MILNOR FIBRATIONS AND THE THOM PROPERTY FOR MAPS $f\bar{g}$

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ABSTRACT. We prove that every map-germ $f\bar{g} : (\mathbb{C}^n, \underline{0}) \to (\mathbb{C}, 0)$ with an isolated critical value at 0 has the Thom $a_{f\bar{g}}$-property. This extends Hironaka's theorem for holomorphic mappings to the case of map-germs $f\bar{g}$ and it implies that every such map-germ has a Milnor-Lê fibration defined on a Milnor tube. One thus has a locally trivial fibration $\phi : \mathbb{S}_\varepsilon \setminus K \to \mathbb{S}^1$ for every sufficiently small sphere around $\underline{0}$, where K is the link of $f\bar{g}$ and in a neighbourhood of K the projection map ϕ is given by $f\bar{g}/|f\bar{g}|$.

INTRODUCTION

Soon after J. Milnor published his book [14], there were several interesting articles about Milnor fibrations for real singularities published by various people, as for instance by E. Looijenga, P. T. Church and K. Lamotke, N. A'Campo, B. Perron, L. Kauffman and W. Neumann, A. Jacquemard and others. More recently, there has been a new wave of interest in the topic and a number of articles have been published by various authors (see for instance [1, 2, 3, 5, 7, 13, 15, 17, 18, 19, 20, 22]).

Unlike the fibration theorem for complex singularities, which holds for every map-germ $(\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$, in the real case one needs to impose stringent conditions to get a fibration on a "Milnor tube", or a fibration on a sphere, as in the holomorphic case.

In [18] we observed that Lê's arguments in [10] for holomorphic mappings extend to every real analytic map germ $(\mathbb{R}^n, \underline{0}) \to (\mathbb{R}^p, 0)$, $n > p$, with an isolated critical value, provided it has the Thom a_f-property and $V := f^{-1}(0)$ has dimension more than 0. Hence one has in that setting a Milnor-Lê fibration:

$$f : N(\varepsilon, \delta) \to \mathbb{D}_\delta \setminus \{0\} \ .$$

Here $N(\varepsilon, \delta)$ denotes a "solid Milnor tube": it is the intersection $f^{-1}(\mathbb{D}_\delta \setminus \{0\}) \cap \mathbb{B}_\varepsilon$, where $\mathbb{B}_\varepsilon$ is a sufficiently small ball around $\underline{0} \in \mathbb{R}^n$ and $\mathbb{D}_\delta$ is a ball in $\mathbb{R}^p$ of radius small enough with respect to ε. This was later completed in [5] (see also [7]), giving necessary and sufficient conditions for one such map-germ to define a Milnor fibration on every small sphere around the origin, with projection map $f/|f|$.

Then, an interesting problem is finding families of map germs $(\mathbb{R}^n, \underline{0}) \to (\mathbb{R}^p, 0)$, $n > p$, having an isolated critical value and the Thom property. This is even better when the given families further have a rich geometry one can use in order to study the topology of the corresponding Milnor fibrations (cf. [3]).

In this article we prove:

Theorem. *Let f, g be holomorphic map germs $(\mathbb{C}^n, \underline{0}) \to (\mathbb{C}, 0)$ such that the map $f\bar{g}$ has an isolated critical value at $0 \in \mathbb{C}$. Then $f\bar{g}$ has the Thom $a_{f\bar{g}}$-property.*

In fact our proof is of a local nature and therefore extends, with same proof, to the case of holomorphic map-germs defined on a complex analytic variety X with an isolated singularity. This result generalizes to higher dimensions the corresponding theorem in [18] for $n = 2$, and it has the following corollaries:

Corollary 1. *Let f, g be holomorphic map-germs defined on a complex analytic variety X with an isolated singularity at a point $\underline{0}$, such that the germ $f\bar{g}$ has an isolated critical value at 0. Then one has a locally trivial fibration*

$$N(\varepsilon, \delta) \xrightarrow{f} \mathbb{D}_\delta \setminus \{0\}, \quad \varepsilon \gg \delta > 0 \text{ sufficiently small},$$

where $N(\varepsilon, \delta) := [(f\bar{g})^{-1}(\mathbb{D}_\delta \setminus \{0\}) \cap \mathbb{B}_\varepsilon]$ is a solid Milnor tube for $f\bar{g}$.

Corollary 2. *Let $\mathcal{L}_X := X \cap \mathbb{S}_\varepsilon$ be the link of X, $V := (f\bar{g})^{-1}(0)$ and $\mathcal{L}_V := \mathcal{L}_X \cap V$ be the link of V. Then one has a locally trivial fibration,*

$$\phi : \mathcal{L}_X \setminus \mathcal{L}_V \longrightarrow \mathbb{S}^1,$$

which restricted to $\mathcal{L}_X \cap N(\varepsilon, \delta)$ is the natural projection $\phi = \frac{f\bar{g}}{|f\bar{g}|}$.

In fact we know from [18] that for $n = 2$ the projection map ϕ in Corollary 2 can be taken to be $\frac{f\bar{g}}{|f\bar{g}|}$ everywhere on $\mathcal{L}_X \setminus \mathcal{L}_V$, not only near the link of V. It would be interesting to know whether or not this statement holds also in higher dimensions. By [5], this is equivalent to asking whether all germs $f\bar{g}$ are d-regular (we refer to [5] for the definition); this is so when $n = 2$, by [18] and [2].

We notice too that holomorphic map-germs actually have the stronger strict Thom w_f-property, by [16] and [4, Theorem 4.3.2], even for functions defined on spaces with non-isolated singularities. We do not know whether or not these statements extend to map-germs $f\bar{g}$ in general. Perhaps this can be proved using D. Massey's work [13] about real analytic Milnor fibrations and a Łojasiewicz inequality.

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1. The theorem

Let U be an open neighbourhood of the origin $\underline{0}$ in $\mathbb{R}^m$ and let $X \subset U$ be a real analytic variety of dimension $n > 0$ with an isolated singularity at 0. Let $\tilde{f} : (U, \underline{0}) \to (\mathbb{R}^k, 0)$ be a real analytic map-germ which is generically a submersion, *i.e.*, its Jacobian matrix $D\tilde{f}$ has rank k on a dense open subset of U. We denote by f the restriction of $\tilde{f}$ to X. As usual, we say that $x \in X \setminus \{0\}$ is *a regular point* of f if $Df_x : T_x X \to \mathbb{R}^k$ is surjective, otherwise x is *a critical point*. A point $y \in \mathbb{R}^k$ is *a regular value* of f if there is no critical point in $f^{-1}(y)$; otherwise y is *a critical value*. We say that f has *an isolated critical value* at $0 \in \mathbb{R}^k$ if there is a neighbourhood $\mathbb{D}_\delta$ of 0 in $\mathbb{R}^k$ so that all points $y \in \mathbb{D}_\delta \setminus \{0\}$ are regular values of f.

Now let U and X be as before, and let $\tilde{f} : (U, \underline{0}) \to (\mathbb{R}^k, 0)$ be a real analytic map-germ such that $f = \tilde{f}|_X$ has an isolated critical value at $0 \in \mathbb{R}^k$. We set $V = f^{-1}(0) = \tilde{f}^{-1}(0) \cap X$. According to [9, 11], there exist Whitney stratifications of U adapted to X and V. Let $(V_\alpha)_{\alpha \in A}$ be such an stratification.

Definition 1.1. The Whitney stratification $(V_\alpha)_{\alpha \in A}$ satisfies *the Thom a_f-condition* with respect to f if for every pair of strata S_α, S_β such that $S_\alpha \subset \overline{S}_\beta$ and $S_\alpha \subset V$, one has that for every sequence of points $\{x_k\} \in S_\beta$ converging to a point x such that the sequence of tangent spaces $T_{x_k}(f^{-1}(f(x_k)) \cap S_\beta)$ has a limit T, one has that T contains the tangent space of S_α at x. We say that f has *the Thom property* if such an stratification exists.

Notice that this condition is automatically satisfied for pairs of strata contained in V, since in that case this regularity condition simply becomes Whitney's (a)-regularity.

Thom's property for complex analytic maps was proved by Hironaka in [9, Section 5 Corollary 1] for all holomorphic maps into $\mathbb{C}$ defined on arbitrary complex analytic varieties. We remark that the critical values of holomorphic maps are automatically isolated, while for real analytic maps into $\mathbb{R}^2$ this is a hypothesis we need to impose. We refer to [18] for examples of maps $f\bar{g}$ with isolated critical values, and also for examples with non-isolated critical values. Hironaka's theorem was an essential ingredient for Lê's fibration theorem in [10]. The corresponding statement was shown by Lê Dũng Tráng to be false in general for complex analytic mappings into $\mathbb{C}^2$ (see Lê's example, for instance in [21, p. 290]). Similarly, there are real analytic map-germs into $\mathbb{R}^2$ which do not have the Thom Property. Here we prove:

Theorem 1.2. *Let $(X, \underline{0})$ be a germ of an n-dimensional complex analytic set with an isolated singularity and let $f, g : (X, \underline{0}) \to (\mathbb{C}, 0)$ be holomorphic map-germs such that $f\bar{g}$ has an isolated critical value at $0 \in \mathbb{C}$. Then the real analytic germ $f\bar{g}$ has the Thom $a_{f\bar{g}}$-property.*

Proof. The proof is inspired by the proof of Pham's theorem given in [8] (Theorem 1.2.1), which concerned holomorphic germs of functions defined on complex analytic subsets of $\mathbb{C}^m$.

We first prove the theorem in the case when the germ of X at $\underline{0}$ is smooth, *i.e.*, $X \cong \mathbb{C}^n$.

Let U be an open neighbourhood of the origin $\underline{0}$ in $\mathbb{C}^n$ so that $f, g : U \to \mathbb{C}$ represent the germs f and g. We identify $\mathbb{C}^{n+1} \cong \mathbb{C}^n \times \mathbb{C}$ and denote by $(z_1, \ldots, z_{n+1})$ the coordinates in $\mathbb{C}^{n+1}$.

Let us denote by V the subset in $\mathbb{C}^n$ with equation $fg = 0$ and by $Sing(f\bar{g})$ the critical locus of $f\bar{g}$. Since $f\bar{g}$ has an isolated critical value at 0, $Sing(f\bar{g})$ is contained in V.

We need the following lemma:

Lemma 1.3. *For each integer $N \geq 1$, let $G = G(N)$ be the subset of $U \times \mathbb{C}$ defined by the equation*

$$F_N(z_1, \cdots z_{n+1}) := (f\bar{g})(z_1, \ldots, z_n) - z_{n+1}^N = 0.$$

Then the singular locus of G is contained in $Sing(f\bar{g}) \times \{0\}$.

Proof. This follows by a straightforward computation of the $2 \times 2(n + 1)$ Jacobian matrix of $f\bar{g} - z_{n+1}^N$. □

Therefore, according to [24] (just as in [8, 1.2.4] for the real analytic case), there exists a Whitney stratification σ_N of G such that $G \cap (\mathbb{C}^n \times \{0\}) = V \times \{0\}$ is a union of strata and such that $G \setminus (V \times \{0\})$ is the union of the strata having dimension $2n$. We assume further that $\underline{0}$ is itself a stratum and that U is chosen small enough so that every other stratum contains $\underline{0}$ in its closure.

Let S_N be the stratification induced by σ_N on $V \times \{0\}$. Adapting the arguments of [8], we will prove that for N sufficiently large, S_N has the Thom condition with respect to $f\bar{g}$. For this we must show that given a sequence of points in $\mathbb{C}^n \setminus V$ which converges to a point x in a stratum in S_N, such that the corresponding sequence of spaces tangent to the fibers of $f\bar{g}$ has a limit T, then T contains the tangent space at x of the corresponding stratum.

For this we will prove that whenever we have a sequence of points $(x_k) = (z_1^{(k)}, \ldots, z_{n+1}^{(k)})$ in $G \setminus (V \times \{0\})$ such that:

(1) $\lim_{k \to \infty} x_k = x \in V \times \{0\}$, and
(2) if we set $t_k = (f\bar{g})(z_1^{(k)}, \ldots, z_n^{(k)})$, we have that the sequence of $(2n-2)$-planes $T_{x_k}\big((f\bar{g})^{-1}(t_k) \times \{z_{n+1}^{(k)}\}\big)$ converges to a limit T,

then, if N is sufficiently large, the space T must contain the tangent space $T_x V_\alpha$ to the strata V_α of S_N containing x. It is clear that this will imply Theorem 1.2 since $G \subset \mathbb{C}^{n+1}$ is a union of fibers of $f\bar{g}$.

We will prove this claim by contradiction. In other words, we assume that there is a sequence (x_k) as above, such that the limit T does not contain the tangent space $T_x V_\alpha$, then we will show that when N is large enough, we necessarily come to a contradiction.

Notice that we can assume that the sequence of $2n$-planes $T_{x_k} G$ converges to a limit τ since the Grassmanian of $2n$-planes in the Euclidian space is a compact manifold.

For each k one has

$$T_{x_k}\left((f\bar{g})^{-1}(t_k) \times \{z_{n+1}^{(k)}\}\right) \subset T_{x_k} G,$$

therefore $T \subset \tau$ and the intersection $\tau \cap (\mathbb{C}^n \times \{0\})$ has real dimension at least $2n-2$. Moreover, as $T_x V_\alpha \not\subset T$, one gets $T_x V_\alpha \neq \tau \cap \mathbb{C}^n \times \{0\}$.

But, since σ_N satisfies Whitney's condition (a), one has $T_x V_\alpha \subset \tau$. This implies that in fact the intersection $\tau \cap (\mathbb{C}^n \times \{0\})$ has real dimension at least $2n-1$. We will show that this is not possible if N is sufficiently large.

According to [12], there exists an open neighbourhood of $\underline{0}$ in $\mathbb{C}^n$ and a real number θ, $0 < \theta < 1$, such that for each $z = (z_1, \ldots, z_n) \in \Omega$ one has :

$$\|(\mathrm{grad}f)(z)\| \geq |f(z)|^\theta \quad \text{and} \quad \|(\mathrm{grad}g)(z)\| \geq |g(z)|^\theta$$

The Jacobian matrix of the map $f\bar{g} - z_{n+1}^N$ with respect to the coordinates $(z_1, \bar{z}_1, z_2, \bar{z}_2, \cdots, z_{n+1}, \overline{z_{n+1}})$ in $\mathbb{R}^{2(n+1)}$ is the $2 \times 2(n+1)$ matrix given in blocks by

$$D(f\bar{g})(z_1, \bar{z}_1, \ldots, z_{n+1}, \overline{z_{n+1}}) = \begin{pmatrix} M_1 & \ldots & M_i & \ldots & M_{n+1} \end{pmatrix},$$

where for each $i = 1, \ldots, n$ the block M_i is:

$$M_i = \begin{pmatrix} \frac{\partial(\Re(f\bar{g}))}{\partial z_i} & \frac{\partial(\Re(f\bar{g}))}{\partial \bar{z}_i} \\ \frac{\partial(\Im(f\bar{g}))}{\partial z_i} & \frac{\partial(\Im(f\bar{g}))}{\partial \bar{z}_i} \end{pmatrix},$$

and

$$M_{n+1} = -\frac{N}{2} \begin{pmatrix} z_{n+1}^{N-1} & \overline{z_{n+1}}^{N-1} \\ \frac{1}{i} z_{n+1}^{N-1} & -\frac{1}{i} \overline{z_{n+1}}^{N-1} \end{pmatrix}.$$

Then an easy computation leads to the following equation for the tangent space $T_{x_k} G$ at $x_k = (z, z_{n+1}) \in G$ (we omit the k in the coordinates in order to simplify the notations) :

$$\sum_{i=1}^n \left(\frac{\partial f}{\partial z_i}(z)\bar{g}(z)v_i + \overline{\frac{\partial g}{\partial z_i}}(z)\bar{f}(z)\overline{v_i} \right) - N z_{n+1}^{N-1} v_{n+1} = 0 .$$

We consider the $2n$-vector appearing in the equation :

$$w_k(z) = \left(\frac{\partial f}{\partial z_1}(z)\bar{g}(z), \overline{\frac{\partial g}{\partial z_1}}(z), \ldots, \frac{\partial f}{\partial z_n}(z)\bar{g}(z), \overline{\frac{\partial g}{\partial z_n}}(z) \right).$$

For simplicity we omit to write that the functions below are evaluated at (z). We have:

$$\left(\frac{\|w_k\|}{N|z_{n+1}|^{N-1}} \right)^2 = \frac{|\bar{g}|^2 \sum_{i=1}^n \left|\frac{\partial f}{\partial z_i}\right|^2 + |\bar{f}|^2 \sum_{i=1}^n \left|\frac{\partial g}{\partial z_i}\right|^2}{N^2 |f\bar{g}|^{2\frac{N-1}{N}}} = \frac{|\bar{g}|^2 \|\mathrm{grad}f\|^2 + |\bar{f}|^2\|\mathrm{grad}g\|^2}{N^2 |f\bar{g}|^{2\frac{N-1}{N}}} .$$

Thus,

$$\left(\frac{\|w_k\|}{N|z_{n+1}|^{N-1}} \right)^2 = \frac{(|\bar{g}||\bar{f}|^\theta)^2 + (|\bar{f}||\bar{g}|^\theta)^2}{N^2 |f\bar{g}|^{2\frac{N-1}{N}}} \geq \frac{2}{N^2} |f\bar{g}|^{\theta - \frac{N-1}{N}} .$$

When N is sufficiently large, *i.e.*, $\theta - \frac{N-1}{N} < 0$, one has :

$$\lim_{k \to \infty} \frac{\|w_k\|}{N|z_{n+1}|^{N-1}} = +\infty \,.$$

Therefore the normalized limit of the vector $(w_k, -N(z_{n+1}^{(k)})^{N-1})$ when $k \to \infty$, is a vector contained in $\mathbb{C}^n \times \{0\}$. Then the $2n$-plane τ contains the complex line $\{\underline{0}\} \times \mathbb{C} \subset \mathbb{C}^n \times \mathbb{C}$. This contradicts the fact that $\tau \cap \mathbb{C}^n \times \{0\}$ has dimension at least $2n - 1$. Thus, if we set $t_k = (f\bar{g})(z_1^{(k)}, \ldots, z_n^{(k)})$, then every sequence of $(2n - 2)$-planes $T_{x_k}\big((f\bar{g})^{-1}(t_k) \times \{z_{n+1}^{(k)}\}\big)$ that converges to a limit T contains the tangent space $T_x V_\alpha$ to the strata V_α of S_N containing x. This completes the proof of the theorem when X is smooth at $\underline{0}$.

When $X \hookrightarrow \mathbb{C}^m$ has an isolated singularity at the origin, we take a Whitney stratification of a neighbourhood U of X in $\mathbb{C}^m$ adapted to X and to $V := (f\bar{g})^{-1}(0)$, and such that $\underline{0}$ is a stratum. We choose U small enough such that any other stratum contains $\bar{0}$ in its closure. Now we consider a sequence of points (x_k) in $X \setminus V$ converging to a point $x \in V$ and such that there is a limit T of the corresponding sequence of spaces tangent to the fibers. If $x = \underline{0}$, then there is nothing to prove since T contains the space tangent to this 0-dimensional stratum. If $x \neq \underline{0}$, then we consider a coordinate chart U_1 for X around x and argue exactly as in the previous case, when X was assumed to be smooth. $\hfill\square$

We now look at the corollaries. We know, by Bertini-Sard's theorem in [23], that there is $\varepsilon > 0$ such that all spheres in $\mathbb{R}^m$ centered at $\underline{0}$ with radius $\leq \varepsilon$ meet transversally each stratum in $\{f\bar{g} = 0\}$. Since $f\bar{g}$ has Thom's $a_{f\bar{g}}$-property, by Theorem 1.2, we get that given $\varepsilon > 0$ as above, there exists $\delta > 0$ sufficiently small with respect to ε such that all fibers $(f\bar{g})^{-1}(t)$ with $|t| \leq \delta$ are transversal to the link $\mathcal{L}_X$. As usual, following the proof of Ehresmann's fibration theorem (see for instance [14, 10, 18]), this implies that one has a locally trivial fibration $N(\varepsilon, \delta) \xrightarrow{f} \mathrm{Image}(f\bar{g}) \subset \mathbb{D}_\delta \setminus \{0\}$, where $N(\varepsilon, \delta) := [(f\bar{g})^{-1}(\mathbb{D}_\delta \setminus \{0\})] \cap \mathbb{B}_\varepsilon$ is a solid Milnor tube for $f\bar{g}$. Thus to complete the proof of Corollary 1 we must show that the image of $f\bar{g}$ covers all of $\mathbb{D}_\delta \setminus \{0\}$. This follows from the lemma below.

Lemma 1.4. *Let X, f and g be as above, so that $f\bar{g}$ is not constant and it has an isolated critical value at $0 = f\bar{g}(\underline{0})$. Then the germ $f\bar{g}$ is locally surjective at $\underline{0}$.*

Proof. If either f or g is constant, the statement in this lemma is a well-known property of holomorphic mappings. So we assume none of these maps is constant, neither is constant the map $f\bar{g}$. We may further assume that f, g have no common factor, for otherwise we may divide both map-germs by that common factor and this will not change the image of the map $f\bar{g}$. We claim that since f and g are both holomorphic, we have that the map-germ

$$(f, g) : \mathbb{C}^n \to \mathbb{C} \times \mathbb{C}$$

is locally surjective for all $n \geq 2$. That is, the image of every neighbourhood of $\underline{0} \in \mathbb{C}^n$ covers a neighbourhood of $(0, 0) \in \mathbb{C} \times \mathbb{C}$. In fact, for $n = 2$ the map germ (f, g) is a finite morphism, which is a finite covering map with ramification locus the discriminant curve; so it is locally surjective. When $n \geq 3$ we may consider a generic complex 2-plane $\mathcal{P}$ in $\mathbb{C}^n$ which is transversal to the fibers of (f, g) and apply the above arguments. Hence $(f, \bar{g})$ is locally surjective, and so is $f\bar{g}$. $\hfill\square$

There are in [6] examples of analytic map-germs $(\mathbb{R}^n, \underline{0}) \xrightarrow{h} (\mathbb{R}^2, 0)$ with an isolated critical value at 0 which are not surjective. The image of h misses a neighbourhood of an arc converging to 0.

The proof of Corollary 2 is just as that of Theorem 1.3 in [18], replacing the Milnor tube $[(f\bar{g})^{-1}(\partial\mathbb{D}_\delta)] \cap \mathbb{B}_\varepsilon$ by the solid Milnor tube $[(f\bar{g})^{-1}(\mathbb{D}_\delta \setminus \{0\})] \cap \mathbb{B}_\varepsilon$, so we leave the details to the reader. (Compare with the first part of the proof of Theorem 1 in [5]).

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