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GRAPHS OF STABLE MAPS FROM CLOSED ORIENTABLE SURFACES TO THE 2-SPHERE

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Abstract. We prove that any bipartite weighted graph can be associated to some stable map from a closed orientable surface to the sphere and obtain necessary and sufficient conditions on a graph to be attached to a fold map of a given degree.

1. Introduction

The local behaviour of stable maps between surfaces was described by Whitney, who determined the typical singularities that these maps may have, namely fold curves with isolated cusps. More recently, the work of T. Ohmoto and F. Aicardi [17], based on the Vassiliev-type isotopy invariants [21], has thrown new light on the study of stable maps from a non local viewpoint. These invariants are related to the behaviour of the branch sets (or apparent contours) of these maps.

In order to investigate the global classification of stable maps from surfaces to the plane, graphs of stable maps were introduced in [12] to provide a combinatorial description of the topology of the singular set (see §2 for the definition). A natural question is to characterize graphs of stable maps (for example they are necessarily bipartite). In [13] the special case of stable maps from the sphere to the plane was studied, with emphasis on fold maps (i.e. those without cusps). The classification of fold maps between manifolds and possible related homotopy principles has been addressed by various authors ([1], [2], [8], [19], [20]). In [13] it was shown that *any tree with zero weights is the graph of a stable map from the sphere to the plane*. On the other hand, the vertices of any tree may be labelled alternately positive and negative (i.e the tree is bipartite). Graphs of fold maps from the sphere to the plane were then characterized as being trees with an equal number of positive and negative vertices. In [14] it was shown that graphs of stable maps of closed orientable surfaces to the plane are precisely non negatively weighted bipartite graphs. As for fold maps, it was shown that the characterization in the spherical case extends to fold maps all of whose regions are planar (this corresponds to the zero weight condition).

Potential applications of stable maps such as the global study of Gauss maps on closed surfaces, or the determination of linking numbers of closed curves in terms of secant maps lead one to consider stable maps and fold maps from surfaces to the sphere of arbitrary degree (the degree zero case being essentially that of maps into the plane). In the present article we characterize graphs of stable maps in this more general setting. The main results are as follows.

1) Any bipartite graph $\mathcal{G}$ with non negatively weighted vertices is the graph of a stable map of a connected orientable and closed (compact and boundaryless) surface into the 2-sphere of

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arbitrary degree. The Euler characteristic of the surface is $2(\chi(\mathcal{G}) - g)$*, where* g *stands for the sum of all the weights in* $\mathcal{G}$*.*

A bipartite graph is said to be *balanced* provided the difference $V^+ - V^-$ between the numbers of positive and negative vertices equals the difference $g^+ - g^-$ between the sums of the weights of the positive and the negative vertices.

2) A bipartite graph can be the graph of some fold map from a closed orientable surface to the plane if and only if it is balanced. Moreover, any bipartite graph can be the graph of some fold map from a closed orientable surface to the sphere of degree $(V^+ - V^-) - (g^+ - g^-)$ *(in particular, the degree of a fold map may be deduced from its weighted graph).*

The basic techniques used here are surgeries on stable maps (together with the corresponding modification of the graph) and Quine's Theorem relating the degree and the number of cusps (with signs) of stable maps between surfaces ([18]). In §3 *reduction* and *extension* of graphs are defined, based on a suitable interpretation of certain codimension one transitions of stable maps ([17]) and used in §4 and §6 in the characterization of graphs of fold maps.

Finally, we notice that the pair given by the graph and the branch set is not enough to determine the isotopy class of a stable map from a surface to the plane or the sphere. As explained in §3, there are examples of non equivalent stable maps sharing both, their graph and their branch set. In order to distinguish between them we need to add some extra information which can be encoded in the form of Blank's words [5, 4, 9, 10] conveniently associated to the curves of the branch set.

2. Stable maps

We first recall some definitions and basic results. Two smooth maps f and g from a surface M to a surface N are said to be C^∞ *right-left equivalent* (simply, *equivalent*) if there are diffeomorphisms, l and k, such that $g \circ l = k \circ f$. The maps f and g are *isotopic* if both the above diffeomorphisms are isotopic to the identity. A map f is said to be *stable* if all maps sufficiently close to f (in the Whitney C^∞-topology) are equivalent to f.

A point of the source surface M is a *non singular point* of f if the map f is a local diffeomorphism around that point, and *singular* otherwise. The singular set Σf of f is the set of singular points of f, and its image $Bf = f(\Sigma f)$ is called the *branch set* of f. By Whitney's theorem [11], for any stable map $f : M \to N$, its singularities are locally of fold type $(x, y) \mapsto (x^2, y)$, and of cusp type $(x, y) \mapsto (x^3 + yx, y)$; Σf is a union of embedded curves on M and Bf is a union of smooth curves on N with transverse double points and possibly many cusp points. The non-singular set (which is immersed into the surface N by the map) consists of finitely many *regions*. Given orientations of the surfaces M and N, a region is *positive* if the map preserves orientation and *negative* otherwise. The singular set is the frontier of each half (positive or negative) of the surface M, i.e. any singular curve lies in the frontier of a positive and a negative region. We denote by M^+ (resp. M^-) the union of all the positive (resp. negative) regions including their boundaries. Clearly, M^+ and M^- meet in their common boundary, the singular set of f.

Topological information of stable maps f may be conveniently encoded in a weighted graph from which the pair $M, \Sigma f$ may be reconstructed (up to diffeomorphism) ([13], [14]). The edges and vertices of the graph correspond (respectively) to the singular curves and the regions (i.e. the connected components of the non-singular set). An edge is incident to a vertex if and only if the singular curve corresponding to the edge lies in the frontier of the region corresponding to the vertex. In other words, given a stable map $f : M \to N$, its graph $\mathcal{G}(f)$ is the *dual graph* of Σf in M. We attach a label to each vertex of the graph, + (or $-$) for positive (resp. negative) regions. Since each component of Σf is the boundary of a positive and of a negative region, the signs of the vertices are assigned alternatively, that is, the graph $\mathcal{G}(f)$ is bipartite. The weight g_v of a vertex v is defined to be the genus of the corresponding region i.e the genus of the closed surface obtained by adding disk to the region, one for each boundary curve. Figure 1 shows different stable maps of zero degree from the torus and bi-torus to the plane and their weighted graphs.

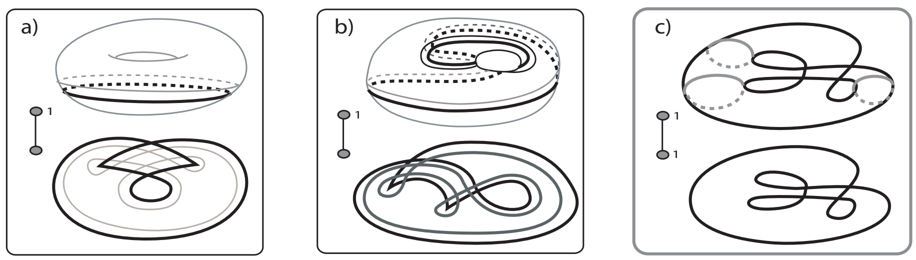

FIGURE 1. Stable maps and their graphs

In the particular case of stable maps from the sphere to the sphere, S. Demoto [7] has studied the isotopy classes corresponding to a graph with a unique edge and 2 vertices. In this case, the branch set is a connected closed curve which may have cusps and/or self-intersections. For $d = deg(f) \geq 2$, Demoto proves that when the branch set has no self-intersections the number of cusps of f is at least $2d$. Example c) in Figure 2 illustrates a map $f : T \to S^2$ with degree 1 whose graph has exactly one edge and the branch set has 4 self-intersections and no cusps. The examples a) and b) in Figure 2 correspond respectively to stable maps from the sphere and the torus to the sphere, whose branch set has no cusps and c) its singular set consists of a unique curve, whereas the second one has degree 1. The corresponding graphs are shown on the left of each picture. As we shall see later, the basic examples displayed in Figures 1 and 2 will take an important role in the proofs of the results of this paper.

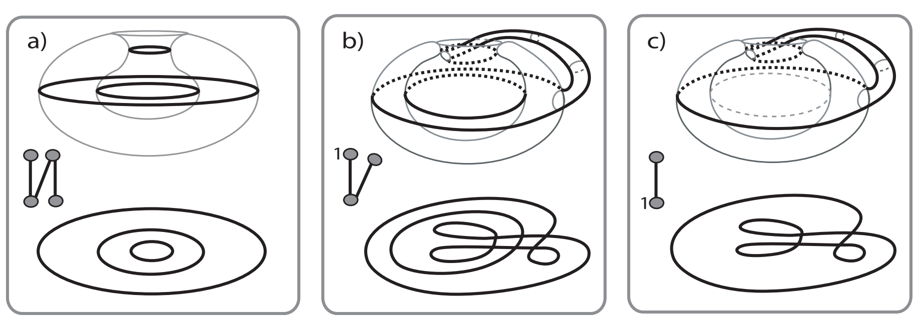

FIGURE 2. Branch sets with 4 self-intersections and no cusps.

We say that the graph $\mathcal{G}(f)$ is of *type* $T(\mathcal{G}) = (m, n, g)$ if it has m edges, n vertices and the total sum of the weights of its vertices is g (called the *total weight* of $\mathcal{G}(f)$). We observe that the following relation holds: $g(M) = \beta_1(\mathcal{G}(f)) + g$, where $g(M)$ denotes the genus of M and $\beta_1(\mathcal{G}(f))$ the $1st$ Betti number of the graph.

A cusp is called positive (resp. negative) if its local mapping degree is $+1$ (resp. -1) with respect to given orientations.

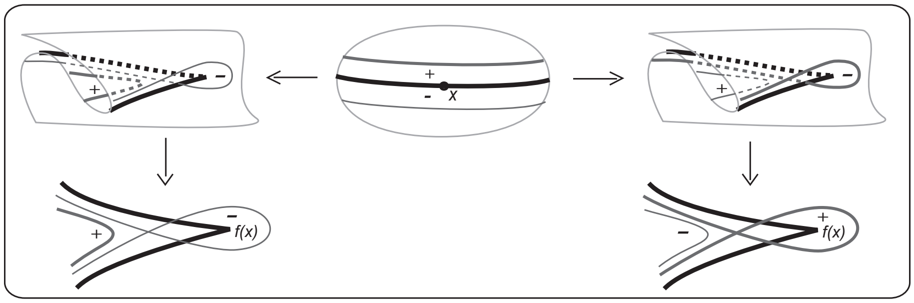

FIGURE 3. Example of negative and positive cusps.

Let f be a stable map between closed surfaces M and N of degree $deg(f)$. In [18] it was shown that

$$\chi(M) - 2\chi(M^-) + C = deg(f)\chi(N),$$

where χ denotes the Euler characteristic and $C = C^+ - C^-$, the number of positive cusps minus the number of negative cusps.

Lemma 2.1. *For a stable map $f : M \to S^2$ with $C = 0$ one has*

$$deg(f) = (V^+ - V^-) - (g^+ - g^-),$$

where V^+ (resp. V^-) is the number of positive (resp. negative) regions and g^+ (resp. g^-) the genus of M^+ (resp. M^-).

Proof: It follows from Quine's formula that $\chi(M) - 2\chi(M^-) = 2deg(f)$. Now, $\chi(M) = \chi(M^+) + \chi(M^-) - \chi(M^+ \cap M^-) = \chi(M^+) + \chi(M^-)$, and thus

$$\chi(M^+) - \chi(M^-) = 2deg(f). \qquad (1)$$

Then the result follows from the relation $\chi(M^\pm) = 2(V^\pm - g^\pm - m)$. $\square$

3. SURGERY OF STABLE MAPS

One way of constructing a stable map is to glue together two stable maps. In particular, in a surgery, a pair of disjoint disks in the surface is removed and replaced by a tube, the map then being extended over the interior of the tube. There are two types of surgery: horizontal and vertical. These were introduced in [14] for stable maps from surfaces to the plane. The extension of these definitions for stable maps between closed surfaces in general is straightforward:

a) Horizontal surgery. Given a stable map h between two surfaces M and N, a *bridge* is an embedded rectangle β in N which meets the branch set Bh in opposite edges (and nowhere else) compatibly with the orientation of the branch set as shown in Figure 4(a) (see [16]). The stable map h_β is constructed as follows. The bridge meets $h(M)$ in two intervals, $h(I)$ and $h(J)$, say. Choose small disks in M one containing I, the other J and replace their interiors by a tube (i.e. an annulus), respecting the orientation of M, so as to obtain an oriented surface. As illustrated in Figure 4(a), the map h may then be extended over the tube to give the required stable map h_β. In particular, if M is the disjoint union of surfaces P and Q and f and g denote the restrictions of h to P and to Q, with I in P and J in Q then we obtain the *horizontal sum* $f +_{hor} g$. In other words $h = f \cup g$ and $(f \cup g)_\beta = f +_{hor} g$.

b) Vertical surgery. In this case we take a connected sum by identifying two small non-singular disks in the domain, one positive and one negative (as in Figure 4 (b)) whose images in

N coincide. The disks are replaced by a tube which is mapped into the plane, with a singular curve running around the middle of the tube. Thus the surgery adds a disjoint embedded curve to the branch set. We denote this sum as $f +_{ver} g$. It is possible also to perform vertical surgery using a bridge, but this will not be needed here. Observe that horizontal (resp. vertical) surgery decreases (resp. increases) the number of edges by one.

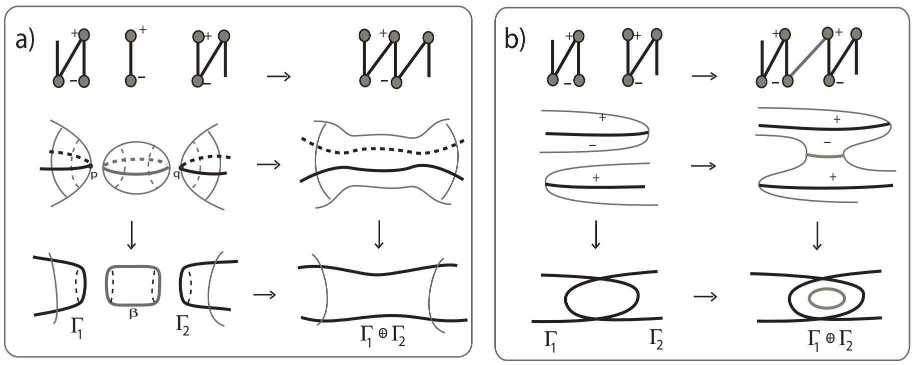

FIGURE 4. Surgeries: (a) horizontal, (b) vertical.

Figure 4 also shows the effects of the surgeries on the graphs. It is easy to see that if $\mathcal{G}_i$ represents the graphs of $f_i, i = 1, 2$ and $\mathcal{G}_1 +_{hor} \mathcal{G}_2$, $\mathcal{G}_1 +_{ver} \mathcal{G}_2$ respectively represent the graphs of $f_1 +_{hor} f_2$ and $f_1 +_{ver} f_2$, then

- $T(\mathcal{G}_1 +_{hor} \mathcal{G}_2) = T(\mathcal{G}_1) + T(\mathcal{G}_2) - (1, 0, 0)$,
- $T(\mathcal{G}_1 +_{ver} \mathcal{G}_2) = T(\mathcal{G}_1) + T(\mathcal{G}_2) + (1, 0, 0)$,

Observe that surgeries do not affect the degree. In particular, the degree of a horizontal or vertical sum of f and g is the sum of the degrees of f and g. In particular, as illustrated in Figure 5, taking the horizontal connected sum of any stable map $f : M \to S^2$ with $g : S^2 \to S^2$ having two cusps depicted below increases the degree of f by one but does not change its graph.

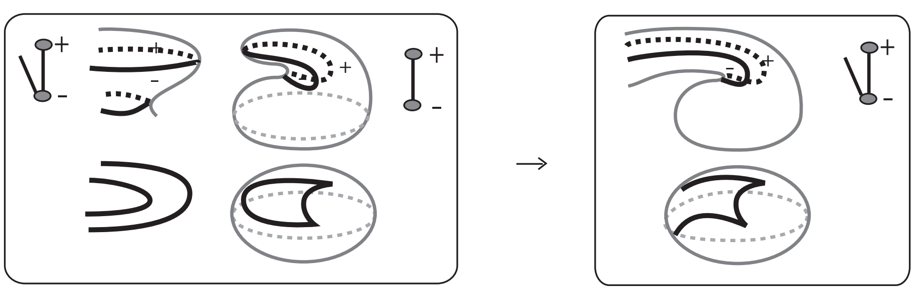

FIGURE 5. Altering the degree and preserving the graph.

c) Transitions. Apart from connected sums we can also use certain transitions in order to alter the graph and/or the branch set of a stable map. A codimension one transition corresponds to a generic homotopy from a given stable map f_0 to another stable map f_1 which is not right-left equivalent to f_0. In other words, this means a path transverse to all the strata of the *the discriminant hypersurface* in the mapping space $C^\infty(M, S^2)$. See [12] or [17] for the description of all the possible transitions. The interesting transitions, from our viewpoint, are those altering the numbers of cusps, or of singular curves, namely the swallowtail, beaks and lips transitions. Figure 6 and 7 show some examples of swallowtail, lips and beaks transitions on a degree one map from the sphere to the sphere. Clearly, the transitions do not alter the degree, for the new map remains in the same pathcomponent of $C^{\infty}(M, S^2)$.

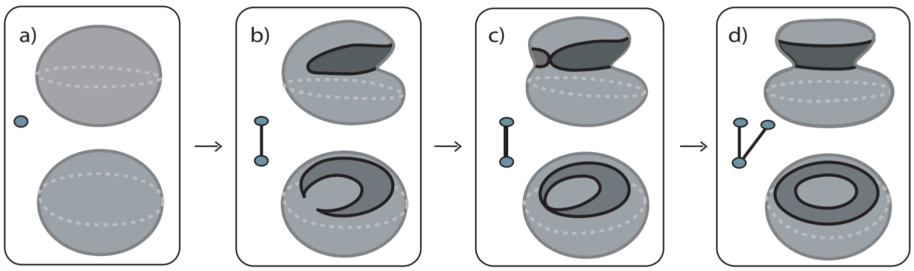

FIGURE 6. Lips $(a \to b)$ and beaks $(b \to d)$ transitions on maps of the sphere.

We shall focus our attention into a special combination of transitions that will be useful in the last section of this paper: The *double beaks+double inverse swallowtail*. This is obtained by successive application of beaks transitions in two nearby segments of neighbouring singular curves (with opposite orientations), followed by successive annihilations of two pairs of cusps (with opposite signs) trough swallowtail transitions. The effects of this homotopy on the graph and branch set are shown in Figure 7. We observe that the total number of singular curves decreases by two, which corresponds to the identification of three successive edges to form one edge of the new graph (referred to as the *reduced graph*). In particular, by means of successive reductions, any odd number of consecutive edges in a tree may be identified to form a single edge in the reduced tree.

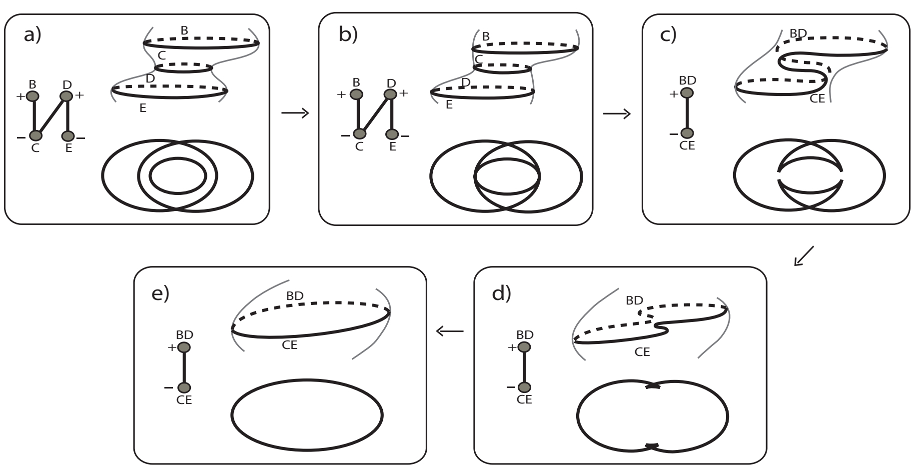

FIGURE 7. Double beaks + double inverse swallowtail: Reduction of a graph .

The inverse homotopy, *double swallowtail+double beaks*, obtained by creating two couples of cusps in a singular curve by means of two swallowtail transitions, followed by a suitable pair of beaks transitions has the effect of replacing an edge by three consecutive edges in a new graph (referred to as the *extended graph*). Note that the extended graph depends on the location where transitions happen. For example, as in Figure 8, given an edge pq, the set $\mathcal{H}$ (resp. $\mathcal{F}$) of edges emanating from p (resp. q) is divided into two subsets $\mathcal{H}_i$ (resp. $\mathcal{F}_i$), $i = 1, 2$, so that these subsets of edges are distributed to created vertices p_1, p_1, p_2, q_2 in the extended graph. Also the weight of p (resp. q) is divided into two weights of p_1, p_2 (resp. q_1, q_2). Conversely, the homotopy in the opposite direction is a reduction of graphs, which gathers edeges and weights. Clearly these homotopies do not affect the degree of a map.

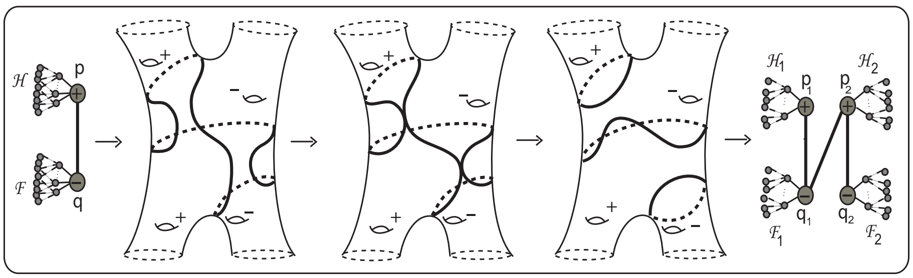

FIGURE 8. Extensions of a graph.

Observe that the graph of any stable map $f : M \to S^2$ is bipartite and that $\chi(M) = 2\chi(\mathcal{G}(f)) - 2g$. In particular, M is the sphere if and only if the graph is a tree with all weights zero. These considerations lead to

Theorem 3.1. *Any bipartite graph with non-negatively weighted vertices is the graph of a stable map of a surface to the sphere of arbitrary degree.*

Proof: It was shown in [14] that any bipartite graph may be realized by a stable map of degree zero from some surface into the sphere. Since the horizontal surgery in Figure 5 does not change the graph the map may be taken to have arbitrary positive degree. To get negative degree compose with the antipodal map of the sphere which does not change the graph. □

Remark 3.2. *We observe that the pair (graph, branch set) is in general not enough to determine the isotopy class of a stable map from a closed surface to the plane or the sphere. A good example of this is obtained from Milnor's example of a plane curve with 6 double points which can be seen as the image of the boundary of a 2-disc by two different immersions. If we define a mapping $f : S^2 \to I\!R^2$ by putting it equal to one of these immersions on the lower hemisphere and to the other on the upper hemisphere, we obtain a fold map from S^2 to $I\!R^2$. On the other hand, by choosing the same immersion on both hemispheres we get a new fold map from S^2 to $I\!R^2$ which can be joined by a smooth family of fold maps to the orthogonal projection of the unit 2-sphere in $I\!R^3$ on the equatorial plane. These two maps although share both, their graph and their apparent contour, are not equivalent* [8]. *We thus need some extra information which is encoded in the set of Blank's words (*[4], [5], [10]*) associated to the curves of the branch set. Once we specify a bijection between the edges of the graph and the curves in the branch set, we can work separately at each vertex by applying the techniques described in* [10] *in order to recover the class of the immersion of a surface with boundary associated to it. A convenient assemblage of these immersions will lead to a stable map class.*

4. FOLD MAPS

In this section we consider fold maps of surfaces into the plane which, of course, are also fold maps into the sphere of degree zero. We recall that a *fold map* is a stable map without cusps, so that the branch set consists of curves immersed in the plane. In [13] it was shown that a necessary and sufficient condition for a graph with zero weights to be the graph of a fold map (of an orientable surface) is that the number of positive and negative vertices be equal. We generalize this fact to the case of graphs with arbitrary weights. In fact it immediately follows from Lemma 2.1 that the graph of fold maps $f : M \to S^2$ of degree zero is balanced, i.e., $V^+ - V^- = g^+ - g^-$. Furthermore, the converse is also true (Theorem 4.2 below). To show this, we begin with the case of trees.

Proposition 4.1. *Any balanced tree is the graph of a fold map of a surface into the plane and hence of degree zero into the sphere.*

Proof: The proof is by induction on the total weight g of the tree. The case of trees of total weight zero was proven in [13]. Let $\mathcal{T}$ be a balanced tree of total weight $g > 0$. Denote by g^+ (resp.g^-) the sum of the weights of the positive (resp. negative) vertices of $\mathcal{T}$. We may suppose that $g^+ > 0$. There are two cases to consider: a) $g^- > 0$ and b) $g^- = 0$.

a) We may choose a positive vertex v of weight $g_1 > 0$ and a negative vertex w of weight $g_2 > 0$ and join them by a path in the tree (necessarily consisting of an odd number of edges). We may assume that vertices of the path have weight zero (otherwise we could choose a shorter such path). Let $\mathcal{T}'$ be the tree obtained by reducing the path to a single edge vw of $\mathcal{T}'$. The tree $\mathcal{T}'$ also has total weight g. An important observation is that reduction leaves g^+, g^- and $V^+ - V^-$ unchanged (though not V^+ or V^-). Thus $\mathcal{T}'$ is also balanced. Let $\mathcal{T}''$ be the tree $\mathcal{T}'$ with the weights g_1 and g_2 replaced by $g_1 - 1$ and $g_2 - 1$ (Figure 9a)). Thus $\mathcal{T}''$ is also balanced. The total weight of $\mathcal{T}''$ is clearly $g - 2$ so that, by induction, it is the graph of a fold map of a surface to the plane. The connected sum (along the singular curve corresponding to the edge vw) of this fold map with the fold map of the bitorus to the plane illustrated in Figure 1 (c) is a fold map with graph $\mathcal{T}'$. By applying a sequence of double swallowtail + double beaks transitions we create a fold map whose graph is $\mathcal{T}$, as required.

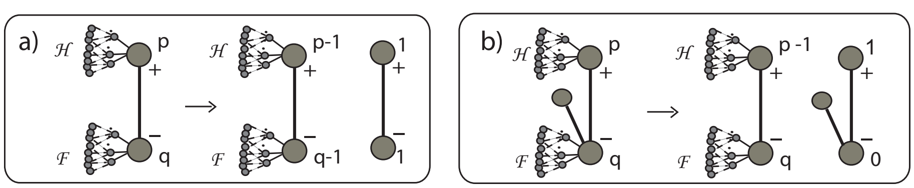

FIGURE 9. Decomposition of trees.

b) In this case, $V^+ - V^- = g^+ > 0$ so that $V^+ > V^-$ and $g^+ < V^+$. Claim: there exists an extreme (i.e. belonging to just one edge) positive vertex of weight zero. Proof of claim: Let L be the number of all positive vertices of weight zero. Then it is easy to see $V^+ - g^+ \leq L$ and by the assumption we have $V^- \leq L$. Now, suppose that there is no extreme positive vertex of weight zero. Fix a negative vertex n and orient all edges of the tree to be bound for n. Then to each positive vertex p of weight zero we may assign a negative vertex z ($\neq n$) so that zp is an edge pointing toward n. Hence $V^- > L$, that makes a contradiction. This proves the claim. Thus we may choose v a positive extreme vertex of weight zero. Now $g^+ > 0$ so there exists a positive vertex w of weight $g_1 > 0$. There is a path from v to w and we may insist that all vertices of this path between v

and w have weight zero. Since both v and w are positive the length of the path is even so we may reduce $\mathcal{T}$ to a tree $\mathcal{T}'$ in which v and w are connected by a path of length two say vuw. As before, $\mathcal{T}'$ is also balanced. Now let$\mathcal{T}''$ be the tree $\mathcal{T}'$ with the edge uv removed and the weight of w reduced by one to $g_1 - 1$ (recall $g_1 > 0$). $\mathcal{T}''$ is clearly also balanced and of total weight $g - 1$. By hypothesis, $\mathcal{T}''$ is the graph of a fold map. Forming the horizontal connected sum with the fold map from the torus to the plane (illustrated in Figure 2b)) yields a fold map whose graph is $\mathcal{T}'$. Finally, as before, a sequence of double swallowtail + double beaks transitions produces a fold map whose graph is $\mathcal{T}$.

Clearly, in both cases the map f is a fold map from the closed surface M with Euler characteristic $\chi(M) = 2 - 2g$ to the plane. □

Theorem 4.2. *Any bipartite balanced graph is the graph of a fold map from a surface to the plane.*

Proof: As above, it is enough to find some map $f : M \to I\!R^2$ whose graph is the given one. Now observe that given any bipartite graph one may obtain a tree with the same vertices by removal of appropriate edges. Moreover, the graph is balanced if and only if the tree is. We then have from Proposition 4.1 that this tree may be realized by a fold map $f : M \to I\!R^2$, where M is a closed surface with genus equal to the sum of all the weights in the tree. Finally we can apply vertical surgeries on f in order to recover the removed edges, where f may be replaced properly via homotopy of fold maps if necesary. □

We remark that a general result due to Y. Eliashberg (Theorem B, [8]) implies that for any closed non necessarily connected curve C separating a closed orientable surface M into pieces M^+ and M^- with common boundary C, there exists a fold map from M to the plane whose singular set is C if and only if $\chi(M^+) = \chi(M^-)$. We saw in [12] that there is a $1-1$ relation between topological classes of curves in a surface M and weighted graphs satisfying the relation $\chi(M) = 2(\chi(\mathcal{G}) - g)$. Since the condition $\chi(M^+) = \chi(M^-)$ amounts to say that the corresponding graph is balanced, we have that Proposition 4.1 can also be obtained from Eliashberg result. Nevertheless, we emphasize that whereas Eliashberg's techniques guarantee the existence of such a map, those presented here furnish a practical method to construct it.

5. Fold maps with prescribed branching data in the plane

It is a well known fact (see [6] or [15]) that the sum of the winding numbers of the boundary curves of a surface immersed in the plane is equal to the Euler characteristic of the surface. Since we can view a fold map from a surface to the plane as a union of immersed surfaces with boundary, with the boundary curves conveniently identified with the singular set of the map, we can apply this result in order to obtain information on the branch set curves of fold maps from closed surfaces to the plane.

Lemma 5.1. *Any branch curve of a fold map $f : M \to I\!R^2$, whose graph is a (weighted) tree has odd winding number (i.e., an even number of double points).*

Proof: Consider the tree with each edge indexed by one plus the winding number of the corresponding branch curve. At any vertex v, the local sum of the indices must be equal to $\chi(R_v)$, where R_v denotes the region represented by v. Since the graph is a tree there is a vertex v_1 which belongs to just one edge e_1. It follows that the index of e_1 must be equal to $\chi(R_{v_1}) + 1 = 2 - 2\omega_1$, where ω_1 is the weight of v_1, and thus even. Removing e_1 we obtain a subtree for which the local sums are also even. By induction on the number of edges of the tree, starting with the case of one edge, the indices of the subtree are all even. In other words, the winding numbers are all odd. □

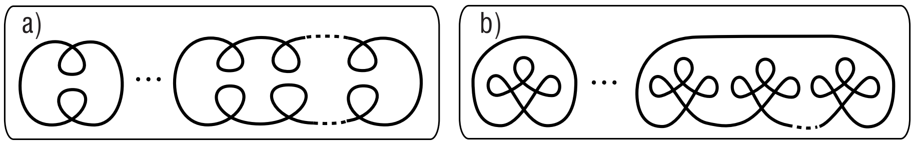

FIGURE 10. Basic plane curves with odd winding numbers.

Figure 10 displays representatives of two different stable isotopy classes (see [3]) with odd winding number. We shall denote them respectively as curves of type $(1,0)$ (10 a)) and $(0,1)$ (10 b)). By a curve of type (a,b) we shall understand a connected sum of a curves of type $(1,0)$ and b curves of type $(0,1)$. We shall refer to these curves as *basic curves.* By a curve of type $(0,0)$ we understand an embedded circle.

Let $\mathcal{T}$ be a weighted tree with vertices $\{v_k\}_{k=1}^n$ and corresponding weights $\{\omega_k\}_{k=1}^n$. We can order the vertices in such a way that $\{v_k\}_{k=1}^r$ are the positive ones and $\{v_k\}_{k=r+1}^n$ the negative. To each edge $v_iv_j, i=1,\cdots,r,\ j=r+1,\cdots,n$, we associate a variable I_{ij}. We write C_k for the sum of the indices I_{kj} for all the edges v_kvj containing v_k.

Lemma 5.2. *The tree $\mathcal{T}$ is balanced if and only if the compatibility conditions*

$$C_k = 2 - 2\omega_k$$

have a unique solution.

Proof: Since $\mathcal{T}$ is a tree the number of edges is $n-1$. The compatibility condition at any vertex v_k is $C_k = 2-2\omega_k$. We thus have a linear system of n equations in $n-1$ variables. On the other hand, we have the conditions,

$$\sum_{i=1}^{r}(C_i - 2 - 2\omega_i) = \sum I_{ij} - 2n = \sum_{j=1+r}^{n}(C_j - 2 - 2\omega_j),$$

where the middle sum runs over all the edges of $\mathcal{T}$.Thus any equation is a consequence of the rest. Now fix a vertex $\star$. For any vertex v_k define d_k to be the length of the (unique) path in the tree between v_k and $\star$. Thus $d_\star = 0,\ \ d_k = 1$ if $v_k\star$ is an edge, for any edge v_iv_j, d_i and d_j differ by one and, for any vertex $v_k \neq \star$ there is a unique edge v_kv_s such that $d_k = d_s+1$. The equation $C_k = 2-2\omega_k$ determines I_{ks} in terms of the other variables i.e. in terms of the I_{ij} for which $d_i = d_j + 1$. For the largest value of d_k, C_k is just I_{ks}, for which $I_{ks} = 2-2\omega_k$ is, of course, the unique solution. Thus the equations $C_k = 2-2\omega_k$ may be solved uniquely for successively smaller values of d_k up to and including $d_1 = 1$. The remaining equation $C_\star = 2-2\omega_*$ is a consequence of the rest. We observe that the solution consists entirely of even integers, corresponding to the fact (already proved) that the winding numbers must all be odd. □

Proposition 5.3. *Any balanced weighted bipartite graph is the graph of a fold map from a surface M to the plane whose branch set consists of basic curves.*

Proof: It is enough to prove the result for a tree, for, given any balanced graph, we may take a maximal tree which will also be balanced. If the tree is the graph of a fold map then by doing vertical surgeries on the fold map we realize the original graph by a fold map. The extra curves introduced into the branch set are all embedded circles hence basic. For a tree the proof goes by induction on the total weight. For zero weighted trees it was shown in [13] by using curves of type $(a, 0), a \in \mathbb{Z}$. Suppose the assertion is true for any balanced tree of total weight g and let $\mathcal{T}$ be a balanced tree with total weight $g+1$. We proceed as in Proposition 4.1 and consider the two cases a) and b) and the corresponding reduced trees. We observe that in both cases, the decomposition of the reduced tree leads to two fold maps:

- f_1, whose branch set is made of a curve of type $(0, 1)$ in case a) and of two curves, one of type $(0, 1)$ and the other of type $(0, 0)$ in case b), and
- f_2, whose graph has total weight lesser than $g+1$ and thus, by the induction hypothesis can be chosen in such a way that all its branch curves are of Type (a, b).

Now observe that their horizontal sum also gives rise to a fold map whose branch curves are of type (a, b). Moreover, the new branch curves produced in the extension process in order to obtain f from $f_1 +_{hor} f_2$ may also be taken in in the family of curves of type (a, b) as can be seen in Figure 11. □

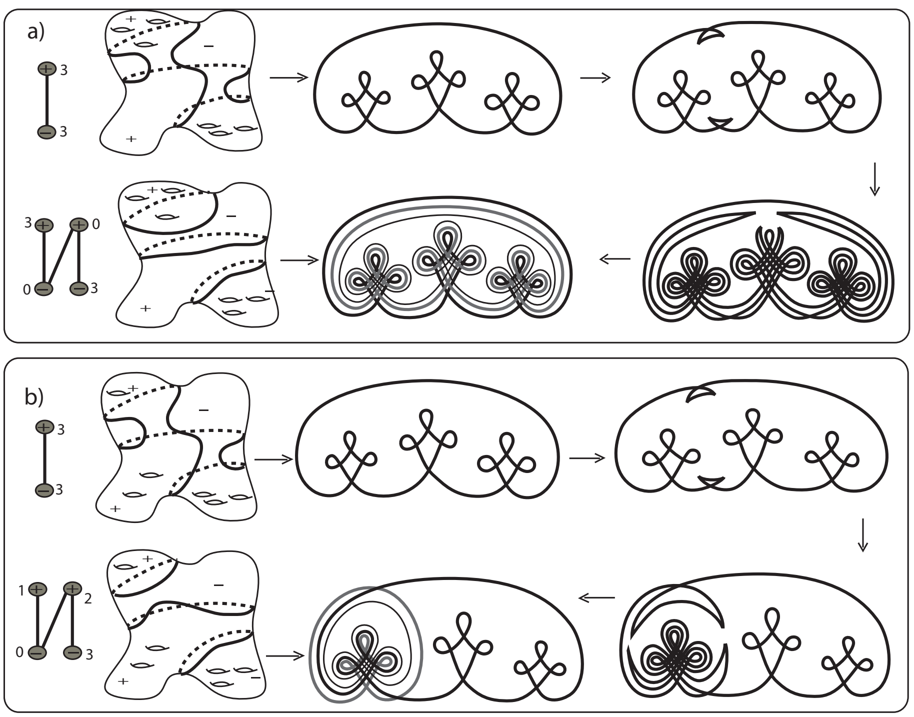

FIGURE 11. Different extensions of a graph.

It can be shown that given a natural number ω and a subset $\{i_1, \cdots, i_k\}$ of odd integers satisfying the relation

$$(i_1 + 1) + \cdots + (i_k + 1) = 2 - 2\omega,$$

we can find an immersion of a surface of genus ω and k boundary components whose respective winding numbers in the plane are $\{i_1, \cdots, i_k\}$. This is proven in a similar way than it was done for discs with holes in [13]. In that case, the family of curves of type $(a, 0)$ was enough to perform all the image curves. Here we must consider all the possible types (a, b), for the curves of type $(0,b)$ contribute to the genus of the considered surface. In fact, for a torus with a unique boundary curve, we can use the curve $(0,1)$ (as in Figure 2a)) and if the curve has genus ω, then the image curves must be chosen of types $\{(a_1,b_1),\cdots,(a_k,b_k)\}$, with $\omega \leq b_1+\cdots+b_k$ for different combinations of these curves defining the image of the boundary of surfaces with non zero genus). Figure 12 illustrates an inductive method for constructing the image of the boundary of immersed regions having k boundary components with total winding number i, for all possible compatible integer sets $(i_1+1,\cdots,i_k+1)$, such that $i=i_1+\cdots+i_k$. This method runs in a similar way to the one used in [13] for fold maps from S^2 to the plane.

In order to construct a fold map corresponding to a given balanced weighted tree we must conveniently assemble different immersed regions whose boundary curves are mapped into a proposed branch set (determined by a given graph).

Proposition 5.4. *Let f be a fold map all whose branch curves are of type (a,b) and suppose that v is an extremal vertex with weight ω. Then the region associated to v has a unique boundary curve whose image by f is of type $(0,\omega)$.*

Proof: Since v is an extreme vertex, there is a unique edge attached to it in the graph. The corresponding branch curve is the image of the boundary of the region R_v represented by v. Supposing that this is a curve of type (a,b), we must have that $a=0$, for curves wit $a\neq 0$ do not satisfy Blank's criterium in order to be the image of the boundary of immersed regions in the plane [9]. On the other hand, the winding number of this branch curve must coincide with the Euler characteristic of R_v, therefore, $1-2\omega = 1-2b$ and we have the required result. □

Remark 5.5. *The results of this section can be transported by stereographic projection to fold maps of degree zero from surfaces to S^2.*

6. Biased graphs and Fold maps

Given an integer number d we say that a bipartite weighted graph is *is biased by d* if the following equality holds

$$V^+ - V^- = g^+ - g^- + d,$$

where V^+ and V^- respectively denote the numbers of vértices with positive, and negative labels, and g^+ and g^- the genus of the corresponding regions.

We shall prove now that any bipartite weighted graph $\mathcal{G}$ can be the graph of some fold map whose degree is equal to the bias of $\mathcal{G}$.

Remark 6.1. *We observe that, as illustrated in Figure 13 below, a curve of type $(0,d), d\geq 0$, can be the branch set of a fold map of degree $d'(\geq 0)$ from the surface of genus $2d''+d'(\geq 0)$ into the sphere, where $d=d'+d''$.*

Any fold map (of a surface into the sphere) has a bipartite graph and degree $(V^+-V^-)-(g^+-g^-)$. Conversely,

Theorem 6.2. *Any bipartite weighted graph may be realized by a fold map (of a surface into the sphere).*

Proof: We prove it first for a tree biased by d and then use vertical surgeries, as above, to extend it to any bipartite graph with bias d. Assume that d is positive (resp. negative). Given such a tree $\mathcal{T}$, let v be one of its vertices that we may suppose is a positive (resp. negative) vertex. Consider a new weighted tree, $\mathcal{T}_d$, obtained from $\mathcal{T}$ by adding d to the weight ω of v. Clearly, $\mathcal{T}_d$ is a balanced tree. Then it follows from Propositions 4.1 and 5.3 that there is a zero

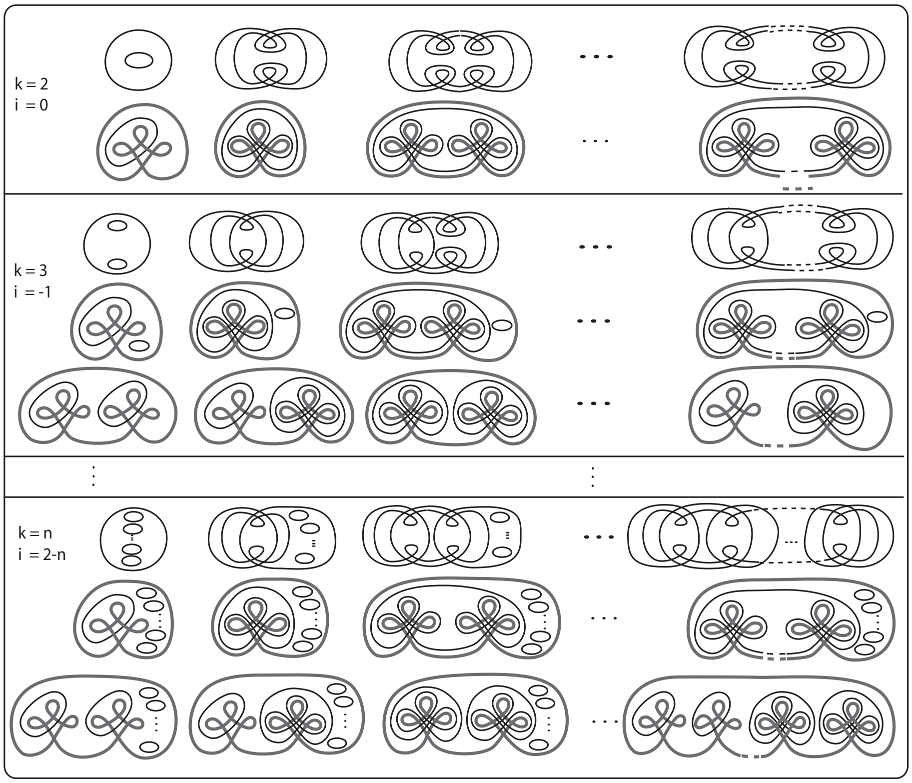

FIGURE 12. Basic curves in the boundary of immersed regions with genus.

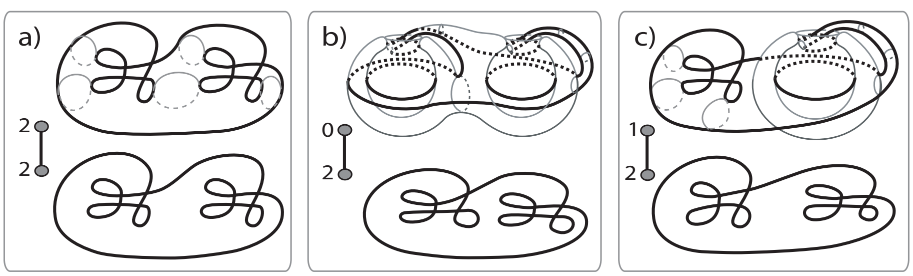

FIGURE 13. genus versus degree.

degree fold map $f : M \to S^2$ whose associated graph is $\mathcal{T}_d$, where $\chi(M) = 2 - 2(g^+ + g^- + d)$ and all the curves in the branch set are of type (a, b). We know from Proposition 5.4 that the branch curve corresponding to the edge e in $\mathcal{T}_d$ must be of type $(0, \omega + d)$. Now, in view of the above remark, we can construct a map $f' : M' \to S^2$ with $\chi(M) = 2 - 2(g^+ + g^-)$, of degree d, without changing the graph and the branch set (see Figure 13. $\square$

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