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THE GEOMETRY OF DOUBLE FOLD MAPS

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Abstract. We study the geometry of a family of singular map germs $(\mathbb{C}^2,0)\to(\mathbb{C}^3,0)$ called *double folds*. As an analogy to David Mond's *fold map germs* of the form

$$f(x,y)=(x,y^2,f_3(x,y)), f_3\in\mathcal{O}_2,$$

double folds are of the form

$$f(x,y)=(x^2,y^2,f_3(x,y)).$$

This family provides lots of interesting germs, such as finitely determined homogeneous corank 2 germs. We also introduce analytic invariants adapted to this family.

1. Introduction

A classification of complex analytic map germs from the plane to 3-space under $\mathcal{A}$-equivalence, that is, changes of coordinates in the source and target, was carried out by David Mond [8]. Like in the work of a taxonomist, Mond's list starts with the simplest singular map germs, the so called *fold maps*. We say that a map germ $f:(\mathbb{C}^2,0)\to(\mathbb{C}^3,0)$ is a fold map if its first two coordinate functions form a *Whitney fold*, $T:(\mathbb{C}^2,0)\to(\mathbb{C}^2,0)$, $T(x,y)=(x,y^2)$. The image of a fold map $f(x,y)=(x,y^2,f_3)$ looks like the graph of the function f_3 'folded' along the OX axis. The third coordinate function of a fold map can be any but, under $\mathcal{A}$-equivalence, we can assume that it is of the form yp, where $p=T^*P$ for some germ P in the ring of germs of functions in two variables $\mathcal{O}_2$. Hence, the normal form of a fold map is

$$f(x,y)=(x,y^2,yp).$$

Fold maps are easy to study because they are germs of corank 1 and because they behave well under the action of the group $G=\{1,i\}$, generated by the reflection $i(x,y)=(x,-y)$. One can see that all lifted double points of a double fold f (that is, pairs $(z,z')\in\mathbb{C}^2\times\mathbb{C}^2$ such that $f(z)=f(z')$ and, if $z=z'$, then f is singular at z) are of the form $(z,i(z))$.

In this work we explore a family which is also related to a group, while it contains lots of interesting corank 2 maps. In general, corank 2 maps are much harder to study than corank 1 ones, but the group action and some ideas lent by the fold case are going to help us. To generate the simplest corank 2 maps for our studies, we can not allow linear terms in f. Thus, we are going to 'fold' twice, once through OX and once through OY axis. We denote $\alpha:(\mathbb{C}^2,0)\to(\mathbb{C}^2,0)$ the *folded hankerchief*

$$\alpha(x,y)=(x^2,y^2).$$

Take the reflections $i_1(x,y)=(-x,y)$ and $i_2(x,y)=(x,-y)$ and the rotation $i_3(x,y)=(-x,-y)$. We write G for the group $\{1,i_1,i_2,i_3\}$. The orbit of any $z\in\mathbb{C}^2$ is $Gz=\alpha^{-1}(\alpha(z))$ and z is a singular point of α if and only if z belongs to $Fix(i_1)\cup Fix(i_2)=OX\cup OY$. Now, related to the group G, we have a family of maps of the form

$$f(x,y)=(x^2,y^2,f_3(x,y)),$$

which we call *double folds*.

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Section 2 covers the basics about double folds. First we compute their multiple point schemes (this was first done by Marar and Nuño-Ballesteros, who introduced double folds in [5]). Then we introduce a decomposition of the multiple point spaces related to the group G. In Section 3 we restrict ourselves to the double fold family and define the notion of DF-stability (and that of SDF-stability). DF-stable singularities are the ones preserved by small perturbations inside the double fold world. We show that the DF-stable singularities are the stable singularities, plus another kind of singularities, namely the standard self tangencies (and also the standard quadruple points in the special double fold case). We introduce an equivalent notion, DF-genericity, to characterize DF-stability in terms of transversality conditions on the facets of the Coxeter complex of the group G. Section 4 deals with DF-stabilizations, where only DF-stable singularities appear. We use these deformations and the decomposition of the multiple point spaces given in 2 to relate certain numbers to double folds. These numbers are candidates for $\mathcal{A}$-invariants (up to a permutation of indices induced by an isomorphism of G). In Section 5 we consider general families of map germs $(\mathbb{C}^n,0)\to(\mathbb{C}^{n+1},0)$, constructed in the same manner as the folds and double folds: choosing a finite map germ $\alpha:\mathbb{C}^n\to\mathbb{C}^n$ and attaching any $(n+1)$-th coordinate function to obtain a map germ of the form (α,f_{n+1}). We find results relating the $\mathcal{A}$-equivalence of this kind of germs to some subgroup of $\mathcal{K}$-equivalence adapted to each α. These results imply that the numbers introduced in section 4 are $\mathcal{A}$-invariant among the finitely determined quasihomogeneous double folds.

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2. Multiple point schemes

Definition 2.1. We call *double fold* (abbreviated as *DF*) any map germ $f:(\mathbb{C}^2,0)\to(\mathbb{C}^3,0)$ of the form $f(x,y)=(x^2,y^2,f_3(x,y))$. The function germ $f_3\in\mathcal{O}_2$ can be written in the form $f_3(x,y) = P_0(x^2,y^2)+xP_1(x^2,y^2)+yP_2(x^2,y^2)+xyP_3(x^2,y^2)$, for some $P_i\in\mathcal{O}_2$. Under $\mathcal{A}$-equivalence, we can eliminate P_0. Then we obtain a *double fold in normal form*

$$f(x,y)=(x^2,y^2,xp_1+yp_2+xyp_3),$$

with $p_i=\alpha^*P_i$, for some $P_i\in\mathcal{O}_2$. We call *special double folds* (abbreviated as *SDF*) the double folds in normal form such that $p_3=0$.

Example 2.2. Fold and double fold families are not exclusive. The cross-cap is usually parameterized as a fold in normal form $(x,y)\mapsto(x,y^2,xy)$, but it can also be regarded as double fold with parameterization $(x,y)\mapsto(x^2,y^2,x+y)$ (see figure 1).

Multiple point spaces were introduced by Mond [9] as a key tool to study map germs

$$(\mathbb{C}^n,0)\to(\mathbb{C}^p,0),\ n<p.$$

Initial papers about map germs $(\mathbb{C}^2,0)\to(\mathbb{C}^3,0)$ (like [7], [8] and [9]) focussed mainly on the case of corank 1, but some recent ones (for instance [5], [6] and the present paper) deal with corank 2 germs. Altough this was done first by Marar and Nuño-Ballesteros, who introduced double folds in [5], we shall sumarize here the computations of some of their multiple point spaces for a better understanding.

Multiple point spaces in the target are computed as described in [10]. Let $f:X\to(\mathbb{C}^{n+1},0)$ be a finite map germ, where X is a n-dimensional Cohen-Macaulay space. Let $f_*\mathcal{O}_X$ denote $\mathcal{O}_X$ as $\mathcal{O}_{n+1}$-module via f. The k-multiple point space in the target is given by the $(k-1)$-th

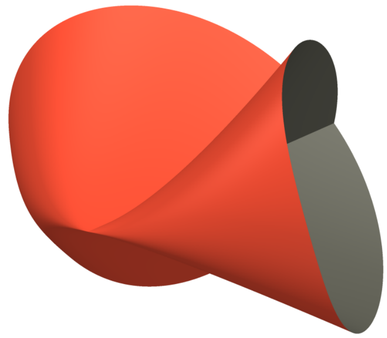

FIGURE 1. The cross-cap is a double fold.

Fitting ideal of the module $f_*\mathcal{O}_X$ defined next: Take a presentation of $f_*\mathcal{O}_X$, that is, an exact sequence

$$\mathcal{O}^p_{n+1} \xrightarrow{\lambda} \mathcal{O}^q_{n+1} \xrightarrow{\varphi} f_*\mathcal{O}_X \longrightarrow 0.$$

The matrix $M(f)$ which represents λ is called a *presentation matrix* for $f_*\mathcal{O}_X$. The k-th Fitting ideal of $f_*\mathcal{O}_X$ is the ideal $F_k(f)$ generated by the minors of size $\min(p,q)-k$ of $M(f)$ if $k<\min(p,q)$, and $F_k(f)=\mathcal{O}_{n+1}$ otherwise. The following method to compute certain presentation matrices can be found in [10, Section 2.2]: Assume $f=(f_1,\dots,f_{n+1}):X\to\mathbb{C}^{n+1}$ is such that $\tilde{f}=(f_1,\dots f_n):X\to(\mathbb{C}^n,0)$ is finite. If $g_1,\dots g_r$ are generators of $\tilde{f}_*\mathcal{O}_X$, then they are generators of $f_*\mathcal{O}_X$ too. Therefore, we obtain an epimorphism $\varphi:\mathcal{O}^r_{n+1}\to\mathcal{O}_X$ which sends the canonical vector e_i to the generator g_i. For any $1\le i\le r$, there exist germs $a_{ij}\in\mathcal{O}_n$, $1\le j\le r$ such that $f_{n+1}g_i=\sum_{j=1}^r \tilde{f}^*a_{ij}g_j$. If $X_1,\dots,X_{n+1}$ denote the variables in $\mathbb{C}^{n+1}$ and δ_{ij} is the Kronecker's delta function, then the matrix $M(f)$ with entries $a_{ij}(X_1,\dots,X_n)-\delta_{ij}X_{n+1}$ is a presentation matrix for $f_*\mathcal{O}_X$.

Given a double fold $f(x,y)=(x^2,y^2,xp_1+yp_2+xyp_3)$, we use the method explained above to find $M(f)$. Take $g_1=1, g_2=x, g_3=y, g_4=xy$ as generators of $\alpha_*\mathcal{O}_2$. For $i=1$, we have $f_3g_1=xp_1+yp_2+xyp_3=0\cdot g_1+\alpha^*P_1g_2+\alpha^*P_2g_3+\alpha^*P_3g_4$. Therefore, the elements of the first column of the matrix are $-Z,P_1,P_2,P_3$. After computing f_3g_i for $i=2,3,4$, we get the matrix

$$M(f)=\begin{pmatrix} -Z & XP_1 & YP_2 & XYP_3\\ P_1 & -Z & YP_3 & YP_2\\ P_2 & XP_3 & -Z & XP_1\\ P_3 & P_2 & P_1 & -Z\end{pmatrix},$$

where P_i represents $P_i(X,Y)$. Since $M(f)$ has size 4×4, f has no points with multiplicity greater than 4. For special double folds, the space of quadruple points in the image is given by the ideal $F_3(f)=\langle P_1(X,Y),P_2(X,Y),Z\rangle$ and $F_2(f)=(F_3(f))^2$. Hence, triple points of special double folds appear concentrated at quadruple points.

We define the source double point space $D(f)$ as the zero locus of the pull back $f^*(F_1(f))$. In the double fold case we have $D(f)=V\big((p_1+yp_3)(p_2+xp_3)(xp_1+yp_2)\big)$. Its defining ideal factorizes as the product of the ideals $I_1:=\langle p_1+yp_3\rangle$, $I_2:=\langle p_2+xp_3\rangle$ and $I_3:=\langle xp_1+yp_2\rangle$. Analogously, the source triple point space, defined as $V(f^*(F_2(f)))$, is given by the product of the ideals $I_{1,2}:=\langle p_1+yp_3,p_2+xp_3\rangle$, $I_{1,3}:=\langle p_1+yp_3,p_2-xp_3\rangle$ and $I_{2,3}:=\langle p_2+xp_3,p_1-yp_3\rangle$. Quadruple points (again with the structure induced by the target) are given by the zeros of

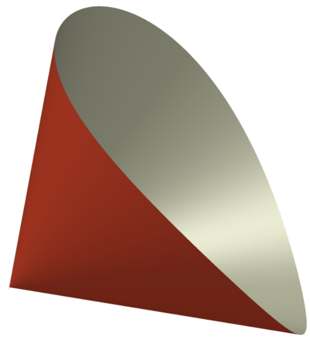

FIGURE 2. The image of a double cone.

$I := \langle p_1, p_2, p_3 \rangle$. We observe the collapse of triple points in the special double fold case: If p_3 equals zero, then the radical of $I_{1,2}I_{1,3}I_{2,3}$ is $\langle p_1, p_2 \rangle$, which is the ideal defining the quadruple point locus.

Definition 2.3. Given a double fold $f = (\alpha, xp_1 + yp_2 + xyp_3)$, we decompose the double point locus as the union of $D_i(f), 1 \leq i \leq 3$, with $D_i(f) := V(I_i)$ and the triple point space as the union of $D_{i,j}(f), 1 \leq i < j \leq 3$, with $D_{i,j} := V(I_{i,j})$. Finally, we denote $D_{1,2,3}(f) = V(I_{1,2,3})$ the quadruple point locus.

Remark 2.4. It's immediate that:

- w belongs to $D_l(f)$ if and only if $i_l(w)$ does so. Moreover $f(w) = f(i_l(w))$.
- w belongs to $D_{l,k}(f)$ if and only if $i_l(w)$ and $i_k(w)$ do so. Moreover
$$f(w) = f(i_l(w)) = f(i_k(w)).$$
- w belongs to $D_{1,2,3}(f)$ if and only if $i_1(w), i_2(w)$ and $i_3(w)$ do so. Moreover
$$f(w) = f(i_1(w)) = f(i_2(w)) = f(i_3(w)).$$

Example 2.5. Take the family $(x, y) \mapsto (x^2, y^2, \lambda_1 x + \lambda_2 y + \lambda_3 xy), \lambda_i \in \mathbb{C}$. Assume $\lambda_3 \neq 0$, then its double points are the following: $D_1(f) = V(\lambda_1 + y\lambda_3)$ is the line $y = -\lambda_1/\lambda_3$, which is obviously i_1-invariant, $D_2(f)$ is the i_2-invariant line $x = -\lambda_2/\lambda_3$ and, if $\lambda_2 \neq 0$, then $D_3(f)$ is the i_3-invariant line $y = -\lambda_1 x/\lambda_2$. We find the triple points where these lines meet:

$$D_{1,2}(f) = \{(-\lambda_2/\lambda_3, -\lambda_1/\lambda_3)\}, \quad D_{1,3}(f) = \{(\lambda_2/\lambda_3, -\lambda_1/\lambda_3)\}$$

and

$$D_{2,3}(f) = \{(-\lambda_2/\lambda_3, \lambda_1/\lambda_3)\}$$

(see figure 3). In the case $\lambda_3 = 0$ we have a special double fold. Thus, its triple points should appear collapsed at quadruple points, with equations $p_1 = p_2 = 0$. Since $p_1 = \lambda_1$ and $p_2 = \lambda_2$, the appearance of quadruple point forces $\lambda_1 = \lambda_2 = 0$ and hence, the map is the folded hankerchief. Another map that fits into this family is the so called *double cone* $(x, y) \mapsto (x^2, y^2, xy)$ (Figure 2). It parameterizes the cone $Z^2 = XY$, but does so in a two-to-one way. Indeed, its double point branch $D_3(f) = V(xp_1 + yp_2) = V(0)$ equals $\mathbb{C}^2$.

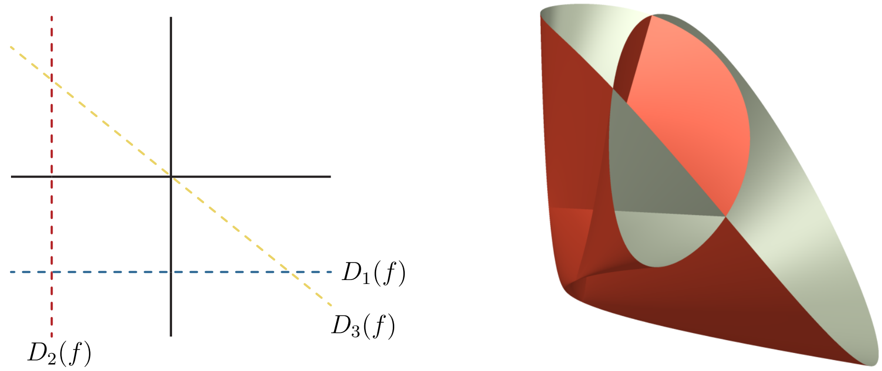

FIGURE 3. Image and double points of a double fold (see Example 2.5).

3. DOUBLE FOLD STABILITY

In this section we study the singularity types which are characteristic of the double folds. By a singularity type we mean an $\mathcal{A}$-equivalence class of multigerms $f : (\mathbb{C}^2, S) \to (\mathbb{C}^3, y)$. A singularity type, represented by f_0, is stable if it appears in any section $f_s, s \in \mathbb{C}$, of any deformation of f_0. It is well known that in the case $\mathbb{C}^2 \to \mathbb{C}^3$ the stable types are transverse double points, triple points and cross-caps. Our goal is to make a version of the concept of stability adapted specifically for double folds. Some types, despite not being stable, are preserved by deformations which occur inside the double fold world. We call them DF-stable types and these deformations DF-deformations. This concept can be adapted to the special double fold case and we shall use the notation (S)DF to refer respectively to both, the double fold and the special double fold case.

Definition 3.1. We call *(S)DF-deformation* of f_0 any germ $F : (\mathbb{C}^2 \times \mathbb{C}, 0) \to (\mathbb{C}^3, 0)$ of the form $F(x, t) = f_t(x)$, such that the germ $f_t : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ is a (special) double fold for all t. We call (S)DF-unfolding any map germ $F : (\mathbb{C}^2 \times \mathbb{C}, 0) \to (\mathbb{C}^3 \times \mathbb{C}, 0)$ of the form $F(x, t) = (f_t(x), t)$ such that $f_t(x)$ is a (S)DF-deformation.

Definition 3.2. We say a multigerm ξ is *(S)DF-stable* if any (S)DF-unfolding F of a multigerm f of type ξ is trivial. That is, if there exist some unfoldings of the identity Ψ, Φ such that $f \times id = \Psi \circ F \circ \Phi$. A (special) double fold $f : U \to \mathbb{C}^3$ is (S)DF-stable if all its multigerms at $f^{-1}(f(w)), w \in U$ are (S)DF-stable.

Remark 3.3. Every stable type is (S)DF-stable.

A priori, it might seem difficult to identify all possible (S)DF-stable maps, but a better understanding of the map α will help us to do so. The map α is the invariant map associated to the Coxeter group G (see [3] for Coxeter group theory). For any Coxeter Group there is a Coxeter complex, in this case $\mathcal{C} := \{\mathbb{C}^2 \setminus (OX \cup OY), OX \setminus \{0\}, OY \setminus \{0\}, \{0\}\}$. The Coxeter complex stratifies the space in a way such that the behavior of the group, and thus that of α, changes whenever we go from a facet to another. Consequently, much information about a double fold is contained in the way its multiple point spaces meet the Coxeter complex. The following proposition is an example of this.

Lemma 3.4. *The germ of a fold $f(x, y) = (x^2, y^2, xp_1 + yp_2 + xyp_3)$ centered at a point $w \in \mathbb{C}^2$ is a cross-cap if and only if one of the three conditions is verified:*

i) $w \in OX \setminus \{0\}$ *and the restricted function* $(p_2 + xp_3)|_{OX}$ *has a simple zero at* w.
ii) $w \in OY \setminus \{0\}$ *and the restricted function* $(p_1 + yp_3)|_{OY}$ *has a simple zero at* w.
iii) $w = 0$ *and* $p_1(w) \neq 0 \neq p_2(w)$.

Proof. A monogerm of map from $\mathbb{C}^2$ to $\mathbb{C}^3$ is a cross-cap if and only if its source double point space is smooth (this follows immediately from [6, Theorem 3.3]). Since cross-caps are singular monogerms, they lie on $OX \cup OY$. Assume first that $w \in OX \setminus \{0\}$. Looking at the 2×2 minors of the differential of f at w it follows that f is singular at w if and only if $p_2 + xp_3$ vanishes at w. Now the source double point space of the germ of f at w is $D_2(f)$, given by the zeros of $p_2 + xp_3$ (notice that, by Remark 2.4, the branches of double points $D_1(f)$ and $D_3(f)$ at $OX \setminus \{0\}$ produce multigerms, not monogerms). Therefore, the double point space of the germ of f at w is smooth if and only if the Milnor number of the germ of function $p_2 + yp_3$ at w equals 0. This happens if and only if at least one of the partial derivatives $\frac{\partial p_2 + xp_3}{\partial x}$ and $\frac{\partial p_2 + xp_3}{\partial y}$ does not vanish at w. Since p_2 and p_3 are functions of x^2 and y^2, we deduce that $\frac{\partial p_2 + xp_3}{\partial y}$ vanishes at OX. Hence, f has a cross-cap at $w \in OX \setminus \{0\}$ if and only $p_2 + xp_3$ vanishes at w and $\frac{\partial p_2 + xp_3}{\partial x}$ does not, that is, if and only if the restriction $(p_2 + xp_3)|_{OX}$ has a simple zero at w. The case $w \in OY \setminus \{0\}$ is analogous. Assume now $w = 0$. The source double point of f is the germ of complex space given by the zeros of $(p_1 + xp_3)(p_2 + p_3)(xp_1 + yp_2)$. The non vanishing of p_1 and p_2 at 0 is a necessary and sufficient condition for this germ of complex space to be smooth. □

Points where the source double point space meets the facets of the Coxeter complex in a generic way are called (S)DF-generic. We shall determine the different possible (S)DF-generic singularities and then show that they are exactly the (S)DF-stable singularities. Let us state the (S)DF-genericity conditions rigorously:

Definition 3.5. Let $f = (\alpha, xp_1 + yp_2 + xyp_3) : U \to \mathbb{C}^3$ be a double fold. We say that a point $w \in \mathbb{C}^2$, that belongs to a facet $C \in \mathcal{C}$, is *DF-generic* if:

1) $(p_1 + yp_3)|_C$, $(p_2 + xp_3)|_C$ and $(xp_1 + yp_2)|_C$ are transverse to $\{0\}$ at w, with the exception $(xp_1 + yp_2)|_{\{0\}}$ (notice that no double fold in canonical form could verify this transversality condition).
2) $(p_1 + yp_3, p_2 + xp_3)|_C$, $(p_1 + yp_3, p_2 - xp_3)|_C$ and $(p_2 + xp_3, p_1 - yp_3)|_C$ are transverse to $\{(0,0)\}$ at w.
3) w is not a quadruple point of f.

A double fold $f : U \to \mathbb{C}^3$ is DF-generic if all points $w \in U$ are DF-generic

Conditions 1) and 2) adapt to the special double fold case just taking $p_3 = 0$ but, since quadruple points are more likely to appear at special double folds (they are the zeros of just two equations in $\mathbb{C}^2$), the SDF genericity conditions don't include condition 3).

Definition 3.6. Let $f = (\alpha, xp_1 + yp_2) : U \to \mathbb{C}^3$ be a special double fold, we say that a point $w \in \mathbb{C}^2$, that belongs to a facet $C \in \mathcal{C}$, is *SDF-generic* if:

1) $p_1|_C$, $p_2|_C$ and $(xp_1 + yp_2)|_C$ are transverse to $\{0\}$ at w, with the exception $(xp_1 + yp_2)|_{\{0\}}$.
2) $(p_1, p_2)|_C$ is transverse to $\{(0,0)\}$ at w.

A special double fold $f : U \to \mathbb{C}^3$ is SDF-generic if all points $w \in U$ are SDF-generic

Remark 3.7. It is immediate from its defining ideals that every point belonging to $D_1(f) \cap OX$ or to $D_2(f) \cap OY$ must belong to $D_3(f)$ too. It is also immediate that $D_3(f)$ always crosses the facet $\{0\}$. Apart from these exceptions, which are inherent to the double fold family, the genericity conditions imply the following more geometric assertion: Given a regular stratification of $D(f)$, the strata have their expected dimension (double points have dimension 1 and triple (quadruple) points have dimension 0) and are transverse to the strata of the Coxeter complex $\mathcal{C}$.

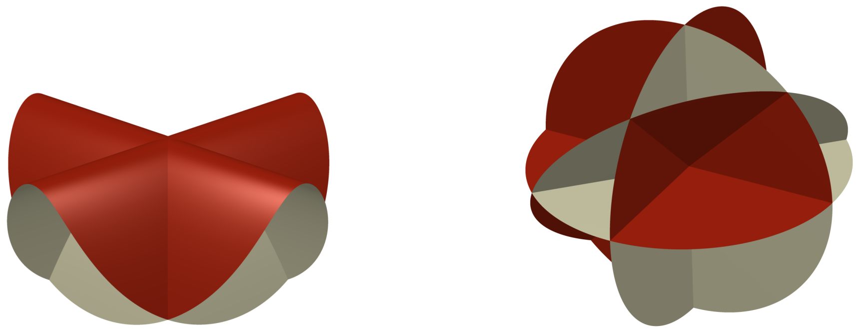

FIGURE 4. Images of a standard self tangency and a standard quadruple point.

Let us introduce our new candidates to be (S)DF-generic multigerms.

Definition 3.8. We call a *standard self tangency* the multigerm formed by two smooth branches with Morse contact. We call a *standard quadruple point* the multigerm formed by four smooth branches such that every three of them meet transversally. These singularities are depicted in Figure 4.

Proposition 3.9. *All standard self tangencies are $\mathcal{A}$-equivalent. All standard quadruple points are $\mathcal{A}$-equivalent.*

Proof. In [12] it is shown that the $\mathcal{A}$-class of a bigerm with smooth branches is determined by the contact type of its branches. Since there is only one contact class of Morse type, all standard self tangencies are equivalent. Let f be a multigerm of standard quadruple point. Any three of its branches form a triple point and there is only one $\mathcal{A}$-class of triple points. Therefore, there exists a change of coordinates that takes f to a multigerm whose branches send (x,y) respectively to $(x,y,0),(x,0,y),(0,x,y)$ and $g(x,y)$ for some regular monogerm g with $\operatorname{Im} g=\{U_1X+U_2Y+U_3Z=0\}$, $U_i\in\mathcal{O}_3$. The plane tangent to $\operatorname{Im} g$ is determined by the equation $t_1X+t_2Y+t_3Z=0$, with $t_i=U_i(0,0)$. If we assume $t_1=0$, then the intersection of the tangent plane with the branches $\{Y=0\}$ and $\{Z=0\}$ is the line $\{Y=Z=0\}$. This contradicts the transversality of these three branches. We deduce $t_1\neq 0$ and, analogously, $t_2\neq 0\neq t_3$. The change $(X,Y,Z)\mapsto(U_1X,U_2Y,U_3Z)$ defines a germ of diffeomorphism that takes our multigerm to the one with image $\{XYZ(X+Y+Z)=0\}$. Now the four branches of our multigerm send (x,y) to $(u_1x,u_2y,0),(u_1x,0,u_3y),(0,u_2x,u_3y)$ and

$$(a_1x+b_1y,a_2x+b_2y,-(a_1+b_1)x-(a_2+b_2)y),$$

where $u_i=U_i\circ f$, and a_1,a_2,b_1,b_2 are some function germs in $\mathcal{O}_2$. We take germs of diffeomorphisms at the source, at the four different points where our multigerm is centered. The first three diffeomorphisms send (x,y) respectively to $(x/u_1,y/u_2)$, $(x/u_1,y/u_3)$ and $(x/u_2,y/u_3)$. The fourth diffeomorphism is the inverse of the germ

$$(x,y)\mapsto(a_1x+b_1y,a_2x+b_2y).$$

These four source coordinate changes take the multigerm to one multigerm defined by four branches sending (x,y) respectively to $(x,y,0),(x,0,y),(0,x,y)$ and $(x,y,-x-y)$. Hence, all germs of standard quadruple point are equivalent. □

Lemma 3.10. *The (S)DF-generic points are regular points, transverse double points, cross-caps, standard self tangencies and triple points (resp. standard quadruple points).*

Proof. Given a (special) double fold f and a point $w = (x_0, y_0) \in \mathbb{C}^2$ satisfying the (S)DF-genericity conditions, we shall determine the type of singularity of the multigerm of f at $f^{-1}(f(w))$. First of all, notice that singular points lie in $OX \cup OY$ and the genericity condition 2) implies that all triple points belong to the facet $\mathbb{C}^2 \setminus (OX \cup OY)$. Hence, from genericity condition 1), together with Lemma 3.4, it follows that all points where f is singular are cross-caps.

Now suppose that f is regular at w and the point w belongs to $D_l(f)$, $1 \leq l \leq 3$. Take the vector fields along f defined by the cross product $\eta := \frac{\partial f}{\partial x} \times \frac{\partial f}{\partial y}$ and $\eta_l = \eta \circ i_l$, for $1 \leq l \leq 3$. The branches of the multigerm of f at w and $i_l w$ are transverse unless $\eta \times \eta_l$ or, equivalently, $\xi_l := (\eta - \eta_l) \times (\eta + \eta_l)$ vanish at w. We study the different cases a), b) and c), where w belongs to $D_1(f), D_2(f)$ and $D_3(f)$ respectively.

Case a) Let w belong to $D_1(f)$, then we have:
$\xi_1|_w = 4x_0y_0(4x_0\frac{\partial(xp_1+xyp_3)}{\partial y}|_w, 4y_0\frac{\partial(xp_1+xyp_3)}{\partial x}|_w, (\frac{\partial(xp_1+xyp_3)}{\partial y}|_w\frac{\partial yp_2}{\partial x}|_w - \frac{\partial(xp_1+xyp_3)}{\partial x}|_w\frac{\partial yp_2}{\partial y}|_w))$.
Suppose first $w \notin OX \cup OY$, then $\xi_1|_w$ vanishes if and only if $\frac{\partial p_1+yp_3}{\partial x}|_w = \frac{\partial p_1+yp_3}{\partial y}|_w = 0$, that is, if and only if $p_1 + yp_3$ is not transverse to $\{0\}$ at w. This is in contradiction with the first genericity condition. Suppose now $w \in OX \cup OY$ and notice $w \notin OY$ because it would be a singular point. Thus, we have $w \in OX \setminus \{0\}$. We claim that the bigerm of f at $(\pm x_0, 0)$ forms a standard self tangency at $(X_0, 0, 0)$, where $X_0 = x_0^2$. The genericity conditions imply that P_1 has a simple zero at $(X_0, 0)$ and P_2 does not vanish at $(X_0, 0)$. Let the germ of $f : \mathbb{C}^2 \to \mathbb{C}^3$ at x_0 parameterize one of the branches and let $\phi : \mathbb{C}^3 \to \mathbb{C}$ be the germ at $(X_0, 0, 0)$ which defines the other branch implicitly. Then, following Montaldi [11], the contact between the branches is given by the $\mathcal{K}$-class of the composition $\phi \circ f$. The branches are given by $(Z^2 \pm \sqrt{X}P_1)^2 - YP_2^2 \pm 2Y\sqrt{X}P_2P_3 - XYP_3^2 = 0$. After choosing the preimage $(x_0, 0)$ and composing we get the function $4x(p_1 + yp_3)(xp_1 + yp_2)$, which is of Morse type in $(x_0, 0)$. Therefore, the multigerm of f at $(\pm x_0, 0)$ is a standard self tangency.

Case b) is symmetric interchanging indices 1 and 2, and OX and OY.

Case c) If $w \in D_3(f)$, then we can assume $w \in D_3(f) \setminus (OX \cup OY)$ because otherwise $w \in D_1(f) \cup D_2(f)$. We have

$$\xi_3|_w = 4x_0y_0\left(4x_0\frac{\partial(xp_1 + yp_2)}{\partial y}|_w,\ -4y_0\frac{\partial(xp_1 + yp_2)}{\partial x}|_w,\right.$$
$$\left.\left(\frac{\partial(xp_1 + yp_2)}{\partial y}|_w\frac{\partial xyp_3}{\partial x}|_w - \frac{\partial(xp_1 + yp_2)}{\partial x}|_w\frac{\partial xyp_3}{\partial y}|_w\right)\right),$$

which vanishes if and only if $\frac{\partial xp_1+yp_2}{\partial x}$ and $\frac{\partial xp_1+yp_2}{\partial y}$ vanish in w, if and only if $xp_1 + yp_2$ is not transverse to $\{0\}$ at w.

As we have seen before, all triple points (and therefore all quadruple points) belong to the facet $\mathbb{C}^2 \setminus (OX \cap OY)$, where the second genericity condition implies that the branches are transverse. Therefore, all triple points are transverse (respectively all quadruple points are standard quadruple points). □

Lemma 3.11. *Every (special) double fold admits a (S)DF-deformation f_t defined in a neighborhood $U \times V$ of $(0,0) \in \mathbb{C}^2 \times \mathbb{C}$ such that, for every $t \in V$, f_t is (S)DF-generic.*

Proof. Let $f = (\alpha, xp_1 + yp_2 + xyp_3)$ be a representative defined at some neighborhood U of the origin. we consider DF-deformations of the form $f_{a,b,c} = (\alpha, x(p_1 + a) + y(p_2 + b) + xy(p_3 + c))$. Denote Δ the analytic space of the points $(a, b, c) \in \mathbb{C}^3$, such that for some point w in U the map $f_{a,b,c}$ does not satisfy all genericity conditions. We claim that Δ is a proper subspace of $\mathbb{C}^3$. Take the first function, $p_1 + yp_3$, of the first condition and any facet of the Coxeter complex $C \in \mathcal{C}$. We consider the map $\psi : C \times \mathbb{C}^3 \to \mathbb{C}$, given by $\psi(w, a, b, c) = p_1(w) + a + y(p_3(w) + c)$. This is clearly a submersion. Therefore, the Basic Transversality Lemma [2, Lemma 4.6] tells us that, for almost every $(a,b,c) \in \mathbb{C}^3$, the map $f_{a,b,c}$ is transverse to 0. We can proceed analogously for all the maps given by the DF-genericity conditions to finally show that, for almost every $(a,b,c) \in \mathbb{C}^3$, all the genericity conditions hold at every point in U. Thus, Δ is a proper subspace. Hence, we can find some particular $(a,b,c) \in \mathbb{C}^3$ and some neighborhood V of 0, such that $t(a,b,c) \notin \mathbb{C}^3$ for any $t \in V$. If we take the DF-deformation

$$f_t(x,y) = (x^2, y^2, x(p_1 + ta) + y(p_2 + tb) + xy(p_3 + tc)$$

defined at $U \times V$, then for any $t \in V$, the map f_t has only DF-generic points at U. The special double fold case is analogous. □

Theorem 3.12. *(S)DF-stable and (S)DF-generic points are the same. As a consequence:*
The DF-stable singularities are

- *Transverse double points, cross-caps and triple points.*
- *Standard self tangencies.*

The SDF-stable singularities are

- *Transverse double points and cross-caps.*
- *Standard self tangencies.*
- *Standard quadruple points.*

Proof. By Lemma 3.11, the DF-stable singularities must be DF-generic. Now take a DF-generic point w of a double fold f. If w is a transverse double point, a cross-cap or a triple point, then it is stable and, hence, DF-stable. Suppose w is a standard self tangency and Let $F = (f_t, t)$ be a DF-unfolding of f. Assume $w \in D_1(f)$. Then, as we have seen in the proof of Lemma 3.10, the point belongs to $OX \setminus \{0\}$, $(p_1 + yp_3)|_{OX}$ has a simple zero at w and the functions $p_2 + xp_3$ and $xp_1 + yp_2$ don't vanish at w. Therefore, there exist a neighborhood $U \times V$ of $(w, 0)$ and a curve of points $w_t \in U \cap OX \setminus \{0\}$, with $t \in V$ and $w_0 = w$, such that $(p_1 + yp_3)|_{OX}$ has a simple zero and the functions $p_2 + xp_3$ and $xp_1 + yp_2$ don't vanish at w_t. All this points are also standard self tangencies and, since they are all $\mathcal{A}$-equivalent by 3.9, they are DF-stable. The proof holds in the special case and is analogous for standard quadruple points. □

4. Counting (S)DF-stable points

A usual way to study germs is to count the number of stable 0-dimensional points of each type which appear in a stabilization of the original germ. One can show that these numbers can be obtained as the dimension (as $\mathbb{C}$-vector space) of certain local algebras related to the different stable 0-dimensional types. We adapt these techniques specifically to (S)DF-deformations and to (S)DF-stable points.

Definition 4.1. We call *(S)DF-stabilization* any (S)DF-deformation F such that there exists a neighborhood $U \times V$ of $(0,0) \in \mathbb{C}^2 \times \mathbb{C}$ such that, for every $t \in V$, f_t is (S)DF-stable.

Remark 4.2. By Lemma 3.11 and Theorem 3.12, every (special) double fold admits a (S)DF-stabilization.

Definition 4.3. For any (special) double fold f we define:

$ST_i(f) = \frac{1}{2} \dim_{\mathbb{C}} \mathcal{O}_1 / j_i^* I_i(f)$, for $i = 1, 2$,
$C_i(f) = \dim_{\mathbb{C}} \mathcal{O}_1 / j_k^* I_i(f)$, for $(i,k) = (1,2), (2,1)$,
$T(f) = \dim_{\mathbb{C}} \mathcal{O}_2 / I_{1,2}(f)$ (in the special DF case: $QD(f) = \frac{1}{4} \dim_{\mathbb{C}} \mathcal{O}_2 / \langle p_1, p_2 \rangle$),

where j_1 and j_2 denote the inclusions of OX and OY into $\mathbb{C}^2$ respectively.

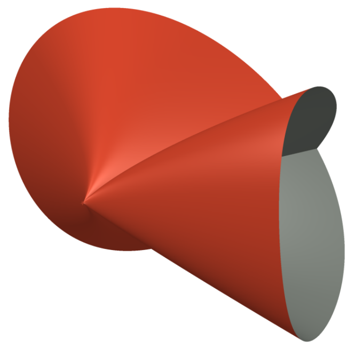

FIGURE 5. A non SDF-stable special double fold (see Example 4.6).

Remark 4.4. We don't include indices for the triple points in different branches because the complex spaces $D_{i,j}(f)$ are all isomorphic, since $\mathcal{O}_2/I_{1,2}(f) \cong \mathcal{O}_2/I_{1,3}(f) \cong \mathcal{O}_2/I_{2,3}(f)$ via the isomorphisms induced by i_1 and i_2.

Proposition 4.5. *Let $ST_i(f), C_i(f)$ and $T(f)$ (respectively $QD(f)$) be finite. Let f_s be a (S)DF-stabilization of f. Then, for a small enough $s \neq 0$, the following equalities hold:*

$ST_i(f) = \#$ *standard self tangencies* $f(D_i(f_s))$,
$C_i(f) = \#$ *cross-caps in* $D_i(f_s) \setminus \{0\}$,
$T(f) = \#$ *triple points of* f_s ($QD(f) = \#$ *standard quadruple points of* f_s).

Proof. Take the zero locus of the different ideals which appear in 4.3. If $ST_i(f), C_i(f)$ and $T(f)$ (respectively $QD(f)$) are finite, then the spaces are 0-dimensional. In this case, the codimension of any of these spaces equals the number of generators of its defining ideal. Hence, the spaces are complete intersection and the Principle of Conservation of Number (see for example [4, Theorem 6.4.7]) applies to them. We only need to check that, if the multigerm of f_s at $f_s^{-1}(f_s(w))$ is (S)DF-generic, then the numbers are 1 if it is the considered singularity and 0 otherwise. ☐

Example 4.6. Take the family of special double folds

$$(x, y) \mapsto (x^2, y^2, x(a_1x^2 + b_1y^2 - c_1) + y(a_2x^2 + b_2y^2 - c_2)).$$

The double points $D_1(f)$ and $D_2(f)$ are given by $a_1x^2 + b_1y^2 = c_1$ and $a_2x^2 + b_2y^2 = c_2$. In the real case, these two spaces collapse to the point 0 if $c_1 = c_2 = 0$. For the germ

$$f(x, y) = (x^2, y^2, x(x^2 + 2y^2) + y(2x^2 + y^2))$$

(Figure 5), we can easily compute $ST_1 = 1/2 \dim_{\mathbb{C}}(\mathcal{O}_1/\langle x^2\rangle) = 1$ and similarly $ST_2 = 1$ and $C_1 = C_2 = 2$. We also have $QD = 1/4 \dim_{\mathbb{C}}(\mathcal{O}_2/\langle 2x^2 + y^2, 2y^2 + x^2\rangle) = 1$. Now take the 2-parameter deformation $f_t = (x^2, y^2, x(x^2 + 2y^2 - t_1) + y(2x^2 + y^2 - t_2))$, where $t = (t_1, t_2)$. We see that, for almost every fixed t with $t_1 \neq 0 \neq t_2$, f_t is a SDF-stable map where we can find (Figure 6) a standard self tangency and two cross-caps along $D_1(f_t) \setminus \{0\}$ and the same on $D_2(f_t) \setminus \{0\}$. We also see the cross-cap at $f_t(0)$ and a standard quadruple point. For these good values of t we can also see that, apart from the restrictions on $D_i(f) \cap D_3(f)$ and $D_3(f) \cap \{0\}$ (see Remark 3.7), the regular stratification of $D(f_t)$ is transverse to every facet of the Coxeter complex.

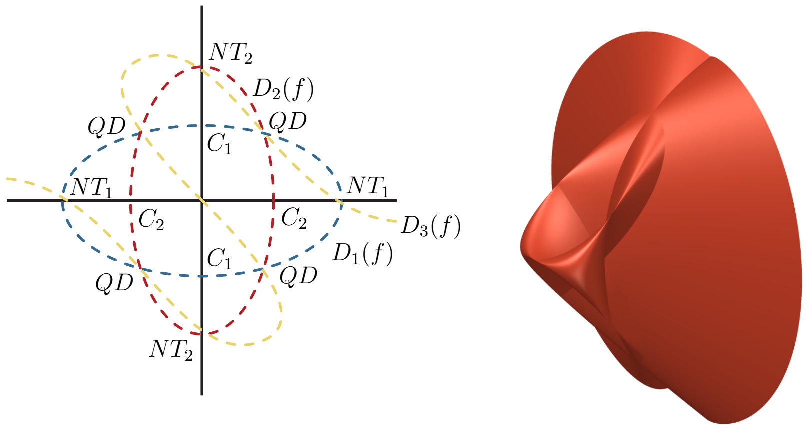

FIGURE 6. A SDF-stable deformation of the surface shown in figure 5.

Example 4.7. If we take the double cone $(x, y) \mapsto (x^2, y^2, xy)$ of Example 2.5, we see easily that $ST_i = 0$, $C_i = \dim_{\mathbb{C}} \mathcal{O}_1/\mathfrak{m}_1 = 1$ for $i = 1, 2$ and $T = \dim_{\mathbb{C}} \mathcal{O}_2/\mathfrak{m}_2$. In fact

$$f_t(x, y) = (x^2, y^2, tx + ty + xy)$$

is a DF-stabilization of the double cone where each section $t \neq 0$ has, as in figure 3, three cross-caps (one in $D_1(f) \setminus \{0\}$, one in $D_2(f) \setminus \{0\}$ and the other at 0) and one triple point.

Remark 4.8. Let $ST(f)$, $C(f)$, $T(f)$ (and respectively $QD(f)$ in the special case) denote the number of standard self tangencies, cross-caps, triple points (and standard quadruple points) respectively that appear taking a (S)DF-stabilization of f. It is known that $C(f)$ and $T(f)$ are well defined $\mathcal{A}$-invariants of f. It is immediate that $Q(f)$ is also invariant, because any map showing a quadruple point can be deformed (outside the special double fold world) into another that shows 4 triple points. It is not clear whether ST is $\mathcal{A}$-invariant or not, but it is easy to see that the numbers with indices $ST_i(f)$ and $C_i(f)$ are not. Given a double fold f, we can interchange x and y at the source and then permute the first two coordinates at the target to obtain a new double fold, say g, such that $ST_1(f) = ST_2(g)$, $ST_2(f) = N_1(g)$, $C_1(f) = C_2(g)$ and $C_2(f) = C_1(g)$. Apart from the permutation of indices 1 and 2 that this change of coordinates produces, examples suggest that changes of coordinates don't make the singularities jump from one space $D_i(f)$ to another one. Therefore, the numbers $ST_i(f)$ and $C_i(f)$ seem to be $\mathcal{A}$-invariant, modulo a simultaneous permutation of all indices 1 and 2 (and that would make ST $\mathcal{A}$-invariant). However, we have only succeeded in showing it for finitely determined quasi homogeneous double folds (Corollary 5.6).

5. $\mathcal{A}$-EQUIVALENCE AND $\mathcal{K}^\alpha$-EQUIVALENCE

The aim of this section is to mimic a result of David Mond [8, Theorem 4.1:1], which shows the coincidence between the $\mathcal{A}$-equivalence of folds $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$, $f(x, y) = (x, y^2, f_3)$ and some easier to use equivalence of the third coordinate function, f_3, defined ad hoc. This equivalence is given by a subgroup of $\mathcal{K}$ called $\mathcal{K}^T$ which behaves well with respect to the Whitney Fold $T(x, y) = (x, y^2)$. We take, instead of the Whitney Fold, any finite mapping $\alpha : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ and consider mappings $(\alpha, f_{n+1}) : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$. We define the group $\mathcal{K}^\alpha$ and the generalization of one direction of Mond's results comes easily: $\mathcal{K}^\alpha$-equivalence for f_{n+1} implies $\mathcal{A}$-equivalence for (α, f_{n+1}).

As usual, we denote $\mathcal{R}_n$ the group of germs of biholomorphism $\varphi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$.

Definition 5.1. Let $\alpha : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ be a finite germ. We define $\mathcal{R}^\alpha$ as the subgroup consisting of the germs $\varphi \in \mathcal{R}_n$ such that there exists a germ $\hat{\varphi} \in \mathcal{R}_n$ such that

$$\hat{\varphi} \circ \alpha = \alpha \circ \varphi.$$

We say that two germs $g, h \in \mathcal{O}_n$ are $\mathcal{K}^\alpha$-*equivalent* if there exist a function $\kappa \in \alpha^*\mathcal{O}_2, \kappa(0) \neq 0$ and a germ of diffeomorphism $\varphi \in \mathcal{R}^\alpha$, such that

$$g = \kappa \cdot h \circ \varphi.$$

Example 5.2. Let $\alpha(x, y) = (x^2, y^2)$, then any diffeomorphism $\varphi \in \mathcal{R}^\alpha$ is of the form

$$\varphi(x, y) = (x\varphi_1, y\varphi_2) \quad \text{or} \quad \varphi(x, y) = (y\varphi_1, x\varphi_2)$$

for some functions $\varphi_1, \varphi_2 \in \alpha^*\mathcal{O}_2$, $\varphi_i(0, 0) \neq 0$. In particular, if $g, h \in \mathbb{C}[x, y]$ are homogeneous $\mathcal{K}^\alpha$-equivalent polynomials, the factors κ and $h \circ \varphi$ are homogeneous. Hence, on one hand, κ is a constant in $\mathbb{C}^*$. On the other hand, since φ is a diffeomorphism, both h and $h \circ \varphi$ are homogeneous of the same degree. We can replace φ by its linear part without changing the composition. Thus, we can assume that φ is of the form $(x, y) \mapsto (ax, by)$ or $(x, y) \mapsto (by, ax)$.

Lemma 5.3. *A diffeomorphism $\varphi \in \mathcal{R}_n$ belongs to $\mathcal{R}^\alpha$ if and only if the algebras $\alpha^*\mathcal{O}_n$ and $(\alpha \circ \varphi)^*\mathcal{O}_n$ are equal.*

Proof. Let $\varphi \in \mathcal{R}^\alpha$ with $\hat{\varphi} \circ \alpha \circ \varphi = \alpha$. Any function $h \circ \alpha \in \alpha^*\mathcal{O}_n$ is equal to

$$(h \circ \hat{\varphi}) \circ \alpha \circ \varphi \in (\alpha \circ \varphi)^*\mathcal{O}_n.$$

Now take $h \circ \alpha \circ \varphi \in (\alpha \circ \varphi)^*\mathcal{O}_n$. This function is equal to $h \circ \hat{\varphi}^{-1} \circ \hat{\varphi} \circ \alpha \circ \varphi = (h \circ \hat{\varphi}^{-1}) \circ \alpha \in \alpha^*\mathcal{O}_n$.

Now suppose that the two sub-algebras above are equal, then there exist some functions $\hat{\varphi}_i$ such that $\alpha_i = \hat{\varphi}_i \circ \alpha \circ \varphi$. Take $\hat{\varphi} = (\hat{\varphi}_1, \ldots, \hat{\varphi}_n)$. Then we have $\alpha = \hat{\varphi} \circ \alpha \circ \varphi$. As α is finite and φ is a biholomorphism, α and $\alpha \circ \varphi$ have the same finite multiplicity. Therefore $\hat{\varphi}$ must have multiplicity 1, and hence is a biholomorphism. □

Theorem 5.4. *Let $\alpha : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ be a finite germ and f_{n+1}, g_{n+1} be two $\mathcal{K}^\alpha$-equivalent functions of $\mathcal{O}_n$, then the map germs $(\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ $f = (\alpha, f_{n+1})$ and $g = (\alpha, g_{n+1})$ are $\mathcal{A}$-equivalent.*

Proof. $f \sim_{\mathcal{K}^\alpha} g$ implies that there exists $\theta_\alpha : (\mathbb{C}^n \times \mathbb{C}, 0) \to (\mathbb{C}, 0)$ of the form

$$\theta_\alpha(X, Z) = \theta(\alpha(X), Z)$$

for some germ of function θ and such that $\theta_\alpha(0, \cdot)$ is a germ of biholomorphism, and there exists $\varphi \in \mathcal{R}_n^\alpha$ such that $g(X) = \theta_\alpha(X, f \circ \varphi(X))$. Since $\varphi \in \mathcal{R}_n^\alpha$, then there exists some germ of biholomorphism $\hat{\varphi}$ such that $\alpha = \hat{\varphi} \circ \alpha \circ \varphi$. We define $\psi_1 : \mathbb{C}^{n+1} \to \mathbb{C}^n$ by $\psi_1 = \hat{\varphi} \circ \pi_1$ and $\psi_2 = \theta \circ (\psi_1, \pi_2)$, where π_i represents the projection over the i-th component of $\mathbb{C}^n \times \mathbb{C}$. Now we define $\psi = (\psi_1, \psi_2) : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^{n+1}, 0)$ and, for every $X \in \mathbb{C}^n$, we have

$$\psi \circ (\alpha, f) \circ \varphi(X) = \big(\hat{\varphi}(\alpha(\varphi(X))), \theta(\hat{\varphi}(\alpha(\varphi(X))), f(\varphi(X)))\big) =$$

$$\big(\alpha(X), \theta_\alpha(X, f(\varphi(X)))\big) = (\alpha, g)(X).$$

As a consequence of $\hat{\varphi}$ and $\theta_\alpha(X, \cdot)$ being biholomorphisms, we have that ψ is a biholomorphism. □

Again, examples suggest that the converse of Theorem 5.4 also holds: $\mathcal{A}$-equivalence of (α, f_{n+1}) and (α, g_{n+1}) implies $\mathcal{K}^\alpha$-equivalence of f_{n+1} and g_{n+1}. However we have not succeed in proving this in general. It was proved by Mond in [8] that it holds when α is the Whitney Fold. We have only succeeded in showing it for finitely determined quasihomogeneous double folds.

It is shown in [5] that any quasihomogeneous double fold must be a homogeneous one. There are only two ways to obtain a homogeneous double fold $f(x,y) = (\alpha, xp_1 + yp_2 + xyp_3)$. One is $p_3 = 0$ and the other $p_1 = p_2 = 0$. Every finitely determined double fold must have a reduced double point space, which is given by $(p_1+yp_3)(p_2+xp_3)(xp_1+yp_2) = 0$. We deduce immediately that every finitely determined quasihomogeneous double fold must be, in fact, a homogeneous special double fold.

Theorem 5.5. *Let $f = (\alpha, f_3)$ and $g = (\alpha, g_3)$ be $\mathcal{A}$-equivalent finitely determined quasihomogeneous double folds , then f_3 and g_3 are $\mathcal{K}^\alpha$-equivalent.*

Proof. Suppose there exist ψ and φ such that $g = \psi \circ f \circ \varphi$. Denote by φ_{i,x_j} the derivative of the i-th component with respect to the variable x_j. Taking into account that $p_1, p_2 \in \mathfrak{m}^2$, the 2-jet of the first two coordinate functions of the equality $g = \psi \circ f \circ \varphi$ gives us

$$\begin{aligned} x^2 &= \psi_{1,X}(\varphi_{1,x}^2 x^2 + \varphi_{1,x}\varphi_{1,y}xy + \varphi_{2,y}^2 y^2) + \psi_{1,Y}(\varphi_{2,x}^2 x^2 + \varphi_{2,x}\varphi_{2,y}xy + \varphi_{2,y}^2 y^2),\\ y^2 &= \psi_{2,X}(\varphi_{1,x}^2 x^2 + \varphi_{1,x}\varphi_{1,y}xy + \varphi_{2,y}^2 y^2) + \psi_{2,Y}(\varphi_{2,x}^2 x^2 + \varphi_{2,x}\varphi_{2,y}xy + \varphi_{2,y}^2 y^2).\end{aligned}$$

Since $d\varphi$ is invertible, we have $\varphi_{1,x}\varphi_{2,y} \neq 0$ or $\varphi_{1,y}\varphi_{2,x} \neq 0$. In the first case from the equations we obtain $\varphi_{1,y} = \varphi_{2,x} = 0$ and, in the second case $\varphi_{1,x} = \varphi_{2,y} = 0$. Suppose we are in the first case (the second one is analogous). Then the differential of φ is of the form $d\varphi(u,v) = (au, bv)$ for some $a, b \in \mathbb{C}^*$.

Notice that w is a source double point of g if and only if it is so for $f \circ \varphi$, if and only if $\varphi(w)$ is a source double point of f. Since f and g are finitely determined, their double point spaces are reduced and thus $\varphi|_{D(g)} : D(g) \to D(f)$ is an isomorphism between complex space germs. We claim that $\varphi|_{D_3(g)}$ is an isomorphism between $D_3(g)$ and $D_3(f)$. We proceed by reduction to the absurd: suppose there is a irreducible component R of $D_3(g)$, such that $\varphi(R) \not\subset D_3(f)$. For example, suppose $\varphi(R) \subset D_1(f)$ (the other case, $\varphi(R) \subset D_2(f)$, is analogous). Since f and g are finitely determined, their diagonal double points are isolated and thus, since $R \subset D_3(g)$ and $\varphi(R) \subset D_1(f)$, we have $\varphi(i_3(R)) = i_1(\varphi(R))$. Let (u,v) be the tangent vector to the curve germ R, we have the equality $d\varphi(i_3(u,v)) = i_1(d\varphi(u,v))$, that is $(-au, -bv) = (-au, bv)$. The last equality implies (u,v) is a horizontal vector. Since g is homogeneous, the equation which defines R is also homogeneous and, thus, it is independent of x. This is implies that y divides $xq_1 + yq_2$, which in turn implies that y divides q_1. Then y^2 divides $q_1q_2(xq_1 + yq_2)$. This is a contradiction, because g is finitely determined and, thus, $D(g) = V(q_1q_2(xq_1 + yq_2))$ must be reduced.

Now we have the isomorphism of complex spaces $\varphi|_{D_3(g)} : D_3(g) \to D_3(f)$, that is, we have the equality $\langle g_3 \rangle = \varphi^* \langle f_3 \rangle$. This implies the existence of a function h, with $h(0,0) \neq 0$, such that $g_3 = h \cdot f_3 \circ \varphi$. Since g_3 y f_3 are homogeneous, we can take the diffeomorphism $\tilde{\varphi} = d\varphi$ and the constant $\kappa = h(0,0) \neq 0$ and get $g_3 = \kappa \cdot f_3 \circ \varphi$. Moreover, as we have seen before, $\tilde{\varphi}$ is a diagonal linear change and thus it belongs to $\mathcal{R}^\alpha$. □

Notice that the $\mathcal{K}^\alpha$-equivalence of f_3 and g_3 splits into two simultaneous equivalences between P_1, P_2 and Q_1, Q_2. In the diagonal case we get an expression

$$xQ_1(x^2, y^2) + yQ_2(x^2, y^2) = \kappa a x P_1(a^2x^2, b^2y^2) + \kappa b y P_2(a^2x^2, b^2y^2).$$

This is equivalent to $Q_1(x,y) = \kappa a P_1(a^2x, b^2y)$ and $Q_2(x,y) = \kappa b P_2(a^2x, b^2y)$. In the antidiagonal case we obtain the expression

$$xQ_1(x^2,y^2) + yQ_2(x^2,y^2) = \kappa ay P_1(a^2y^2, b^2x^2) + \kappa bx P_2(a^2y^2, b^2x^2),$$

which is equivalent to $Q_1(x,y) = \kappa b P_2(a^2y, b^2x)$ and $Q_2(x,y) = \kappa a P_1(a^2y, b^2x)$. Now the next corollary follows immediately.

Corollary 5.6. *Let f and g be two $\mathcal{A}$-equivalent quasihomogeneous finitely determined special double folds, then:*

$ST_i(f) = ST_j(g)$,
$C_i(f) = C_i(g)$,
$QD(f) = QD(g)$,
$\mu(D_i(f)) = \mu(D_j(g))$,

where $j = i$ in the diagonal case, and in the antidiagonal the pairs (i,j) are $(1,2)$, $(2,1)$, $(3,3)$.

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