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CLASSICAL ZARISKI PAIRS

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ABSTRACT. We compute the fundamental groups of all irreducible plane sextics constituting classical Zariski pairs.

1. Introduction

A *classical Zariski pair* is a pair of irreducible plane sextics that share the same combinatorial type of singularities but differ by the Alexander polynomial [10]. The first example of such a pair was constructed by O. Zariski [13]. Then, it was shown in [4] that the curves constituting a classical Zariski pair have simple singularities only and, within each pair, the Alexander polynomial of one of the curves is $t^2 - t + 1$, whereas the polynomial of the other curve is trivial. The former curve is called *abundant*, and the latter *non-abundant*. The abundant curve is necessarily of *torus type*, *i.e.*, its equation can be represented in the form $f_2^3 + f_3^2 = 0$, where f_2 and f_3 are homogeneous polynomials of degree 2 and 3, respectively.

A complete classification of classical Zariski pairs up to equisingular deformation was recently obtained by A. Özgüner [1]. Altogether, there are 51 pairs, one of them being in fact a triple (assuming that the complex orientations of both $\mathbb{P}^2$ and of complex curves are taken into account): the non-abundant curves with the set of singularities $\mathbf{E}_6 \oplus \mathbf{A}_{11} \oplus \mathbf{A}_1$ form two distinct complex conjugate deformation families. The purpose of this note is to compute the fundamental groups of (the complements of) the curves constituting classical Zariski pairs. We prove the following theorem.

1.0.1. **Theorem.** *Within each classical Zariski pair, the fundamental group of the abundant (respectively, non-abundant) curve is* $\mathbb{B}_3/(\sigma_1\sigma_2)^3$ *(respectively,* $\mathbb{Z}_6$*).*

This theorem is proved in Section 4, using the list of [1] and a case by case analysis. In fact, most groups are already known, see [2], [5], [8], [3], and [9], and the few missing curves can be obtained by perturbing the set of singularities $\mathbf{A}_{17} \oplus 2\mathbf{A}_1$. The construction and the computation of the fundamental group are found in Sections 2 (the non-abundant curves) and 3 (the abundant curves).

2. The curve not of torus type

2.1. Up to projective transformation, there is a unique curve $C \subset \mathbb{P}^2$ with the set of singularities $\mathbf{A}_{17} \oplus 2\mathbf{A}_1$ and not of torus type, see [11]; its transcendental lattice is $\begin{bmatrix} 4 & 2 \\ 2 & 10 \end{bmatrix}$. (In the case under consideration, the transcendental lattice can be defined as the orthogonal complement $NS(\tilde{Y})^\perp \subset H_2(\tilde{Y})$, where $\tilde{Y}$ is the minimal resolution of singularities of the double plane ramified at C. Recall that $\tilde{Y}$ is a $K3$-surface.) After nine blow-ups, the curve transforms to the union of two of the three type $\tilde{\mathbf{A}}_0^*$ fibers in a Jacobian rational elliptic surface with the combinatorial

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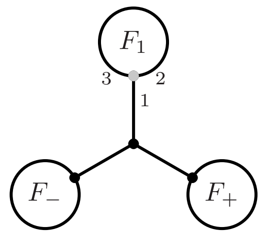

Figure 1. The skeleton Sk of $\bar{B}$

type of singular fibers $\tilde{\mathbf{A}}_8 \oplus 3\tilde{\mathbf{A}}_0^*$ (in Kodaira's notation, one fiber of type I_9 and three fibers of type I_1). For the equation, consider the pencil of cubics given by

$$f_b(x, y) := b(-x^2 - xy^2 + y) + (x^3 - xy + y^3) = 0, \qquad b \in \mathbb{P}^1,$$

and take two fibers corresponding to two distinct roots of $b^3 = 1/27$. (All three roots give rise to nodal cubics, which are the three type $\tilde{\mathbf{A}}_0^*$ fibers in the elliptic pencil above. The curve corresponding to $b = \epsilon/3$, $\epsilon^3 = 1$, has a node at $x = (2/5)\epsilon^{-1}$, $y = (1/5)\epsilon$. The type $\tilde{\mathbf{A}}_8$ fiber blows down to the nodal cubic $\{f_0 = 0\}$.)

2.1.1. **Lemma.** *For the curve C as in 2.1, one has*

$$\pi_1(\mathbb{P}^2 \smallsetminus C) = \langle p, \gamma_+ \mid p^9 = 1,\ \gamma_+^{-1} p \gamma_+ = p^4 \rangle.$$

Proof. Consider the trigonal curve $\bar{B} \subset \Sigma_2$ with a type $\mathbf{A}_8$ singular point. Its skeleton Sk, see [7], is shown in Figure 1.

Let F_1, $F_\pm$ be the type $\tilde{\mathbf{A}}_0^*$ singular fibers of $\bar{B}$ (vertical tangents), and let F_∞ be the type $\tilde{\mathbf{A}}_8$ fiber. (Recall that F_1, $F_\pm$ are located inside the small loops in Figure 1, whereas F_∞ is inside the outer region.) Consider the minimal resolution of the double covering $\tilde{X} \to \Sigma_2$ ramified at $\bar{B}$ and the exceptional section $E \subset \Sigma_2$, and denote by tildes the pull-backs of the fibers in $\tilde{X}$.

Consider the nonsingular fiber F over the $\bullet$-vertex v of Sk next to F_1 (shown in grey in Figure 1), denote $\pi_F := \pi_1(F \smallsetminus (\bar{B} \cup E))$, and pick a canonical basis $\{\alpha_1, \alpha_2, \alpha_3\}$ for π_F defined by the marking of Sk at v shown in Figure 1, see [7]. Then the fundamental group $\tilde{\pi}_F := \pi_1(\tilde{F} \smallsetminus E)$ of the punctured torus $\tilde{F} \smallsetminus E$ is obtained from π_F by adding the relations $\alpha_1^2 = \alpha_2^2 = \alpha_3^2 = 1$ and passing to the kernel of the homomorphism $\pi_F \to \mathbb{Z}_2$, $\alpha_1, \alpha_2, \alpha_3 \mapsto 1$. Hence, $\tilde{\pi}_F$ is the free group generated by

$$p := \alpha_1\alpha_2 = (\alpha_2\alpha_1)^{-1} \quad \text{and} \quad q := (\alpha_3\alpha_2) = (\alpha_2\alpha_3)^{-1}.$$

Start with the group

$$G_1 = \pi_1(\tilde{X} \smallsetminus (E \cup \tilde{F}_+ \cup \tilde{F}_- \cup \tilde{F}_\infty))$$

and compute it applying Zariski–van Kampen's approach [12] to the elliptic pencil on $\tilde{X}$. Let γ_1, $\gamma_\pm$ be the generators of the free group

$$\pi_1(\mathbb{P}^1 \smallsetminus (F_1 \cup F_+ \cup F_- \cup F_\infty), F)$$

represented by the shortest loops in Sk starting at v and circumventing the corresponding fibers in the counterclockwise direction. (We identify fibers of the ruling and their projections to the base.) Fix a closed disk Δ in the base and consider a *proper section* over Δ, *i.e.*, a topological section of the ruling disjoint from the fiberwise convex hull of $\bar{B}$, see [7]. Using this proper section, one can lift these generators to $\Sigma_2 \smallsetminus (\bar{B} \cup E)$ and to $\tilde{X} \smallsetminus E$. Using the same proper section, define the braid monodromies $m_1, m_\pm \in \operatorname{Aut}\pi_F$ and their lifts $\tilde m_1, \tilde m_\pm \in \operatorname{Aut}\tilde\pi_F$. In this notation, the group G_1 has the following presentation, *cf.* [12]:

$$G_1 = \bigl\langle p, q, \gamma_+, \gamma_- \bigm| p = \tilde m_1(p),\ q = \tilde m_1(q),\ \gamma_\pm^{-1} p \gamma_\pm = \tilde m_\pm(p),\ \gamma_\pm^{-1} q \gamma_\pm = \tilde m_\pm(q) \bigr\rangle.$$

The braid monodromy is computed as explained in [7]; for $\bar B$ it is

$$m_1 = \sigma_2, \quad m_+ = \sigma_1^{-3}\sigma_2\sigma_1^3, \quad m_- = \sigma_1^{-1}\sigma_2^2\sigma_1\sigma_2^{-2}\sigma_1,$$

where σ_1, σ_2 are the Artin generators of $\mathbb{B}_3$ (we assume that the braid group $\mathbb{B}_3$ acts on π_F from the left), and in terms of p and q it takes the form

$$\begin{gathered}\tilde m_1\colon p \mapsto pq, \quad q \mapsto q;\\ \tilde m_+\colon p \mapsto pqp^3, \quad q \mapsto p^{-4}q^{-1}p^{-4}q^{-1}p^{-1};\\ \tilde m_-\colon p \mapsto (pq)^2(p^2q)^2p, \quad q \mapsto p^{-1}q^{-1}(p^{-2}q^{-1})^3p^{-1}q^{-1}p^{-1}.\end{gathered}$$

The very first relation $p = pq$ implies $q = 1$. Hence also $\tilde m_\pm(q) = 1$ and $p^9 = 1$. Thus, one has

$$G_1 = \bigl\langle p, \gamma_+, \gamma_- \bigm| p^9 = 1,\ \gamma_+^{-1}p\gamma_+ = p^4,\ \gamma_-^{-1}p\gamma_- = p^7 \bigr\rangle. \tag{2.1.2}$$

In order to pass to the group $\pi_1(\mathbb{P}^2 \smallsetminus B)$, we need to patch back in one of the nine irreducible components of the type $\tilde{\mathbf{A}}_8$ fiber F_∞. (The component to be patched in is the proper transform of the nodal curve $\{f_0(x,y) = 0\}$.) This operation adds to (**2.1.2**) an additional relation $[\partial\tilde\Gamma] = 1$, where $\tilde\Gamma$ is a small holomorphic disk in $\tilde X$ transversal to the component in question. Using a proper section again, one can see that in G_1 there is a relation $[\partial\tilde\Gamma]^{-1}p^? = \gamma_-\gamma_+$, where $p^?$ is merely an element of the group $\tilde\pi_F$ of the fiber (modulo the relations in G_1), which we do not bother to compute. Adding the extra relation $[\partial\tilde\Gamma] = 1$ to (**2.1.2**) and eliminating γ_-, one arrives at the presentation announced in the statement. (Note that $7 = 4^{-1} \bmod 9$, hence the order of p remains 9.) □

2.1.3. **Corollary.** *The commutant of the group $\pi_1(\mathbb{P}^2 \smallsetminus C)$ as in Lemma 2.1.1 is a central subgroup of order* 3.

Proof. The commutant is normally generated by the commutator $p^{-1}\gamma_+^{-1}p\gamma_+ = p^3$; it is a central element of order 3. □

2.1.4. **Corollary.** *For any irreducible perturbation C' of the curve C as in 2.1, one has $\pi_1(\mathbb{P}^2 \smallsetminus C') = \mathbb{Z}_6$.*

Proof. Let $G = \pi_1(\mathbb{P}^2 \smallsetminus C')$. Due to Corollary 2.1.3, the commutant $[G,G]$ is a quotient of $\mathbb{Z}_3$, hence either $\mathbb{Z}_3$ or $\{1\}$. Furthermore, $[G,G] \subset G$ is a central subgroup. On the other hand, since C is irreducible, $G/[G,G] = \mathbb{Z}_6$, and any central extension

$$\{1\} \to \mathbb{Z}_3 \to G \to \mathbb{Z}_6 \to \{1\}$$

of the cyclic group $\mathbb{Z}_6$ would be abelian. □

3. The curve of torus type

3.1. Up to projective transformation, there is a unique torus type curve $C \subset \mathbb{P}^2$ with the set of singularities $\mathbf{A}_{17} \oplus 2\mathbf{A}_1$, see [11]; its transcendental lattice is $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$. Similar to 2.1, this curve blows up to the union of the two type $\tilde{\mathbf{A}}_0^*$ fibers in a Jacobian rational elliptic surface with the combinatorial type of singular fibers $\tilde{\mathbf{E}}_8 \oplus 2\tilde{\mathbf{A}}_0^*$ (in Kodaira's notation, one fiber of type II^* and two fibers of type I_1). The curve can be given by the equation

$$f(x,y) := (y^3 + y^2 + x^2)\Bigl(y^3 + y^2 + x^2 - \frac{4}{27}\Bigr) = 0,$$

and its torus structure is

$$f(x,y) = \Big(y^3 + y^2 + x^2 - \frac{2}{27}\Big)^2 + \Big(\frac{\sqrt[3]{4}}{9}\Big)^3.$$

3.1.1. **Lemma.** *Let C be a curve as in 3.1, and let U be a Milnor ball about the type $\mathbf{A}_{17}$ singular point of C. Then the homomorphism $\pi_1(U \smallsetminus C) \to \pi_1(\mathbb{P}^2 \smallsetminus C)$ induced by the inclusion $U \hookrightarrow \mathbb{P}^2$ is surjective.*

Proof. In the coordinates $\tilde{y} = y/x$, $\tilde{z} = 1/x$, the curve is given by the equation

$$(\tilde{y}^3 + \tilde{y}^2\tilde{z} + \tilde{z})\Big(\tilde{y}^3 + \tilde{y}^2\tilde{z} + \tilde{z} - \frac{4}{27}\tilde{z}^3\Big) = 0,$$

the type $\mathbf{A}_{17}$ singular point is at the origin, and each component is inflection tangent to the line $\{\tilde{z} = 0\}$ at this point. To compute the group, apply Zariski–van Kampen theorem [12] to the vertical pencil $\{\tilde{z} = \text{const}\}$, choosing for the reference a generic fiber $F = \{\tilde{z} = \epsilon\}$ close to the origin. On the one hand, one has an epimorphism $\pi_1(F \smallsetminus C) \twoheadrightarrow \pi_1(\mathbb{P}^2 \smallsetminus C)$. On the other hand, the intersection $C \cap \{\tilde{z} = 0\}$ consists of a single 6-fold point; hence, if ϵ is small enough, all six points of the intersection $C \cap F$ belong to U and the generators of $\pi_1(F \smallsetminus C)$ can be chosen inside U. □

3.1.2. **Corollary.** *Let C' be a perturbation of the curve C as in 3.1 with the set of singularities $\mathbf{A}_{14} \oplus \mathbf{A}_2 \oplus 2\mathbf{A}_1$. Then $\pi_1(\mathbb{P}^2 \smallsetminus C') = \mathbb{B}_3/(\sigma_1\sigma_2)^3$.*

Proof. Let U be as in Lemma 3.1.1. Then $\pi_1(U \smallsetminus C') = \mathbb{B}_3$ and, due to the lemma, there is an epimorphism $\mathbb{B}_3 \twoheadrightarrow \pi_1(\mathbb{P}^2 \smallsetminus C')$. Since C' is necessarily irreducible and of torus type (so that the abelianization of $\pi_1(\mathbb{P}^2 \smallsetminus C')$ is $\mathbb{Z}_6$ and $\pi_1(\mathbb{P}^2 \smallsetminus C')$ factors to $\mathbb{B}_3/(\sigma_1\sigma_2)^3$), the latter epimorphism factors through an isomorphism $\mathbb{B}_3/(\sigma_1\sigma_2)^3 \cong \pi_1(\mathbb{P}^2 \smallsetminus C')$. □

3.1.3. **Remark.** The other irreducible perturbations of C that are of torus type are considered elsewhere, see [5]. Their groups are also $\mathbb{B}_3/(\sigma_1\sigma_2)^3$.

4. Proof of Theorem 1.0.1

4.1. The groups of all but one sextics of torus type occurring in classical Zariski pairs are known, see [5] for a 'map' and further references; all groups are $\mathbb{B}_3/(\sigma_1\sigma_2)^3$. The only missing curve has the set of singularities $\mathbf{A}_{14} \oplus \mathbf{A}_2 \oplus 2\mathbf{A}_1$. Such a curve can be obtained by a perturbation from a reducible sextic of torus type with the set of singularities $\mathbf{A}_{17} \oplus 2\mathbf{A}_1$ (see Proposition 5.1.1 in [6]), and its group is given by Corollary 3.1.2.

4.2. The fundamental groups of most non-abundant sextics appearing in classical Zariski pairs are computed in [5], [8], [3], with a considerable contribution from [9]. According to [3], unknown are the groups of the curves with the sets of singularities

$$\mathbf{A}_{17} \oplus \mathbf{A}_1, \quad \mathbf{A}_{14} \oplus \mathbf{A}_2 \oplus 2\mathbf{A}_1, \quad 2\mathbf{A}_8 \oplus 2\mathbf{A}_1, \quad 2\mathbf{A}_8 \oplus \mathbf{A}_1.$$

The first curve can be obtained by a perturbation from a sextic with a single type $\mathbf{A}_{19}$ singular point. According to [2], its group is abelian. The three other curves are perturbations of the curve C constructed in 2.1, and their groups are abelian due to Corollary 2.1.4. (Note that the perturbations exist due to Proposition 5.1.1 in [6], and the resulting curves are unique up to equisingular deformation due to [1].) □

4.2.1. **Remark.** A curve C as in 2.1 can also be perturbed to a sextic with the set of singularities $\mathbf{A}_{17} \oplus \mathbf{A}_1$, but the result is reducible.

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