\eta' := \eta'(\omega, C, a) := \eta\big(\omega', \{z, C'\}, a\big).$$

This is well defined, since C' and z are algebraic integral curves of ω'.

We get the following relation between η and η':

Proposition 4.9. *Let ω be a differential form with algebraic integral curve C. Let $a \in U_{X \neq 0}$ be a point in the chart at infinity such that $\omega(a) = 0$. If*

$$\eta'(\omega, C, a) = (k_z : k : w)$$

for some projective coordinates $(k_z : k : w) \in \mathbb{P}^2$, then

$$\eta(\omega, C, a) = \big(k - (\deg C) k_z : w - (\deg \omega + 2) k_z\big).$$

In other words, $\eta \in \mathbb{P}^1$ is the linear projection of $\eta' \in \mathbb{P}^2$ from the point

$$(1 : \deg C : \deg \omega + 2)$$

onto the line $\{k_z = 0\} \subset \mathbb{P}^2$, identified with $\mathbb{P}^1$ via $(0 : k : w) \mapsto (k : w)$.

Proof. By definition we have

$$(k_z : k : w) = \big(K_z(a) : K_{C'}(a) : d\omega'(a)\big)$$

Using Proposition 4.7 we can compute

$$\begin{aligned}
\eta(\omega, C, a) &= \big(K_C^h(a) : (d\omega)^h(a)\big) \\
&= \big(K_C'(a) : (d\omega)'(a)\big) \\
&= \big(K_{C'}(a) - (\deg C)K_z(a) : d\omega'(a) - (\deg \omega + 2)K_z(a)\big) \\
&= \big(k - (\deg C)k_z : w - (\deg \omega + 2)k_z\big).
\end{aligned}$$

□

Remark 4.10. If $a \in U_{X\neq 0}$ lies *not* on the line at infinity $L_\infty = \{z = 0\}$, then

$$K_z(a) = 0.$$

Hence

$$\eta'(\omega, C, a) = (0 : k : w),$$

and the projection formula of Proposition 4.9 reduces to

$$\eta(\omega, C, a) = (k - (\deg C) \cdot 0 : \ w - (\deg \omega + 2) \cdot 0) = (k : w).$$

In other words, away from L_∞ we may identify η with the last two coordinates of η'. Geometrically this just says that in this case η' already lies on the line $k_z = 0$ and is therefore unchanged by the projection.

Remark 4.11 (Extension to several integral curves)**.** Proposition 4.9 extends verbatim to a collection of integral curves $\{C_1, \dots, C_r\}$. In particular, $\eta(\omega, \{C_i\}, a) \in \mathbb{P}^r$ is the projection of $\eta'(\omega, \{C_i\}, a) \in \mathbb{P}^{r+1}$ from the point

$$(1 : \deg C_1 : \dots : \deg C_r : \deg \omega + 2) \in \mathbb{P}^{r+1}.$$

to the line $k_z = 0$. The proof is identical to the case $r = 1$.

Remark 4.12. Combining Proposition 4.9 with Theorem 3.5, we see that for quasi-homogeneous singularities of $C \cup \{z = 0\}$ one can compute η' from the local geometry, and recover η by incorporating the global information encoded in the degrees of the integral curves.

Example 4.13. Let ω be a differential form with integral curve C which is triply tangent to the line at infinity at a point a. In this situation $C \cap \{z = 0\}$ has an A_3 singularity at a and a local equation is $z(z - y^3) = 0$. This equation is quasi homogeneous with respect to $\deg_{\mathrm{a}} z = 3, \deg_{\mathrm{a}} y = 1$. It follows that

$$\eta' = (\deg_a z : \deg_a C : \deg_a y_a + \deg_a z_a) = (3 : 3 : 4).$$

Projecting this form $(1 : \deg C : \deg \omega + 2)$ gives

$$\eta = \big(3 - 3(\deg C) : 4 - 3(\deg \omega + 2)\big).$$

Notice that only the degree of C and ω are used in this formula, but neither the equation of ω nor the cofactor K_C of C. If for example $\deg C = 3$ and $\deg \omega = 1$ we obtain

$$\eta = (-6 : -5) = (6 : 5).$$

Let us compare this to a direct computation for an explicit example: The differential form $\omega = 3x\,\mathrm{dy} - 2y\,\mathrm{dx}$ is has an integral curve $C = x^2 - y^3$ with cofactor $K_C = 6\,\mathrm{dx}\wedge\mathrm{dy}$ since

$$dC \wedge \omega = (2x\,\mathrm{dx} - 3y^2\,\mathrm{dy}) \wedge (3x\,\mathrm{dy} - 2y\,\mathrm{dx}) = 6(x^2 - y^3)\,\mathrm{dx}\wedge\mathrm{dy}\,.$$

We have $C^h = X^2Z - Y^3$ and $C' = z - y^3$ and hence C' has $\{z = 0\}$ as a triple tangent. So the geometry of ω and C agrees with the above assumptions.

Now $d\omega = 3\,\mathrm{dx}\wedge\mathrm{dy} - 2\,\mathrm{dy}\wedge\mathrm{dx} = 5\,\mathrm{dx}\wedge\mathrm{dy}$ and we can compute

$$\eta = \big(K_C(a) : d\omega(a)\big) = (6 : 5).$$

This agrees with the value predicted by the geometry.

We now arrive at the main result of the paper: a concrete criterion for Darboux integrability that is computable purely from the geometry of $C \cup L_\infty$, without any further knowledge of ω. The main improvement over [4] is that the present form also incorporates quasi-homogeneous singularities *on* the line at infinity (via the projection step for η'), and thus applies in substantially more cases.

Before stating the Theorem, we package the relevant η'-data into a matrix.

Definition 4.14. Let $C = C_1 \cup \cdots \cup C_r$ be an algebraic integral curve of a differential form ω, and let $P_1, \ldots, P_k$ be the η-geometric points of $C \cup L_\infty$.

We define $M_{\eta'}$ to be the matrix

$$M_{\eta'} \;=\; \begin{pmatrix} K_z(P_1) & K_{C_1}(P_1) & \cdots & K_{C_r}(P_1) & d\omega(P_1) \\ \vdots & \vdots & & \vdots & \vdots \\ K_z(P_k) & K_{C_1}(P_k) & \cdots & K_{C_r}(P_k) & d\omega(P_k) \\ 1 & \deg C_1 & \cdots & \deg C_r & \deg\omega + 2 \end{pmatrix},$$

where the j-th row is computed in local coordinates at P_j and is therefore defined only up to multiplication by a nonzero scalar. The rank of $M_{\eta'}$ is nevertheless independent of the choice of local coordinates.

By Definition 4.8, the j-th row is a representative of

$$\eta'_j := \eta'(\omega, \{C_1, \ldots, C_r\}, P_j),$$

and is by Theorem 3.5 determined by the local geometry of $C \cup L_\infty$ at P_j. The last row is the projection center from Remark 4.11.

Theorem 4.15. *Let $C = C_1 \cup \cdots \cup C_r$ be an algebraic integral curve of a differential form ω, and let $M_{\eta'}$ be as in Definition 4.14.*

If $\operatorname{rank} M_{\eta'} \le r + 1$ and the η-geometric points of $C \cup L_\infty$ do not lie on any curve of degree $\deg\omega - 1$, then ω is Darboux integrable.

Proof. By the rank condition there exists a hyperplane in $\mathbb{P}^{r+1}$ that contains all points η'_j *and* the projection center. Because the hyperplane contains the center, its image under the linear projection $\pi\colon \mathbb{P}^{r+1} \dashrightarrow \mathbb{P}^r$ is again a hyperplane; hence the projected points $\pi(\eta'_j)$ all lie on a hyperplane in $\mathbb{P}^r$.

By Remark 4.11 we have $\pi(\eta'_j) = \eta(\omega, \{C_1, \ldots, C_r\}, P_j)$. It follows, that there exist coefficients $(\lambda_1, \ldots, \lambda_r, \lambda_{r+1}) \neq 0$ such that

$$\sum_{i=1}^{r} \lambda_i\, K^h_{C_i}(P_j) \;+\; \lambda_{r+1}\, d\omega^h(P_j) \;=\; 0 \qquad (j = 1, \ldots, k).$$

Write

$$\sum_{i=1}^{r} \lambda_i \, K^h_{C_i} \; + \; \lambda_{r+1} \, d\omega^h \; =: \; q \, \mathrm{dx} \wedge \mathrm{dy},$$

with q a homogeneous polynomial of degree $\deg \omega - 1$. Then $q(P_j) = 0$ for all j. By hypothesis, no nonzero polynomial of degree $\deg \omega - 1$ vanishes on all P_j, hence $q \equiv 0$. This yields a nontrivial linear relation among the cofactors and $d\omega$, so ω is Darboux integrable. so ω is Darboux integrable. □

5. Zeros outside of integral curves

Consider a differential form ω with an integral curve C. A necessary condition for the existence of a center of ω at a point $a \in \mathbb{C}^2$ is $\omega(a) = d\omega(a) = 0$. Notice that for quasi-homogeneous singularities a of C we have $\omega(a) = 0$ but usually $d\omega(a) \neq 0$, see Theorem 3.5. In this section we will concern ourselves with the question where and under what conditions points with $\omega(a) = d\omega(a) = 0$ exist.

Proposition 5.1. *Consider a differential form ω with an integral curve C and cofactor K_C. If $a \notin C$ is a zero of ω then $K_C(a) = 0$.*

Proof. Since C is an integral curve of ω we have

$$dC \wedge \omega = C \cdot K_C$$

where K_C is the cofactor of C. Since $\omega(a) = 0$ but $C(a) \neq 0$ we must have $K_C(a) = 0$ □

We now use Darboux' condition for the existence of an integrating factor:

Proposition 5.2. *Let ω be a differential form, $C_1, \ldots, C_r$ algebraic integral curves of ω, $K_1, \ldots, K_r$ their cofactors and a a zero of ω with $a \notin C_i$ for $i = 1 \ldots r$. Assume that*

$$\sum \alpha_i K_i = -d\omega$$

for appropriate $\alpha_i \in \mathbb{C}$. Then $d\omega(a) = 0$.

Proof. By the previous Proposition we have $K_i(a) = 0$ for $i = 1 \ldots r$ and therefore

$$d\omega(a) = -\sum \alpha_i K_i(a) = 0.$$

□

We want to estimate the number of zeros of ω that lie outside an integral curve C using only the geometry of C. For this, we use some standard notation from vector bundle theory on $\mathbb{P}^2$. For $n \in \mathbb{Z}$, we denote by $\mathcal{O}(n)$ the usual line bundle on $\mathbb{P}^2$. If E is a vector bundle on $\mathbb{P}^2$, we write $c_1(E)$ and $c_2(E)$ for its first and second Chern classes. For a finite sheaf or finite scheme Q, we write $\deg(Q)$ for its length. With this notation, we need the following standard Chern class computation:

Lemma 5.3. *Let*

$$0 \longrightarrow K \longrightarrow E \longrightarrow L \longrightarrow Q \longrightarrow 0$$

be an exact sequence of coherent sheaves on $\mathbb{P}^2$, where

(1) E is a vector bundle of rank r,
(2) L is a line bundle,
(3) Q is zero-dimensional of length $\ell = \deg(Q)$.

Then K is a vector bundle of rank $r-1$, and its Chern classes satisfy

$$c_1(K) = c_1(E) - c_1(L),$$
$$c_2(K) = c_2(E) - c_1(E)\, c_1(L) + c_1(L)^2 - \ell.$$

Proof. Since E and L are locally free, K is a second syzygy sheaf of Q. As Q has finite support on the smooth surface $\mathbb{P}^2$, any second syzygy is locally free: see [18, II, §1, Def. 1.1.5 & Thm. 1.1.6]. Thus K is a vector bundle of rank $r-1$.

Exactness gives the alternating Whitney identity for total Chern classes

$$c(K)\, c(L) = c(E)\, c(Q).$$

Here $c(L) = 1 + c_1(L)$ since L is a line bundle, and since Q is zero-dimensional of length ℓ, we have $c(Q) = 1 - \ell\,[\mathrm{pt}]$. Comparing degrees 1 and 2 yields

$$c_1(K) + c_1(L) = c_1(E),$$
$$c_2(K) + c_1(K)c_1(L) = c_2(E) - \ell,$$

and hence

$$c_1(K) = c_1(E) - c_1(L)$$
$$c_2(K) = c_2(E) - c_1(E)c_1(L) + c_1(L)^2 - \ell$$

□

With this we can compute our formula:

Proposition 5.4. *Let C be an algebraic integral curve of a differential form ω, and let X be the finite scheme defined by $C_x = C_y = C = 0$. Then ω has at most*

$$(\deg\omega)^2 + (\deg C)(\deg C - \deg\omega - 1) - \deg X$$

zeros outside of C.

Proof. Let a be a zero of $\omega = P\,\mathrm{dx} + Q\,\mathrm{dy}$ outside of C, i.e $P(a) = Q(a) = 0$ and $C(a) \neq 0$. Since C is an integral curve of ω we have

$$dC \wedge \omega = CK_C$$

where $K_C = k_C\,\mathrm{dx}\wedge\mathrm{dy}$ is the cofactor of C. Since $\omega(a) = 0$ and $C(a) \neq 0$ we must also have $k_C(a) = 0$.

Hence every zero $a \notin C$ of ω lies in the zero scheme

$$Y := V(P, Q, k_C).$$

In particular, the number of zeros of ω outside C, counted without multiplicity, is bounded above by $\deg Y$.

Consider the exact sequence

$$0 \to \mathcal{K} \to \underbrace{\mathcal{O}(s) \oplus \mathcal{O}(s) \oplus \mathcal{O}(s-1)}_{=:E} \xrightarrow{(C_x,\, C_y,\, C)} \underbrace{\mathcal{O}(s+d-1)}_{=:L} \to \mathcal{O}_X \to 0,$$

with $s = \deg\omega$ and $d = \deg C$. From

$$dC \wedge \omega = CK_C \quad \Longleftrightarrow \quad C_xQ - C_yP - Ck_C = 0$$

we see that $(Q, -P, -k_C)^t \colon \mathcal{O} \to E$ factors through $\mathcal{K}$. Hence it defines a section of $\mathcal{K}$ whose zero scheme is Y. Applying Lemma 5.3 to this exact sequence, we see that $\mathcal{K}$ is a vector bundle and hence, if Y is zero-dimensional, $\deg Y = c_2(\mathcal{K})$.

To apply the formula of Lemma 5.3 we compute the Chern classes of E and L in our current situation:

$$\begin{aligned}c_1(L) &= s+d-1\\ c_1(E) &= 3s-1\\ c_2(E) &= s^2+2s(s-1) = 3s^2-2s.\end{aligned}$$

We therefore have

$$\begin{aligned}c_2(\mathcal{K}) &= c_2(E) - c_1(E)c_1(L) + c_1(L)^2 - \ell\\ &= (3s^2-2s) - (3s-1)(s+d-1) + (s+d-1)^2 - \deg X\\ &= s^2 - sd + d^2 - d - \deg X\\ &= s^2 + d(d-s-1) - \deg X.\end{aligned}$$

□

Example 5.5. If C is a sextic with $\deg X = 20$ then there exists an ω of degree 3 with C as integral curve. The proposition above gives that there is at most

$$3^2 + 6(6-3-1) - 20 = 1$$

zero of ω outside of C.

We now count the number of zeros of ω at infinity.

Proposition 5.6. *Let C be an integral curve of a differential form ω and X_∞ the finite scheme defined by $C_x = C_y = C = z = 0$ over some field k. Assume furthermore that*

$$\deg X_\infty \le \deg C - \deg \omega - 2$$

and that the characteristic of k does not divide $\deg C$. Then ω has, counted with multiplicity, exactly

$$\deg \omega - 1$$

zeros at infinity.

Proof. Since C is an integral curve of ω, and writing $K_C = k_C \,\mathrm{dx}\wedge\mathrm{dy}$, we have

$$(C_x, C_y, C)\cdot(Q, -P, -k_C)^t = 0.$$

Denote by the subscript ∞ the restriction to the line at infinity $\{z=0\}$. Then

$$(Q_\infty, -P_\infty, -k_{C,\infty})^t$$

is a syzygy of $(C_{\infty,x}, C_{\infty,y}, C_\infty)$ over $k[x,y]$.

We claim that the columns of

$$\begin{pmatrix} x & \frac{1}{g}C_{\infty,y}\\ y & -\frac{1}{g}C_{\infty,x}\\ -\deg C & 0\end{pmatrix} \qquad \text{with } \; g := \gcd(C_{\infty,x}, C_{\infty,y})$$

generate the $k[x,y]$-module $\operatorname{Syz}(C_{\infty,x}, C_{\infty,y}, C_\infty)$. Indeed, the first column comes from the Euler relation

$$xC_{\infty,x} + yC_{\infty,y} = (\deg C)C_\infty.$$

Since $\deg C \neq 0$ in k, we may assume that all further generators of the syzygy module have zero in the third entry. All such syzygies are between $C_{\infty,x}$ and $C_{\infty,y}$ only and are therefore generated by the second column.

Thus $(Q_\infty, -P_\infty, -k_{C,\infty})^t$ is a $k[x,y]$-linear combination of these two columns. Note that $\deg g = \deg X_\infty$, so the degree of the second column equals

$$\deg(C) - 1 - \deg X_\infty \geq \deg \omega + 1$$

by our assumption on X_∞. Therefore the expression for

$$(Q_\infty, -P_\infty, -k_{C,\infty})^t$$

must involve only the first column. The corresponding coefficient then has degree $\deg \omega - 1$, and hence ω has $\deg \omega - 1$ zeros at infinity (counted with multiplicity). □

Example 5.7. Let $C \subset \mathbb{P}^2$ be a maximal sextic, i.e. a sextic with simple singularities and maximal Milnor number 19. Assume in addition that the line at infinity meets C with multiplicity 2 and transversely elsewhere. In this situation the modified Tjurina number of C is 20, and differential forms ω of degree 3 with integral curve C exist. By Example 5.5 there is at most one zero of ω outside C.

Now $\deg X_\infty = 1$ and $\deg C - \deg \omega - 2 = 1$; therefore, by Proposition 5.6, we have at least

$$\deg \omega - 1 = 2$$

zeros on the line at infinity. Generically we expect at most one of them at the multiplicity-2 point and none at the transverse intersections. Hence, generically, the second point lies outside C but on the line at infinity. It follows that, even if we can prove that ω admits a Darboux integrating factor with respect to C, we expect no centers of ω outside C and the line at infinity. For this reason we do not consider maximal sextics in our constructions.

Example 5.8. Let $C \subset \mathbb{P}^2$ be a submaximal sextic, i.e. a sextic with simple singularities and Milnor number 18. Assume in addition that the line at infinity meets C with multiplicities $(2,2,1,1)$ or $(3,1,1,1)$. In this situation the modified Tjurina number of C is again 20, and differential 1-forms ω of degree 3 with integral curve C exist. Again, we expect only one zero of ω outside C, *but* here $\deg X_\infty = 2$ and $\deg C - \deg \omega - 2 = 1$, so Proposition 5.6 does not apply. Indeed, in examples the expected zero lies away from the line at infinity. All constructions in this article are of this type.

6. Review of results in finite characteristic

In this section we discuss the center problem introduced by Poincaré in [20, 21]. After recalling its classical formulation, we summarize the finite-characteristic results obtained by our group in [8, 25].

Definition 6.1. Let $\omega = P\,\mathrm{dx} + Q\,\mathrm{dy}$ be a polynomial differential form over $\mathbb{R}$ such that

$$P = -y + \text{ higher order terms}, \qquad Q = x + \text{ higher order terms}.$$

We say that the origin is a *center* of ω if there exists a punctured neighborhood of the origin in $\mathbb{R}^2$ such that every integral curve of ω in that neighborhood is closed. Otherwise, the origin is called a *focus*.

The condition of having a center can be expressed algebraically in terms of focal values:

Definition 6.2. Let $\omega = P\,\mathrm{dx} + Q\,\mathrm{dy}$ be a polynomial differential form over a field K of characteristic 0 such that

$$P = -y + \text{ higher order terms}, \qquad Q = x + \text{ higher order terms}.$$

Frommer's algorithm recursively constructs a formal power series

$$F = x^2 + y^2 + \text{ higher order terms } \in K[[x,y]]$$

such that, in degree $2j+2$, all terms of

$$F_xP + F_yQ$$

can be eliminated except for a term of the form

$$s_j(P,Q)\big(x^{2j+2} + y^{2j+2}\big).$$

The *jth focal value* of ω is the scalar $s_j(P,Q) \in K$. Moreover, as a function of the coefficients of P and Q, $s_j(P,Q)$ is given by a polynomial with rational coefficients.

The vanishing of all focal values is equivalent to the existence of a formal first integral F satisfying $dF = \mu\,\omega$ for some formal unit μ; see Frommer [16]. For polynomial systems, such a formal first integral is in fact analytic; see [2, pp. 33–36]. In particular, over $\mathbb{R}$ the origin is a center if and only if all focal values vanish.

This motivates the following construction.

Definition 6.3. On the 14-dimensional space W of normalized cubic differential forms we consider the focal polynomials

$$s_j \in \mathbb{Q}[p_{ij}, q_{ij}].$$

Let $I_\infty = (s_1, s_2, \dots)$ be the *Bautin ideal* [1], and for $j \geq 1$ let $I_j = (s_1, \dots, s_j)$ be its truncations. We denote their vanishing sets by

$$Z_j = V(I_j) \subset W, \qquad Z_\infty = V(I_\infty) = \bigcap_{j\geq 1} Z_j.$$

By the discussion above, a form $\omega \in W$ lies in Z_∞ if and only if ω has a center at 0. We therefore call Z_∞ the *center variety of cubic systems*, or simply the *center variety*.

The geometry of the cubic center variety is still far from being understood. Even the number and codimensions of its irreducible components are unknown. Darboux integrability yields explicit first integrals whenever sufficiently many invariant algebraic curves are present and thus produces families of differential forms having a center. Although this does not account for all centers, it explains important parts of the center variety.

One way to obtain information about the global structure of the center variety is to study its reduction modulo a prime.

To pass to finite characteristic, one has to understand for which primes the focal values remain defined. One can show that no prime factor occurring in any of the denominators of s_j is larger than $2j+2$. Therefore, over a finite field $\mathbb{F}_p$, the focal values s_j are well defined for $j \leq (p-3)/2$. In the following, we work over $\mathbb{F}_{29}$ and use X_{13} as an approximation to the center variety over this field.

In [6] Jakob Kröker developed a very fast C++ implementation of Frommer's algorithm and tested, for (almost) all points of $(\mathbb{F}_{29})^{14}$, whether they lie on X_{13}. He also computed the codimension of the Zariski tangent space of X_{13} at each point found. This required about 11 CPU-years on a compute cluster, and the results are collected in a database available at [7].

Using only the number of points found one can heuristically estimate the number of irreducible components of X_{13} in each codimension. Indeed a variety of dimension d has on the order of p^d points over $\mathbb{F}_p$ (a very rough heuristic inspired by the Weil conjectures, see [28],[14]). Using this heuristic, Kröker [8] obtains the estimates collected in the second column of Table 3. In that computation, points are bucketed by the codimension r of the Zariski tangent space. At singular points of X_{13} the codimension of the tangent space drops below codimension of the containing

codimension r	by counting points	by interpolation	known
5	$\geq 1.00 \pm 0.00$	1	1
6	$\geq 1.96 \pm 0.00$	2	2
7	$\geq 3.84 \pm 0.00$	4	4
8	$\geq 3.83 \pm 0.00$	4	4
9	$\geq 11.97 \pm 0.04$	14	10
10	$\geq 32.85 \pm 0.37$	41	4
11	$\geq 77.50 \pm 3.07$	98	5

TABLE 3. Estimated numbers of irreducible components of $X_{13} \subset \mathbb{F}_{29}^{14}$ that contain a smooth $\mathbb{F}_{29}$-rational point of X_{13}.

component, so such points are assigned to lower–codimension buckets. This depresses the counts in higher codimensions; thus the second column should be read as a conservative lower bound.

Based on Kröker's database, Johannes Steiner [25] computed a heuristic decomposition of X_{13} over $\mathbb{F}_{29}$, together with low-degree candidate equations for the conjectured components. The method lifts smooth points of X_{13} to higher-order local solutions (jets) via a Hensel-type procedure and then uses interpolation to recover low-degree polynomials vanishing on these jets (a finite-field analogue of [24]). Counting distinct candidate ideals, he conjectures the component counts in the third column of Table 3. Concrete low-degree generators for the candidate ideals are available at [9].

Definition 6.4. Let $\omega \in (\mathbb{F}_{29})^{14}$ be a normalized differential form. If $\omega \in X_{13}$, let r be the codimension of the Zariski tangent space of X_{13} at ω.

We say that ω has *Steiner type* r_i if $\omega \in V_{r,i} \cap X_{13}$, where $V_{r,i}$ is Steiner's heuristic component as defined in [25]. Note that a differential form may have several Steiner types, since the $V_{r,i}$ intersect.

If ω has integer coefficients, we say it has Steiner type r_i if its reduction mod 29 can be normalized to a form of Steiner type r_i. Again, a differential form may have several Steiner types, since different normalization centers of ω can yield different types.

Johannes Steiner also assigned a selection of Jakob Kröker's solutions to his heuristic components $V_{r,i}$ and, for each, computed all algebraic integral curves of degree at most 6 having at least one smooth $\mathbb{F}_{29}$-rational point. This yields a treasure trove of geometric information about these components, which we have used in [4] and will also use here.

To our knowledge this is the only study that treats the entire 14-dimensional parameter space of normalized degree-3 differential forms and the full ideal generated by the first thirteen focal values. There are, however, caveats:

(1) It is not proved that the heuristic components $V_{r,i}$ are genuine components of X_{13} in characteristic 29, although Steiner's tests provide strong evidence (with one doubtful case).
(2) Even if they are components of X_{13} in characteristic 29, it is unknown whether they are components of X_∞ in that characteristic (i.e., whether they admit a center). Many examples are Darboux integrable with low-degree invariant curves, but not all.
(3) Even if they are components of X_∞ in characteristic 29, persistence to characteristic 0 is unknown. The fourth column of Table 3 lists the irreducible components in characteristic 0 known to us ([30], [31], [3], [8], [26],[4]); all these components are of Steiner type, but many predicted Steiner types remain unrealized in characteristic 0.

7. Constructions

In this section we use the methods developed in this paper to construct several codimension 11 components of the degree 3 center variety; most of which are, to our best knowledge, new. We collect the needed ingredients in the following proposition.

Proposition 7.1 (Blueprint for constructing codim-11 components)**.** *Assume there exist*

(1) a plane sextic $C = C_1 \cup \cdots \cup C_r$ with simple singularities whose Milnor numbers add up to 18 (i.e. C is submaximal),
(2) a line L_∞ bitangent to C (after a projective change of coordinates we take L_∞ to be the line at infinity),
(3) a degree-3 differential 1-form ω having C as an integral curve (existence of ω follows from (1)–(2)),

such that the following additional properties hold.

(a) The matrix $M_{\eta'}$ from Theorem 4.15 has rank at most $r+1$.
(b) The points η-geometric points of $C \cup L_\infty$ do not lie on a conic.
(c) ω has a zero outside $C \cup L_\infty$.
(d) For some prime p, after reducing the coefficients of ω modulo p and normalizing for Frommer's algorithm, the first 13 focal values vanish (this is automatic from (a) and (b)) and the associated 13×13 Jacobian has rank 11 over $\mathbb{F}_p$ (this is not).

Assume finally that

(e) there exists an irreducible 1-dimensional family of plane sextics containing C such that
  (i) the family is not constant up to projective equivalence,
  (ii) every member C' has total Milnor number at least 18,
  (iii) the line L_∞ is at least bitangent to every member C',
  (iv) every member has the same component structure
$$C' = C'_1 \cup \cdots \cup C'_r$$
  as C.

Then there is a codimension-11 component of the degree-3 center variety whose general member is Darboux-integrable, with an invariant sextic of the type described in (1)–(2).

Proof. Let $\mathcal{C}$ be the equisingular locus of plane sextics satisfying (1)–(2), and put $\mathcal{H}$ for the vector space of degree-3 polynomial 1-forms $P\,\mathrm{d}x + Q\,\mathrm{d}y$. Set

$$\mathcal{W} := \{(\omega', C') \in \mathcal{H} \times \mathcal{C} \mid C' \text{ is an integral curve of } \omega'\}.$$

Let $\pi_1 : \mathcal{W} \to \mathcal{H}$ and $\pi_2 : \mathcal{W} \to \mathcal{C}$ be the projections to the first and second factor, respectively. For $C' \in \mathcal{C}$, the fiber of π_2 is the affine vector space

$$\pi_2^{-1}(C') = \mathcal{V}_{C'}(3) = \{\omega' \in \mathcal{H} \mid C' \text{ is an integral curve of } \omega'\}.$$

By Theorem 2.8 we have

$$\dim \mathcal{V}_{C'}(3) \geq \binom{3-6+3}{2} + \binom{3+1}{2} - (6-1)^2 + \deg X \;=\; \deg X - 19.$$

Since C' has total Milnor number 18 by (1) and is bitangent to L_∞ by (2), we get $\deg X = 18 + 2 = 20$, hence $\dim \mathcal{V}_{C'}(3) \geq 1$. In particular, π_2 is surjective.

By (1)–(3) we have a point $(\omega, C) \in \mathcal{W}$. Let $\mathcal{W}_{irr}$ be the irreducible component of $\mathcal{W}$ containing (ω, C).

Property (a) holds on $\mathcal{W}_{irr}$, since the η–matrix depends only on the local singularity types of $C \cup L_\infty$ and on the fixed component decomposition, which are constant along $\mathcal{W}$. Property (b) is Zariski open and holds at (ω, C) by assumption, hence it holds on a nonempty open subset of $\mathcal{W}_{irr}$.

By Example 5.8 the number of zeros of ω in $\mathbb{P}^2 \setminus (C \cup L_\infty)$ is at most 1 and is lower semicontinuous for forms satisfying (1)–(3). Hence the locus where (c) holds is Zariski open in $\mathcal{W}_{irr}$; it contains (ω, C) and is therefore nonempty.

Let $\mathcal{W}^0_{irr} \subset \mathcal{W}_{irr}$ be the nonempty Zariski-open locus where (b) and (c) hold. For any $(\omega', C') \in \mathcal{W}^0_{irr}$, condition (a) gives $\operatorname{rank} M_{\eta'} \leq r+1$, and (b) says that the η-geometric points do not lie on a conic, i.e. on a curve of degree $\deg \omega' - 1 = 2$. Thus the hypotheses of Theorem 4.15 are satisfied, and ω' is Darboux integrable.

Let $\mathcal{H}_{norm} \subset \mathcal{H}$ be the affine subspace of differential 1-forms that vanish at the origin and whose linear part is $x\,\mathrm{dx} + y\,\mathrm{dy}$. This space has dimension 14. Now let

$$\mathcal{W}_{norm} := \mathcal{W}^0_{irr} \cap (\mathcal{H}_{norm} \times \mathcal{C})$$

be the locally closed subvariety of $\mathcal{W}^0_{irr}$ whose differential forms are normalized and $\mathcal{C}_{norm} := \pi_2(\mathcal{W}_{norm})$ the sextic curves that occur as integral curves of normalized differential forms in $\mathcal{W}_{norm}$.

To see that $\mathcal{W}_{norm}$ is nonempty let (ω_0, C_0) be the pair obtained from (ω, C) by translating the zero from (c) to the origin. All properties of the proposition are invariant under translation, so $(\omega_0, C_0) \in \mathcal{W}^0_{irr}$. Since ω_0 is Darboux integrable and the origin is not on the integral curve C_0, the 2×2 matrix S of linear coefficients of ω_0 is symmetric. Hence there exists $A \in \mathrm{GL}_2$ with $A^\mathsf{T} S A = I$, and after the linear change of variables $(x, y) \mapsto A^{-1}(x, y)$ we obtain $(\omega_{\mathrm{norm}}, C_{\mathrm{norm}}) \in \mathcal{W}_{norm}$.

It follows that $\pi_1(\mathcal{W}_{norm})$ is a family of normalized Darboux–integrable degree-3 differential 1-forms with a sextic integral curve satisfying (1)–(3) and (a)–(c). Property (d) then implies that $\pi_1(\mathcal{W}_{norm})$ lies in a component Y of the center variety having codimension at least 11 in the 14-dimensional affine space of all normalized degree-3 differential forms $\mathcal{H}_{norm}$, i.e. it has dimension at most 3.

Let $\mathcal{D} \subset \mathcal{C}$ be the PGL_3-saturation of the family from (e), i.e. the union of all projective images of its members. On $\mathcal{D}$, the condition of having exactly the singularities prescribed in (1)–(2) is Zariski open. Since $C \in \mathcal{D}$ satisfies (1)–(2), this open subset is nonempty. Set $\mathcal{D}_{norm} := \mathcal{D} \cap \mathcal{C}_{norm}$. Restricting the fibration $\mathcal{W}_{norm} \to \mathcal{C}_{norm}$ over $\mathcal{D}_{norm}$ yields a subfamily

$$\mathcal{V}_{norm} \longrightarrow \mathcal{D}_{norm}.$$

Then $\pi_1(\mathcal{V}_{norm})$ is also a family of normalized Darboux–integrable degree-3 differential 1-forms with a sextic integral curve satisfying (1)–(3) and (a)–(c). We now estimate $\dim \pi_1(\mathcal{V}_{norm})$.

Recall that the space of normalized degree-3 forms $\mathcal{H}_{norm}$ is invariant under rotations about the origin and under scalings

$$\omega_{\mathrm{norm}}(x, y) \;\longmapsto\; \frac{1}{\lambda^2}\,\omega_{\mathrm{norm}}(\lambda x, \lambda y).$$

The only fixed form under this 2-dimensional group is $x\,\mathrm{dx} + y\,\mathrm{dy}$; every other form has a 2-dimensional orbit, and forms on the same orbit have projectively equivalent integral curves. By normalizing the differential forms of the curves given in (e) we obtain a 1-dimensional family of forms in $\pi_1(\mathcal{V}_{norm})$ whose integral curves are not projectively equivalent. The saturation of this family by rotations and scalings still lies inside $\pi_1(\mathcal{V}_{norm})$ and has dimension at least $1 + 2 = 3$. Since the ambient space of normalized forms $\mathcal{H}_{norm}$ has dimension 14, it follows that

$$\operatorname{codim} \pi_1(\mathcal{V}_{norm}) \leq \; 14 - 3 \; = \; 11.$$

Steiner type	**geometry**	**# η-pts**	**ref.**
11_2	A_2+A_4-quartic + contact conic	6	7.1
11_{18}	A_1+A_4-quartic + two bitangents	6	7.3
11_{25}	$3A_2$-quartic + $3,3$-contact conic	7	7.2
11_{27}	A_1+A_4-quartic + two bitangents	6	7.3
11_{53}	$3A_1$-quartic + a flex and a hyperflex line	7	7.4
11_{59}	A_1+A_4-quartic + two bitangents	6	7.3

TABLE 4. Conjectured codimension-11 components of the center variety over $\mathbb{F}_{29}$ that contain examples satisfying (1)–(3), and (a)–(d) of Proposition 7.1, together with a short geometric description of the corresponding invariant sextic.

From (d) we already know that $\pi_1(\mathcal{W}_{norm})$ is contained in a component Y of the center variety with $\operatorname{codim} Y \geq 11$. Because $\pi_1(\mathcal{V}_{norm}) \subset \pi_1(\mathcal{W}_{norm}) \subset Y$, we have

$$11 \leq \ \operatorname{codim} Y \leq \operatorname{codim} \pi_1(\mathcal{W}_{norm}) \leq \operatorname{codim} \pi_1(\mathcal{V}_{norm}) \ \leq \ 11,$$

so all three codimensions are equal to 11. In particular, $Y = \overline{\pi_1(\mathcal{W}_{norm})} = \overline{\pi_1(\mathcal{V}_{norm})}$ is an irreducible codimension-11 component of the degree-3 center variety whose generic element $(\omega_{\mathrm{norm}}, C_{\mathrm{norm}})$ satisfies (1)–(3) and (a)–(e). □

To apply this proposition one could, in principle, proceed as follows:

(1) Find all possible singularity baskets and component structures that satisfy (1) and (2). This yields a large but finite list of cases.
(2) Select those with at least six η-geometric points.
(3) Select those for which $M_{\eta'}$ drops rank.
(4) Construct a 1-dimensional family of examples.
(5) Check (b) and (c) for a single example.
(6) Reduce modulo a suitable finite field, normalize, and check (d).

The main bottlenecks are step (4), which typically requires substantial hand work, and step (6), which often fails. In that case one has constructed a codimension-11 family that is not itself a component but lies inside a larger component. At present we cannot predict from the configuration whether this happens.

In this paper we use a shortcut. Jakob Kröker [7] found many differential forms over the finite field $\mathbb{F}_{29}$ that satisfy (d). Johannes Steiner [25] computed the $\mathbb{F}_{29}$-rational integral curves of degree at most 6 for many of these examples. We scanned these examples for configurations satisfying (1), (2), and (a); the resulting list is given in Table 5. Only for these configurations did we attempt (and solve) the manual construction step in characteristic 0.

We present our constructions as follows:

(1) **Configuration.** Propose a configuration of components and singularities that satisfies (1) and (2).
(2) **Darboux integrability.** Compute $M_{\eta'}$ and verify (a).

Steiner type	degrees	# η-geom. pts	construction
11_2	$\{2,4\}$	6	7.1
11_{18}	$\{1,1,4\}$	6	7.3
11_{25}	$\{2,4\}$	7	7.2
11_{27}	$\{1,1,4\}$	6	7.3
11_{53}	$\{1,1,4\}$	7	7.4
11_{59}	$\{1,1,4\}$	6	7.3

TABLE 5. Conjectured codimension-11 components of the center variety over $\mathbb{F}_{29}$ that contain examples satisfying (1)–(3), and (a)–(d) of Proposition 7.1, each with an invariant sextic whose component degrees are shown.

(3) **Construction.** Construct a 1-dimensional family of curves (e). For one curve C in the family, compute ω; check (b) and (c).
(4) **Reduction modulo p.** Reduce mod p, normalize, and check (d).

The computations are carried out in `Macaulay2`; the scripts are available online [5].

The first two constructions (Steiner types 11_2 and 11_{25}) start with a maximal quartic C_4 ($\mu = 6$) and a conic C_2 having prescribed contact multiplicities with C_4: $\{2,2,2,2\}$ in the first case and $\{3,3,1,1\}$ in the second. A contact of multiplicity m between two distinct components contributes $2m-1$ to the total Milnor number of the union (this applies when the contact occurs along a single branch at a simple singularity). Hence $\mu = 6 + 4 \cdot 3 = 18$ in the first case and $\mu = 6 + 2 \cdot 5 + 2 \cdot 1 = 18$ in the second. Thus $C_6 = C_4 \cup C_2$ is submaximal in both cases.

7.1. $A_2 + A_4$-quartic & contact conic (Steiner type 11_2).

7.1.1. *Configuration.* Let $C_6 = C_4 \cup C_2$, where C_4 is a quartic and C_2 is a smooth conic. Assume the following hold:

(i) C_4 has an A_2 singularity at a point A and an A_4 singularity at a point B.
(ii) C_2 intersects C_4 with multiplicity 2 at four distinct points, one of them being B. Denote the others by U, V, W.
(iii) The line at infinity L_∞ passes through A with multiplicity 2 and is tangent to C_4 at a further smooth point T.

If no further singularities occur, then C_6 is submaximal: its singularities are A_2 at A, D_7 at B and A_3 at the smooth contact points U, V, W. Hence the total Milnor number is $2+3+3+3+7 = 18$. Such a configuration would satisfy (1) and (2) of Proposition 7.1.

7.1.2. *Darboux integrability.* The η-geometric points of $C_6 \cup L_\infty$ are A, B, T, U, V, W. Looking up the values of η' for the corresponding simple singularities in Table 2, we obtain

$$M_{\eta'} = \begin{pmatrix} 0 & 10 & 2 & 7 \\ 2 & 6 & 0 & 5 \\ 2 & 2 & 0 & 3 \\ 0 & 2 & 2 & 3 \\ 0 & 2 & 2 & 3 \\ 0 & 2 & 2 & 3 \\ 1 & 4 & 2 & 5 \end{pmatrix} \quad \begin{array}{c} (D_7 \text{ at } A) \\ (D_5 \text{ at } B) \\ (A_3 \text{ at } T) \\ (A_3 \text{ at } U) \\ (A_3 \text{ at } V) \\ (A_3 \text{ at } W) \\ (\text{the projection center}) \end{array}$$

where the columns are $(K_z, K_{C_4}, K_{C_2}, d\omega)$. We put 0 in the column of any component that does not pass through the given point.

We observe that $M_{\eta'} \cdot (2, 1, 2, -2)^{\mathrm{t}} = 0$, so the rank of M is at most 3 and condition (a) is satisfied. Hence each ω having this configuration of integral curves would be Darboux-integrable.

7.1.3. *Explicit construction.* It remains to realize the configuration described above and show that it lies in a family as in (e). Classically, any plane quartic carrying an A_2 and an A_4 singularity is projectively equivalent to

$$C_4\colon (xy - z^2)^2 - xz^3 = 0,$$

and in these coordinates the A_2 lies at $A = (1:0:0)$ while the A_4 lies at $B = (0:1:0)$; see [17, page 25]. A direct computation shows that there is a unique line through the B which is tangent to C_4 at a smooth point; it is

$$L\colon 4y + z = 0,$$

with tangency point $T = (16:-1:4)$.

The space of conics in $\mathbb{P}^2$ is $\mathbb{P}^5$, and each imposed tangency to a given curve contributes a single condition. Hence the family of contact conics for C_4 is at least 1-dimensional. It remains to exhibit one example with exactly the tangencies described above. For this we recall the following classical construction for contact curves. Let

$$M = \begin{pmatrix} F & G \\ G & H \end{pmatrix}$$

a homogeneous symmetric matrix. Then $F = 0$ is contact to $\det M = 0$. Indeed on $F = 0$ we have $\det M = G^2$ and therefore all intersection points are of multiplicity 2.

Here we look at

$$M = \begin{pmatrix} -x^2 - 4xy - 2xz - 4yz + 3z^2 & 2xy + xz + 4yz - z^2 \\ 2xy + xz + 4yz - z^2 & -4y\,z - z^2 \end{pmatrix}$$

Since $\det M = -4C_4$ we have that

$$C_2\colon\; -x^2 - 4xy - 2xz - 4yz + 3z^2 = 0$$

is contact to C_4. On also checks that C_2 is smooth, that the four contact points are distinct, and that one of them is B. Furthermore one can check that the six η-geometric points are not contained in a conic. This proves that a curve as in (1), (2) exist and that is satisfies (a), (b) and (e).

We now construct ω as in (3) and check (c): Applying the substitution $z \mapsto z - 4y$ sends the bitangent line L to the line at infinity $z = 0$, and using computer algebra one finds a unique (up to scaling) degree-3 differential 1-form ω for which both C_4 and C_2 are integral curves. This ω has a zero at $(71:10:51)$; after translating so that this zero is at the origin, an affine expression is:

$$\begin{aligned}
\omega = (&867x^2y - 81498xy^2 - 194208y^3 \\
&+ 170x^2 + 6868xy - 64702y^2 \\
&+ 145x + 3450y)dx \\
+(&1734x^3 + 112710x^2y - 138720xy^2 - 2663424y^3 \\
&+ 5066x^2 + 372130xy + 209984y^2 \\
&+ 3450x + 279380y)dy
\end{aligned}$$

Its linear part is symmetric, hence ω has a center at $(0, 0)$.

7.1.4. *Reduction modulo p.* Over the finite field $\mathbb{F}_{29}$ this differential form can be normalized to

$$(x^3+4x^2y+3xy^2-2y^3-4x^2+14xy+6y^2+x)\,\mathrm{dx}$$
$$+(-14x^3+5x^2y-2y^3-3x^2+11xy+3y^2+y)\,\mathrm{dy}\,.$$

With Frommer's Algorithm we can compute (as a sanity check) that the first 13 focal values vanish, and that the Jacobian matrix of the first 13 focal values has rank 11. This shows (d) and we can apply Proposition 7.1 to obtain a codim 11 component of the center variety in degree 3.

7.2. $3A_2$-quartic & $3,3$-contact conic (Steiner type 11_{25}).

7.2.1. *Configuration.* Let $C_6 = C_4 \cup C_2$, where C_4 is a quartic and C_2 is a smooth conic. Assume the following hold:

(i) C_4 has three A_2 singularities at points U, V, W.
(ii) C_2 has contact of order 3 with C_4 at two distinct points A, B, and meets C_4 transversely at two further points R, S.
(iii) The line at infinity L_∞ passes through S with multiplicity 2 and is tangent to C_4 at a further smooth point T.
(iv) The points A, B, R, S, T, U, V, W do not lie on a conic.

If no further singularities occur, then C_6 is submaximal: its singularities are A_2 at U, V, W, A_5 at A, B, and A_1 at R, S. Hence the total Milnor number is $2+2+2+5+5+1+1 = 18$. Such a configuration satisfies (1) and (2) of Proposition 7.1.

7.2.2. *Darboux integrability.* The η-geometric points are U, V, W, A, B, S, T. Looking up the quasi-homogeneous values of η' at these points (in this order) yields

$$\begin{pmatrix} 0 & 6 & 0 & 5 \\ 0 & 6 & 0 & 5 \\ 0 & 6 & 0 & 5 \\ 0 & 3 & 3 & 4 \\ 0 & 3 & 3 & 4 \\ 1 & 1 & 1 & 2 \\ 2 & 2 & 0 & 3 \\ 1 & 4 & 2 & 5 \end{pmatrix} \qquad \begin{matrix} (A_2 \text{ at } U) \\ (A_2 \text{ at } V) \\ (A_2 \text{ at } W) \\ (A_5 \text{ at } A) \\ (A_5 \text{ at } B) \\ (D_4 \text{ at } S) \\ (A_3 \text{ at } T) \\ (\text{projection center}) \end{matrix}$$

A quick check yields $M_\eta \cdot (4,5,3,-6)^t = 0$ and hence $\operatorname{rank} M_\eta \le 3$. This proves (a).

7.2.3. *Explicit construction.* It remains to realize the configuration described above and to show that it lies in a family as in (e). Classically, any plane quartic with three A_2 singularities is projectively equivalent (after a projective change of coordinates) to

$$C_4:\ x^2y^2+y^2z^2+z^2x^2-2xyz(x+y+z)=0,$$

and in these coordinates the A_2 singularities U, V, W lie at the coordinate points; see [17, page 14]. A direct computation shows that

$$C_2:\ 9x^2-80xy-432y^2-80xz+1880yz-432z^2=0$$

is a smooth conic meeting C_4 with intersection multiplicity 3 at $A = (36{:}9{:}4)$ and $B = (36{:}4{:}9)$, and transversely at $R = (4{:}9{:}36)$ and $S = (4{:}36{:}9)$. Furthermore,

$$L:\ 27x+125y-512z=0$$

meets C_4 and C_2 transversely at S and is tangent to C_4 at the point $T = (1600 : -576 : -225)$. One checks that the points U, V, W, A, B, R, S, T do not lie on a conic. Thus these curves realize the configuration described above.

To see that this configuration lies in a family as in (e), we do a dimension count. The space of conics in $\mathbb{P}^2$ has dimension 5, and the existence of two points of multiplicity-3 contact with C_4 imposes $2 \cdot 2 = 4$ conditions. The two remaining simple intersections impose no further condition. Hence C_2 lies in a 1-dimensional family with at least the prescribed contacts. Furthermore one can check that the six η-geometric points are not contained in a conic. This proves that a curve as in (1), (2) exist and that it satisfies (a), (b) and (e).

We now construct ω as in (3) and check (c): Applying the substitution $z \mapsto z + \frac{27x+125y}{512}$ sends the bitangent line L to the line at infinity $z = 0$, and using computer algebra one finds a unique (up to scaling) degree-3 differential 1-form ω for which both C_4 and C_2 are integral curves. This ω has a zero at $(256 : -256 : 273)$; after translating so that this zero is at the origin, an affine expression is:

$$\begin{aligned}\omega = (&-38233377x^3 + 1624359555x^2y - 7790988387xy^2 + 8443514625y^3\\ &- 2731921920x^2 + 33127564288xy + 1632816640y^2 - 33512488960x\\ &- 17154703360y)dx\\ -(&439348455x^3 - 1706788629x^2y - 462700875xy^2 + 916085625y^3\\ &+ 9614212608x^2 - 34486000640xy + 4299742720y^2 + 17154703360x-\\ &27556577280y)dy\end{aligned}$$

Its linear part is symmetric, hence ω has a center at $(0, 0)$.

7.2.4. *Reduction modulo p.* Over the finite field $\mathbb{F}_{29}$ this can be normalized to

$$\begin{aligned}&(-8x^3 - 13x^2y + 5xy^2 - 11y^3 - 10x^2 - 2xy + 14y^2 + x)\,\mathrm{dx}\\ &+ (14x^3 + 10x^2y + 12xy^2 - 9y^3 - 2x^2 + 14xy - 14y^2 + y)\,\mathrm{dy}\,.\end{aligned}$$

With Frommer's Algorithm we can compute that the first 13 focal values vanish, and that the Jacobian matrix of the first 13 focal values has rank 11. This shows (d) and we can apply Proposition 7.1 to obtain a codim 11 component of the center variety in degree 3.

7.3. A_1+A_4-quartic & two bitangents (Steiner types 11_{18}, 11_{27}, 11_{59}).

7.3.1. *Configuration.* Let C_4 be a quartic, and let L, L', L'' be (generalized) bitangents—i.e., lines meeting C_4 with intersection multiplicity 2 at two distinct points. Assume

(i) C_4 has an A_1 singularity at A and an A_4 singularity at B;
(ii) L intersects C_4 in two smooth points S and T;
(iii) L' intersects C_4 in A and a further smooth point U
(iv) L'' intersects C_4 in B and a further smooth point V

We now have three possibilities for the choice of the line at infinity:

(1) $L_\infty = L$ and $C_6 = C_4 \cup L' \cup L''$;
(2) $L_\infty = L'$ and $C_6 = C_4 \cup L \cup L''$;
(3) $L_\infty = L''$ and $C_6 = C_4 \cup L \cup L'$.

In all three cases the configuration satisfies (1)–(2) of Proposition 7.1.

7.3.2. *Darboux integrability.* In all three cases $C_6 \cup L_\infty = C_4 \cup L \cup L' \cup L''$, and the η-geometric points of $C_6 \cup L_\infty$ are A, B, S, T, U, V. Looking up the values of η' for the corresponding simple singularities in Table 2, we obtain:

$$M_{\eta'} = \begin{pmatrix} 0 & 1 & 0 & 2 & 2 \\ 0 & 0 & 2 & 10 & 7 \\ 2 & 0 & 0 & 2 & 3 \\ 2 & 0 & 0 & 2 & 3 \\ 0 & 2 & 0 & 2 & 3 \\ 0 & 0 & 2 & 2 & 3 \\ 1 & 1 & 1 & 4 & 5 \end{pmatrix} \qquad \begin{array}{c} (D_4 \text{ at } A) \\ (D_7 \text{ at } B) \\ (A_3 \text{ at } S) \\ (A_3 \text{ at } T) \\ (A_3 \text{ at } U) \\ (A_3 \text{ at } V) \\ \text{(projection center)} \end{array}$$

where the columns are $(K_L, K_{L'}, K_{L''}, K_{C_4}, d\omega)$. Note that

$$M_\eta \cdot (2,2,2,1,-2)^\mathsf{T} = 0,$$

hence $\operatorname{rank} M_{\eta'} \leq 4$. Moreover, choosing L, L', or L'' as the line at infinity merely permutes the first three columns, so the rank of $M_{\eta'}$ is unchanged. This proves (a) in all three cases.

7.3.3. *Explicit construction.* It remains to realize the configuration described above and to show that it lies in a family as in (e). Classically, any plane quartic with an A_1 and an A_4 singularity is projectively equivalent to

$$C_4 : \ (xz + y^2)^2 + xy^3 - \lambda\, x^2 yz = 0, \qquad \lambda \neq 0, -1,$$

with the A_1 at $A = (1:0:0)$ and the A_4 at $B = (0:0:1)$; see [17, page 25] after a projective change of coordinates. A direct computation shows that, for $\lambda \neq -1, 0$, the following lines are (generalized) bitangents satisfying (ii)–(iv):

$$\begin{aligned} L :&\ \lambda^2 x + 4(\lambda - 1)y - 16z = 0, \\ L' :&\ y + 4(\lambda + 1)z = 0, \\ L'' :&\ \lambda^2 x - 4(\lambda + 1)y = 0. \end{aligned}$$

Thus the configuration occurs in a 1-parameter family as in (e). From now on we fix $\lambda = 1$ and compute the differential forms as in (3) for the three choices of L_∞.

Sending L to infinity via the linear change $z \mapsto \frac{x-z}{16}$ we obtain

$$\begin{aligned} \omega = (&-3x^3 + 26x^2y - 64xy^2 - 416y^3 + 6x^2 + 14xy - 16y^2 - 3x - 16y)\,\mathrm{d}x \\ &+ (16x^3 + 16x^2y - 256xy^2 - 768y^3 - 56x^2 - 16xy - 128y^2 + 40x)\,\mathrm{d}y, \end{aligned}$$

with a zero (off the invariant curves) at $(28:-9:40)$.

Sending L' to infinity via $z \mapsto -\frac{y-z}{8}$ we obtain

$$\begin{aligned} \omega' = (&-83xy^2 - 136y^3 + 38xy + 56y^2 - 3x - 16y)\,\mathrm{d}x \\ &+ (6x^3 + 63x^2y - 56xy^2 - 256y^3 - 35x^2 - 88xy + 64y^2 + 40x)\,\mathrm{d}y, \end{aligned}$$

with a zero at $(63:-18:260)$.

Finally, sending L'' to infinity via $(y,z) \mapsto \left(\frac{x-z}{8},\, y\right)$ we obtain

$$\begin{aligned} \omega'' = (&-15x^3 - 1664x^2y + 34816xy^2 - 131072y^3 \\ &+ 33x^2 + 1088xy - 2048y^2 - 21x - 192y + 3)\,\mathrm{d}x \\ &+ (1264x^3 - 4096x^2y - 196608xy^2 - 1808x^2 \\ &+ 5120xy + 592x - 1024y - 48)\,\mathrm{d}y, \end{aligned}$$

with a zero at $(664:-189:5850)$.

7.3.4. *Reduction modulo p.* Over the finite field $\mathbb{F}_{29}$ all three differential forms can be normalized for Frommer's algorithm. In each case the first 13 focal values vanish, and the Jacobian matrix of the first 13 focal values has rank 11. This verifies (d), so Proposition 7.1 yields three codimension-11 components of the degree-3 center variety. We also find that ω, ω', and ω'' have Steiner type 11_{58}, 11_{27}, and 11_{18}, respectively.

7.4. $3A_1$–quartic with a flex line and a hyperflex line (Steiner type 11_{53}).

7.4.1. *Configuration.* Let $C_6 = C_4 \cup L_{\text{flex}} \cup L_{\text{hyper}}$, where C_4 is a quartic and $L_{\text{flex}}, L_{\text{hyper}}$ are lines. Assume:

(i) C_4 has three A_1 singularities at points U, V, W.
(ii) C_4 has a flex (inflection) point at A, and L_{flex} is its flex line; that is, L_{flex} meets C_4 with multiplicity 3 at A and multiplicity 1 at a further point B.
(iii) C_4 has a hyperflex at B, and L_{hyper} is the hyperflex line at B; that is, L_{hyper} meets C_4 with multiplicity 4 at B.
(iv) C_4 admits a bitangent L_∞ with (smooth) contact points S and T.

If no further singularities occur, then C_6 is submaximal: its singularities are A_1 at U, V, W, A_5 at A, and D_{10} at B, so the total Milnor number is $3 + 5 + 10 = 18$. Hence the configuration satisfies *(1)* and *(2)* of Proposition 7.1.

7.4.2. *Darboux integrability.* The η-geometric points are U, V, W, A, B, S, T. Looking up the quasi-homogeneous values of η' at these points (in this order) yields

$$\begin{pmatrix} 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 6 & 0 & 6 & 8 \\ 0 & 2 & 8 & 8 & 10 \\ 2 & 0 & 0 & 2 & 3 \\ 2 & 0 & 0 & 2 & 3 \\ 1 & 1 & 1 & 4 & 5 \end{pmatrix} \quad \begin{array}{c} (A_1 \text{ at } U) \\ (A_1 \text{ at } V) \\ (A_1 \text{ at } W) \\ (A_5 \text{ at } A) \\ (D_{10} \text{ at } B) \\ (A_3 \text{ at } S) \\ (A_3 \text{ at } T) \\ (\text{projection center}) \end{array}$$

with column order $(K_z,\, K_{L_{\text{flex}}},\, K_{L_{\text{hyper}}},\, K_{C_4},\, d\omega)$. Zeros indicate components not passing through the given point. Moreover,

$$M_{\eta'} \cdot (3,\, 2,\, 1,\, 6,\, -6)^{\mathsf{T}} = 0,$$

hence $\operatorname{rank} M_{\eta'} \le 4$. This proves (a).

7.4.3. *Explicit construction.* It remains to realize the configuration above and to exhibit a 1–parameter family as in (e). Consider the morphism

$$\varphi : \mathbb{P}^1 \longrightarrow \mathbb{P}^2$$

defined by

$$(s:t) \mapsto \big(s^4 : st^3 : (s-t)^2(s+\lambda t)^2\big), \qquad \lambda \in \mathbb{C} \setminus \{-1, 0, 1\}.$$

Let C_4 be the image of φ, and set

$$A = \varphi(1{:}0), \quad B = \varphi(0{:}1), \quad S = \varphi(1{:}1), \quad T = \varphi(\lambda{:}-1).$$

Then:

(i) C_4 is rational; by the genus formula a general member is expected to have three A_1–singularities.
(ii) $L_{\text{flex}} = \{y = 0\}$ is a flex line at A and meets C_4 again (transversely) at B.

(iii) $L_{\text{hyper}} = \{x = 0\}$ is a hyperflex line at B.
(iv) $L_\infty = \{z = 0\}$ is bitangent to C_4 at S and T.

Thus we obtain a 1–parameter family as in (e). For $\lambda = 4$ one checks that

$$C_6 = C_4 \cup L_{\text{flex}} \cup L_{\text{hyper}}$$

has exactly the singularities listed in the configuration, and that the η-geometric points U, V, W, A, B, S, T do not lie on a conic, establishing (b).

Using computer algebra, one finds a unique (up to scaling) degree-3 differential 1-form

$$\begin{aligned}\omega = &(-91x^2y - 819xy^2 - 4160y^3 - 13xy - 240y^2 - 4y)\,\mathrm{dx}\\ &+(52x^3 - 1638x^2y + 6656xy^2 - 68x^2 + 672xy + 16x)\,\mathrm{dy}\end{aligned}$$

for which C_6 is an integral curve, as in (3). The form ω has a zero at $(28{:}-12{:}325)$ lying outside $C_6 \cup L_\infty$, which establishes (c).

7.4.4. *Reduction modulo p.* Over the finite field $\mathbb{F}_{29}$, after reduction and normalization at the zero found above, ω becomes

$$\begin{aligned}&\bigl(-9x^2y + 5xy^2 - 4y^3 - 10x^2 + 10xy + 2y^2 + x\bigr)\,dx\\ +&\bigl(x^3 + 3x^2y + 9xy^2 - 3y^3 - 6x^2 - 10xy + 10y^2 + y\bigr)\,dy.\end{aligned}$$

Using Frommer's algorithm, the first 13 focal values vanish, and the Jacobian matrix of these 13 focal values has rank 11. This verifies (d), and by Proposition 7.1 we obtain a codimension-11 component of the degree-3 center variety.

7.5. **Cuspidal cubic and a 3-tangent star (Steiner types 11_{50}, 11_{64}).** As a final construction, we show that two of the components in [26] are recovered by our blueprint. Noticing that these example fits seamlessly into the η framework was a key motivation for the present systematic study of submaximal sextics with a bitangent.

7.5.1. *Configuration.* Let $C_6 = C_{\text{cusp}} \cup C_{\text{star}}$ be the union of two plane cubics. Assume:

(i) C_{cusp} has a cusp at A.
(ii) C_{star} has a D_4–singularity at B, i.e. $C_{\text{star}} = L_1 \cup L_2 \cup L_3$ is the union of three distinct lines concurrent at B.
(iii) For $i = 1, 2, 3$, the line L_i meets C_{cusp} with intersection multiplicity 2 at a smooth point P_i and transversely at a second smooth point Q_i.
(iv) The line at infinity L_∞ passes through two of the tangency points, say Q_1 and Q_2, and is transverse to both branches there.

If no further singularities occur, then C_6 is submaximal: its singularities are A_2 at A, D_4 at B, A_3 at each P_i, and A_1 at each Q_i, hence the total Milnor number is

$$\mu(C_6) = 2 + 4 + 3 \cdot 3 + 3 \cdot 1 = 18.$$

In addition, L_∞ is a generalized bitangent to C_6 so the configuration satisfies (1)–(2) of Proposition 7.1.

7.5.2. *Darboux integrability.* The η-geometric points are A, B, P_1, P_2, P_3, Q_1, Q_2. Looking up the quasi-homogeneous values of η' at these points (in this order) yields

$$\begin{pmatrix} 0 & 0 & 6 & 5 \\ 0 & 3 & 0 & 2 \\ 0 & 2 & 2 & 3 \\ 1 & 1 & 1 & 2 \\ 1 & 3 & 3 & 5 \end{pmatrix} \qquad \begin{matrix} (A_2 \text{ at } A) \\ (D_4 \text{ at } B) \\ (A_3 \text{ at } P_1, P_2, P_3) \\ (D_4 \text{ at } Q_1, Q_2) \\ (\text{projection center}) \end{matrix}$$

with column order (K_z, $K_{C_{\mathrm{star}}}$, $K_{C_{\mathrm{cusp}}}$, $d\omega$). Moreover,

$$M_{\eta'} \cdot (3, 4, 5, -6)^{\mathsf{T}} = 0,$$

hence $\operatorname{rank} M_{\eta'} \leq 4$. This proves (a).

7.5.3. *Explicit construction.* These configurations occur in a natural family. Fix a cuspidal cubic C_{cusp}, and let B be a general point. Projection from B induces a degree-3 map

$$\pi : \widetilde{C_{\mathrm{cusp}}} \to \mathbb{P}^1;$$

since $\widetilde{C_{\mathrm{cusp}}} \cong \mathbb{P}^1$, the Riemann–Hurwitz formula gives $\deg R_\pi = 4$. Equivalently, there are exactly four lines through B meeting C_{cusp} with multiplicity two: one is the line through the cusp A, and the other three are simple tangents at smooth points P_1, P_2, P_3. Let L_1, L_2, L_3 denote these tangents, and write Q_i for the third (transverse) intersection of L_i with C_{cusp}. Set $C_{\mathrm{star}} := L_1 \cup L_2 \cup L_3$, and choose L_∞ to be the line through two of the third intersection points, say Q_1 and Q_2 (for a general choice, L_∞ is transverse to the branches there).

As B varies, this yields a 2–dimensional family of such configurations; modulo projectivities preserving C_{cusp} (a 1–dimensional group), the projective equivalence classes form a 1–dimensional family. This proves (e). Conditions (b) and (c) hold for a general choice and can be checked explicitly for a sample member, exactly as in the previous examples.

7.5.4. *Reduction modulo p.* A subtle field-of-definition issue appears here. Over an algebraically closed field every differential form with symmetric linear part can be normalized (i.e. brought to linear part $x\,dx + y\,dy$). Over nonclosed fields such as $\mathbb{R}$ or $\mathbb{F}_{29}$ this need not be possible, because it amounts to diagonalizing a symmetric 2×2 matrix over the base field.

In the previous examples we circumvented this by choosing a member of the family whose linear part *is* diagonalizable over the base field. In the present family, however, diagonalizability over $\mathbb{F}_{29}$ occurs precisely when the tangent lines L_1 and L_2, are *conjugate over* $\mathbb{F}_{29}$ (each defined only over a quadratic extension, while their union $L_1 \cup L_2$ is defined over $\mathbb{F}_{29}$). The same phenomenon occurs over $\mathbb{R}$ (cf. [26]), where one only obtains a real equation for $L_1 \cup L_2$. (The parallel behavior over $\mathbb{R}$ and $\mathbb{F}_{29}$ is not structural; it need not persist over other finite fields.)

Respecting this constraint, we nevertheless find examples over $\mathbb{F}_{29}$ that can be normalized and satisfy condition (d); these are of Steiner type 11_{64}. Because in this type examples only L_3 is rational over $\mathbb{F}_{29}$, only this lines appears in the Kröker–Steiner database, and for the same reason this case is absent from Table 5.

7.5.5. *Switching bitangents.* As in Example 7.3, we may also take one of the generalized bitangents L_1, L_2, L_3 as the line at infinity. Over $\mathbb{C}$ this yields two further families: one obtained from L_1 (equivalently from L_2, since L_1 and L_2 are conjugate) and one from L_3. Over $\mathbb{R}$ and over $\mathbb{F}_{29}$ the first family does not occur, because the line at infinity must be defined over the base field. The second does occur and gives a family of Steiner type 11_{50}.

Remark 7.1. After the first version of this paper was posted on the arXiv, Joan Torregrosa pointed out to the author that, except for the first example, the differential forms displayed in this section yield complex (but not real) centers for the parameter choices shown. Since each construction comes in families, one can choose other parameters so that real centers occur (with the exception of the Steiner type 11_{25}). For the reader's convenience, explicit differential forms with a real center are listed below; ω_i denotes the form corresponding to the Steiner type 11_i.

$$\begin{aligned}
\omega_{18} =&(-46875x^3 - 10350000x^2y + 1607040000xy^2 - 29859840000y^3\\
&+ 314000x^2 + 892800xy + 481075200y^2 - 718976x + 2727936y)\,\mathrm{dx} +\\
&(13500000x^3 - 699840000x^2y - 44789760000xy^2 - 52790400x^2\\
&+ 2280960000xy - 7166361600y^2 + 2727936x - 31850496y)\,\mathrm{dy}\\
\omega_{27} =&(-2550xy^2 - 6300y^3 - 4170xy - 9420y^2 - 1632x - 2532y)\,\mathrm{dx} +\\
&(75x^3 + 1175x^2y - 3300xy^2 - 14400y^3 - 860x^2 - 20340xy\\
&- 33480y^2 - 2532x - 4932y)\,\mathrm{dy}\\
\omega_{53} =&(-16151527x^2y - 496082615xy^2 - 2491949880y^3 + 212660x^2\\
&+ 7459809xy + 33402810y^2 - 12220x - 7800y)\,\mathrm{dx} +\\
&(9229444x^3 - 992165230x^2y + 3987119808xy^2 + 1938244x^2\\
&+ 90410880xy - 15748992y^2 - 7800x - 341640y)\,\mathrm{dy}\\
\omega_{59} =&(-38025x^3 + 1284400x^2y - 11356800xy^2 - 49753600y^3 - 470340x^2\\
&+ 15624960xy + 40684800y^2 - 5476608x - 7723008y)\,\mathrm{dx} +\\
&(439400x^3 - 3244800x^2y - 60569600xy^2 - 155750400y^3 - 5314400x^2\\
&+ 76477440xy + 147363840y^2 - 7723008x - 13713408y)\,\mathrm{dy}
\end{aligned}$$

References

[1] N. N. Bautin. On the number of limit cycles appearing with variation of the coefficients from an equilibrium state of the type of a focus or a center. *Mat. Sbornik N.S.*, 30(72):181–196, 1952.

[2] Y. N. Bibikov. *Local theory of nonlinear analytic ordinary differential equations*, volume 702 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-New York, 1979. DOI: 10.1007/BFb0064649

[3] Y. L. Bondar′and A. P. Sadovskiĭ. On a theorem of Żołądek. *Differ. Uravn.*, 44(2):263–265, 287, 2008.

[4] H.-C. von Bothmer. New Solutions to the Poincaré Center Problem in Degree 3. Available at arXiv:2409.01751, 2024.

[5] H.-C. von Bothmer. Supplementary Macaulay2 scripts for the paper "Darboux Integrability via Singularities of Invariant Curves at Infinity". Available at DOI: :10.5281/zenodo.17280680, 2025.

[6] H.-C. von Bothmer and J. Kröker. A improved C++ program for calculating focal values in characteristic p. Available at http://sourceforge.net/projects/centerfocus/, 2009.

[7] H.-C. von Bothmer and J. Kröker. A database of differential forms in characteristic p. Available at http://centerfocus.de/, 2010.

[8] H.-C. von Bothmer and J. Kröker. A survey of the Poincaré center problem in degree 3 using finite field heuristics. Available at arXiv:1012.3402, 2010.

[9] H.-C. von Bothmer and J. Kröker. centerfocus: Macaulay2 packages for studying the poincaré center problem. Available at `https://github.com/bothmer/centerfocus`, 2024.

[10] E. Brieskorn and H. Knörrer. *Plane Algebraic Curves*. Birkhäuser Basel, 1986. DOI: 10.1007/978-3-0348-5097-1

[11] C. Christopher, J. Llibre, C. Pantazi, and S. Walcher. Inverse problems for invariant algebraic curves: explicit computations. *Proc. Roy. Soc. Edinburgh Sect. A*, 139(2):287–302, 2009. DOI: 10.1017/S0308210507001175

[12] C. Christopher, J. Llibre, and J. V. Pereira. Multiplicity of invariant algebraic curves in polynomial vector fields. *Pacific J. Math.*, 229(1):63–117, 2007. DOI: 10.2140/pjm.2007.229.63

[13] G. Darboux. Mémoire sur les équations différentielles algébriques du permier ordre et du premier degré (Mélanges). *Bull. Sci. Math.*, 2:60–96;123–144; 151–200, 1878.

[14] P. Deligne. La conjecture de Weil, I. *Publ. Math. IHES*, 43:273–307, 1974. DOI: 10.1007/BF02684373

[15] M. H. Dulac. Determination et integration d'une certaine classe d'equations differentielles ayant pour point singulier un centre. *Bull. Sci. Math.*, 32:230–252, 1908.

[16] M. Frommer. Über das Auftreten von Wirbeln und Strudeln (geschlossener und spiraliger Integralkurven) in der Umgebung rationaler Unbestimmtheitsstellen. *Math. Ann.*, 109:395–424, 1934. DOI: 10.1007/BF01449147

[17] C.-M. Hui. *Plane Quartic Curves*. PhD thesis, University of Liverpool, Liverpool, UK, 1979.

[18] C. Okonek, M. Schneider, and H. Spindler. *Vector bundles on complex projective spaces*. Modern Birkhäuser Classics. Birkhäuser/Springer Basel AG, Basel, 2011. Corrected reprint of the 1988 edition, With an appendix by S. I. Gelfand.

[19] U. Persson. *Double sextics and singular K-3 surfaces*, pages 262–328. Springer Berlin Heidelberg, 1985. DOI: 10.1007/BFb0075003

[20] H. Poincaré. Mémoire sur les courbes définies par une équation différentielle. *Journal de Mathématiques Pures et Appliquées*, 3e série, 7:375–422, 1881.

[21] H. Poincaré. Mémoire sur les courbes définies par une équation différentielle. *Journal de Mathématiques Pures et Appliquées*, 3e série, 8:251–296, 1882.

[22] H. Poincaré. Sur les courbes définies par les équations différentielles. *Journal de Mathématiques Pures et Appliquées*, 4e série, 1:167–244, 1885. DOI: 10.1002/mmnd.48018850129

[23] K. Saito. Quasihomogene isolierte Singularitäten von Hyperflächen. *Invent. Math.*, 14:123–142, 1971. DOI: 10.1007/BF01405360

[24] A. J. Sommese, J. Verschelde, and C. W. Wampler. Numerical decomposition of the solution sets of polynomial systems into irreducible components. *SIAM J. Numer. Anal.*, 38(6):2022–2046, 2001. DOI: 10.1137/S0036142900372549

[25] J. Steiner. Untersuchung der Komponenten der Zentrumsvarietät durch Interpolation. Bachelor Thesis, Georg-August-Universität Göttingen, January 2011.

[26] J. Torregrosa. Cubic planar vector fields with high local cyclicity. *São Paulo J. Math. Sci.*, 19(1):Paper No. 4, 16, 2025. DOI: 10.1007/s40863-024-00486-9

[27] T. Urabe. Dynkin graphs and combinations of singularities on plane sextic curves. In *Singularities (Iowa City, IA, 1986)*, volume 90 of *Contemp. Math.*, pages 295–316. Amer. Math. Soc., Providence, RI, 1989.

[28] A. Weil. Number of solutions of equations over finite fields. *Bull. Amer. Math. Soc.*, 55:497–508, 1949. DOI: 10.1090/S0002-9904-1949-09219-4

[29] J.-G. Yang. Sextic curves with simple singularities. *Tohoku Math. J. (2)*, 48(2):203–227, 1996. DOI: 10.2748/tmj/1178225377

[30] H. Żołądek. The classification of reversible cubic systems with center. *Topol. Methods Nonlinear Anal.*, 4(1):79–136, 1994. DOI: 10.12775/TMNA.1994.024

[31] H. Żołądek. Remarks on: "The classification of reversible cubic systems with center" [Topol. Methods Nonlinear Anal. **4** (1994), no. 1, 79–136; MR1321810 (96m:34057)]. *Topol. Methods Nonlinear Anal.*, 8(2):335–342 (1997), 1996. DOI: 10.12775/TMNA.1994.024

Hans-Christian von Bothmer, Fachbereich Mathematik der Universität Hamburg, Bundesstrasse 55, 20146 Hamburg, Germany

Email address: `hans.christian.v.bothmer@uni-hamburg.de`