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SOME REMARKS ABOUT THE TOPOLOGY OF CORANK 2 MAP GERMS FROM $\mathbb{R}^2$ TO $\mathbb{R}^2$

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ABSTRACT. Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a finitely determined map germ. The link of f is obtained by taking a small enough representative $f : U \subset \mathbb{R}^2 \to \mathbb{R}^2$ and the intersection of its image with a small enough sphere S^1_ϵ centered at the origin in $\mathbb{R}^2$. We will use Gauss words to classify topologically corank 2 map germs. In particular, we will center our attention in map germs that belong to the Thom-Boardman class $\Sigma^{2,0}$.

1. INTRODUCTION

In a previous paper [9] we defined the Gauss word, which is a complete topological invariant for a finitely determined map germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ and we used it to classify corank 1 map germs. The following logical step is to try to extend this classification to germs of corank 2. This classification is also motivated by the fact that, as we will see in proposition 3.7, some examples of links are not realizable by corank 1 map germs, even if $|\deg(f)| \leq 1$ (see figure 1).

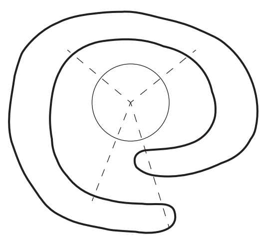

FIGURE 1.

This classification was completed for the $\Sigma^{2,0}$ Thom-Boardman class in the case of $\mathcal{K}$ - equivalence following Mather's techniques of classification (see for example [5]) and Nishimura proved in [10] that, dealing with $\mathcal{K}$ - $\mathcal{C}^0$ - classes, the absolute value of $\deg(f)$ becomes a complete topological invariant. In the complex case, we can find related results in [7] and a full clasification for weighted homogeneous map germs from $\mathbb{C}^2$ to $\mathbb{C}^2$ in an article of T.Gaffney and D.Mond in [4].

The fact that we are not able to consider our germs as 1-parameter unfoldings of functions, as we did in the corank 1 case, makes things to become much more complex. The absolute value of the topological degree does not have to be necessarily less or equal than 1 and although our Gauss words continue being a complete topological invariant, since their links are not constituted as the union of 2 curves (as we did in [9]) the simplifications of letters are not allowed anymore.

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In this work, we will classify corank 2 map germs with some additional convenient restrictions. Firstly we will suppose that f is of type $\Sigma^{2,0}$ (that is, f has corank 2, but the pair (f, df) has corank 0 at the origin). Departing from this point, we will establish a prenormal form of this kind of germs by using their $\mathcal{A}^2$-classes and Eisenbud-Levine formula([2]) will let us to compute their topological degree. As final step we will consider particular cases and under some restrictions on the number of monomials which appear in the second coordinate germ, we will obtain the different topological classes that we have in each case.

2. The link of a finitely determined map germ

We say that two smooth maps $f : M \to N$ and $g : M' \to N'$ between smooth manifolds are $\mathcal{A}$-equivalent if there exist diffeomorphisms $\phi : M \to M'$ and $\psi : N \to N'$ such that $g = \psi \circ f \circ \phi^{-1}$. If ϕ, ψ are homeomorphisms instead of diffeomorphisms, then we say that f, g are topologically equivalent.

In the same way, two smooth map germs $f, g : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ are $\mathcal{A}$-equivalent if there exist diffeomorphism germs $\phi, \psi : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ such that $g = \psi \circ f \circ \phi^{-1}$. If ϕ, ψ are homeomorphisms instead of diffeomorphisms, then we say that f, g are topologically equivalent.

We say that $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ is k-determined if for any map germ g with the same k-jet, we have that g is $\mathcal{A}$-equivalent to f. We say that f is finitely determined if it is k-determined for some k.

Let $f : U \to V$ be a smooth proper map, where $U, V \subset \mathbb{R}^2$ are open subsets. We denote by $S(f) = \{p \in U : Jf_p = 0\}$ the singular set of f, where Jf is the Jacobian determinant. It is a consequence of the Whitney's work [12] that f is *stable* if and only if the following two conditions hold:

(1) 0 is a regular value of Jf, so that $S(f)$ is a smooth curve in U.
(2) The restriction $f|_{S(f)} : S(f) \to V$ is an immersion with only transverse double points, except at isolated points, where it has simple cusps.

We denote $\Delta(f) = f(S(f))$ and we define $X(f)$ as the closure of $f^{-1}(\Delta(f)) \setminus S(f)$. If f is stable, then $S(f)$ is a smooth plane curve and $\Delta(f), X(f)$ are plane curves whose only singularities are simple cusps or transverse double points.

Given a finitely determined map germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$, if it is real analytic, we can consider its complexification $\hat{f} : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$. It is well known that $\hat{f}$ is also finitely determined as a complex analytic map germ. Then, by the Mather-Gaffney geometric criterion [11], it has an isolated instability. In other words, we can find a small enough representative $\hat{f} : U \to V$, where U, V are open sets, such that

(1) $\hat{f}^{-1}(0) = \{0\}$,
(2) the restriction $\hat{f}|_{U \setminus \{0\}}$ is stable.

From the condition (2), both the cusps and the double folds are isolated points in $U \setminus \{0\}$. By the curve selection lemma [6], we deduce that they are also isolated in U. Thus, we can shrink the neighbourhood U if necessary and get a representative such that $\hat{f}|_{U \setminus \{0\}}$ is stable with only simple folds. Coming back to the real map f, we have the following immediate consequence.

Corollary 2.1. *Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a finitely determined map germ. Then there is a representative $f : U \to V$, where $U, V \subset \mathbb{R}^2$ are open sets, such that*

(1) $f^{-1}(0) = \{0\}$,
(2) $f : U \to V$ *is proper,*
(3) *the restriction $f|_{U \setminus \{0\}}$ is stable with only simple folds.*

We finish this section with an important result due to Fukuda [3], which tell us that any finitely determined map germ, $f:(\mathbb{R}^n,0)\to(\mathbb{R}^p,0)$, with $n\le p$, has a conic structure over its link. In order to simplify the notation, we only state the result in our case $n=p=2$.

Given $\epsilon>0$, we denote:

$$S^1_\epsilon=\{x\in\mathbb{R}^2:\|x\|^2=\epsilon\},\quad D^2_\epsilon=\{x\in\mathbb{R}^2:\|x\|^2\le\epsilon\}.$$

and given a map germ $f:(\mathbb{R}^2,0)\to(\mathbb{R}^2,0)$ we consider a representative $f:U\to V$ and put:

$$\widetilde{S}^1_\epsilon=f^{-1}(S^1_\epsilon),\quad \widetilde{D}^2_\epsilon=f^{-1}(D^2_\epsilon).$$

Theorem 2.2. *Let $f:(\mathbb{R}^2,0)\to(\mathbb{R}^2,0)$ be a finitely determined map germ. Then, up to $\mathcal{A}$-equivalence, there is a representative $f:U\to V$ and $\epsilon_0>0$, such that, for any ϵ with $0<\epsilon\le\epsilon_0$ we have:*

(1) *$\widetilde{S}^1_\epsilon$ is diffeomorphic to S^1.*
(2) *The map $f|_{\widetilde{S}^1_\epsilon}:\widetilde{S}^1_\epsilon\to S^1_\epsilon$ is stable, in other words, it is a Morse function all of whose critical values are distinct.*
(3) *$f|_{\widetilde{D}^2_\epsilon}$ is topologically equivalent to the cone of $f|_{\widetilde{S}^1_\epsilon}$.*

Definition 2.3. Let $f:(\mathbb{R}^2,0)\to(\mathbb{R}^2,0)$ be a finitely determined map germ. We say that the stable map $f|_{\widetilde{S}^1_\epsilon}:\widetilde{S}^1_\epsilon\to S^1_\epsilon$ is the *link* of f, where f is a representative such that (1), (2) and (3) of theorem 2.2 hold for any ϵ with $0<\epsilon\le\epsilon_0$. This link is well defined, up to $\mathcal{A}$-equivalence.

Since any finitely determined map germ is topologically equivalent to the cone of its link, we have the following immediate consequence.

Corollary 2.4. *Two finitely determined map germs $f,g:(\mathbb{R}^2,0)\to(\mathbb{R}^2,0)$ are topologically equivalent if their associated links are topologically equivalent.*

3. Gauss words

In this section we recall briefly (for more information and examples see [9]) how we define an adapted version of the Gauss word in our particular case of study and some consequences of such definition.

Definition 3.1. Let $\gamma:S^1\to S^1$ be a stable map, that is, such that all its singularities are of Morse type and its critical values are distinct. We fix orientations in each S^1 and we also choose base points $z_0\in S^1$ in the source and $a_0\in S^1$ in the target.

Suppose that γ has r critical values labeled by r letters $a_1,\dots,a_r\in S^1$ and let us denote their inverse images by $z_1,\dots,z_k\in S^1$. We assume they are ordered such that $a_0\le a_1<\dots<a_r$ and $z_0\le z_1<\dots<z_k$ and following the orientation of each S^1.

We define a map $\sigma:\{1,\dots,k\}\to\{a_1,\dots,a_r,\overline{a}_1,\dots,\overline{a}_r\}$ in the following way: given $i\in\{1,\dots,k\}$, then $\gamma(z_i)=a_j$ for some $j\in\{1,\dots,r\}$; we define $\sigma(i)=a_j$, if z_i is a regular point and $\sigma(i)=\overline{a}_j$, if z_i is a singular point (i.e., the bar $\overline{a}_j$ is used to distinguish whether the inverse image of the critical value is regular or singular). We call *Gauss word* to the sequence $\sigma(1)\dots\sigma(k)$.

For instance, the link of the cusp $f(x,y)=(x,xy+y^3)$ has two critical values with four inverse images and the associated Gauss word is $a\overline{b}\overline{a}b$ (see figure 2).

It is obvious that the Gauss word is not uniquely determined, since it depends on the chosen orientations and base points in each S^1. Different choices will produce the following changes in the Gauss word:

(1) a cyclic permutation in the letters $a_1,\dots,a_r$;
(2) a cyclic permutation in the sequence $\sigma(1)\dots\sigma(k)$;

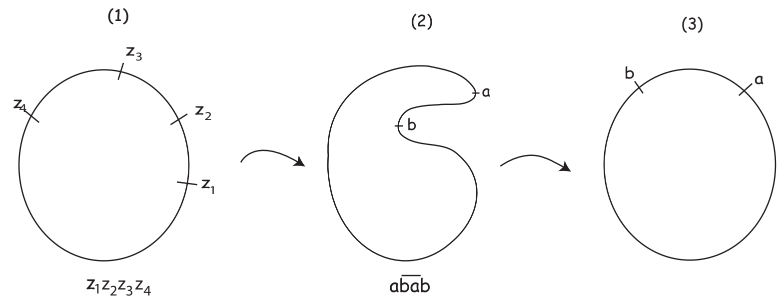


FIGURE 2.

(3) a reversion in the set of the letters $a_1, \ldots, a_r$;
(4) a reversion in the sequence $\sigma(1) \ldots \sigma(k)$.

We say that two Gauss words are equivalent if they are related through these four operations. Under this equivalence, the Gauss word is now well defined.

In order to simplify the notation, given a stable map $\gamma : S^1 \to S^1$, we denote by $w(\gamma)$ the associated Gauss word and by $\simeq$ the equivalence relation between Gauss words. We also denote by $\deg(\gamma)$ the topological degree. Then, we can state the main result of this section (see [9]).

Theorem 3.2. *Let $\gamma, \delta : S^1 \to S^1$ be two stable maps. Then γ, δ are topologically equivalent if and only if*

$$\begin{cases} w(\gamma) \simeq w(\delta), & \text{if } \gamma, \delta \text{ are singular,} \\ \\ |\deg(\gamma)| = |\deg(\delta)|, & \text{if } \gamma, \delta \text{ are regular.} \end{cases}$$

Given a finitely determined map germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$, we denote by $w(f)$ the Gauss word of its link and by $\deg(f)$ the local topological degree.

If $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ is a finitely determined map germ, then we can compute Gauss word of the link of f just by looking at the relative position of the branches of the three curves $S(f)$, $\Delta(f)$ and $X(f)$.

Example 3.3. Let us consider the finitely determined map germ $f(x, y) = (x, y^3 - x^2y)$. The discriminant $\Delta(f)$ has a tree structure with one vertex at the origin and 4 adjacent edges labeled by 4 letters $a_1, \ldots, a_4$. Analogously, $S(f) \cup X(f)$ has also a tree structure with one vertex at the origin and 8 adjacent edges labeled by $Z_1, \ldots, Z_8$. We assume that the edges are well ordered $a_1 < \cdots < a_4$ and $Z_1 < \cdots < Z_8$ with respect to the chosen base points and orientations in the source and the target. We define the map $\sigma : \{1, \ldots, 8\} \to \{a_1, \ldots, a_4, \overline{a}_1, \ldots, \overline{a}_4\}$ in the following way: given $i \in \{1, \ldots, 8\}$, then $\gamma(Z_i) = a_j$ for some $j \in \{1, \ldots, 4\}$; we define $\sigma(i) = a_j$, if $Z_i \subset X(f)$ and $\sigma(i) = \overline{a}_j$, if $Z_i \subset S(f)$. Then, $\sigma(1) \ldots \sigma(8)$ is equal to the Gauss word of the link of f, obtaining in this case the word $a_1\overline{a}_2\overline{a}_1a_2a_3\overline{a}_4\overline{a}_3a_4$ (see figure 3).

As a direct consequence, we have the following corollary.

Corollary 3.4. *Let $f, g : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be two finitely determined map germs. Then, if f and g are topologically equivalent, their links are topologically equivalent.*

Now,by using theorem 3.2 and corollaries 2.4 and 3.4 we can state the following result:

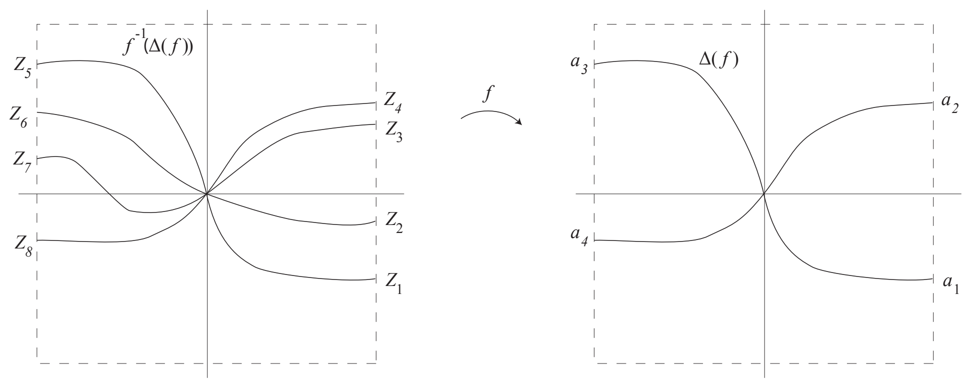


Figure 3.

Corollary 3.5. *Let $f, g : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be two finitely determined map germs. Then f, g are topologically equivalent if and only if*

$$\begin{cases} w(f) \simeq w(g), & \text{if } f, g \text{ are singular outside the origin,} \\ \\ |\deg(f)| = |\deg(g)|, & \text{if } f, g \text{ are regular outside the origin.} \end{cases}$$

Remark 3.6. If f is regular outside the origin and $|\deg(f)| = r$, then f is topologically equivalent to the germ $z \to z^r$, with $z = x + iy$.

Before finishing this section, let us state a result that will give us a necessary condition that a stable map $\gamma : S^1 \longrightarrow S^1$ should verify to be the link of a corank 1 map germ.

Proposition 3.7. *Any finitely determined map germ $f : (\mathbb{R}^2, 0) \longrightarrow (\mathbb{R}^2, 0)$ of corank 1, with link γ, verifies that*

$$\operatorname{mult}(\gamma) = \begin{cases} 0, & \text{if } \deg(f) = 0, \\ 1, & \text{if } \deg(f) = \pm 1. \end{cases}$$

We will define the multiplicity of a stable map $\gamma : S^1 \longrightarrow S^1$ as $\operatorname{mult}(\gamma) = min_{p \in S^1} \operatorname{mult}(p)$, with $\operatorname{mult}(p) = \#\gamma^{-1}(p)$.

Proof. The three possible values of the topological degree of f are a consequence of a known result (see for example [9]). Let us suppose that $f(x, y) = (x, g_x(y))$, with

$$g_x(y) = y^n + a_{n-2}(x)y^{n-2} + \cdots + a_1(x)y.$$

If $\deg(f) = 0$, n is even, $n - 1$ is odd and, as a consequence, the both curves g_x^+, g_x^-, that will form the link of f will have both an odd number of folds. Thus, γ will not be surjective and $\operatorname{mult}(\gamma) = 0$. If $\deg(f) = \pm 1$, n is odd, $n - 1$ is even and the union of both partial curves will completely fill S^1, so $\operatorname{mult}(\gamma) = 1$. □

4. Topological classification of map germs of type $\Sigma^{2,0}$

In this section of the chapter we will classify corank 2 map germs, $f : (\mathbb{R}^2, 0) \longrightarrow (\mathbb{R}^2, 0)$ which are of type $\Sigma^{2,0}$.

First of all we will state a result that will give us two prenormal forms of map germs of this type.

Theorem 4.1. *Let $f : (\mathbb{R}^2, 0) \longrightarrow (\mathbb{R}^2, 0)$ a corank 2 map germ of type $\Sigma^{2,0}$. Then, f can be written in one of the following prenormal forms:*

(1) $(xy, g(x, y))$
(2) $(x^2 + y^2, h(x, y))$,

where $g, h \in \mathcal{M}_2^2$

Proof. Firstly, we know that if we consider a map germ f of type $\Sigma^{2,0}$, its 2-jet $j^2 f(0)$ is situated in one of the following $\mathcal{A}^2$-classes (see for example [5]):

$$(xy, x^2 + y^2), \quad (xy, x^2), \quad (xy, 0), \quad (x^2 + y^2, 0).$$

Therefore, f will present one of the following forms:

(1) $(xy + a(x, y), b(x, y))$, with $a(x, y) \in \mathcal{M}_2^3$, $b(x, y) \in \mathcal{M}_2^2$
(2) $(x^2 + y^2 + c(x, y), d(x, y))$, with $c(x, y) \in \mathcal{M}_2^3$, $d(x, y) \in \mathcal{M}_2^2$

By applying Morse's lemma we know that if we consider a function germ $f : (\mathbb{R}^2, 0) \longrightarrow (\mathbb{R}, 0)$ of the form $f(x, y) = u(x, y) + v(x, y)$, with $u(x, y)$ being a non degenerate quadratic form and with $v(x, y) \in \mathcal{M}_2^3$, we can choose a suitable change of coordinates

$$\begin{array}{rccc} \alpha : & (\mathbb{R}^2, 0) & \longrightarrow & (\mathbb{R}^2, 0) \\ & (x, y) & \rightarrow & (X, Y) \end{array}$$

such that $u = f \circ \alpha^{-1}$.

As we have a non degenerate quadratic form in the first component, if we apply this change of coordinates in (1) and (2), we arrive to the desired result. □

The first step to classify topologically this kind of germs will be to compute their topological degree. Taking it into account, we state and prove the following result.

Proposition 4.2. *Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a finitely determined map germ of type $\Sigma^{2,0}$.*

(1) *If $f(x, y) = (xy, g(x, y))$, f can have degree $0, \pm 1$ or ± 2.*
(2) *If $f(x, y) = (x^2 + y^2, h(x, y))$, f has degree 0.*

Proof. let us prove first (2). If our germ f has as first component $x^2 + y^2$ it is not surjective. Then, $\deg(f) = 0$.

For (1), we can suppose, without loss of generality, that

$$g(x, y) = ax^p + by^q + k(x, y)$$

where

$$p, q \geq 2, \qquad a, b > 0, \qquad \text{and} \qquad k(x, y) \in \langle xy \rangle.$$

As we know that $(xy, g(x, y))$ is $\mathcal{K}$-equivalent to $(xy, ax^p + by^q)$ and that the topological degree is a $\mathcal{K}$-invariant we only need to compute the topological degree of $(xy, ax^p + by^q)$. We will do it by applying Eisenbud-Levine's formula ([2]), given by

$$\deg(f) = \text{sign}\langle , \rangle_\varphi,$$

the signature of the quadratic form associated to a linear function $\varphi : Q(f) \to \mathbb{R}$ defined conveniently, with

$$Q(f) = \frac{\mathcal{E}_2}{\langle f_1, f_2 \rangle}.$$

Thus, we have that

$$Q(f) = \frac{\mathcal{E}_2}{\langle xy, ax^p + by^q \rangle}$$

and a basis of this space will be given by

$$\{1, x, x^2, \ldots, x^{p-1}, y, y^2, \ldots, y^{q-1}, J(f)\}$$

with $J(f) = qby^q - pax^p$.

We define the map

$$\begin{array}{rccc} \varphi: & Q(f) & \longrightarrow & \mathbb{R} \\ & J(f) & \to & 1 \\ & [1] & \to & 0 \\ & [x] & \to & 0 \\ & [y] & \to & 0 \\ & \vdots & \vdots & \vdots \\ & [x^{p-1}] & \to & 0 \\ & [y^{q-1}] & \to & 0 \end{array}$$

We will suppose that $a = b = \pm 1$, generalizing the result later.

The matrix of

$$\begin{array}{rccc} \langle , \rangle_\varphi : & Q(f) \times Q(f) & \longrightarrow & \mathbb{R} \\ & (p, q) & \to & \varphi(pq) \end{array}$$

with respect to the basis of $Q(f)$ is

$$A = \begin{array}{c} \\ 1 \\ x \\ x^2 \\ \vdots \\ x^{p-1} \\ y \\ y^2 \\ \vdots \\ y^{q-1} \\ J \end{array} \begin{array}{c} \begin{array}{cccccccccc} 1 & x & x^2 & \cdots & x^{p-1} & y & y^2 & \cdots & y^{q-1} & J \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & \mp\frac{1}{p+q} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & \mp\frac{1}{p+q} & 0 & \cdots & 0 & 0 & 0\cdots & 0 & 0 & \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \pm\frac{1}{p+q} & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \pm\frac{1}{p+q} & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \end{array}$$

taking into account the following facts:

- Each element of the form $x^i y^j \in \langle xy \rangle$ and as a consequence is 0 in $Q(f)$
- Each element of the form $Jx^i, Jy^j, x^{p+i}, y^{q+j}$ can be written as linear combination of the components of f, that is, they are 0 in $Q(f)$
- The elements x^p and y^q can be written in the following form:
  - $x^p = \frac{\mp qy^q \pm px^p \pm q(\pm x^p \pm y^q)}{\pm(p+q)} = \frac{\mp 1}{p+q} J \pm \frac{q}{p+q}(\pm x^p \pm y^q)$, with $\varphi(x^p) = \mp\frac{1}{p+q}$
  - $y^q = \frac{\pm qy^q \mp px^p \pm p(\pm x^p \pm y^q)}{\pm(p+q)} = \frac{\pm 1}{p+q} J \pm \frac{p}{p+q}(\pm x^p \pm y^q)$, with $\varphi(y^q) = \pm\frac{1}{p+q}$

Therefore, by computing the determinant of the matrix $(xI - A)$ we obtain the following characteristic polynomials, depending on the parity of p and q:

- If p and q are odd, $\det(xI - A) = (x^2 - 1)(x^2 - \frac{1}{(p+q)^2})^{\frac{p+q-2}{2}}$ and, as a consequence, $\deg(f) = \operatorname{sign}\langle , \rangle_\varphi = 0$.
- If p and q are even, $\det(xI - A) = (x^2 - 1)(x^2 - \frac{1}{(p+q)^2})^{\frac{p+q-4}{2}}(x \mp \frac{1}{p+q})(x \pm \frac{1}{p+q})$ and, as a consequence, $\deg(f) = \operatorname{sign}\langle , \rangle_\varphi = \begin{cases} 0, & \text{if } ab > 0, \\ \pm 2, & \text{if } ab < 0. \end{cases}$

- If p and q have different parity, $\det(xI - A) = (x^2-1)(x^2 - \frac{1}{(p+q)^2})^{\frac{p+q-3}{2}}(x \pm \frac{1}{p+q})$ and, as a consequence, $\deg(f) = \mathrm{sign}\langle,\rangle_\varphi = \pm 1$.

Let us see now that we are able to generalize this result for any $a, b \in \mathbb{R}$, with $a, b \neq 0$. We will prove this by constructing a homotopy.

Let $f_0(x,y) = (xy, ax^p + by^q)$, with $a > 0$ (analogous for $a < 0$), $f_1(x,y) = (xy, x^p + by^q)$ and we consider the family

$$f_t(x,y) = (xy, ((1-t)a + t)x^p + by^q),$$

with $t \in [0,1]$.

If we prove that for any t, $f_t^{-1}(0) = \{0\}$ and that if $t = 0$, $f_t = f_0$ and if $t = 1$, $f_t = f_1$, we will have that f_0 and f_1 are homotopic and, as a consequence, $\deg(f_0) = \deg(f_1)$.

As $(1-t)a + t \neq 0$ for any t, we will have that if we want that both terms of f_t vanish, x and y must be 0. Then, for any t, $f_t^{-1}(0) = \{0\}$. On the other hand, by substituting, if $t = 0$, $f_t(x,y) = (xy, ax^p + by^q) = f_0(x,y)$ and if $t = 1$, $f_t(x,y) = (xy, x^p + by^q) = f_1(x,y)$. Then, f_0 and f_1 are homotopic and $\deg(f_0) = \deg(f_1)$.

Analogously, we will have that $\deg(xy, x^p + by^q) = \deg(xy, x^p + y^q)$ if $b > 0$. Then,

$$\deg(xy, ax^p + by^q) = \deg(xy, x^p + y^q).$$

□

Now, putting together theorem 4.1 and proposition 4.2, we have the following corollary.

Corollary 4.3. *Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a finitely determined map germ of type $\Sigma^{2,0}$. Then,* $|\deg(f)| \leq 2$.

Proof. If f is of type $\Sigma^{2,0}$, by theorem 4.1 it can be written in the form $(xy, g(x,y))$ or in the form $(x^2 + y^2, h(x,y))$, and we have just seen that the absolute value of their topological degree is less or equal than 2. □

Before starting to compute the different topological classes of this kind of germs, we should remember the concepts of admissible weights and weighted degrees of a weighted homogeneous map germ which were introduced by Gaffney and Mond in [4] and will be very helpful for us in our classification.

Definition 4.4. Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a weighted homogeneous map germ. We will say that its weights w_1, w_2 and its weighted degrees d_1, d_2 are *admissible* if they verify the two following conditions:

(1) $(w_1, w_2) = (d_1, d_2) = 1$
(2) $w_1 = w_2 = 1$ (homogeneous case) or $d_1 = k_1 w_1 w_2,\ d_2 = k_2 w_1 w_2 + w_1 + w_2$ (type 1) or $d_1 = k_1 w_1 w_2 + w_1,\ d_2 = k_2 w_1 w_2 + w_2$ (*type* 2).

Once we have introduced this concept, let us see its relation with finitely determined map germs.

Proposition 4.5. *Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a weighted homogeneous finitely determined map germ. Then, w_1, w_2, d_1, d_2 must be admissible.*

Proof. Given a weighted homogeneous finitely determined map germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$, since it is real analytic, we can consider its complexification $\hat{f} : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$. It is well known that $\hat{f}$ is also weighted homogeneous and finitely determined as a complex analytic map germ.Then, by applying [Proposition 5.3, 4], its weights w_1, w_2 and weighted degrees d_1, d_2 must be admissible for $\hat{f}$, and, as a consequence, for f. □

Remark 4.6. Let us see how we apply this result to a finitely determined map germ in our particular case of study.

- $f(x,y) = (xy, g(x,y))$

  If f is weighted homogeneous, that is,

  $$g(x,y) = \sum_{i=0}^{p} a_i (x^{w_2})^i (y^{w_1})^{p-i}$$

  we must have that $(w_1, w_2) = (w_1 + w_2, pw_1w_2) = 1$. Departing from the basis that w_1, w_2 must be relatively primes, we have the following consequences, according to the value of p.
  - If $p = 1$, f is generically finitely determined.
  - If $p = 2$, f won't be finitely determined if w_1 and w_2 are odd.
  - If $p = 3$, f won't be finitely determined if $w_1 + w_2 = 3k$, with $k \in \mathbb{N}$.
  - In general, if $p = t_1^{\alpha_1} \dots t_m^{\alpha_m}$, f won't be finitely determined if there exists i such that $w_1 + w_2 = kt_i$, with $1 \leq i \leq m$ and $k \in \mathbb{N}$.
- $f(x,y) = (x^2 + y^2, h(x,y))$

  Because of the first component, we are only able to study this kind of germ in the homogeneous case $w_1 = w_2 = 1$, with

  $$h(x,y) = \sum_{i=0}^{p} a_i x^i y^{p-i}$$

  and $(2,p) = 1$. We arrive quickly to the conclusion that if $p = 2k$, f won't be finitely determined.

4.1. **Germs with prenormal form** $(xy, g(x,y))$**.** We consider the special case of weighted homogeneous map germs, that is,

$$g(x,y) = \sum_{i=0}^{p} a_i (x^{w_2})^i (y^{w_1})^{p-i},$$

with $(w_1 + w_2, pw_1w_2)$ being the weighted degrees of our germ and $(w_1, w_2) = 1$. We also suppose that $p \leq 3$. Then, the following results will give us a complete topological classification of these particular cases.

Theorem 4.7. *($p = 1$) Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a finitely determined map germ of corank 2 of the form $f(x,y) = (xy, ax^{w_2} + by^{w_1})$. Then,*

(1) *if w_1, w_2 are odd, f is topologically equivalent to the fold (x, y^2),*
(2) *if w_1, w_2 have different parity, f is topologically equivalent to the cusp $(x, xy + y^3)$.*

Proof. Let us prove first (1).

If w_1, w_2 are odd, we know by the proof of theorem 4.2 that $\deg(f) = 0$. In addition to this, if we compute its singular set, we get the equation $w_1 by^{w_1} - w_2 ax^{w_2} = 0$. Since this equation is irreducible, we can conclude that $S(f)$, and, as a consequence $\Delta(f)$, only present a single branch.

Let us see that we are going to have a single topological class which is the class of the fold. To prove this is enough to see that for any $a, b \in \mathbb{R} \setminus \{0\}$ there are points where f does not have any inverse image.

Let us consider the point $(1,0)$. We get the equations $xy = 1$ and $ax^{w_2} + by^{w_1} = 0$, obtaining that

$$y = \left(\frac{-a}{b}\right)^{1/(w_1+w_2)}.$$

Thus, if $ab > 0$ f does not have any inverse image and the result is proved (see figure 4).

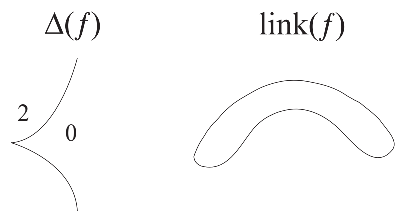


FIGURE 4.

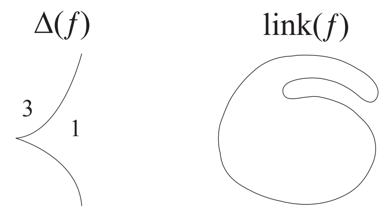


FIGURE 5.

Analogously, if we take now the point $(-1, 0)$ we have that every map germ f with $ab < 0$ does not have any inverse image either and we arrive to the conclusion again that we have a single configuration of inverse images in the discriminant curve, which is the one of the fold.

If w_1 and w_2 are of distinct parity, applying an analogous procedure as in (1) to prove the existence of points with a single inverse image, we obtain the desired result (see figure 5).

□

Theorem 4.8. *($p = 2$) Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a finitely determined map germ of corank 2 of the form $f(x, y) = (xy, ax^{2w_2} + bx^{w_2}y^{w_1} + cy^{2w_1})$. Then,*

1 *if w_1, w_2 have the same parity, f is not finitely determined,*

2 *if w_1, w_2 have distinct parity, we have three cases,*
- *if $(w_1 - w_2)^2b^2 + 16w_1w_2ac > 0$,*
  - *f is topologically equivalent to the map germ $(xy, x^2 + xy^2 + y^4)$ if $ac > 0$*
  - *f is topologically equivalent to the map germ $(xy, x^2 + 20xy^2 - y^4)$ if $ac < 0$*
- *if $(w_1 - w_2)^2b^2 + 16w_1w_2ac < 0$, f is topologically equivalent to the map germ $(xy, x^2 + xy^2 - y^4)$*
- *if $(w_1 - w_2)^2b^2 + 16w_1w_2ac = 0$, f is not finitely determined.*

Proof. If w_1, w_2 are both even or odd the result follows from remark 4.6. Let us suppose that w_1 and w_2 have different parity. The Jacobian determinant is given by

$$J(f) = -2w_2ax^{2w_2} + b(w_1 - w_2)x^{w_2}y^{w_1} + 2w_1cy^{2w_1},$$

that can be factorized in the form

$$-2w_2(x^{w_2} - \lambda_1y^{w_1})(x^{w_2} - \lambda_2y^{w_1}),$$

with $\lambda_i = \lambda_i(a, b, c, w_1, w_2) \in \mathbb{C}$, $i = 1, 2$. These λ_i are obtained by solving the quadratic equation given by the Jacobian determinant, whose discriminant is

$$(w_1 - w_2)^2b^2 + 16w_1w_2ac = 0.$$

Then, if this discriminant is positive we have two different real solutions for λ_i and as a consequence two branches in our singular set $S(f)$, if it is negative our singular set is empty outside of the origin and in the case that the discriminant vanishes, $\lambda_1 = \lambda_2$ and f won't be finitely determined. If the discriminant is negative, by remark 3.6 and proposition 4.2, taking into account that ac must be necessarily negative, we have that f will be topologically equivalent to the germ $(xy, x^2 - y^2)$. Since this germ is not finitely determined we can choose another member of this topological class that is finitely determined. let us take, for example, $(xy, x^2 + xy^2 - y^4)$.

Thus, we center our attention in the case $(w_1 - w_2)^2b^2 + 16w_1w_2ac > 0$. If we call

$$C_i \equiv x^{w_2} - \lambda_iy^{w_1} = 0$$

for $i = 1, 2$, and apply the coordinate changes

$$\begin{cases} x = \alpha t^{w_1} \\ y = \beta t^{w_2} \end{cases}$$

we have that

$$f|_{C_i}(t) = (\alpha\beta t^{w_1+w_2}, (a\lambda_i^2 + b\lambda_i + c)t^{2w_1w_2}),$$

whose derivative never vanishes out of 0 and it will present double folds if and only if $\alpha\beta = 0$, which is impossible. Let us observe that these curves are going to be symmetrical with respect to the y-axis (figure 6). From this point, we must consider two different cases:

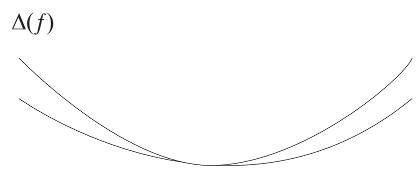


FIGURE 6.

- If $ac > 0$, by proposition 4.2 we know that $\deg(f) = 0$. Taking into account that our discriminant set has 2 branches and the link of f can't have more than one connected component, if we are able to prove that for any b we have points with no inverse images, we finish.

  If we consider the point $(0, -1)$ we obtain the equations

  $$xy = 0 \quad \text{and} \quad ax^{2w_2} + bx^{w_2}y^{w_1} + cy^{2w_1} = -1,$$

  getting the equality $y = \left(\frac{-1}{c}\right)^{1/(2w_1)}$ if $x = 0$ and $x = \left(\frac{-1}{a}\right)^{1/(2w_2)}$ if $y = 0$. In both cases if a and c are positive the equalities don't have any real solution. Thus, f does not present any inverse image (see figure 7).

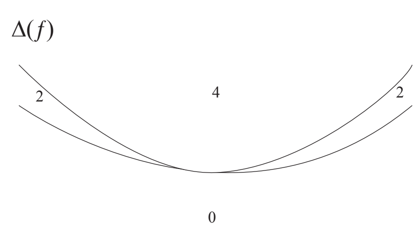


FIGURE 7.

Considering the point $(0, 1)$ and applying a totally analogous procedure we arrive to the conclusion that if $a, c < 0$ f does not present any inverse image either (see figure 8). Then, we have in both cases a single configuration of inverse images in the discriminant, obtaining the associated link and Gauss word that appear in figure 9. Thus, f is topologically equivalent to the known corank 1 normal form $(x, y^4 - xy^2 - x^2y)$. If we want to take a normal form of corank 2 we can choose, for example, $(xy, x^2 + xy^2 + y^4)$.

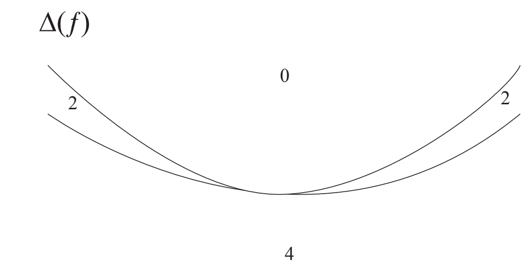


FIGURE 8.

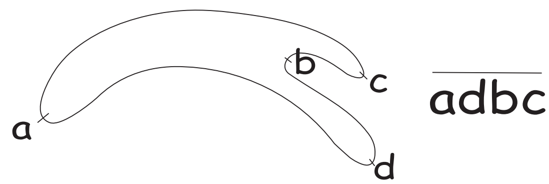


FIGURE 9.

- If $ac < 0$, using again proposition 4.2, we know that $\deg(f) = \pm 2$. Taking into account that we are dealing with a map germ whose discriminant only has two branches if we are able to prove that the maximum number of inverse images of f is 4 we finish.

  Let us consider the equations

$$xy = d \quad \text{and} \quad ax^{2w_2} + bx^{w_2}y^{w_1} + cy^{2w_1} = e,$$

with $(d,e) \in \mathbb{R}^2$. From here, we get the equality

$$cy^{2(w_1+w_2)} + bd^{w_2}y^{w_1+w_2} - ey^{2w_2} + ad^{2w_2} = 0.$$

Applying Descartes method and using the hypothesis $ac < 0$ we arrive to the conclusion that we can have three sign changes for $y > 0$ in the best of the cases and since all the exponents are even except $w_1 + w_2$, this is the only term whose sign is going to change when we consider $y < 0$. Then, we will have in this last case a single inverse image and a total of 4 inverse images, as we wanted to prove.

Thus, the only possible configuration of the inverse images in the discriminant of a map germ of this type will be the one that appears in figure 10, having its correspondent associated link and Gauss word (figure 11).

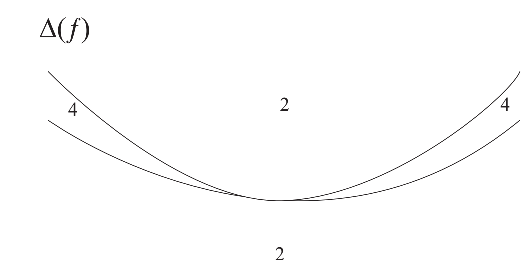


FIGURE 10.

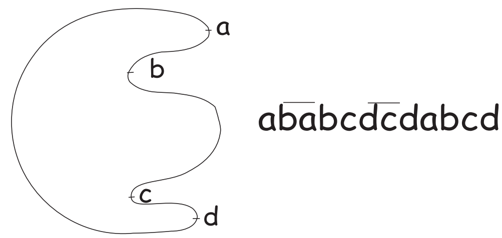


FIGURE 11.

To finish, let us choose a representative of this topological class, for example,

$$(xy, x^2 + 20xy^2 - y^4).$$

$\square$

Theorem 4.9. *($p = 3$) Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a finitely determined map germ of corank 2 of the form $f(x, y) = (xy, ax^{3w_2} + bx^{2w_2}y^{w_1} + cx^{w_2}y^{2w_1} + dy^{3w_1})$. let us denote by*

$$A = -3w_2a\frac{(2w_1 - w_2)c}{3} - \left(\frac{(w_1 - 2w_2)b}{3}\right)^2,$$

$$B = -3w_2a3w_1d - \frac{(w_1 - 2w_2)b}{3}\frac{(2w_1 - w_2)c}{3},$$

$$C = \frac{(w_1 - 2w_2)b}{3}3w_1d - \left(\frac{(2w_1 - w_2)c}{3}\right)^2.$$

Then:

(1) *Let us suppose that w_1, w_2 have different parity,*
- *if $B^2 - 4AC > 0$, f is topologically equivalent to the simple cusp $(x, xy + y^3)$,*
- *if $B^2 - 4AC < 0$, f is topologically equivalent to one of the map germs that appear in table 1,*
- *if $B^2 - 4AC = 0$, f is not finitely determined.*

(2) *In the case that w_1, w_2 are both odd,*
- *if $B^2 - 4AC > 0$, f is topologically equivalent to one of the map germs that appear in the table 2,*
- *if $B^2 - 4AC < 0$, f is topologically equivalent to one of the map germs that appear in table 3,*
- *if $B^2 - 4AC = 0$, f is not finitely determined.*

Proof. If we compute the Jacobian determinant of f we get

$$Jf(x, y) = -3w_2ax^{3w_2} + (w_1 - 2w_2)bx^{2w_2}y^{w_1} + (2w_1 - w_2)cx^{w_2}y^{2w_1} + 3w_1dy^{3w_1}.$$

Let us realize that if we make the coordinate changes

$$\begin{cases} \overline{x} = x^{w_2} \\ \\ \overline{y} = y^{w_1} \end{cases}$$

we get the cubic form

$$Jf(\overline{x}, \overline{y}) = -3w_2a\overline{x}^3 + (w_1 - 2w_2)b\overline{x}^2\overline{y} + (2w_1 - w_2)c\overline{x}\overline{y}^2 + 3w_1d\overline{y}^3.$$

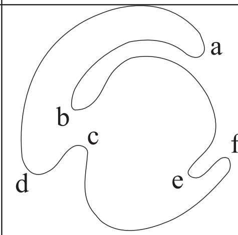
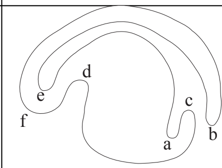
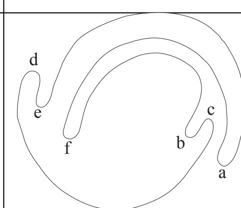
Degree	*Germ*	*Associated link*
1	$(xy, x^6 + 7x^4y^3 + 8x^2y^6 + y^9)$	 $a\overline{ba}bc\overline{dc}de\overline{fe}f$
	$(xy, x^6 + 2x^4y^3 + 9x^2y^6 + y^9)$	 $ab\bar{c}b\bar{a}bcd\bar{e}dc\bar{b}cde\bar{f}e\bar{d}ef$
	$(xy, x^6 - x^4y^3 + 7x^2y^6 + y^9)$	 $ab\overline{cb}cde\bar{f}edcb\bar{a}bcd\overline{ed}ef$

TABLE 1.

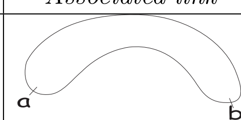
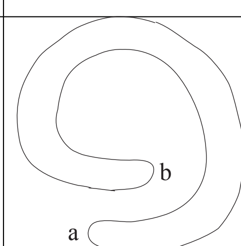
Degree	*Germ*	*Associated link*
0	(x, y^2)	 $\overline{ab}$
	$(xy, x^3 - x^2y^3 - xy^6 + y^9)$	 $\bar{a}ba\bar{b}ab$

TABLE 2.

From this point we apply a known result (see for example [5]) which tell us that a cubic form will be of symbolic, hyperbolic, parabolic or elliptic type if and only if its associated quadratic

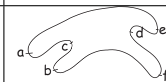
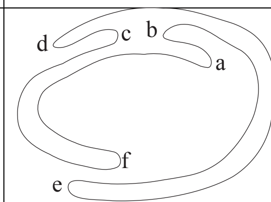
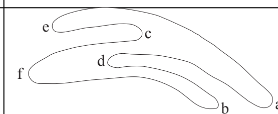
Degree	*Germ*	*Associated link*
0	$(xy, x^3 - x^2y^3 + 3xy^6 + y^9)$	 $c\overline{acb}cd\overline{fde}d$
	$(xy, x^3 - 6x^2y^3 + 4xy^6 + y^9)$	 $abc\overline{dc}de\overline{f}edcb\overline{ab}af\overline{e}f$
	$(xy, x^3 + 6x^2y^3 + 6xy^6 + y^9)$	 $\overline{a}bcd\overline{e}d\overline{c}de\overline{f}edc\overline{b}c\overline{d}cb$

TABLE 3.

form obtained by computing the Hessian determinant is of symbolic, hyperbolic, parabolic or elliptic type respectively. Thus, if we compute the Hessian determinant of $Jf(\overline{x}, \overline{y})$ we get the quadratic form $A\overline{x}^2 + B\overline{xy} + C\overline{y}^2$ with A, B, C depending on the values of the initial coefficients a, b, c, d and of the weights w_1, w_2 and undoing the coordinate changes we made earlier we get the function $Ax^{2w_2} + Bx^{w_2}y^{w_1} + Cy^{2w_1}$ which we will use to determine the different cases of study. Therefore, we have the following possibilities:

(1) Let us suppose that w_1 and w_2 have different parity. Firstly, if we consider as we did in the case $p = 2$ the coordinate changes

$$\begin{cases} x = \alpha t^{w_1} \\ y = \beta t^{w_2} \end{cases}$$

together with the image of the restriction of f to each one of the curves of the singular set, C_i, we get

$$f_{|C_i(t)}(t) = (\alpha\beta t^{w_1+w_2}, (a\lambda_i^3 + b\lambda_i^2 + c\lambda_i + d)t^{3w_1w_2}),$$

realizing that each one of these branches is symmetric with respect to the y-axis. Now, let us see the different configurations of inverse images that we can have in the discriminant, in order to obtain the distinct topological classes. As first step we will prove that $\#f^{-1}(z) \leq 5$, $\forall z \in \mathbb{R}^2$.

Let us take a point $(e, f) \in \mathbb{R}^2$ and let us consider the equations

$$\begin{cases} xy = e \\ ax^{3w_2} + bx^{2w_2}y^{w_1} + cx^{w_2}y^{2w_1} + dy^{3w_1} = f. \end{cases}$$

Taking in the first equation $x = \frac{e}{y}$, with $y \neq 0$ and substituting we get

$$a(\frac{e}{y})^{3w_2} + b(\frac{e}{y})^{2w_2}y^{w_1} + c(\frac{e}{y})^{w_2}y^{2w_1} + dy^{3w_1} = f.$$

As last step we multiply both sides of the equation by y^{3w_2}, obtaining the final equation

$$dy^{3(w_1+w_2)} + ce^{w_2}y^{2(w_1+w_2)} - fy^{3w_2} + be^{2w_2}y^{w_1+w_2} + ae^{3w_2} = 0.$$

Now, putting in order the monomials according to their weighted degree and taking into account that the order of appearance of $(c, -f, b)$ can suffer variations due to the different values of (w_1, w_2), we apply Descartes rule of signs. Since we are working with a a polynomial consisting of 5 monomials, the worst configuration (with a biggest number of inverse images) will be given by $+-+-+$. Then, we will have at most 4 inverse images for $y > 0$ or $y < 0$ indistinctly(let us take $y > 0$). If $y < 0$, taking into account the parity of the weighted degrees of the monomials, we have the configuration $---++$ (or $--+++$, depending on the parity of w_2), obtaining a single inverse image and a total of 5 inverse images as we wanted to prove. If $(c, -f, b)$ would appear in a distinct order, by applying an analogous procedure we would arrive to the same result.

Secondly, we are going to prove that our germ f is always going to have points with a single inverse image and points with 3 inverse images. To do this we take a point $(0, f) \in \mathbb{R}^2$ and consider the equations

$$\begin{cases} xy = 0 \\ ax^{3w_2} + bx^{2w_2}y^{w_1} + cx^{w_2}y^{2w_1} + dy^{3w_1} = f. \end{cases}$$

Since xy vanishes, x or y must be 0 and using the second equation we get in the first case $y = \left(\frac{f}{d}\right)^{1/(3w_1)}$ and in the second case $x = \left(\frac{f}{a}\right)^{1/(3w_2)}$. Therefore, if w_1 is even and w_2 is odd we will have 3 inverse images if $fd > 0$ and a single one if $fd < 0$; analogously, if w_1 is odd and w_2 is even we will have 3 inverse images if $fa > 0$ and a single one if $fa < 0$. Then, from this point, what we know for sure is that the sectors of our bifurcation set in the image of f created by the discriminant curves that contain the y-axis are going to have one of them 3 inverse images and the other, a single one.

With all these previous calculations we are now in conditions to obtain the different topological classes.

- If $B^2 - 4AC > 0$, we have a single branch in our singular set and as a consequence, the only possible configuration of inverse images in its single discriminant curve is the one that appear in figure 12, which is clearly identified with the Gauss word and the link of the simple cusp (figure 13). Then, f is topologically equivalent to the simple cusp.

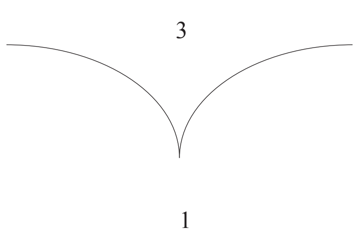


FIGURE 12.

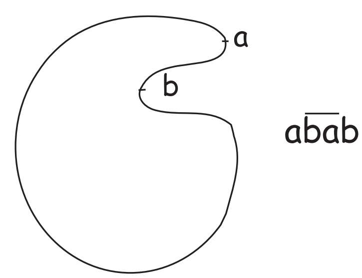


FIGURE 13.

- If $B^2 - 4AC < 0$ we have three branches and two possible configurations of inverse images in the discriminant curves (figure 14), obtaining in the first case the associated link and Gauss word that appears in figure 15 , with normal form

$(xy, x^6 + 7x^4y^3 + 8x^2y^6 + y^9)$ and in the second case the two different topological classes that appear in figure 16, having as normal forms $(xy, x^6+2x^4y^3+9x^2y^6+y^9)$ and $(xy, x^6 - x^4y^3 + 7x^2y^6 + y^9)$ respectively.

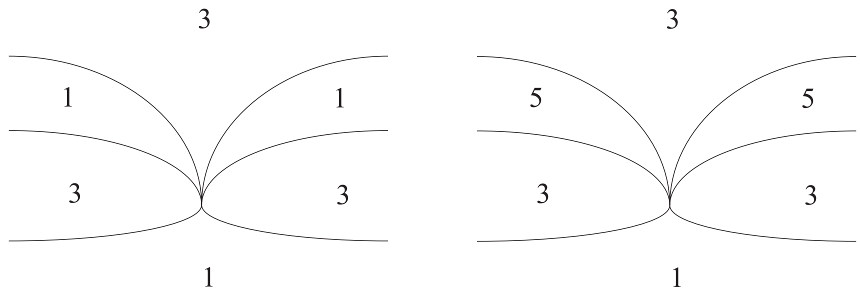
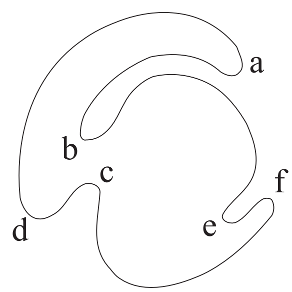
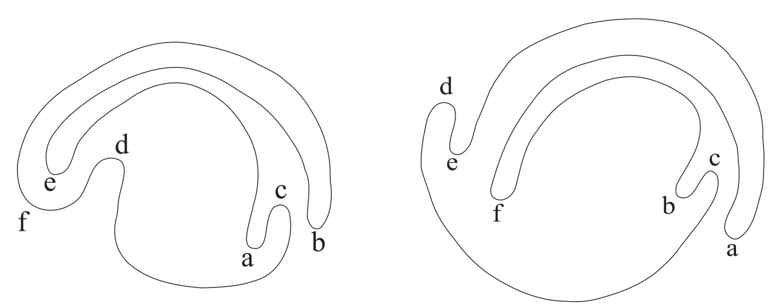


FIGURE 14.



a$\overline{ba}$bc$\overline{dc}$de$\overline{fe}$f

FIGURE 15.



abc̄bābcdēdc$\overline{b}$cdef̄ed̄ef abc$\overline{cb}$cde$\overline{f}$edcbābcd$\overline{ed}$ef

FIGURE 16.

- If $B^2 - 4AC = 0$ we will have a non reduced component in our singular set and f won't be finitely determined.

(2) If w_1 and w_2 are odd, we consider again the coordinate changes

$$\begin{cases} x = \alpha t^{w_1} \\ y = \beta t^{w_2} \end{cases}$$

together with the image of the restriction of f to each one of the curves of the singular set, C_i. In this case, these images are symmetric with respect to the x-axis. Now, let us see the different configurations of inverse images that we can have in the discriminant, in order to obtain the distinct topological classes. Firstly, we will prove that $\# f^{-1}(z) \leq 6$, $\forall z \in \mathbb{R}^2$.

Following a totally analogous procedure to the case of weights with different parity, taking a point $(e, f) \in \mathbb{R}^2$ we arrive to the equation

$$dy^{3(w_1+w_2)} + ce^{w_2}y^{2(w_1+w_2)} - fy^{3w_2} + be^{2w_1}y^{w_1+w_2} + ae^{3w_2} = 0$$

and applying Descartes method we conclude that points situated in the image of f are going to present 6 inverse images at most.

Let us see now that f is always going to have points with 2 inverse images in the y-axis. To prove this, we consider a point $(0, f) \in \mathbb{R}^2$. Since the first component of f must vanish we get the equalities $y = \left(\frac{f}{d}\right)^{1/(3w_1)}$, with a single inverse image $\left(0, \left(\frac{f}{d}\right)^{1/(3w_1)}\right)$ and $x = \left(\frac{f}{a}\right)^{1/(3w_2)}$, with a single inverse image $\left(\left(\frac{f}{a}\right)^{1/(3w_2)}, 0\right)$, getting a total of 2 inverse images, as we wanted to prove.

With all these previous remarks we are in conditions of giving a restricted list of the possible distribution of inverse images that we can have in the discriminant curves.

- If $B^2 - 4AC > 0$ our singular set and as a consequence the discriminant has a single real branch. Therefore, we only have two possible distributions of inverse images (figure 17), getting in the first case the link and Gauss word that appear in the left hand side of figure 18, with the associated normal form of the fold (x, y^2), and in the last case the one that appear in the right hand side of figure 18, with the associated normal form $(xy, x^3 - x^2y^3 - xy^6 + y^9)$.

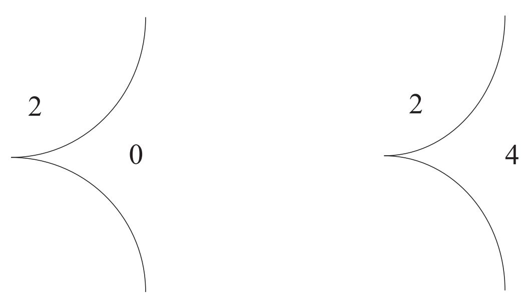


FIGURE 17.

- If $B^2 - 4AC < 0$ our singular set, and as a consequence the discriminant, has 3 distinct real branches and the initial number of possible configurations of inverse images in the discriminant is much bigger (see figure 19). Let us see that (d) and (e) can't occur.

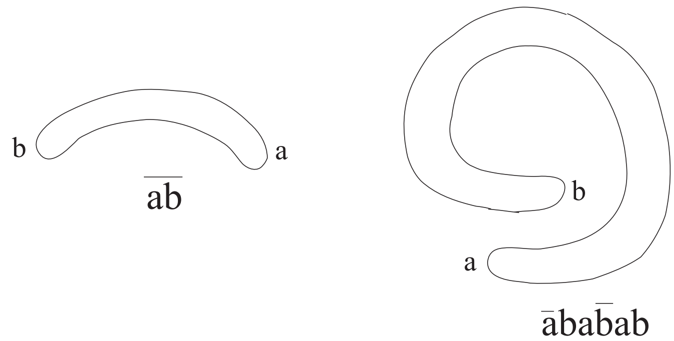


FIGURE 18.

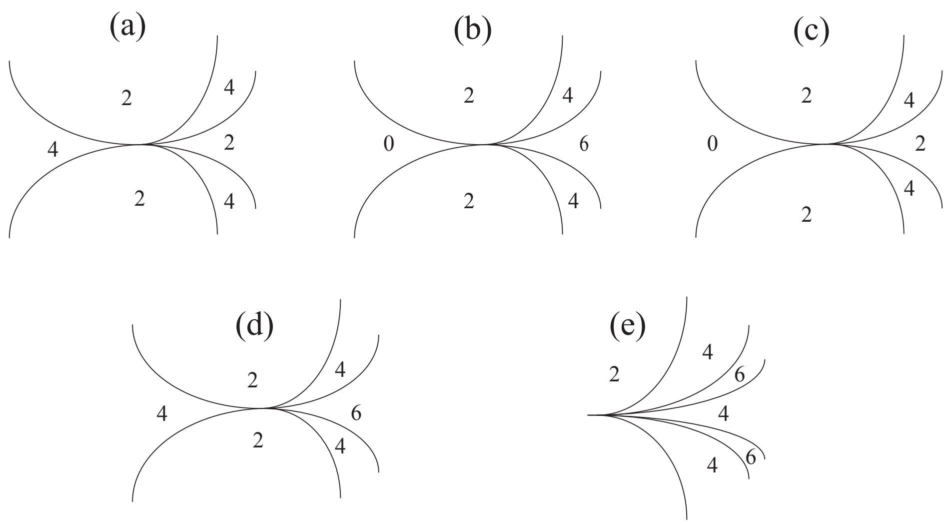


FIGURE 19.

If we had the configuration of (d), we would have points of the form $(e, 0) \in \mathbb{R}^2$ with 6 inverse images. let us suppose that $e > 0$. We obtain the equation

$$dy^{3(w_1+w_2)} + ce^{w_2}y^{2(w_1+w_2)} + be^{2w_1}y^{w_1+w_2} + ae^{3w_2} = 0.$$

If we apply Descartes method to this polynomial, the only possible signs configuration to get 6 inverse images is $+-+-$ for $y > 0$, obtaining $+-+-$ for $y < 0$. If (d) was possible, taking a point of the form $(e, 0)$ with $e < 0$ we should have 4 inverse images. But this is impossible because applying again Descartes method and using the sign of coefficients (a, b, c, d) we have had to choose to obtain 6 inverse images when $e > 0$ we obtain a signs configuration of the form $++++$ for any y. Therefore, we have just arrived to a contradiction and the configuration (d) is not possible.
To prove that (e) is not possible either we will choose a point of the form

$$(e^{w_1+w_2}, te^{3w_1w_2}) \in \mathbb{R}^2,$$

that is, a point of a generic cusp and we consider the equations

$$\begin{cases} xy = e^{w_1+w_2} \\ ax^{3w_2} + bx^{2w_2}y^{w_1} + cx^{w_2}y^{2w_1} + dy^{3w_1} = te^{3w_1w_2}. \end{cases}$$

If we suppose that $y \neq 0$, we can take $x = \frac{e^{w_1+w_2}}{y}$ and by substituting in the second equation and multiplying both terms by y^{3w_2} we have

$$a(e^{w_1+w_2})^{3w_2} + b(e^{w_1+w_2})^{2w_2}y^{w_1+w_2} + c(e^{w_1+w_2})^{w_2}y^{2(w_1+w_2)}$$

$$-te^{w_1w_2}y^{3w_2} + dy^{3(w_1+w_2)} = 0,$$

that is, a polynomial constituted by 5 monomials and where, applying Descartes method, we are going to have in the worst of the cases 4 sign changes, and as a consequence, 4 inverse images for $e > 0$ and $e < 0$. Then, (e) is not possible.
Thus, we only have 3 possible configurations $((a), (b)$ and $(c))$ obtaining for each one a single topological class given by its correspondent associated link and Gauss word (see figure 20).

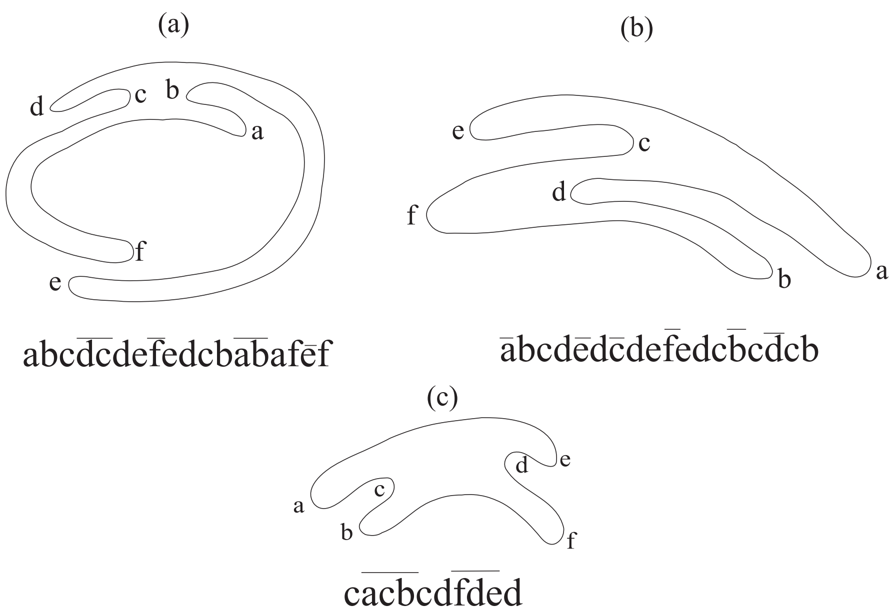


FIGURE 20.

To finish we associate to (a) the normal form $(xy, x^3 - 6x^2y^3 + 4xy^6 + y^9)$, to (b) $(xy, x^3 + 6x^2y^3 + 6xy^6 + y^9)$ and to (c) $(xy, x^3 - x^2y^3 + 3xy^6 + y^9)$.

- If we consider the remaining case, $B^2 - 4AC = 0$, following and analogous argument to the case of weights with different parity we conclude that f won't be finitely determined.

$\square$

4.2. **Germs with prenormal form $(x^2+y^2, h(x,y))$.** As we did with germs with prenormal form $(xy, g(x,y))$, we will suppose that $h(x,y)$ is a weighted homogeneous polynomial, that is,

$$h(x,y)=\sum_{i=0}^{p} b_i(x^{w_2})^i(y^{w_1})^{p-i},$$

although, in general, f won't be weighted homogeneous. We distinguish two different cases, according to the parity of p.

4.3. $\underline{p=2k}$**.** The following theorem will give us the classification of all germs of this type.

Theorem 4.10. *Let f be of type $\Sigma^{2,0}$,*

$$f(x,y)=(x^2+y^2,\sum_{i=0}^{p} b_i(x^{w_2})^i(y^{w_1})^{p-i}),$$

with $p=2k$. Then, f is not finitely determined.

Proof. We will prove it for $p=2$, being analogous for the remaining cases.

If w_1 or w_2 are greater than 1, when we compute the Jacobian determinant of f we obtain a expression of the form $2yA$ or $2xB$ with A,B depending on w_1,w_2,x,y. In the first case, we have the curve $y=0$ in the singular set, getting an image (x^2, ax^{2w_2}) that clearly presents double points. If we have $x=0$ by an analogous procedure we arrive to the same conclusion.

If $w_1=w_2=1$ we have branches of the form $x=\lambda y$ in the singular set, and as a consequence, each one of the discriminant curves will have the form $((\lambda y)^2+y^2, a(\lambda y)^2+b(\lambda y)y+cy^2)$ that present double points of the form $y_1=-y_2$. Then, f is not finitely determined either. □

4.4. **General case.** Firstly, we will see that if one of the weights is even and the other is different from 1, f won't be finitely determined.

Theorem 4.11. *Let f be of type $\Sigma^{2,0}$,*

$$f(x,y)=(x^2+y^2,\sum_{i=0}^{p} b_i(x^{w_2})^i(y^{w_1})^{p-i}).$$

Then, if w_1 or w_2 is even, with the other weight being greater than 1, f is not finitely determined.

Proof. Let us suppose that w_1 is even and $w_2>1$. If we compute the Jacobian determinant of f we get

$$Jf(x,y)=2x\sum_{i=0}^{p-1} w_1(p-i)b_i(x^{w_2})^i(y^{w_1})^{p-i-1}-2y\sum_{i=1}^{p} w_2 i b_i(x^{w_2})^{i-1}(y^{w_1})^{p-i}.$$

Since $w_2>1$ we can get one x out of the second summation, obtaining

$$Jf(x,y)=2x(\sum_{i=0}^{p-1} w_1(p-i)b_i(x^{w_2})^i(y^{w_1})^{p-i-1}-y\sum_{i=1}^{p} w_2 i b_i(x^{w_2})^{i-2}(y^{w_1})^{p-i}.$$

Therefore, one of the branches of $S(f)$ will always be given by the equation $x=0$ and

$$f_{|x=0}(y)=(y^2,y^{pw_1}),$$

that will always present double points of the form $y_1=-y_2$. Thus, f is not finitely determined. □

Let us see now what happen when both weights are odd. We will give some particular results about it.

Theorem 4.12. *($p = 1$) Let f be of type $\Sigma^{2,0}$, $f(x, y) = (x^2 + y^2, ax^{w_2} + by^{w_1})$, with w_1, w_2 both odd. Then, f is topologically equivalent to the germ $(x^2 + y^2, x^3 + y^5)$.*

Proof. Let us suppose that w_1, w_2 are both odd and greater than 1 (if one of them was 1, f wouldn't be of type $\Sigma^{2,0}$ anymore). In this case

$$Jf(x, y) = 2xy(w_1 by^{w_1-2} - w_2 ax^{w_2-2}),$$

obtaining that our singular set $S(f)$ will have 3 branches, $x = 0$, $y = 0$ and $y^{w_1-2} = \frac{w_2 ax^{w_2-2}}{w_1 b}$. In the first two f does not present any problem. Let us see that it does not present any problem in the third one either. To see this we make the coordinates change $\begin{cases} x = \alpha t^{w_1-2} \\ y = \beta t^{w_2-2} \end{cases}$ with $\beta = (w_2 a)^{1/(w_1-2)} \in \mathbb{C}$ and $\alpha = (w_1 b)^{1/(w_2-2)} \in \mathbb{C}$. We have that

$$f|_{y^{w_1-2}=\frac{w_2 ax^{w_2-2}}{w_1 b}}(t) = (A(t), B(t)),$$

with $A(t) = \alpha^2 t^{2(w_1-2)} + \beta^2 t^{2(w_2-2)}$ and $B(t) = a\alpha^{w_2} t^{w_2(w_1-2)} + b\beta^{w_1} t^{w_1(w_2-2)}$. It is clear that although $A(t)$ is going to present double points of the form $t_1 = -t_2$, it is not going to happen with $B(t)$. Then, f is finitely determined.

Thus, $\Delta(f)$ will have three branches and we can only have two possible configurations (see figure 21):

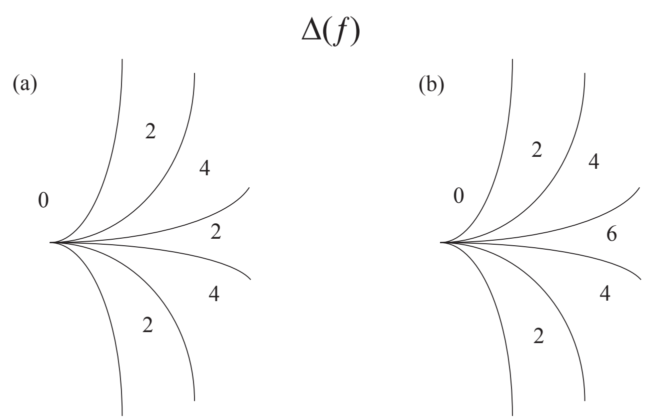


FIGURE 21.

Let us see that (b) is not possible. To prove this we consider a point $(e, 0) \in \mathbb{R}^2$ and we will prove by Descartes method that it will present at most 2 inverse images. We have the equations

$$\begin{cases} x^2 + y^2 = e \\ ax^{w_2} + by^{w_1} = 0 \end{cases}$$

obtaining a single equation of the form $A(y^{\frac{w_1}{w_2}})^2 + y^2 = e$ with $A > 0$. We consider the coordinate change $y = z^{w_2}$ in order to be able to work with integer exponents and we get $Az^{2w_1} + z^{2w_2} - e = 0$ that, applying Descartes method will always present at most 1 root if $z > 0$ and 1 root if $z < 0$, having a total of 2 roots z_1 and z_2 and as a consequence y_1 and y_2. Therefore, the only possible configuration is given by (a) and, since the 3 branches of the singular set are symmetric with respect to the origin of coordinates the only possible topological class is the associated to the link and Gauss word of figure 22.

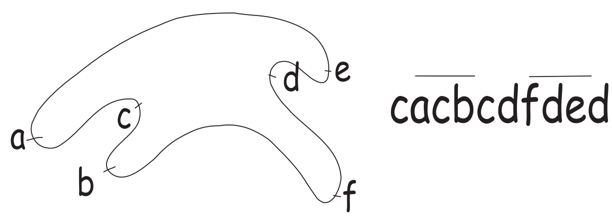


FIGURE 22.

Then, f is topologically equivalent to $(x, y^4 - x^2y^2 - \frac{1}{4}x^3y)$ and to the corank 2 normal form $(x^2 + y^2, x^3 + y^5)$.

$\square$

Theorem 4.13. *($p = 3$, homogeneous case) Let f be of type of $\Sigma^{2,0}$, $f(x, y) = (x^2 + y^2, ax^3 + bx^2y + cxy^2 + dy^3)$. Then, if we denote by*

$$A = b(\frac{3d - 2b}{3}) - (\frac{2c - 3a}{3})^2,$$
$$B = -bc - \frac{(2c - 3a)(3d - 2b)}{9},$$
$$C = \frac{c(3a - 2c)}{3} - (\frac{3d - 2b}{3})^2$$

we have that

(1) *if $B^2 - 4AC > 0$, f is topologically equivalent to the fold,*
(2) *if $B^2 - 4AC < 0$, f is topologically equivalent to one of the germs that appear in table 4,*

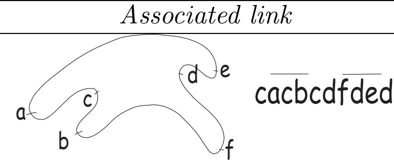
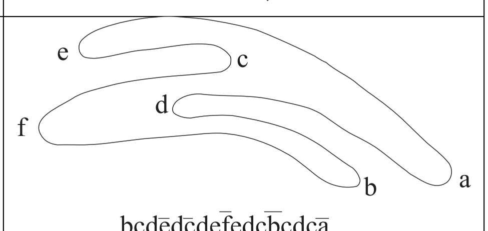
<table>
<tr><th>Degree</th><th>Germ</th><th>Associated link</th></tr>
<tr><td rowspan="2">0</td><td>$(x^2 + y^2, x^3 + y^5)$</td><td>
</td></tr>
<tr><td>$(x^2 + y^2, x^3 + x^2y - 3xy^2 + y^3)$</td><td>
</td></tr>
</table>

TABLE 4.

(3) *if $B^2 - 4AC = 0$, f is not finitely determined.*

Proof. Applying the result used earlier for map germs of the form $(xy, g(x,y))$ in the case $p = 3$ we obtain coefficients $A = A(a,,b,c,d)$, $B = B(a,b,c,d)$ and $C = C(a,b,c,d)$ such that $Jf(x,y)$ will present a symbolic, elliptical, hyperbolic or parabolic quadratic form if and only if $Ax^2 + Bxy + Cy^2$ presents a symbolic, elliptical, hyperbolic or parabolic quadratic form. Therefore, we have several cases:

(1) If $B^2 - 4AC > 0$, $S(f)$ presents a single branch $x = \lambda y$ whose image will be, as happen with all the germs of this form, symmetric with respect to the x-axis. Since the only possible configuration of inverse images is the one that appears in figure 23, f will be topologically equivalent to the fold.

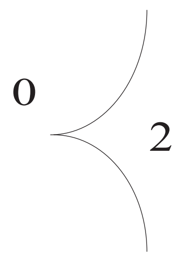


FIGURE 23.

(2) If $B^2 - 4AC < 0$, $S(f)$ will present three distinct real branches, obtaining in the discriminant the possible configurations of figure 24 and from each one of them a single topological class, symmetric with respect to the origin of coordinates. In case (a) we have the associated link and Gauss word of figure 25, taking as normal form $(x^2 + y^2, x^3 + y^5)$ and in case (b) we obtain the link of figure 26, taking as normal form $(x^2 + y^2, x^3 + x^2y - 3xy^2 + y^3)$.

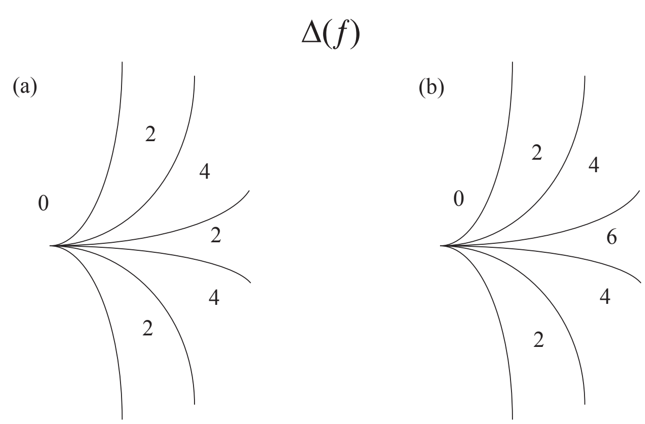


FIGURE 24.

(3) If $B^2 - 4AC = 0$ we obtain a non reduced component in $S(f)$. Then, f is not finitely determined.

□

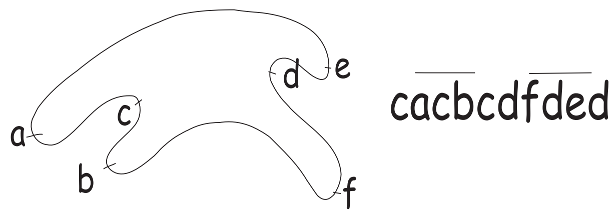


FIGURE 25.

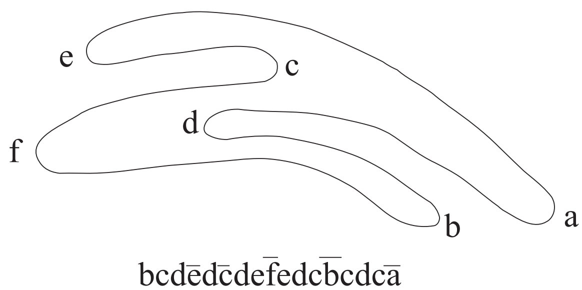


FIGURE 26.

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