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THE JUMP OF THE MILNOR NUMBER OF QUASIHOMOGENEOUS SINGULARITIES FOR LINEAR DEFORMATIONS

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Abstract. The jump of the Milnor number of an isolated singularity f_0 is the minimal non-zero difference between the Milnor numbers of f_0 and one of its deformations f_s. We determinate the jump of quasihomogeneous singularities in the class of linear deformations.

1. Introduction

One of the important problems in singularity theory is the adjacency problem: when a singularity (or a class of singularities) can be deformed to another one. In other words whether a "type" of a singularity may be changed to another "type" by an arbitrarily small deformation. A simpler problem is to find how some invariants of singularities may change by an arbitrarily small deformation. In the article we study such a change of the Milnor number for isolated plane curve singularities. We are interested in finding the smallest positive change under some class of deformations – we will call it the jump of the Milnor number of a given singularity.

We start from basic definitions. They are given in n-dimensional case, but further we will focus on only the **plane curve singularities**. Let $f_0 : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be an **isolated singularity** or in short **singularity**. We define a **deformation of the singularity** f_0 as a germ of a holomorphic function $f : (\mathbb{C} \times \mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ such that

(1) $f(0, z) = f_0(z)$,
(2) $f(s, 0) = 0$.

The deformation $f(s, z)$ of the singularity f_0 will be treated as a family (f_s) of function germs, taking $f_s(z) := f(s, z)$. For the sufficiently small s we can define the **Milnor number of f_s at** 0 by

$$\mu_s := \mu(f_s) = \dim_{\mathbb{C}} \mathcal{O}_n/(\nabla f_s),$$

where $\mathcal{O}_n$ is the ring of holomorphic function germs at 0, and (∇f_s) is the ideal in $\mathcal{O}_n$ generated by $\frac{\partial f_s}{\partial z_1}, \dots, \frac{\partial f_s}{\partial z_n}$.

The Milnor number is upper semi-continuous in the Zariski topology in families of singularities ([GLS06], Theorem 2.6 I and Proposition 2.57 II), so there exists an open neighbourhood $0 \in S$ such that

(1) $\mu_s =$ const. for $s \in S \setminus \{0\}$,
(2) $\mu_0 \geq \mu_s$ for $s \in S$.

The constant difference $\mu_0 - \mu_s$ (for $s \in S$) will be called **the jump of the deformation** (f_s) and denoted by $\lambda((f_s))$. **The jump of the Milnor number of the singularity** f_0 is the smallest non-zero value among all the jumps of deformations of the singularity f_0. It will be denoted by $\lambda(f_0)$.

Many authors have considered what values the jump of the Milnor number can take. One of the first general result was obtained by Sabir Gusein-Zade ([GZ93]). In his work he proved that there exist singularities f_0 for which $\lambda(f_0) > 1$ and that for any irreducible plane curve singularity

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f_0 we have $\lambda(f_0) = 1$. Later, S. Brzostowski, T. Krasiński and J. Walewska in [BKW21] proved that for the particular reducible singularities $f_0^n(x,y) = x^n + y^n$, $n \geq 2$, we have $\lambda(f_0) = \left[\frac{n}{2}\right]$. Determining the jump of a singularity is difficult because it is not a topological invariant ([BK14], [dPW95] Section 7.3).

A simpler problem is to determinate the jump when we limit ourselves to specific classes of deformations. For non-degenerate deformations (it means each element of the family (f_s) is a non-degenerate singularity in the Kouchnirenko sense [Kou76]) the jump (denoted by $\lambda^{nd}(f_0)$) was considered in [Bod07], [Wal13], [BKW21], [KW19].

In this paper we consider the jump of the Milnor number for **linear deformations** of f_0 i.e. deformations of the form $f_s = f_0 + sg$, where g is a holomorphic function in the neighbourhood of 0 such that $g(0) = 0$. We will denote the jump of f_0 for this class of deformations by $\lambda^{lin}(f_0)$. The main result is a formula for the jump of the Milnor number $\lambda^{lin}(f_0)$ for quasihomogeneous plane curve singularities. The simpler problem of homogeneous singularities was treated in [Zak17].

In generic case (the general precise result is given in Theorem 5.1) the formula is as follows

Theorem. *If $f_0(x,y) = a_{p,0}x^p + \ldots + a_{0,q}y^q$ is a quasihomogeneous isolated singularity and $3 \leq p \leq q$ then*

$$\lambda^{lin}(f_0) = \begin{cases} p-2, & \text{if } p = q \\ p-1, & \text{if } p \neq q \text{ and } p|q \\ GCD(p,q), & \text{if } p \neq q \text{ and } p\nmid q \end{cases}. \tag{1}$$

The first case concerns the homogeneous singularity. We illustrate the result with two examples.

Example 1.1. *For the homogeneous singularities $f_0(x,y) = x^n + y^n$, where $n \geq 3$, the various types of jumps are different:*

$$\lambda(f_0) = \left[\frac{n}{2}\right], \lambda^{lin}(f_0) = n-2, \lambda^{nd}(f_0) = n-1.$$

If we put for example $n = 5$ then:

$$\lambda(f_0) = 2, \lambda^{lin}(f_0) = 3, \lambda^{nd}(f_0) = 4.$$

Example 1.2. *For the quasihomogeneous singularity $f_0(x,y)$, that is not homogeneous, we have $\lambda^{nd}(f_0) = \lambda^{lin}(f_0)$ because formulas (1) and in Theorem 10 in* [Wal13] *are the same. However, constructions given in* [Wal13] *for non-degenerate case, and in Theorem 4.1 for linear case may give different deformations realizing this jump. For example, for the quasihomogeneous singularity $f_0(x,y) = x^6 + y^9$ we have $\lambda^{nd}(f_0) = \lambda^{lin}(f_0) = 3$ and*

(1) $f_s(x,y) = x^6 + y^9 + sx^5y$ *– the non-degenerate deformation,*
(2) $f_s(x,y) = x^6 + y^9 + sxy(y^3 + x^2)^2$ *– the linear deformation.*

To get the main result the **Enriques diagrams** will be used. To any singularity we assign a weighted Enriques diagram (D, ν) which represents the whole resolution process of this singularity ([CA00] Chapter 3.9). It is a tree with two types of edges. M. Alberich-Carramiñana and J. Roé ([ACR05] Theorem 1.3, Remark 1.4) gave a necessary and sufficient condition for two Enriques diagrams of singularities to be linear adjacent. It means that one singularity is a linear deformation of another. They used a wider class of Enriques diagrams, so-called abstract Enriques diagrams, which are described in Section 2.

2. Abstract Enriques diagrams

Information about abstract Enriques diagrams can be found in [ACR05] and [KP99]. Moreover in my previous paper [Zak17], in which I gave the estimation of $\lambda^{lin}(f_0)$ for homogeneous singularities, abstract Enriques diagrams are described in more details with examples. The formula for λ^{lin} for homogeneous singularities is in my PhD thesis [Zak19] (in Polish).

Definition 2.1 ([ACR05]). *An* ***abstract Enriques diagram*** *(in short an Enriques diagram) is a rooted tree D with binary relation between vertices, called proximity, which satisfies:*

(1) *The root is proximate to no vertex.*
(2) *Every vertex that is not the root is proximate to its immediate predecessor.*
(3) *No vertex is proximate to more than two vertices.*
(4) *If a vertex Q is proximate to two vertices, then one of them is the immediate predecessor of Q and it is proximate to the other.*
(5) *Given two vertices P, Q with Q proximate to P, there is at most one vertex proximate to both of them.*

The fact that Q is proximate to P in D we will denote by $Q \xrightarrow{D} P$ or in short $Q \to P$. The vertices which are proximate to two points are called **satellite**, the other vertices are called **free**. The vertex is **a leaf** if it has no successor. To show graphically the proximity relation, Enriques diagrams are drawn according to the following rules:

(1) If Q is a free successor of P, then the edge going from P to Q is curved (not necessarily tangent).
(2) The sequence of edges connecting a maximal succession of vertices proximate to the same vertex P are shaped into a line segment, orthogonal to the edge joining P to the first vertex of the sequence (if it is also straight).

The example of an abstract Enriques diagram is shown in Figure 1.

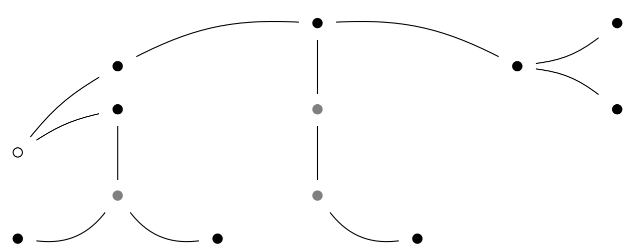

FIGURE 1. The abstract Enriques diagram. Satellite vertices are marked in gray. The root is white.

We will now introduce few basic notations that are needed in later chapters. First, we define weights on vertices of an abstract Enriques diagrams which correspond, in particular case of plane curve singularities, to the orders of the proper transforms of the function describing the singularity.

A **weight function** is any function $\nu : D \to \mathbb{Z}$. A pair (D, ν), where D is an abstract Enriques diagram and ν a weight function, is called a **weighted Enriques diagram**. A **consistent Enriques diagram** is a weighted Enriques diagram such that for all $P \in D$

$$\nu(P) \geq \sum_{Q \to P} \nu(Q). \tag{2}$$

A **complete Enriques diagram** is a weighted Enriques diagram such that for all not-leaf $P \in D$ the equality in (2) holds and for all leaves $P \in D$ it is a free vertex with weight 1 not proximate to another free vertex with weight 1. To the weight function ν of a weighted diagram D we associate a **system of values** on D, which is another map $\mathrm{ord}_\nu : D \to \mathbb{Z}$, defined recursively as

$$\mathrm{ord}_\nu(P) := \begin{cases} \nu(P), & \text{if } P \text{ is the root,} \\ \nu(P) + \sum\limits_{P \to Q} \mathrm{ord}_\nu(Q), & \text{otherwise.} \end{cases}$$

For any consistent (D, ν) we define **the Milnor number of** (D, ν) by

$$\mu((D, \nu)) := \sum_{P \in D} \nu(P)(\nu(P) - 1) + 1 - r_D,$$

where $r_D := \sum_{P \in D} r_D(P)$, $r_D(P) := \left(\nu(P) - \sum_{Q \to P} \nu(Q)\right)$ for every $P \in D$.

A **subdiagram** of an abstract Enriques diagram D is a subtree $D_0 \subset D$ with the same proximity relation such that if $Q \in D_0$ then its predecessor belongs to D_0.

In the class of consistent weighted Enriques diagrams, we introduce equivalence relation. We say that consistent weighted diagrams (D, ν) and (D', ν') are **equivalent** if they differ at most in free vertices of weight 1. The equivalence class of (D, ν) is denoted by $[(D, \nu)]$ and called the **type** of (D, ν). Of course, the Milnor number is invariant in the class $[(D, \nu)]$.

A **minimal Enriques diagram** is a consistent Enriques diagram (D, ν) with:

(1) no free vertices of weight 0,
(2) no free vertices of weight 1 except for these such $P \in D$ for which there exists a satellite vertex $Q \in D$ satisfying $Q \to P$.

It is easy to see ([Zak17], Theorem 2.12) that

Theorem 2.2. *Let (D, ν) be a consistent weighted diagram. There exists exactly one minimal diagram which belongs to $[(D, \nu)]$.*

The theory of Enriques diagrams has its roots in the theory of plane curve singularities. The embedded resolution of a plane curve singularity using blow-ups can be explicitly presented as a complete Enriques diagram. A precise description can be found in [CA00] Chapter 3.8 and Chapter 3.9. Two plane curve singularities are embedded topologically equivalent if and only if their Enriques diagrams are isomorphic (as graphs). For the Enriques diagram of a plane curve singularity, the weight function represents the orders of the consecutive proper transforms while the system of values – the orders of the total transforms of the function defining the singularity. Also the Milnor number of the Enriques diagram coincides with the Milnor number of the corresponding singularity. We need only the next fact which easily follows from these results.

Theorem 2.3 ([CA00] Theorem 3.8.6)**.** *There exists a bijection between minimal Enriques diagrams and topological types of singularities.*

In the paper [ACR05], M. Alberich-Carramiñana and J. Roé gave a necessary and sufficient condition for two Enriques diagrams of singularities to be linear adjacent. This is the key result we will use in the sequel. First we give definitions.

Definition 2.4. *Let (D, ν) and (D', ν') be weighted Enriques diagrams, with (D', ν') consistent. We will write $(D', \nu') \geq (D, \nu)$ when there exist isomorphic subdiagrams $D_0 \subset D$, $D'_0 \subset D'$ with an isomorphism (that preserves proximity relations)*

$$i : D_0 \to D'_0$$

such that the new weight function $\kappa_{\nu'} : D \to \mathbb{Z}$ for D, defined by

$$\kappa_{\nu'}(P) := \begin{cases} \nu'(i(P)), & P \in D_0 \\ 0, & P \notin D_0 \end{cases}$$

satisfies

$$\mathrm{ord}_\nu(P) \leq \mathrm{ord}_{\kappa_{\nu'}}(P)$$

for any $P \in D$.

Definition 2.5. *Let $[(D,\nu)]$ and $[(\widetilde{D},\widetilde{\nu})]$ be types of Enriques diagrams. $[(\widetilde{D},\widetilde{\nu})]$ is **linear adjacent** to $[(D,\nu)]$ if there exists a consistent Enriques diagram $(D',\nu') \in [(\widetilde{D},\widetilde{\nu})]$ such that $(D',\nu') \geq (D_{min},\nu_{min})$, where (D_{min},ν_{min}) is the minimal diagram of type $[(D,\nu)]$.*

Theorem 2.6 ([ACR05] Theorem 1.3 and Remark 1.4)**.** *Let $[(D,\nu)]$ and $[(\widetilde{D},\widetilde{\nu})]$ be types of consistent Enriques diagrams. The following conditions are equivalent:*

(1) *$[(\widetilde{D},\widetilde{\nu})]$ is linear adjacent to $[(D,\nu)]$.*
(2) *For every singularity f_0 whose Enriques diagram belongs to $[(\widetilde{D},\widetilde{\nu})]$, there exists a linear deformation (f_s) of f_0 such that the Enriques diagram of a generic element f_s belongs to $[(D,\nu)]$.*
(3) *There exists a singularity f_0 whose Enriques diagram belongs to $[(\widetilde{D},\widetilde{\nu})]$ and a linear deformation (f_s) of f_0 such that the Enriques diagram of a generic element f_s belongs to $[(D,\nu)]$.*

This theorem was also formulated using prime divisors by J. Fernández de Bobadilla, M. Pe Pereira and P. Popescu-Pampu in Theorem 3.25 ([dBPPP17]).

Theorems 2.3 and 2.6 imply the following corollary:

Corollary 2.7. *$\lambda^{lin}(f_0)$ is a topological invariant.*

3. Enriques diagrams of quasihomogeneous singularities

Let $f_0(x,y)=\sum_{i,j\in\mathbb{N}} a_{i,j}x^iy^j$ be an isolated singularity. It is known that f_0 is reduced in the ring $\mathbb{C}\{x,y\}$ of convergent series. The singularity f_0 is called **quasihomogeneous**, if there exist $w_x, w_y \in \mathbb{N}$ and a number $W\in\mathbb{N}$ such that, for every $(i,j)\in \operatorname{supp}(f_0)$, it holds $iw_x + jw_y = W$, where $\operatorname{supp}(f_0) := \{(i,j)\in\mathbb{N} : a_{i,j}\neq 0\}$. Without loss of generality, f_0 can be expressed as

$$f_0(x,y) = x^ky^l(x^p + \ldots + \gamma_{i,j}x^iy^j + \ldots + \gamma_{0,q}y^q), \quad k,l\in\{0,1\}, p\leq q, k+l+p\geq 2, \tag{3}$$

and for every term $\gamma_{i,j}x^iy^j$, $\gamma_{i,j}\neq 0$, the equality $(i+k)w_x + (j+l)w_y = W$ holds.

Then after simple rescaling the variables $x\mapsto x', y\mapsto \frac{y'}{\sqrt[q+l]{\gamma_{0,q}}}$, that does not change the Milnor number of f_0, we may assume f_0 has the form:

$$f_0(x,y) = x^ky^l(x^p + \ldots + \gamma_{i,j}x^iy^j + \ldots + y^q), \quad k,l\in\{0,1\}, p\leq q, k+l+p\geq 2, \tag{4}$$

In the case $p=q$ we get a homogeneous singularity.

Since f_0 is reduced and quasihomogeneous in two variables, we can represent f_0 as a product of irreducible factors

$$f_0(x,y) = x^ky^l\prod_{i=1}^{\tilde{d}}(x^r+\alpha_iy^s), \quad \alpha_i\neq 0, \alpha_i\neq\alpha_j \text{ for } i\neq j, \tag{5}$$

where $\tilde{d} = \operatorname{GCD}(p,q)$, $r=\frac{p}{\tilde{d}}$, $s=\frac{q}{\tilde{d}}$, $\operatorname{GCD}(r,s)=1$. By this form of quasihomogeneous singularity and by the resolution process of singularities (more details in [CA00] Chapter 3.7) the Enriques diagram of any quasihomogeneous singularity can be easily described.

In fact, let assume first that $k=l=0$. If $r=s$ then singularity (5) is homogeneous and hence $r=s=1$ and $p=q=\tilde{d}$. So $f_0(x,y)=\prod_{i=1}^{\tilde{d}}(x+\alpha_iy)$ for some $\alpha_i\neq 0, \alpha_i\neq\alpha_j$ for $i\neq j$. Then one blowing up resolves the singularity and the Enriques diagram of f_0 is shown in Figure 2. Now assume $r<s$ (the case $s<r$ is analogous). So $f_0(x,y)=\prod_{i=1}^{\tilde{d}}(x^r+\alpha_iy^s)$, $r<s$, $\operatorname{GCD}(r,s)=1$. Hence the singularity f_0 has the unique tangent line $\{x=0\}$. Then after one blowing up the proper transform of this singularity is described in the coordinates $(x',y')=(\frac{x}{y},y)$ by the polynomial $\prod_{i=1}^{\tilde{d}}(x'^r+\alpha_iy'^{s-r})$. This singularity has also the unique tangent line (either

$\{x = 0\}$ if $r < s - r$ or $\{y = 0\}$ if $r > s - r$) except the case $r = 1$ and $s = 2$. In the exceptional case we get a homogeneous singularity. In the first case (only one tangent line) after finite number of blowing ups we also get a homogenous singularity. In both cases we always get a homogenous singularity for which the next blowing up gives its resolution. According to the above description we may describe the Enriques diagram (D, ν) of f_0 (see Figure 3). The first edges (from R_1 to some R_m) are curved and next ones (from R_m to R_t) are straight. The diagram (D, ν) has $\tilde{d}$ leaves. Moreover this is a complete Enriques diagram. If $p|q$ then $t = \frac{q}{p}$. In particular if f_0 is homogeneous then $t = 1$.

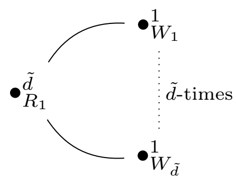

FIGURE 2. The Enriques diagram of a homogeneous singularity of order $\tilde{d}$.

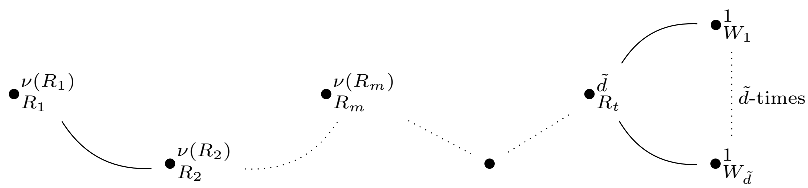

FIGURE 3. The Enriques diagram of a quasihomogeneous singularity f_0 for $k = l = 0$.

If $k = 1$ or $l = 1$ then we proceed analogously as above with small modification. We have to add one or two leaves to the Enriques diagram in Figure 3 to appropriate vertices. If $l = 1$ i.e. there is the factor y in the factorization (5) of f_0, we add a leaf T_1 with weight 1 to the root R_1 (Figure 4(a)). If $k = 1$ i.e. there is the factor x in the factorization (5) of f_0, we add such a leaf T_2 to the last free vertex among $R_1, \ldots, R_t$ i.e. to R_m in Figure 3. Two possible cases $R_m \neq R_t$ and $R_m = R_t$ are presented in Figure 4(b) and 4(c), respectively.

For $t, d \in \mathbb{N}$ we define the set H_d^t as the set of the abstract Enriques diagrams (D, ν) satisfying conditions:

(1) (D, ν) is a minimal diagram,
(2) the elements of D is a sequence $\{R_1, \ldots, R_t\}$ such that R_i is a successor of R_{i-1} for $i \in \{2, \ldots, t\}$ (a bamboo from R_1 to R_t),
(3) $\nu(R_t) = d$.

From the above construction of the Enriques diagrams of a quasihomogeneous singularity (5) we see that its minimal diagram belongs to some H_d^t. We denote the subset of H_d^t corresponding to quasihomogeneous singularities by Q_d^t. This means for every diagram (D, ν) from Q_d^t, $t > 1$ there exists a singularity (4) with the same Enriques diagram as (D, ν) such that either $d = GCD(p, q) = \tilde{d}$ (if p does not divide q) or $d = GCD(p, q) + k = \tilde{d} + k$ (if p divides q). For $t = 1$ the set Q_d^t represents homogeneous singularities with $d = GCD(p, q) + k + l = \tilde{d} + k + l$. For $d = 1$ and $p > 1$, Q_d^t represents irreducible curves with one characteristic exponent $\langle p, q \rangle$, along with the additional factor of the transversal line y (if $l = 1$) or the maximal contact line x

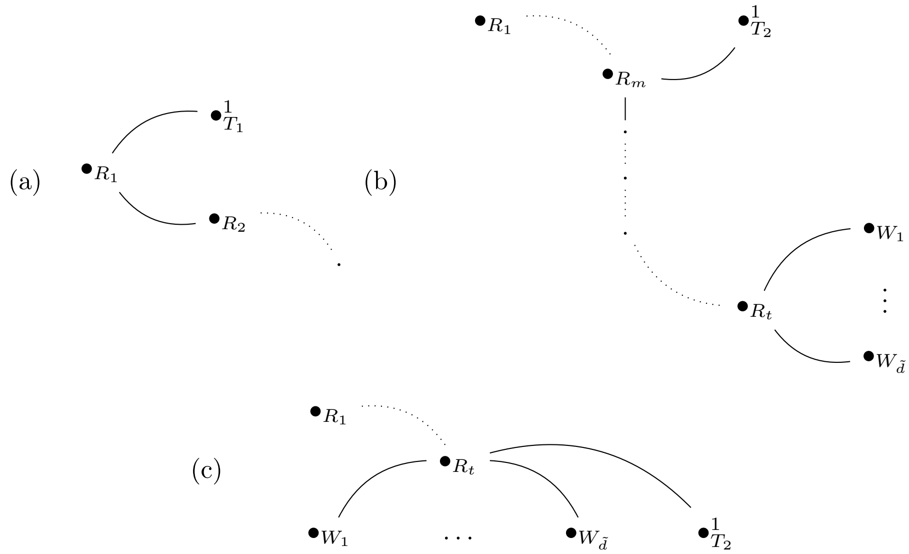

FIGURE 4. The Enriques diagrams of quasihomogeneous singularities. In the figure (a) $l = 1$, while in (b) and (c) $k = 1$. Case (c) holds if $p|q$.

(if $k = 1$); for $d = 1$ and $p = 1$, Q_d^t represents smooth branches along with the additional factor of the transversal line y (if $l = 1$).

It is easy to show the abstract Enriques diagrams which belong to Q_d^t have the following properties.

Theorem 3.1. *If a weighted Enriques diagram (D, ν) belongs to Q_d^t $(t \neq 1)$ then*

(1) $\nu(R_1) \leq \sum_{R_i \to R_1} \nu(R_i) + 1$,

(2) *if R_k is the first satellite vertex for some $k \in 3, \dots, t$ then* $\nu(R_{k-1}) \leq \sum_{R_i \to R_{k-1}} \nu(R_i) + 1$,

(3) *for any $k = 3, \dots, t$ such that R_k is not the first satellite vertex, we have*

$$\nu(R_{k-1}) = \sum_{R_i \to R_{k-1}} \nu(R_i).$$

The subset Q_d^t is a proper subset of H_d^t, for example the minimal Enriques diagram of the singularity $f_0(x, y) = (x^2 - y^2)(x^6 - y^9)$ belongs to $H_d^t \setminus Q_d^t$.

For any $(D, \nu) \in H_d^t$ we define w_D as the number of vertices which R_t is proximate to. If (D, ν) is the Enriques diagram of singularity (5), then obviously

$$w_D = \begin{cases} 0, & \text{if } p = q \\ 1, & \text{if } p \neq q \text{ and } p|q \\ 2, & \text{if } p \neq q \text{ and } p \nmid q \end{cases} . \tag{6}$$

In fact, when $p = q$ then $R_t = R_1$ and hence $w_D = 0$. If $q = mp$, $m > 1$, then after $m - 1$ blowing ups we get R_t and the all vertices in this process are free and hence $w_D = 1$. If $p \nmid q$ we get the first vertices in blowing ups are free (up to $[\frac{q}{p}] + 1$) and next are satellite, so $w_D = 2$.

4. Estimation of the Milnor number for abstract Enriques diagrams

In this section we will estimate the Milnor number of these diagrams to which diagrams from Q_d^t are linear adjacent. Precisely, for any $(D,\nu) \in Q_d^t$ we will find the maximum in the set

$$\{\mu((E,\lambda)) : [(D,\nu)] \text{ is linear adjacent to } [(E,\lambda)], (E,\lambda) \notin [(D,\nu)]\}, \tag{7}$$

where $d,t \in \mathbb{N}$ and $dt > 1$. If $dt = 1$ then $H_1^1 = Q_1^1$ represents a smooth curve (by our definition it is not a singularity). We will show that this maximum equals

$$\begin{array}{ll} \mu((D,\nu)) - 1, & \text{if } d = 1 \\ \mu((D,\nu)) - 1, & \text{if } d = 2, w_D = 0 \\ \mu((D,\nu)) - w_D, & \text{if } d = 2, w_D \neq 0 \\ \mu((D,\nu)) - (d - 2 + w_D), & \text{if } d \geq 3 \end{array}.$$

We will start from the easier part i.e. we will find the Enriques diagrams which realize these values. This theorem will be proved even for any $(D,\nu) \in H_d^t$ (not only for $(D,\nu) \in Q_d^t$).

Theorem 4.1. *Let $d,t \in \mathbb{N}$, $dt > 1$ and (D,ν) be an Enriques diagram from H_d^t. There exists a minimal Enriques diagram $(E_D,\lambda_D) \notin [(D,\nu)]$ such that $[(D,\nu)]$ is linear adjacent to $[(E_D,\lambda_D)]$ and*

$$\mu\left((E_D,\lambda_D)\right) = \begin{cases} \mu((D,\nu)) - 1, & \text{if } d = 1 \\ \mu((D,\nu)) - 1, & \text{if } d = 2, w_D = 0 \\ \mu((D,\nu)) - w_D, & \text{if } d = 2, w_D \neq 0 \\ \mu((D,\nu)) - (d - 2 + w_D), & \text{if } d \geq 3 \end{cases}. \tag{8}$$

Proof. The minimal diagram (D,ν) is shown in Figure 5. We will define the diagram (E_D,λ_D)

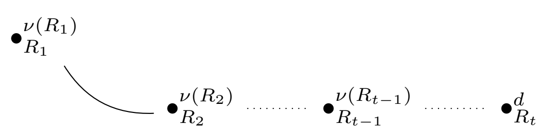

Figure 5. The minimal Enriques diagram (D,ν).

by a modification of (D,ν). If $d = 1$ we remove only the last vertex from (D,ν) (Figure 6(a)) and this will be (E_D,λ_D). If $d = 2$ and R_t is the root, then E_D consists of only one vertex with weight 1. If $d = 2$ and R_t is not the root we change the weight of the last vertex to 1 and add one additional vertex W with weight 1, so that $W \to R_t, R_{t-1}$ (Figure 6(b)) and this is (E_D,λ_D). If $d \geq 3$ we change the weight of the last vertex to $d-1$ and add new vertices $U, W_1, \ldots, W_{d-3}$ (if $d = 3$ there are no W_i vertices), all proximate to R_t. The weights of new vertices are: $\lambda_D(U) = 2$, $\lambda_D(W_i) = 1$ (for $i = 1,\ldots,d-3$). The proximity relation of the new vertices is (Figure 6(c))

$$\begin{aligned} &W_{d-3} \to W_{d-4}, R_t \\ &\ldots \\ &W_2 \to W_1, R_t \\ &W_1 \to U, R_t \\ &U \to R_t. \end{aligned}$$

It is easy to check that each (E_D,λ_D) is a minimal (and hence consistent) diagram and that $(E_D,\lambda_D) \notin [(D,\nu)]$. Moreover, there exists a consistent Enriques diagram $(D',\nu') \notin [(D,\nu)]$ such that $(D',\nu') \geq (E_D,\lambda_D)$.

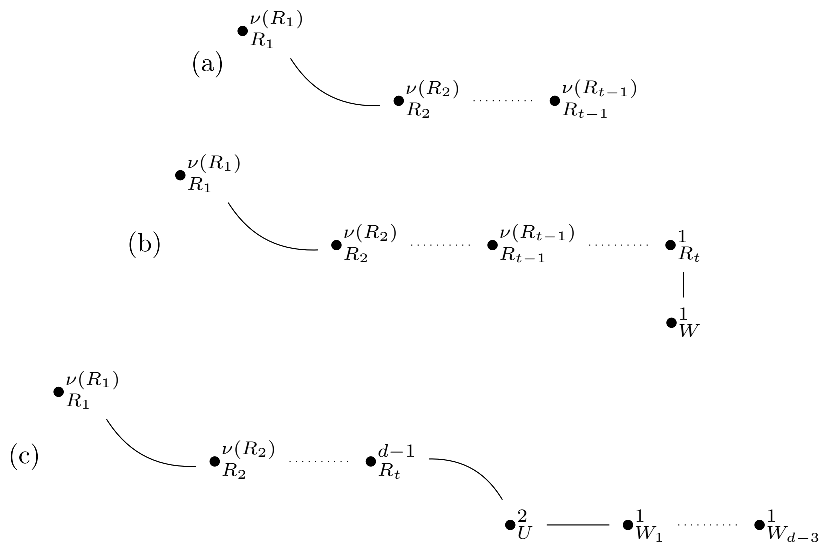

FIGURE 6. The Enriques diagram (E_D, λ_D).

Indeed, as above we should consider three cases: $d = 1$, $d = 2$, $d \geq 3$. We give details in the case $d \geq 3$ as the remaining cases are similar.

Let $(D', \nu') \in [(D, \nu)]$ be a consistent Enriques diagram, such that D' has one additional free vertex U (Figure 7), then $D' \subset E_D$. Now, we have to show that for every $P \in D'$, we have $\text{ord}_{\lambda_D}(P) \leq \text{ord}_{\kappa_{\nu'}}(P)$, where $\kappa_{\nu'}(P) = 0$ for $P \notin D'$.

- If $P \in D \setminus \{R_t\}$ then $\text{ord}_{\lambda_D}(P) = \text{ord}_{\kappa_{\nu'}}(P)$.
- For $P = R_t$, we have

$$\text{ord}_{\lambda_D}(R_t) = \sum_{R_t \to P} \text{ord}_{\lambda_D}(P) + \lambda_D(R_t) = \text{ord}_{\kappa_{\nu'}}(P) + \nu(P) - 1 =$$

$$\text{ord}_{\kappa_{\nu'}}(P) + \kappa_{\nu'}(P) - 1 = \text{ord}_{\kappa_{\nu'}}(R_t) - 1 \leq \text{ord}_{\kappa_{\nu'}}(R_t).$$

- For $P = U$,

$$\text{ord}_{\lambda_D}(U) = \text{ord}_{\lambda_D}(R_t) + \lambda_D(U) = \text{ord}_{\kappa_{\nu'}}(R_t) - 1 + 2 =$$

$$\text{ord}_{\kappa_{\nu'}}(R_t) + 1 = \text{ord}_{\kappa_{\nu'}}(R_t) + \kappa_{\nu'}(U) = \text{ord}_{\kappa_{\nu'}}(U).$$

- For any $P = W_i \to V, R_t$ $(i = 1, \ldots, d-3)$, where $V = W_{i-1}$ or $V = U$,

$$\text{ord}_{\lambda_D}(W_i) = \text{ord}_{\lambda_D}(R_t) + \text{ord}_{\lambda_D}(V) + \lambda_D(W_i) = \text{ord}_{\kappa_{\nu'}}(R_t) - 1 + \text{ord}_{\kappa_{\nu'}}(V) + 1 =$$

$$\text{ord}_{\kappa_{\nu'}}(R_t) + \text{ord}_{\kappa_{\nu'}}(V) + 0 = \text{ord}_{\kappa_{\nu'}}(R_t) + \text{ord}_{\kappa_{\nu'}}(V) + \kappa_{\nu'}(W_i) = \text{ord}_{\kappa_{\nu'}}(W_i).$$

Thus $[(D, \nu)]$ is linear adjacent to $[(E_D, \lambda_D)]$. Now we may compute the Milnor number of (E_D, λ_D). It is easy to notice that

$$r_{E_D} = \begin{cases} r_D + 1, & \text{if } d = 1 \\ r_D - 1, & \text{if } d = 2, w_D = 0 \\ r_D - 2 + w_D, & \text{if } d = 2, w_D \neq 0 \\ r_D - d + 2 + w_D, & \text{if } d \geq 3 \end{cases}$$

and then after simply calculation we get (8). □

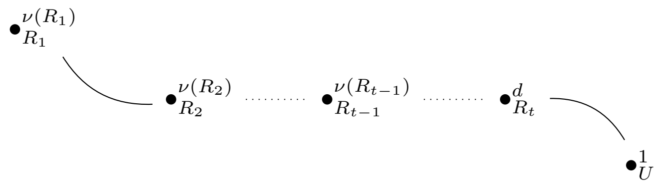

FIGURE 7. The Enriques diagram (D', ν').

To show that the diagram from Theorem 4.1 realizes the maximum in (7), it is enough to prove that for every $(D, \nu) \in Q_d^t$ all diagrams $(\tilde{D}, \tilde{\nu})$ such that $[(D, \nu)]$ is linear adjacent to $[(\tilde{D}, \tilde{\nu})]$ have not greater Milnor numbers than the diagram (E_D, λ_D) constructed for (D, ν) in Theorem 4.1. Of course, we may consider only $(\tilde{D}, \tilde{\nu})$ which have the type different from (D, ν). We do this in a series of lemmas in which we consecutively assume:

(1) Case - there is no subdiagram of $\tilde{D}$ isomorphic (as rooted tree with preserving shapes of edges but not weights) to D (Lemma 4.2);
(2) Case - there is a subdiagram of $\tilde{D}$ isomorphic to D,
   (a) Subcase - $d > 2$,
      (i) The inequality
$$\sum_{P \text{ successor of } i^{-1}(R_t)} \min(2, \tilde{\nu}(P)) + z \leq \nu(R_t) - 1,$$
where z is the number of vertices proximate to $i^{-1}(R_t)$ that are not its successors, holds (Lemma 4.3);
      (ii) The opposite inequality
$$\sum_{P \text{ successor of } i^{-1}(R_t)} \min(2, \tilde{\nu}(P)) + z > \nu(R_t) - 1,$$
where z is the number of vertices proximate to $i^{-1}(R_t)$ that are not its successors, holds (Lemma 4.4);
   (b) Subcase - $d = 2$ (Lemma 4.5);
   (c) Subcase - $d = 1$ (Lemma 4.6).

From the construction of E_D, we have that $E_D = D \setminus \{R_t\}$ for $d = 1$ and $D \subset E_D$ for $d \geq 2$. Then, we can consider both weights ν and λ_D on E_D (for $d = 1$) and on D (for $d \geq 2$).

We start with the case (1) that there is no subdiagram of $\tilde{D}$ isomorphic to D.

Lemma 4.2. *Let $d, t \in \mathbb{N}$, $(D, \nu) \in Q_d^t$ and let $(\tilde{D}, \tilde{\nu})$ be an arbitrary Enriques diagram such $[(D, \nu)]$ is linear adjacent to $[(\tilde{D}, \tilde{\nu})]$. If there is no subdiagram of $\tilde{D}$ isomorphic to D, then*
$$\mu(\tilde{D}, \tilde{\nu})) \leq \begin{cases} \mu((D, \nu)) - 1, & \text{if } d = 1 \\ \mu((D, \nu)) - 1, & \text{if } d = 2, w_D = 0 \\ \mu((D, \nu)) - w_D, & \text{if } d = 2, w_D \neq 0 \\ \mu((D, \nu)) - (d - 2 + w_D), & \text{if } d \geq 3 \end{cases}.$$

Proof. Firstly, without loss of generality we may assume that $(\tilde{D}, \tilde{\nu})$ is a minimal Enriques diagram. Now, we will construct another diagram (E, λ) such that $[(E, \lambda)]$ is linear adjacent to $[(\tilde{D}, \tilde{\nu})]$ and $\mu((E, \lambda)) = \mu((E_D, \lambda_D))$. Since $[(D, \nu)]$ is linear adjacent to $[(\tilde{D}, \tilde{\nu})]$ there exist a consistent Enriques diagram $(D', \nu') \in [(D, \nu)]$ such that $(D', \nu') \geq (\tilde{D}, \tilde{\nu})$, two subdiagrams $\tilde{D}_0 \subset \tilde{D}$, $D'_0 \subset D'$ and an isomorphism $i : D'_0 \to \tilde{D}_0$. Since there is no subdiagram of $\tilde{D}$ isomorphic to D, we have $R_t \notin D'_0$. Let (E_D, λ_D) be the diagram from Theorem 4.1 constructed for (D, ν). We get a diagram $(E, \lambda) \in [(E_D, \lambda_D)]$ as a modification of E_D (analogous to the construction of D' from D). Then D'_0 is also a subdiagram of E, because E and D' "differ" after R_{t-1}. For every $P \in D'_0$ we have $\nu'(P) = \lambda(P)$. Therefore, consequently for every $P \in \tilde{D}_0$ it holds $\kappa_\lambda(P) = \kappa_{\nu'}(P)$. This implies that $[(E,\lambda)]$ is linear adjacent to $[(\tilde{D},\tilde{\nu})]$, so for every singularity f_0 whose Enriques diagram belong to $[(E,\lambda)]$, there exists a linear deformation (f_s) of f_0 such that the Enriques diagram of a generic element f_s belongs to $[(\tilde{D},\tilde{\nu})]$ (Theorem 2.6). Because the Milnor number is upper semi-continuous ([GLS06] Theorem 2.6) then for sufficiently small s, we have $\mu(f_s) \le \mu(f_0)$. Therefore

$$\mu((\tilde{D},\tilde{\nu})) = \mu(f_s) \le \mu(f_0) = \mu\left((E,\lambda)\right) = \mu\left((E_D,\lambda_D)\right). \quad \square$$

In the next lemmas we will consider the case (2) that there exists subdiagram of $\tilde{D}$ isomorphic to D. First, the two lemmas for the subcase (2a) i.e. $d > 2$.

Lemma 4.3. *Let $d,t \in \mathbb{N}$, $d \ge 3$, $(D,\nu) \in Q_t^d$ and let $(\tilde{D},\tilde{\nu}) \notin [(D,\nu)]$ be an arbitrary minimal Enriques diagram such $[(D,\nu)]$ is linear adjacent to $[(\tilde{D},\tilde{\nu})]$. If*

(1) *there exist a subdiagram $\tilde{D}_0 \subset \tilde{D}$ and an isomorphism $i : \tilde{D}_0 \to D$ (not necessarily preserving the weights),*

(2)
$$\sum_{P \text{ successor of } i^{-1}(R_t)} \min(2,\tilde{\nu}(P)) + z \le \nu(R_t) - 1,$$
where z is the number of vertices proximate to $i^{-1}(R_t)$ that are not its successors,

then
$$\mu((\tilde{D},\tilde{\nu})) \le \mu((D,\nu)) - (d - 2 + w_D).$$

Proof. We may assume that $(\tilde{D},\tilde{\nu})$ is a minimal diagram. Notice that

$$ord_{\tilde{\nu}}(i^{-1}(R_t)) < ord_\nu(R_t). \tag{9}$$

Indeed, let us assume that $ord_{\tilde{\nu}}(i^{-1}(R_t)) = ord_\nu(R_t)$. We prove by induction (using Theorem 3.1) with respect to the number of satellite vertices in D that then $\tilde{\nu}(i^{-1}(R_j)) = \nu(R_j)$ for $j = 1,\dots,t$. From this we get that $(\tilde{D},\tilde{\nu}) \in [(D,\nu)]$, which is impossible.

Let us pass to the construction of (E,λ) such that $[(E,\lambda)]$ is linear adjacent to $[(\tilde{D},\tilde{\nu})]$ and $\mu\left((E,\lambda)\right) \le \mu((D,\nu)) - (d-2+w_D)$. We do this in two steps, first we construct (E',λ') and then after some simple modification of (E',λ') we get (E,λ).

Let $\{S_1,\dots,S_m\}$ be the set of vertices proximate to $i^{-1}(R_t)$ in $\tilde{D}$. We will construct (E',λ').

- $E' = \{Q_1,\dots,Q_t,U_1,\dots,U_m\}$,
- $\lambda'(Q_i) = \nu(R_i)$ for $i = 1,\dots,t-1$,
- $\lambda'(Q_t) = \nu(R_t) - 1$,
- $\lambda'(U_i) = \min(2,\tilde{\nu}(S_i))$ for S_i that are free ($i \in \{1,\dots,m\}$),
- $\lambda'(U_i) = 1$ for S_i that are not free ($i \in \{1,\dots,m\}$),
- $Q_i \xrightarrow{E'} Q_j \Leftrightarrow R_i \xrightarrow{D} R_j$ for $i,j \in \{1,\dots,t\}$,
- $U_i \xrightarrow{E'} U_j \Leftrightarrow S_i \xrightarrow{\tilde{D}} S_j$ for $i,j \in \{1,\dots,m\}$,
- $S_i \xrightarrow{\tilde{D}} i^{-1}(R_t) \Leftrightarrow U_i \xrightarrow{E'} Q_t$ for $i \in \{1,\dots,m\}$,
- $U_i \xrightarrow{E'} Q_t$ for $i = 1,\dots,m$.

The diagram (E',λ') is consistent due to the second condition in the assumption. Since $d \ge 3$, its Milnor number can be easily estimated by

$$\begin{aligned}\mu((E',\lambda')) &= \mu((D,\nu)) - (d-2+w_D) - d^2 + 3d - 2 - x = \\ &= \mu((D,\nu)) - (d-2+w_D) - (d-1)(d-2) - x \le \mu((D,\nu)) - (d-2+w_D),\end{aligned}$$

where x is the number of successors of $i^{-1}(R_t)$ in $\tilde{D}$ with weight 1. Because $[(D,\nu)]$ is linear adjacent to $[(\tilde{D},\tilde{\nu})]$, there there exists $(D',\nu') \in [(D,\nu)]$ such that $(D',\nu') \geq (\tilde{D},\tilde{\nu})$. We can modify (E',λ') to get $(E,\lambda) \in [(E',\lambda')]$ (analogous to the construction of D' from D). Since $(D',\nu') \geq (\tilde{D},\tilde{\nu})$ then exist subsets $D_0 \subset \tilde{D}$, $D'_0 \subset D'$ and a function $\kappa_{\nu'}$ on $\tilde{D}$ such that for $P \in \tilde{D}$ we have $\mathrm{ord}_{\tilde{\nu}}(P) \leq \mathrm{ord}_{\kappa_{\nu'}}(P)$. Because $D'_0 \subset D' \subset E'$ we can define function $\kappa_{\lambda'}$ using the same isomorphic subsets D_0, D'_0. Then for every $P \in \tilde{D} \setminus \{i^{-1}(R_t)\}$ we have

$$\mathrm{ord}_{\kappa_{\lambda'}}(P) = \mathrm{ord}_{\kappa_{\nu'}}(P) \geq \mathrm{ord}_{\tilde{\nu}}(P).$$

If $R_t \notin D'_0$ we also have

$$\mathrm{ord}_{\kappa_{\lambda'}}(i^{-1}(R_t)) = \mathrm{ord}_{\kappa_{\nu'}}(i^{-1}(R_t)) \geq \mathrm{ord}_{\tilde{\nu}}(i^{-1}(R_t)).$$

If $R_t \in D'_0$ then from (9)

$$\mathrm{ord}_{\kappa_{\lambda'}}(i^{-1}(R_t)) = \mathrm{ord}_{\kappa_{\nu'}}(i^{-1}(R_t)) - 1 = \mathrm{ord}_{\nu}(R_t) - 1 \geq \mathrm{ord}_{\tilde{\nu}}(i^{-1}(R_t)).$$

From these facts we get that $(E',\lambda') \geq (\tilde{D},\tilde{\nu})$. This gives that $[(E,\lambda)]$ is linear adjacent to $[(\tilde{D},\tilde{\nu})]$, so for every singularity f_0 whose Enriques diagram belong to $[(E,\lambda)]$, there exists a linear deformation (f_s) of f_0 such that the Enriques diagram of a generic element f_s belongs to $[(\tilde{D},\tilde{\nu})]$ (Theorem 2.6). Because the Milnor number is upper semi-continuous ([GLS06] Theorem 2.6) then for sufficiently small s, we have $\mu(f_s) \leq \mu(f_0)$. Therefore

$$\mu((\tilde{D},\tilde{\nu})) \leq \mu\left((E,\lambda)\right) = \mu\left((E',\lambda')\right) \leq \mu((D,\nu)) - (d - 2 + w_D). \quad \square$$

Now, we will consider the opposite situation to the second condition in Lemma 4.3.

Lemma 4.4. *Let $d,t \in \mathbb{N}$, $d \geq 2$, $(D,\nu) \in Q^t_d$ and let $(\tilde{D},\tilde{\nu}) \notin [(D,\nu)]$ be an arbitrary minimal Enriques diagram such that $[(D,\nu)]$ is linear adjacent to $[(\tilde{D},\tilde{\nu})]$. Let us assume there exist a subdiagram $\tilde{D}_0 \subset \tilde{D}$ and an isomorphism $i : \tilde{D}_0 \to D$ such that*

$$\sum_{P \text{ successor of } i^{-1}(R_t)} \min(2, \tilde{\nu}(P)) + z > \nu(R_t) - 1,$$

where z is number of vertices proximate to $i^{-1}(R_t)$ that are not its successors. Then

$$\mu((\tilde{D},\tilde{\nu})) \leq \mu((D,\nu)) - (d - 2 + w_D).$$

Proof. Since $\tilde{\nu}(i^{-1}(R_t)) \leq \nu(R_t)$ (because $(\tilde{D},\tilde{\nu}) \notin [(D,\nu)]$) and from the consistency of $[(\tilde{D},\tilde{\nu})]$

$$\sum_{P \text{ successor of } i^{-1}(R_t)} \min(2, \tilde{\nu}(P)) + z \leq \tilde{\nu}(i^{-1}(R_t)),$$

we get

$$\nu(R_t) \leq \sum_{P \text{ successor of } i^{-1}(R_t)} \min(2, \tilde{\nu}(P)) + z \leq \tilde{\nu}(i^{-1}(R_t)) \leq \nu(R_t).$$

Then we get the equality $\nu(R_t) = \tilde{\nu}(i^{-1}(R_t))$. This follows that $\nu(R_j) = \tilde{\nu}(i^{-1}(R_j))$ for $j = t_0 + 1, \ldots, t$, where R_{t_0} is the last free vertex in (D,ν) and for the rest we have $\nu(R_j) - \tilde{\nu}(i^{-1}(R_j)) \in \{0,1,2\}$. Because $(\tilde{D},\tilde{\nu}) \notin [(D,\nu)]$, in $(\tilde{D},\tilde{\nu})$ there are no vertices after R_j ($j = 1, \ldots, t-1$) and all the successors of $i^{-1}(R_t)$ have weight 2 at most. If after them there is a vertex of weight 2, it has to be free. Then, after the $i^{-1}(R_t)$ in $\tilde{D}$ we can have a "new branch" with vertices of weight 2 and the length of such "new branch" is limited by $l := \mathrm{ord}_{\nu}(R_t) - \mathrm{ord}_{\tilde{\nu}}(i^{-1}(R_t))$. The number of such branches b is less that $\frac{d}{2}$, since $(\tilde{D},\tilde{\nu})$ is consistent. Moreover $r(D) - r(\tilde{D}) < \nu(R_1) - d$.

The Milnor number of $(\tilde{D}, \tilde{\nu})$ can be estimated by:

$$\mu((\tilde{D},\tilde{\nu})) \leq \mu((D,\nu))+$$
$$\sum_{j=1}^{k_0} \big(\tilde{\nu}(i^{-1}(R_j))\left(\tilde{\nu}(i^{-1}(R_j))-1\right) - \nu(R_j)\left(\nu(R_j)-1\right)\big) + lb2 + r(D) - r(\tilde{D}) \leq$$
$$\mu((D,\nu)) + \sum_{j=1}^{k_0} \big(\tilde{\nu}(i^{-1}(R_j))\left(\tilde{\nu}(i^{-1}(R_j))-1\right) - \nu(R_j)\left(\nu(R_j)-1\right)\big) +$$
$$d\left(\mathrm{ord}_{\nu}(R_t) - \mathrm{ord}_{\tilde{\nu}}(i^{-1}(R_t))\right) + (\nu(R_1) - d - 1) \leq \mu((D,\nu)) - (d-2+w_D).$$

For the last inequality, we have to consider two possible cases:

- there exists $j_0 \in \{1,\ldots,k_0\}$, such that $\tilde{\nu}(i^{-1}(R_j)) = \nu(R_j) - 1$ for $j = 1,\ldots,j_0$, $\tilde{\nu}(i^{-1}(R_j)) = \nu(R_j)$ for $j = j_0+1,\ldots,k_0$,
- $\tilde{\nu}(i^{-1}(R_1)) = \nu(R_1) - 2$, $\tilde{\nu}(i^{-1}(R_j)) = \nu(R_j) - 1$ for $j = 2,\ldots,k_0$.

In each case, regarding the definition of w_D, we easily get the required estimationa. □

In the next lemma we consider the subcase (2b) i.e.$d = 2$.

Lemma 4.5. *Let $k \in \mathbb{N}$, $(D,\nu) \in Q_2^t$ and let $(\tilde{D},\tilde{\nu}) \notin [(D,\nu)]$ be an arbitrary minimal Enriques diagram such that $[(D,\nu)]$ is linear adjacent to $[(\tilde{D},\tilde{\nu})]$. Then*

$$\mu((\tilde{D},\tilde{\nu})) \leq \begin{cases} \mu((D,\nu)) - 1, & \text{if } w_D = 0 \\ \mu((D,\nu)) - w_D, & \text{if } w_D \neq 0 \end{cases}$$

Proof. If $w_D = 0$ then the only diagram $(\tilde{D},\tilde{\nu}) \notin [(D,\nu)]$ such that $[(D,\nu)]$ is linear adjacent to $[(\tilde{D},\tilde{\nu})]$ is (E_D,λ_D). Then $\mu((\tilde{D},\tilde{\nu})) = \mu((E_D,\lambda_D)) = \mu((D,\nu)) - 1$.

Let assume that $w_D \neq 0$. If there is no subdiagram of $\tilde{D}$ isomorphic to D we can apply Lemma 4.2. If there exist subdiagrams $\tilde{D}_0 \subset \tilde{D}$, $D_0 \subset D$ and an isomorphism $i : \tilde{D}_0 \to D$, then

$$\sum_{P \text{ successor of } i^{-1}(R_t)} \min(2,\tilde{\nu}(P)) + z = \nu(R_t),$$

where z is the number of vertices proximate to $i^{-1}(R_t)$ that are not its successors. Then from Lemma 4.4 we get $\mu((\tilde{D},\tilde{\nu})) \leq \mu((D,\nu)) - w_D$. □

The last lemma is for the last subcase (2c) $d = 1$ and it is easy to prove.

Lemma 4.6. *Let $k \in \mathbb{N}$, $(D,\nu) \in Q_1^t$ and let $(\tilde{D},\tilde{\nu}) \notin [(D,\nu)]$ be an arbitrary minimal Enriques diagram such that $[(D,\nu)]$ is linear adjacent to $[(\tilde{D},\tilde{\nu})]$. If there exist subdiagrams $\tilde{D}_0 \subset \tilde{D}$, $D_0 \subset D$ and an isomorphism $i : \tilde{D}_0 \to D$, then $\mu((\tilde{D},\tilde{\nu})) \leq \mu((D,\nu)) - 1$.*

Now, we can formulate the main result (that indeed the diagram (E_D,λ_D) from Theorem 4.1 realizes the maximum in (7)). This theorem is a consequence of previous lemmas.

Theorem 4.7. *Let $d,t \in \mathbb{N}$, $dt > 1$, $(D,\nu) \in Q_d^t$ and let $(\tilde{D},\tilde{\nu}) \notin [(D,\nu)]$ be an arbitrary Enriques diagram such that $[(D,\nu)]$ is linear adjacent to $[(\tilde{D},\tilde{\nu})]$. Then*

$$\mu((\tilde{D},\tilde{\nu})) \leq \begin{cases} \mu((D,\nu)) - 1, & \text{if } d = 1 \\ \mu((D,\nu)) - 1, & \text{if } d = 2, w_D = 0 \\ \mu((D,\nu)) - w_D, & \text{if } d = 2, w_D \neq 0 \\ \mu((D,\nu)) - (d-2+w_D), & \text{if } d \geq 3 \end{cases}.$$

5. Formula for jump of the Milnor number of quasihomogeneous singularity for linear deformations

In this section we apply Theorems 4.1 and 4.7 to the Enriques diagrams of a quasihomogeneous singularity.

As a consequence of these theorems and the construction of the Enriques diagrams of quasihomogeneous singularities we can formulate the following facts.

Theorem 5.1. *For any quasihomogeneous singularity f_0 of form (4) the jump of Milnor number of f_0 for linear deformations is*

$$\lambda^{lin}(f_0) = \begin{cases} 1, & \text{if } d = 1 \\ 1, & \text{if } d = 2, w_{f_0} = 0 \\ w_{f_0}, & \text{if } d = 2, w_{f_0} \neq 0 \\ d - 2 + w_{f_0}, & \text{if } d \geq 3 \end{cases}, \tag{10}$$

where:

- *if $p = q$ then $d = k + l + p$ and $w_{f_0} = 0$,*
- *if $p \neq q$ and $p|q$ then $d = k + p$ and $w_{f_0} = 1$,*
- *if $p \neq q$ and $p\nmid q$ then $d = GCD(p, q)$ and $w_{f_0} = 2$.*

Proof. Let f_0 be a quasihomogeneous singularity and (D, ν) its Enriques diagram. From Theorem 4.1 there exists diagram $(E_D, \lambda_D) \notin [(D, \nu)]$ such that $[(D, \nu)]$ is linear adjacent to $[(E_D, \lambda_D)]$ and

$$\mu\left((E_D, \lambda_D)\right) = \begin{cases} \mu((D, \nu)) - 1, & \text{if } d = 1 \\ \mu((D, \nu)) - 1, & \text{if } d = 2, w_D = 0 \\ \mu((D, \nu)) - w_D, & \text{if } d = 2, w_D \neq 0 \\ \mu((D, \nu)) - (d - 2 + w_D), & \text{if } d \geq 3 \end{cases}.$$

Since (E_D, λ_D) is minimal Theorem 2.3 and Theorem 2.6 give

$$\lambda^{lin}(f_0) \leq \begin{cases} 1, & \text{if } d = 1 \\ 1, & \text{if } d = 2, w_D = 0 \\ w_D, & \text{if } d = 2, w_D \neq 0 \\ d - 2 + w_D, & \text{if } d \geq 3 \end{cases}. \tag{11}$$

From Theorem 4.7 for any Enriques diagram $(\tilde{D}, \tilde{\nu}) \notin [(D, \nu)]$ such that $[(D, \nu)]$ is linear adjacent to $[(\tilde{D}, \tilde{\nu})]$ we have $\mu((\tilde{D}, \tilde{\nu})) \leq \mu\left((E_D, \lambda_D)\right)$. It gives the opposite inequality in (11) and as a consequence we get (10), because $w_D = w_{f_0}$. □

Taking into account the simple characterization (6) of w_D, we get a more effective formula.

Corollary 5.2. *Let f_0 be a quasihomogeneous singularity of form (4). Then*

(1) *If $p = q$ i.e. f_0 is a homogeneous singularity then*

$$\lambda^{lin}(f_0) = \begin{cases} 1, & \text{if } k + l + p = 2 \\ k + l + p - 2, & \text{if } k + l + p \geq 3 \end{cases}.$$

(2) *If $p \neq q$ and $p|q$ then*

$$\lambda^{lin}(f_0) = \begin{cases} 1, & \text{if } p + k \leq 2 \\ p + k - 1, & \text{if } p + k \geq 3 \end{cases}.$$

(3) *If $p \neq q$ and $p\nmid q$ then*

$$\lambda^{lin}(f_0) = GCD(p, q).$$

If we consider only the "standard" quasihomogeneous singularities i.e. $k = l = 0$ in (4), we get a very simple formula for the jump.

Corollary 5.3. *Let f_0 be a quasihomogeneous singularity defined in (4) and $k = l = 0$. Then*

(1) *If $p = q$ i.e. f_0 is a homogeneous singularity then*

$$\lambda^{lin}(f_0) = \begin{cases} 1, & \text{if } p = 2 \\ p-2, & \text{if } p \geq 3 \end{cases}.$$

(2) *If $p \neq q$ then*

$$\lambda^{lin}(f_0) = \begin{cases} p-1, & \text{if } p|q \\ GCD(p,q), & \text{if } p\not| q \end{cases}.$$

Example 5.4. *Let's consider the singularity from Example 1.2 i.e. $f_0(x,y) = x^6 + y^9$. Its minimal Enriques diagram is shown in Figure 8(a). The minimal Enriques diagram realizing the $\lambda^{lin}(f_0)$ (constructed in Theorem 4.1) is shown in Figure 8(b). A linear deformation having this diagram is $f_s(x,y) = f_0(x,y) + sxy(y^3 + x^2)^2$.*

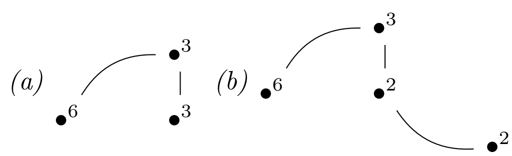

FIGURE 8. The minimal Enriques diagrams of f_0 and f_s

Remark 5.5. *It is not an easy task to write down an explicit formula of the deformation from the constructed Enriques diagram. Obviously, in specific case it can be done (as in Example 5.4).*

As a corollary we give a formula for the jump of Milnor number for **semi-quasihomogeneous singularities** i.e. singularities of the form $f_0 = f_0' + g$, where f_0' is a quasihomogeneous singularity with respect to some weights (w_x, w_y) and $\mathrm{ord}_{(w_x,w_y)} g > \mathrm{ord}_{(w_x,w_y)} f_0'$.

Corollary 5.6. *For any semi-quasihomogeneous singularity f_0*

$$\lambda^{lin}(f_0) = \lambda^{lin}(f_0').$$

Proof. It suffices to notice that Enriques diagrams of f_0 and f_0' have the same type. □

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