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(CO)TORSION OF EXTERIOR POWERS OF DIFFERENTIALS OVER COMPLETE INTERSECTIONS

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Abstract. The main goal of this paper is to establish a generalized Lipman-Zariski result in characteristic zero for complete intersection germs whose singular locus has codimension at least three, generalizing the corresponding result of Graf for hypersurfaces. More precisely, we prove that the condition that the sheaf of *reflexive* Kähler differential p-forms is free implies smoothness. The proof given here rests on recognizing the torsion, as was known previously, and the cotorsion as homologies of the generalized Koszul complexes constructed by Kirby, Buchsbaum-Eisenbud, Lebelt, and Bruns-Vetter and applying certain rigidity and symmetry results based on work of Lebelt and Rodicio, yielding also a different proof of Graf's result. To make the paper accessible to both complex analysts and algebraic geometers we include full descriptions of the necessary background.

Introduction

Let (X, x) be a germ of a normal complete intersection variety of positive dimension d, either a complex analytic variety locally embedded in $\mathbb{C}^n$ or an algebraic variety over an algebraically closed field of characteristic zero. Consider the sheaf Ω^p_X of Kähler differential p-forms on X and let $\Omega^p_{X,x}$ be the corresponding stalk at x. The purpose of this paper is to discuss some rigidity results on the vanishing of torsion and cotorsion of $\Omega^p_{X,\,x}$, to clarify the precise ranges of p values for which these do not vanish, and finally to use these results to prove a generalized Lipman-Zariski result for complete intersection germs.

The geometric version of the Lipman-Zariski Conjecture (LZ) states that if a variety over a field of characteristic zero has a locally free tangent sheaf it must be smooth [72]. Despite progress in the past 50 years towards its resolution, the conjecture is still open (for more information see Section 4). For normal varieties, the conjecture is intimately related to the vanishing of the local torsion and cotorsion of Ω^1_X since these are the obstructions for $\Omega^1_{X,x}$ to be isomorphic to $(\Omega^1_{X,x})^{**}$, the dual of $T_{X,x} = (\Omega^1_{X,x})^* = \mathrm{Hom}_{\mathscr{O}_{X,x}}(\Omega^1_{X,x}, \mathscr{O}_{X,x})$, via the exact sequence of stalks

$$0 \longrightarrow \mathrm{tor}\ \Omega^1_{X,x} \longrightarrow \Omega^1_{X,x} \longrightarrow (\Omega^1_{X,x})^{**} \longrightarrow \mathrm{cotor}\ \Omega^1_{X,x} \longrightarrow 0.$$

Motivated by the (LZ) conjecture, Graf recently proposed studying the following problem. Set the notation $\Omega^{[p]}_X = (\Omega^p_X)^{**}$ for the sheaf of reflexive Kähler differential p-forms. Of course, the case $p = 1$ is the (LZ) conjecture.

Generalized Lipman-Zariski Problem (GLZ) [32, Question 1.2]: *Let X be a d-dimensional complex variety. Under what assumptions on X and for which values of p with $1 \leqslant p \leqslant d-1$ does local freeness of $\Omega^{[p]}_X$ imply that X is smooth?*

Graf provided a solution to the (GLZ) problem: he proved that the implication holds for any p with $1 \leqslant p \leqslant d-1$ for any hypersurface X whose singular locus has codimension at least

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three [32, Thm 1.8]. However, Graf also found examples for which the implication in the (GLZ) problem fails, namely, he constructed isolated terminal singularities (X, x) for which the reflexive sheaves $\Omega^{[p]}_{X,x}$ are free for most values of p [32, Prop 1.5].

In addition, he proposed analogues with stronger assumptions: the *weak generalized Lipman-Zariski problem* (WGLZ) assuming local freeness of $\Omega^p_X/(\operatorname{tor}\Omega^p_X)$ and the *very weak generalized Lipman-Zariski problem* (VWGLZ)[1] assuming local freeness of Ω^p_X. He provided solutions to both questions, showing that the implication in (WGLZ) holds for normal hypersurface singularities for any $1 \leqslant p \leqslant d$ [32, Thm 1.9] and in (VWGLZ) for all varieties for any $1 \leqslant p \leqslant e$ where e is the local embedding dimension [32, Thm 1.10]; see Section 4 for a discussion of these.

Graf conjectures that his solution to the (GLZ) should also be valid for complete intersections under the same restriction on the codimension of the singular locus. He also wondered whether his hypersurface result holds without this assumption. As the question of smoothness is of a local nature, one may study these questions locally. In this paper, we prove the following result which provides a solution to the (GLZ) problem for complete intersections and at the same time an affirmative answer to the conjecture by Graf as formulated above.

Theorem 4.1. *Let X be either a complex analytic variety or an algebraic variety over an algebraically closed field of characteristic zero. Assume that X is complete intersection at a point $x \in X$ of dimension $\dim_x X = d > 0$. Let the codimension of the singular locus of X at x be at least three[2]. If the sheaf $\Omega^{[p]}_X$ of reflexive Kähler differential p-forms is free at x for some p with $1 \leqslant p \leqslant d-1$, then x is a smooth point of X.*

We note that for $p = 1$, the case of the Lipman-Zariski conjecture, this was mostly known before for complete intersection singularities (and for much more general ones as described in Section 4). Indeed, when X is a hypersurface[3] (resp., projective complete intersection) it was proved by Scheja-Storch [91] (resp., Moen [77], Hochster [51]) without any restriction on the codimension of the singular locus. For X an affine complete intersection variety over a field of characteristic zero, it was recently announced by Källström [59, Cor 1.3(1)] and [58]. It is worth mentioning that his proof also does not impose any restriction on the codimension of the singular locus and has a different flavor.

Graf's proof in the hypersurface case rested on obtaining some results on the vanishing of the torsion and cotorsion of the sheaf of Kähler differentials Ω^p_X, as does our proof of the general case. The study of the torsion and cotorsion in the modules of Kähler differential p-forms in connection to regularity questions of affine rings has a long history; when $p = 1$ see Kunz's book [67, Chapter 7 p. 135-136, App. D] and more generally Kunz-Waldi's book [69, Chapter 5]. Set $d = \dim_x X$ and $k = \operatorname{codim}_x \operatorname{Sing} X$. Lipman (see also Suzuki [96]) showed that for an affine complete intersection germ over a field of characteristic zero, $\Omega^1_{X,x}$ is *torsion-free* if and only if X is *normal* at x, while $\Omega^1_{X,x}$ is *reflexive* if and only if $k \geqslant 3$ [72, Prop 8.1].[4] In the 1970s Vetter proved in [99, Satz 4] that for a non-smooth reduced complete intersection singularity and for

[1]The (VWGLZ) problem will play critical role in our solution of the (GLZ) problem for complete intersections.

[2]For $x \in X$, the codimension of the singular locus of X at x is defined as
$\operatorname{codim}_x \operatorname{Sing} X := \dim_x X - \dim_x \operatorname{Sing} X$.

[3]Scheja and Storch proved the (LZ) conjecture for hypersurface rings R. A local ring (R, m) is called a hypersurface ring if and only if $\dim_{R/m}(\frac{m}{m^2}) \leqslant 1 + \dim R$. Recall $\dim_{R/m}(\frac{m}{m^2})$ is the embedding dimension of R.

[4]Knighten [66, Sec 4, Exs] and Greb and Rollenske [42], while studying differentials on quotient varieties, observed that if we consider certain non complete intersection varieties X the sheaf of Kähler differentials Ω^1_X can have both torsion and cotorsion.

$1 \leqslant p \leqslant d$, the module $\Omega^p_{X,x}$ is torsion-free (resp., reflexive) if $p < k$ (resp., $p < k-1$) and has torsion (resp., cotorsion) if $p = k$ (resp., if $p = \max\{1, k-1\}$). Lebelt strengthened Vetter's result by showing that if $\Omega^p_{X,x}$ is torsion free (resp., reflexive) then $p < k$ (resp., $p < k-1$); cf. Satz 0 and Bemerkung page 197 in [70]. Greuel commented further in [43, Prop 1.11 (ii)] that $\Omega^p_{X,x}$ is torsion for any $p > d$, that is, $\Omega^p_{X,x} = \operatorname{tor}\Omega^p_{X,x}$ (this had been observed earlier by Ferrari for any reduced complex analytic space, see the proof of [30, Prop 1.1]). This fact forces $(\Omega^p_{X,x})^{**} = 0$ since $d > 0$ and hence yields $\operatorname{cotor}\Omega^p_{X,x} = 0$ for $p > d$. These results left open the question of cotorsion in the range $k - 1 < p \leqslant d$ (assuming $k > 1$).

In the case of normal hypersurface singularities, Graf recovered Lebelt's torsion-freeness and reflexivity results for $\Omega^p_{X,x}$. He did this by showing that the vanishing of torsion and cotorsion of $\Omega^p_{X,\,x}$ is rigid [32, Theorem 1.11], as described below. Graf's proof is novel in that it involves utilizing the residue map. In the same paper he also proves several other results concerning special cases of the generalized Lipman-Zariski problem for more general rings.

For a normal hypersurface X embedded in an open connected subset Y of $\mathbb{C}^{d+1}$ and described as the zero set of a non-constant function f defined on Y, Lebelt and Graf considered the following complex $(K,\, a^p)$ of sheaves

$$K\colon\quad 0 \longrightarrow \Omega^0_Y|_X \longrightarrow \Omega^1_Y|_X \longrightarrow \cdots \longrightarrow \Omega^p_Y|_X \xrightarrow{a^p} \Omega^{p+1}_Y|_X \longrightarrow \cdots \longrightarrow \Omega^{d+1}_Y|_X \longrightarrow 0$$

with differential given by $a^p(g) = df \wedge g$ for $g \in \Omega^p_Y|_X$. Graf showed that the cohomologies of the complex K measure the torsion and cotorsion of Ω^p_X, namely, that

$$\operatorname{tor}\Omega^p_X \cong \mathrm{H}^p(K) \quad \text{for all } 1 \leqslant p \leqslant d+1$$
$$\operatorname{cotor}\Omega^p_X \cong \mathrm{H}^{p+1}(K) \quad \text{for all } 0 \leqslant p \leqslant d$$

and that K is locally a Koszul complex and hence satisfies rigidity on vanishing of its cohomology, that is, $\mathrm{H}^{p+1}(K) = 0$ implies $\mathrm{H}^p(K) = 0$. The new observation (relative to Lebelt's results) in Graf's Theorem 1.11 is the description of the cotorsion of Ω^p_X as the $(p+1)$-cohomology group[5] of the complex (K, a^p) which consequentially yields that Ω^p_X is reflexive if $\operatorname{cotor}\Omega^p_X = 0$ (Theorem 1.11.3 in [32]) and that $\operatorname{cotor}\Omega^p_X \neq 0$ when $k - 1 \leqslant p \leqslant d$.

In this paper we extend Graf's results to complete intersection singularities of higher codimension as follows.

Corollary 3.1. *Let X be either a complex analytic variety or an algebraic variety over an algebraically closed field of characteristic zero. Assume that X is complete intersection at a normal point $x \in X$ of dimension $\dim_x X = d > 0$. Suppose x is a singular point of X. Then for $1 \leqslant p \leqslant d$ one has*

$$\operatorname{tor}\Omega^p_{X,x} \neq 0, \quad \textit{if and only if} \quad k \leqslant p \leqslant d$$
$$\operatorname{cotor}\Omega^p_{X,x} \neq 0, \quad \textit{if and only if} \quad k-1 \leqslant p \leqslant d$$

where k is the codimension of the singular locus of X at x.

As mentioned earlier Vetter showed in [99] that the torsion of $\Omega^k_{X,x}$ and the cotorsion of $\Omega^{k-1}_{X,x}$ does not vanish. The non-vanishing of the torsion of $\Omega^p_{X,x}$ when $k \leqslant p \leqslant d$ was first observed by Lebelt [70, Satz 0]. The torsion also does not vanish if $d+1 \leqslant p \leqslant \operatorname{embdim}_x X$ (see Remark 3.2).

[5] According to Graf, the isomorphism between $\operatorname{tor}\Omega^{p+1}_X$ and $\operatorname{cotor}\Omega^p_X$ appeared in a less explicit form for isolated hypersurface singularities in [44, Lem 2.4 Bemerkung]. In addition, when X is a normal, pure dimensional hypersurface this isomorphism has been observed by Kersken (Section 1.5 in [63, Sec 1.5]) in the midst of clarifying the relationships between the sheaf of regular meromorphic forms she introduced and other kinds of sheaves of differential forms. Further generalizations can be found in the book by Kunz and Waldi [69].

What is new here is that our result identifies the range for which the cotorsion is not zero and in particular yields the conclusion for any $p \leqslant d$ that $\Omega^p_{X,x}$ is reflexive if its cotorsion vanishes. For the case $p = d$ this was known; in fact, $\operatorname{cotor} \Omega^d_{X,x}$ vanishes if and only if x is a smooth point (see, e.g., [69, Cor 5.22]).

In contrast to Graf's proof in the hypersurface case, ours uses classic sequences due to Lebelt, Kirby, Buchsbaum-Eisenbud, and Bruns-Vetter, some rigidity results of Lebelt and Rodicio (Corollary 2.4) which can be traced back to work of Eagon-Northcott and Buchsbaum-Rim, and a symmetry result (Corollary 2.5) that to the best of our knowledge has not appeared in the literature before. To use these, we also prove an analogous result to Graf's that the torsion and cotorsion of $\Omega^p_{X,x}$ can be identified as certain homologies of these complexes (Lemma 2.8 in our paper); for the former this was known as far back as Lebelt.

We recall some earlier results on the description of torsion and cotorsion of Kähler p-forms on complete intersection singularities. For the torsion of $\Omega^1_{X,x}$ on a hypersurface singularity see [67, Exer., p. 341] or [21, Theorem 1]. Rim investigated the torsion and cotorsion of $\Omega^d_{X,x}$ for certain complete intersection singularities, see [86, Cor. 1.3 (i)]. When x is an isolated complete intersection singularity, Greuel identified the torsion of $\Omega^d_{X,x}$ as the first homology group of an appropriate Koszul complex and showed that $\dim_{\mathbb{C}} \operatorname{tor}(\Omega^d_{X,x}) = \dim_{\mathbb{C}} \mathcal{O}_{\operatorname{Sing} X,x}$, cf. Proposition 1.11(iii) in [43] (see also [13, Satz 4]). For a reduced hypersurface X, Kersken gave an alternate characterization of the torsion of Ω^p_X [63, (1.5)]. Michler gave a fairly explicit description of the torsion of Ω^d_X for a reduced affine/quasi-homogeneous hypersurface X with an isolated singularity at 0, [75], [76]. Mond, in [78, Section 3], studied the torsion module of Ω^p_X when X is a free (also almost free) divisor in a complex manifold. Aleksandrov described the torsion of Ω^p_X for germs $(X, 0)$ of reduced hypersurface in $\mathbb{C}^{d+1}$(see [1]; also [2] for related matters). Lastly, Yau expressed the cotorsion of $\Omega^d_{X,x}$ for an isolated hypersurface singularity (X, x) in $\mathbb{C}^{d+1}$ as $\Omega^{d+1}_{X,x}$ in [97, Thm 2.8] (see also [69, Chapter 1]).

The paper is organized as follows. In Section 1, we review some known results from commutative and homological algebra that will be needed in the proofs. In particular we will recall the construction of generalized Koszul complexes $\mathscr{D}_p$ (due to Buchsbaum-Eisenbud, Kirby, Lebelt, Bruns-Vetter and others). Section 2 contains the proofs of the rigidity and symmetry properties of the complexes $\mathscr{D}_p$ that are instrumental in the proofs of Corollary 3.1 and Theorem 4.1. In Section 3 we prove our results on the torsion and cotorsion of the p-differentials, also required for our proof of Theorem 4.1. In Section 4 we recall the history of the Lipman-Zariski conjecture and its generalizations, we solve the generalized Lipman-Zariski problem for complete intersection germs whose singular locus has codimension at least 3, and we discuss various weak versions of the same problem; we also discuss relationships between the classical Lipman-Zariski conjecture and the generalized Lipman-Zariski problem.

1. Background

In this section, we first list the two main settings for which we obtain our results and then recall in full detail the relevant background material in the following order: the concepts of torsion and cotorsion, the various kinds of differentials with which we are concerned, and the full construction of the generalized Koszul complexes that we use and their properties.

Much of this material may be known to the reader, but we include it here for two reasons: (1) we hope to make this paper easy to read for both complex analysts and algebraic geometers and (2) we wish to fix notation for the generalized Koszul complexes, which are built in two different ways in the literature (beginning with either of the two dual versions of a map of free modules) and written in many possible equivalent forms.

After fixing the setting for the paper, we review some background material on differentials and on torsion and cotorsion. Then we recall some classic sequences constructed by Lebelt, Buchsbaum-Eisenbud, Kirby, and Bruns-Vetter for studying these over complete intersection rings.

Throughout the paper, all rings will be assumed to be commutative with 1 and all modules unitary. For any ring Q, recall the notation $\operatorname{Spec} Q = \{P \mid P \text{ is a prime ideal in } Q\}$.

1.1. **Setting.** We work in either one of the following two possible settings, one analytic and one algebraic. In either case, set $d = \dim_x X$, the dimension of X at x, and $n = \operatorname{embdim}_x X$, the embedding dimension of X at x, and set $c = n - d$, the codimension of X at x. We will assume that $d > 0$ throughout.

1. Let X be a reduced, irreducible complex analytic variety with structure sheaf $\mathcal{O}_X$. We say X is a complete intersection at $x \in X$, if the stalk $\mathcal{O}_{X,x}$ is isomorphic as a local $\mathbb{C}$-algebra to a quotient of a convergent power series ring $\mathbb{C}\{z_1, \dots z_n\}$ by an ideal generated by $n - \dim_x X$ elements, which necessarily form a regular sequence.
2. Let X be an algebraic variety over an algebraically closed field k of characteristic 0 and with structure sheaf $\mathcal{O}_X$. By variety we mean an integral, separated scheme of finite type over k, but see Main assumption below. We say X is a complete intersection at $x \in X$ if the stalk $\mathcal{O}_{X,x}$ is isomorphic as a k-algebra to a quotient of a regular local ring of dimension n by an ideal generated by $n - \dim_x X$ elements, which necessarily form a regular sequence.

In either context, a point x in X is said to be *normal* if the corresponding ring $\mathcal{O}_{X,x}$ is a normal local ring (hence also a domain). Our results will usually have this assumption.

Main assumption: Since the question we are addressing is local, we assume from now on that X is embedded either (in the first case) in some open connected subset $Y \subseteq \mathbb{C}^n$ or (in the second case) in some affine space $Y = \mathbb{A}^n$ over k as the zero set of holomorphic, respectively regular, functions $f_1, \dots, f_c$ on Y that form a regular sequence (in either case).

1.2. **On torsion and cotorsion of modules and sheaves.** We follow the treatment by Bruns-Herzog in Section 1.4 of [12]. Let R be a commutative ring with identity and M be an R-module. The *dual* M^* of M is by definition the R-module $\operatorname{Hom}_R(M, R)$; its bidual $(M^*)^*$ is denoted by M^{**}. The bilinear map $\phi\colon M \times M^* \to R$ sending $(x, \alpha) \in M \times M^*$ to $\alpha(x)$ induces a natural homomorphism $e_M\colon M \to M^{**}$ called the *evaluation map*. One says that M is *torsionless* if e_M is injective and M is *reflexive* if e_M is bijective. If e_M fails to be surjective we say that M has *cotorsion* and we denote the cotorsion by $\operatorname{cotor} M := \operatorname{coker} e_M$.

The *torsion submodule* of M is defined as the kernel of the natural map $i : M \to M \otimes Q(R)$ where $Q(R)$ is the total quotient ring of R. More precisely we have

$$\operatorname{tor} M = \ker i = \{m \in M \mid rm = 0, \text{for some non-zerodivisor } r \in R\}$$

and one says that M is *torsion-free* if $\operatorname{tor} M = 0$, and M is a *torsion module* if $\operatorname{tor} M = M$. Clearly the torsion submodule $\operatorname{tor} M$ is contained in $\ker e_M$. If R is a Noetherian domain and M is finitely generated, one can show that $\ker e_M = \operatorname{tor} M$, and hence in that setting, the concepts of being torsionless and being torsion-free are equivalent. A proof that $\ker e_M = \operatorname{tor} M$ in the above setting can be found in Remark, page 70 in [39]. If R is only a reduced, Noetherian ring and M is finitely generated the two kernels are still equal to each other; see [72, §8] or [90][6].

We shall apply these notions to $(X, \mathcal{O}_X)$, a reduced complex analytic or algebraic variety, and the stalk at a point $x \in X$ of a coherent analytic or algebraic sheaf $\mathcal{F}$ viewed as an $\mathcal{O}_{X,x}$-module. For a coherent sheaf $\mathcal{F}$, define the torsion and cotorsion sheaves, $\operatorname{tor}\mathcal{F}$ and $\operatorname{cotor}\mathcal{F}$, as the kernel and cokernel sheaves of the natural map $e_{\mathcal{F}}\colon \mathcal{F} \to \mathcal{F}^{**}$. At the stalk level, this gives an exact sequence of $\mathcal{O}_{X,x}$-modules

$$0 \longrightarrow \operatorname{tor}\mathcal{F}_x \longrightarrow \mathcal{F}_x \xrightarrow{e_{\mathcal{F}_x}} (\mathcal{F}^{**})_x \longrightarrow \operatorname{cotor}\mathcal{F}_x \longrightarrow 0.$$

where the middle map is the evaluation map $e_{\mathcal{F}_x}$ upon the identification $(\mathcal{F}^{**})_x \cong (\mathcal{F}_x)^{**}$.

1.3. **Differentials.** We will be concerned mainly with the following three types of sheaves of differential forms on X for each $p \geqslant 0$, each defined further below:

- the sheaf of Kähler differential p-forms: Ω^p_X
- the sheaf of reflexive differential p-forms: $\Omega^{[p]}_X$
- the sheaf of Ferrari differential p-forms: $\check{\Omega}^p_X$

For other types of differential forms on singular spaces and how they relate to each other, one can consult the works of Reiffen and Vetter [85], Ferrari, [29],[30], Yau [97], Kersken ([61], [62], [63]), Jörder [57], Huber-Jörder [53], Samuelsson Kalm [88] and the references therein to related work of Griffiths, Barlet, Henkin-Passare. Properties of Kähler and reflexive differentials can be found in the books by Kunz [67], by Kunz and Waldi [69], and by Berger, Kiehl, Kunz, and Nastold [7] and in the papers [33, §3], [41], [60] and the references therein.

Kähler differential forms

Let X be either a complex analytic variety or an algebraic variety over a field of characteristic zero with X embedded in a smooth Y as the zero set of holomorphic, respectively, regular, functions $f_1, \ldots, f_c$ on Y. We follow the presentation in [32, §2.C] in order that our definitions work equally well in both the algebraic and the analytic settings. Let Ω^1_X be the *sheaf of Kähler differentials* on X: if Ω^1_Y is the sheaf of Kähler differentials on Y, one sets

$$\Omega^1_X = \Omega^1_Y|_X \,/(df_1, \ldots, df_c)$$

This definition is independent of the choice of embedding of X in Y. In the analytic case, this was shown by [36, Thm 1.2]. In the algebraic case, this module can be defined in a more intrinsic way; see, for example, [45, Ch II.8] or [74, §25]. For details of the definition of Ω^1_Y, $\Omega^1_Y|_X$, and df_i in each setting, see these references as well.

For any integer $p \geqslant 0$, define the *sheaf of differential p-forms*, or *p-differentials*, as the exterior power

$$\Omega^p_X = \bigwedge^p \Omega^1_X.$$

In particular, one defines $\Omega^0_X \cong \mathcal{O}_X$.

Denoting the ideal sheaf of $(f_1, \ldots, f_c)$ by $\mathcal{I}$, one obtains the standard exact sequence (called the conormal sequence)

$$\mathcal{I}/\mathcal{I}^2 \xrightarrow{d} \Omega^1_Y|_X \longrightarrow \Omega^1_X \longrightarrow 0.$$

[6] Additionally, Vasconcelos showed in [98, Thm 3.1 App] that in a commutative Noetherian ring R that has no embedded primes and such that R_p is a Gorenstein ring for every minimal prime p, the concepts of being torsionless and torsion-free module are equivalent for a finitely generated R-module.

As Y is smooth and we are working locally, we may also take Y small enough to assume that the sheaf Ω^1_Y is a free $\mathcal{O}_Y$-module of rank n on $dz_1, \dots, dz_n$ for local coordinates $z_1, \dots, z_n$, hence the same is true for $\Omega^1_Y|_X$ over $\mathcal{O}_X$. So one obtains an exact sequence

$$\mathcal{I}/\mathcal{I}^2 \xrightarrow{d} \mathcal{O}_X^{\oplus n} \longrightarrow \Omega^1_X \longrightarrow 0.$$

In this paper, we work with the corresponding sequence of stalks since the Lipman-Zariski conjecture and its generalizations are local in nature.

Assume now that X is a complete intersection at x, that is, assume that $f_1, \dots, f_c$ is a regular sequence on $\mathcal{O}_{Y,x}$, as in the setting of this paper. Then the stalk $\mathcal{I}_x/\mathcal{I}_x^2$ is a free $\mathcal{O}_{X,x}$-module of rank c on the images of $f_1, \dots, f_c$, yielding the exact sequence

$$\mathcal{O}_{X,x}^{\oplus c} \xrightarrow{[a_{ij}]} \mathcal{O}_{X,x}^{\oplus n} \longrightarrow \Omega^1_{X,x} \longrightarrow 0.$$

where $A = [a_{ij}]$ is the matrix given by expressing $df_j = \sum_{i=1}^n a_{ij} dz_i$. Whenever the classic Jacobian matrix $J = [\partial f_i / \partial z_j]$ makes sense, the matrix A can be taken to be its transpose since we are in the characteristic 0 setting.

As we will be assuming X is normal, hence reduced, the map d_x on stalks is injective; see [28, Thm 2], [27, Ex 16.17(a)] for the algebraic case, or [92, Lem 2.1 and §4, Ex 1], [2, Section 10, pages 20-21], or [37, Chapter 2, page 104] for the analytic setting. So the sequence on stalks becomes

$$0 \longrightarrow \mathcal{O}_{X,x}^{\oplus c} \xrightarrow{[a_{ij}]} \mathcal{O}_{X,x}^{\oplus n} \longrightarrow \Omega^1_{X,x} \longrightarrow 0 \tag{1.1}$$

Note that this gives a finite free resolution of the $\mathcal{O}_{X,x}$-module $\Omega^1_{X,x}$. It is to this map of free modules that one applies the construction of the generalized Koszul complexes described in Section 1.4.

Remark 1.2. We make a few additional comments comparing the algebro-geometric and analytic cases since the latter of these has some more subtle finiteness conditions. In the algebro-geometric case, the stalk $\Omega^1_{X,x}$ is really the module of Kähler differentials $\Omega^1_{\mathcal{O}_{X,x}/k}$: For any finitely generated k-algebra R, $\Omega^1_{R/k}$ is defined as the universal object presenting the functor of all k-derivations from R to all R-modules M, finitely generated or not; see, for example, [45, Ch II.8] or [74, §25].

In the analytic case, the stalks $\Omega^1_{X,x}$ are really the *universally finite* Kähler differentials $\hat{\Omega}^1_{\mathcal{O}_{X,x}/\mathbb{C}}$ by [38, Sätze 4 and 5 in III.4]; see also [7, §2] for a detailed discussion. The notion of *universally finite* derivations was introduced to remedy the situation that, for power series algebras R over a field k of characteristic zero, the module of Kähler differentials is not finitely generated over k. This notion agrees with the usual one in the case of finitely generated k-algebras. We recall the definition, following [32, §2.C] and [68, Ch 11]: For any finitely generated k-algebra R, $\hat{\Omega}^1_{R/k}$ is defined as the universal object presenting the functor of all k-derivations from R to all finitely generated R-modules M. Furthermore, when R is the completion of a finitely generated k-algebra R' with respect to some maximal ideal, this module is the completion of the usual module of Kähler differentials $\Omega^1_{R'|k}$; see [27, Ex 16.14(d)].

Reflexive differential forms

For X either a complex analytic variety or an algebraic variety over a field of characteristic zero, let Ω^1_X be the sheaf of Kähler differentials and $\mathcal{T}_X = (\Omega^1_X)^* := \mathcal{H}om_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)$ be the tangent sheaf on X. The basic properties of reflexive differential forms are described below; we follow [33, §3].

The module of *reflexive differential p-forms* is defined to be the reflexive hull

$$\Omega_X^{[p]} := (\Omega_X^p)^{**}$$

of the sheaf of Kähler p-forms. Sections of $\Omega_X^{[p]}$ are called *reflexive differential forms.*

Remark 1.3. In [66, Chapter I], Knighten gave an equivalent description of the sheaf of reflexive 1-forms. She studied the sheaf of Zariski differentials and then proved that the sheaf of Zariski differentials is isomorphic to the sheaf of reflexive 1-forms (Corollary 3 in [66]).

Remark 1.4. Let X_{reg} denote the smooth locus of the variety X and $j\colon X_{\mathrm{reg}} \hookrightarrow X$ its open embedding into X. If X is normal and irreducible, then the reflexive differentials can be obtained via restriction to the open set $U = X_{\mathrm{reg}}$

$$\Omega_X^{[p]} = j_*\left(\Omega_X^p|_U\right) = j_*(\Omega^p_{X_{\mathrm{reg}}}).$$

In the algebraic context see [46, Prop 1.6]. In the analytic setting and for X only a normal complex space see [93, Prop 7].

If X is only reduced, then the torsion of the p-differentials can also be obtained as the kernel of this restriction map, that is,

$$\operatorname{tor}\Omega_X^p = \ker\left(\Omega_X^p \longrightarrow j_*\left(\Omega_X^p|_U\right)\right)$$

We give a quick proof in the normal case. There it follows from the following commutative diagram

$$\begin{tikzcd}
\Omega_X^p \arrow[r] \arrow[d] & (\Omega_X^p)^{**} \arrow[d] \\
j_*(\Omega_X^p|_U) \arrow[r] & j_*((\Omega_X^p)^{**}|_U)
\end{tikzcd}$$

since the right hand arrow is an isomorphism by [93, Remarques 2), page 367] (as $\Omega_X^{[p]} \cong (\Omega_X^{[p]})^{**}$ is already reflexive) and the bottom arrow is an isomorphism since the sheaves Ω_X^p and $(\Omega_X^p)^{**}$ agree on the regular locus. For the generically smooth, reducible case, see [30, Prop 1.1] or [60, App A] for reduced quasi-projective schemes.

Remark 1.5. For normal varieties of pure dimension d, $\Omega_X^{[d]}$ is equal to Grothendieck's dualizing sheaf ω_X, cf. [37, Ch II, Cor 5.32] in the analytic case and cf. [68, Chapter 9, Theorem 9.7, Corollary 9.8] in the algebraic case.

For a singular space, unlike the sheaves of Kähler differentials, the reflexive sheaves $\Omega_X^{[p]}$ have a well-defined perfect pairing. Graf showed the following; compare to our Lemma 2.11 in the general setting.

Lemma 1.6. *([32, Lemma 3.2]) For X a normal complex space of dimension d and for any $0 \leqslant p \leqslant d$, the pairing*

$$\Omega_X^{[p]} \times \Omega_X^{[d-p]} \longrightarrow \Omega_X^{[d]} \tag{1.7}$$

induced by the wedge product, is a perfect pairing, that is, there is an isomorphism

$$\Omega_X^{[p]} \cong \mathcal{H}om_{\mathcal{O}_X}(\Omega_X^{[d-p]}, \Omega_X^{[d]}).$$

If $\Omega_X^{[d]}$ is free (for example when X is Gorenstein) and $\Omega_X^{[d-p]}$ is free, then the lemma above implies that $\Omega_X^{[p]}$ will be free. This observation will be useful in Section 4 when we relate the classical and generalized Lipman-Zariski problems.

Ferrari differential forms

For X either a reduced complex analytic variety or an algebraic variety over a field of characteristic zero, consider the torsion-free sheaf

$$\check{\Omega}^p_X = \Omega^p_X / \operatorname{tor} \Omega^p_X$$

where the torsion subsheaf is as defined in Section 1.2 or in the normal case as in the equivalent version given in Remark 1.4.

We conclude this section with some historical remarks. The sheaves $\check{\Omega}^p_X$ were studied by Ferrari in the early 1970s in the complex analytic setting, that is, for a reduced complex analytic variety X of an open set $U \subset \mathbb{C}^n$. Ferrari initially considered a new sheaf of germs of holomorphic differential p-forms on X and denoted it by $\tilde{\Omega}^p_X$. In order to define this sheaf we need some notation. We follow Ferrari [29, Section 1]. Let Ω^p_U be the sheaf of holomorphic p-forms on U. Let $j\colon X_{\mathrm{reg}} \hookrightarrow U$ denote the natural embedding of the smooth locus of X into U and for an open subset $V \subseteq U$, let $j^\bullet_V : \Gamma(V, \Omega^p_U) \to \Gamma(V \cap X_{\mathrm{reg}}, \Omega^p_{X_{\mathrm{reg}}})$ be the restriction homomorphism. If $f \in \Gamma(V, \Omega^p_U)$ is a holomorphic p-form, Ferrari defines the restriction of $f_{|X}$ to be $j^\bullet_V f$, where $j^\bullet_V$ is as above. Then, he defines f to vanish on X if $j^\bullet_V f = 0$.

Set for each open $V \subset U$,

$$\mathscr{H}^p_U(V) := \{f \in \Gamma(V, \Omega^p_U);\ \ j^\bullet_V f = 0 \ \text{ on } V \cap X_{\mathrm{reg}}\}$$

Let $\mathscr{H}^p_U$ the subsheaf of Ω^p_U consisting of holomorphic differential p-forms that "vanish on X". Ferrari considered the quotient sheaf

$$\tilde{\Omega}^p_X := (\Omega^p_U / \mathscr{H}^p_U)_{|X},$$

and showed in [29] that that these sheaves are coherent, they vanish if $p > \dim_{\mathbb{C}} X$ and have a well-defined pull-back theory under holomorphic maps between complex spaces.

In a sequel paper [30] Ferrari defined the torsion $\tau\mathcal{F}$ of a coherent analytic sheaf $\mathcal{F}$ on a reduced but possibly reducible complex space X to be the kernel of the natural map $e_{\mathcal{F}} : \mathcal{F} \to \mathcal{F}^{**}$. As he mentions in the paragraph after Definizione (6.1) in [30], since $\mathcal{O}_{X,x}$ is a reduced Noetherian ring and $\mathcal{F}_x$ is an $\mathcal{O}_{X,x}$ module of finite type, one has

$$\tau(\mathcal{F})_x = \{\sigma \in \mathcal{F}_x;\ \text{there exists } \ \lambda \in \mathcal{O}_{X,x} \text{ non-zero divisor such that } \lambda\,\sigma = 0\}.$$

The latter set is $\operatorname{tor}(\Omega^p_{X,x})$ according to the definition in Section 1.2.

There exists a natural surjective map from the sheaf of Kähler p-forms to the sheaf of Ferrari p-forms $\pi : \Omega^p_X \to \tilde{\Omega}^p_X$, and an associated short exact sequence

$$0 \to \ker \pi \to \Omega^p_X \xrightarrow{\pi} \tilde{\Omega}^p_X \to 0. \tag{1.8}$$

Ferrari proved via a series of Lemmas 2.1, 3.1, 4.1 in [30] that the kernel of π is a torsion sheaf and consequentially that $\ker \pi = \tau(\Omega^p_X) = \operatorname{tor}(\Omega^p_X)$. Taking this latter observation into account, we obtain from (1.8) that $\check{\Omega}^p_X = \tilde{\Omega}^p_X$. Also, since the sheaves $\tilde{\Omega}^p_X$ vanish for $p > \dim_{\mathbb{C}} X$ one can conclude that for these values of p, the sheaves of Kähler p-forms are torsion sheaves.

1.4. **Generalized Koszul complexes $\mathscr{D}_p(f)$.** We now recall the construction of certain complexes generalizing the Eagon-Northcott ([25], [26]) and Buchsbaum-Rim complexes ([19]) and, in some cases, containing resolutions of the module of p-differentials. These were constructed by Lebelt, Kirby, Buchsbaum-Eisenbud, Weyman, Bruns-Vetter and others (one could read the introduction in [18], Appendix A2.6 in [27] and [15, Ch 2, §E] for information regarding the history and applications of these constructions, and also the works of Weyman [100], Herzog-Martinskovsky [49], Bruns-Vetter [16], Ichim-Vetter [54], [55] for further generalizations).

Let R be a Noetherian ring, and let

$$f\colon F \to G$$

be a homomorphism of free R-modules with $\operatorname{rank} F = c$ and $\operatorname{rank} G = n$. Assume that $c \leqslant n$, and set $d = n - c$. The complexes $\mathscr{D}_p(f)$ constructed from this map involve the various exterior, symmetric and divided powers of these free modules. If F and G are only locally free, it is also possible to construct such complexes with a bit more care (see the footnotes in [27, App A]), and to obtain global versions for complexes of bundles on some non-affine spaces, but as our results are for local singularities we shall not need these.

Consider the exterior algebra of G and the symmetric algebra of $F^* = \operatorname{Hom}_R(F, R)$

$$\bigwedge = \bigwedge G = \bigoplus_{k \geqslant 0} \bigwedge^k G. \qquad\qquad S = S(F^*) = \bigoplus_{k \geqslant 0} S_k(F^*)$$

and its graded dual, the divided power algebra

$$D = D(F) = \bigoplus_{k \geqslant 0} D_k F \stackrel{\text{def}}{=} \text{gr-Hom}_R(S(F^*), R)$$

that is with

$$D_k(F) = (S_k(F^*))^*$$

The graded pieces are, respectively, the exterior powers of G and the symmetric powers of F^* and the divided powers of F. Clearly D is an S-module via the contravariant component of Hom. We shall not describe its algebra structure as we do not need it.

We note that some authors define the divided powers simply as the duals of the symmetric powers, that is, via the formula $D_k F = (S_k(F))^*$ instead of $D_k F = (S_k(F^*))^*$, but we choose to use the latter version since it is (covariantly) functorial; see for example [27, §A2.6].

The generalized Koszul complexes $\mathscr{D}_p(f)$ are built in two portions which are then glued together to form the full complexes. There are two classic constructions of these complexes, the second of which gave the first description of the entire complexes:

1. In the construction due to Lebelt and to Bruns and Vetter in [70], [13], and [14], which builds on Vetter's work in [99], the left-hand portions are built first from the map f and glued to appropriate duals, which form the right-hand portions (although in their book [15] they construct them as in (2) below).
2. In the construction due to Kirby and to Buchsbaum and Eisenbud in [65] and [18] (see also [27, §A2.6]), which builds on work of Eagon and Northcott in [26] and Buchsbaum and Rim in [20], the complexes forming the right-hand portions are built first from the map $f^* = \operatorname{Hom}_R(f, R)$ and then glued to appropriate duals, which form the left-hand portions. These give the same resulting sequences.

In this paper, we shall follow the first construction since we are interested in the fact that under certain conditions and for certain values of p the left-hand portions provide resolutions of p-th exterior power of coker f. On the other hand, one may note that the second construction has a natural beauty in that the right-hand portions are the graded strands of a Koszul complex for the symmetric algebra. We recall the second construction as well in 1.6 and compare the two in 1.7. For a more general exposition one could read Section 3 in Ichim-Vetter's paper [54]. Of interesting note, the gluing map is given from yet another point of view in [49] in certain cases.

We warn the reader that our numbering of the complexes also follows construction (1) rather than (2). Note that all the unadorned tensor products below are taken to be over the ring R.

Left-hand portion.

First we describe the left-hand portion of each complex $\mathscr{D}_p(f)$. For each $p \geqslant 0$, let $\mathscr{C}_p(f)$ be the complex

$$\mathscr{C}_p(f)\colon\quad 0 \to D_p \otimes \bigwedge^{0} \xrightarrow{\partial} D_{p-1} \otimes \bigwedge^{1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} D_1 \otimes \bigwedge^{p-1} \xrightarrow{\partial} D_0 \otimes \bigwedge^{p}, \tag{1.9}$$

where $D_k = D_k(F) = (S_k(F^*))^*$, $\bigwedge^k = \bigwedge^k G$, and the differential

$$\partial\colon D_k(F) \otimes \bigwedge^{p-k} G \to D_{k-1}(F) \otimes \bigwedge^{p-k+1} G$$

is given on a basis for D_k as follows: Choosing a basis for $b_1, \ldots, b_c$ for F, let

$$\partial({b_1}^{(j_1)} \ldots {b_c}^{(j_c)} \otimes \alpha) = \sum_{i=1}^{c} {b_1}^{(j_1)} \ldots {b_i}^{(j_i-1)} \ldots {b_c}^{(j_c)} \otimes \alpha \wedge f(b_i)$$

for $j_1 + j_2 + \cdots + j_c = k$ and $\alpha \in \bigwedge^{p-k}$.

Remark 1.10. In characteristic zero, where symmetric and divided powers are isomorphic, this complex is isomorphic to the pth exterior power of the short complex $F \xrightarrow{f} G$ situated in homological degrees 1 and 0. For the definition of the exterior power of a complex, see [11], for example.

Right-hand portion.

The right-hand portion of the complex $\mathscr{D}_p(f)$ is simply the dual $\mathscr{C}_{d-p}(f)^* = \operatorname{Hom}_R(\mathscr{C}_{d-p}(f), R)$ of the complex $\mathscr{C}_{d-p}(f)$. Upon identifying the duals $(D_k(F))^*$ with the symmetric powers $S_k = S_k(F^*)$, it can be written as

$$S_0 \otimes (\bigwedge^{d-p})^* \to S_1 \otimes (\bigwedge^{d-p-1})^* \to \cdots \to S_{d-p-1} \otimes (\bigwedge^{1})^* \to S_{d-p} \otimes (\bigwedge^{0})^* \to 0 \tag{1.11}$$

with differential

$$\partial\colon S_k(F^*) \otimes (\bigwedge^{d-p-k} G)^* \to S_{k+1}(F^*) \otimes (\bigwedge^{d-p-k-1} G)^*$$

given on a basis for S_k as follows: choosing a basis for $e_1, \ldots, e_n$ for G and denoting the dual basis elements for each $(\bigwedge^{\ell})^*$ by $(e_{i_1} \wedge \cdots \wedge e_{i_\ell})^*$ with $i_1 < \cdots < i_\ell$, let

$$\partial\left(\beta \otimes (e_{i_1} \wedge \cdots \wedge e_{i_k})^*\right) = \sum_{j=1}^{\ell} (\beta f^*(e_{i_j}^*)) \otimes (e_{i_1} \wedge \cdots \wedge \widehat{e_{i_j}} \wedge \cdots \wedge e_{i_k})^*$$

for $\beta \in S_k(F^*)$, where $f^*\colon G^* \to F^*$ is the dual of the map f and $\widehat{e_{i_j}}$ means that e_{i_j} is omitted. Note that, up to isomorphism, this is simply a graded strand of the Koszul complex of the elements $f^*(e_1^*), \ldots, f^*(e_n^*)$ on the symmetric algebra $S(F^*)$ for any basis $e_1, \ldots, e_n$ of G.

Remark 1.12. Note that by left exactness of Hom the kernel at the left end of this complex is the R-module $(\bigwedge^{d-p}(\operatorname{coker} f))^*$.

Gluing to obtain the entire complex.

Finally, for $0 \leqslant p \leqslant d$, the two portions are glued together via a gluing map

$$D_0(F) \otimes \bigwedge^{p} G \xrightarrow{\nu_p} S_0(F^*) \otimes (\bigwedge^{d-p} G)^*$$

defined as follows. First note that since $D_0 \cong S_0 \cong R$, we need only find a map $\bigwedge^p \xrightarrow{\nu_p} (\bigwedge^{d-p})^*$. Since F and G are free of rank c and n, respectively, we may fix orientations for F and G, that is, generators $\gamma \in \bigwedge^c F \cong R$ and $\delta \in \bigwedge^n G \cong R$. In the proof of Lemma 2.8, when we discuss the well-defined pairing, δ is the isomorphism map, not a generator of $\wedge^n G$. Consider (by abuse of notation) the image $f(\gamma) \in \bigwedge^c(G)$ under the map $\bigwedge^c f\colon \bigwedge^c F \to \bigwedge^c G$ induced by $f\colon F \to G$. The pairing given by the composition

$$\bigwedge^{p} \times \bigwedge^{d-p} \xrightarrow{\mu} \bigwedge^{d} \xrightarrow{f(\gamma)} \bigwedge^{n} \xrightarrow{\cong} R, \tag{1.13}$$

where the first map is multiplication in the exterior algebra, the second is multiplication by $f(\gamma)$, and the third is the isomorphism that the orientation δ determines, yields the desired map $\bigwedge^p \xrightarrow{\nu_p} (\bigwedge^{d-p})^*$. Note that for different choices of orientation the resulting map ν_p will differ only by multiplication by a unit and so will not affect the homology modules.

Gluing (1.9) and (1.11), we arrive for $0 \leqslant p \leqslant d$ at the full complex $\mathscr{D}_p = \mathscr{D}_p(f)$

$$\mathscr{D}_p(f)\colon \quad 0 \to D_p \otimes \bigwedge^{0} \to \cdots \to D_0 \otimes \bigwedge^{p} \xrightarrow{\nu_p} S_0 \otimes (\bigwedge^{d-p})^* \to \cdots \to S_{d-p} \otimes (\bigwedge^{0})^* \to 0 \tag{1.14}$$

Following [14], for values of p outside that range, we define the complexes as follows.

- For $p > d$ we set $\mathscr{D}_p(f) := \mathscr{C}_p(f)$, that is, using only the left-hand portion, and
- For $p < 0$ we define $\mathscr{D}_p(f) := (\mathscr{D}_{d-p}(f))^*$, yielding only a right-hand portion.

Here is a display of the sequences $\mathscr{D}_p(f)$ for the range $-2 \leqslant p \leqslant d+2$, with gluing maps ν_p arranged along the middle column.

$$\vdots$$

$$\mathscr{D}_{d+2}(f): \qquad 0 \to D_{d+2} \otimes \bigwedge^{0} \to \cdots\cdots\cdots\cdots\cdots \to D_0 \otimes \bigwedge^{d+2} \longrightarrow 0$$

$$\mathscr{D}_{d+1}(f): \qquad 0 \to D_{d+1} \otimes \bigwedge^{0} \to \cdots\cdots\cdots\cdots \to D_0 \otimes \bigwedge^{d+1} \longrightarrow 0$$

$$\mathscr{D}_{d}(f): \qquad 0 \to D_{d} \otimes \bigwedge^{0} \to \cdots\cdots\cdots \to D_0 \otimes \bigwedge^{d} \xrightarrow{\nu_d} S_0 \otimes (\bigwedge^{0})^* \to 0$$

$$\mathscr{D}_{d-1}(f): \qquad 0 \to D_{d-1} \otimes \bigwedge^{0} \to \cdots\cdots \to D_0 \otimes \bigwedge^{d-1} \xrightarrow{\nu_{d-1}} S_0 \otimes (\bigwedge^{1})^* \to S_1 \otimes (\bigwedge^{0})^* \to 0$$

$$\vdots$$

$$\mathscr{D}_{p}(f): \qquad 0 \to D_{p} \otimes \bigwedge^{0} \to \cdots \to D_0 \otimes \bigwedge^{p} \xrightarrow{\nu_p} S_0 \otimes (\bigwedge^{d-p})^* \to \cdots \to S_{d-p} \otimes (\bigwedge^{0})^* \to 0$$

$$\vdots$$

$$\mathscr{D}_{1}(f): \qquad 0 \to D_{1} \otimes \bigwedge^{0} \to D_0 \otimes \bigwedge^{1} \xrightarrow{\nu_1} S_0 \otimes (\bigwedge^{d-1})^* \to \cdots\cdots \to S_{d-1} \otimes (\bigwedge^{0})^* \to 0$$

$$\mathscr{D}_{0}(f): \qquad 0 \to D_{0} \otimes \bigwedge^{0} \xrightarrow{\nu_0} S_0 \otimes (\bigwedge^{d})^* \to \cdots\cdots\cdots \to S_{d} \otimes (\bigwedge^{0})^* \to 0$$

$$\mathscr{D}_{-1}(f): \qquad 0 \longrightarrow S_0 \otimes (\bigwedge^{d+1})^* \to \cdots\cdots\cdots\cdots \to S_{d+1} \otimes (\bigwedge^{0})^* \to 0$$

$$\mathscr{D}_{-2}(f): \qquad 0 \longrightarrow S_0 \otimes (\bigwedge^{d+2})^* \to \cdots\cdots\cdots\cdots\cdots \to S_{d+2} \otimes (\bigwedge^{0})^* \to 0$$

$$\vdots$$

1.5. **Homological indexing.** We will use the following homological degree indexing for each complex as it is the most natural one for both the classic comparison to a Koszul complex given in Lemma 2.4 and the symmetry property developed in Corollary 2.5. (Note that different sources use different indexing.) We will consider position 0 to be at the right end of each complex. For example, for $p \leqslant d$, this means that position 0 is occupied by $S_{d-p} \otimes (\bigwedge^0)^*$ and the degree increases to the left.

For the range of indices $-1 \leqslant p \leqslant d+1$, the complex therefore has length exactly $d+1$, with the right end at position 0 and left end at position $d+1$. This is crucial for Corollary 2.5 on the symmetry in vanishing of homology.

Alternate form.

An alternate way to view the complex is to rewrite the terms in the right-hand portion of the complex using the isomorphisms $\bigwedge^k \cong (\bigwedge^{n-k})^*$. In this way the complex becomes

$$0 \to D_p \otimes \bigwedge^{0} \to \cdots \to D_0 \otimes \bigwedge^{p} \xrightarrow{\nu_p} S_0 \otimes \bigwedge^{p+c} \to \cdots \to S_{d-p} \otimes \bigwedge^{n} \to 0 \tag{1.15}$$

In fact, if one uses the isomorphism obtained from the orientation $\delta \in \bigwedge^n(G) \cong R$ that was chosen for the gluing, then a simple computation yields that the gluing map in this new complex becomes simply multiplication by the element $f(\gamma)$. More precisely, one has

$$\nu_p(r \otimes \omega) = r \otimes (\omega \wedge f(\gamma)).$$

1.6. **Dual construction.** Here we review the construction of complexes by Kirby and Buchsbaum and Eisenbud in [65] and [18] (see also [17]) beginning instead with a map from a free R-module of greater rank to one of lesser rank. In fact, Buchsbaum and Eisenbud construct a doubly-indexed family of complexes $\mathbf{L_p^q}$ associated to it, of which the complexes $\mathbf{L_p^1}$ correspond to the ones appearing in this paper. We will denote these by $\mathcal{C}^p(f^*)$.

More explicitly, begin with the map $f^*\colon G^* \to F^*$ that is the dual of the one considered above in the Lebelt-Bruns-Vetter construction. For the right portions, one constructs complexes $\mathscr{B}_p(f^*)$ associated to f^* of the form

$$\mathscr{B}_p(f^*)\colon\quad S_0F^* \otimes \bigwedge^{p} G^* \xrightarrow{\tilde\partial} S_1F^* \otimes \bigwedge^{p-1} G^* \xrightarrow{\tilde\partial} \cdots \xrightarrow{\tilde\partial} S_{p-1}F^* \otimes \bigwedge^{1} G^* \xrightarrow{\tilde\partial} S_pF^* \otimes \bigwedge^{0} G^* \to 0 \tag{1.16}$$

with differential $\tilde\partial$ defined as follows: Choose a basis $\{y_i\}$ for F^* and let $\{\hat y_i\}$ be the corresponding dual basis for F. Then $f(\hat y_i) \in G$ acts on $\bigwedge G^*$. The map $\tilde\partial$ takes an element

$$m \otimes h \in S_{p-i}F^* \otimes \bigwedge^{i} G^*$$

to the element

$$\sum y_i m \otimes f(\hat y_i)(h) \in S_{p-i+1}F^* \otimes \bigwedge^{i-1} G^*.$$

Next, for $0 \leqslant p \leqslant d$, similarly to the construction of the complexes $\mathscr{D}_p(f)$, one splices the R-duals of the complexes $\mathscr{B}_{d-p}(f^*)$ on the left to the complexes $\mathscr{B}_p(f^*)$ on the right to produce a family of complexes $\mathcal{C}^p(f^*)$ called the Buchsbaum-Eisenbud-Kirby generalized Koszul complexes (see Appendix A2.6.1 in [27]).

$$\mathcal{C}^p(f^*)\colon\quad 0 \to (S_{d-p}F^*)^* \xrightarrow{\partial^*} \cdots \xrightarrow{\partial^*} (\bigwedge^{d-p} G^*)^* \xrightarrow{\tilde\nu_i} \bigwedge^{p} G^* \xrightarrow{\partial} \cdots \xrightarrow{\partial} S_pF^* \to 0 \tag{1.17}$$

For $p < 0$ and $p > d$ one defines $\mathcal{C}^p(f^*)$ to be the appropriate half-complexes $\mathscr{B}_{d-p}(f^*)$ and $\mathscr{B}_p(f^*)$, respectively.

1.7. **Comparison of the two constructions and duality.** Comparing the complexes (1.14) and (1.17), one can verify that for all p, there is a natural isomorphism

$$\mathscr{D}_p(f) \cong \mathcal{C}^{d-p}(f^*)$$

between the Lebelt-Bruns-Vetter complexes and the Buchsbaum-Eisenbud-Kirby complexes. In addition, they each satisfy a duality as follows.

$$(\mathscr{D}_p(f))^* \cong \mathscr{D}_{d-p}(f) \qquad (\mathcal{C}^p(f^*))^* \cong \mathcal{C}^{d-p}(f^*)$$

Composing with the latter, one gets an isomorphism $\mathscr{D}_p(f) \cong (\mathcal{C}^p(f^*))^*$. We will use these isomorphisms when translating the classic acyclicity results in Section 1.9 given in the sources for the complexes $\mathcal{C}^p$ to facts about the complexes $\mathscr{D}_p$ which we use in this paper.

1.8. **Basic properties of the generalized Koszul complexes.** We summarize certain well-known key features of the complexes $\mathscr{D}_p(f)$ that arise from a homomorphism

$$f\colon F \to G$$

with $\operatorname{rank} F = c < n = \operatorname{rank} G$, respectively, of the complexes $\mathcal{C}^p(f^*)$ that arise from

$$f^*\colon G^* \to F^*.$$

Set $d = n - c$.

(1) The original map f appears as the first nontrivial map in the complex $\mathscr{D}_1(f)$ upon making the following identifications

$$F \cong D_1 \otimes \bigwedge^{0} \longrightarrow D_0 \otimes \bigwedge^{1} \cong G.$$

Likewise, the last nontrivial map in $\mathscr{D}_{d-1}(f)$ is its dual f^*.

(2) For $p \geqslant 1$ the last map in the left hand portion of the complex $\mathscr{D}_p(f)$ gives a presentation of the exterior powers of the cokernel of f, that is, there is an exact sequence

$$D_1 \otimes \bigwedge^{p-1} \xrightarrow{\partial} D_0 \otimes \bigwedge^{p} \longrightarrow \bigwedge^{p}(\operatorname{coker} f) \longrightarrow 0.$$

Similarly, for $p \leqslant d-1$ the last map in the full complex $\mathscr{D}_p(f)$ gives a presentation of the symmetric powers of the cokernel of f^*, that is, there is an exact sequence

$$S_{d-p-1} \otimes (\bigwedge^{1})^* \xrightarrow{\partial} S_{d-p} \otimes (\bigwedge^{0})^* \longrightarrow S_{d-p}(\operatorname{coker} f^*) \longrightarrow 0.$$

See also Remark 1.12.

(3) With the identifications $D_0 \otimes \bigwedge^0 \cong R \cong S_0 \otimes (\bigwedge^0)^*$, the maps ν_0 and $\nu_d = (\nu_0)^*$ are given up to isomorphism by the maximal minors of the original map f.

(4) Let $I_c(f)$ denote the ideal generated by the $c \times c$ minors of the matrix of f with respect to any bases of R^c and R^n. If $\mathfrak{p}$ is a prime ideal in R such that $\mathfrak{p} \not\supseteq I_c(f)$, then $\mathscr{D}_p(f) \otimes R_{\mathfrak{p}}$ are split exact for all $p \in \mathbb{Z}$ by [14, Lem 1.2] or [13, Prop 1].

In particular, in the setting of this paper with f giving a presentation of $\Omega^1_{X,x}$ as described in part (5) below, one concludes from

$$V(I_c(f)) = \operatorname{Sing} R := \{\mathfrak{p} \in \operatorname{Spec} R \mid R_{\mathfrak{p}} \text{ is not regular}\}$$

that the homology $\mathrm{H}(\mathscr{D}_p(f))$ of the complex $\mathscr{D}_p(f)$ is supported on $\operatorname{Sing} R$.

(5) Suppose now that we are in the setting of this paper, as given in Section 1.1. Taking the stalks at x and letting $R = \mathcal{O}_{X,x}$ be the local complete intersection ring, we obtain the exact sequence (1.1)

$$0 \longrightarrow R^c \xrightarrow{f} R^n \longrightarrow \Omega^1_{X,x} \longrightarrow 0.$$

Since R is Cohen-Macaulay, the grade of the determinantal ideal $I_c(f)$ is then equal to

$$\begin{aligned}\operatorname{grade} I_c(f) &= \operatorname{ht} I_c(f) = \dim R - \dim R/I_c(f)\\ &= \dim R - \dim \operatorname{Sing} R = \operatorname{codim} \operatorname{Sing} R.\end{aligned}$$

Therefore, for all $p \leqslant \operatorname{codim} \operatorname{Sing} R$ the left-hand portion of $\mathscr{D}_p(f)$ is acyclic by [14, Prop 2.1] and hence a resolution of $\bigwedge^p(\operatorname{coker} f)$ by (2) above.

1.9. **Acyclicity properties of the generalized Koszul complexes.** In certain cases, the complexes $\mathscr{D}_p(f)$ (or equivalently $\mathcal{C}^{d-p}(f^*)$) are known to be acyclic, that is, they are resolutions of the cokernels at the right ends. Such results were proved by groups on both sides of the ocean, first by Buchsbaum and Eisenbud in [18] and later by Bruns and Vetter in [14], as well as earlier by Lebelt for his partial complexes. The well-known special case where the map is given by a matrix of indeterminates over a polynomial ring goes back even further in some cases: For the Eagon-Northcott complexes this was established in [26], and for the Buchsbaum-Rim complexes in [20], each building on Northcott's result that the depth of the ideal generated by the maximal minors equals the maximal possible value.

In Section 2 we will need these acyclicity results. We will use the version given in the following theorem that appears in [15, Thm 2.16]. Part (a) of the theorem below, including acyclicity statements when $p = -1$ and $p = d + 1$, was first proved by Buchsbaum and Eisenbud, and can be found in [27, Thm A2.10(c)].

Note that in the statement below we have translated their results to our notation using the following two facts. The complexes $\mathscr{D}_p(f)$ are isomorphic to $\mathcal{C}^{d-p}(f^*)$ as described in (1.7) above. The complexes $\mathscr{D}_p(f)$ that we use are isomorphic to the complexes $\mathcal{D}_{d-p}(f^*)$ appearing in [15, Thm 2.16].

Theorem 1.18 (Buchsbaum-Eisenbud, Bruns-Vetter)**.** *Let R be a Noetherian ring, $f\colon F \to G$ a homomorphism of finitely generated free R-modules with* $\operatorname{rank} F = c$ *and* $\operatorname{rank} G = n$*. Set $d = n - c$. Suppose $c \leqslant n$ and* $\operatorname{grade}(I_c(f)) = n - c + 1 = d + 1$*. Then the following hold:*

- (a) *The complexes $\mathscr{D}_p(f)$ are acyclic for $0 \leqslant p \leqslant d$.*
- (b) *The complex $\mathscr{D}_p(f)$ resolves $R/I_c(f)$ for $p = d$ and $S_{d-p}(\operatorname{coker} f^*)$ for $p = 0, \dots, d-1$.*
- (c) *The R-modules $R/I_c(f)$ and $S_{d-p}(\operatorname{coker} f^*)$ for $p = 0, \cdots, d-1$ are perfect.*

In Theorem 1.18 a key assumption is that the grade of $I_c(f)$ is the maximal possible. When the map f is given by a matrix of indeterminates this was established by Northcott's grade result below [81, Prop 1 and 2]. As Northcott's result is fundamental in our proofs of various properties of the complexes, we mention it here.

Theorem 1.19 (Northcott)**.** *Let S be a Noetherian ring and $U = (u_{ij})$ be an $s \times r$ matrix of indeterminates with $s \leqslant r$. Let $I_s(U)$ denote the ideal of the polynomial ring $S[\{u_{ij}\}]$ generated by the maximal minors of U. Then $I_s(U)$ is a proper ideal, and it has grade equal to $r - s + 1$. Furthermore, $I_s(U)$ is perfect; in particular, all prime ideals belonging to $I_s(U)$ also have grade $r - s + 1$. If S is also a domain then $I_s(U)$ is a prime ideal.*

2. Rigidity and symmetry

Throughout this section, let R be a Noetherian ring and let

$$f\colon F \to G$$

be an injective homomorphism of free R-modules such that $\operatorname{rank} F = c$ and $\operatorname{rank} G = n$ with $d = n - c > 0$. Consider the generalized Koszul complexes $\mathscr{D}_p(f)$ described in Section 1.4.

The proof of Corollary 3.1 has as its main ingredients two properties of $\mathscr{D}_p(f)$. The first property is the rigidity of each complex, that is, the property that if some homology module vanishes then so do the higher indexed ones. The second property, which we call symmetry, relates the complexes to each other. It says that corresponding homologies of the complexes have the same support and hence vanish simultaneously.

Both properties derive from the fact that the complexes $\mathscr{D}_p(f)$ can be viewed as certain closely related generic Koszul complexes by using a deformation to the generic case. This idea first appeared in Eagon-Northcott's work [25], [26] and in Buchsbaum-Rim's work [20]. Lebelt used it to prove the rigidity of the complexes that he had constructed in [70]. For the complexes $\mathscr{D}_p(f)$, Rodicio gave a proof of the rigidity in [87] using the same idea as Lebelt. The deformation to the generic case has also been applied by Hayasaka and Hyry to study questions related to Buchsbaum-Rim multiplicity of a parameter ideal; see [47]. We include a proof below for completeness and as we need the details for our symmetry result. Then we prove the symmetry result, which as far as we know is not in the literature. We note that although the deformation involves a choice of bases, the consequential rigidity and symmetry results are independent of this choice.

2.1. **Deformation to generic case.**

We recall the deformation to the generic case as in [25, §4], [20, §2], [70, Satz B], [87, §2]. Fixing bases of F and G, suppose that the matrix of f is given by $A = (a_{ij})$ with $a_{ij} \in R$ and so in these bases f is given as

$$R^c \xrightarrow{A} R^n.$$

Now we build the corresponding generic map. Let $X = (x_{ij})$ be an $n \times c$ matrix of indeterminates and $R[X]$ be the corresponding polynomial ring. Consider the linear map given by multiplication by the matrix X

$$R[X]^c \xrightarrow{X} R[X]^n.$$

Via the surjective ring map $\pi\colon R[X] \to R$ induced by the assignment $\pi(x_{ij}) = a_{ij}$, one can view R as a quotient of the ring $R[X]$ by the ideal generated by the regular sequence $\{x_{ij} - a_{ij}\}$. Clearly, the map X is a lift along π of the map f in the sense that there is an isomorphism

$$X \otimes_{R[X]} R \cong f \tag{2.2}$$

where R is viewed as an $R[X]$-module via π.

Lastly, consider the cokernel of multiplication by the transpose of X

$$R[X]^n \xrightarrow{X^T} R[X]^c \to \operatorname{coker} X^{\mathrm{T}} \to 0$$

which the reader will recognize as the homology at the right end of the generalized Koszul complex $\mathscr{D}_{d-1}(X)$ built from the generic map X of free $R[X]$-modules.

We begin with the reinterpretations of the complexes as Koszul complexes over the generic ring from [70, Satz B] and extended in [87, Thm 2.3]), with roots in [25], [20], and [26].

Lemma 2.3 (Lebelt, Rodicio)**.** *Let R be a Noetherian ring, and let $f\colon F \to G$ be an injective homomorphism of free R-modules such that* $\operatorname{rank} F = c$ *and* $\operatorname{rank} G = n$ *with $d = n - c$.*

For each p with $-1 \leqslant p \leqslant d+1$ the complex $\mathscr{D}_p$ associated to the map f is quasi-isomorphic (via a series of two maps) to the Koszul complex

$$K(\{x_{ij} - a_{ij}\}; W_p)$$

where, in the notation of Section 2.1, the $R[X]$-module W_p is given by

$$W_p = \begin{cases} S_{d-p}(\operatorname{coker} X^T), & p < d \\ R[X]/I_c(X), & p = d \\ \bigwedge^{d+1}(\operatorname{coker} X), & p = d+1 \end{cases}$$

and $A = (a_{ij})$ is any matrix for the map f.

Proof. Let $\mathscr{D}_p(f)$ and $\mathscr{D}_p(X)$ denote the complexes from Section 1.4 associated to the maps f and X, respectively; these are complexes of free modules over R and $R[X]$, respectively. We consider $\mathscr{D}_p(f)$ as an $R[X]$-module via the map sending x_{ij} to a_{ij}. As the construction of the complexes $\mathscr{D}_p$ is functorial, the isomorphism (2.2) yields an isomorphism of complexes of $R[X]$-modules

$$\mathscr{D}_p(f) \cong \mathscr{D}_p(X) \otimes_{R[X]} R.$$

Letting K denote the Koszul complex $K(\{x_{ij} - a_{ij}\}; R[X])$, we have a series of quasi-isomorphisms

$$\mathscr{D}_p(X) \otimes_{R[X]} R \xrightarrow{\simeq} \mathscr{D}_p(X) \otimes_{R[X]} K \xrightarrow{\simeq} W_p \otimes_{R[X]} K.$$

The first is due to the fact that $x_{ij} - a_{ij}$ is a regular sequence on $R[X]$ and hence the Koszul complex K provides a free $R[X]$-resolution of R. The second is due to the fact that each generic complex $\mathscr{D}_p(X)$ is acyclic for this range of p values and thus a resolution of its 0th homology; see Theorem 1.18 or [27, Thm A2.10(c)] (applied to $X^T : R[X]^n \to R[X]^c$). Note that $\mathscr{D}_p(X)$ and K are composed of free $R[X]$-modules and hence tensoring with them preserves quasi-isomorphisms. We are done as $W_p \otimes_{R[X]} K$ is the desired Koszul complex. Finally, that the 0th homology of $\mathscr{D}_p(X)$ is isomorphic to W_p follows from Section 1.8. ☐

Since Koszul complexes are well-known to be rigid, one gets the results [70, Satz B] and [87, Thm 2.3] as an immediate consequence:

Corollary 2.4 (Rigidity – Lebelt and Rodicio)**.** *Let R and $f : F \to G$ be as above. For each p with $-1 \leqslant p \leqslant d+1$ the complex $\mathscr{D}_p$ associated to the map f is rigid. In other words, if one has $\mathrm{H}_i(\mathscr{D}_p) = 0$ for some $i \geqslant 0$, then $\mathrm{H}_j(\mathscr{D}_p) = 0$ for all $j \geqslant i$.*

The next consequence of Lemma 2.3 is the following symmetry property. The reverse implication in the last result (that is, the vanishing) was first observed by Bruns-Vetter, cf. [14, Prop 2.1] (see also [54, Thm 5.2]).

Corollary 2.5 (Symmetry)**.** *Let R be a Noetherian Cohen-Macaulay ring, and let $f : F \to G$ be an injective homomorphism of free R-modules such that $\operatorname{rank} F = c$ and $\operatorname{rank} G = n$ with $d = n - c > 0$. For all p, q with $-1 \leqslant p, q \leqslant d$ and all i, the complexes associated to f satisfy*

$$\operatorname{Supp} \mathrm{H}_i(\mathscr{D}_p) = \operatorname{Supp} \mathrm{H}_i(\mathscr{D}_q).$$

In particular, one has

$$\mathrm{H}_i(\mathscr{D}_p) = 0 \iff \mathrm{H}_i(\mathscr{D}_q) = 0.$$

If, furthermore, $(R, \mathfrak{m})$ is local and f is minimal (that is, $\operatorname{im}(f) \subseteq \mathfrak{m}G$), then

$$\mathrm{H}_i(\mathscr{D}_p) = 0 \iff i > n - c + 1 - \operatorname{ht} I_c(f),$$

where $I_c(f)$ denotes the ideal generated by the $c \times c$ minors of any matrix for f.

Proof. Let p be any integer with $-1 \leqslant p \leqslant d$. By Lemma 2.3, we have that

$$\mathrm{H}_i(\mathscr{D}_p) = \mathrm{H}_i(K(\{x_{ij} - a_{ij}\}; W_p))$$

where $W_p = S_{d-p}(\operatorname{coker} X^{\mathrm{T}})$ if $p < d$, $W_d = R[X]/I_c(X)$, and $A = (a_{ij})$ is any matrix for f. We need to show that the supports of these Koszul homology modules are independent of the value of p in this range.

Each W_p is an $R[X]/I_c(X)$-module. To see this, consider the generic complexes $\mathscr{D}_p(X)$ associated to the map $R[X]^c \xrightarrow{X} R[X]^n$ given by multiplication by the matrix X as described in (2.1). Note that W_p is the 0th homology of $\mathscr{D}_p(X)$. In fact, all the homologies of $\mathscr{D}_p(X)$ are known to be annihilated by $I_c(X)$ (see, e.g., [65, Theorem 1]). Furthermore, since X is generic, one has $\sqrt{\operatorname{Ann} W_p} = \sqrt{I_c(X)}$ (e.g., [87, Lemma 2.7]) and therefore an equality

$$\operatorname{Supp} W_p = V(\operatorname{Ann} W_p) = V(I_c(X)) = \operatorname{Supp} R[X]/I_c(X) \tag{2.6}$$

where, recall that for an ideal I in a ring S one defines $V(I)$ to be the set of all prime ideals in S containing I. In particular, for any $\mathfrak{p} \in \operatorname{Spec} R[X]/I_c(X)$ one has

$$\dim(W_p)_{\mathfrak{p}} = \dim(R[X]/I_c(X))_{\mathfrak{p}} \tag{2.7}$$

for all p in this range.

Let us remark first that the ring $S = R[X]/I_c(X)$ is Cohen-Macaulay. Since R is Cohen-Macaulay, so is $R[X]$. The quotient ring S is perfect by Theorem 1.18(c) and Theorem 1.19. Then [12, Thm 2.1.5(a)] implies that S is also Cohen-Macaulay.

Our next goal is to verify that each W_p is Cohen-Macaulay over $R[X]$ and hence by (2.6) maximal Cohen-Macaulay over the quotient ring $S = R[X]/I_c(X)$. As noted in the proof of Lemma 2.3, the properties in Section 1.8 give that W_p is the 0th homology of the complex $\mathscr{D}_p(X)$ associated to the map $R[X]^c \xrightarrow{X} R[X]^n$. We apply Bruns and Vetter's result Theorem 1.18 to this map, noting that $\operatorname{grade}(I_c(X)) = n - c + 1$ by Northcott's result Theorem 1.19, to obtain its acyclicity. As noted before, the acyclicity can equivalently be obtained from the original result of Buchsbaum-Eisenbud as described in [27, Thm A2.10(c)], which includes also the case $p = -1$. Therefore we have that S is Cohen-Macaulay and that each complex $\mathscr{D}_p(X)$ is acyclic and hence a finite graded free resolution of W_p.

The Cohen-Macaulayness of W_p follows from the acyclicity of the complexes $\mathscr{D}_p(X)$ by the following standard argument: We use the terminology $*$-maximal and $*$-local from [12, §1.5]. A module M is Cohen-Macaulay if its localizations at all prime ideals in $\operatorname{Supp} M$ are so. First, applying [12, Thm 1.5.8(b) and Thm 1.5.9] reduces the problem from general prime ideals to homogeneous ones. Second, it suffices to check after localizing at the $*$-maximal ideals, which are all of the form $\mathfrak{n} = \mathfrak{m} + (X_{ij})$ for the various maximal ideals $\mathfrak{m}$ of R. Doing so, one gets a free resolution $\mathscr{D}_p(X)_{\mathfrak{n}}$ of the localization $(W_p)_{\mathfrak{n}}$ over the $*$-local ring $R[X]_{\mathfrak{n}}$. In fact, this localized resolution is also minimal by the property that each differential ∂ of the complex $E. = \mathscr{D}_p(X)$ satisfies $\operatorname{im}(\partial_k) \subseteq (X_{ij})E_{k-1}$ and since $(X_{ij}) \subseteq \mathfrak{n}$.

The image of $\mathfrak{n}$ in S will also be denoted by $\mathfrak{n}$. By the argument above one has equalities

$$\operatorname{pd}_{R[X]_{\mathfrak{n}}}(W_p)_{\mathfrak{n}} = n - c + 1 = \operatorname{pd}_{R[X]_{\mathfrak{n}}} S_{\mathfrak{n}}$$

where the second equality follows from the case of $p = d$ where one has $W_d = S$ (also the complex $\mathscr{D}_d(X)_{\mathfrak{n}}$ is really the localization of the well-known Eagon-Northcott resolution; see Theorem 2 in [25] or page 600 in [27]).

Applying the Auslander-Buchsbaum formula to $(W_p)_{\mathfrak{n}}$ yields

$$\operatorname{depth}_{R[X]_{\mathfrak{n}}}(W_p)_{\mathfrak{n}} + \operatorname{pd}_{R[X]_{\mathfrak{n}}}(W_p)_{\mathfrak{n}} = \operatorname{depth} R[X]_{\mathfrak{n}}.$$

Applying the Auslander-Buchsbaum formula to $S_{\mathfrak{n}}$ yields

$$\operatorname{depth}_{R[X]_{\mathfrak{n}}} S_{\mathfrak{n}} + \operatorname{pd}_{R[X]_{\mathfrak{n}}} S_{\mathfrak{n}} = \operatorname{depth} R[X]_{\mathfrak{n}}.$$

Subtracting gives

$$\operatorname{depth}_{R[X]_{\mathfrak{n}}}(W_p)_{\mathfrak{n}} = \operatorname{depth}_{R[X]_{\mathfrak{n}}} S_{\mathfrak{n}}.$$

However, both depths can be measured over the ring $S_{\mathfrak{n}}$ instead yielding

$$\operatorname{depth}_{S_{\mathfrak{n}}}(W_p)_{\mathfrak{n}} = \operatorname{depth}_{S_{\mathfrak{n}}} S_{\mathfrak{n}} = \dim S_{\mathfrak{n}}$$

where the last equality is due to the fact that S is Cohen-Macaulay. So indeed $(W_p)_{\mathfrak{n}}$ is Cohen-Macaulay. Therefore localizations of W_p at other homogeneous prime ideals are also Cohen-Macaulay. Hence W_p is a Cohen-Macaulay module, as desired.

Next, recall, as stated at the beginning of the proof, to prove the equality of supports in the statement, it is enough to prove that the support of $\mathrm{H}_i(K(\{x_{ij}-a_{ij}\}; W_p))$ is independent of p. Recall that an R-module is considered as an $R[X]$-module via the map sending x_{ij} to a_{ij}. As such, the isomorphism at the beginning of the proof is as $R[X]$-modules, and hence it suffices to work over the ring S. Take any $\mathfrak{p} \in \operatorname{Spec} S$. Let $\mathfrak{J} = (\{\overline{x_{ij} - a_{ij}}\})$ denote the ideal in S generated by the images of the elements $x_{ij} - a_{ij}$ and $\mathfrak{J}_{\mathfrak{p}}$ its localization as an ideal in $S_{\mathfrak{p}}$. Consider the localized homology groups

$$\mathrm{H}_i(K(\{x_{ij}-a_{ij}\}; W_p))_{\mathfrak{p}} = \mathrm{H}_i(K(\{x_{ij} - a_{ij}\}; (W_p)_{\mathfrak{p}}))$$

By the depth sensitivity of the Koszul complex (cf. [12, Theorem 1.6.17]), the indices i for which this homology vanishes depend only on the number $g_p = \operatorname{grade}(\mathfrak{J}_{\mathfrak{p}}, (W_p)_{\mathfrak{p}})$. More precisely, it vanishes if and only if $i > nc - g_p$. To complete the claim on supports, we show that this grade equals $g = \operatorname{grade}(\mathfrak{J}_{\mathfrak{p}}, S_{\mathfrak{p}})$, independent of the index p.

If $\mathfrak{J}_{\mathfrak{p}}$ is the unit ideal, then all the Koszul homologies vanish and we are done. So, we may assume that $\mathfrak{J}_{\mathfrak{p}}$ is proper and so $\mathfrak{J} \subseteq \mathfrak{p}$. Now, by [12, Prop 1.2.10(a)] one has

$$\begin{aligned} g_p &= \operatorname{grade}(\mathfrak{J}_{\mathfrak{p}}, (W_p)_{\mathfrak{p}}) = \inf\{\operatorname{depth}(W_p)_{\mathfrak{q}} \mid \mathfrak{p} \supseteq \mathfrak{q} \supseteq \mathfrak{J}\} \\ g &= \operatorname{grade}(\mathfrak{J}_{\mathfrak{p}}, S_{\mathfrak{p}}) = \inf\{\operatorname{depth} S_{\mathfrak{q}} \mid \mathfrak{p} \supseteq \mathfrak{q} \supseteq \mathfrak{J}\} \end{aligned}$$

noting that the prime ideals containing $\mathfrak{J}_{\mathfrak{p}}$ are all of the form $\mathfrak{q}_{\mathfrak{p}}$ with $\mathfrak{p} \supseteq \mathfrak{q} \supseteq \mathfrak{J}$ and that $((W_p)_{\mathfrak{p}})_{\mathfrak{q}_{\mathfrak{p}}} \cong (W_p)_{\mathfrak{q}}$. But, as $(W_p)_{\mathfrak{p}}$ is maximal Cohen-Macaulay over $S_{\mathfrak{p}}$, these quantities are equal.

In particular, when R is local with maximal ideal $\mathfrak{m}$, we can calculate this grade precisely: Again, denote the image of $\mathfrak{m}$ in S also by $\mathfrak{m}$ and the image of x_{ij} also by x_{ij}. Recall that $\mathfrak{J} \subseteq \operatorname{Ann} \mathrm{H}_i(\mathscr{D}_p)$ since $\mathscr{D}_p$ is isomorphic to the Koszul complex $K(\{x_{ij}-a_{ij}\}, W_p)$ whose homology is annihilated by $\{x_{ij} - a_{ij}\}$, the generators of $\mathfrak{J}$ (see [12, Prop 1.6.5(b)], for example). So one has $\operatorname{Supp} \mathrm{H}_i(\mathscr{D}_p) \subseteq V(\mathfrak{J})$. As $\mathfrak{p} = \mathfrak{m} + (x_{ij})$ is the unique maximal ideal of S containing $\mathfrak{J}$, we may therefore localize at $\mathfrak{p}$ to calculate the support of $\mathscr{D}_p$ or determine its vanishing. Then, since S is Cohen-Macaulay, standard results (cf. [74, Thm 17.4(i)]) yield equalities

$$\begin{aligned} g = \operatorname{grade}(\mathfrak{J}_{\mathfrak{p}}, S_{\mathfrak{p}}) = \operatorname{ht} \mathfrak{J}_{\mathfrak{p}} &= \dim S_{\mathfrak{p}} - \dim S_{\mathfrak{p}}/\mathfrak{J}_{\mathfrak{p}} \\ &= \dim R + nc - (n - c + 1) - \dim S_{\mathfrak{p}}/\mathfrak{J}_{\mathfrak{p}} \\ &= \dim R + nc - (n - c + 1) - \dim R/I_c(f) \\ &= nc - (n - c + 1) + \operatorname{ht} I_c(f) \end{aligned}$$

where ht denotes height, or codimension and the equality in the second line is by Theorem 1.19 and the equality in the third line is due to the isomorphism $S_{\mathfrak{p}}/\mathfrak{J}_{\mathfrak{p}} \cong R/I_c(f)$. Recalling that

$$\mathrm{H}_i(\mathscr{D}_p) = 0 \iff i > (nc - g)$$

completes the proof. □

We end this section by showing that the homologies of the complexes $\mathscr{D}_p$ in the positions flanking the gluing map give the torsion and cotorsion of the exterior powers of the cokernel of the map f. In [70, Lemma 2] Lebelt shows that the torsion can be identified with a particular homology of some complexes that he constructs there. In his subsequent paper, [71, p.5, Beispiele (ii)], he shows that his complexes are isomorphic to the complexes $\mathscr{C}_p$ together with the first gluing map. The cotorsion statement is surely known too, but we give a proof here for lack of a suitable reference. For $p > d$, note that the exterior power is a torsion module, hence its cotorsion equals 0, and so the homologies flanking the map ν_p trivially give its torsion and cotorsion.

Recall that an R-module M is said to satisfy Serre's condition S_n if the inequality

$$\operatorname{depth} M_{\mathfrak{p}} \geqslant \min\{n, \dim R_{\mathfrak{p}}\}$$

holds for every prime ideal $\mathfrak{p}$ in R.

Lemma 2.8 (Torsion/Cotorsion)**.** *Let R be a Noetherian local normal ring and $f\colon F \to G$ be an injective homomorphism of free R-modules of ranks c and n, respectively, with $c < n$, and set $d = n - c$. Set $N = \operatorname{coker} f$ and assume that N is locally free in codimension 1. For each p with $0 \leqslant p \leqslant d$, one has*

$$\mathrm{H}_i(\mathscr{D}_p) \cong \begin{cases} \operatorname{tor} \bigwedge^p N & i = d-p+1 \\ \operatorname{cotor} \bigwedge^p N & i = d-p. \end{cases}$$

Remark 2.9. When $p = d$ and $N = \Omega^1_{X,x}$, a stalk of the sheaf of Kähler 1-forms of a normal hypersurface X embedded in an open subset of $\mathbb{C}^{d+1}$, Graf and Yau (the latter in the case of Stein hypersurfaces with isolated singularities) showed that $\operatorname{cotor} \Omega^d_{X,x} \cong \Omega^{d+1}_{X,x}$, while Lemma 2.8 yields $\operatorname{cotor} \Omega^d_{X,x} = \mathrm{H}_0(\mathscr{D}_d) = R/I_1(f)$ for $R = \mathcal{O}_{X,x}$ and $f\colon R^1 \to R^{d+1}$ the Jacobian presentation of $\Omega^1_{X,x}$ (see also [69, Chapter 1]).

Proof. Let $0 \leqslant p \leqslant d$. Consider the following picture of the middle portion of the complex $\mathscr{D}_p$ with the gluing map labelled ν_p and the adjacent differentials labelled α and β.

$$\begin{tikzcd}[column sep=small] \cdots \arrow[r, "\alpha"] & \bigwedge^p \otimes D_0 \arrow[rrr, "\nu_p"] \arrow[dr, "\pi_p"'] & & & (\bigwedge^{d-p})^* \otimes S_0 \arrow[r, "\beta"] & \cdots \\ & & \bigwedge^p N \arrow[r, "\theta_1"] & (\bigwedge^{d-p} N)^* \arrow[ur, "(\pi_{n-p})^*"'] & & \end{tikzcd} \tag{2.10}$$

Indicated below the first row are the cokernel of α and the kernel of β, as well as the induced map θ_1 between them; see Section 1.8 (2). For the latter note also that Hom is left exact and that β is the dual of the differential $\bigwedge^{d-p-1} \otimes D_1 \to \bigwedge^{d-p} \otimes D_0$ in $\mathscr{D}_{d-p}$.

By construction, it is clear that the two homologies of the first row of the diagram above, that is, $\mathrm{H}_i(\mathscr{D}_p)$ for $i = d-p+1$ and $i = d-p$, are isomorphic to the kernel and cokernel of θ_1, respectively. Next we identify those with the desired torsion and cotorsion modules of $\bigwedge^p N$.

First note that the gluing map ν_p comes from the pairing $\bigwedge^p \times \bigwedge^{d-p} \to R$ given in (1.13). The map θ_1 comes from the induced pairing $\bigwedge^p N \times \bigwedge^{d-p} N \to R$ on the quotient modules. This pairing can be seen to be well-defined as follows: Recall that N is defined via the exact sequence $F \xrightarrow{f} G \to N \to 0$; taking p-th exterior powers one obtains an exact sequence of the form $F \otimes \bigwedge^{p-1} G \xrightarrow{\Psi} \bigwedge^p G \to \bigwedge^p N \to 0$, where $\Psi(x \otimes \omega) = f(x) \wedge \omega$ for $x \in F$ and $\omega \in \bigwedge^{p-1} G$. Based on this observation $\bigwedge^p N = \{[\kappa]; \kappa \in \bigwedge^p G\}$ where for $\kappa, \kappa' \in \bigwedge^p G$ we define $[\kappa] = [\kappa']$ if and only if $\kappa - \kappa' = \sum_{i=1}^c f(b_i) \wedge \sigma_i$ for some $\sigma_i \in \bigwedge^{p-1} G$ and $\{b_1, \cdots, b_c\}$ a basis of F. The pairing $\bigwedge^p N \times \bigwedge^{d-p} N \to R$ is defined by sending a pair of equivalence classes $([\kappa], [l])$ to $\delta(\kappa \wedge l \wedge f(b_1) \wedge f(b_2) \wedge \cdots \wedge f(b_c))$, where δ is an orientation on G as in Section 1.4. It is clear from the description of the equivalence classes that the above pairing is well-defined. It is elementary that for any such pairing $\bigwedge^p N \times \bigwedge^{d-p} N \to R$ one gets a commutative triangle

$$\begin{tikzcd}
& (\bigwedge^p N)^{**} \arrow[dr, "\theta_2^*"] & \\
\bigwedge^p N \arrow[ur, "e_{\bigwedge^p N}"] \arrow[rr, "\theta_1"] & & (\bigwedge^{d-p} N)^*
\end{tikzcd}$$

where θ_2^* is the dual of the other map $\theta_2\colon \bigwedge^{d-p} N \to (\bigwedge^p N)^*$ induced by the pairing and $e_{\bigwedge^p N}$ is the natural evaluation map. By the lemma below, the map θ_2^* is an isomorphism. Hence the kernel and cokernel of θ_1 are the same as those of $e_{\bigwedge^p N}$, which are the torsion and cotorsion of $\bigwedge^p N$ since R is a normal local Noetherian ring. $\square$

Lemma 2.11. *In the notation and with the hypotheses above, the map*

$$\theta_2^*\colon (\bigwedge^p N)^{**} \to (\bigwedge^{d-p} N)^*$$

is an isomorphism.

Proof. Consider the map $\theta_2\colon \bigwedge^{d-p} N \to (\bigwedge^p N)^*$, and let $\mathfrak{p}$ be a prime ideal of height at most 1. By hypothesis, $N_\mathfrak{p}$ is free over $R_\mathfrak{p}$, necessarily of rank d since the sequence

$$0 \to F_\mathfrak{p} \to G_\mathfrak{p} \to N_\mathfrak{p} \to 0$$

is exact. So the pairing is perfect there and the maps $(\theta_2)_\mathfrak{p}$ and $(\theta_2^*)_\mathfrak{p}$ are isomorphisms. We are done by Lemma 2.12 applied to the map θ_2^* if we note that the dual W^* of any R-module W satisfies S_2 since R does (for the last statement see [94, Lemma 15.21.11, Tag 0AUY] or [98, Proposition 1.3] and the references therein). $\square$

The following fact is well-known and the proof is an elementary application of depth-counting after localization.

Lemma 2.12. *Let M and N be R-modules which satisfy S_2 and S_1, respectively. Assume that the R-homomorphism $\varphi\colon M \to N$ gives an isomorphism when localized at any prime ideal $\mathfrak{p}$ such that* $\operatorname{ht}\mathfrak{p} \leqslant 1$*. Then φ is an isomorphism.*

Proof. One can see this in two steps, noting that any primes in the supports of the kernel K and cokernel C of φ must have height at least two by hypothesis. First, if K is nonzero, localizing at a minimal prime $\mathfrak{p}$ in the support of K provides an inclusion $K_\mathfrak{p} \subseteq M_\mathfrak{p}$ of a nonzero module of finite length into one of positive depth, which is impossible. Next, if C is nonzero, localizing at a minimal prime $\mathfrak{p}$ in the support of C and depth counting along the exact sequence $0 \to M_\mathfrak{p} \to N_\mathfrak{p} \to C_\mathfrak{p} \to 0$ yield a contradiction. $\square$

3. Torsion and cotorsion of p-differentials

With the tools developed in Section 2, we are now in a position to describe the precise range of non-vanishing of the torsion and cotorsion of $\Omega^p_{X,x}$ for complete intersection singularities. We note that the non-vanishing of the torsion of $\Omega^p_{X,x}$ when $k \leqslant p \leqslant d$ was first observed by Lebelt [70, Satz 0]. What is new here is that our result identifies the range for which the cotorsion is not zero and in particular yields the conclusion that $\Omega^p_{X,x}$ is reflexive if its cotorsion vanishes.

Corollary 3.1. *Let X be either a complex analytic variety or an algebraic variety over an algebraically closed field of characteristic zero. Assume that X is complete intersection at a* *normal point $x \in X$ of dimension $\dim_x X = d > 0$. Suppose x is a singular point of X. Then for $1 \leqslant p \leqslant d$ one has*

$$\begin{aligned} \operatorname{tor} \Omega^p_{X,x} \neq 0, &\quad \text{if and only if} \quad k \leqslant p \leqslant d \\ \operatorname{cotor} \Omega^p_{X,x} \neq 0, &\quad \text{if and only if} \quad k-1 \leqslant p \leqslant d \end{aligned}$$

where k is the codimension of the singular locus of X at x.

Remark 3.2. The torsion and cotorsion for values of p above the dimension are well known for any normal variety: If n denotes the embedding dimension of $\mathcal{O}_{X,x}$, then $\Omega^1_{X,x}$ is generated minimally by n elements. Furthermore, recall that normality implies that $\Omega^1_{X,x}$ is locally free of rank d at primes of height at most 1. This implies that for $p > d$ for the localizations of its p-th exterior power at such primes equal 0, and hence the support of $\Omega^p_{X,x}$ lies in codimension 2 or higher and the exterior power is torsion (by normality there is a nonzero divisor that annihilates it). On the other hand, as $\Omega^p_{X,x}$ has $\binom{n}{p}$ minimal generators, it vanishes if and only if $p > n$. In summary, one has

$$\begin{aligned} \operatorname{tor} \Omega^p_{X,x} &\neq 0 \text{ for } d+1 \leqslant p \leqslant n \\ \operatorname{cotor} \Omega^p_{X,x} &= 0 \text{ for } d+1 \leqslant p \leqslant n \\ \Omega^p_{X,x} &= 0 \text{ for } p > n \end{aligned}$$

As mentioned in the introduction, for torsion the non-vanishing in Corollary 3.1 was known to Lebelt. We present a proof for both the torsion and cotorsion based instead on our symmetry result.

Proof. We use the set-up and notation of Sections 1.1 and 1.3. Setting $R = \mathcal{O}_{X,x}$, the exact sequence from (1.1) becomes

$$0 \longrightarrow R^c \xrightarrow{f} R^n \longrightarrow \Omega^1_{X,x} \longrightarrow 0$$

and consider the generalized Koszul complexes $\mathscr{D}_p = \mathscr{D}_p(f)$ associated to the map f. Since the singularity is studied locally, we may assume that the matrix of f has its entries in the maximal ideal $\mathfrak{m}$ of R.

Note first that $k = \operatorname{ht} I_c(f)$ since the singular locus is given by $V(I_c(f))$.

By Corollary 2.5 and recalling that $d = n - c$, we have

$$\mathrm{H}_i(\mathscr{D}_p) = 0 \iff i > d + 1 - k,$$

Next we apply Lemma 2.8 to the R-module $N = \Omega^1_{X,x}$, noting that it is locally free in codimension 1 since $\mathcal{O}_{X,x}$ is normal of characteristic 0 and hence smooth in codimension 1. Therefore, we get

$$\mathrm{H}_i(\mathscr{D}_p) \cong \begin{cases} \operatorname{tor} \Omega^p_{X,x} & i = d - p + 1 \\ \operatorname{cotor} \Omega^p_{X,x} & i = d - p \end{cases}$$

since by definition $\Omega^p_{X,x} = \bigwedge^p \Omega^1_{X,x}$. Putting these together gives the desired result. □

4. A generalized Lipman-Zariski problem

In this section we briefly recall the Lipman-Zariski conjecture and some key developments towards its resolution (the conjecture is still open). Then we provide solutions to the generalized Lipman-Zariski problems posed by Graf for local complete intersection germs (X, x) whose singular locus has codimension at least three at x. We also review relationships between the Lipman-Zariski conjecture and the generalized Lipman-Zariski problems. Last we illustrate how a solution of the generalized Lipman-Zariski problem would imply the Lipman-Zariski conjecture for the same variety.

4.1. **On the Lipman-Zariski conjecture.** In his very interesting 1965 paper, Lipman stated the following question (originally posed to him by Zariski):

Lipman-Zariski Conjecture ([72, Intro]). Let x be a closed point on an algebraic variety X over a field k of characteristic 0. Let R be the local ring of x in X and let $\Omega^1_{R/k}$ be the module of k-differentials of R. If the dual module of k-derivations $\mathrm{Der}_k(R,R) := \mathrm{Hom}_R(\Omega^1_{R/k}, R)$ is a free R-module, is x a non-singular point of X?

Lipman investigated in *ibid.* the consequences of $\mathrm{Der}_k(R,R)$ being free at x. Assuming that, he proved that x has to be a *normal* point of X. He also obtained a bound on the codimension of the singular locus in a neighborhood of x. Furthermore he discovered an interesting criterion for determining when the module of k-derivations $\mathrm{Der}_k(R,R)$ is free (cf. Proposition 6.2 in *ibid.*)

The papers by Källstrom [59] and Biswas, Gurjar and Kolte [9] contain references and commentaries on the Lipman-Zariski Conjecture. We would like to summarize some of the key developments. The conjecture was shown to be true early on by

- Zariski and Lipman [72, §7] when x is the vertex of a cone in affine 3-space, or when x is the origin on a surface in $\mathbb{C}^3$ described by the equation $z^n = f(x,y)$ with $f(0,0) = 0$,
- Mount and Villamayor for irreducible affine algebraic curves over a field of characteristic 0 (Mount and Villamayor proved the Nakai conjecture for such curves in [79]. See Section 7 in Becker [6] on how the Nakai conjecture implies the Lipman-Zariski conjecture, which was also observed by Rego [84], and see Section 6 in [6] for some information regarding the Nakai conjecture[7]),
- Scheja and Storch for hypersurface singularities [91],
- Moen and Hochster for reduced projective complete intersections [77], [51], and
- Hochster and Platte when X is the spectrum of a graded ring [52], [83],
- Herzog for certain reduced Cohen-Macaulay algebras of dimension at least three [48, Satz 4.2]

Let us also remark that Hochster observed in [52, Remark 2] that if there exists a Cohen-Macaulay counterexample to the (LZ) conjecture, then there exists a Gorenstein counterexample as well. In the same paper, Hochster proposed studying complete intersection surfaces in $\mathbb{C}^4$ with isolated singularities (having in mind the possibility of a counterexample to the (LZ) conjecture).

In the 1980s the (LZ) conjecture was confirmed by

- Steenbrink and van Straten [95] for isolated singularities in higher dimensional complex analytic varieties (dimension bigger than 2),
- Flenner [31] for varieties with non-isolated singularities whose singular locus has codimension at least 3 (these latter two cases were proven using Hodge theory type results; for a somewhat alternate proof see also Kersken [64, (3.3), (3.5)]), and
- Platte [82] and Ishibashi [56] for certain invariant rings under finite group actions.

More recently, there has been new progress as follows.

- Källström announced a proof of the conjecture for affine complete intersections over a field of characteristic 0, cf. [59, Cor 1.3(1)]. We have been unable to follow the proofs of Lemma 2.3(2) and of Proposition 2.2 (in particular the validity of base change in claim (ii) in the proof of the latter) in his paper. He has posted a new version of his 2010 preprint [58] circumventing the usage of Proposition 2.2.

[7]The interested reader may search on the web for the latest developments on the proof of the Nakai conjecture.

- Biswas, Gurjar and Kolte confirmed the (LZ) conjecture for several classes of normal algebraic surfaces; cf. [9, Thm 1.2]. Theorem 1.2(5) in *ibid.* hinges on knowing that the Lipman-Zariski conjecture is true for complete intersections.
- Graf and Bergner-Graf recently proved the conjecture for normal surface singularities of low genus [34], [8].

In [9], the authors mention work by Becker [6, Sec 8] that reduces the proof of the Lipman-Zariski conjecture to proving it in the case of varieties with isolated singularities. And as Steenbrink and van Straten had proved it for higher dimensional isolated singularities the validity of the conjecture depends on whether it holds for surfaces with isolated singularities.

For singularities arising in the minimal model program, progress in establishing the Lipman-Zariski conjecture has been made recently (cf. [23], [35], [40], [50] and the references therein).

4.2. **On Graf's generalized Lipman-Zariski problem.** In the following result, we provide a solution to the (GLZ) problem for complete intersection germs (X, x) of higher codimension whose singular locus has codimension at least three, giving an affirmative answer to the conjecture by Graf in [32]. The rigidity results obtained earlier allow us to modify appropriately Graf's solution of the (GLZ) problem in the hypersurface case to the complete intersection case.

Theorem 4.1. *Let X be either a complex analytic variety or an algebraic variety over an algebraically closed field of characteristic zero. Assume that X is complete intersection at a point $x \in X$ of dimension $\dim_x X = d > 0$. Let the codimension of the singular locus of X at x be at least three. If the sheaf $\Omega_X^{[p]}$ of reflexive Kähler differential p-forms is free at x for some p with $1 \leqslant p \leqslant d-1$, then x is a smooth point of X.*

Proof. Let $R = \mathcal{O}_{X,x}$. We use the set-up/notation of Sections 1.1 and 1.3. Starting with the fundamental exact sequence

$$0 \longrightarrow R^c \xrightarrow{f} R^n \longrightarrow \Omega^1_{X,\,x} \longrightarrow 0$$

consider the generalized Koszul complexes $\mathscr{D}_p = \mathscr{D}_p(f)$ associated to the map f.

Replacing N by $\Omega^1_{X,\,x}$ in the diagram (2.10) from the proof of Lemma 2.8, we obtain the following diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & \bigwedge^p \otimes D_0 \arrow[rr, "\nu_p"] \arrow[dr, "\pi_p"'] & & (\bigwedge^{d-p})^* \otimes S_0 \arrow[r, "\partial_{d-p}"] & (\bigwedge^{d-p-1})^* \otimes S_1 \arrow[r] & \cdots \\
& & \Omega^p_{X,x} \xrightarrow{\theta_1} (\Omega^{d-p}_{X,x})^* \cong \Omega^{[p]}_{X,x} \arrow[ur, "(\pi_{n-p})^*"'] & & &
\end{tikzcd}$$

where π_p is the projection to the coker ∂.

We will assume that x is a singular point of X and obtain a contradiction. Let $\mathcal{H} := \mathrm{H}_{d-p-1}(\mathscr{D}_p)$ be the $(d-p-1)$ homology of the complex $\mathscr{D}_p$. First we prove that $\mathcal{H} = 0$.

Suppose that $\mathcal{H} \neq 0$. Since $\operatorname{Supp} \mathcal{H} \subseteq \operatorname{Sing} R$ by (1.8.4) and the codimension of the singular locus of X at x is at least three, we have $\operatorname{ht}(\mathfrak{p}) \geqslant 3$ for any minimal prime $\mathfrak{p}$ in $\operatorname{Supp}(\mathcal{H}) = V(\operatorname{Ann}(\mathcal{H}))$. By assumption $\Omega^{[p]}_{X,x}$ is free, so, localizing at a minimal prime $\mathfrak{p}$ in $\operatorname{Supp}(\mathcal{H})$, from the right-hand portion of the first row of the above diagram, we obtain an exact sequence of free modules

$$0 \to (\Omega^{[p]}_{X,x})_{\mathfrak{p}} \xrightarrow{(\pi_{n-p})^*} (\bigwedge^{d-p})^*_{\mathfrak{p}} \xrightarrow{(\partial_{d-p})_{\mathfrak{p}}} (\bigwedge^{d-p-1})^*_{\mathfrak{p}} \otimes S_1 \longrightarrow (\operatorname{coker} \partial_{d-p})_{\mathfrak{p}} \to 0.$$

But then the projective dimension of $(\operatorname{coker}\partial_{d-p})_{\mathfrak{p}}$ would be less than or equal to 2 and thus $\operatorname{depth}((\operatorname{coker}\partial_{d-p})_{\mathfrak{p}}) \geqslant 1$ by the Auslander-Buchsbaum formula.

There exists an exact sequence

$$0 \longrightarrow \mathcal{H} \longrightarrow \operatorname{coker}\partial_{d-p} \longrightarrow \operatorname{im}\partial_{d-p-1} \to 0.$$

If we localize this sequence at a minimal prime in the support of $\mathcal{H}$, we obtain a short exact sequence

$$0 \longrightarrow \mathcal{H}_{\mathfrak{p}} \longrightarrow (\operatorname{coker}\partial_{d-p})_{\mathfrak{p}} \longrightarrow (\operatorname{im}\partial_{d-p-1})_{\mathfrak{p}} \to 0.$$

If these modules were non-zero then by Corollary 18.6 b) in [27]

$$\operatorname{depth}(\mathcal{H}_{\mathfrak{p}}) \geqslant \min\left(\operatorname{depth}(\operatorname{coker}\partial_{d-p})_{\mathfrak{p}}, \operatorname{depth}((\operatorname{im}\partial_{d-p-1})_{\mathfrak{p}}) + 1\right) \geqslant 1. \tag{4.2}$$

Note that, as $\mathcal{H}_{\mathfrak{p}} \neq 0$, we know that $(\operatorname{coker}\partial_{d-p})_{\mathfrak{p}} \neq 0$. If $(\operatorname{im}\partial_{d-p-1})_{\mathfrak{p}} = 0$, then $\mathcal{H}_{\mathfrak{p}} \cong (\operatorname{coker}\partial_{d-p})_{\mathfrak{p}}$. But the latter has depth bigger or equal to 1, hence $\operatorname{depth}(\mathcal{H}_{\mathfrak{p}}) \geqslant 1$.

On the other hand, since $\mathfrak{p}$ is minimal among the primes containing $\operatorname{Ann}\mathcal{H}$, by Corollary 2.18 in [27] we know that the $R_{\mathfrak{p}}$ module $\mathcal{H}_{\mathfrak{p}}$ is a non-zero module of finite length. This implies[8] that $\operatorname{Supp}(\mathcal{H}_{\mathfrak{p}}) = \{\mathfrak{p}\, R_{\mathfrak{p}}\} = \operatorname{Ass}(\mathcal{H}_{\mathfrak{p}})$ (by Corollary 9.3.14 in [80]). Since the maximal ideal is associated to $\mathcal{H}_{\mathfrak{p}}$ we know there exists a nonzero $x \in \mathcal{H}_{\mathfrak{p}}$ that is annihilated by the maximal ideal. Therefore $\operatorname{depth}(\mathcal{H}_{\mathfrak{p}}) = 0$ and we have a contradiction to (4.2). Therefore $\mathcal{H} = 0$, that is, $\mathrm{H}_{d-p-1}(\mathscr{D}_p) = 0$.

Next, by Corollary 2.4, this implies that $\mathrm{H}_{d-p}(\mathscr{D}_p) = 0 = \mathrm{H}_{d-p+1}(\mathscr{D}_p)$. Since $N = \Omega^1_{X,\,x}$ is locally free by the assumption on the singular locus of X, Lemma 2.8 applies to yield that $\operatorname{tor}(\Omega^p_{X,x}) = 0 = \operatorname{cotor}(\Omega^p_{X,x})$ and hence that $\Omega^p_{X,x} \cong \Omega^{[p]}_{X,x}$. Since $\Omega^{[p]}_{X,x}$ is free by assumption, by the previous isomorphism we know that $\Omega^p_{X,x}$ is free. But if $\Omega^p_{X,x}$ is free, Graf's Theorem 4.3 below yields that x is a smooth point of X. □

It is well-known that freeness of Ω^1_X at $x \in X$ implies that x is a non-singular point of X (cf. Theorem 1.12, Chapter II in [37] when X is a complex analytic space or Theorem 8.8, Chapter II in [45] when X is an algebraic scheme over a perfect field). Graf generalized the above smoothness criterion in [32, Theorem 1.10] and called it the Very Weak Generalized Lipman-Zariski Theorem (VWGLZ).

Theorem 4.3 (Graf)**.** *Let (X, x) be any singularity of embedding dimension e. If Ω^p_X is free for some p, $1 \leqslant p \leqslant e$, then x is a non-singular point of X.*

Graf also proposed studying a weaker version of the (GLZ) problem:

Weak generalized Lipman-Zariski problem (WGLZ) [32, Question 1.3]: *Let X be a d-dimensional complex variety such that the sheaf $\check{\Omega}^p_X$ of Kähler p-forms modulo torsion is locally free for some p with $1 \leqslant p \leqslant d$. Under what assumptions on X and for which values of p, does local freeness of $\check{\Omega}^p_X$ imply that X is smooth?*

Let us remark that as $\check{\Omega}^p_X \hookrightarrow \Omega^{[p]}_X$ and the two sheaves agree on X_{reg}, it is clear that if the implication in the (GLZ) problem holds for a normal variety and some value of p with $1 \leqslant p \leqslant d-1$, then the implication in the (WGLZ) problem will also hold for this variety and this value of p. Graf proved that the (WGLZ) problem has a solution for hypersurfaces [32, Theorem 1.9]. Similarly to how Graf does so in the hypersurface case, we can also use the rigidity result Corollary 2.4 to provide the following solution to the (WGLZ) problem, also without any restriction on the codimension of the singular locus.

[8] If R is a Noetherian ring and M a finite R-module, then M has finite length if and only if $\operatorname{Supp} M$ consists of maximal ideals if and only if $\operatorname{Ass} M$ consists of maximal ideals if and only if $\operatorname{Ass} M = \operatorname{Supp} M$

Corollary 4.4 (WGLZ for complete intersections)**.** *Let X be either a complex analytic variety or an algebraic variety over an algebraically closed field of characteristic zero. Assume that X is complete intersection at a normal point $x \in X$ of dimension $\dim_x X = d > 0$. If $\check{\Omega}^p_X$ is free at x for some p with $1 \leqslant p \leqslant d$, then x is a non-singular point of X.*

Proof. The proof is identical to Graf's proof for his solution to the weak generalized Lipman-Zariski problem for normal hypersurface singularities, cf. [32, Thm 1.9]. Assume that $\check{\Omega}^p_{X,x}$ is free for some p with $1 \leqslant p \leqslant d$. Recall that normality implies that $\Omega^1_{X,x}$ is locally free at primes of height at most 1.

There exists an injective map $\bar{\theta}\colon \check{\Omega}^p_{X,x} \to \Omega^{[p]}_{X,x}$ that is therefore an isomorphism at primes of height at most 1. By Lemma 2.12 we can conclude that $\Omega^{[p]}_{X,x} \cong \check{\Omega}^p_{X,x}$. Then

$$\operatorname{cotor} \Omega^p_{X,x} = \operatorname{cotor} \check{\Omega}^p_{X,x} = 0$$

But then $\mathrm{H}_{d-p}(\mathscr{D}_p) = 0$ and so by Corollary 2.4 one has $\mathrm{H}_{d-p+1}(\mathscr{D}_p) = 0$. Applying Lemma 2.8, using again the assumption that $\Omega^1_{X,x}$ is locally free at primes of height at most 1, one obtains that $\operatorname{tor} \Omega^p_{X,x} = 0$. Hence $\Omega^p_{X,x} \cong \Omega^{[p]}_{X,x}$. But $\Omega^{[p]}_{X,x} \cong \check{\Omega}^p_{X,x}$ is free by assumption, hence $\Omega^p_{X,x}$ is free. Then x is a non-singular point of X by Theorem 4.3 since $d \leqslant e$. $\square$

Remark 4.5. Some special cases of (WGLZ) were known before. When $p = 1$ and X an affine complete intersection, Corollary 4.4 was proved by Lipman [73, Theorem] without any assumption on the codimension of the singular locus. In the algebraic setting, Scheja and Storch also showed that if k is a field of characteristic zero and R is a Noetherian k-algebra with a universally-finite k-derivation, $\mathfrak{p}$ is a prime ideal in R and $\hat{\Omega}^1_{R/k}$ is the module of universally finite k-differentials then, if $(\hat{\Omega}^1_{R/k})_{\mathfrak{p}}/(\operatorname{tor} \hat{\Omega}^1_{R/k})_{\mathfrak{p}}$ is free, then $R_{\mathfrak{p}}$ is regular [91, Satz 9.2].

4.3. **Relationship between (GLZ) and (LZ).** On a normal complex variety X of pure dimension d which is Gorenstein (i.e., $\Omega^{[d]}_X$ is locally free and X is Cohen-Macaulay), the classical (LZ) conjecture is equivalent to the statement of the (GLZ) problem for $p = d-1$ by Lemma 1.6. Dropping the Gorenstein assumption, Graf showed among other things in Theorem 1.6 in [32], that for a d-dimensional normal germ (X, x) of a complex variety whose singular locus has codimension at least three, if $\Omega^{[d-1]}_X$ is free, then $x \in X$ is the cone over the r-th Veronese embedding of $\mathbb{P}^{d-1}$ for some integer r that divides $d - 1$. Therefore there are only finitely many examples showing that the conclusion to the (GLZ) problem for $p = d - 1$ fails under the previous assumptions.

A solution to the (GLZ) problem will imply the (LZ) conjecture. We restrict our attention to normal varieties X over an algebraically closed field of characteristic zero. As the variety is normal we know that $\Omega^{[p]}_{X,x} = j_*(\Omega^p_{X_{\mathrm{reg}}})_x$. There exists a natural homomorphism[9]

$$\psi : \wedge^p \Omega^{[1]}_{X,x} \to \Omega^{[p]}_{X,x}.$$

Suppose for some normal variety X and for some p with $1 \leqslant p \leqslant \dim_x X - 1$ we know that if $\Omega^{[p]}_{X,x}$ is free then $x \in X$ is non-singular. We will show that the (LZ) conjecture holds for X. Suppose that $T_{X,x}$ is free. Then $\Omega^{[1]}_{X,x} = (T_{X,x})^*$ is free and consequentially $\wedge^p \Omega^{[1]}_{X,x}$ is free and

[9]This homomorphism has been recently used by Du, Gao and Yau [24] to construct a new invariant of isolated singularities and by de Fernex [22] in his study of smooth solutions to the complex Plateau problem.

thus reflexive. But then the map ψ is an isomorphism by Lemma 2.12 since X is normal and both sheaves are reflexive and agree on X_{reg}. Therefore $\Omega^{[p]}_{X,x}$ is free, and so by the assumption we know that $x \in X$ is a non-singular point of X.

4.4. **Further remarks on (GLZ).** Here are a few final notes.

(i) Graf showed in [32, Prop 1.5] that the answer to the (GLZ) problem is negative in general. He constructed isolated terminal singularities (they are actually quotient singularities) for which $\Omega^{[p]}_X$ is free for various values of $p > 1$. These germs are not complete intersections (but they are germs of normal varieties).

(ii) In the proof of the (GLZ) (resp., (WGLZ)) problem for complete intersection germs (X, x) with codimension of the singular locus at least 3 (resp., normal) Graf (and we) made heavily use of the fact that the germs (X, x) were normal. In [72] Lipman proved that if the tangent sheaf is free at x, then the variety X is normal at x. Could it be possible that $\Omega^{[p]}_X$ being free at x would imply that X is normal at x?

(iii) In the proof of the (GLZ) problem for complete intersection germs (X, x) we assumed that the codimension of the singular locus of X is at least three. It would be interesting to see whether this problem holds true if the codimension of the singular locus of X is two, as is the case when (X, x) is normal.

4.5. **A variation of the generalized Lipman-Zariski problem.** One could propose another generalization of the Lipman-Zariski conjecture, using a different kind of sheaves of differential p-forms on X. Let $\pi\colon M \to X$ be an an appropriate desingularization of X with M a complex manifold. Consider the sheaves $\pi_*\Omega^p_M$. Under what assumptions on X and for which values of p with $1 \leqslant p \leqslant d-1$ does local freeness of $(\pi_*\Omega^p_M)$ imply that X is smooth? Kersken showed in [64, 2.1 Satz] that for a reduced, pure-dimensional complex space X, the condition that the stalk of $\pi_*\Omega^1_M$ at x is free is equivalent to x being a smooth point of X. A different proof was given recently by Samuelsson Kalm and Sera [89, Theorem 1.1 (i)]. For normal Kawamata log-terminal singularities this problem is equivalent to the generalized Lipman-Zariski problem (by [40] the sheaves $\pi_*\Omega^p_M$ are reflexive and thus equal to $\Omega^{[p]}_X$).

Samuelsson Kalm and Sera, inspired by work of Barlet ([4], [5]), have studied another variation of the LZ conjecture (see [89, Theorem 1.1 (ii)]). For other regularity results in terms of sheaves related to differentials, see [3] and [10].

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