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NEARBY CYCLES FOR PARITY SHEAVES ON A DIVISOR WITH SIMPLE NORMAL CROSSINGS

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Abstract. The first author recently introduced a "nearby cycles formalism" in the framework of chain complexes of parity sheaves. In this paper, we compute this functor in two related settings: (i) affine space, stratified by the action of a torus, and (ii) the global Schubert variety associated to the first fundamental coweight of the group PGL_n. The latter is a parity-sheaf analogue of Gaitsgory's central sheaf construction.

1. Introduction

In [A], the first author introduced a "nearby cycles formalism" associated to an algebraic map $f : X \to \mathbb{A}^1$ in the framework of (chain complexes) of parity sheaves. This entails the construction of a functor

$$\Psi_f : D^{\mathrm{mix}}_{\mathbb{G}_{\mathrm{m}}}(X_\eta, \Bbbk) \to D^{\mathrm{mix}}(X_0, \Bbbk)$$

with properties resembling those of the classical unipotent nearby cycles functor [B, R], including a canonical nilpotent endomorphism $\mathrm{N}_\Psi : \Psi_f(\mathcal{F}) \to \Psi_f(\mathcal{F})\langle 2\rangle$, called the *monodromy endomorphism*. (See Section 2 below for a review of the notation and setup.) It is expected that this functor will make it possible to adapt Gaitsgory's construction of "central sheaves" [Ga] to the setting of the mixed modular derived category [AR1], which has found numerous applications in modular geometric representation theory (see [AR3, §7.1]).

In this paper, we compute the first nontrivial examples of the nearby cycles functor Ψ_f, in the following two related settings:

- $X = \mathbb{A}^n$, and $f : X \to \mathbb{A}^1$ is the map $f(x_1, \ldots, x_n) = x_1 \cdots x_n$. In this case, the special fiber X_0 is the union of the coordinate hyperplanes in $\mathbb{A}^n$, and hence a divisor with simple normal crossings.
- $X = \overline{\mathcal{G}r}_{\check\varpi_1}$, the "global Schubert variety" associated to the first fundamental coweight $\check\varpi_1$ for the group PGL_n, as defined in [Z]. This space is equipped with a map $f : X \to \mathbb{A}^1$ such that $f^{-1}(t)$ for any $t \neq 0$ is identified with the minuscule Schubert variety $\mathrm{Gr}_{\check\varpi_1}$ in the affine Grassmannian for PGL_n, isomorphic to $\mathbb{P}^{n-1}$. The special fiber X_0 is a subset of the affine flag variety Fl, known as the "central degeneration of $\mathrm{Gr}_{\check\varpi_1}$."

These two cases are closely related: there is an open affine subset of $\overline{\mathcal{G}r}_{\check\varpi_1}$ that can be identified with $\mathbb{A}^n$ in a way that is compatible with the map to $\mathbb{A}^1$. This fact is used in a crucial way in this paper: we compute the nearby cycles complex on $\mathbb{A}^n$ directly from the definition, and then we use this open embedding to deduce the result on $\overline{\mathcal{G}r}_{\check\varpi_1}$.

An explicit description of the nearby cycles object $\Psi_f(\underline{\Bbbk}_{X_\eta}\{n\})$ is given in Sections 6 and 9. From this description, one can see that $\Psi_f(\underline{\Bbbk}_{X_\eta}\{n\})$ is, in fact, a perverse sheaf. (Unlike in the classical case, the t-exactness of the mixed nearby cycles functor of [A] is not known in general.)

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Moreover, the canonical nilpotent endomorphism $\mathrm{N}_\Psi : \Psi_f(\underline{\Bbbk}_{X_\eta}\{n\}) \to \Psi_f(\underline{\Bbbk}_{X_\eta}\{n\})\langle 2\rangle$ gives rise to a filtration

$$M_\bullet \Psi_f(\underline{\Bbbk}_{X_\eta}\{n\}),$$

called the *monodromy filtration*. (See Section 10 for details.)

The following result describes the associated graded of this filtration. One can also read off multiplicities of composition factors from this statement.

Theorem 1.1. *Let X denote either $\mathbb{A}^n$ or $\overline{\mathcal{G}r}_{\check{\varpi}_1}$, as above. The associated graded of the monodromy filtration on the mixed perverse sheaf $\Psi_f(\underline{\Bbbk}_{X_\eta}\{n\})$ is given by*

$$\mathrm{gr}^M_k \Psi_f(\underline{\Bbbk}_{X_\eta}\{n\}) = \bigoplus_{\substack{p,q\geq 0\\ p-q=k}} \bigoplus_{\substack{I\subsetneq[n]\\ |I|=n-1-p-q}} \underline{\Bbbk}_{\overline{X_I}}\{|I|\}\langle k-1\rangle.$$

In particular, each component of this associated graded is pure, so the monodromy filtration coincides with (a shift of) the *weight filtration* on $\Psi_f(\underline{\Bbbk}_{X_\eta}\{n\})$ (in the sense of [AR2]). For an analogous statement in the context of classical (ℓ-adic) nearby cycles, see [I, §3.4], as well as [RZ, S]. See also [G2, Proposition 9.1].

Contents. Section 2 gives a brief review of the nearby cycles formalism from [A]. In Section 3, we fix notation for the case of $\mathbb{A}^n$, and in Sections 4–6, we carry out the nearby cycles calculation in this case. In Sections 7 and 8, we study the geometry of $\overline{\mathcal{G}r}_{\check{\varpi}_1}$. The nearby cycles calculation in this case is done in Section 9. Section 10 is devoted to the study of the monodromy filtration (on either $\mathbb{A}^n$ or $\overline{\mathcal{G}r}_{\check{\varpi}_1}$). Finally, Section 11 contains a few explicit examples.

Acknowledgments. The complex of parity sheaves $\mathcal{Z}^{\overline{\mathcal{G}r}_{\check{\varpi}_1}}$ described in Section 9 (see also Section 11) has been discovered and studied independently by B. Elias [E] from a rather different perspective. We are grateful to him for keeping us informed about his work.

2. Background on the nearby cycles formalism

2.1. **Graded parity sheaves.** Let $\Bbbk$ be a complete local principal ideal domain. Throughout the paper, we will consider sheaves with coefficients in $\Bbbk$. Let R denote the $\mathbb{G}_\mathrm{m}$-equivariant cohomology of a point:

$$R = \mathsf{H}^\bullet_{\mathbb{G}_\mathrm{m}}(\mathrm{pt};\Bbbk) = \Bbbk[\xi]. \tag{2.1}$$

Here, $\xi \in \mathsf{H}^2_{\mathbb{G}_\mathrm{m}}(\mathrm{pt};\Bbbk)$ is the canonical generator, as in [A, §2.2]. We regard this as a bigraded ring by setting $\deg\xi = (2,2)$.

Let X be a complex algebraic variety, and let H be an algebraic group acting on X. Suppose that there is an action of $\mathbb{G}_\mathrm{m}$ on H by group automorphisms, so that we may form the group $\mathbb{G}_\mathrm{m} \ltimes H$, and that the H-action on X extends to an action of $\mathbb{G}_\mathrm{m} \ltimes H$. Assume that X is equipped with a fixed algebraic stratification $(X_s)_{s\in\mathscr{S}}$ satisfying the assumptions of [A, §2.2]. In particular, for each stratum X_s, there is a unique (up to isomorphism and shift) indecomposable $\mathbb{G}_\mathrm{m} \ltimes H$-equivariant parity sheaf[1] supported on $\overline{X_s}$.

As in [A], it is useful to distinguish the $\mathbb{G}_\mathrm{m}$-equivariance from the H-equivariance. For this reason, the additive category of $\mathbb{G}_\mathrm{m} \ltimes H$-equivariant parity sheaves on X is denoted

[1]We recall that in [JMW], a distinction is made between "parity complexes" and "parity sheaves"; the latter are indecomposable by definition. We prefer to call all these objects "parity sheaves," and to use the adjective "indecomposable" when needed.

by $\mathrm{Parity}_{\mathbb{G}_\mathrm{m}}(X/H, \Bbbk)$. Following [AR1, A], we denote the cohomological shift functor in $\mathrm{Parity}_{\mathbb{G}_\mathrm{m}}(X/H, \Bbbk)$ by $\{1\}$. A *graded parity sheaf* is a formal expression of the form

$$\mathcal{F} = \bigoplus_{i\in\mathbb{Z}} \mathcal{F}^i[-i],$$

where $\mathcal{F}^i \in \mathrm{Parity}_{\mathbb{G}_\mathrm{m}}(X/H, \Bbbk)$, and where only finitely many of the $\mathcal{F}^i$ are nonzero. Recall that for a graded parity sheaf, the *Tate twist* $\langle 1\rangle$ is defined by

$$\mathcal{F}\langle 1\rangle = \mathcal{F}\{-1\}[1].$$

If $\mathcal{F}$ and $\mathcal{G}$ are graded parity sheaves, we define $\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})$ to be the bigraded $\Bbbk$-module given by

$$\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})^i_j = \bigoplus_{\substack{p,q\in\mathbb{Z}\\ q-p=i-j}} \mathrm{Hom}(\mathcal{F}^p, \mathcal{G}^q\{j\}).$$

This is naturally a bigraded R-module. A *morphism* of graded parity sheaves $\phi : \mathcal{F} \to \mathcal{G}$ is just an element $\phi \in \underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})^0_0$. Note that a homogeneous element $\psi \in \underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})^i_j$ of bidegree (i, j) can be thought of as a morphism $\psi : \mathcal{F} \to \mathcal{G}[i]\langle -j\rangle$.

2.2. **Three derived categories.** Let r and $\bar{\xi}$ be indeterminates, and let

$$R^\vee = \Bbbk[\mathsf{r}] \qquad \text{and} \qquad \Lambda = \Bbbk[\bar{\xi}]/(\bar{\xi}^2).$$

We regard these as bigraded rings by setting $\deg \mathsf{r} = (0, -2)$ and $\deg \bar{\xi} = (1, 2)$.

The theory developed in [A] involves three triangulated categories, briefly summarized in the table below. In all three, an object is a pair $(\mathcal{F}, \delta)$, where $\mathcal{F}$ is a graded parity sheaf, and δ (called the "differential") is an element of bidegree $(1, 0)$ in some bigraded $\Bbbk$-module, satisfying some condition.

Category	*Differentials live in...*	*and satisfy...*
$D^{\mathrm{mix}}_{\mathbb{G}_\mathrm{m}}(X/H, \Bbbk)$	$\underline{\mathrm{End}}(\mathcal{F})$	$\delta^2 = 0$
$D^{\mathrm{mix}}(X/H, \Bbbk)$	$\Lambda \otimes \underline{\mathrm{End}}(\mathcal{F})$	$\delta^2 + \kappa(\delta) = 0$
$D^{\mathrm{mix}}_{\mathrm{mon}}(X/H, \Bbbk)$	$R^\vee \otimes \underline{\mathrm{End}}(\mathcal{F})$	$\delta^2 = \mathsf{r}\xi \cdot \mathrm{id}_\mathcal{F}$

In the second row, κ is a certain map that satisfies $\kappa(\bar{\xi} \cdot \mathrm{id}) = \xi \cdot \mathrm{id}$ and obeys the Leibniz rule. See [A, §3] for further details on all three of these categories.

We remark that Hom-groups in $D^{\mathrm{mix}}_{\mathrm{mon}}(X/H, \Bbbk)$ inherit an action of $R^\vee$. In particular, every object $\mathcal{F} \in D^{\mathrm{mix}}_{\mathrm{mon}}(X/H, \Bbbk)$ carries a canonical endomorphism

$$\mathsf{r} \cdot \mathrm{id}_\mathcal{F} : \mathcal{F} \to \mathcal{F}\langle 2\rangle,$$

and all morphisms in $D^{\mathrm{mix}}_{\mathrm{mon}}(X/H, \Bbbk)$ commute with r. See [A, Definition 3.5].

According to [A, Proposition 5.5], there is a fully faithful functor

$$\mathrm{Mon} : D^{\mathrm{mix}}(X/H, \Bbbk) \to D^{\mathrm{mix}}_{\mathrm{mon}}(X/H, \Bbbk).$$

An explicit formula for this functor can be found in [A, Eq. (5.5)].

The categories $D^{\mathrm{mix}}_{\mathbb{G}_\mathrm{m}}(X/H, \Bbbk)$ and $D^{\mathrm{mix}}(X/H, \Bbbk)$ admit a *perverse t-structure*. Their hearts are denoted by $\mathrm{Perv}^{\mathrm{mix}}_{\mathbb{G}_\mathrm{m}}(X/H, \Bbbk)$ and $\mathrm{Perv}^{\mathrm{mix}}(X/H, \Bbbk)$, respectively.

2.3. **The nearby cycles functor.** Now let $f : X \to \mathbb{A}^1$ be a $\mathbb{G}_\mathrm{m}$-equivariant map, where $\mathbb{G}_\mathrm{m}$ acts on $\mathbb{A}^1$ by the natural scaling action. Let $X_0 = f^{-1}(0)$, and let $X_\eta = f^{-1}(\mathbb{A}^1 \smallsetminus \{0\})$. Assume that each stratum of X is contained in either X_0 or X_η, and that

$$\mathsf{H}^\bullet_{\mathbb{G}_\mathrm{m} \ltimes H}(X_s, \Bbbk) \text{ is free as an } R\text{-module} \qquad \text{for all } X_s \subset X_0. \tag{2.2}$$

We let

$$\mathbf{i} : X_0 \hookrightarrow X \qquad \text{and} \qquad \mathbf{j} : X_\eta \hookrightarrow X$$

be the inclusion maps. By [A, Theorem 8.4], the condition (2.2) implies that

$$\mathrm{Mon} : D^{\mathrm{mix}}(X_0/H, \Bbbk) \xrightarrow{\sim} D^{\mathrm{mix}}_{\mathrm{mon}}(X_0/H, \Bbbk) \tag{2.3}$$

is an equivalence of categories. (In contrast, on X_η, Mon is never an equivalence.) The main content of [A] is the construction of a functor

$$\Psi_f : D^{\mathrm{mix}}_{\mathbb{G}_\mathrm{m}}(X_\eta/H, \Bbbk) \to D^{\mathrm{mix}}(X_0/H, \Bbbk),$$

together with a natural nilpotent endomorphism $\mathrm{N} : \Psi_f(\mathcal{F}) \to \Psi_f(\mathcal{F})\langle 2\rangle$. Explicitly, the functor is given by the formula

$$\Psi_f(\mathcal{F}) = \mathrm{Mon}^{-1}\mathbf{i}^*\mathbf{j}_*\mathcal{J}(\mathcal{F})\langle -2\rangle.$$

For a discussion of pullback and push-forward functors in this setting, see [A, §6] (and also [AR1, §2]). The notation $\mathcal{J} : D^{\mathrm{mix}}_{\mathbb{G}_\mathrm{m}}(X_\eta/H, \Bbbk) \to D^{\mathrm{mix}}_{\mathrm{mon}}(X_\eta/H, \Bbbk)$ is used for for the “pro-unipotent Jordan block functor” as defined in [A, §9].

3. Parity sheaves on affine space

We will use the notation $[n] = \{1, \dots, n\}$. For $I \subset [n]$, let

$$(\mathbb{A}^n)_I = \{(x_1, \dots, x_n) \in \mathbb{A}^n \mid x_i = 0 \text{ if and only if } i \notin I\}.$$

The collection of subvarieties $\{(\mathbb{A}^n)_I\}_{I\subset[n]}$ constitutes a stratification of $\mathbb{A}^n$. Note that $(\mathbb{A}^n)_{[n]}$ is an open dense subset of $\mathbb{A}^n$, and $(\mathbb{A}^n)_\varnothing$ is just the origin.

Let $T = \mathbb{G}_\mathrm{m}^{n-1}$, and let $\hat{T} = T \times \mathbb{G}_\mathrm{m}$. Throughout, the “last” copy of $\mathbb{G}_\mathrm{m}$ in $\hat{T}$ will play a different conceptual role from the first $n-1$ copies, and the notation will reflect that. Let

$$\alpha_1, \dots, \alpha_{n-1}, \xi : \hat{T} \to \mathbb{G}_\mathrm{m}$$

be the characters given by

$$\alpha_i(t_1, \dots, t_{n-1}, z) = t_i, \qquad \xi(t_1, \dots, t_{n-1}, z) = z.$$

Define a character $\alpha_n : \hat{T} \to \mathbb{G}_\mathrm{m}$ by

$$\alpha_n = \xi - \alpha_1 - \cdots - \alpha_{n-1}. \tag{3.1}$$

Let $\hat{T}$ act on $\mathbb{A}^n$ with weights $\alpha_1, \dots, \alpha_{n-1}, \alpha_n$. In other words,

$$(t_1, \dots, t_{n-1}, z) \cdot (x_1, \dots, x_n) = (t_1x_1, t_2x_2, \dots, t_{n-1}x_{n-1}, t_1^{-1}t_2^{-1}\cdots t_{n-1}^{-1}zx_n).$$

The set $\{\alpha_1, \dots, \alpha_{n-1}, \xi\}$ is a $\mathbb{Z}$-basis for the character lattice $X_*(\hat{T})$. We have

$$\mathsf{H}^\bullet_{\hat{T}}(\mathrm{pt}; \Bbbk) = \Bbbk[\alpha_1, \dots, \alpha_{n-1}, \xi],$$

where the generators $\alpha_1, \dots, \alpha_{n-1}, \xi$ all have degree 2. This ring is an algebra over the ring $R = \Bbbk[\xi]$ from (2.1). Of course, $\{\alpha_1, \dots, \alpha_{n-1}, \alpha_n\}$ is another basis for $X_*(\hat{T})$, and another set of generators for $\mathsf{H}^\bullet_{\hat{T}}(\mathrm{pt}; \Bbbk)$. It is sometimes convenient to use this basis instead.

Let $f : \mathbb{A}^n \to \mathbb{A}^1$ be the map $f(x_1, \dots, x_n) = x_1x_2\cdots x_n$. Let $\hat{T}$ act on $\mathbb{A}^1$ via the character ξ. Then f is $\hat{T}$-equivariant. We have

$$(\mathbb{A}^n)_0 = f^{-1}(0) = \bigcup_{I\subsetneq[n]} (\mathbb{A}^n)_I \qquad \text{and} \qquad (\mathbb{A}^n)_\eta = (\mathbb{A}^n)_{[n]}.$$

The following lemma says that condition (2.2) holds.

Lemma 3.1. *For each subset $I \subsetneq [n]$, $\mathsf{H}^\bullet_{\hat{T}}((\mathbb{A}^n)_I; \Bbbk)$ is free as an R-module.*

Proof. Let $T_I = \cap_{i\in I} \ker \alpha_i$. Elementary considerations show that

$$\mathsf{H}^\bullet_{\hat{T}}((\mathbb{A}^n)_I;\Bbbk) \cong \mathsf{H}^\bullet_{T_I}(\mathrm{pt};\Bbbk) \cong \Bbbk[\alpha_1,\ldots,\alpha_n]/(\{\alpha_i \mid i \in I\}).$$

Since $I \neq [n]$ by assumption, this ring is free over R. □

We now introduce notation for parity sheaves on $\mathbb{A}^n$. Of course, for each $I \subset [n]$, the closure $\overline{(\mathbb{A}^n)_I}$ is an affine space of dimension $|I|$. In particular, $\overline{(\mathbb{A}^n)_I}$ is smooth, so the constant sheaf is a parity sheaf. We introduce the notation

$$\mathcal{E}(I) = \underline{\Bbbk}_{\overline{(\mathbb{A}^n)_I}}\{|I|\}.$$

This is a ($\hat{T}$-equivariant) perverse parity sheaf. If $i \in I$, there is a canonical morphism

$$\dot{\epsilon}_i : \mathcal{E}(I) \to \mathcal{E}(I \smallsetminus \{i\})\{1\}$$

induced by $*$-restriction and adjunction, and another canonical morphism

$$\dot{\eta}_i : \mathcal{E}(I \smallsetminus \{i\})\{-1\} \to \mathcal{E}(I)$$

induced by !-restriction and adjunction. (We may occasionally write $\mathcal{E}^{\mathbb{A}^n}(I)$, $\dot{\epsilon}_i^{\mathbb{A}^n}$, or $\dot{\eta}_i^{\mathbb{A}^n}$ to avoid confusion with the notation to be introduced in Section 8.)

Lemma 3.2. *Let $I \subset [n]$.*

(1) *If $i \notin I$, then $\dot{\epsilon}_i\dot{\eta}_i = \alpha_i \cdot \mathrm{id} : \mathcal{E}(I) \to \mathcal{E}(I)\{2\}$.*
(2) *If $i \in I$, then $\dot{\eta}_i\dot{\epsilon}_i = \alpha_i \cdot \mathrm{id} : \mathcal{E}(I) \to \mathcal{E}(I)\{2\}$.*

Proof. We first consider the special case where $n = 1$. For part (1), the statement is nonempty only when $I = \varnothing$. In this case, $\mathcal{E}(I)$ is the skyscraper sheaf on the point $\{0\} \subset \mathbb{A}^1$, and the map $\dot{\epsilon}_1\dot{\eta}_1 : \mathcal{E}(I) \to \mathcal{E}(I)\{2\}$ can be regarded as an element of $\mathsf{H}^2_T(\mathrm{pt},\Bbbk) \cong \Bbbk[\alpha_1]$. It is well known that this element can be identified with the T-equivariant Euler class of the vector bundle $\mathbb{A}^1 \to \mathrm{pt}$, and, moreover, that this equivariant Euler class is precisely the character of the T-action on $\mathbb{A}^1$: see, for instance, [AtB, §3]. That is, $\dot{\epsilon}_1\dot{\eta}_1 = \alpha_1 \cdot \mathrm{id}$.

For part (2), consider the map

$$\begin{aligned} \phi = \dot{\epsilon}_1 \circ (-) \circ \dot{\eta}_1 : \mathrm{Hom}(\underline{\Bbbk}_{\mathbb{A}^1}, \underline{\Bbbk}_{\mathbb{A}^1}\{k\}) &\to \mathrm{Hom}(\underline{\Bbbk}_{\mathrm{pt}}, \underline{\Bbbk}_{\mathrm{pt}}\{k+2\}) \\ \text{or} \qquad \mathsf{H}^k_T(\mathbb{A}^1,\Bbbk) &\to \mathsf{H}^{k+2}_T(\mathrm{pt},\Bbbk). \end{aligned}$$

Recall that $\mathsf{H}^\bullet_T(\mathbb{A}^1,\Bbbk)$ is a free $\mathsf{H}^\bullet_T(\mathrm{pt},\Bbbk)$-module of rank 1, and that the map ϕ is a homomorphism of $\mathsf{H}^\bullet_T(\mathrm{pt},\Bbbk)$-modules. By part (1), $\phi(\mathrm{id}) = \alpha_1$, and $\phi(\dot{\eta}_1\dot{\epsilon}_1) = \alpha_1^2$. Since $\mathsf{H}^\bullet_T(\mathrm{pt},\Bbbk)$ is a domain, ϕ is injective, and we deduce that $\dot{\eta}_1\dot{\epsilon}_1 = \alpha_1 \cdot \mathrm{id}$.

The lemma for general n follows from the $n = 1$ case by taking suitable external tensor products. □

Proposition 3.3. *For any $I \subset [n]$, we have*

$$\sum_{i\notin I} \dot{\epsilon}_i \circ \dot{\eta}_i + \sum_{i\in I} \dot{\eta}_i \circ \dot{\epsilon}_i = \xi \cdot \mathrm{id}_{\mathcal{E}(I)}.$$

Proof. This follows immediately from Lemma 3.2 and (3.1). □

4. Direct sums of parity sheaves

This section contains a number of technical lemmas about maps between various direct sums of Tate twists of the parity sheaves $\mathcal{E}(I)$. Most of the calculations in this section involve morphisms of parity sheaves or graded parity sheaves, as discussed in Section 2.1.

Later in this section, we will encounter some formulas involving the indeterminate r. It will be convenient to treat these on the same footing as ordinary morphisms of graded parity sheaves. We adopt the convention that a homogeneous element

$$\phi \in (R^\vee \otimes \underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G}))^0_0 \tag{4.1}$$

of bidegree $(0,0)$ may simply be called a "morphism" $\phi : \mathcal{F} \to \mathcal{G}$.

4.1. First round of direct sums.

For $k \in \{0, 1, \dots, n\}$, let

$$\mathcal{E}^\oplus_k = \bigoplus_{\substack{I \subset [n] \\ |I| = k}} \mathcal{E}(I).$$

For $1 \le k \le n$, define $\epsilon : \mathcal{E}^\oplus_k \to \mathcal{E}^\oplus_{k-1}\{1\}$ by

$$\epsilon = \sum_{\substack{I \subset [n],\ |I|=k \\ i \in I}} (-1)^{|\{j | 1 \le j < i \text{ and } j \notin I\}|} (\dot\epsilon_i : \mathcal{E}(I) \to \mathcal{E}(I \smallsetminus \{i\})\{1\}).$$

Similarly, define $\eta : \mathcal{E}^\oplus_{k-1}\{-1\} \to \mathcal{E}^\oplus_k$ by

$$\eta = \sum_{\substack{I \subset [n],\ |I|=k \\ i \in I}} (-1)^{|\{j | 1 \le j < i \text{ and } j \notin I\}|} (\dot\eta_i : \mathcal{E}(I \smallsetminus \{i\})\{-1\} \to \mathcal{E}(I)).$$

It is sometimes convenient to (implicitly) allow the notation $\mathcal{E}^\oplus_{-1} = \mathcal{E}^\oplus_{n+1} = 0$. We also understand $\epsilon : \mathcal{E}^\oplus_0 \to \mathcal{E}^\oplus_{-1}$ and $\mathcal{E}^\oplus_{n+1} \to \mathcal{E}^\oplus_n$ to be the zero maps, and likewise for η. These conventions make it possible to state the following lemma without worrying about special cases.

Lemma 4.1. (1) *We have $\epsilon \circ \epsilon = 0$ and $\eta \circ \eta = 0$.*
(2) *We have $\epsilon\eta + \eta\epsilon = \xi \cdot \mathrm{id}$.*

Proof. The first assertion follows easily from the formulas. For the second, using Proposition 3.3, we have

$$\epsilon\eta + \eta\epsilon = \sum_{|I|=k} \left(\sum_{i \in I} \dot\eta_i \dot\epsilon_i + \sum_{i \notin I} \dot\epsilon_i \dot\eta_i \right) = \sum_{|I|=k} \xi \cdot \mathrm{id}_{\mathcal{E}(I)} = \xi \cdot \mathrm{id}_{\mathcal{E}^\oplus_k}. \qquad \square$$

Next, for $0 \le i \le n-1$, define an object $\mathrm{E}^\oplus_i$ by

$$\mathrm{E}^\oplus_i = \mathcal{E}^\oplus_i \langle -n+i+1 \rangle \oplus \mathcal{E}^\oplus_i \langle -n+i+3 \rangle \oplus \cdots \oplus \mathcal{E}^\oplus_i \langle n-i-3 \rangle \oplus \mathcal{E}^\oplus_i \langle n-i-1 \rangle.$$

This object has $n-i$ summands. (We may sometimes consider the object $\mathrm{E}^\oplus_n = 0$ as well.) Define a map $N : \mathrm{E}^\oplus_i \to \mathrm{E}^\oplus_i\langle 2 \rangle$ or

$$N : \mathcal{E}^\oplus_i \langle -n+i+1 \rangle \oplus \cdots \oplus \mathcal{E}^\oplus_i \langle n-i-1 \rangle \to \mathcal{E}^\oplus_i \langle -n+i+3 \rangle \oplus \cdots \oplus \mathcal{E}^\oplus_i \langle n-i+1 \rangle$$

by

$$N = \begin{bmatrix} 0 & \mathrm{id} & & & \\ & 0 & \mathrm{id} & & \\ & & \ddots & & \\ & & & 0 & \mathrm{id} \\ & & & & 0 \end{bmatrix}.$$

Also, let $\underline{\epsilon} : \mathrm{E}_i^{\oplus} \to \mathrm{E}_{i-1}^{\oplus}[1]$ and $\underline{\eta} : \mathrm{E}_{i-1}^{\oplus}[-1] \to \mathrm{E}_i^{\oplus}$ be the maps given by

$$\underline{\epsilon} = \begin{bmatrix} \epsilon & & & \\ & \epsilon & & \\ & & \ddots & \\ & & & \epsilon \\ 0 & 0 & \cdots & 0 \end{bmatrix} \qquad \text{and} \qquad \underline{\eta} = \begin{bmatrix} 0 & \eta & & & \\ 0 & & \eta & & \\ \vdots & & & \ddots & \\ 0 & & & & \eta \end{bmatrix}$$

Lemma 4.2. (1) *We have* $\underline{\epsilon} \circ \underline{\epsilon} = 0$ *and* $\underline{\eta} \circ \underline{\eta} = 0$.
(2) *We have* $\underline{\epsilon} \circ N = N \circ \underline{\epsilon}$ *and* $\underline{\eta} \circ N = N \circ \underline{\eta}$.
(3) *We have* $\underline{\epsilon}\underline{\eta} + \underline{\eta}\underline{\epsilon} = \xi \cdot N$.

Proof. This follows easily from Lemma 4.1. □

Remark 4.3. Note that when applied to $\mathrm{E}_{n-1}^{\oplus}$, Lemma 4.2(3) reduces to the equation

$$\begin{bmatrix} 0 \end{bmatrix}\begin{bmatrix} 0 \end{bmatrix} + \begin{bmatrix} 0 & \eta \end{bmatrix}\begin{bmatrix} \epsilon \\ 0 \end{bmatrix} = \xi \cdot \begin{bmatrix} 0 \end{bmatrix}.$$

Thus, Lemma 4.2(3) has nontrivial content only for $\mathrm{E}_i^{\oplus}$ with $i \leq n-2$. One may check this statement relies on Proposition 3.3 only for $|I| \leq n-2$. For the significance of this observation, see Section 9.2.

Define maps

$$\iota^{\lhd} : \mathcal{E}_i^{\oplus}\langle -n+i \rangle \to \mathrm{E}_i^{\oplus}\langle -1 \rangle, \quad \iota^{\rhd} : \mathcal{E}_i^{\oplus}\langle n-i \rangle \to \mathrm{E}_i^{\oplus}\langle 1 \rangle, \quad \text{and} \quad \epsilon^{\rhd} : \mathcal{E}_i^{\oplus}\langle n-i \rangle \to \mathrm{E}_{i-1}^{\oplus}\langle -1 \rangle[1]$$

by

$$\iota^{\lhd} = \begin{bmatrix} \mathrm{id} \\ \mathsf{r} \\ \vdots \\ \mathsf{r}^{n-i-1} \end{bmatrix}, \qquad \iota^{\rhd} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \mathrm{id} \end{bmatrix}, \qquad \epsilon^{\rhd} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \epsilon \end{bmatrix}.$$

(See (4.1) for the interpretation of maps involving the indeterminate r.) We also let

$$\rho = \mathsf{r}^{n-i} : \mathcal{E}_i^{\oplus}\langle -n+i \rangle \to \mathcal{E}_i^{\oplus}\langle n-i \rangle.$$

Lemma 4.4. *We have:*

$$\begin{array}{llll} \epsilon^{\rhd}\epsilon = \underline{\epsilon}\epsilon^{\rhd} = 0 & \iota^{\rhd}\epsilon = \epsilon^{\rhd} & \iota^{\rhd}\eta = \underline{\eta}\iota^{\rhd} & \iota^{\rhd}\rho + N\iota^{\lhd} = \mathsf{r}\iota^{\lhd} \\ \epsilon^{\rhd}\eta + \underline{\eta}\epsilon^{\rhd} = \xi\iota^{\rhd} & \underline{\epsilon}\iota^{\rhd} = N\epsilon^{\rhd} & \mathsf{r}\iota^{\lhd}\eta = \underline{\eta}\iota^{\lhd} & \epsilon^{\rhd}\rho + \underline{\epsilon}\iota^{\lhd} = \iota^{\lhd}\epsilon \end{array}$$

Proof. These equations are all straightforward matrix calculations from the definitions above. □

Lastly, define maps

$$p^{\lhd} : \mathrm{E}_i^{\oplus}\langle -1 \rangle \to \mathcal{E}_i^{\oplus}\langle -n+i \rangle, \quad p^{\rhd} : \mathrm{E}_i^{\oplus}\langle 1 \rangle \to \mathcal{E}_i^{\oplus}\langle n-i \rangle, \quad \text{and} \quad \eta^{\lhd} : \mathrm{E}_{i-1}^{\oplus}\langle 1 \rangle[-1] \to \mathcal{E}_i^{\oplus}\langle -n+i \rangle$$

by

$$p^{\lhd} = \begin{bmatrix} \mathrm{id} & 0 & \cdots & 0 \end{bmatrix}, \qquad p^{\rhd} = \begin{bmatrix} \mathsf{r}^{n-i-1} & \cdots & \mathsf{r} & \mathrm{id} \end{bmatrix}, \qquad \eta^{\lhd} = \begin{bmatrix} \eta & 0 & \cdots & 0 \end{bmatrix}.$$

Lemma 4.5. *We have*

$$\begin{array}{llll} \eta^{\lhd}\underline{\eta} = \eta\eta^{\lhd} = 0 & \eta^{\lhd} = \eta p^{\lhd} & p^{\lhd}\underline{\epsilon} = \epsilon p^{\lhd} & p^{\rhd}N + \rho p^{\lhd} = \mathsf{r}p^{\rhd} \\ \eta^{\lhd}\underline{\epsilon} + \epsilon\eta^{\lhd} = \xi p^{\lhd} & \eta^{\lhd}N = p^{\lhd}\underline{\eta} & p^{\rhd}\underline{\epsilon} = \mathsf{r}\epsilon p^{\rhd} & p^{\rhd}\underline{\eta} + \rho\eta^{\lhd} = \eta p^{\rhd} \end{array}$$

Proof. Similar to Lemma 4.4. □

4.2. **Second round of direct sums.** Let

$$\boldsymbol{\mathcal{E}}^{\lhd} = \mathcal{E}_0^{\oplus}\langle -n\rangle \oplus \mathcal{E}_1^{\oplus}\langle 1-n\rangle \oplus \cdots \oplus \mathcal{E}_n^{\oplus},$$
$$\boldsymbol{\mathcal{E}}^{\rhd} = \mathcal{E}_0^{\oplus}\langle n\rangle \oplus \mathcal{E}_1^{\oplus}\langle n-1\rangle \oplus \cdots \oplus \mathcal{E}_n^{\oplus}.$$

Define $\boldsymbol{\epsilon} : \boldsymbol{\mathcal{E}}^{\lhd} \to \boldsymbol{\mathcal{E}}^{\lhd}[1]$ and $\boldsymbol{\eta} : \boldsymbol{\mathcal{E}}^{\lhd} \to \boldsymbol{\mathcal{E}}^{\lhd}\langle -2\rangle[1]$ by

$$\boldsymbol{\epsilon} = \begin{bmatrix} 0 & \epsilon & & & \\ & 0 & \epsilon & & \\ & & \ddots & & \\ & & & 0 & \epsilon \\ & & & & 0 \end{bmatrix}, \qquad \boldsymbol{\eta} = \begin{bmatrix} 0 & & & & \\ \eta & 0 & & & \\ & & \ddots & & \\ & & \eta & 0 & \\ & & & \eta & 0 \end{bmatrix}.$$

The same matrices also define maps $\boldsymbol{\epsilon} : \boldsymbol{\mathcal{E}}^{\rhd} \to \boldsymbol{\mathcal{E}}^{\rhd}\langle -2\rangle[1]$ and $\boldsymbol{\eta} : \boldsymbol{\mathcal{E}}^{\rhd} \to \boldsymbol{\mathcal{E}}^{\rhd}[1]$. It will be clear from context whether we are working with $\boldsymbol{\mathcal{E}}^{\lhd}$ or $\boldsymbol{\mathcal{E}}^{\rhd}$, so no confusion should result from this overloading of notation.

Lemma 4.6. (1) *We have* $\boldsymbol{\epsilon} \circ \boldsymbol{\epsilon} = 0$ *and* $\boldsymbol{\eta} \circ \boldsymbol{\eta} = 0$.
(2) *We have* $\boldsymbol{\epsilon\eta} + \boldsymbol{\eta\epsilon} = \xi \cdot \mathrm{id}$.

Proof. This follows easily from Lemma 4.1. (Note that in the special case $|I| = 0$, $\epsilon\eta = \xi \cdot \mathrm{id}$, and for $|I| = n$, $\eta\epsilon = \xi \cdot \mathrm{id}$.) □

Next, let

$$\mathbf{E}^{\diamondsuit} = \mathrm{E}_0^{\oplus} \oplus \mathrm{E}_1^{\oplus} \oplus \cdots \oplus \mathrm{E}_{n-2}^{\oplus} \oplus \mathrm{E}_{n-1}^{\oplus}.$$

(Recall that $\mathrm{E}_n^{\oplus} = 0$. The object $\mathbf{E}^{\diamondsuit}$ has only n summands, in contrast with $\boldsymbol{\mathcal{E}}^{\lhd}$ and $\boldsymbol{\mathcal{E}}^{\rhd}$, which have $n+1$ summands each.) Define maps $\boldsymbol{N} : \mathbf{E}^{\diamondsuit} \to \mathbf{E}^{\diamondsuit}\langle 2\rangle$, $\underline{\boldsymbol{\epsilon}} : \mathbf{E}^{\diamondsuit} \to \mathbf{E}^{\diamondsuit}[1]$, and $\underline{\boldsymbol{\eta}} : \mathbf{E}^{\diamondsuit} \to \mathbf{E}^{\diamondsuit}[1]$ by

$$\boldsymbol{N} = \begin{bmatrix} N & & & \\ & N & & \\ & & \ddots & \\ & & & N \end{bmatrix}, \qquad \underline{\boldsymbol{\epsilon}} = \begin{bmatrix} 0 & \underline{\epsilon} & & & \\ & 0 & \underline{\epsilon} & & \\ & & \ddots & & \\ & & & 0 & \underline{\epsilon} \\ & & & & 0 \end{bmatrix}, \qquad \underline{\boldsymbol{\eta}} = \begin{bmatrix} 0 & & & & \\ \underline{\eta} & 0 & & & \\ & & \ddots & & \\ & & \underline{\eta} & 0 & \\ & & & \underline{\eta} & 0 \end{bmatrix}.$$

Lemma 4.7. (1) *We have* $\underline{\boldsymbol{\epsilon}} \circ \underline{\boldsymbol{\epsilon}} = 0$ *and* $\underline{\boldsymbol{\eta}} \circ \underline{\boldsymbol{\eta}} = 0$.
(2) *We have* $\underline{\boldsymbol{\epsilon}} \circ \boldsymbol{N} = \boldsymbol{N} \circ \underline{\boldsymbol{\epsilon}}$ *and* $\underline{\boldsymbol{\eta}} \circ \boldsymbol{N} = \boldsymbol{N} \circ \underline{\boldsymbol{\eta}}$.
(3) *We have* $\underline{\boldsymbol{\epsilon}}\underline{\boldsymbol{\eta}} + \underline{\boldsymbol{\eta}}\underline{\boldsymbol{\epsilon}} = \xi \cdot \boldsymbol{N}$.

Proof. This follows easily from Lemma 4.2. □

Remark 4.8. In particular, the proof of Lemma 4.7(3), like that of Lemma 4.2(3) (see Remark 4.3), relies on Proposition 3.3 only for $|I| \le n-2$.

We now introduce maps $\boldsymbol{\iota}^{\lhd} : \boldsymbol{\mathcal{E}}^{\lhd} \to \mathbf{E}^{\diamondsuit}\langle -1\rangle$, $\boldsymbol{\iota}^{\rhd} : \boldsymbol{\mathcal{E}}^{\rhd} \to \mathbf{E}^{\diamondsuit}\langle 1\rangle$, and $\boldsymbol{\epsilon}^{\rhd} : \boldsymbol{\mathcal{E}}^{\rhd} \to \mathbf{E}^{\diamondsuit}\langle -1\rangle[1]$ as follows:

$$\boldsymbol{\iota}^{\lhd} = \begin{bmatrix} \iota^{\lhd} & & \cdots & & 0 \\ & \iota^{\lhd} & \cdots & & 0 \\ & & \ddots & & \\ & & & \iota^{\lhd} & 0 \end{bmatrix}, \qquad \boldsymbol{\iota}^{\rhd} = \begin{bmatrix} \iota^{\rhd} & & \cdots & & 0 \\ & \iota^{\rhd} & \cdots & & 0 \\ & & \ddots & & \\ & & & \iota^{\rhd} & 0 \end{bmatrix}, \qquad \boldsymbol{\epsilon}^{\rhd} = \begin{bmatrix} 0 & \epsilon^{\rhd} & & & \\ 0 & & \epsilon^{\rhd} & & \\ \vdots & & & \ddots & \\ 0 & & & & \epsilon^{\rhd} \end{bmatrix}.$$

We also let $\boldsymbol{\rho} : \boldsymbol{\mathcal{E}}^{\lhd} \to \boldsymbol{\mathcal{E}}^{\rhd}$ be the map given by

$$\boldsymbol{\rho} = \begin{bmatrix} \mathsf{r}^n & & & & \\ & \mathsf{r}^{n-1} & & & \\ & & \ddots & & \\ & & & \mathsf{r} & \\ & & & & \mathrm{id} \end{bmatrix}.$$

The following lemma is immediate from the definitions.

Lemma 4.9. *We have* $\boldsymbol{\rho\epsilon} = \mathsf{r}\boldsymbol{\epsilon\rho}$ *and* $\boldsymbol{\eta\rho} = \mathsf{r}\boldsymbol{\rho\eta}$.

Lemma 4.10. *We have:*

$$\begin{array}{llll} \boldsymbol{\epsilon}^{\triangleright}\boldsymbol{\epsilon} = \underline{\boldsymbol{\epsilon}}\boldsymbol{\epsilon}^{\triangleright} = 0 & \boldsymbol{\iota}^{\triangleright}\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^{\triangleright} & \boldsymbol{\iota}^{\triangleright}\boldsymbol{\eta} = \underline{\boldsymbol{\eta}}\boldsymbol{\iota}^{\triangleright} & \boldsymbol{\iota}^{\triangleright}\boldsymbol{\rho} + \boldsymbol{N}\boldsymbol{\iota}^{\triangleleft} = \mathsf{r}\boldsymbol{\iota}^{\triangleleft} \\ \boldsymbol{\epsilon}^{\triangleright}\boldsymbol{\eta} + \underline{\boldsymbol{\eta}}\boldsymbol{\epsilon}^{\triangleright} = \xi\boldsymbol{\iota}^{\triangleright} & \underline{\boldsymbol{\epsilon}}\boldsymbol{\iota}^{\triangleright} = \boldsymbol{N}\boldsymbol{\epsilon}^{\triangleright} & \mathsf{r}\boldsymbol{\iota}^{\triangleleft}\boldsymbol{\eta} = \underline{\boldsymbol{\eta}}\boldsymbol{\iota}^{\triangleleft} & \boldsymbol{\epsilon}^{\triangleright}\boldsymbol{\rho} + \underline{\boldsymbol{\epsilon}}\boldsymbol{\iota}^{\triangleleft} = \boldsymbol{\iota}^{\triangleleft}\boldsymbol{\epsilon} \end{array}$$

Proof. These equations are all straightforward matrix calculations using Lemma 4.4. □

We conclude this section with maps

$$\boldsymbol{p}^{\triangleleft} : \mathbf{E}^{\diamondsuit}\langle -1\rangle \to \boldsymbol{\mathcal{E}}^{\triangleleft}, \quad \boldsymbol{p}^{\triangleright} : \mathbf{E}^{\diamondsuit}\langle 1\rangle \to \boldsymbol{\mathcal{E}}^{\triangleright}, \quad \text{and} \quad \boldsymbol{\eta}^{\triangleleft} : \mathbf{E}^{\diamondsuit}\langle 1\rangle[-1] \to \boldsymbol{\mathcal{E}}^{\triangleleft}$$

defined as follows:

$$\boldsymbol{p}^{\triangleleft} = \begin{bmatrix} p^{\triangleleft} & & & \\ & p^{\triangleleft} & & \\ & & \ddots & \\ & & & p^{\triangleleft} \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \qquad \boldsymbol{p}^{\triangleright} = \begin{bmatrix} p^{\triangleright} & & & \\ & p^{\triangleright} & & \\ & & \ddots & \\ & & & p^{\triangleright} \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \qquad \boldsymbol{\eta}^{\triangleleft} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \eta^{\triangleleft} & & & \\ & \eta^{\triangleleft} & & \\ & & \ddots & \\ & & & \eta^{\triangleleft} \end{bmatrix}.$$

Lemma 4.11. *We have*

$$\begin{array}{llll} \boldsymbol{\eta}^{\triangleleft}\underline{\boldsymbol{\eta}} = \boldsymbol{\eta}\boldsymbol{\eta}^{\triangleleft} = 0 & \boldsymbol{\eta}^{\triangleleft} = \boldsymbol{\eta}\boldsymbol{p}^{\triangleleft} & \boldsymbol{p}^{\triangleleft}\underline{\boldsymbol{\epsilon}} = \boldsymbol{\epsilon}\boldsymbol{p}^{\triangleleft} & \boldsymbol{p}^{\triangleright}\boldsymbol{N} + \boldsymbol{\rho}\boldsymbol{p}^{\triangleleft} = \mathsf{r}\boldsymbol{p}^{\triangleright} \\ \boldsymbol{\eta}^{\triangleleft}\underline{\boldsymbol{\epsilon}} + \boldsymbol{\epsilon}\boldsymbol{\eta}^{\triangleleft} = \xi\boldsymbol{p}^{\triangleleft} & \boldsymbol{\eta}^{\triangleleft}\boldsymbol{N} = \boldsymbol{p}^{\triangleleft}\underline{\boldsymbol{\eta}} & \boldsymbol{p}^{\triangleright}\underline{\boldsymbol{\epsilon}} = \mathsf{r}\boldsymbol{\epsilon}\boldsymbol{p}^{\triangleright} & \boldsymbol{p}^{\triangleright}\underline{\boldsymbol{\eta}} + \boldsymbol{\rho}\boldsymbol{\eta}^{\triangleleft} = \boldsymbol{\eta}\boldsymbol{p}^{\triangleright} \end{array}$$

Proof. Similar to Lemma 4.10. □

5. Push-forwards from the generic part

In this section, we will use the objects from Section 4.2 to carry out some sheaf-theoretic computations. We introduce the notation

$$\mathcal{E}_{\eta} = \underline{\Bbbk}_{(\mathbb{A}^n)_{\eta}}\{n\} \in \mathrm{Parity}_{\mathbb{G}_\mathrm{m}}((\mathbb{A}^n)_{\eta}/T, \Bbbk).$$

Proposition 5.1. *In* $D^{\mathrm{mix}}_{\mathbb{G}_\mathrm{m}}(\mathbb{A}^n/T, \Bbbk)$ *or* $D^{\mathrm{mix}}(\mathbb{A}^n/T, \Bbbk)$*, we have*

$$\mathbf{j}_!\mathcal{E}_{\eta} \cong \boldsymbol{\mathcal{E}}^{\triangleleft} \underset{\leftarrow [1]}{\circlearrowleft} \boldsymbol{\epsilon} \qquad \textit{and} \qquad \mathbf{j}_*\mathcal{E}_{\eta} \cong \boldsymbol{\mathcal{E}}^{\triangleright} \underset{\leftarrow [1]}{\circlearrowleft} \boldsymbol{\eta}\,.$$

Remark 5.2. For a version of Proposition 5.1 that looks more like a "classical" chain complex, one can go down to the level of objects from Section 4.1. In this language, $\mathbf{j}_!\mathcal{E}_{\eta}$ is given by

$$\mathcal{E}^{\oplus}_n \xrightarrow[{[1]}]{\epsilon} \mathcal{E}^{\oplus}_{n-1}\langle -1\rangle \xrightarrow[{[1]}]{\epsilon} \cdots \xrightarrow[{[1]}]{\epsilon} \mathcal{E}^{\oplus}_1\langle 1-n\rangle \xrightarrow[{[1]}]{\epsilon} \mathcal{E}^{\oplus}_0\langle -n\rangle\ ,$$

and $\mathbf{j}_*\mathcal{E}_{\eta}$ is given by

$$\mathcal{E}^{\oplus}_0\langle n\rangle \xrightarrow[{[1]}]{\eta} \mathcal{E}^{\oplus}_1\langle n-1\rangle \xrightarrow[{[1]}]{\eta} \cdots \xrightarrow[{[1]}]{\eta} \mathcal{E}^{\oplus}_{n-1}\langle 1\rangle \xrightarrow[{[1]}]{\eta} \mathcal{E}^{\oplus}_n\ .$$

Proof. We will prove the statement for $\mathbf{j}_!\mathcal{E}_{\eta}$. The proof for $\mathbf{j}_*\mathcal{E}_{\eta}$ is similar. We proceed by induction on n. For $n = 1$, we must check that

$$\mathbf{j}_!\mathcal{E}_{\eta} \cong \mathcal{E}([1]) \xrightarrow[{[1]}]{\epsilon} \mathcal{E}(\varnothing)\langle -1\rangle\ .$$

This holds by [A, Examples 3.1 and 6.5] (see also [AR2, §A.1]).

We now turn to the general case. Observe that if $\mathcal{F}$ is a parity sheaf on $\mathbb{A}^{n-1}$ and $\mathcal{G}$ is a parity sheaf on $\mathbb{A}^1$, then $\mathcal{F} \boxtimes \mathcal{G}$ is a parity sheaf on $\mathbb{A}^n$. There is therefore a well-defined functor

$$\boxtimes : D^{\mathrm{mix}}(\mathbb{A}^1/\mathbb{G}_\mathrm{m}, \Bbbk) \times D^{\mathrm{mix}}(\mathbb{A}^{n-1}/\mathbb{G}^{n-1}_\mathrm{m}, \Bbbk) \to D^{\mathrm{mix}}(\mathbb{A}^n/\mathbb{G}^n_\mathrm{m}, \Bbbk).$$

For convenience, let us label strata of $\mathbb{A}^{n-1}$ by subsets of $\{2, \ldots, n\}$. For any $I \subset [n]$, we identify $(\mathbb{A}^n)_I$ with $(\mathbb{A}^1)_{I\cap\{1\}} \times (\mathbb{A}^{n-1})_{I\cap\{2,\ldots,n\}}$. We claim that for any $\mathcal{F} \in D^{\mathrm{mix}}(\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}, \Bbbk)$ and $\mathcal{G} \in D^{\mathrm{mix}}(\mathbb{A}^{n-1}/\mathbb{G}_{\mathrm{m}}^{n-1}, \Bbbk)$ and any $I \subset [n]$, we have

$$(\mathcal{F} \boxtimes \mathcal{G})|_{(\mathbb{A}^n)_I} \cong \mathcal{F}|_{\mathbb{A}^1_{I\cap\{1\}}} \boxtimes \mathcal{G}|_{\mathbb{A}^{n-1}_{I\cap\{2,\ldots,n\}}}. \tag{5.1}$$

It is enough to check this in the special case where $\mathcal{F}$ and $\mathcal{G}$ are parity sheaves. In that case, the assertion is clear.

Let $U' \subset \mathbb{A}^1$ and $U'' \subset \mathbb{A}^{n-1}$ be the open sets consisting of points all of whose coordinates are nonzero, and let $\mathbf{j}' : U' \hookrightarrow \mathbb{A}^1$ and $\mathbf{j}'' : U'' \hookrightarrow \mathbb{A}^{n-1}$ be the inclusion maps. More generally, we will use $'$ and $''$ below to indicate the $\mathbb{A}^1$- or $\mathbb{A}^{n-1}$-analogues of objects defined for $\mathbb{A}^n$. We claim that

$$\mathbf{j}_!\mathcal{E}_\eta \cong (\mathbf{j}'_!\mathcal{E}'_\eta) \boxtimes (\mathbf{j}''_!\mathcal{E}''_\eta). \tag{5.2}$$

Both sides have the same restriction to U, so it is enough to show that the restriction of the right-hand side to any stratum in the complement of U is 0. This follows from (5.1).

By (5.2), $\mathbf{j}_!\mathcal{E}_\eta$ is given by the total complex of the following double complex:

$$\begin{tikzcd}
\mathcal{E}(\varnothing) \boxtimes \mathcal{E}^{\oplus\prime\prime}_{n-1} \arrow[r, "-\mathrm{id}\boxtimes\epsilon''", "{[1]}"'] & \mathcal{E}(\varnothing) \boxtimes \mathcal{E}^{\oplus\prime\prime}_{n-2}\langle -1\rangle \arrow[r, "-\mathrm{id}\boxtimes\epsilon''", "{[1]}"'] & \cdots \arrow[r, "-\mathrm{id}\boxtimes\epsilon''", "{[1]}"'] & \mathcal{E}(\varnothing) \boxtimes \mathcal{E}^{\oplus}_{0}\langle -n+1\rangle \\
\mathcal{E}([1]) \boxtimes \mathcal{E}^{\oplus\prime\prime}_{n-1} \arrow[u, "\epsilon'\boxtimes\mathrm{id}", "{[1]}"'] \arrow[r, "\mathrm{id}\boxtimes\epsilon''", "{[1]}"'] & \mathcal{E}([1]) \boxtimes \mathcal{E}^{\oplus\prime\prime}_{n-2}\langle -1\rangle \arrow[u, "\epsilon'\boxtimes\mathrm{id}", "{[1]}"'] \arrow[r, "\mathrm{id}\boxtimes\epsilon''", "{[1]}"'] & \cdots \arrow[r, "\mathrm{id}\boxtimes\epsilon''", "{[1]}"'] & \mathcal{E}([1]) \boxtimes \mathcal{E}^{\oplus}_{0}\langle -n+1\rangle \arrow[u, "\epsilon'\boxtimes\mathrm{id}", "{[1]}"']
\end{tikzcd} \tag{5.3}$$

To finish the proof, we observe that we may identify

$$\mathcal{E}^{\oplus}_i \cong (\mathcal{E}(\varnothing) \boxtimes \mathcal{E}^{\oplus\prime\prime}_i) \oplus (\mathcal{E}([1]) \boxtimes \mathcal{E}^{\oplus\prime\prime}_{i-1}).$$

Moreover, with respect to this identification, and taking into account the signs in the definition of ϵ, we can write $\epsilon : \mathcal{E}^{\oplus}_i \to \mathcal{E}^{\oplus}_{i-1}\{1\}$ as

$$\epsilon = \begin{bmatrix} -\mathrm{id}\boxtimes\epsilon'' & \epsilon'\boxtimes\mathrm{id} \\ & \mathrm{id}\boxtimes\epsilon'' \end{bmatrix}.$$

We conclude that the total complex of (5.3) can be identified with the complex described in Remark 5.2. $\square$

For the next proposition, we recall that the functor $\mathcal{J} : D^{\mathrm{mix}}_{\mathbb{G}_{\mathrm{m}}}((\mathbb{A}^n)_\eta/T) \to D^{\mathrm{mix}}_{\mathrm{mon}}((\mathbb{A}^n)_\eta/T)$ from [A, §9] is defined by simply regarding an object $(\mathcal{F}, \delta) \in D^{\mathrm{mix}}_{\mathbb{G}_{\mathrm{m}}}((\mathbb{A}^n)_\eta/T)$ as an object of $D^{\mathrm{mix}}_{\mathrm{mon}}((\mathbb{A}^n)_\eta/T)$. (This makes sense because the element $\xi \in R$ acts by 0 on all $\underline{\mathrm{Hom}}$-spaces of parity sheaves on $(\mathbb{A}^n)_\eta$.)

Proposition 5.3. *In $D^{\mathrm{mix}}_{\mathrm{mon}}(\mathbb{A}^n/T, \Bbbk)$, we have*

$$\mathbf{j}_!\mathcal{J}(\mathcal{E}_\eta) \cong \begin{tikzcd} \boldsymbol{\mathcal{E}}^{\lhd} \arrow[loop right, "{[1]}"', "\boldsymbol{\epsilon}+\mathsf{r}\boldsymbol{\eta}"] \end{tikzcd} \qquad \text{and} \qquad \mathbf{j}_*\mathcal{J}(\mathcal{E}_\eta) \cong \begin{tikzcd} \boldsymbol{\mathcal{E}}^{\rhd} \arrow[loop right, "{[1]}"', "\boldsymbol{\eta}+\mathsf{r}\boldsymbol{\epsilon}"] \end{tikzcd}.$$

Proof. We will prove the statement for $\mathbf{j}_!\mathcal{J}(\mathcal{E}_\eta)$. The proof for $\mathbf{j}_*\mathcal{J}(\mathcal{E}_\eta)$ is similar. Let

$$\delta = \boldsymbol{\epsilon} + \mathsf{r}\boldsymbol{\eta} : \boldsymbol{\mathcal{E}}^{\lhd} \to \boldsymbol{\mathcal{E}}^{\lhd}[1],$$

and let

$$\mathcal{F} = (\boldsymbol{\mathcal{E}}^{\lhd}, \delta) = \begin{tikzcd} \boldsymbol{\mathcal{E}}^{\lhd} \arrow[loop right, "{[1]}"', "\boldsymbol{\epsilon}+\mathsf{r}\boldsymbol{\eta}"] \end{tikzcd} \in D^{\mathrm{mix}}_{\mathrm{mon}}(\mathbb{A}^n/T, \Bbbk).$$

(By Lemma 4.6, we have $\delta^2 = \mathsf{r}\xi \cdot \mathrm{id}$, so this is indeed a well-defined object of $D^{\mathrm{mix}}_{\mathrm{mon}}(\mathbb{A}^n/T, \Bbbk)$.) Consider the morphism $\mathsf{r} \cdot \mathrm{id} : \mathcal{F}\langle -2\rangle \to \mathcal{F}$. By construction, the cone of this map is the object $\mathcal{G}$ given by

$$\mathcal{G} = \begin{tikzcd} \boldsymbol{\mathcal{E}}^{\triangleleft} \arrow[loop right, "{\boldsymbol{\epsilon}+\mathsf{r}\boldsymbol{\eta}}", "{[1]}"'] \\ \boldsymbol{\mathcal{E}}^{\triangleleft}\langle -2\rangle[1] \arrow[u, "{[1]}", "\mathsf{r}"'] \arrow[loop right, "{-\boldsymbol{\epsilon}-\mathsf{r}\boldsymbol{\eta}}", "{[1]}"'] \end{tikzcd}.$$

On the other hand, consider $\mathbf{j}_!\mathcal{E}_\eta \in D^{\mathrm{mix}}(\mathbb{A}^n/T, \Bbbk)$. Using Proposition 5.1 and the definition of Mon, we find that $\mathrm{Mon}(\mathbf{j}_!\mathcal{E}_\eta)$ is given by

$$\mathrm{Mon}(\mathbf{j}_!\mathcal{E}_\eta) = \begin{tikzcd} \boldsymbol{\mathcal{E}}^{\triangleleft} \arrow[loop right, "\boldsymbol{\epsilon}", "{[1]}"'] \arrow[d, shift right, "{[1]}"', "\xi"] \\ \boldsymbol{\mathcal{E}}^{\triangleleft}\langle -2\rangle[1] \arrow[u, shift right, "{[1]}"', "\mathsf{r}"] \arrow[loop right, "-\boldsymbol{\epsilon}", "{[1]}"'] \end{tikzcd}.$$

Define a chain map $f : \mathcal{G} \to \mathrm{Mon}(\mathbf{j}_!\mathcal{E}_\eta)$ by the diagram

$$\begin{tikzcd} \boldsymbol{\mathcal{E}}^{\triangleleft} \arrow[loop left, "{\boldsymbol{\epsilon}+\mathsf{r}\boldsymbol{\eta}}", "{[1]}"'] \arrow[r, "\mathrm{id}"] \arrow[dr, "\boldsymbol{\eta}"] & \boldsymbol{\mathcal{E}}^{\triangleleft} \arrow[loop right, "\boldsymbol{\epsilon}", "{[1]}"'] \arrow[d, shift right, "{[1]}"', "\xi"] \\ \boldsymbol{\mathcal{E}}^{\triangleleft}\langle -2\rangle[1] \arrow[loop left, "{-\boldsymbol{\epsilon}-\mathsf{r}\boldsymbol{\eta}}", "{[1]}"'] \arrow[u, "{[1]}", "\mathsf{r}"'] \arrow[r, "\mathrm{id}"] & \boldsymbol{\mathcal{E}}^{\triangleleft}\langle -2\rangle[1] \arrow[u, shift right, "{[1]}"', "\mathsf{r}"] \arrow[loop right, "-\boldsymbol{\epsilon}", "{[1]}"'] \end{tikzcd}.$$

To check that this really is a chain map, we must show that f commutes with the differentials: in other words, we must verify that

$$\begin{bmatrix} \mathrm{id} & \\ \boldsymbol{\eta} & \mathrm{id} \end{bmatrix} \begin{bmatrix} \boldsymbol{\epsilon}+\mathsf{r}\boldsymbol{\eta} & \mathsf{r} \\ & -\boldsymbol{\epsilon}-\mathsf{r}\boldsymbol{\eta} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\epsilon} & \mathsf{r} \\ \xi & -\boldsymbol{\epsilon} \end{bmatrix} \begin{bmatrix} \mathrm{id} & \\ \boldsymbol{\eta} & \mathrm{id} \end{bmatrix}.$$

This holds by Lemma 4.6.

Next, define $\bar{f} : \mathrm{Mon}(\mathbf{j}_!\mathcal{E}_\eta) \to \mathcal{G}$ by the diagram

$$\begin{tikzcd} \boldsymbol{\mathcal{E}}^{\triangleleft} \arrow[loop left, "\boldsymbol{\epsilon}", "{[1]}"'] \arrow[d, shift right, "{[1]}"', "\xi"] \arrow[r, "\mathrm{id}"] \arrow[dr, "-\boldsymbol{\eta}"] & \boldsymbol{\mathcal{E}}^{\triangleleft} \arrow[loop right, "{\boldsymbol{\epsilon}+\mathsf{r}\boldsymbol{\eta}}", "{[1]}"'] \\ \boldsymbol{\mathcal{E}}^{\triangleleft}\langle -2\rangle[1] \arrow[loop left, "-\boldsymbol{\epsilon}", "{[1]}"'] \arrow[u, shift right, "{[1]}"', "\mathsf{r}"] \arrow[r, "\mathrm{id}"] & \boldsymbol{\mathcal{E}}^{\triangleleft}\langle -2\rangle[1] \arrow[u, "{[1]}", "\mathsf{r}"'] \arrow[loop right, "{-\boldsymbol{\epsilon}-\mathsf{r}\boldsymbol{\eta}}", "{[1]}"'] \end{tikzcd}.$$

A similar calculation shows that $\bar{f}$ is again a chain map. Moreover, $f \circ \bar{f}$ and $\bar{f} \circ f$ are both identity maps.

In other words, f and $\bar{f}$ show us that $\mathcal{G} \cong \mathrm{Mon}(\mathbf{j}_!\mathcal{E}_\eta)$ in $D^{\mathrm{mix}}_{\mathrm{mon}}(\mathbb{A}^n/T, \Bbbk)$. From the definition of $\mathcal{G}$, we see that there is a distinguished triangle

$$\mathcal{F}\langle -2\rangle \xrightarrow{\mathsf{r}} \mathcal{F} \to \mathrm{Mon}(\mathbf{j}_!\mathcal{E}_\eta) \to .$$

Now apply $\mathbf{i}^*$ to this triangle. Since Mon commutes with the recollement structure by [A, Proposition 6.7], we have

$$\mathbf{i}^*\mathrm{Mon}(\mathbf{j}_!\mathcal{E}_\eta) \cong \mathrm{Mon}(\mathbf{i}^*\mathbf{j}_!\mathcal{E}_\eta) = 0.$$

It follows that $\mathsf{r} : \mathbf{i}^*\mathcal{F}\langle -2\rangle \to \mathbf{i}^*\mathcal{F}$ is an isomorphism in $D^{\mathrm{mix}}_{\mathrm{mon}}((\mathbb{A}^n)_0/T, \Bbbk)$. On the other hand, since $D^{\mathrm{mix}}_{\mathrm{mon}}((\mathbb{A}^n)_0/T, \Bbbk) \cong D^{\mathrm{mix}}((\mathbb{A}^n)_0/T, \Bbbk)$ (see (2.3) or [A, Theorem 8.4]), it is also a nilpotent map, as in [A, Remarks 3.3 and 5.6]. Since r is both nilpotent and an isomorphism, we must have

$$\mathbf{i}^*\mathcal{F} = 0. \tag{5.4}$$

It is clear by construction that

$$\mathbf{j}^*\mathcal{F} \cong \mathcal{J}(\mathcal{E}_\eta). \tag{5.5}$$

The two conditions (5.4) and (5.5) uniquely characterize $\mathbf{j}_!\mathcal{J}(\mathcal{E}_\eta)$, so we are done. ☐

Proposition 5.4. *The object* $\mathbf{i}_*\mathbf{i}^*\mathbf{j}_*\mathcal{J}(\mathcal{E}_\eta) \in D^{\mathrm{mix}}_{\mathrm{mon}}(\mathbb{A}^n/T, \Bbbk)$ *is given by*

$$\begin{tikzcd} \boldsymbol{\mathcal{E}}^{\triangleright} \arrow[loop right, "{[1]}", "\boldsymbol{\eta}+\mathsf{r}\boldsymbol{\epsilon}"'] \\ \boldsymbol{\mathcal{E}}^{\triangleleft}[1] \arrow[u, "{[1]}"', "\boldsymbol{\rho}"] \arrow[loop right, "{[1]}", "-\boldsymbol{\epsilon}-\mathsf{r}\boldsymbol{\eta}"'] \end{tikzcd}$$

Proof. The proposition is equivalent to the claim that the diagram above depicts the cone of the canonical map $\mathbf{j}_!\mathcal{J}(\mathcal{E}_\eta) \to \mathbf{j}_*\mathcal{J}(\mathcal{E}_\eta)$. It follows from Lemma 4.9 that the matrix $\boldsymbol{\rho}$ defines a chain map $\phi : \mathbf{j}_!\mathcal{J}(\mathcal{E}_\eta) \to \mathbf{j}_*\mathcal{J}(\mathcal{E}_\eta)$. The object depicted above is evidently the cone of ϕ. Since $\phi|_{(\mathbb{A}^n)_\eta}$ is the identity map $\mathcal{J}(\mathcal{E}_\eta) \to \mathcal{J}(\mathcal{E}_\eta)$, ϕ must be the canonical map $\mathbf{j}_!\mathcal{J}(\mathcal{E}_\eta) \to \mathbf{j}_*\mathcal{J}(\mathcal{E}_\eta)$. ☐

6. The nearby cycles sheaf for affine space

Let $\boldsymbol{\mathcal{Z}} \in D^{\mathrm{mix}}((\mathbb{A}^n)_0/T, \Bbbk)$ be the object given by

$$\boldsymbol{\mathcal{Z}} = \begin{tikzcd} \mathbf{E}^{\diamondsuit} \arrow[loop right, "{[1]}", "\underline{\boldsymbol{\epsilon}}+\underline{\boldsymbol{\eta}}-\bar{\xi}\boldsymbol{N}"'] \end{tikzcd}.$$

To check that $\boldsymbol{\mathcal{Z}}$ is a well-defined object of $D^{\mathrm{mix}}((\mathbb{A}^n)_0/T, \Bbbk)$, we must show that

$$(\underline{\boldsymbol{\epsilon}} + \underline{\boldsymbol{\eta}} - \bar{\xi}\boldsymbol{N})^2 + \kappa(\underline{\boldsymbol{\epsilon}} + \underline{\boldsymbol{\eta}} - \bar{\xi}\boldsymbol{N}) = 0.$$

To prove this, recall that $\bar{\xi}^2 = 0$, and that $\bar{\xi}$ supercommutes with all elements of $\Lambda \otimes \underline{\mathrm{End}}(\boldsymbol{\mathcal{Z}})$. Recall also that $\kappa(\underline{\boldsymbol{\epsilon}} + \underline{\boldsymbol{\eta}} - \bar{\xi}\boldsymbol{N}) = -\xi\boldsymbol{N}$ (see Section 2.2 or [A, §3] for the definition). Using Lemma 4.7, we find that

$$\begin{aligned}(\underline{\boldsymbol{\epsilon}} + \underline{\boldsymbol{\eta}} - \bar{\xi}\boldsymbol{N})^2 &+ \kappa(\underline{\boldsymbol{\epsilon}} + \underline{\boldsymbol{\eta}} - \bar{\xi}\boldsymbol{N}) \\ &= \underline{\boldsymbol{\epsilon}}^2 + \underline{\boldsymbol{\eta}}^2 + \bar{\xi}^2\boldsymbol{N}^2 + \underline{\boldsymbol{\epsilon}}\underline{\boldsymbol{\eta}} + \bar{\xi}\underline{\boldsymbol{\epsilon}}\boldsymbol{N} + \underline{\boldsymbol{\eta}}\underline{\boldsymbol{\epsilon}} + \bar{\xi}\underline{\boldsymbol{\eta}}\boldsymbol{N} - \bar{\xi}\boldsymbol{N}\underline{\boldsymbol{\epsilon}} - \bar{\xi}\boldsymbol{N}\underline{\boldsymbol{\eta}} - \xi\boldsymbol{N} = 0,\end{aligned}$$

as desired.

Remark 6.1. Observe that (as in Remarks 4.3 and 4.8) the fact that $\boldsymbol{\mathcal{Z}}$ is well-defined relies on Proposition 3.3 only for $|I| \le n-2$.

We will also need to work with the object $\mathrm{Mon}(\boldsymbol{\mathcal{Z}}) \in D^{\mathrm{mix}}_{\mathrm{mon}}((\mathbb{A}^n)_0/T, \Bbbk)$. Applying the explicit formula from [A, Eq. (5.5)], we obtain

$$\mathrm{Mon}(\boldsymbol{\mathcal{Z}}) = \begin{tikzcd} \mathbf{E}^{\diamondsuit} \arrow[loop right, "{[1]}", "\underline{\boldsymbol{\epsilon}}+\underline{\boldsymbol{\eta}}"'] \arrow[d, shift right, "{[1]}", "\xi"'] \\ \mathbf{E}^{\diamondsuit}\langle -2\rangle[1] \arrow[u, shift right, "{[1]}"', "\mathsf{r}-\boldsymbol{N}"] \arrow[loop right, "{[1]}", "-\underline{\boldsymbol{\epsilon}}-\underline{\boldsymbol{\eta}}"'] \end{tikzcd}$$

Proposition 6.2. *There is a chain map* $\boldsymbol{\iota}:\mathbf{i}_*\mathbf{i}^*\mathbf{j}_*\mathcal{J}(\mathcal{E}_\eta)\to\mathrm{Mon}(\boldsymbol{\mathcal{Z}})\langle 1\rangle$ *given by*

$$\begin{tikzcd}
\boldsymbol{\mathcal{E}}^{\triangleright} \arrow[loop left, "{[1]}", "\boldsymbol{\eta}+\mathsf{r}\boldsymbol{\epsilon}"'] \arrow[r, "\boldsymbol{\iota}^{\triangleright}"] \arrow[dr, "\boldsymbol{\epsilon}^{\triangleright}"] & \mathbf{E}^{\diamondsuit}\langle 1\rangle \arrow[loop right, "{[1]}"', "\underline{\boldsymbol{\epsilon}}+\underline{\boldsymbol{\eta}}"] \arrow[d, shift right, "\xi"', "{[1]}"] \\
\boldsymbol{\mathcal{E}}^{\triangleleft}[1] \arrow[loop left, "{[1]}", "-\boldsymbol{\epsilon}-\mathsf{r}\boldsymbol{\eta}"'] \arrow[u, "\boldsymbol{\rho}", "{[1]}"'] \arrow[r, "\boldsymbol{\iota}^{\triangleleft}"] & \mathbf{E}^{\diamondsuit}\langle -1\rangle[1] \arrow[loop right, "{[1]}"', "-\underline{\boldsymbol{\epsilon}}-\underline{\boldsymbol{\eta}}"] \arrow[u, shift right, "\mathsf{r}-\boldsymbol{N}"', "{[1]}"]
\end{tikzcd}$$

Proof. Let δ_1 be the differential of $\mathbf{i}_*\mathbf{i}^*\mathbf{j}_*\mathcal{J}(\mathcal{E}_\eta)$, and let δ_2 be the differential of $\mathrm{Mon}(\boldsymbol{\mathcal{Z}})\langle 1\rangle$. Regarding both these objects as direct sums of two terms as depicted above, these differentials and the map $\boldsymbol{\iota}$ are given by

$$\delta_1=\begin{bmatrix}\boldsymbol{\eta}+\mathsf{r}\boldsymbol{\epsilon} & \boldsymbol{\rho}\\ & -\boldsymbol{\epsilon}-\mathsf{r}\boldsymbol{\eta}\end{bmatrix},\qquad \delta_2=\begin{bmatrix}\underline{\boldsymbol{\epsilon}}+\underline{\boldsymbol{\eta}} & \mathsf{r}-\boldsymbol{N}\\ \xi & -\underline{\boldsymbol{\epsilon}}-\underline{\boldsymbol{\eta}}\end{bmatrix},\qquad \boldsymbol{\iota}=\begin{bmatrix}\boldsymbol{\iota}^{\triangleright} & \\ \boldsymbol{\epsilon}^{\triangleright} & \boldsymbol{\iota}^{\triangleleft}\end{bmatrix}.$$

We must show that $\boldsymbol{\iota}\delta_1=\delta_2\boldsymbol{\iota}$, or in other words, that

$$\begin{bmatrix}\boldsymbol{\iota}^{\triangleright}\boldsymbol{\eta}+\mathsf{r}\boldsymbol{\iota}^{\triangleright}\boldsymbol{\epsilon} & \boldsymbol{\iota}^{\triangleright}\boldsymbol{\rho}\\ \boldsymbol{\epsilon}^{\triangleright}\boldsymbol{\eta}+\mathsf{r}\boldsymbol{\epsilon}^{\triangleright}\boldsymbol{\epsilon} & \boldsymbol{\epsilon}^{\triangleright}\boldsymbol{\rho}-\boldsymbol{\iota}^{\triangleleft}\boldsymbol{\epsilon}-\mathsf{r}\boldsymbol{\iota}^{\triangleleft}\boldsymbol{\eta}\end{bmatrix}=\begin{bmatrix}\underline{\boldsymbol{\epsilon}}\boldsymbol{\iota}^{\triangleright}+\underline{\boldsymbol{\eta}}\boldsymbol{\iota}^{\triangleright}+\mathsf{r}\boldsymbol{\epsilon}^{\triangleright}-\boldsymbol{N}\boldsymbol{\epsilon}^{\triangleright} & \mathsf{r}\boldsymbol{\iota}^{\triangleleft}-\boldsymbol{N}\boldsymbol{\iota}^{\triangleleft}\\ \xi\boldsymbol{\iota}^{\triangleright}-\underline{\boldsymbol{\epsilon}}\boldsymbol{\epsilon}^{\triangleright}-\underline{\boldsymbol{\eta}}\boldsymbol{\epsilon}^{\triangleright} & -\underline{\boldsymbol{\epsilon}}\boldsymbol{\iota}^{\triangleleft}-\underline{\boldsymbol{\eta}}\boldsymbol{\iota}^{\triangleleft}\end{bmatrix}.$$

This follows from Lemma 4.4. ☐

Proposition 6.3. *There is a chain map* $\boldsymbol{p}:\mathrm{Mon}(\boldsymbol{\mathcal{Z}})\langle 1\rangle\to\mathbf{i}_*\mathbf{i}^*\mathbf{j}_*\mathcal{J}(\mathcal{E}_\eta)$ *given by*

$$\begin{tikzcd}
\mathbf{E}^{\diamondsuit}\langle 1\rangle \arrow[loop left, "{[1]}", "\underline{\boldsymbol{\epsilon}}+\underline{\boldsymbol{\eta}}"'] \arrow[r, "\boldsymbol{p}^{\triangleright}"] \arrow[dr, "-\boldsymbol{\eta}^{\triangleleft}"] \arrow[d, shift right, "\xi"', "{[1]}"] & \boldsymbol{\mathcal{E}}^{\triangleright} \arrow[loop right, "{[1]}"', "\boldsymbol{\eta}+\mathsf{r}\boldsymbol{\epsilon}"] \\
\mathbf{E}^{\diamondsuit}\langle -1\rangle[1] \arrow[loop left, "{[1]}", "-\underline{\boldsymbol{\epsilon}}-\underline{\boldsymbol{\eta}}"'] \arrow[u, shift right, "\mathsf{r}-\boldsymbol{N}"', "{[1]}"] \arrow[r, "\boldsymbol{p}^{\triangleleft}"] & \boldsymbol{\mathcal{E}}^{\triangleleft}[1] \arrow[loop right, "{[1]}"', "-\boldsymbol{\epsilon}-\mathsf{r}\boldsymbol{\eta}"] \arrow[u, "\boldsymbol{\rho}", "{[1]}"']
\end{tikzcd}$$

Proof. Let δ_1 and δ_2 be as in the proof of Proposition 6.2. We must show that $\boldsymbol{p}\delta_2=\delta_1\boldsymbol{p}$, where

$$\boldsymbol{p}=\begin{bmatrix}\boldsymbol{p}^{\triangleright} & \\ -\boldsymbol{\eta}^{\triangleleft} & \boldsymbol{p}^{\triangleleft}\end{bmatrix}.$$

In other words, we must show that

$$\begin{bmatrix}\boldsymbol{p}^{\triangleright}\underline{\boldsymbol{\epsilon}}+\boldsymbol{p}^{\triangleright}\underline{\boldsymbol{\eta}} & \mathsf{r}\boldsymbol{p}^{\triangleright}-\boldsymbol{p}^{\triangleright}\boldsymbol{N}\\ -\boldsymbol{\eta}^{\triangleleft}\underline{\boldsymbol{\epsilon}}-\boldsymbol{\eta}^{\triangleleft}\underline{\boldsymbol{\eta}}+\xi\boldsymbol{p}^{\triangleleft} & -\mathsf{r}\boldsymbol{\eta}^{\triangleleft}+\boldsymbol{\eta}^{\triangleleft}\boldsymbol{N}-\boldsymbol{p}^{\triangleleft}\underline{\boldsymbol{\epsilon}}-\boldsymbol{p}^{\triangleleft}\underline{\boldsymbol{\eta}}\end{bmatrix}=\begin{bmatrix}\boldsymbol{\eta}\boldsymbol{p}^{\triangleright}+\mathsf{r}\boldsymbol{\epsilon}\boldsymbol{p}^{\triangleright}-\boldsymbol{\rho}\boldsymbol{\eta}^{\triangleleft} & \boldsymbol{\rho}\boldsymbol{p}^{\triangleleft}\\ \boldsymbol{\epsilon}\boldsymbol{\eta}^{\triangleleft}+\mathsf{r}\boldsymbol{\eta}\boldsymbol{\eta}^{\triangleleft} & -\boldsymbol{\epsilon}\boldsymbol{p}^{\triangleleft}-\mathsf{r}\boldsymbol{\eta}\boldsymbol{p}^{\triangleleft}\end{bmatrix}.$$

This follows from Lemma 4.5. ☐

Lemma 6.4. *There is a null-homotopic chain map* $\mathrm{Mon}(\boldsymbol{\mathcal{Z}})\to\mathrm{Mon}(\boldsymbol{\mathcal{Z}})\langle 2\rangle$ *given by*

$$\begin{bmatrix}\mathsf{r}-\boldsymbol{N} & \\ & \mathsf{r}-\boldsymbol{N}\end{bmatrix}:\mathrm{Mon}(\boldsymbol{\mathcal{Z}})\to\mathrm{Mon}(\boldsymbol{\mathcal{Z}})\langle 2\rangle.$$

Proof. We continue to let δ_2 be the differential of $\mathrm{Mon}(\boldsymbol{\mathcal{Z}})$, as in the proof of Proposition 6.2. It is easy to check that $\left[\begin{smallmatrix}\mathsf{r}-\boldsymbol{N} & \\ & \mathsf{r}-\boldsymbol{N}\end{smallmatrix}\right]$ commutes with δ_2, so it is a chain map. Next, consider the map $h=\left[\begin{smallmatrix}0 & 0\\ \mathrm{id} & 0\end{smallmatrix}\right]$, shown as a dotted line below.

$$\begin{tikzcd}
\mathbf{E}^{\diamondsuit} \arrow[loop left, "{[1]}", "\underline{\boldsymbol{\epsilon}}+\underline{\boldsymbol{\eta}}"'] \arrow[r, "\mathsf{r}-\boldsymbol{N}"] \arrow[dr, dashed, "h=\mathrm{id}"] \arrow[d, shift right, "\xi"', "{[1]}"] & \mathbf{E}^{\diamondsuit}\langle 2\rangle \arrow[loop right, "{[1]}"', "\underline{\boldsymbol{\epsilon}}+\underline{\boldsymbol{\eta}}"] \arrow[d, shift right, "\xi"', "{[1]}"] \\
\mathbf{E}^{\diamondsuit}\langle -2\rangle[1] \arrow[loop left, "{[1]}", "-\underline{\boldsymbol{\epsilon}}-\underline{\boldsymbol{\eta}}"'] \arrow[u, shift right, "\mathsf{r}-\boldsymbol{N}"', "{[1]}"] \arrow[r, "\mathsf{r}-\boldsymbol{N}"] & \mathbf{E}^{\diamondsuit}[1] \arrow[loop right, "{[1]}"', "-\underline{\boldsymbol{\epsilon}}-\underline{\boldsymbol{\eta}}"] \arrow[u, shift right, "\mathsf{r}-\boldsymbol{N}"', "{[1]}"]
\end{tikzcd}$$

We then have

$$\delta_2 h + h\delta_2 = \begin{bmatrix} \underline{\boldsymbol{\epsilon}}+\underline{\boldsymbol{\eta}} & \mathsf{r}-\boldsymbol{N} \\ \xi & -\underline{\boldsymbol{\epsilon}}-\underline{\boldsymbol{\eta}} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ \mathrm{id} & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \mathrm{id} & 0 \end{bmatrix} \begin{bmatrix} \underline{\boldsymbol{\epsilon}}+\underline{\boldsymbol{\eta}} & \mathsf{r}-\boldsymbol{N} \\ \xi & -\underline{\boldsymbol{\epsilon}}-\underline{\boldsymbol{\eta}} \end{bmatrix} = \begin{bmatrix} \mathsf{r}-\boldsymbol{N} & 0 \\ 0 & \mathsf{r}-\boldsymbol{N} \end{bmatrix},$$

so our chain map is null-homotopic, as claimed. □

Theorem 6.5. *On $\mathbb{A}^n$, we have $\Psi_f(\mathcal{E}_\eta) \cong \boldsymbol{\mathcal{Z}}\langle -1\rangle$. In particular, $\Psi_f(\mathcal{E}_\eta)$ is a perverse sheaf. The monodromy endomorphism is given by the map $\boldsymbol{N} : \boldsymbol{\mathcal{Z}} \to \boldsymbol{\mathcal{Z}}\langle 2\rangle$.*

Proof. For the first assertion, since Mon is fully faithful, it is enough to show that

$$\mathrm{Mon}(\Psi_f(\mathcal{E}_\eta))\langle 2\rangle \cong \mathrm{Mon}(\boldsymbol{\mathcal{Z}})\langle 1\rangle.$$

The object $\mathrm{Mon}(\Psi_f(\mathcal{E}_\eta))\langle 2\rangle \cong \mathbf{i}_*\mathbf{i}^*\mathbf{j}_*\mathcal{J}(\mathcal{E}_\eta)$ has been described in Proposition 5.4, and in Propositions 6.2 and 6.3, we have constructed two maps

$$\mathrm{Mon}(\Psi_f(\mathcal{E}_\eta))\langle 2\rangle \underset{\boldsymbol{p}}{\overset{\boldsymbol{\iota}}{\rightleftarrows}} \mathrm{Mon}(\boldsymbol{\mathcal{Z}})\langle 1\rangle\ .$$

It remains to show that $\boldsymbol{\iota}$ and $\boldsymbol{p}$ are isomorphisms in $D^{\mathrm{mix}}_{\mathrm{mon}}(\mathbb{A}^n/T, \Bbbk)$.

Let $q^{\lhd} : \boldsymbol{\mathcal{E}}^{\lhd} \to \boldsymbol{\mathcal{E}}^{\lhd}$, $q^{\rhd} : \boldsymbol{\mathcal{E}}^{\rhd} \to \boldsymbol{\mathcal{E}}^{\rhd}$, and $g : \boldsymbol{\mathcal{E}}^{\rhd} \to \boldsymbol{\mathcal{E}}^{\lhd}$ all be defined by the matrix

$$\begin{bmatrix} 0 & & & \\ & \ddots & & \\ & & 0 & \\ & & & \mathrm{id} \end{bmatrix}.$$

In other words, all three of these maps can be thought of as "projection onto the summand $\mathcal{E}_n^{\oplus}$." It is straightforward to check the following equalities:

$$\begin{aligned} q^{\lhd} &= \mathrm{id} - \boldsymbol{p}^{\lhd}\boldsymbol{\iota}^{\lhd} & \qquad g\boldsymbol{\rho} &= q^{\lhd} & \qquad g\boldsymbol{\epsilon} &= 0 \\ q^{\rhd} &= \mathrm{id} - \boldsymbol{p}^{\rhd}\boldsymbol{\iota}^{\rhd} & \qquad \boldsymbol{\rho} g &= q^{\rhd} & \qquad \boldsymbol{\eta} g &= 0 \end{aligned}$$

As in Proposition 6.2, let δ_1 be the differential of $\mathbf{i}_*\mathbf{i}^*\mathbf{j}_*\mathcal{J}(\mathcal{E}_\eta)$, and let δ_2 be the differential of $\mathrm{Mon}(\boldsymbol{\mathcal{Z}})\langle 1\rangle$. We claim that

$$\delta_1 \begin{bmatrix} 0 & 0 \\ g & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ g & 0 \end{bmatrix} \delta_1 = \mathrm{id} - \boldsymbol{p}\boldsymbol{\iota}. \tag{6.1}$$

Indeed, the left-hand side is given by

$$\begin{bmatrix} \boldsymbol{\eta}+\mathsf{r}\boldsymbol{\epsilon} & \boldsymbol{\rho} \\ 0 & -\boldsymbol{\epsilon}-\mathsf{r}\boldsymbol{\eta} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ g & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ g & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\eta}+\mathsf{r}\boldsymbol{\epsilon} & \boldsymbol{\rho} \\ 0 & -\boldsymbol{\epsilon}-\mathsf{r}\boldsymbol{\eta} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\rho} g & 0 \\ g\boldsymbol{\eta}+\mathsf{r}g\boldsymbol{\epsilon}-\boldsymbol{\epsilon}g-\mathsf{r}\boldsymbol{\eta}g & g\boldsymbol{\rho} \end{bmatrix} = \begin{bmatrix} q^{\rhd} & 0 \\ g\boldsymbol{\eta}-\boldsymbol{\epsilon}g & q^{\lhd} \end{bmatrix},$$

while the right-hand side is given by

$$\mathrm{id} - \boldsymbol{p}\boldsymbol{\iota} = \begin{bmatrix} \mathrm{id}-\boldsymbol{p}^{\rhd}\boldsymbol{\iota}^{\rhd} & 0 \\ \boldsymbol{\eta}^{\lhd}\boldsymbol{\iota}^{\rhd}-\boldsymbol{p}^{\lhd}\boldsymbol{\epsilon}^{\rhd} & \mathrm{id}-\boldsymbol{p}^{\lhd}\boldsymbol{\iota}^{\lhd} \end{bmatrix} = \begin{bmatrix} q^{\rhd} & 0 \\ \boldsymbol{\eta}^{\lhd}\boldsymbol{\iota}^{\rhd}-\boldsymbol{p}^{\lhd}\boldsymbol{\epsilon}^{\rhd} & q^{\lhd} \end{bmatrix}.$$

To finish the proof of (6.1), one must show that $g\boldsymbol{\eta} - \boldsymbol{\epsilon}g = \boldsymbol{\eta}^{\lhd}\boldsymbol{\iota}^{\rhd} - \boldsymbol{p}^{\lhd}\boldsymbol{\epsilon}^{\rhd}$. This is again a routine matrix calculation.

Next, let $h : \mathrm{E}_i^{\oplus}\langle 1\rangle \to \mathrm{E}_i^{\oplus}\langle -1\rangle$ be the map given by

$$h = \begin{bmatrix} 0 & & & & & & \\ \mathrm{id} & 0 & & & & & \\ \mathsf{r} & \mathrm{id} & 0 & & & & \\ \mathsf{r}^2 & \mathsf{r} & \mathrm{id} & 0 & & & \\ \vdots & & & & \ddots & & \\ \mathsf{r}^{n-i-3} & & & \cdots & \mathrm{id} & 0 & \\ \mathsf{r}^{n-i-2} & \mathsf{r}^{n-i-3} & & \cdots & \mathsf{r} & \mathrm{id} & 0 \end{bmatrix}.$$

Easy matrix calculations show that

$$\begin{aligned} \mathsf{r}h - Nh &= \iota^{\rhd}p^{\rhd} - \mathrm{id} & \qquad h\underline{\epsilon} - \underline{\epsilon}h &= \epsilon^{\rhd}p^{\rhd} \\ \mathsf{r}h - hN &= \iota^{\lhd}p^{\lhd} - \mathrm{id} & \qquad h\underline{\eta} - \underline{\eta}h &= -\iota^{\lhd}\eta^{\lhd} \end{aligned}$$

Next, let $\boldsymbol{h} : \mathbf{E}^{\diamondsuit}\langle 1\rangle \to \mathbf{E}^{\diamondsuit}\langle -1\rangle$ be given by

$$\boldsymbol{h} = \begin{bmatrix} h & & & \\ & h & & \\ & & \ddots & \\ & & & h \end{bmatrix}.$$

Using the calculations above, one can show that

$$\begin{aligned} \mathsf{r}\boldsymbol{h} - \boldsymbol{N}\boldsymbol{h} &= \boldsymbol{\iota}^{\triangleright}\boldsymbol{p}^{\triangleright} - \mathrm{id} & \qquad h\underline{\boldsymbol{\epsilon}} - \underline{\boldsymbol{\epsilon}}\boldsymbol{h} &= \boldsymbol{\epsilon}^{\triangleright}\boldsymbol{p}^{\triangleright} \\ \mathsf{r}\boldsymbol{h} - \boldsymbol{h}\boldsymbol{N} &= \boldsymbol{\iota}^{\triangleleft}\boldsymbol{p}^{\triangleleft} - \mathrm{id} & \qquad h\underline{\boldsymbol{\eta}} - \underline{\boldsymbol{\eta}}\boldsymbol{h} &= -\boldsymbol{\iota}^{\triangleleft}\boldsymbol{\eta}^{\triangleleft} \end{aligned}$$

We claim that

$$\delta_2 \left[\begin{smallmatrix} 0 & 0 \\ \boldsymbol{h} & 0 \end{smallmatrix}\right] + \left[\begin{smallmatrix} 0 & 0 \\ \boldsymbol{h} & 0 \end{smallmatrix}\right] \delta_2 = \boldsymbol{\iota}\boldsymbol{p} - \mathrm{id} \tag{6.2}$$

The left-hand side is given by

$$\begin{bmatrix} \underline{\boldsymbol{\epsilon}}+\underline{\boldsymbol{\eta}} & \mathsf{r}-\boldsymbol{N} \\ \xi & -\underline{\boldsymbol{\epsilon}}-\underline{\boldsymbol{\eta}} \end{bmatrix} \left[\begin{smallmatrix} 0 & 0 \\ \boldsymbol{h} & 0 \end{smallmatrix}\right] + \left[\begin{smallmatrix} 0 & 0 \\ \boldsymbol{h} & 0 \end{smallmatrix}\right] \begin{bmatrix} \underline{\boldsymbol{\epsilon}}+\underline{\boldsymbol{\eta}} & \mathsf{r}-\boldsymbol{N} \\ \xi & -\underline{\boldsymbol{\epsilon}}-\underline{\boldsymbol{\eta}} \end{bmatrix} = \begin{bmatrix} \mathsf{r}\boldsymbol{h}-\boldsymbol{N}\boldsymbol{h} & 0 \\ \boldsymbol{h}\underline{\boldsymbol{\epsilon}}+\boldsymbol{h}\underline{\boldsymbol{\eta}}-\underline{\boldsymbol{\epsilon}}\boldsymbol{h}-\underline{\boldsymbol{\eta}}\boldsymbol{h} & \mathsf{r}\boldsymbol{h}-\boldsymbol{h}\boldsymbol{N} \end{bmatrix},$$

while the right-hand side is given by

$$\boldsymbol{\iota}\boldsymbol{p} - \mathrm{id} = \begin{bmatrix} \boldsymbol{\iota}^{\triangleright}\boldsymbol{p}^{\triangleright}-\mathrm{id} & 0 \\ \boldsymbol{\epsilon}^{\triangleright}\boldsymbol{p}^{\triangleright}-\boldsymbol{\iota}^{\triangleleft}\boldsymbol{\eta}^{\triangleleft} & \boldsymbol{\iota}^{\triangleleft}\boldsymbol{p}^{\triangleleft}-\mathrm{id} \end{bmatrix}.$$

The equality (6.2) then follows from the calculations above.

The two equations (6.1) and (6.2) show that $\boldsymbol{p}\boldsymbol{\iota}$ and $\boldsymbol{\iota}\boldsymbol{p}$ are both chain-homotopic to identity maps in $D^{\mathrm{mix}}_{\mathrm{mon}}(\mathbb{A}^n/T, \Bbbk)$. In other words, $\boldsymbol{p}$ and $\boldsymbol{\iota}$ are isomorphisms in $D^{\mathrm{mix}}_{\mathrm{mon}}(\mathbb{A}^n/T, \Bbbk)$, as desired.

Since the underlying graded parity sheaf $\mathbf{E}^{\diamondsuit}$ of $\boldsymbol{\mathcal{Z}}$ is a direct sum of objects of the form $\mathcal{E}(I)\langle k\rangle$, each of which is perverse, we conclude that $\boldsymbol{\mathcal{Z}}$ itself is a mixed perverse sheaf.

Finally, it remains to describe the monodromy endomorphism $\mathrm{N}_\Psi : \boldsymbol{\mathcal{Z}} \to \boldsymbol{\mathcal{Z}}\langle 2\rangle$. By definition (see [A, Definition 3.5]), the map $\mathrm{Mon}(\mathrm{N}_\Psi) : \mathrm{Mon}(\boldsymbol{\mathcal{Z}}) \to \mathrm{Mon}(\boldsymbol{\mathcal{Z}})\langle 2\rangle$ is given by $\mathsf{r}\cdot\mathrm{id}$. By Lemma 6.4, this map is homotopic (i.e., equal in $D^{\mathrm{mix}}_{\mathrm{mon}}(\mathbb{A}^n/T, \Bbbk)$) to $\mathrm{Mon}(\boldsymbol{N})$. We conclude that $\mathrm{N}_\Psi = \boldsymbol{N}$. ☐

7. Background on the affine flag variety

This section contains notation and preliminaries related to the affine Weyl group and the affine flag variety of PGL_n.

7.1. **The extended affine Weyl group.** Let W be the Weyl group of PGL_n, identified with the symmetric group on $\{1, \dots, n\}$. It is generated by the simple reflections $s_1, \dots, s_{n-1}$, where s_i is the permutation of $[n] = \{1, 2, \dots, n\}$ that exchanges i and $i+1$.

Let $\mathbf{Y}$ be the coweight lattice of PGL_n. We identify $\mathbf{Y}$ explicitly as

$$\mathbf{Y} = \mathbb{Z}^n/\mathbb{Z}\cdot(1, \dots, 1).$$

Let $\check\Phi \subset \mathbf{Y}$ be the set of coroots, given by $\check\Phi = \{e_i - e_j | i, j \in [n],\ i \neq j\}$, where $\{e_i\}_{i\in[n]}$ denotes the standard basis in $\mathbb{Z}^n$. The coroot lattice $\mathbb{Z}\check\Phi \subset \mathbf{Y}$ is then identified with the image of the set $\{(y_1, \dots, y_n) \in \mathbb{Z}^n \mid \sum y_i \equiv 0 \pmod n\}$. Let

$$W_{\mathrm{aff}} = W \ltimes \mathbb{Z}\check\Phi \qquad \text{and} \qquad W_{\mathrm{ext}} = W \ltimes \mathbf{Y}$$

be the affine Weyl group and the extended affine Weyl group, respectively. For $\lambda \in \mathbf{Y}$, we write t_λ for the corresponding element of W_{ext}. Let $\check\alpha_0 = (1, 0, \dots, 0, -1)$ be the highest coroot, and let $s_{\alpha_0} \in W$ be the reflection with respect to this root (i.e., the permutation that exchanges 1 and n). Recall that W_{aff} is a Coxeter group; it is generated by W together with the affine simple reflection

$$s_n = s_{\alpha_0} t_{-\alpha_0}.$$

(The affine simple reflection is usually denoted by s_0, but we will use s_n because it will allow for uniformity of notation with the setting of Section 3.)

The group W_{ext} is not a Coxeter group, but it nevertheless makes sense to speak of the lengths of elements in W_{ext}, following [IM]. Let

$$\omega = s_1 s_2 \cdots s_{n-1} t_{(0,\ldots,0,1)}.$$

This is an element of length 0 of order n. In fact, the set $\{1, \omega, \omega^2, \ldots, \omega^{n-1}\}$ is the set of all elements of length 0. We adopt the convention that a *reduced expression* for $w \in W_{\mathrm{ext}}$ is an expression of the form

$$w = s_{i_1} \cdots s_{i_k} \omega^m,$$

where $0 \le m < n$, and where $s_{i_1} \cdots s_{i_k}$ is a reduced expression in W_{aff}. A *subexpression* of this expression is a word obtained by omitting some of the simple reflections (but without changing the power of ω).

7.2. **The affine flag variety.** Let $B \subset \mathrm{PGL}_n(\mathbb{C}((t)))$ be the usual ("upper-triangular") Iwahori subgroup, and let $\mathrm{Fl} = \mathrm{PGL}_n(\mathbb{C}((t)))/B$ be the affine flag variety for PGL_n. It is well known that the B-orbits on Fl are parametrized by W_{ext}. For $w \in W_{\mathrm{ext}}$, let Fl_w denote the corresponding B-orbit.

Lemma 7.1. *Let $w \in W_{\mathrm{ext}}$, and let $w = s_{i_1} \cdots s_{i_k} \omega^m$ be a reduced expression. Assume that $k < n$, and that no two of the simple reflections $s_{i_1}, \ldots, s_{i_k}$ are equal. Then the Bott–Samelson resolution*

$$\pi : P_{s_{i_1}} \times^B P_{s_{i_2}} \times^B \cdots P_{s_{i_k}} \times^B \mathrm{Fl}_{\omega^m} \to \overline{\mathrm{Fl}_w}$$

is a bijection.

We will later see that this map is actually an isomorphism of varieties.

Proof sketch. We may reduce to the case where $m = 0$, so that we are working in W_{aff} instead of W_{ext}. The Bott–Samelson map is always surjective; we just need to prove that it is injective. Consider the standard basis $\{T_w : w \in W_{\mathrm{aff}}\}$ for the affine Hecke algebra. Under our assumptions, every subexpression of $s_{i_1} \cdots s_{i_k}$ is reduced. This observation implies that

$$(T_{s_{i_1}} + 1)(T_{s_{i_2}} + 1) \cdots (T_{s_{i_k}} + 1) = \sum_{\substack{u \text{ a subexpression} \\ \text{of } s_{i_1} s_{i_2} \cdots s_{i_k}}} T_u.$$

It is well known that for any $u \le w$, the coefficient of T_u on the right-hand side above encodes the cohomology of the fiber of the Bott–Samelson resolution over any point of Fl_u. Since these coefficients are all 1, the fibers are single points □

Let $\mathbb{G}_{\mathrm{m}}$ act on $\mathbb{C}((t))$ by scaling the indeterminate t (the "loop rotation action"). Then one can form the semidirect product $\mathbb{G}_{\mathrm{m}} \ltimes B$, and this group acts on Fl. It is well known that the orbits of $\mathbb{G}_{\mathrm{m}} \ltimes B$ on Fl are the same as those of B. Let $\mathrm{Parity}_{\mathbb{G}_{\mathrm{m}} \ltimes B}(\mathrm{Fl}, \Bbbk)$ be the category of $\mathbb{G}_{\mathrm{m}} \ltimes B$-equivariant parity sheaves on Fl. For any $w \in W_{\mathrm{ext}}$, there is a unique indecomposable object $\mathcal{E}_w \in \mathrm{Parity}_{\mathbb{G}_{\mathrm{m}} \ltimes B}(\mathrm{Fl})$ that is supported on $\overline{\mathrm{Fl}_w}$ and that satisfies

$$\mathcal{E}_w|_{\mathrm{Fl}_w} \cong \underline{\Bbbk}_{\mathrm{Fl}_w}\{\dim \mathrm{Fl}_w\}.$$

In particular, for $w = \omega^m$, the variety $\mathrm{Fl}_{\omega^m} = \overline{\mathrm{Fl}_{\omega^m}}$ is a single point, and $\mathcal{E}_{\omega^m}$ is a skyscraper sheaf.

The category $\mathrm{Parity}_{\mathbb{G}_{\mathrm{m}} \ltimes B}(\mathrm{Fl}, \Bbbk)$ is equipped with a monoidal structure given by the convolution product, denoted by $\star$. The skyscraper sheaf at the identity element $\mathcal{E}_1$ is the unit for this product. For any $w \in W_{\mathrm{ext}}$, if $w = s_{i_1} \cdots s_{i_k} \omega^m$ is a reduced expression, then $\mathcal{E}_w$ is a direct summand (with multiplicity 1) of

$$\mathcal{E}_{s_{i_1}} \star \cdots \star \mathcal{E}_{s_{i_k}} \star \mathcal{E}_{\omega^m}.$$

Moreover, this convolution product is canonically (see [RW, §10.2]) isomorphic to $\pi_* \underline{\Bbbk}\{\dim \mathrm{Fl}_w\}$, where π is the Bott–Samelson resolution of $\overline{\mathrm{Fl}_w}$ corresponding to the given reduced expression.

For any simple reflection s, there are counit and unit maps between $\mathcal{E}_s$ and shifts of $\mathcal{E}_1$. Following [EW], we denote these maps by

$$\underset{s}{\bullet} : \mathcal{E}_s \to \mathcal{E}_1\{1\} \qquad \text{and} \qquad \overset{s}{\bullet} : \mathcal{E}_1\{-1\} \to \mathcal{E}_s,$$

respectively.

7.3. **The first fundamental coweight and admissible elements.** Let

$$\check{\varpi}_1 = \check{\varpi}_1^{(1)} = (1, 0, \dots, 0) \in \mathbf{Y}$$

be the first fundamental coweight. Its orbit under W consists of the coweights

$$\check{\varpi}_1^{(i)} = (0, \dots, 0, 1, 0, \dots, 0) \quad (1 \text{ in the } i\text{th coordinate}), \qquad 1 \le i \le n.$$

For brevity, we will denote the corresponding elements of W_{ext} by

$$\mathsf{t}^i = t_{\check{\varpi}_1^{(i)}} \qquad \text{for } 1 \le i \le n.$$

One can check that

$$\begin{aligned}
\mathsf{t}^1 &= s_n s_{n-1} \cdots s_3 s_2 \omega \\
\mathsf{t}^2 &= s_1 s_n \cdots s_4 s_3 \omega \\
&\;\vdots \\
\mathsf{t}^i &= s_{i-1} s_{i-2} \cdots s_{i+2} s_{i+1} \omega \\
&\;\vdots \\
\mathsf{t}^n &= s_{n-1} s_{n-2} \cdots s_2 s_1 \omega
\end{aligned} \tag{7.1}$$

(In the expression for t^i, the subscripts on the simple reflections are to be understood modulo n.) In fact, these are reduced expressions for $\mathsf{t}^1, \dots, \mathsf{t}^n$.

Definition 7.2. An element $w \in W_{\mathrm{ext}}$ is said to be $\check{\varpi}_1$-*admissible* if there is some $i \in \{1, \dots, n\}$ such that $w \le \mathsf{t}^i$ in the Bruhat order.

We will classify the $\check{\varpi}_1$-admissible elements using the following notion.

Let $I \subsetneq [n]$ be a proper subset. List its elements in some order as

$$i_1, i_2, \dots, i_k.$$

This order is called an *acceptable order* on I if the following conditions hold:

1. If $i_1 < n$, then $i_1 + 1 \notin I$. If $i_1 = n$, then $1 \notin I$.
2. Exactly one of the following inequalities is false:

$$i_1 > i_2 > \cdots > i_k > i_1.$$

Here is a more intuitive description of this notion. Consider the affine Dynkin diagram of type $\tilde{A}_{n-1}$:

$$\begin{tikzcd}[column sep=small] & & n \arrow[dll, dash] \arrow[drr, dash] & & \\ 1 \arrow[r, dash] & 2 \arrow[r, dash] & \cdots \arrow[r, dash] & n-2 \arrow[r, dash] & n-1 \end{tikzcd} \tag{7.2}$$

An acceptable order on I is an order obtained by listing its elements in clockwise order, starting from an element whose immediate counterclockwise neighbor does not belong to I.

Lemma 7.3. *Let $I \subsetneq [n]$ be a proper subset.*

(1) *Choose an acceptable order $i_1, i_2, \ldots, i_k$ on I, and let*

$$w_I = s_{i_1} s_{i_2} \cdots s_{i_k} \omega \in W_{\mathrm{ext}}. \tag{7.3}$$

This element is independent of the choice of acceptable order.

(2) *The assignment $I \mapsto w_I$ gives a bijection*

$$\{\textit{proper subsets of } [n]\} \overset{\sim}{\leftrightarrow} \{\check{\varpi}_1\textit{-admissible elements of } W_{\mathrm{ext}}\}. \tag{7.4}$$

Proof. (1) Suppose we have another acceptable order on I. This order must be of the form $i_{t+1}, i_{t+2}, \ldots, i_k, i_1, i_2, \ldots, i_t$, for some t with $1 \le t < k$ and with $i_{t+1} + 1 \notin I$ (or $1 \notin I$, in the case where $i_{t+1} = n$). Our assumptions imply that every simple reflection in the set $\{s_{i_1}, \ldots, s_{i_t}\}$ commutes with every simple reflection in the set $\{s_{i_{t+1}}, \ldots, s_{i_k}\}$. It follows that w_I is independent of the choice of acceptable order.

(2) Every $\check{\varpi}_1$-admissible element is given by a subexpression of some expression in (7.1). It is clear that any such subexpression is of the form considered in (7.3), so the map (7.4) is surjective.

Next, let $I, I' \subsetneq [n]$. Choose acceptable orders for both I and I'. The corresponding expressions for w_I and $w_{I'}$ from (7.3) are reduced. If $w_I = w_{I'}$, then (by Matsumoto's theorem) the reduced expression for w_I can be changed into that for $w_{I'}$ by applying a sequence of braid relations. But since there are no repeated simple reflections in (7.3), the only possible braid relations that can be applied are those of the form $s_i s_j = s_j s_i$, where i and j are not neighbors in (7.2). This implies that $I = I'$, so the map (7.4) is injective. $\square$

In view of Lemma 7.3, we introduce the notation

$$\mathrm{Fl}_I = \mathrm{Fl}_{w_I} \qquad \text{for any } I \subsetneq [n].$$

8. Parity sheaves on the global Schubert variety

Following [PZ, Z], one can associate to any reductive group G and any dominant coweight λ a space $\overline{\mathcal{G}r}_\lambda$, called a "global Schubert variety." This variety is equipped with a map to $\mathbb{A}^1$. Its generic fiber is a subvariety of the affine Grassmannian of G, and its special fiber is a subvariety of the affine flag variety of G.

In this section, we study the geometry of this variety in the special case where the group is $G = \mathrm{PGL}_n$ and the coweight is $\lambda = \check{\varpi}_1$.

8.1. **The global Schubert variety $\overline{\mathcal{G}r}_{\check{\varpi}_1}$.** We begin by giving a concrete description of the variety $\overline{\mathcal{G}r}_{\check{\varpi}_1}$. This description comes from [G1, §4.1] (where it is called the "standard local model"). For a discussion of how the two settings are related, see [PZ, §7.2.1] and [PRS].

For $y \in \mathbb{A}^1$ and $i \in [n]$, let $g_i(y) : \mathbb{C}^n \to \mathbb{C}^n$ be the linear map given by

$$g_i(y) = \begin{bmatrix} 1 &&&&&& \\ & \ddots &&&&& \\ && 1 &&&& \\ &&& y &&& \\ &&&& 1 && \\ &&&&& \ddots & \\ &&&&&& 1 \end{bmatrix},$$

where the "y" appears as the ith entry on the diagonal. Let

$$\overline{\mathcal{G}r}_{\check{\varpi}_1} = \{(y, L_1, \ldots, L_n) \in \mathbb{A}^1 \times \mathbb{P}^{n-1} \times \cdots \times \mathbb{P}^{n-1} \mid \\ g_1(y)(L_1) \subset L_2, \ldots, g_{n-1}(y)(L_{n-1}) \subset L_n, g_n(y)(L_n) \subset L_1\}.$$

Projection onto the first coordinate gives us a map

$$f : \overline{\mathcal{G}r}_{\check{\varpi}_1} \to \mathbb{A}^1.$$

Of course, if $y \neq 0$, then $g_i(y)$ is invertible, so the lines $L_2, \ldots, L_n$ are all determined by L_1 alone. We deduce that

$$f^{-1}(y) \cong \mathbb{P}^{n-1} \cong \mathrm{Gr}_{\check{\varpi}_1} \qquad \text{if } y \neq 0.$$

On the other hand, the special fiber $f^{-1}(0)$ can be embedded as a closed subvariety of the affine flag variety Fl. See [G1, §4.2] for an explicit description of this embedding. Via this embedding, according to [Z, Theorem 3], we have

$$f^{-1}(0) = \bigcup_{\substack{w \in W_{\mathrm{ext}} \\ w \text{ is } \check{\varpi}_1\text{-admissible}}} \mathrm{Fl}_w.$$

Let $T \subset \mathrm{PGL}_n$ be the maximal torus consisting of diagonal matrices. The natural action of T on $\mathbb{P}^{n-1}$ commutes with each of the $g_i(y)$'s, so there is an induced action of T on $\overline{\mathcal{G}r}_{\check{\varpi}_1}$ given by

$$t \cdot (y, L_1, \ldots, L_n) = (y, tL_1, tL_2, \ldots, tL_n).$$

Next, we define an action of $\mathbb{G}_{\mathrm{m}}$ on $\overline{\mathcal{G}r}_{\check{\varpi}_1}$ by letting $z \in \mathbb{G}_{\mathrm{m}}$ act by

$$z \cdot (y, L_1, \ldots, L_n) = (zy, L_1, g_1(z)L_2, g_1(z)g_2(z)L_3, \ldots, g_1(z)\cdots g_{n-1}(z)L_n).$$

The actions of T and $\mathbb{G}_{\mathrm{m}}$ commute, so there is an action of $\hat{T} = T \times \mathbb{G}_{\mathrm{m}}$ on $\overline{\mathcal{G}r}_{\check{\varpi}_1}$. We define characters $\alpha_1, \ldots, \alpha_{n-1}, \xi : \hat{T} \to \mathbb{G}_{\mathrm{m}}$ by

$$\alpha_i\left(\begin{bmatrix} y_1 && \\ & \ddots & \\ && y_n \end{bmatrix}, z\right) = y_{i+1}y_i^{-1}, \qquad \xi\left(\begin{bmatrix} y_1 && \\ & \ddots & \\ && y_n \end{bmatrix}, z\right) = z.$$

These formulas agree with those in Section 3 after making the change in coordinates $t_i = y_{i+1}y_i^{-1}$. Note that the restrictions of $\alpha_1, \ldots, \alpha_n$ to $T \subset \hat{T}$ are the negatives of the usual simple roots of PGL_n. Alternatively, they are the simple roots with respect to which the upper-triangular Iwahori subgroup B may be thought of as "negative." We also let

$$\alpha_n = \xi - \alpha_1 - \cdots - \alpha_{n-1} : \left(\begin{bmatrix} y_1 && \\ & \ddots & \\ && y_n \end{bmatrix}, z\right) \mapsto zy_1y_n^{-1}.$$

8.2. **An open affine subset of $\overline{\mathcal{G}r}_{\breve{\varpi}_1}$.** Suppose $1 \le j,k \le n$ Define a map $p_{j,k} : \mathbb{A}^n \to \mathbb{A}^1$ by

$$p_{j,k}(x_1,\dots,x_n) = \begin{cases} 1 & \text{if } k-j=-1 \text{ or } k-j=n-1 \\ x_j x_{j+1}\cdots x_k & \text{if } j \le k \text{ and } k-j < n-1, \\ x_j x_{j+1}\cdots x_n x_1 x_2 \cdots x_k & \text{if } k < j \text{ and } k-j < -1. \end{cases}$$

In other words $p_{j,k}$ is the product of variables starting at x_j and proceeding counterclockwise around (7.2) up to x_k, except when this rule would give us the product of all the variables, in which case we instead take the empty product. Next, let $u_k : \mathbb{A}^n \to \mathbb{P}^{n-1}$ be the map given by

$$u_k = [p_{k,n} : p_{k,1} : p_{k,2} : \cdots : p_{k,n-1}],$$

and then let $u : \mathbb{A}^n \to \overline{\mathcal{G}r}_{\breve{\varpi}_1}$ be the map given by

$$u(x_1,\dots,x_n) = (x_1x_2\cdots x_n, u_1, u_2, \cdots, u_n).$$

For example, when $n=4$, u is given by

$$\begin{aligned} u(x_1,x_2,x_3,x_4) = (x_1x_2x_3x_4, [1 : x_1 : x_1x_2 : x_1x_2x_3], [x_2x_3x_4 : 1 : x_2 : x_2x_3], \\ [x_3x_4 : x_3x_4x_1 : 1 : x_3], [x_4 : x_4x_1 : x_4x_1x_2 : 1]). \end{aligned}$$

It is straightforward to check that u does indeed take values in $\overline{\mathcal{G}r}_{\breve{\varpi}_1}$, i.e., that

$$g_i(x_1\cdots x_n)\cdot u_i \subset u_{i+1}.$$

Moreover, if we let $\hat{T}$ act on $\mathbb{A}^n$ by

$$t\cdot(x_1,\dots,x_n) = (\alpha_1(t)x_1,\dots,\alpha_n(t)x_n),$$

then the map u is $\hat{T}$-equivariant.

Lemma 8.1. *The map $u : \mathbb{A}^n \to \overline{\mathcal{G}r}_{\breve{\varpi}_1}$ is an open embedding. Its image meets every B-orbit in $f^{-1}(0)$. Indeed, we have*

$$u^{-1}((\overline{\mathcal{G}r}_{\breve{\varpi}_1})_\eta) = (\mathbb{A}^n)_\eta \qquad \textit{and} \qquad u^{-1}(\mathrm{Fl}_I) = (\mathbb{A}^n)_I \quad \textit{for } I \subsetneq [n].$$

Proof sketch. Given a point $(y, L_1,\dots,L_n) \in \overline{\mathcal{G}r}_{\breve{\varpi}_1}$, write each line L_i in homogeneous coordinates as $L_i = [a_{i1} : a_{i2} : \cdots : a_{in}]$. Let $U \subset \overline{\mathcal{G}r}_{\breve{\varpi}_1}$ be the open subset consisting of points $(y, L_1,\dots,L_n)$ such that $a_{ii} \ne 0$ for all $i \in [n]$. It is easy to see from the formula that $u : \mathbb{A}^n \to \overline{\mathcal{G}r}_{\breve{\varpi}_1}$ actually takes values in U. On the other hand, there is a map $v : U \to \mathbb{A}^n$ given by

$$v(y, L_1,\dots,L_n) = \left(\tfrac{a_{12}}{a_{11}}, \tfrac{a_{23}}{a_{22}}, \dots, \tfrac{a_{n-1,n}}{a_{n-1,n-1}}, \tfrac{a_{n1}}{a_{nn}}\right).$$

Elementary calculations show that u and v are inverse to one another.

For the claim that U meets every B-orbit in $f^{-1}(0)$, see [G1, Proposition 4.5(ii)]. $\square$

Corollary 8.2. *The variety $\overline{\mathcal{G}r}_{\breve{\varpi}_1}$ is smooth. For any $I \subsetneq [n]$, the Schubert variety $\overline{\mathrm{Fl}_{w_I}}$ is smooth.*

Proof. The singular locus of $\overline{\mathcal{G}r}_{\breve{\varpi}_1}$ must be contained in the special fiber $f^{-1}(0)$, and it must be stable under the Iwahori subgroup B. Since the open set U from the proof of Lemma 8.1 meets every B-orbit in $f^{-1}(0)$, it must meet the singular locus if the latter is nonempty. But U is smooth, so the singular locus is empty. The same reasoning shows that Schubert varieties associated to $\breve{\varpi}_1$-admissible elements of W_{ext} are smooth. $\square$

Corollary 8.3. *Let $I \subsetneq [n]$, and let $i_1, \dots, i_k$ be an acceptable order on I. The Bott–Samelson resolution*

$$\pi : P_{s_{i_1}} \times^B P_{s_{i_2}} \times^B \cdots P_{s_{i_k}} \times^B \mathrm{Fl}_\omega \to \overline{\mathrm{Fl}_{w_I}}$$

is an isomorphism of varieties.

Proof. It is a well-known consequence of Zariski's main theorem that a bijective map between smooth complex varieties is an isomorphism, so this follows from Lemma 7.1 and Corollary 8.2. □

8.3. **Parity sheaves.** Observe that the collection $\{\mathrm{Fl}_I\}_{I \subsetneq [n]} \cup \{(\overline{\mathcal{G}r}_{\varpi_1})_\eta\}$ constitutes a stratification of $\overline{\mathcal{G}r}_{\varpi_1}$. By Corollary 8.2, the closure of every stratum is smooth, so the constant sheaf is a parity sheaf. We introduce the notation

$$\mathcal{E}(I) = \begin{cases} \underline{\Bbbk}_{\overline{\mathrm{Fl}_I}}\{|I|\} & \text{if } I \subsetneq [n], \\ \underline{\Bbbk}_{\overline{\mathcal{G}r}_{\varpi_1}}\{n\} & \text{if } I = [n]. \end{cases}$$

This is a perverse parity sheaf. If $i \in I$, there is a canonical morphism

$$\dot{\epsilon}_i : \mathcal{E}(I) \to \mathcal{E}(I \smallsetminus \{i\})\{1\}$$

induced by $*$-restriction and adjunction, and another canonical morphism

$$\dot{\eta}_i : \mathcal{E}(I \smallsetminus \{i\})\{-1\} \to \mathcal{E}(I)$$

induced by !-restriction and adjunction. We may occasionally write $\mathcal{E}^{\overline{\mathcal{G}r}_{\varpi_1}}(I)$, $\dot{\epsilon}_i^{\overline{\mathcal{G}r}_{\varpi_1}}$, or $\dot{\eta}_i^{\overline{\mathcal{G}r}_{\varpi_1}}$ to avoid ambiguity with the notation from Section 3.

Lemma 8.4. *We have $u^*\mathcal{E}^{\overline{\mathcal{G}r}_{\varpi_1}}(I) \cong \mathcal{E}^{\mathbb{A}^n}(I)$. Moreover, $u^*\dot{\epsilon}_i^{\overline{\mathcal{G}r}_{\varpi_1}}$ can be identified with $\dot{\epsilon}_i^{\mathbb{A}^n}$, and $u^*\dot{\eta}_i^{\overline{\mathcal{G}r}_{\varpi_1}}$ can be identified with $\dot{\eta}_i^{\mathbb{A}^n}$.*

Proof. This follows immediately from Lemma 8.1. □

The following lemma relates these objects to the convolution structure discussed in Section 7.2.

Lemma 8.5. *Let $I \subsetneq [n]$, and let $i_1, \dots, i_k$ be an acceptable order on I. Then there is a canonical isomorphism*

$$\mathcal{E}_{s_{i_1}} \star \cdots \star \mathcal{E}_{s_{i_k}} \star \mathcal{E}_\omega \cong \mathcal{E}(I). \tag{8.1}$$

Moreover, via this identification, for any $i_t \in I$, we have

$$\begin{aligned} \mathrm{id}_{\mathcal{E}_{s_{i_1}}} \star \cdots \star \mathrm{id}_{\mathcal{E}_{s_{i_{t-1}}}} \star \underset{s_{i_t}}{\uparrow} \star \mathrm{id}_{\mathcal{E}_{s_{i_{t+1}}}} \star \cdots \star \mathrm{id}_{\mathcal{E}_{s_{i_k}}} \star \mathrm{id}_{\mathcal{E}_\omega} &= \dot{\epsilon}_{i_t}, \\ \mathrm{id}_{\mathcal{E}_{s_{i_1}}} \star \cdots \star \mathrm{id}_{\mathcal{E}_{s_{i_{t-1}}}} \star \overset{s_{i_t}}{\downarrow} \star \mathrm{id}_{\mathcal{E}_{s_{i_{t+1}}}} \star \cdots \star \mathrm{id}_{\mathcal{E}_{s_{i_k}}} \star \mathrm{id}_{\mathcal{E}_\omega} &= \dot{\eta}_{i_t}. \end{aligned} \tag{8.2}$$

Proof. The left-hand side of (8.1) is canonically (see [RW, §10.2]) isomorphic to $\pi_*\underline{\Bbbk}\{\dim \mathrm{Fl}_{w_I}\}$, where π is the Bott–Samelson resolution indicated in the bottom row of (8.3) below. The isomorphism (8.1) then follows from Corollary 8.3.

$$\begin{tikzcd} P_{s_{i_1}} \times^B \cdots \times^B P_{s_{i_{t-1}}} \times^B B \times^B P_{s_{i_{t+1}}} \times^B \cdots \times^B P_{s_{i_k}} \times^B \mathrm{Fl}_\omega \arrow[r] \arrow[d, hook] & \overline{X_{I \smallsetminus \{i_t\}}} \arrow[d, hook] \\ P_{s_{i_1}} \times^B \cdots \times^B P_{s_{i_{t-1}}} \times^B P_{s_{i_t}} \times^B P_{s_{i_{t+1}}} \times^B \cdots \times^B P_{s_{i_k}} \times^B \mathrm{Fl}_\omega \arrow[r] & \overline{X_I} \end{tikzcd} \tag{8.3}$$

That corollary also tells us that the top horizontal map is also an isomorphism. Both vertical maps are closed embeddings. The left-hand sides of (8.2) are induced by $*$- or !-restriction and adjunction in the left-hand column of (8.3), while the right-hand sides of (8.2) are defined analogously using the right-hand column of (8.3). The isomorphisms in (8.2) follow. $\square$

Proposition 8.6. *For any $I \subset [n]$ with $|I| \leq n-2$, we have*

$$\sum_{i \notin I} \dot{\epsilon}_i \circ \dot{\eta}_i + \sum_{i \in I} \dot{\eta}_i \circ \dot{\epsilon}_i = \xi \cdot \mathrm{id}_{\mathcal{E}(I)}.$$

Proof. A proper subset $B \subsetneq [n]$ is said to be a *block* if it admits a unique acceptable order. (In other words, a block is a sequence of consecutive labels in (7.2), reading in clockwise order.) Given a nonempty block B with acceptable order $B = (b_1, \dots, b_k)$, we define the *core* of B to be the block $B^\circ = (b_1, \dots, b_{k-1})$, and its *tail* to be the remaining integer b_k.

Let B be a nonempty block, with core B° and tail j. We begin by proving the following auxiliary statement, an equality of maps $\mathcal{E}(B^\circ) \to \mathcal{E}(B^\circ)\{2\}$:

$$\dot{\epsilon}_j \circ \dot{\eta}_j + \sum_{i \in B^\circ} \dot{\eta}_i \circ \dot{\epsilon}_i = \Big(\sum_{b \in B} \alpha_b \Big) \cdot \mathrm{id}_{\mathcal{E}(B^\circ)}. \tag{8.4}$$

We proceed by induction on the number of elements in B°. If B° is empty (so that $\mathcal{E}(B^\circ) = \mathcal{E}_\omega$), then by Lemma 8.5, the left-hand side of (8.4) reduces to

$$\dot{\epsilon}_j \circ \dot{\eta}_j = (\underset{s_j}{\overset{\bullet}{|}} \star \mathrm{id}_{\mathcal{E}_\omega}) \circ (\overset{s_j}{\underset{\bullet}{|}} \star \mathrm{id}_{\mathcal{E}_\omega}) = \alpha_j \cdot \mathrm{id}_{\mathcal{E}(B^\circ)},$$

where the last step follows from [EW, §1.4.1].

Now, assume that B° is nonempty. Let b_1 be its first element (in the acceptable order), and let $B' = B \smallsetminus \{b_1\}$. Of course, B' is still a block, and B and B' have the same tail. We have

$$\mathcal{E}(B^\circ) \cong \mathcal{E}_{s_{b_1}} \star \mathcal{E}(B'^\circ).$$

In the following calculation, we will label some of the maps with superscripts to indicate the domain. We have

$$\begin{aligned} &\dot{\epsilon}_j \circ \dot{\eta}_j^{B^\circ} + \sum_{i \in B^\circ} \dot{\eta}_i \circ \dot{\epsilon}_i^{B^\circ} \\ &\quad = \mathrm{id}_{\mathcal{E}_{s_{b_1}}} \star (\dot{\epsilon}_j \circ \dot{\eta}_j^{B'^\circ}) + \Big(\sum_{i \in B'^\circ} \mathrm{id}_{\mathcal{E}_{s_{b_1}}} \star (\dot{\eta}_i \circ \dot{\epsilon}_i^{B'^\circ}) \Big) + \dot{\eta}_{b_1} \circ \dot{\epsilon}_{b_1}^{B^\circ} \\ &\quad = \sum_{b \in B'} \mathrm{id}_{\mathcal{E}_{s_{b_1}}} \star (\alpha_b \cdot \mathrm{id}_{\mathcal{E}(B'^\circ)}) + (\overset{s_{b_1}}{\underset{\bullet}{|}} \star \mathrm{id}_{\mathcal{E}(B'^\circ)}) \circ (\underset{s_{b_1}}{\overset{\bullet}{|}} \star \mathrm{id}_{\mathcal{E}(B'^\circ)}). \end{aligned} \tag{8.5}$$

Let b_2 be the first element of B' (in the acceptable order). The simple reflections s_{b_1} and s_{b_2} do not commute (b_1 and b_2 are adjacent labels in (7.2)), but s_{b_1} commutes with s_b for all $b \in B' \smallsetminus \{b_2\}$. According to [EW, §1.4.1], we have

$$\mathrm{id}_{\mathcal{E}_{s_{b_1}}} \star \alpha_b = \begin{cases} \alpha_b \star \mathrm{id}_{\mathcal{E}_{s_{b_1}}} & \text{if } b \neq b_2, \\ (\alpha_{b_1} + \alpha_{b_2}) \star \mathrm{id}_{\mathcal{E}_{s_{b_1}}} - \overset{s_{b_1}}{\underset{\bullet}{|}} \circ \underset{s_{b_1}}{\overset{\bullet}{|}} & \text{if } b = b_2. \end{cases} \tag{8.6}$$

Combining (8.5) and (8.6), we obtain (8.4).

We can generalize (8.4) as follows. Let $I \subsetneq [n]$ be a subset such that $I \cap B = B^\circ$, and such that I admits an acceptable order starting with B°. The same calculation as above shows that

$$\dot{\epsilon}_j \circ \dot{\eta}_j + \sum_{i \in B^\circ} \dot{\eta}_i \circ \dot{\epsilon}_i = \Big(\sum_{b \in B} \alpha_b \Big) \cdot \mathrm{id}_{\mathcal{E}(I)}. \tag{8.7}$$

We now return to the main statement of the proposition. It is easily seen that that there is a unique way to write the set $[n]$ as a disjoint union of blocks

$$[n] = B_1 \sqcup \cdots \sqcup B_r \qquad \text{such that} \qquad I = B_1^\circ \cup \cdots \cup B_r^\circ.$$

(This is where the assumption that $|I| \leq n-2$ is required: this block decomposition does not exist if $|I| > n-2$.) Assume that these blocks are numbered in such a way that if we list the elements of B_1°, then those of B_2°, etc., according to the acceptable order for each block, the resulting list is in an acceptable order on I. Moreover, if we cyclically permute the blocks, say as

$$B_t, B_{t+1}, \ldots, B_r, B_1, \ldots, B_{t-1},$$

and then list the elements in their cores as above, we again obtain an acceptable order on I.

Let j_t denote the tail of B_t. We have

$$\sum_{i \notin I} \dot{\epsilon}_i \circ \dot{\eta}_i + \sum_{i \in I} \dot{\eta}_i \circ \dot{\epsilon}_i = \sum_{t=1}^{r} \Big(\dot{\epsilon}_{j_t} \circ \dot{\eta}_{j_t} + \sum_{i \in B_t^\circ} \dot{\eta}_i \circ \dot{\epsilon}_i \Big).$$

For each t, I admits an acceptable order starting with $B_\circ^t$, so we can apply (8.7) to the right-hand side above. We conclude that

$$\sum_{i \notin I} \dot{\epsilon}_i \circ \dot{\eta}_i + \sum_{i \in I} \dot{\eta}_i \circ \dot{\epsilon}_i = \sum_{t=1}^{r} \sum_{b \in B_t} \alpha_b \cdot \mathrm{id}_{\mathcal{E}(I)} = \xi \cdot \mathrm{id}_{\mathcal{E}(I)},$$

as desired. $\square$

9. The nearby cycles sheaf on $\overline{\mathcal{G}r}_{\check{\varpi}_1}$

Let us introduce the notation

$$\mathcal{E}_\eta = \underline{\Bbbk}_{(\overline{\mathcal{G}r}_{\check{\varpi}_1})_\eta}\{n\} \in \mathrm{Parity}_{\mathbb{G}_m}((\overline{\mathcal{G}r}_{\check{\varpi}_1})_\eta/T, \Bbbk).$$

The goal of this section is to compute $\Psi_f(\mathcal{E}_\eta)$. Note that compared to Proposition 3.3, Proposition 8.6 is missing a few cases. Unfortunately, the calculations in Sections 4–6 make use of all the cases of Proposition 3.3, so we cannot simply copy those computations for $\overline{\mathcal{G}r}_{\check{\varpi}_1}$.

Instead, we take a roundabout approach. Let

$$u_0 : (\mathbb{A}^n)_0 \to (\overline{\mathcal{G}r}_{\check{\varpi}_1})_0 \qquad \text{and} \qquad u_\eta : (\mathbb{A}^n)_\eta \to (\overline{\mathcal{G}r}_{\check{\varpi}_1})_\eta$$

be the restrictions of the map $u : \mathbb{A}^n \to \overline{\mathcal{G}r}_{\check{\varpi}_1}$ from Lemma 8.1. We will first show that u_0^* is fully faithful on perverse sheaves, and we will then use this to show that the desired result on $\overline{\mathcal{G}r}_{\check{\varpi}_1}$ can be deduced from Theorem 6.5.

9.1. Mixed perverse sheaves on $\mathbb{A}^n$ and $\overline{\mathcal{G}r}_{\check{\varpi}_1}$.

Recall that $\Bbbk$ is either a field or a complete discrete valuation ring. In the case where $\Bbbk$ is not a field, let π be a generator of its maximal ideal.

Throughout this subsection, we will treat the parity sheaves $\mathcal{E}(I)$ as T-equivariant objects, and we will work in the T-equivariant derived category. However, the same statements hold in the $\hat{T}$-equivariant setting, with the same proofs.

In the case where $\Bbbk$ is not a field, for each $I \subset [n]$, we set

$$\bar{\mathcal{E}}(I) = \mathrm{cone}(\mathcal{E}(I) \xrightarrow{\pi \cdot \mathrm{id}} \mathcal{E}(I)) \qquad \text{in } D^{\mathrm{mix}}(\overline{\mathcal{G}r}_{\check{\varpi}_1}/T, \Bbbk).$$

Lemma 9.1. (1) *If $\Bbbk$ is a field, every perverse sheaf $\mathcal{F} \in \mathrm{Perv}^{\mathrm{mix}}(\overline{\mathcal{G}r}_{\check{\varpi}_1}/T, \Bbbk)$ admits a finite filtration whose subquotients are of the form $\mathcal{E}(I)\langle k\rangle$ for some $I \subset [n]$ and some $k \in \mathbb{Z}$.*

(2) *If $\Bbbk$ is not a field, every perverse sheaf $\mathcal{F} \in \mathrm{Perv}^{\mathrm{mix}}(\overline{\mathcal{G}r}_{\check{\varpi}_1}/T, \Bbbk)$ admits a finite filtration whose subquotients are of the form $\mathcal{E}(I)\langle k\rangle$ or $\bar{\mathcal{E}}(I)\langle k\rangle$ for some $I \subset [n]$ and some $k \in \mathbb{Z}$.*

Proof. We have $\mathcal{E}(I) \cong \mathrm{IC}((\overline{\mathcal{G}r}_{\check{\varpi}_1})_I, \Bbbk)$, and if $\Bbbk$ is not a field, we also have

$$\bar{\mathcal{E}}(I) \cong \mathrm{IC}((\overline{\mathcal{G}r}_{\check{\varpi}_1})_I, \Bbbk/(\pi)).$$

Part (1) is just a restatement of the fact that when $\Bbbk$ is a field, every mixed perverse sheaf has finite length.

When $\Bbbk$ is a complete discrete valuation ring, perverse sheaves need not have finite length, but a mixed variant of [RSW, Lemma 2.1.4 and Remark 2.1.5] implies that every mixed perverse sheaf admits a finite filtration of the desired form. □

Lemma 9.2. *Let $I, J \subsetneq [n]$.*

(1) *The following natural map is an isomorphism for $i = 0, 1$:*

$$\mathrm{Hom}(\mathcal{E}(I), \mathcal{E}(J)\langle k\rangle[i]) \to \mathrm{Hom}(u^*\mathcal{E}(I), u^*\mathcal{E}(J)\langle k\rangle[i]).$$

(2) *Suppose that $\Bbbk$ is not a field. Then the following maps are isomorphisms for $i = 0, 1$:*

$$\mathrm{Hom}(\mathcal{E}(I), \bar{\mathcal{E}}(J)\langle k\rangle[i]) \to \mathrm{Hom}(u^*\mathcal{E}(I), u^*\bar{\mathcal{E}}(J)\langle k\rangle[i]), \tag{9.1}$$

$$\mathrm{Hom}(\bar{\mathcal{E}}(I), \mathcal{E}(J)\langle k\rangle[i]) \to \mathrm{Hom}(u^*\bar{\mathcal{E}}(I), u^*\mathcal{E}(J)\langle k\rangle[i]), \tag{9.2}$$

$$\mathrm{Hom}(\bar{\mathcal{E}}(I), \bar{\mathcal{E}}(J)\langle k\rangle[i]) \to \mathrm{Hom}(u^*\bar{\mathcal{E}}(I), u^*\bar{\mathcal{E}}(J)\langle k\rangle[i]). \tag{9.3}$$

Proof. (1) Let $K = I \cap J$, and let $h : \overline{\mathrm{Fl}_K} \to (\overline{\mathcal{G}r}_{\check{\varpi}_1})_0$ be the inclusion map. The intersection of the supports of $\mathcal{E}(I)$ and $\mathcal{E}(J)$ is precisely $\overline{\mathrm{Fl}_K}$, so there is a natural isomorphism

$$\mathrm{Hom}(h^*\mathcal{E}(I), h^!\mathcal{E}(J)\langle k\rangle[i]) \xrightarrow{\sim} \mathrm{Hom}(\mathcal{E}(I), \mathcal{E}(J)\langle k\rangle[i]).$$

Similarly, if we let $h' : \overline{(\mathbb{A}^n)_K} \to \mathbb{A}^n$ be the inclusion map, there is a natural isomorphism

$$\mathrm{Hom}((h')^*u^*\mathcal{E}(I), (h')^!u^*\mathcal{E}(J)[i]) \xrightarrow{\sim} \mathrm{Hom}(u^*\mathcal{E}(I), u^*\mathcal{E}(J)\langle k\rangle[i]).$$

The lemma thus reduces to the study of the natural map

$$\mathrm{Hom}(h^*\mathcal{E}(I), h^!\mathcal{E}(J)\langle k\rangle[i]) \to \mathrm{Hom}((h')^*u^*\mathcal{E}(I), (h')^!u^*\mathcal{E}(J)\langle k\rangle[i]). \tag{9.4}$$

Let $r = |I \smallsetminus K|$, and let $s = |J \smallsetminus K|$. Observe that

$$h^*\mathcal{E}(I) \cong \mathcal{E}(K)\langle -r\rangle[r] \qquad \text{and} \qquad h^!\mathcal{E}(J) \cong \mathcal{E}(K)\langle s\rangle[-s]$$

Analogous statements hold on $\mathbb{A}^n$, so (9.4) further reduces to the study of the map

$$\mathrm{Hom}(\mathcal{E}(K), \mathcal{E}(K)\langle k+r+s\rangle[i-r-s]) \to \mathrm{Hom}(u^*\mathcal{E}(K), u^*\mathcal{E}(K)\langle k+r+s\rangle[i-r-s]). \tag{9.5}$$

If $i \neq -k$, both sides vanish, so there is nothing to prove. Assume from now on that $i = -k$. Then these Hom-groups can be computed inside the category of parity sheaves, or inside the ordinary (nonmixed) derived category. The map (9.5) can thus be identified with the first map in the following sequence:

$$\mathsf{H}_T^{i-r-s}(\underline{\Bbbk}_{\overline{\mathrm{Fl}_K}}) \to \mathsf{H}_T^{i-r-s}(\underline{\Bbbk}_{\overline{(\mathbb{A}^n)_K}}) \to \mathsf{H}_T^{i-r-s}(\underline{\Bbbk}_{(\mathbb{A}^n)_\varnothing}). \tag{9.6}$$

The last term is the cohomology of the stalk of $\underline{\Bbbk}_{\overline{\mathrm{Fl}_K}}$ at the point Fl_ω. The composition of the maps in (9.6) is surjective by [FW, Theorem 5.7(2) and Proposition 7.1]. Since the T-fixed point $(\mathbb{A}^n)_\varnothing$ is attractive, the second map in (9.6) is an isomorphism. We conclude that (9.5) is surjective.

We wish to prove that (9.5) is an isomorphism for $i = 0, 1$. Of course, both sides are free $\Bbbk$-modules, so it is enough to prove that they have the same rank. Recall that $r, s \geq 0$. For $i = k = 0$, both sides of (9.5) vanish unless $r = s = 0$, and in that case both sides have rank 1.

If $i = -k = 1$, then both sides vanish unless $r + s \leq 1$. In fact, they also vanish when $r = s = 0$ by parity considerations. We therefore must have $r + s = 1$ and $i - r - s = 0$, so again, both sides of (9.5) have rank 1.

(2) Consider the diagram

$$
\begin{array}{ccc}
\downarrow & & \downarrow \\
\mathrm{Hom}(\mathcal{E}(I), \mathcal{E}(J)\langle k\rangle[i]) & \longrightarrow & \mathrm{Hom}(u^*\mathcal{E}(I), u^*\mathcal{E}(J)[i]) \\
{\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\
\mathrm{Hom}(\mathcal{E}(I), \mathcal{E}(J)\langle k\rangle[i]) & \longrightarrow & \mathrm{Hom}(u^*\mathcal{E}(I), u^*\mathcal{E}(J)\langle k\rangle[i]) \\
\downarrow & & \downarrow \\
\mathrm{Hom}(\mathcal{E}(I), \bar{\mathcal{E}}(J)\langle k\rangle[i]) & \longrightarrow & \mathrm{Hom}(u^*\mathcal{E}(I), u^*\bar{\mathcal{E}}(J)\langle k\rangle[i]) \\
\downarrow & & \downarrow \\
\mathrm{Hom}(\mathcal{E}(I), \mathcal{E}(J)\langle k\rangle[i+1]) & \longrightarrow & \mathrm{Hom}(u^*\mathcal{E}(I), u^*\mathcal{E}(J)\langle k\rangle[i+1]) \\
\downarrow & & \downarrow
\end{array}
$$

For $i = 0, 1$, the terms in the fourth row vanish, and the horizontal maps in the first two rows are isomorphisms by part (1). It follows that the third horizontal map is an isomorphism. We have proved (9.1).

The proof of (9.2) is similar, using the diagram

$$
\begin{array}{ccc}
\downarrow & & \downarrow \\
\mathrm{Hom}(\mathcal{E}(I), \mathcal{E}(J)\langle k\rangle[i-1]) & \longrightarrow & \mathrm{Hom}(u^*\mathcal{E}(I), u^*\mathcal{E}(J)\langle k\rangle[i-1]) \\
\downarrow & & \downarrow \\
\mathrm{Hom}(\bar{\mathcal{E}}(I), \mathcal{E}(J)\langle k\rangle[i]) & \longrightarrow & \mathrm{Hom}(u^*\bar{\mathcal{E}}(I), u^*\mathcal{E}(J)\langle k\rangle[i]) \\
\downarrow & & \downarrow \\
\mathrm{Hom}(\mathcal{E}(I), \mathcal{E}(J)\langle k\rangle[i]) & \longrightarrow & \mathrm{Hom}(u^*\mathcal{E}(I), u^*\mathcal{E}(J)\langle k\rangle[i]) \\
{\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\
\mathrm{Hom}(\mathcal{E}(I), \mathcal{E}(J)\langle k\rangle[i]) & \longrightarrow & \mathrm{Hom}(u^*\mathcal{E}(I), u^*\mathcal{E}(J)\langle k\rangle[i]) \\
\downarrow & & \downarrow
\end{array}
\tag{9.7}
$$

Finally, the isomorphism (9.3) follows from (9.1) using a commutative diagram very similar to (9.7). ☐

Proposition 9.3. *The functor $u_0^* : \mathrm{Perv}((\overline{\mathcal{G}r}_{\check{\varpi}_1})_0/T, \Bbbk) \to \mathrm{Perv}((\mathbb{A}^n)_0/T, \Bbbk)$ is fully faithful.*

Proof. Since $(\mathbb{A}^n)_0$ meets every stratum in $(\overline{\mathcal{G}r}_{\check{\varpi}_1})_0$, the functor u_0^* kills no nonzero object. It follows immediately that u_0^* is faithful.

We will prove that u_0^* is also full in the case where $\Bbbk$ is not a field. The proof in the field case is easier; the appropriate modifications are left to the reader.

For any $\mathcal{F} \in \mathrm{Perv}((\overline{\mathcal{G}r}_{\check{\varpi}_1})_0/T, \Bbbk)$, we first claim that

$$\mathrm{Hom}(\mathcal{E}(I), \mathcal{F}) \to \mathrm{Hom}(u_0^*\mathcal{E}(I), u_0^*\mathcal{F}) \tag{9.8}$$

is an isomorphism. The proof is by induction on the length of a filtration of $\mathcal{F}$ as in Lemma 9.1. If $\mathcal{F}$ itself is isomorphic to some $\mathcal{E}(J)\langle k\rangle$ or $\bar{\mathcal{E}}(J)\langle k\rangle$, the claim holds by (the $i = 0$ case of) Lemma 9.2. For general $\mathcal{F}$, the claim follows by a five-lemma argument involving the $i = 1$ case of Lemma 9.2.

Next, we claim that for any $\mathcal{F} \in \mathrm{Perv}((\overline{\mathcal{G}r}_{\check{\varpi}_1})_0/T, \Bbbk)$, the map

$$\mathrm{Hom}(\mathcal{E}(I), \mathcal{F}[1]) \to \mathrm{Hom}(u_0^*\mathcal{E}(I), u_0^*\mathcal{F}[1]) \tag{9.9}$$

is injective. Again, if $\mathcal{F}$ is isomorphic to $\mathcal{E}(J)\langle k\rangle$ or $\bar{\mathcal{E}}(J)\langle k\rangle$, the claim holds by Lemma 9.2. The general case follows by induction, the four-lemma, and Lemma 9.2 again.

It remains to show that for all $\mathcal{G} \in \mathrm{Perv}((\overline{\mathcal{G}r}_{\check{\varpi}_1})_0/T, \Bbbk)$, the map

$$\mathrm{Hom}(\mathcal{G}, \mathcal{F}) \to \mathrm{Hom}(u_0^*\mathcal{G}, u_0^*\mathcal{F}) \tag{9.10}$$

is an isomorphism. This holds by induction on the length of a filtration of $\mathcal{G}$ as in Lemma 9.1, using (9.8), (9.9), and the four-lemma. $\square$

9.2. **The nearby cycles sheaf.** It makes sense to copy the definitions of direct sums of parity sheaves from Section 4: we may speak, for instance, of (analogues of) $\mathrm{E}_i^{\oplus}$ or $\mathbf{E}^{\diamondsuit}$ on $\overline{\mathcal{G}r}_{\check{\varpi}_1}$. We do not know whether all the lemmas from Section 4 hold on $\overline{\mathcal{G}r}_{\check{\varpi}_1}$, but there are a few steps in the calculation that rely only on those cases of Proposition 3.3 that overlap with Proposition 8.6: see Remarks 4.3, 4.8, and 6.1.

In particular, by Remark 6.1, it makes sense to consider the object

$$\boldsymbol{\mathcal{Z}} = \mathbf{E}^{\diamondsuit} \underset{\leftarrow [1]}{\circlearrowleft}^{\underline{\boldsymbol{\epsilon}}+\underline{\boldsymbol{\eta}}-\bar{\xi}\boldsymbol{N}}$$

in $D^{\mathrm{mix}}((\overline{\mathcal{G}r}_{\check{\varpi}_1})_0/T, \Bbbk)$. We will sometimes write $\boldsymbol{\mathcal{Z}}^{\mathbb{A}^n}$ and $\boldsymbol{\mathcal{Z}}^{\overline{\mathcal{G}r}_{\check{\varpi}_1}}$ to distinguish the object defined here from that defined in Section 6.

Lemma 9.4. *For any $\mathcal{F} \in D^{\mathrm{mix}}_{\mathbb{G}_{\mathrm{m}}}((\overline{\mathcal{G}r}_{\check{\varpi}_1})_\eta/T, \Bbbk)$, there is a natural isomorphism*

$$u_0^*\Psi_f(\mathcal{F}) \cong \Psi_f(u_\eta^*\mathcal{F}).$$

Moreover, this isomorphism is compatible with the monodromy endomorphisms.

This lemma is an instance of a very general fact about the commutativity of nearby cycles and restriction to an open subset.

Proof. Let $\mathbf{i} : (\overline{\mathcal{G}r}_{\check{\varpi}_1})_0 \hookrightarrow \overline{\mathcal{G}r}_{\check{\varpi}_1}$ and $\mathbf{j} : (\overline{\mathcal{G}r}_{\check{\varpi}_1})_\eta \hookrightarrow \overline{\mathcal{G}r}_{\check{\varpi}_1}$ be the inclusion maps, and let $\mathbf{i}'$ and $\mathbf{j}'$ be their analogues for $\mathbb{A}^n$. It is an exercise in the recollement formalism to show that $u^* \circ \mathbf{j}_* \cong \mathbf{j}'_* \circ u_\eta^*$ and that $u_0^* \circ \mathbf{i}^* \cong (\mathbf{i}')^* \circ u^*$. It follows immediately from the definitions that u_0^* commutes with Mon, and that u_η^* commutes with $\mathcal{J}$. The result follows. $\square$

Theorem 9.5. *On $\overline{\mathcal{G}r}_{\check{\varpi}_1}$, we have $\Psi_f(\mathcal{E}_\eta) \cong \boldsymbol{\mathcal{Z}}\langle -1\rangle$. In particular, $\Psi_f(\mathcal{E}_\eta)$ is a perverse sheaf. The monodromy endomorphism is given by the map $\boldsymbol{N} : \boldsymbol{\mathcal{Z}} \to \boldsymbol{\mathcal{Z}}\langle 2\rangle$.*

Proof. In the body of this proof, all sheaves will be labelled with a superscript indicating the variety on which they live. By Lemma 9.4, we have

$$u_0^*\Psi_f(\mathcal{E}_\eta^{\overline{\mathcal{G}r}_{\check{\varpi}_1}}) \cong \Psi_f(u_\eta^*\mathcal{E}_\eta^{\overline{\mathcal{G}r}_{\check{\varpi}_1}}) \cong \Psi_f(\mathcal{E}_\eta^{\mathbb{A}^n}). \tag{9.11}$$

On the other hand, it follows from Lemma 8.4 that

$$u_0^*\boldsymbol{\mathcal{Z}}^{\overline{\mathcal{G}r}_{\check{\varpi}_1}} \cong \boldsymbol{\mathcal{Z}}^{\mathbb{A}^n}.$$

Combining these observations with Theorem 6.5, we see that

$$u_0^*\Psi_f(\mathcal{E}_\eta^{\overline{\mathcal{G}r}_{\check{\varpi}_1}}) \cong u_0^*\boldsymbol{\mathcal{Z}}^{\overline{\mathcal{G}r}_{\check{\varpi}_1}}\langle -1\rangle.$$

Theorem 6.5 also tells us that the right-hand side of (9.11) is perverse. Since u_0^* is t-exact and kills no nonzero perverse sheaf, we see that $\Psi_f(\mathcal{E}_\eta^{\overline{\mathcal{G}r}_{\check{\varpi}_1}})$ must be perverse as well. Then, by Proposition 9.3, we conclude that $\Psi_f(\mathcal{E}_\eta^{\overline{\mathcal{G}r}_{\check{\varpi}_1}}) \cong \boldsymbol{\mathcal{Z}}^{\overline{\mathcal{G}r}_{\check{\varpi}_1}}\langle -1\rangle$. Since (9.11) identifies the monodromy endomorphisms on both sides, the description of this map in Theorem 6.5 remains valid on $\overline{\mathcal{G}r}_{\check{\varpi}_1}$. $\square$

10. The monodromy filtration

The discussion in this section applies to both $\mathbb{A}^n$ and $\overline{\mathcal{G}r}_{\check{\varpi}_1}$. The nilpotent endomorphism $\boldsymbol{N} : \boldsymbol{\mathcal{Z}} \to \boldsymbol{\mathcal{Z}}\langle 2\rangle$ determines a canonical filtration on $\boldsymbol{\mathcal{Z}}$, as described in the following lemma. This filtration is called the *monodromy filtration*.

Lemma 10.1. *There is a unique increasing filtration $M_\bullet\boldsymbol{\mathcal{Z}}$ on $\boldsymbol{\mathcal{Z}}$ with the following properties:*

(1) *For all i, we have $\boldsymbol{N}(M_i\boldsymbol{\mathcal{Z}}) \subset (M_{i-2}\boldsymbol{\mathcal{Z}})\langle 2\rangle$.*
(2) *For $i \geq 0$, the map $\boldsymbol{N}^i : \boldsymbol{\mathcal{Z}} \to \boldsymbol{\mathcal{Z}}\langle 2i\rangle$ induces an isomorphism*

$$\mathrm{gr}_i^M \boldsymbol{\mathcal{Z}} \xrightarrow{\sim} \mathrm{gr}_{-i}^M \boldsymbol{\mathcal{Z}}\langle 2i\rangle.$$

The analogous statement for a nilpotent operator on a vector space is well known: see, for instance, [SZ, Proposition 2.1], from which the following proof is adapted.

Proof. Let $m > 0$ be such that $\boldsymbol{N}^m = 0$. (Of course, such an m exists by [A, Remark 5.6].) If the desired filtration exists, it must have the following properties:

(1) $M_k\boldsymbol{\mathcal{Z}} = 0$ for all $k \leq -m$.
(2) For $i \geq 0$, $M_i\boldsymbol{\mathcal{Z}}$ is the preimage under $\boldsymbol{N}^{i+1}$ of $M_{-i-2}\boldsymbol{\mathcal{Z}}\langle 2+2i\rangle$.
(3) For $i > 0$, $M_{-i}\boldsymbol{\mathcal{Z}} = \boldsymbol{N}^i(M_i\boldsymbol{\mathcal{Z}})\langle -2i\rangle$.

The second condition above is a restatement of the fact that $\boldsymbol{N}^{i+1} : \mathrm{gr}_{i+1}^M \boldsymbol{\mathcal{Z}} \to \mathrm{gr}_{-i-1}^M \boldsymbol{\mathcal{Z}}\langle 2i+2\rangle$ is injective, and the third corresponds to the fact that $\boldsymbol{N}^i : \mathrm{gr}_i^M \boldsymbol{\mathcal{Z}} \to \mathrm{gr}_{-i}^M \boldsymbol{\mathcal{Z}}\langle 2i\rangle$ is surjective.

But it is now easy to see that the three conditions above actually determine a unique filtration on $\boldsymbol{\mathcal{Z}}$. ☐

As explained in [SZ, Remark 2.3] (see also [I, §3.4]), there is an explicit formula for the monodromy filtration in terms the kernel and image filtrations: we have

$$M_k\boldsymbol{\mathcal{Z}} = \sum_{p-q=k} (\ker \boldsymbol{N}^{p+1}) \cap (\mathrm{im}\, \boldsymbol{N}^q\langle -2q\rangle).$$

Theorem 10.2. *The associated graded of the monodromy filtration on $\boldsymbol{\mathcal{Z}}$ is given by*

$$\mathrm{gr}_k^M \boldsymbol{\mathcal{Z}} \cong \bigoplus_{\substack{r,s\geq 0\\ r-s=w}} \mathcal{E}_{n-1-r-s}^{\oplus}\langle w\rangle.$$

Proof. We begin with a calculation on the underlying graded parity sheaf $\mathbf{E}^\diamondsuit$ of $\boldsymbol{\mathcal{Z}}$. Unpacking the definition of $\mathbf{E}^\diamondsuit$, we have

$$\mathbf{E}^\diamondsuit = \bigoplus_{i=0}^{n-1} \mathrm{E}_i^{\oplus} = \bigoplus_{i=0}^{n-1}\bigoplus_{j=1}^{n-i} \mathcal{E}_i^{\oplus}\langle -n+i-1+2j\rangle.$$

Let us rewrite this sum by making the following substitutions: let $k = -n+i-1+2j$, $q = n-i-j$, and $p = j-1$. Then $-n+1 \leq k \leq n-1$. We have $q \geq 0$ and $p \geq 0$, $p-q = k$, and $p+q = n-i-1$. Then

$$\mathbf{E}^\diamondsuit = \bigoplus_{k=-n+1}^{n-1} \bigoplus_{\substack{p,q\geq 0\\ p-q=k}} \mathcal{E}_{n-1-p-q}^{\oplus}\langle k\rangle.$$

Next, the map $\boldsymbol{N}$ is the direct sum of operators $N = N_{\mathrm{E}_i^{\oplus}} : \mathrm{E}_i^{\oplus} \to \mathrm{E}_i^{\oplus}\langle 2\rangle$ for $i = 0, \ldots, n-1$. Since $\mathbf{E}^\diamondsuit$ is a (mixed) perverse sheaf, it makes sense to consider the kernels and images of these operators. From the definitions, we have

$$\ker N^{p+1}_{\mathrm{E}_i^\oplus} = \bigoplus_{j=1}^{p+1} \mathcal{E}_i^\oplus\langle -n+i-1+2j\rangle, \qquad \operatorname{im} N^q_{\mathrm{E}_i^\oplus\langle -2q\rangle} = \bigoplus_{j=1}^{n-i-q} \mathcal{E}_i^\oplus\langle -n+i-1+2j\rangle$$

$$(\ker N^{p+1}_{\mathrm{E}_i^\oplus}) \cap (\operatorname{im} N^q_{\mathrm{E}_i^\oplus\langle -2q\rangle}) = \bigoplus_{j=1}^{\min\{p+1,n-i-q\}} \mathcal{E}_i^\oplus\langle -n+i-1+2j\rangle.$$

Now let p and q vary, subject to the constraint that $p-q=k$. Let $a = \min\{p+1, n-i-q\}$, and let $b = \max\{p+1, n-i-q\}$. We have $a \le b$ and $a+b = n-i+1+k$, so $2a \le n-i+1+k$. We conclude that

$$\sum_{p-q=k} (\ker N^{p+1}_{\mathrm{E}_i^\oplus}) \cap (\operatorname{im} N^q_{\mathrm{E}_i^\oplus\langle -2q\rangle}) = \bigoplus_{j=1}^{\lfloor (n-i+1+k)/2\rfloor} \mathcal{E}_i^\oplus\langle -n+i-1+2j\rangle.$$

Let $r = j-1$ and $s = n-i-j$. The conditions $1 \le j \le (n-i+1+k)/2$ are equivalent to $r \ge 0$ and $r-s \le k$.

$$\sum_{p-q=k} (\ker N^{p+1}_{\mathrm{E}_i^\oplus}) \cap (\operatorname{im} N^q_{\mathrm{E}_i^\oplus\langle -2q\rangle}) = \bigoplus_{w\le k} \bigoplus_{\substack{r,s\ge 0\\ r+s=n-i-1\\ r-s=w}} \mathcal{E}_i^\oplus\langle w\rangle.$$

Now take the sum over all i. We conclude that

$$\sum_{p-q=k} (\ker \boldsymbol{N}^{p+1}) \cap (\operatorname{im} \boldsymbol{N}^q\langle -2q\rangle) = \bigoplus_{w\le k} \bigoplus_{\substack{r,s\ge 0\\ r-s=w}} \mathcal{E}^\oplus_{n-1-r-s}\langle w\rangle.$$

Let $M_k\mathbf{E}^\diamondsuit$ denote this graded parity sheaf. Degree considerations show that the differential on $\boldsymbol{\mathcal{Z}}$ induces a differential on $M_k\mathbf{E}^\diamondsuit$, so we obtain a well-defined mixed perverse sheaf $M_k\boldsymbol{\mathcal{Z}}$. The resulting filtration $M_\bullet\boldsymbol{\mathcal{Z}}$ of $\boldsymbol{\mathcal{Z}}$ satisfies the conditions of Lemma 10.1, so it must be the monodromy filtration. The formula for the associated graded is immediate from this description. $\square$

11. Examples

In this section, we unpack the definition of $\boldsymbol{\mathcal{Z}}$ and write it down explicitly for $n \le 3$. For $n = 1$, we have $\mathbf{E}^\diamondsuit = \mathrm{E}_0^\oplus = \mathcal{E}_0^\oplus = \mathcal{E}(\varnothing)$. The maps $\boldsymbol{N}$, $\underline{\boldsymbol{\epsilon}}$, and $\underline{\boldsymbol{\eta}}$ are zero, so we just have

$$\boldsymbol{\mathcal{Z}} = \mathcal{E}(\varnothing)$$

with zero differential.

For $n = 2$, we have

$$\begin{aligned}\mathbf{E}^\diamondsuit &= \mathrm{E}_0^\oplus \oplus \mathrm{E}_1^\oplus = (\mathcal{E}_0^\oplus\langle -1\rangle \oplus \mathcal{E}_0^\oplus\langle 1\rangle) \oplus \mathcal{E}_1^\oplus \\ &= (\mathcal{E}(\varnothing)\langle -1\rangle \oplus \mathcal{E}(\varnothing)\langle 1\rangle) \oplus \mathcal{E}(\{1\}) \oplus \mathcal{E}(\{2\}).\end{aligned}$$

We arrange these summands in order by Tate twist, and then expand the definitions of $\boldsymbol{N}$, $\underline{\boldsymbol{\epsilon}}$, and $\underline{\boldsymbol{\eta}}$ to obtain

$$\boldsymbol{\mathcal{Z}} = \begin{tikzcd} \mathcal{E}(\varnothing)\langle 1\rangle \arrow[r, "{[1]}", "{\left[\begin{smallmatrix}\dot\eta_1\\ -\dot\eta_2\end{smallmatrix}\right]}"'] \arrow[rr, bend left, "{-\bar\xi\cdot\mathrm{id}}", "{[1]}"'] & (\mathcal{E}(1)\oplus\mathcal{E}(2)) \arrow[r, "{[1]}", "{[\,\dot\epsilon_1\ \ -\dot\epsilon_2\,]}"'] & \mathcal{E}(\varnothing)\langle -1\rangle \end{tikzcd}$$

Alternatively, we may work with *parity sequences* (as defined in [AMRW]) in place of graded parity sheaves. In this language, the shifts [1] along the arrows are absorbed into the objects, and the whole picture looks more like a "classical" chain complex of parity sheaves. (Objects in mixed modular derived categories in [AR1, AR2, AMRW] were drawn in this way.) For $n = 2$, our object $\boldsymbol{\mathcal{Z}}$ looks like

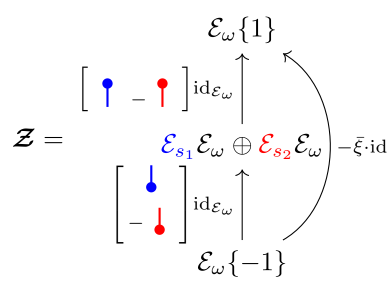
$$\boldsymbol{\mathcal{Z}} = \begin{tikzcd}
\mathcal{E}(\varnothing)\{1\} \\
\mathcal{E}(1)\oplus\mathcal{E}(2) \arrow[u, "{[\,\dot\epsilon_1\ -\dot\epsilon_2\,]}"] \\
\mathcal{E}(\varnothing)\{-1\} \arrow[u, "{\left[\begin{smallmatrix}\dot\eta_1\\ -\dot\eta_2\end{smallmatrix}\right]}"] \arrow[uu, bend right, "-\bar\xi\cdot\mathrm{id}"']
\end{tikzcd}$$

For $n = 3$, we have

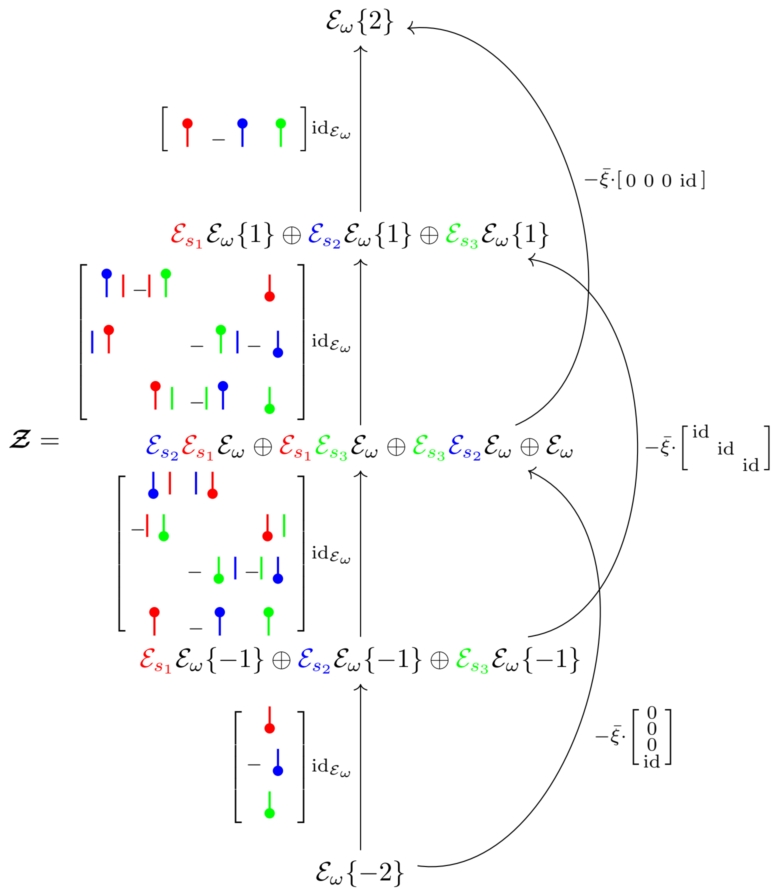
$$\boldsymbol{\mathcal{Z}} = \begin{tikzcd}
\mathcal{E}(\varnothing)\{2\} \\
\mathcal{E}(1)\{1\}\oplus\mathcal{E}(2)\{1\}\oplus\mathcal{E}(3)\{1\} \arrow[u, "{[\,\dot\epsilon_1\ -\dot\epsilon_2\ \dot\epsilon_3\,]}"] \\
\mathcal{E}(1,2)\oplus\mathcal{E}(1,3)\oplus\mathcal{E}(2,3)\oplus\mathcal{E}(\varnothing) \arrow[u, "{\left[\begin{smallmatrix}\dot\epsilon_2 & -\dot\epsilon_3 & & \dot\eta_1\\ \dot\epsilon_1 & & -\dot\epsilon_3 & -\dot\eta_2\\ & \dot\epsilon_1 & -\dot\epsilon_2 & \dot\eta_3\end{smallmatrix}\right]}"] \arrow[uu, bend right, "{-\bar\xi\cdot[\,0\ 0\ 0\ \mathrm{id}\,]}"'] \\
\mathcal{E}(1)\{-1\}\oplus\mathcal{E}(2)\{-1\}\oplus\mathcal{E}(3)\{-1\} \arrow[u, "{\left[\begin{smallmatrix}\dot\eta_2 & \dot\eta_1 & \\ -\dot\eta_3 & & \dot\eta_1\\ & -\dot\eta_3 & -\dot\eta_2\\ \dot\epsilon_1 & -\dot\epsilon_2 & \dot\epsilon_3\end{smallmatrix}\right]}"] \arrow[uu, bend right, "{-\bar\xi\cdot\left[\begin{smallmatrix}\mathrm{id} & & \\ & \mathrm{id} & \\ & & \mathrm{id}\end{smallmatrix}\right]}"'] \\
\mathcal{E}(\varnothing)\{-2\} \arrow[u, "{\left[\begin{smallmatrix}\dot\eta_1\\ -\dot\eta_2\\ \dot\eta_3\end{smallmatrix}\right]}"] \arrow[uu, bend right, "{-\bar\xi\cdot\left[\begin{smallmatrix}0\\0\\0\\\mathrm{id}\end{smallmatrix}\right]}"']
\end{tikzcd}$$

Finally, on $\overline{\mathcal{G}r}_{\check\varpi_1}$, we can redraw these complexes using the Elias–Williamson calculus to indicate the maps in the differentials. We will use different colors for the various simple reflections. For brevity, we omit the convolution symbol $\star$.

For $n = 2$, using blue for s_1 and red for s_2, we have



For $n = 3$, we use red for s_1, blue for s_2, and green for s_3. We have



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