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GENERIC SPACE CURVES, GEOMETRY AND NUMEROLOGY

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Abstract. A projective curve $\Gamma \in P^3(C)$ defines a stratification of P^3 according to the types of the singularities of the projection of Γ from the variable point. In this paper we calculate the degrees of these strata, assuming that Γ is projection-generic in the sense of [8].

We use geometrical properties of the stratifications of P^3 and of the blow-up B_Γ of P^3 along Γ (with exceptional set denoted E_Γ) introduced in [9] to introduce several auxiliary curves: more precisely, there are three 2-dimensional strata: the surface of tangents to Γ, the surface of T-secants (i.e. lines joining two points with coplanar tangents), and the surface of 3-secants. We obtain three plane curves by intersecting these with a generic plane. Three curves in E_Γ were introduced in [9]. We also have three curves in $\Gamma \times \Gamma$ closely related to them. Our numerical results are obtained by applying the genus and related formulae to these curves.

Introduction

A projective curve $\Gamma \subset P^3(\mathbb{C})$ defines a stratification of P^3 according to the types of the singularities of the projection of Γ from the variable point. The object of this paper is to calculate the degrees of (the closures of) these strata, and the number of special points of each type on Γ, in terms of the degree d and genus g of Γ. We will assume throughout that Γ is projection-generic in the sense of [8]. This has two advantages: in [9] we obtained local normal forms for this stratification; and we will see that we obtain precise answers, without needing to interpret our numbers as being counted with multiplicities.

Our techniques are essentially classical. We will use geometrical properties of the stratifications of P^3 and of the blow-up B_Γ of P^3 along Γ (with exceptional set denoted E_Γ) introduced in [9] to introduce several auxiliary curves: the study of these and of their interrelations is itself of some interest. More precisely, there are three 2-dimensional strata: the surface of tangents to Γ, the surface of T-secants (i.e. lines joining two points with coplanar tangents), and the surface of 3-secants. We obtain 3 plane curves Π_* by intersecting these with a generic plane. Three curves E_* in E_Γ were introduced in [9]. We also have three curves T_* in $\Gamma \times \Gamma$ closely related to them. Our numerical results are obtained by applying the genus and related formulae to these curves.

We introduce our notation in §1, and describe the singularities of the auxiliary curves in §2 (in the case of the T_* the proofs are deferred to §7). In §3 we begin calculations, and in Lemma 3.2 express many of our degrees in terms of parameters k_i. In §4 we analyse the correspondences T_*, give the degrees of the 2-dimensional strata in Proposition 5.1, and calculate the k_i in Proposition 4.2. In §5 we analyse the E_* and in Proposition 5.2 complete the count of points of special types on Γ. In §6 we analyse the Π_*, and determine the degrees of the remaining curve strata, which we list in Theorem 6.1. After some experiment, I have settled on collecting all these formulae by powers of g, since it turns out that in all but two cases, the term not involving g factors over $\mathbb{Q}[d]$ into linear factors.

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1. Recall of results and notations of [8]

If Γ is a smooth space curve, the projections Γ_P of Γ from a variable point P of space give a 3-parameter family of maps to the plane. In [8], we analysed this situation in the real C^∞ case, and gave explicit genericity conditions, (PG1)-(PG6) below, defining a set of space curves which is open and dense in the family of all such curves. These conditions also make sense in the complex case, and (as in [9]) we will assume them throughout the paper. Among projective algebraic curves, it is not clear that those satisfying the conditions form a dense set, though this should not be hard to establish at least for rational curves of high enough degree. In practice, this fails for degree 1 or 2, and seems to hold for degrees ≥ 3.

The projection along a line L through P has a singular point if L meets Γ in more than 1 point, or if L is tangent to Γ. Call a line meeting Γ in r points an $r-$secant; it is a $T-r-$secant if the tangents at 2 of the points are coplanar (we omit r if $r = 2$). Write $T_Q\Gamma$ for the tangent at a point $Q \in \Gamma$, and $O_Q\Gamma$ for the osculating plane at Q.

Hypothesis (PG1) is that the family of projections of Γ from points not on Γ is generic in the sense that it versally unfolds the singularities of any curve of the set. More precisely, the induced map is transverse to each stratum; thus the unfolding is versal in each case except that of the X_9 stratum (quadruple points), where we only have topological versality.

It follows that the curvature of Γ is non-zero, but the torsion may vanish: if it vanishes at Q, we call Q a stall. Equivalently, here the local intersection number of Γ with $O_Q\Gamma$ exceeds 3; the hypothesis implies that it is at most 4.

It also follows that for $P \in P^3\setminus\Gamma$, the types of singularities of the projection have codimension ≤ 3, and the points P such that the sum of the codimensions of singularities of Γ_P is c form smooth $(3-c)$-dimensional manifolds which regularly stratify $P^3 \setminus \Gamma$; normal forms are given by model versal unfoldings of the singularities that occur (except for the X_9 stratum; a precise normal form for this case was given in [9, Lemma 7.2]).

We partition $P^3 \setminus \Gamma$ as follows. If Σ denotes a list of singularities, $S^o(\Sigma)$ consists of points P such that Σ is the list of singularities of Γ_P. Define also

$S(\Sigma)$ is the closure of $S^o(\Sigma)$ in P^3, and

$n(\Sigma)$ is the degree of $S(\Sigma)$.

We will calculate these degrees in all cases where $S(\Sigma)$ has dimension 1 or 2.

For a codimension 0 set of projections we have only normal crossing (A_1) singularities. Apart from these, in codimension 1, Γ_P can have

a cusp (A_2) if, for some $Q \in \Gamma$, $P \in T_Q\Gamma$;

a tacnode (A_3) if P lies on a T-secant QR of Γ; or

a triple point (D_4) if P lies on a 3-secant QRS of Γ.

For a codimension 2 set of points P, Γ_P can have two codimension 1 singularities, or one of A_4, A_5, D_5, D_6 or X_9 (in Arnold's notation, but here we have maps $\mathbb{C} \to \mathbb{C}^2$, not functions on $\mathbb{C}^2$). In codimension 3 we have A_6, A_7, D_8 and combinations of singularities of lower codimension.

In particular, any T-secant contains a unique point, its T-centre, projection of Γ from which gives an A_5, rather than an A_3 singularity; for a T-3-secant a D_8 rather than a D_6.

We also contemplate projections of Γ from points of itself. Since Γ has nowhere zero curvature, each projection $\Gamma_P := \pi_P(\Gamma)$ with $P \in \Gamma$ is well defined and is again given by a smooth map. We can, at least locally, regard $\{\Gamma_P\}$ as a 1-parameter family of parametrised plane curves. Hypothesis (PG2) is that the family of projections Γ_P of Γ from points $P \in \Gamma$ has generic singularities.

To fit these into the family of projections from points $P \notin \Gamma$, write $\pi_\Gamma : B_\Gamma \to P^3$ for the blow-up along Γ and E_Γ for the exceptional set; thus a point of E_Γ is a pair (P, Π) with $P \in \Gamma$

and Π a plane through $T_P\Gamma$. There is a natural projection $\pi_E : E_\Gamma \to \Gamma$, which is a fibre bundle with fibre a projective line. Now define a family of curves $\{\Phi_z : z \in B_\Gamma\}$ by:

if $z \notin E_\Gamma$, so $z \in B_\Gamma \setminus \Gamma$, set $\Phi_z := \Gamma_z$,

if $z = (P, \Pi) \in E_\Gamma$, set $\Phi_z := \Gamma_P \cup L$, where $L := \pi_P(\Pi)$.

Thus the line L goes through the point $Y_P := \pi_P(T_P\Gamma)$. This is a flat family: near any point there is a smooth function whose zero set meets the fibre over z in Φ_z.

If Φ, Φ' are plane curves and $P \in \Phi \cap \Phi'$, define $\kappa_P(\Phi, \Phi')$ to be the local intersection number at P minus 1; write $\kappa(\Phi, \Phi')$ for the sum over all $P \in \Phi \cap \Phi'$. We will call $P \in \Gamma$ a *special point* if either Γ_P fails to have normal crossings or, for some line L through Y_P, $\kappa(L, \Gamma_P) \geq 2$: condition (PG4) is that we always have $\kappa(L, \Gamma_P) \leq 2$.

We next list the types of special point on Γ and our notation for them (we follow [9] rather than [8]). If Γ_P itself fails to have normal crossings, it has a singular point Z_P. There are three cases, according to the type of the singularity.

α: type A_2: $P \in T_Q\Gamma$ for some $Q \in \Gamma$,

β: type A_3: we have a T-3-secant $P(QR)$,

γ: type D_4: we have a 4-secant $PQRS$.

Hypothesis (PG3) is that for cases α, β and γ, Y_PZ_P is transverse to Γ_P at all points.

We say P has type δ if Y_P is a double point on Γ_P, i.e. if $Q \in T_P\Gamma$ for some $Q \neq P$.

If Γ_P has normal crossings and $\Gamma_P \cup L$ does not, then $\kappa(L, \Gamma_P) > 0$. Excluding cases α, β, γ, δ, we have $\kappa(L, \Gamma_P) = 1$ if either

a: L touches Γ_P at Y_P ($\Pi = O_P\Gamma$),

b: L touches Γ_P elsewhere (Π contains a tangent line $T_Q\Gamma$), or

c: L passes through a node of Γ_P (Π contains a trisecant PQR).

These cases occur when the point (P, Π) lies on certain curves in E_Γ. We denote these curves by E_a, E_b, E_c respectively.

In all cases, $\kappa(L, \Gamma_P) \leq 2$. The cases $\kappa(L, \Gamma_P) = 2$ are enumerated as follows, where $q, r, \ldots$ denote the images of $Q, R, \ldots$ under projection from P.

ab: L touches Γ_P at p and q: $T_Q\Gamma \subset O_P\Gamma = \Pi$.

ac: L touches Γ_P at p and goes through a double point $r = s$: $PRS \subset O_P\Gamma = \Pi$.

bb: L touches Γ_P at q and r: $T_P\Gamma$, $T_Q\Gamma$ and $T_R\Gamma$ lie in Π.

bc: L touches Γ_P at q and passes through a double point $r = s$: Π contains $T_P\Gamma$, $T_Q\Gamma$ and the trisecant PRS.

cc: L_P passes through double points $q = r$ and $s = t$: Π contains $T_P\Gamma$ and the trisecants PQR and PST.

a_2: L_P is an inflexional tangent at p: P is a stall on Γ, $\Pi = O_P\Gamma$.

b_2: L_P is an inflexional tangent at q: $T_P\Gamma \subset O_Q\Gamma = \Pi$.

c_2: L is tangent at q to a double point $q = r$ of Γ_P: we have a T-trisecant PQR with $T_P\Gamma$, $T_Q\Gamma$ both in Π.

Hypothesis (PG5) states that the curve E_c has transverse intersections with E_a, E_b and E_c at $S(ac)$, $S(bc)$ and $S(cc)$ respectively.

Finally, hypothesis (PG6) is that for no $P \in \Gamma$ can we have more than one of the cases α, β, γ, δ, *ab*, *ac*, *bb*, *bc*, *cc*, a_2, b_2, c_2.

If $P \in \Gamma$ is not of type α, β, γ or δ, there is at most one line L through Y_P with $\kappa(L, \Gamma_P) \geq 2$. Thus if there is a special point of E_Γ in $\pi_E^{-1}(P)$, it is unique and we denote its type by the same symbol as above.

If Γ_P has a singular point Z_P, and (P, Π) such that $L = Y_PZ_P$, we also denote the type of (P, Π) by the same letter as for P. For P of type δ we distinguish (P, Π) of type δ_1, with

$\Pi = O_P\Gamma$ and type δ_2, when Π passes through $T_Q\Gamma$: in each of these, L touches Γ_P at Y_P.

If X is any of α, β, γ, δ, ab, ac, bb, bc, cc, a_2, b_2, c_2 we write $S(X)$ for the set of points of Γ of type X, and $\#(X)$ for its cardinality. We calculate all the numbers $\#(X)$ below. We extend the stratification of $P^3 \setminus \Gamma$ to P^3 by declaring the strata in Γ to be the $S(X)$ just defined, and the rest of Γ to be a single stratum. We proved in [9] that this stratification is regular, and obtained normal forms at all points of Γ. We will make crucial use of these in this paper.

We now define our auxiliary curves. First, we have the three curves E_a, E_b and E_c in E_Γ defined above.

Next we have three curves in $\Gamma \times \Gamma$:
$(P, Q) \in T_a$: if $P \in O_Q\Gamma$,
$(P, Q) \in T_b$: if PQ is a T-secant,
$(P, Q) \in T_c$: if PQ is a trisecant.
For $Y \subset \Gamma \times \Gamma$, denote by Y^t the image of Y under interchange of factors.
There are just three 2-dimensional strata in P^3, which we will denote by $A := S(A_2)$, the surface of tangents to Γ; by $B := S(A_3)$, the surface of T-secants; and by $C := S(D_4)$ the surface of trisecants. Each of these is, by definition, a ruled surface. Choose a plane Π_0 transverse to all strata of the stratification of P^3, and define three curves in Π_0 by
$\Pi_a := \Pi_0 \cap A$, $\Pi_b := \Pi_0 \cap B$, $\Pi_c := \Pi_0 \cap C$.

2. Singularities of the auxiliary curves

We first consider the curves in E_Γ, which were analysed in our previous work. We are interested in singularities of the curves in relation to the projection π_E, which is a submersion on Γ. The result is as follows.

Lemma 2.1. [9, Theorem 1.2, Addendum 4.3, Lemma 4.4, Addendum 5.2] *The curves E_a, E_b and E_c in E_Γ are smooth, disjoint and submerse on Γ except as below.*
At a point of type ab, ac, bb, bc or cc, the two curves meet transversely.
At $S(a_2)$, E_a and E_b touch (simply).
At $S(b_2)$, E_b has a cusp with non-vertical tangent.
At $S(c_2)$, E_b and E_c touch.
At $S(\alpha)$, E_b and E_c meet transversely.
At $S(\beta)$, E_c touches the fibre.
At $S(\gamma)$, E_c has 3 transverse branches.
At $S(\delta_1)$, E_a and E_c meet transversely.
At $S(\delta_2)$, E_b touches the fibre.

Next we consider the plane sections Π_a, Π_b, Π_c of A, B and C. These have the same degrees as the corresponding surfaces, and have singularities only where the plane meets a singular set of the surface. If Γ has degree d, Π meets Γ in d points, and for any curve stratum $S(\Sigma)$ in $P^3 \setminus \Gamma$, Π meets $S(\Sigma)$ transversely in $n(\Sigma)$ points, and the local picture of strata at each is given by that in a versal unfolding of Σ. Denote by m_a, m_b and m_c the respective multiplicities of A, B and C along Γ.

We recall that B is the surface of tangents to $S(A_5)$, so has a cuspidal edge along $S(A_5)$. We do not need to list intersection points of these plane sections, since we obtain the same information from the mutual intersections of the surfaces A, B and C.

Lemma 2.2. *The curve Π_a has d A_2 singularities and $n(2A_2)$ A_1 singularities.*

The curve Π_b *has* $n(2A_3)$ A_1 *singularities,* $n(A_5)$ A_2 *singularities, and* d *ordinary singular points of multiplicity* m_b.

The curve Π_c *has* $n(2D_4)$ A_1 *singularities,* $n(X_9)$ X_9 *singularities, and* d *ordinary singular points of multiplicity* m_c.

To describe the singularities of the curves in $\Gamma \times \Gamma$ we need further notations. First we define notation for the points of $\Gamma \times \Gamma$ which play a rôle: these are related to special lines $PQ \in P^3$.

A special line of type A_4 is tangent to Γ at a stall P: a point of $S(a_2)$. Define $(P, P) \in W_1$, and W_1' to consist of points (Q, P) with $Q \neq P$, $Q \in O_P\Gamma$.

A special line of type D_5 is a secant PQ with $Q \in T_P\Gamma$: P has type δ, Q has type α. Define $(P, Q) \in W_2$, $(P, P) \in W_2'$.

A special line of type D_6 is a T-trisecant PQR with $T_Q\Gamma$, $T_R\Gamma$ coplanar: P has type β, Q, R have type c_2. Define $(P, Q) \in W_3$, $(Q, R) \in W_3'$.

A special line of type X_9 is a 4-secant $PQRS$: P, Q, R, S have type γ. Define $(P, Q) \in W_4$.

We also have special lines PQ where $T_P\Gamma \subset O_Q\Gamma$: P has type b_2, Q has type ab. Define $(P, Q) \in W_5$.

Finally, we have special lines which are trisecants $PQR \subset O_P\Gamma$: P has type ac. Here Q, R are not special in the sense of [8], but the projections Γ_Q, Γ_R each have a flecnode. Define $(P, Q) \in W_6$.

For $T \subset \Gamma \times \Gamma$, write $I_1(T)$ for the set of singular points of projection on the first factor (classically known as coincidence points), and $I_2(T)$ for singular points of the second projection, so that $I_1(T^t) = (I_2(T))^t$. We call a point $(P, Q) \in I_1(T)$ simple if T is smooth at (P, Q) and the local intersection number of T with $\{P\} \times \Gamma$ is 2; similarly for $I_2(T)$. We now describe these points, and also the intersections of the T_* with the diagonal $\Delta \subset \Gamma \times \Gamma$ (classically known as united points) and with each other. The proofs will require detailed calculations, which we defer to §7.

Theorem 2.3. *(i) We have* $T_a \cap \Delta = T_b \cap \Delta = W_1$, $T_c \cap \Delta = W_2'$. *At* W_1, *the tangent to* T_a *is* $3t_p + t_q = 0$, *and to* T_b *is* $t_p + t_q = 0$.

(ii) We have $I_1(T_a) = W_5^t$, $I_2(T_a) = W_1'^t \cup W_2$, $I_1(T_b) = W_2 \cup W_5$, $I_1(T_c) = W_2^t \cup W_3 \cup W_4$, $I_2(T_c) = W_2 \cup W_3^t \cup W_4$. *All coincidence points except* W_4 *are simple.*

(iii) The curves T_a *and* T_b *are smooth; the singularities of* T_c *are simple nodes at points of type* W_4, *with 2 transverse branches, each tangent to neither fibre.*

(iv) We have $T_a \cap T_b = W_1 \cup W_2^t \cup W_5$, $T_a \cap T_c = W_2^t \cup W_6$, *and* $T_b \cap T_c = W_2 \cup W_2^t \cup W_3'$.

(v) The intersection number at each of these common points is $+1$, *except that at* W_2^t *the intersection number of* T_a^t *and* T_c *is 2.*

3. Preliminaries

We denote the degrees of the 2-dimensional strata by $d_a = n(A_2)$, $d_b = n(A_3)$ and $d_c = n(D_4)$. Denote also by m_a, m_b and m_c the respective multiplicities of A, B and C along Γ.

Lemma 3.1. *The multiplicities along* Γ *in* P^3 *are* $m_a = 2$, $m_b = 2(d - 3 + g)$ *and* $m_c = \frac{1}{2}(d - 2)(d - 3) - g$.

Proof. Since, as is well known, the tangent surface A has a cuspidal edge along Γ, $m_a = 2$.

The projection of Γ from a general point P of itself is a plane curve Γ_P with degree $d - 1$ and genus g, whose only singularities are simple nodes (type A_1). It follows from Plücker's formulae that such a curve has class $2(d - 2 + g)$ and that the number of nodes is $\frac{1}{2}(d - 2)(d - 3) - g$.

Now T-secants of Γ through P project to tangents from Y_P to Γ_P, hence there are just $2(d - 3 + g)$ of them; and trisecants through P project to lines joining Y_P to nodes of Γ_P. The result follows. □

The strata $S(A_4)$, $S(D_5)$, $S(D_6)$, $S(X_9)$ are unions of straight lines: write k_1, k_2, k_3, k_4 for the numbers of these lines, k_5 for the number of tangents to Γ which lie in an osculating plane at a different point, and k_6 for the number of trisecants PQR lying in the osculating plane $O_P\Gamma$. Several of our degrees can easily be expressed in terms of the k_i.

Lemma 3.2. *We have*

(i) $\#(a_2) = n(A_4) = n(A_6) = \#(W_1) = k_1$, *and* $\#(W_1') = (d-4)k_1$.
(ii) $\#(\alpha) = \#(\delta) = n(D_5) = \#(W_2) = \#(W_2') = k_2$.
(iii) $\#(\beta) = n(D_6) = n(D_8) = k_3$, $\#(c_2) = \#(W_3) = \#(W_3') = 2k_3$.
(iv) $\#(\gamma) = 4n(X_9) = 4k_4$, $\#(W_4) = 12k_4$.
(v) $\#(ab) = \#(b_2) = \#(W_5) = k_5$.
(vi) $\#(ac) = k_6$ *and* $\#(W_6) = 2k_6$.

Proof. (i) holds since $S(A_4)$ consists of k_1 lines, each of which is a tangent at a stall P, and contains a unique point of $S(A_6)$, and there are $\#(a_2)$ stalls. Moreover, each contributes one point $(P, P) \in W_1$, and $O_P\Gamma$ has intersection number 4 with Γ at P, hence there are $d-4$ further intersections Q with $(Q, P) \in W_1'$ (note that $O_P\Gamma$ cannot be tangent at a further point).

(ii) holds since $S(D_5)$ consists of k_2 lines, each of which is a tangent at a point $P \in S(\delta)$, meeting Γ again at a point $Q \in S(\alpha)$; it contributes one point (P, Q) to W_2 and one point (Q, Q) to W_2'.

(iii) holds since $S(D_6)$ consists of k_3 lines, each of which is a T-trisecant, meeting Γ in two points P, Q with coplanar tangents, each in $S(c_2)$, and one other point $R \in S(\beta)$, and contains a unique point of $S(D_8)$. It also contributes 2 points (R, P), (R, Q) to W_3 and 2 points (P, Q), (Q, P) to W_3'.

(iv) holds since $S(X_9)$ consists of k_4 4-secants $PQRS$, each of which meets Γ in 4 points of $S(\gamma)$. Any ordered pair from $PQRS$ gives a point of W_4.

(v) holds since k_5 counts the secants PQ with $T_P\Gamma \subset O_Q\Gamma$, and the point $P \in S(b_2)$, $Q \in S(ab)$, and $(P, Q) \in W_5$.

(vi) holds since there are k_6 trisecants $PQR \subset O_P\Gamma$, and $P \in S(ac)$, (P, Q) and (P, R) belong to W_6. □

We will make frequent use of the genus formula for a curve on an algebraic surface. The following version is the most convenient for us, since it does not assume the curve irreducible:

If M is a reduced curve on a smooth surface S with canonical class K_S, we have

$$[M].([M] + K_S) = \mu(M) - \chi(M).$$

Here $\mu(M)$ denotes the total Milnor number and $\chi(M)$ the (topological) Euler characteristic of the Riemann surface M. This formula is easily deduced from the traditional version (see e.g. [1, 1.15]). It also follows from a routine topological argument (see e.g. [7, Theorem 6.4.1]) that the numbers $\mu(M_t) - \chi(M_t)$ are constant in a family of curves M_t.

In particular, if M is a plane curve of degree d, we obtain the Plücker relation $\mu(M)-\chi(M) = d(d-3)$.

Next we need the Plücker relations for space curves which are given in [3, p.270]. Let Δ be a reduced and non-planar space curve (it need not be projection-generic). We have invariants:

$\chi(\Delta)$, the Euler characteristic of Δ,
$r_0(\Delta)$, the degree, the number of points in which Δ meets a general plane,
$r_1(\Delta)$, the rank, the number of tangent lines to Δ meeting a general line,
$r_2(\Delta)$, the class, the number of osculating planes of Δ containing a general point.

At a point P where, in some local co-ordinates, we have local parametrisations

$$x_1 = a_1 t^{b_1} + \dots, \quad x_2 = a_2 t^{b_2} + \dots, \quad x_3 = a_3 t^{b_3} + \dots,$$

with the $a_i \neq 0$ and $0 < b_1 < b_2 < b_3$, we set $b_0 := 0$ and define

$$s_i(P) := b_{i+1} - b_i - 1 \quad (i = 0, 1, 2).$$

At all but finitely many $P \in \Delta$, $s_0 = s_1 = s_2 = 0$, so we can define

$$s_i(\Delta) = \sum_{P \in \Delta} s_i(P).$$

In fact, in the cases arising below, we do not encounter points P with $\sum_i s_i(P) > 1$. We call s_0 the number of cusps, s_1 the number of flexes, and s_2 the number of stalls.

The set of osculating planes to Δ is a curve $\Delta^\vee$ in the dual projective space $P^\vee$. We call $\Delta^\vee$ the *dual* curve of Δ. The elementary projective characters of $\Delta^\vee$ are $r_i^\vee = r_{2-i}$, $s_i^\vee = s_{2-i}$ $(i = 0, 1, 2)$.

The following are partial analogues for space curves of the Plücker formulae.

Lemma 3.3. [5, Cor.5.3, p.491] [3, p.270] *For Δ a reduced and non-planar space curve, we have* $-\chi(\Delta) - s_0 = -2r_0 + r_1$, $-\chi(\Delta) - s_1 = r_0 - 2r_1 + r_2$, $-\chi(\Delta) - s_2 = r_1 - 2r_2$.

Proof. We claim that projecting Δ from a general point P gives a plane curve with Euler characteristic $-\chi(\Delta)$, degree r_0, class r_1, with s_0 cusps and $s_1 + r_2$ flexes. Applying the Plücker formulas to this projection gives the first two relations; the same argument for the dual curve yields the third.

To justify the claim, note that if $P \notin \Delta$, projection does not change the degree. If P lies on no tangent, projection introduces no new cusp. A general P lies on the osculating planes at just r_2 distinct ordinary points; thus projecting from P adds r_2 to the number of flexes. A general line through a general point P is a general line; through it pass r_1 tangent planes to C, so its projected image lies on r_1 tangents to C_P. □

Apply Lemma 3.3 to the curve Γ. Here $s_0 = s_1 = 0$ since Γ is smoothly embedded and the curvature does not vanish, and $r_0 = d$. Since Γ is smooth and connected of genus g, $\chi(\Gamma) = 2-2g$. It thus follows from the lemma that $r_1 = 2d + 2g - 2$, $r_2 = 3d + 3g - 6$ and $s_2 = 4d + 12g - 12$. Now it follows from the definitions that $d_a = r_1$ and $k_1 = s_2$. Hence we have

$$d_a = 2d - 2 + 2g, \quad k_1 = 4d - 12 + 12g. \tag{1}$$

4. Correspondences in $\Gamma \times \Gamma$

In this section we evaluate the constants k_i by studying the correspondences T_*.

In general, a curve $T \subset \Gamma \times \Gamma$ is called a *correspondence* on Γ. We denote the degrees of the projections on the factors by $d_1(T)$, $d_2(T)$, thus $d_1(T^t) = d_2(T)$. We need the notion of *valence*: see e.g. [3, pp 284] for further details.

For $P \in \Gamma$, write $T(P) = \{Q \mid (P, Q) \in T\}$: we can consider this as a divisor on Γ if we count multiplicities appropriately. Then T has valence k if the linear equivalence class of $T(P) + kP$ is independent of P. We will denote the valence of T by $v(T)$; we have $v(T^t) = v(T)$. For the above cases, we have

Lemma 4.1. *(compare* [3, pp 291-5]*)*
T_a *has* $d_1(T_a) = d - 3$, $d_2(T_a) = 3d + 6g - 9$ *and* $v(T_a) = 3$.
T_b *has* $d_1(T_b) = m_b = 2d + 2g - 6$ *and* $v(T_b) = 4$.
T_c *has* $d_1(T_c) = 2m_c = (d - 2)(d - 3) - 2g$ *and* $v(T_c) = d - 4$.

Proof. Consider the projection π_P of Γ from P with image Γ_P; recall that Y_P denotes the image of the tangent at P. As in the proof of Lemma 3.1, Γ_P has degree $d-1$ and genus g, and hence by Lemma 3.3 has class $2(d-2+g)$, $m_c = \frac{1}{2}(d-2)(d-3)-g$ nodes, and $3(d+2g-3)$ flexes. Now

$(P,Q) \in T_a$ if $\pi_P(Q)$ is a flex of Γ_P,
$(P,Q) \in T_a^t$ if $\pi_P(Q)$ lies on the tangent at Y_P,
$(P,Q) \in T_b$ if Y_P lies on the tangent to Γ_P at $\pi_P(Q)$,
$(P,Q) \in T_c$ if $\pi_P(Q)$ is a node of Γ_P.

Hence $d_1(T_a)$ is the number of further intersections with Γ_P of the tangent at Y_P, so is 2 less than the degree; $d_2(T_a)$ is equal to the number of flexes of Γ_P; $d_1(T_b)$ is the number of tangents from Y_P, which is equal to the class, diminished by 2 to allow for the tangent at Y_P itself; and $d_1(T_c)$ is equal to double the number of nodes of Γ_P, since each contributes two points Q.

For the valences we argue following [3, p 295].

For T_a, consider the projection $\pi_P : \Gamma \to P^2$ from P. The canonical class $K_\Gamma = \pi_P^*(-3H_{P^2}) + T_a(P)$, and $\pi_P^* H_{P^2} = H_{P^3} - P$, so $T_a(P) + 3P = K_\Gamma + 3H_{P^3}$.

For T_b, consider the projection $\pi_L : \Gamma \to P^1$ from $T_P\Gamma$. Then the canonical class $K_\Gamma = \pi_L^*(-2H_{P^1}) + T_b(P)$, and $\pi_L^* H_{P^1} = H_{P^3} - 2P$, so $T_b(P) + 4P = K_\Gamma + 2H_{P^3}$.

For T_c, as on [3, p 291] we have $K_\Gamma = \pi_P^*((d-4)H_{P^2}) - D$, where D is the preimage of the double points of Γ_P, and hence is $T_c(P)$. Again using $\pi_P^* H_{P^2} = H_{P^3} - P$, we find $T_c(P) + (d-4)P = (d-4)H_{P^3} - K_\Gamma$, giving valence $(d-4)$. □

It is shown on [3, p.285] that a correspondence T with valency has the divisor class of

$$(d_1(T) + v(T))(* \times \Gamma) + (d_2(T) + v(T))(\Gamma \times *) - v(T)\Delta(\Gamma),$$

where $\Delta(\Gamma)$ denotes the diagonal. Since the diagonal has self-intersection number $2-2g$, we can now calculate all intersection numbers. In particular,

$$T.\Delta(\Gamma) = d_1(T) + d_2(T) + 2gv(T), \tag{2}$$

$$T.T' = d_1(T)d_2(T') + d_2(T)d_1(T') - 2gv(T)v(T'). \tag{3}$$

We also apply the genus formula. Since the canonical class is $(2g-2)(* \times \Gamma + \Gamma \times *)$, this gives

$$-\chi(T) = 2d_1(T)d_2(T) + (2g-2)(d_1(T) + d_2(T)) - 2gv(T)^2 - \mu(T). \tag{4}$$

We can now count the intersections of T_a, T_b and T_c with Δ in two ways: they are enumerated in Theorem 2.3 (i) and shown to have multiplicity 1, and then counted in Lemma 3.2, giving the numbers k_1, k_1 and k_2; or we can use (2), with the values given by Lemma 4.1. The first two confirm the calculation $k_1 = 4d + 12g - 12$ of (1); the third gives

$$k_2 = 2(d-2)(d-3) + 2g(d-6). \tag{5}$$

We can also obtain $\chi(T)$ in two ways. Projecting on the first factor gives $d_1(T)\chi(\Gamma)$, diminished by the effect of ramification, thus if the coincidence points are simple, we obtain $d_1(T)(2-2g) - I_1(T)$. Now coincidence points of T_a, T_b and T_c were enumerated in Theorem 2.3 (ii) and shown to be simple, and then counted in Lemma 3.2. Secondly, we can use (4), with the values given by Lemma 4.1. Applying this to T_a, T_a^t and T_b yields

$-\chi(T_a) = (d-3)(2g-2) + k_5$,
$-\chi(T_a) = (3d+6g-9)(2g-2) + (d-4)k_1 + k_2$,
$-\chi(T_a) = 2(d-3)(3d+6g-9) + (2g-2)(4d+6g-12) - 18g$;
$-\chi(T_b) = (2d+2g-6)(2g-2) + k_2 + k_5$,

$-\chi(T_b) = 2(2d+2g-6)^2 + (2g-2)(4d+4g-12) - 32g.$

In view of the values of k_1 and k_2 given by (1) and (5), the second and third of these equations yield the same value; comparing with the first then gives

$$k_5 = 6(d-3)(d-4) + 6g(3d-14) + 12g^2; \tag{6}$$

and now the other equations both give the same value for $\chi(T_b)$.

Similarly, the intersection numbers $T_a.T_b$, $T_a.T_c$ and $T_b.T_c$ can be computed either using Theorem 2.3 (iv) and (v), and Lemma 3.2 or using (3), with the values given by Lemma 4.1. Comparing the results gives

$k_1 + k_2 + k_5 = (4d+6g-12)(2d+2g-6) - 24g,$

$2k_2 + 2k_6 = (4d+6g-12)((d-2)(d-3)-2g) - 6g(d-4),$

$2k_2 + 2k_3 = 2(2d+2g-6)((d-2)(d-3)-2g) - 8g(d-4).$

Here the first is an identity in view of the known values of k_1, k_2 and k_5; the others yield values for k_6 and k_3.

Finally, applying the same procedure as for T_a and T_b to T_c, but now taking account of the fact that $\mu(T_c) = 12k_4$, gives

$-\chi(T_c) = ((d-2)(d-3)-2g)(2g-2) + k_2 + k_3 + 12k_4,$

$-\chi(T_c) = 2((d-2)(d-3)-2g)^2 + (4g-4)((d-2)(d-3)-2g) - 2g(d-4)^2 - 12k_4.$

Here substituting the known values of k_2 and k_3 allows us to solve for k_4. Collecting our results gives

Proposition 4.2. *We have*

$k_1 = 4(d-3) + 12g,$

$k_2 = 2(d-2)(d-3) + 2g(d-6),$

$k_3 = 2(d-2)(d-3)(d-4) + 2g(d^2-10d+26) - 4g^2,$

$k_4 = \frac{1}{12}(d-2)(d-3)^2(d-4) - \frac{1}{2}g(d^2-7d+13) + \frac{1}{2}g^2,$

$k_5 = 6(d-3)(d-4) + 6g(3d-14) + 12g^2,$

$k_6 = 2(d-2)(d-3)(d-4) + 3g(d^2-8d+18) - 6g^2.$

We also have

$\chi(T_a) = -2(d-3)(3d-13) - 10g(2d-9) - 12g^2,$

$\chi(T_b) = -8(d-3)(d-4) - 8g(3d-14) - 16g^2,$

$\chi(T_c) = -(d+1)(d-2)(d-3)(d-4) + 6g(d-5) + 6g^2.$

The number k_1 of stalls comes from the Plücker relations. The number k_2 of tangents meeting Γ again was first given by Cayley [2], and the number k_4 of 4-secants was first given by Salmon 1868; with a fuller proof given by Zeuthen [11]. In [6] a formula for k_2 is given for arbitrary curves; applying this to the dual curve $\Gamma^\vee$ yields a formula for $k_5(\Gamma)$. The formulae for k_3 and k_6 appear to be new.

5. Curves in E_Γ

In this section, by studying the curves E_a, E_b and E_c, we complete the evaluation of numbers of types of special points on Γ.

At a general point of each of these curves, the projection π_E induces a submersion on Γ. The list of exceptions was given in Lemma 2.1. In particular, the projection of E_a on Γ is an isomorphism. The degrees of the projections of E_b and E_c to Γ coincide with the multiplicities along Γ of the surfaces B and C, hence are equal to m_b and m_c respectively. Applying Lemma 2.1, we obtain formulae for the Euler characteristics of E_b and E_c and for the mutual intersection numbers as follows.

$$
(7)\qquad
\begin{array}{lclcl}
E_a.E_b & = & \#(ab)+2\#(a_2) & = & \#(ab)+2k_1. \\
E_a.E_c & = & \#(ac)+\#(\delta) & = & \#(ac)+k_2. \\
E_b.E_c & = & \#(bc)+\#(\alpha)+2\#(c_2) & = & \#(bc)+k_2+4k_3. \\
\chi(E_b) & = & m_b\chi(\Gamma)-\#(bb)-\#(b_2)-\#(\delta) & = & m_b(2-2g)-\#(bb)-k_5-k_2. \\
\chi(E_c) & = & m_c\chi(\Gamma)-\#(cc)-\#(\beta)-2\#(\gamma) & = & m_c(2-2g)-\#(cc)-k_3-8k_4.
\end{array}
$$

The group of divisors on B_Γ is free on the classes $[H]$ of a (pulled back) plane and $[E]$ of E_Γ. The strict transform $\hat{A}$ of A is obtained from the total transform by subtracting $[E]$ multiplied by the multiplicity m_a of $S(A_2)$ along Γ; similarly for B and C. Thus

$$
(8)\qquad [\hat{A}] = d_a[H]-m_a[E],\quad [\hat{B}] = d_b[H]-m_b[E],\quad [\hat{C}] = d_c[H]-m_c[E].
$$

Taking intersections with E_Γ defines a map from divisors on B_Γ to those on E_Γ. Since the blow up of a point in a surface gives a curve of self-intersection -1, the self-intersection of E_Γ in B_Γ has the class $-[D]$, where D is the class of a section of the bundle $E_\Gamma \to \Gamma$. Denote by $[F]$ the class in E_Γ of a fibre. Then since a plane meets Γ in d points, the trace of $[H]$ on E_Γ is $d[F]$.

The surface E_Γ is a P^1-bundle over Γ, associated to a plane bundle E. This situation is described in Beauville [1, III,18]: the group of divisors of E_Γ is free on $[D]$ and $[F]$, we have

$$
(9)\qquad [D].[D] = k_0,\quad [D].[F] = 1,\quad [F].[F] = 0,
$$

where $k_0 = deg\ E$, and the canonical class is $K_E = -2[D] + (deg\ E + 2g - 2)[F]$.

In fact, we have $k_0 = -4d - 2g + 2$. To see this, we can apply the adjunction formula to the blow-up $B_\Gamma \to P^3$ to see that K_E is the pullback of $K_P + 2[E] = -4[H] + 2[E]$, hence is $-4d[F] - 2[D]$.

According to [9, Corollary 7.3.1], the surface A touches E_Γ along the curve a, and also meets it in the fibres over the points of $S(\alpha)$; B meets E_Γ in b, the fibres over $S(\beta)$, and fibres over $S(\delta)$ counted twice; and C meets E_Γ in c and the fibres over $S(\gamma)$. It follows from (8) by taking traces on E (recall that $m_a = 2$) that we have divisors

$$
(10)\qquad [E_a] = [D] + c_a[F],\quad [E_b] = m_b[D] + c_b[F],\quad [E_c] = m_c[D] + c_c[F],
$$

where

$$
\begin{array}{lclcl}
c_a & = & \frac{1}{2}(dd_a - \#(\alpha)) & = & \frac{1}{2}(dd_a - k_2), \\
c_b & = & d_b d - \#(\beta) - 2\#(\delta) & = & dd_b - k_3 - 2k_2, \\
c_c & = & d_c d - \#(\gamma) & = & dd_c - 4k_4.
\end{array}
$$

Using (9) and (10) gives formulae for the mutual intersection numbers of E_a, E_b and E_c alternative to those of (7). In particular,

$\#(ab) + 2k_1 = [E_a].[E_b] = k_0 m_b + \frac{1}{2}m_b(dd_a - k_2) + d_b d - k_3 - 2k_2,$

$\#(ac) + k_2 = [E_a].[E_c] = k_0 m_c + \frac{1}{2}m_c(dd_a - k_2) + d_c d - 4k_4.$

We have already calculated m_b and m_c in Lemma 3.1, d_a in (1); by Lemma 3.2, $\#(ab) = k_5$ and $\#(ac) = k_6$, and the values of the k_i are given in Proposition 4.2. Substituting these enables us to complete the calculation of the degrees of 2-dimensional strata.

Proposition 5.1. *We have*

$d_a = 2d - 2 + 2g,$

$d_b = 2(d-1)(d-3) + 2g(d-3)$

$d_c = \frac{1}{3}(d-1)(d-2)(d-3) - g(d-2).$

Now by Lemma 2.1, E_a is isomorphic to Γ, so has genus g; E_b has $\#(bb)$ simple (A_1) nodes and $\#(b_2)$ simple (A_2) cusps; E_c has $\#(cc)$ simple nodes and $\#(\gamma)$ triple points (type D_4). Thus, first,

$-\chi(E_a) = \{(\frac{1}{2}d_a d - \#(\alpha))[F] + [D]\}.\{(\frac{1}{2}d_a d - \#(\alpha) - 4d)[F] - [D]\}$,
which indeed reduces, substituting from (9), to $2g-2$. Then we have
$-\chi(E_b) = [E_b].([E_b] + K_E) - \#(bb) - 2\#(b_2)$,
$-\chi(E_c) = [E_c].([E_c] + K_E) - \#(cc) - 4\#(\gamma)$,
and hence formulae alternative to those of (7); comparing the two and substituting known values completes the count of special points of the various types.

Proposition 5.2. *We have*
$\#(\alpha) = \#(\delta) = 2(d-2)(d-3) + 2g(d-6)$,
$\#(\beta) = 2(d-2)(d-3)(d-4) + 2g(d^2 - 10d + 26) - 4g^2$,
$\#(\gamma) = \frac{1}{3}(d-2)(d-3)^2(d-4) - 2g(d^2 - 7d + 13) + 2g^2$,
$\#(a_2) = 4(d-3) + 12g$,
$\#(b_2) = 6(d-3)(d-4) + 6g(3d-14) + 12g^2$,
$\#(c_2) = 4(d-2)(d-3)(d-4) + 4g(d^2 - 10d + 26) - 8g^2$,
$\#(ab) = 6(d-3)(d-4) + 6g(3d-14) + 12g^2$,
$\#(ac) = 2(d-2)(d-3)(d-4) + 3g(d^2 - 8d + 18) - 6g^2$,
$\#(bb) = 4(d-5)(d-3)(d-4) + 4g(3d^2 - 30d + 77) + 12(d-6)g^2 + 4g^3$,
$\#(bc) = 3(d-5)(d-2)(d-3)(d-4) + g(5d^3 - 65d^2 + 288d - 448) + (2d^2 - 26d + 92)g^2 - 4g^3$,
$\#(cc) = \frac{1}{4}(d-2)(d-3)(d-4)(d-5)(2d-3) + \frac{1}{4}g(d^4 - 24d^3 + 177d^2 - 502d + 468) - (d^2 - 10d + 28)g^2 + g^3$.

We also have
$\chi(E_b) = -4(d-3)^2(d-4) - 4g(3d^2 - 24d + 49) - 4g^2(3d-14) - 4g^3$,
$\chi(E_c) = -\frac{1}{12}(d-2)(d-3)(6d^3 - 55d^2 + 169d - 192) - \frac{1}{4}g(d^4 - 24d^3 + 173d^2 - 490d + 500) + g^2(d^2 - 10d + 30) - g^3$.
However the Euler characteristics of the normalised curves (which give the genera) are given by $\chi(\tilde{E}_b) = \chi(E_b) + \#(bb)$ and $\chi(\tilde{E}_c) = \chi(E_c) + \#(cc) + 8k_4$, which lead to the simpler formulae

$$\begin{aligned} \chi(\tilde{E}_b) &= -8(d-3)(d-4) - 8g(3d-14) - 16g^2, \\ \chi(\tilde{E}_c) &= -(d-2)(d-3)(2d-9) - g(3d^2 - 25d + 60) + 6g^2. \end{aligned} \tag{11}$$

The degree d_c of the surface C of trisecants was first given by Cayley [2], with a full proof by Zeuthen [11]. The number of tritangent planes (equal to $\frac{1}{3}\#(bb)$) was also given by Zeuthen [11]. The formulae for d_b, $\#(bc)$ and $\#(cc)$ appear to be new.

6. Degrees of curve strata

In this section we complete the calculation of the degrees of the 1-dimensional strata. We first state the result; the formulae will be obtained in stages through the section.

Theorem 6.1. *We have*
$n(A_4) = 4(d-3) + 12g$,
$n(A_5) = 6(d-3)^3 + 12g(d-5) + 6g^2$,
$n(D_5) = 2(d-2)(d-3) + 2g(d-6)$,
$n(D_6) = 2(d-2)(d-3)(d-4) + 2g(d^2 - 10d + 26) - 4g^2$,
$n(X_9) = \frac{1}{12}(d-2)(d-3)^2(d-4) - \frac{1}{2}g(d^2 - 7d + 13) + \frac{1}{2}g^2$,
$n(2A_2) = 2(d-1+g)(d-3+g) = 2(d-1)(d-3) + 4g(d-2) + 2g^2$,
$n(A_2A_3) = 2d(d-3)(2d-7) + 2g(4d^2 - 19d + 6) + 4g^2(d-3)$,
$n(A_2D_4) = \frac{1}{3}(d-2)(d-3)(d-4)(2d+1) + \frac{2}{3}g(d^3 - 9d^2 + 20d + 6) - 2g^2(d-2)$,
$n(2A_3) = 2(d+1)(d-3)^2(d-4) + 4g(d^3 - 8d^2 + 13d + 16) + 2g^2(d^2 - 7d + 4)$,
$n(A_3D_4) = \frac{1}{3}(d-2)(d-3)(d-4)(2d^2 - 5d - 9) + \frac{1}{3}g(2d^4 - 27d^3 + 103d^2 - 66d - 204) - 2g^2(d^2 - 6d + 2)$,
$n(2D_4) = \frac{1}{72}(d-2)(d-3)(d-4)(d-5)(4d^2 - d - 12) - \frac{1}{6}g(d-3)(d-5)(2d^2 - 3d - 8) + \frac{1}{2}dg^2(d-5)$.

The values of $n(A_4)$, $n(D_5)$, $n(D_6)$ and $n(X_9)$ were denoted k_1, k_2, k_3 and k_4 and calculated in Proposition 4.2. The degrees of curves of intersection of two strata can be evaluated as follows.

Lemma 6.2. [9, Lemma 2.1] *Along $S(A_4)$, $S(A_2)$ and $S(A_3)$ are smooth, and intersect with multiplicity 2; along $S(A_5)$, $S(A_3)$ has a cuspidal edge; along $S(D_5)$, $S(A_2), S(A_3)$ and $S(D_4)$ are all smooth, and any two of them meet transversely; along $S(D_6)$, $S(A_3)$ and $S(D_4)$ are smooth, and intersect with multiplicity 2; and along $S(X_9)$, $S(D_4)$ has 4 branches, any two of which are transversal.*

It follows that intersections of cycles are given by

$$\begin{array}{rcl} [A].[B] & = & S(A_2A_3) + 2S(A_4) + S(D_5) + 2m_b[\Gamma], \\ [A].[C] & = & S(A_2D_4) + S(D_5) + 2m_c[\Gamma], \\ [B].[C] & = & S(A_3D_4) + S(D_5) + 2S(D_6) + m_bm_c[\Gamma]. \end{array} \tag{12}$$

Taking degrees, we find

$d_ad_b = n(A_2A_3) + 2k_1 + k_2 + 2dm_b$,
$d_ad_c = n(A_2D_4) + k_2 + 2dm_c$,
$d_bd_c = n(A_3D_4) + k_2 + 2k_3 + dm_bm_c$;

and substituting the values already obtained now yields the values of $n(A_2A_3)$, $n(A_2D_4)$ and $n(A_3D_4)$.

Next we consider the generic plane sections Π_a, Π_b, Π_c of the respective surfaces A, B and C. These have the same degrees as the corresponding surfaces, and have singularities given by Lemma 2.2. Applying the Plücker formula, we obtain

$-\chi(\Pi_a) = d_a(d_a-3) - 2d - n(2A_2)$,
$-\chi(\Pi_b) = d_b(d_b-3) - n(2A_3) - 2n(A_5) - d(m_b-1)^2$,
$-\chi(\Pi_c) = d_c(d_c-3) - n(2D_4) - 9n(X_9) - d(m_c-1)^2$.

However, it will be more convenient to use instead the normalisations of these curves, and here we have

$$\begin{array}{rcl} -\chi(\tilde{\Pi}_a) & = & d_a(d_a-3) - 2d - 2n(2A_2), \\ -\chi(\tilde{\Pi}_b) & = & d_b(d_b-3) - 2n(2A_3) - 2n(A_5) - dm_b(m_b-1), \\ -\chi(\tilde{\Pi}_c) & = & d_c(d_c-3) - 2n(2D_4) - 12n(X_9) - dm_c(m_c-1). \end{array} \tag{13}$$

To obtain alternative formulae, we first observe that $\tilde{\Pi}_a$ can be identified with Γ itself, so $\chi(\tilde{\Pi}_a) = 2-2g$.

Next we compare T_b, E_b and Π_b. A general T-secant PQ of Γ determines 2 points (P,Q), $(Q,P) \in T_b$, 2 points (P,Π), $(Q,\Pi') \in E_b$ and a single point $PQ \cap \Pi_0$ of Π_b. There are various exceptions to this, but the number of exceptions decreases if we compare instead the normalisations T_b (already normal), $\tilde{E}_b$ and $\tilde{\Pi}_b$: the only special cases now are the k_1 tangents at stalls P, which yield a single point in each of T_b, $\tilde{E}_b$ and $\tilde{\Pi}_b$. Hence $\chi(T_b) = \chi(\tilde{E}_b) = 2\chi(\tilde{\Pi}_b) - k_1$.

The equality is confirmed by our calculations, and we obtain

$$\chi(\tilde{\Pi}_b) = -2(d-3)(2d-9) - 2g(6d-31) - 8g^2. \tag{14}$$

Similarly for the third case, a general trisecant PQR of Γ gives rise to 6 points of T_c, 3 points of E_c and a single point of Π_c. Again using the normalisations, we find that the only exceptions are (a) each of k_2 tangents $T_P\Gamma$ meeting Γ again in Q, giving only 3 points of $\tilde{T}_c$ and 2 points of $\tilde{E}_c$, and (b) each of k_4 4-secants $PQRS$ of Γ, giving 24 points of $\tilde{T}_c$ (12 points of T_c), 12 points of $\tilde{E}_c$ (4 points of E_c) and 4 points of $\tilde{\Pi}_c$ (1 point of Π_c). Hence $\chi(\tilde{T}_c) = \chi(T_c) + 12k_4 = 2\chi(\tilde{E}_c) - k_2$, giving

$$\chi(\tilde{T}_c) = -4(d-2)(d-3)(d-4) - 6g(d^2 - 8d + 18) + 12g^2,$$

and $\chi(\tilde{E}_c) = 3\chi(\tilde{\Pi}_c) - k_2$, giving

$$\chi(\tilde{\Pi}_c) = -\tfrac{1}{3}(d-2)(d-3)(2d-11) - g(d^2-9d+24) + 2g^2.$$

Substituting in (13) now yields the values of $n(2A_2)$ and $n(2D_4)$ and the equation

$$n(2A_3) + n(A_5) = 2(d-3)^2(d^2-3d-1) + 4g(d^3-8d^2+16d+1) + 2g^2(d^2-7d+7). \tag{15}$$

Lemma 6.3. *If $\Delta^\vee$ is the dual curve to Δ, then $S(2A_2)(\Delta^\vee)$ is the dual curve to $S(A_5)(\Delta)$.*

Proof. We can define $S(2A_2)(\Delta^\vee)$ as the set of planes through a pair of coplanar tangents of $\Delta^\vee$, or of Δ. But this is just the set of tangent planes of $S(A_3)(\Delta)$, and hence, of $S(A_5)(\Delta)$. □

We next study the curve $S(A_5)$, which from now on we denote by F. Recall that each T-secant of Γ touches F at its T-centre, thus the tangent surface to F is $S(A_3) = B$.

Theorem 6.4. *For Γ projection-generic, the curve F has no flexes, it has stalls only at $S(b_2)$, and has cusps only at $S(A_7)$ and $S(\delta)$.*

Proof. The only 0-dimensional strata lying on F are the compound singularities $S(A_2A_5)$, $S(A_3A_5)$, $S(D_4A_5)$, and $S(A_7)$, $S(D_8)$ outside Γ, and $S(b_2)$, $S(\delta)$ on Γ. By [9, §2], F is smooth at the compound singularities and at $S(D_8)$ and is cusped at $S(A_7)$. By the normal forms of [9, §6], F is smooth at $S(b_2)$ and is cusped at $S(\delta)$.

It follows in the generic case by [4] and in general by specialisation that the tangent line to a curve at a flex is singular on its tangent surface. But for Γ projection-generic, the singular locus of B is $\Gamma \cup F \cup S(2A_3)$. Now Γ cannot contain a straight line. If F or $S(2A_3)$ contained a line L, L could not itself be a T-secant (the T-trisecants form $S(D_6)$). For each T-secant meeting L, its T-plane contains the tangent line to $S(A_3)$, hence contains L. Thus the tangent lines to Γ at the end points of the T-secant meet L. We claim that this implies that Γ is planar: a contradiction. For take L as $y = z = 0$ in $\mathbb{C}^3$, and take a local parameter t on Γ. Since the tangent at (x, y, z) meets L, $(dy/dt)/(dz/dt) = y/z$. Hence $d(y/z)/dt = 0$, thus y/z is constant along Γ. Thus the singular locus of $S(A_3)$ contains no straight line, so F has no flex.

Any stall of F lies on the self-intersection curve $S(2A_3) \cup \Gamma$ of the tangent surface B of F. Now $S(2A_3)$ meets F only in $S(A_7)$ which gives cusps, not stalls on $S(A_5)$. The curve Γ itself meets F in $S(b_2) \cup S(\delta)$. By [9, Proposition 8.4], B has a cuspidal cross-cap at a point of $S(b_2)$, hence such a point is indeed a stall on F. By [9, Theorem 9.2], at a point of $S(\delta)$, B has a swallowtail singularity, and by [9, Corollary 9.2.1], the local parameters of F at such a point are $s_0 = 1$, $s_1 = s_2 = 0$, so it does not count as a stall. □

Thus we have projective characters $s_1(F) = 0$, $s_2(F) = \#(b_2) = k_5$, $r_1(F) = d_b$. Moreover, since each T-secant meets F in just one point, we can identify the normalisation $\tilde{F}$ with $\tilde{\Pi}_b$, so have $\chi(F) = \chi(\tilde{\Pi}_b)$, which was calculated in (14). The Plücker relations of Lemma 3.3 imply (since $s_1(F) = 0$)

$$r_0(F) = \tfrac{1}{2}(3r_1(F) - s_2(F) - 3\chi(F)) = \tfrac{1}{2}(3d_b - k_5 - 3\chi(\tilde{\Pi}_b)),$$

and this gives the degree $n(A_5)$ of $F = S(A_5)$. The value of $n(2A_3)$ now follows from (15).

We also have $s_0(F) = 2r_1(F) - s_2(F) - 4\chi(F) = 2d_b - k_5 - 4\chi(\tilde{\Pi}_b)$, and Theorem 6.4 gives $s_0(F) = k_2 + n(A_7)$, so we obtain

$$n(A_7) = 12(d-3)(d-4) + 4g(8d-41) + 20g^2. \tag{16}$$

A formula for the degree of the curve $S(2A_2)$ (there called the *nodal curve*), for arbitrary space curves, can be found in [6], in the form $\frac{1}{2}\{r_1(r_1-1) - r_2 - 3(r_0+s_1)\}$. One can also calculate $n(2A_3)$ by applying this formula to F, but must then note that this nodal curve has to be interpreted as containing Γ with multiplicity $\binom{m_b}{2}$ as well as $S(2A_3)$. The other formulae in this section are new.

7. Calculations in $\Gamma \times \Gamma$

In this section we prove Theorem 2.3. We first prove most of (i) in Lemma 7.1, and all of (iv) in Lemma 7.2. The assertions in (ii) and (iii) require calculations, which we give in Lemmas 7.3, 7.4 and 7.5, for the respective curves T_a, T_b and T_c. We complete the proof of (v) in Proposition 7.7.

In these arguments we will make explicit use of the condition of projection genericity, particularly (PG6), which implies in particular that the subsets $W_1 \dots W_6$ of $\Gamma \times \Gamma$ are mutually disjoint. We also need the following consequences of (PG1):

at any stall $P \in \Gamma$ we have $s_2(P) = 1$,

there is no T-3-secant with T-centre on Γ,

the cross-ratio of the planes through a 4-secant containing the 4 tangent lines is *not* equal to the cross-ratio of the 4 points on the line.

Before starting our calculations, we note that the situation of T_b and T_c was also considered in [3, pp 290-297]. However precise conditions for counting multiplicities were passed over there, and of the hypotheses actually listed on p.291, the absence of 5-secants, T-4-secants and flexes (points with $s_1(P) > 0$) follow from (PG1), and the condition that no osculating 2-plane contain a tangent line is not generic.

The correspondences T_a and T_b are among those studied in [10], and the results concerning them are given there. The proofs below are more direct than those of [10] for this special case. Of the hypotheses of the other paper, that $s_0(P) + s_1(P) + s_2(P) \leq 1$ for each $P \in \Gamma$ follows from our hypotheses $s_0(P) = s_1(P) = 0$ and $s_2(P) \leq 1$; (PG6) implies all the other conditions (in fact we just need $S(a_2)$, $S(\alpha)$, $S(\delta)$, $S(ab)$ and $S(b_2)$ disjoint).

Lemma 7.1. *We have $T_a \cap \Delta(\Gamma) = T_b \cap \Delta(\Gamma) = W_1$, $T_c \cap \Delta(\Gamma) = W_2'$.*

Proof. If $(P,Q) \in T_a$ we have $P \in O_Q\Gamma$. Conversely, the plane $O_Q\Gamma$ has intersection number d with Γ, and the point Q accounts for 3; for the other points P, we have $(P,Q) \in T_a$. If also $P = Q$, the intersection number at Q is 4, so Q is a stall.

If $(P,P) \in T_b$, there is an intersection of $T_Y\Gamma_P$ with Γ_P at Y_P additional to that expected, i.e. Y_P is a flex of Γ_P, so again P is a stall of Γ.

If $(P,P) \in T_c$, then there is a line meeting Γ twice in P and once elsewhere. This must be the tangent at P, so $P \in S(\delta)$. □

Lemma 7.2. *We have $T_a \cap T_b = W_1 \cup W_2^t \cup W_5$, $T_a \cap T_c = W_2^t \cup W_6$, and $T_b \cap T_c = W_2 \cup W_2^t \cup W_3'$.*

Proof. Intersections on the diagonal are dealt with by Lemma 7.1, so consider pairs $P \neq Q$.

If $(P,Q) \in T_a \cap T_b$, then $P \in O_Q\Gamma$. If $P \in T_Q\Gamma$ then $(P,Q) \in W_2^t$; if not, $T_P\Gamma$ meets $T_Q\Gamma$ in a point different from P in $O_Q\Gamma$, so is contained in this plane, hence $(P,Q) \in W_5$.

If $(P,Q) \in T_a \cap T_c$, then $PQ \subset O_Q\Gamma$ and PQ is a trisecant PQR. If $R = P$, we have $Q \in T_P\Gamma \subset O_Q\Gamma$, so $Q \in S(\delta) \cap S(b_2)$, contradicting projection genericity. If $R = Q$, $P \in T_Q\Gamma$, so $(P,Q) \in W_2^t$. If P, Q, R are distinct, then Q has type ac, hence $(P,Q) \in W_6$.

If $(P,Q) \in T_b \cap T_c$, then again PQ is a trisecant PQR. If $R = P$ then $Q \in T_P\Gamma$, so $(P,Q) \in W_2$. If $R = Q$ then $P \in T_Q\Gamma$, so $(P,Q) \in W_2^t$. Otherwise, $(P,Q) \in W_3'$. □

For the next results, we need direct calculations of the low order terms in the expansions of the curves at the special points. Parts of these calculations appeared in a preliminary version of [9], where they were used to establish the local structure of the curves E_a, E_b and E_c in E_Γ with respect to each other and to the projection π_E: however in the final version of [9], the local structure is obtained from the main versality results.

We work throughout in affine 3-space, and take up the notation and calculations of [8, Proposition 6.15]: denote a typical point by $X = (x, x', x'')$; points of Γ are denoted $P = (p, p', p'')$, Q, R etc. We regard the co-ordinates p, p', p'' as functions of a local parameter t_p on Γ which vanishes at P (we omit the subscript p if there is no ambiguity). Their Taylor expansions are denoted $p = \sum_0^\infty p_r t_p^r$, $p' = \sum_0^\infty p'_r t_p^r$, etc.

Successive derivatives of the vector P with respect to t_p are denoted by suffices: $P_1, P_2, \ldots$. Thus at $t_p = 0$ we have $P_r = r!(p_r, p'_r, p''_r)$. However, we denote by P_0 the result of substituting $t_p = 0$ in P.

We take co-ordinates with P_0 at the origin, with tangent along the $x''-$axis. Since Γ is smooth at P, $p''_1 \neq 0$. We may take x'' (scaled by p''_1, which we retain to preserve homogeneity in our formulae) as local co-ordinate at P, so $p''_r = 0$ for $r \neq 1$. When $Q_0 \neq P_0$, we also suppose Q_0 in the plane $x' = 0$, so $q'_0 = 0$. We will expand an equation for the correspondence T_* as Taylor series in t_p and t_q. If the terms of degree ≤ 1 are $at_p + bt_q = 0$, then we have a coincidence point $I_1(T)$ if $b = 0$ (or $I_2(T)$ if $a = 0$, a singular point of T if $a = b = 0$), and it is simple iff the coefficient of t_q^2 is non-zero.

Lemma 7.3. *The curve T_a is smooth at all points. We have $I_1(T_a) = W_5^t$ and $I_2(T_a) = W_1'^t \cup W_2$. All coincidence points are simple. At a united point in W_1, the tangent is $3t_p + t_q = 0$.*

Proof. We have $(P, Q) \in T_a$ if $Q \in O_P\Gamma$, so $P - Q, P_1$ and P_2 are coplanar. Since $(P_0, Q_0) \in T_a$, we have $p'_2 = 0$; as the curvature does not vanish at P_0, $p_2 \neq 0$. We have

$$\Delta_0 := [P - Q, P_1, P_2] = \begin{vmatrix} -q_0 + \ldots & -q'_1 t_q + \ldots & -q''_0 + \ldots \\ 2p_2 t_p + \ldots & 3p'_3 t_p^2 + \ldots & p''_1 \\ 2p_2 + \ldots & 6p'_3 t_p + \ldots & 0 \end{vmatrix};$$

the terms of degree 1 in t_p and t_q are $6q_0 p'_3 p''_1 t_p - 2p_2 q'_1 p''_1 t_q$.

Thus for $I_1(T_a)$ we have $q'_1 = 0$, so $T_{Q_0}\Gamma \subset O_{P_0}\Gamma$ and $(P_0, Q_0) \in W_5^t$.
For $I_2(T_a)$ we have either

$q_0 = 0$:, so $Q_0 \in T_{P_0}$ and $(P_0, Q_0) \in W_2$, or

$p'_3 = 0$, so P_0 is a stall on Γ and $(P_0, Q_0) \in W_1'^t$;

we cannot have both, for this would imply $P_0 \in S(\delta) \cap S(a_2)$. This gives the coincidence points as stated, and proves smoothness.

In the W_5^t case, to obtain the coefficient of t_q^2, we set $t_p = 0$ in the determinant: the coefficient is $-2p_2 q'_2 p''_1$, which does not vanish since if $q'_2 = 0$, Q_0 would be a stall, so $Q_0 \in S(ab) \cap S(a_2)$.

In the other cases, we need the coefficient of t_p^2. If $p'_3 = 0$, the coefficient is $-12q_0 p'_4 p''_1$, which cannot vanish else we would have $s_2(P) \geq 2$; if $q_0 = 0$, the coefficient is $6p_2 p'_3 q''_0$, and this is non-zero as $q''_0 = 0$ implies $Q_0 = P_0$.

We also need to consider the case $Q_0 = P_0$. Here P and Q are given by the same parametrisation, i.e. with the same coefficients $p_r \ldots$ but with different parameters t_p, t_q. Since we have a stall, $p'_3 = 0$ and $p'_4 \neq 0$, so the determinant reduces to

$$\begin{vmatrix} p_2(t_p^2 - t_q^2) + \ldots & p'_4(t_p^4 - t_q^4) + \ldots & p''_1(t_p - t_q) \\ 2p_2 t_p + \ldots & 4p'_4 t_p^3 + \ldots & p''_1 \\ 2p_2 + \ldots & 12p'_4 t_p^2 + \ldots & 0 \end{vmatrix}.$$

A factor $(t_p - t_q)^3$ can be removed; when this is done, the terms of least degree reduce to $2p_2 p'_4 p''_1(3t_p + t_q)$; in particular, the curve is smooth. □

Lemma 7.4. *The curve T_b is smooth at all points. At a united point, the tangent is given by $t_q = -t_p$. The set $I_1(T_b) = W_2 \cup W_5$. All coincidence points are simple.*

Proof. First consider a neighbourhood of the point defined by a T-secant P_0Q_0, with $P_0 \neq Q_0$. Since $(P_0, Q_0) \in T_b$, we have $q_1' = 0$. Now $(P, Q) \in T_b$ if $P - Q$, P_1 and Q_1 are coplanar, so $\Delta_b := [P - Q, P_1, Q_1] = 0$. We have

$$\Delta_b = \begin{vmatrix} p_2t_p^2 + \ldots - q_0 - \ldots & p_2't_p^2 + \ldots - q_2't_q^2 + \ldots & p_1''t_p - q_0'' - \ldots \\ 2p_2t_p + \ldots & 2p_2't_p + \ldots & p_1'' \\ q_1 + \ldots & 2q_2't_q + \ldots & q_1'' + \ldots \end{vmatrix}.$$

The linear terms in the expansion of Δ_b are $-2p_2'(-q_1q_0'' + q_0q_1'')t_p + 2q_0q_2'p_1''t_q$. For $I_1(T_b)$, either

$q_0 = 0$, so $Q_0 \in T_{P_0}\Gamma$ and $(P_0, Q_0) \in W_2$, or

$q_2' = 0$, thus $O_{Q_0}\Gamma$ is $x' = 0$ so $T_{P_0}\Gamma \subset O_{Q_0}\Gamma$ and $(P_0, Q_0) \in W_5$.

We cannot have both, else we would have $P_0 \in S(\delta) \cap S(b_2)$. As a check, we note that for $I_2(T_b)$ we have the transposed cases:

$(-q_1q_0'' + q_0q_1'') = 0$; since $q_1' = 0$, this is the condition for $P_0 \in T_{Q_0}\Gamma$, so $P_0 \in S(\alpha)$, $(P_0, Q_0) \in W_2^t$, or

$p_2' = 0$, thus $O_{P_0}\Gamma$ is $x' = 0$, so here $T_{Q_0} \subset O_{P_0}$ and $(P_0, Q_0) \in W_5^t$.

This proves smoothness of T_b. As before, we check the second degree terms:

if $q_0 = 0$, the coefficient is $-q_1q_2'p_1''$, and $q_1 \neq 0$ for otherwise $T_{P_0}\Gamma = T_{Q_0}\Gamma$,

if $q_2' = 0$, the coefficient is $-3q_0q_3'p_1''$, and $q_3' \neq 0$, for otherwise Q_0 is a stall.

We also see that when $p_2' = 0$, the coefficient of t_p^2 is $3p_3'(q_1q_0'' - q_0q_1'')$.

Again we must also consider the case $P_0 = Q_0$. We already saw in [8, Lemma 6.11 (ii)] that in this case, (as we saw in Lemma 7.1) P_0 must be a stall, so $(P_0, P_0) \in W_1$, and also that for the two parameters we have, to first order, $t_p + t_q = 0$. □

Lemma 7.5. *At a point $(P, Q) \in T_c$, the curve is smooth and transverse to neither fibre except as follows. We have $I_1(T_c) = W_2^t \cup W_3 \cup W_4$, $I_2(T_c) = W_2 \cup W_3^t \cup W_4$. Points of $W_2^t \cup W_3$ are simple coincidence points. Points of W_4 are double points, with 2 transverse branches, each tangent to neither fibre.*

Proof. We use the same notation as before, and take the trisecant $P_0Q_0R_0$ to lie on $x' = 0$. First suppose P_0, Q_0, R_0 are distinct, so we may suppose $0 = p_0''$, q_0'' and r_0'' also distinct. Since the points are collinear, $\frac{q_0}{q_0''} = \frac{r_0}{r_0''} = \lambda$, say.

The condition for collinearity of P, Q, R is that the matrix

$$\begin{pmatrix} 1 & p & p' & p'' \\ 1 & q & q' & q'' \\ 1 & r & r' & r'' \end{pmatrix} \tag{17}$$

have rank 2. Since the first and last columns are independent for P_0, Q_0, R_0 and hence nearby, it suffices to equate to zero the determinants formed by omitting the third and second columns, which we denote respectively by Δ_c and Δ_c'.

For the terms of order at most 1 in t_p, t_q, t_r it suffices to consider

$$\begin{pmatrix} 1 & 0 & 0 & p_1''t_p \\ 1 & q_0 + q_1t_q & q_1't_q & q_0'' + q_1''t_q \\ 1 & r_0 + r_1t_r & r_1't_r & r_0'' + r_1''t_r \end{pmatrix}.$$

The terms of degree at most 1 in t_p, t_q and t_r are:

Δ_c: $p_1''(r_0 - q_0)t_p + (q_1 - \lambda q_1'')r_0''t_q + (\lambda r_1'' - r_1)q_0''t_r$,

Δ_c': $q_1'r_0''t_q - r_1'q_0''t_r$.

If $q_1' = 0$, $T_{P_0}\Gamma$ and $T_{Q_0}\Gamma$ both lie in the plane $x' = 0$, so we have a T-3-secant and $(P_0, Q_0) \in W_3'$: here $t_r = 0$ and $p_1''(r_0 - q_0)t_p + (q_1 - \lambda q_1'')r_0''t_q = 0$.

Similarly if $r'_1 = 0$, we have $t_q = 0$ (we cannot have $q'_1 = r'_1 = 0$), $T_{P_0}\Gamma$ and $T_{R_0}\Gamma$ are coplanar: here $(P_0, Q_0) \in W_3^t \subset I_2(T_c)$.

Otherwise, we can eliminate t_r to get $0 = r'_1 p''_1 (r_0 - q_0) t_p + \xi r''_0 t_q$, where $\xi = (q_1 - \lambda q''_1) r'_1 - (r_1 - \lambda r''_1) q'_1$.

Thus $\xi = 0$ is the condition for $T_{Q_0}\Gamma$ and $T_{R_0}\Gamma$ to be coplanar: in this case $(P_0, Q_0) \in W_3 \subset I_1(T_c)$.

Thus if $(P_0, Q_0) \in W_3$, the coefficient of t_q vanishes. We now claim that it follows from the fact that P_0 is not the T-centre of Q_0R_0 that the coefficient C in $t_p = Ct_q^2$ is non-zero. As direct calculation is messy, we proceed a little differently.

Write the co-ordinates of Q as (q, q', q''). Projecting from P_0 (the origin) to $x'' = c$ gives $(cq/q'', cq'/q'')$; similarly for R. Since the second co-ordinates have leading terms $(cq'_1/q''_0)t_q$ and $(cr'_1/r''_0)t_r$, we can solve $q'/q'' = r'/r''$ for t_r in terms of t_q. The order of contact of the projected curves is now the order of the difference of the first co-ordinates, hence the order of $qr'' - q''r$. Since P_0 is not the T-centre of Q_0R_0, this order is 2.

On the other hand, we have $\Delta_c = (qr'' - q''r) - p(r'' - q'') + p''_1 t_p (r - q)$ and $\Delta'_c = (q'r'' - q''r') - p'(r'' - q'') + p''_1 t_p (r' - q')$. Since p, p' each have order at least 2, we can ignore the terms involving p, p'. Substituting the above solution for t_r thus makes Δ'_c vanish to order at least 2. So to this order, the equation $\Delta_c = 0$ yields $t_p = (qr'' - q''r)/p''_1(q - r)$ which, since the denominator is non-vanishing, has order precisely 2.

Next we treat the case when P_0, Q_0 and R_0 are not all distinct. Suppose $R_0 = P_0$: then $Q_0 \in T_{P_0}\Gamma$, so $q_0 = 0$. Here $P_0 \in S(\delta)$ and $(P_0, Q_0) \in W_2$; the 3-secants near $T_{P_0}\Gamma$ give a branch of T_c. We take co-ordinates so that the plane $O_{P_0}\Gamma$ is given by $x' = 0$, and so $p'_2 = 0$. Since $P_0 \notin S(a_2)$, $p'_3 \neq 0$. Since $P_0 \notin S(ab)$, $T_{Q_0}\Gamma \not\subset O_{P_0}\Gamma$, so $p_2 q'_1 \neq 0$.

In the matrix (17) we subtract the first row from the third, and divide the result by $t_r - t_p$ giving, to first order, $(0, p_2(t_p + t_r), 0, p''_1)$. Terms of order ≤ 2 in the matrix are:

$$\begin{pmatrix} 1 & 0 & 0 & p''_1 t_p \\ 1 & q_1 t_q + q_2 t_q^2 & q'_1 t_q + q'_2 t_q^2 & q''_0 + q''_1 t_q + q''_2 t_q^2 \\ 0 & p_2(t_p + t_r) + p_3(t_p^2 + t_p t_r + t_r^2) & p'_3(t_p^2 + t_p t_r + t_r^2) & p''_1 \end{pmatrix}$$

Denote the minors corresponding to Δ_c and Δ'_c by Δ_1 and Δ'_1. These have first order terms $q_1 p''_1 t_q - p_2 q''_0 (t_p + t_r)$ and $q'_1 p''_1 t_q$ respectively, so to first order we have $0 = t_q = t_p + t_r$ and $(P_0, Q_0) \in I_2(T_c)$.

Assign weight 1 to t_p and t_r, 2 to t_q: then up to weight 2 we have $\Delta'_1 = p''_1 q'_1 t_q - q''_0 p'_3 (t_p^2 + t_p t_r + t_r^2)$, so to that order $t_q = \frac{q''_0 p'_3}{p''_1 q'_1} t_p^2$, with non-zero coefficient. Hence the coincidence point is simple.

We have now shown that at each point of T_c we have a smoothly immersed curve. Double points can only occur if P_0Q_0 lies in two trisecants, or more accurately, defines a 4-secant $P_0Q_0R_0S_0$, thus $(P_0, Q_0) \in W_4$. It remains to show that the two branches at such a point are not tangent. With the above notation, we had $0 = r'_1 p''_1 (r_0 - q_0) t_p + \xi r''_0 t_q$, where $\xi = (q_1 - \lambda q''_1) r'_1 - (r_1 - \lambda r''_1) q'_1$. Thus the condition for tangency of the two branches is

$$\{(q_1 - \lambda q''_1) r'_1 - (r_1 - \lambda r''_1) q'_1\} r''_0 s'_1 p''_1 (s_0 - q_0) = \{(q_1 - \lambda q''_1) s'_1 - (s_1 - \lambda s''_1) q'_1\} s''_0 r'_1 p''_1 (r_0 - q_0).$$

Substituting $q_0 = \lambda q''_0$, $r_0 = \lambda r''_0$, $s_0 = \lambda s''_0$, and dividing both sides by $\lambda p''_1 q'_1 r'_1 s'_1$, this reduces to

$$\left\{\frac{q_1 - \lambda q''_1}{q'_1} - \frac{r_1 - \lambda r''_1}{r'_1}\right\} r''_0 (s''_0 - q''_0) = \left\{\frac{q_1 - \lambda q''_1}{q'_1} - \frac{s_1 - \lambda s''_1}{s'_1}\right\} s''_0 (r''_0 - q''_0).$$

Now the points P_0, Q_0, R_0, S_0 lie on the line $x' = 0, x = \lambda x''$, with co-ordinates $0, q''_0, r''_0, s''_0$, and the tangents to Γ at these points lie in the planes $x' = \mu(x - \lambda x'')$, where the corresponding values of μ are $0, \frac{q_1'}{q_1-\lambda q_1''}, \frac{r_1'}{r_1-\lambda r_1''}, \frac{s_1'}{s_1-\lambda s_1''}$. The above equality requires the cross-ratios $(0, q_0'', r_0'', s_0'')$ and $(0, \frac{q_1'}{q_1-\lambda q_1''}, \frac{r_1'}{r_1-\lambda r_1''}, \frac{s_1'}{s_1-\lambda s_1''})$ to be equal. But projection genericity implies that they are not. $\square$

Before treating intersection numbers in $\Gamma\times\Gamma$ we introduce a map which, in some cases, enables us to deduce them from intersection numbers in E_Γ (which were given in Lemma 2.1). Define $\Phi : \Gamma \times \Gamma \rightsquigarrow E_\Gamma$ by $(P, Q) \mapsto (P, \Pi)$, where Π is the plane through $T_P\Gamma$ and Q. This is defined except if $Q \in T_P\Gamma$, i.e. except on the diagonal $\Delta(\Gamma)$ and W_2. We have $\Phi(T_a^t) = E_a$, $\Phi(T_b) = E_b$ and $\Phi(T_c) = E_c$; the images of points of other types are given by

$$\begin{array}{c|cccccccc} x & W_1' & W_2^t & W_3 & W_3' & W_4 & W_5 & W_5^t & W_6 \\ \Phi(x) & a_2 & \alpha & \beta & c_2 & \gamma & b_2 & ab & ac \end{array}.$$

Lemma 7.6. *The restriction of Φ to the complement of $\Delta(\Gamma) \cup T_b$ is a submersion. Along T_b, the map is a simple fold, except at points of W_5.*

Proof. With co-ordinates as above, Π is the plane through the x''−axis and (q, q', q''), which has initial position $(q_0, 0, q_0'')$. Thus for t_q small, the angle made by Π is $q_1't_q/q_0$. Hence the map is a local submersion if $q_1' \neq 0$, i.e. if $(P_0, Q_0) \notin T_b$.

For the second assertion we may suppose $q_0 \neq 0$, $q_0' = 0$, $q_1' = 0$. Since the point P is given by the first projection, it suffices to keep $P = P_0$ fixed, and see how Π varies with Q. Here the angle is $q_2't_q^2/q_0$ (modulo higher terms), so is a non-zero multiple of t_q^2 except if $q_2' = 0$, i.e. $(P_0, Q_0) \in W_5$. $\square$

Proposition 7.7. *The intersection number at each of the common points of Lemma 7.2 is* 1*, except that at W_2^t the intersection number of T_a^t and T_c is 2.*

Proof. We treat the cases in turn.

$W_1 \subset T_a \cap T_a^t \cap T_b$. We have seen that at such a point, to first order, T_a is given by $3t_p + t_q = 0$, hence T_a^t by $t_p + 3t_q = 0$, while T_b is given by $t_p + t_q = 0$.

$W_2 \subset T_a^t \cap T_b \cap T_c$. We have $W_2 \subset I_2(T_a^t) \cap I_1(T_b) \cap I_2(T_c)$, so at these points T_b is transverse to the others. From the above calculations, the least order terms at W_2 are

for T_a^t we have $t_q = \frac{3p_3'q_0''}{q_1'p_1''}t_p^2$.

for T_c we have $t_q = \frac{p_3'q_0''}{q_1'p_1''}t_p^2$.

Since $p_3'q_0'' \neq 0$, T_a^t and T_c have intersection number 2.

$W_3' \subset T_b \cap T_c$: the case of a T-3-secant $(P_0Q_0)R_0$. Here we apply Lemma 7.6. Since E_b and E_c have simple tangency at a point of $S(c_2)$, it follows since Φ is a simple fold along T_b that the pre-images T_b and T_c are transverse.

$W_5 \subset T_a \cap T_b$. Here transversality holds since $W_5 \subset I_1(T_b) \cap I_2(T_a)$.

$W_6 \subset T_a \cap T_c$. Here since E_a and E_c are transverse at ac, it follows from Lemma 7.6 that T_a and T_c are transverse at W_6. $\square$

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