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LINKS OF SINGULARITIES UP TO REGULAR HOMOTOPY

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ABSTRACT. We classify links of the singularities $x^2 + y^2 + z^2 + v^{2d} = 0$ in $(\mathbb{C}^4, 0)$ up to regular homotopies precomposed with diffeomorphisms of $S^3 \times S^2$. Let us denote the link of this singularity by L_d and denote by i_d the inclusion $L_d \subset S^7$. We show that for arbitrary diffeomorphisms $\varphi_d : S^3 \times S^2 \longrightarrow L_d$ the compositions $i_d \circ \varphi_d$ are image regularly homotopic for two different values of d, $d = d_1$ and $d = d_2$, if and only if $d_1 \equiv d_2 \mod 2$.

1. INTRODUCTION

It is well-known that the infinite number of Brieskorn equations in $\mathbb{C}^5$

$$z_1^{6k-1} + z_2^3 + z_3^2 + z_4^2 + z_5^2 = 0, \quad \text{(intersected with } S^9 = \{\Sigma|z_i|^2 = 1\}\text{)}$$

describe the finite number of homotopy spheres. Why do we have infinitely many equations for a finite number of homotopy spheres? The answer was given in [E-Sz]: These equations give all the embeddings of these homotopy spheres in S^9 up to regular homotopy.

The present paper grew out from an attempt to investigate the analogous question for the equations

$$(*) \qquad x^2 + y^2 + z^2 + v^k = 0.$$

It was proved in [K-N] that the links of the singularities $(*)$ are S^5 or $S^3 \times S^2$ depending on the parity of k. Again we have infinite number of equations for both diffeomorphism types of links. So it seems natural to pose the analogous

Question: What are the differences between the links for different values of k of the same parity? Do they represent different immersions up to regular homotopy?

For k odd, when the link is S^5, the question about the regular homotopy turns out to be trivial, since any two immersions of S^5 to S^7 are regularly homotopic. (By Smale's result, see [S1], the set of regular homotopy classes of immersions $S^5 \longrightarrow S^7$ can be identified with $\pi_5(SO_7)$. The later group is trivial by Bott's result [B].)

The situation is quite different for k even. Put $k = 2d$ and let us denote by X_d the algebraic variety defined by the equation $(*)$, by L_d its link, and by i_d the inclusion $L_d \hookrightarrow S^7$. In this case the question on regular homotopy classes of i_d turns out to be not well-posed.

It is true that L_d is diffeomorphic to $S^3 \times S^2$ for any d, but the question about the regular homotopy makes sense only after having given a concrete diffeomorphism $\varphi_d : S^3 \times S^2 \longrightarrow L_d$, and only then we can ask about the regular homotopy classes

$$i_d \circ \varphi_d : S^3 \times S^2 \longrightarrow S^7.$$

(In the case of Brieskorn equations precomposing an immersion $f : \Sigma^7 \longrightarrow S^9$ with an orientation preserving self-diffeomorphism of the homotopy sphere Σ^7 does not change the regular homotopy class of the immersion f. This is not so for the manifold $S^3 \times S^2$.)

Definition (see [P]). Given manifolds M, N, and two immersions f_0 and f_1 from M to N, we say that f_0 and f_1 are *image-regular homotopic* if there is a self-diffeomorphism φ of M such that f_1 is regularly homotopic to $f_0 \circ \varphi$.

Notation:

1) $I(M,N)$ will denote the image-regular homotopy classes of immersions of M to N. The image regular homotopy class of an immersion f will be denoted by $\mathrm{im}\,[f]$.

2) Recall that an immersion is called framed if its normal bundle is trivialized. Fr-Imm (M,N) will denote the framed regular homotopy classes of framed immersions of M to N.

In the case when the immersion f is framed $\mathrm{reg}\,[f]$ will denote its framed regular homotopy class.

Remark. Note that for the inclusions $i_d : L_d \subset S^7$ their regular homotopy classes $\mathrm{reg}\,[i_d]$ are not well-defined, but their image regular homotopy classes $\mathrm{im}\,[i_d]$ are well-defined.

Formulation of the results

Theorem 1. *For any simply connected, stably parallelizable,* 5*-dimensional manifold* M^5 *the framed regular homotopy classes of framed immersions in* S^7 *can be identified with* $H^3(M;\mathbb{Z})$*, i.e.*

$$\text{Fr-Imm}\,(M^5, S^7) = H^3(M;\mathbb{Z}).$$

Corollary. *In particular,*

$$\text{Fr-Imm}\,(S^3 \times S^2, S^7) = \mathbb{Z}.$$

Theorem 2. *The set* $I(S^3 \times S^2, S^7)$ *of image-regular homotopy classes of framed immersions* $S^3 \times S^2 \longrightarrow S^7$ *can be identified with* $\mathbb{Z}_2$*.*

Theorem 3. *The inclusions* $i_d : \ L_d \hookrightarrow S^7$ *for* $d = d_1$ *and* d_2 *represent the same element in* $I(S^3 \times S^2, S^7) = \mathbb{Z}_2$ *(i.e.* $\mathrm{im}\,[i_{d_1}] = \mathrm{im}\,[i_{d_2}]$*) if and only if* $d_1 \equiv d_2 \mod 2$.

Remark. The identifications in the above Theorems arise only after we have fixed a parallelization of the manifolds (or a stable parallelization). (Different parallelizations provide different identifications. For the Corollary these identifications differ by an affine shift $x \mapsto x + a$, where $a \in \pi_3(SO) = \mathbb{Z}$ is the difference of the two parallelizations. Similarly, in Theorem 2, a is replaced by a mod 2 in $\mathbb{Z}_2 = \pi_3(SO)/\mathrm{im}\, j_*(\pi_3(SO_3))$, where j is the inclusion $j : SO_3 \subset SO$. Now we describe a concrete stable parallelization of $S^3 \times S^2$ we shall use.

Hence, we want to choose a trivialization of the stable tangent bundle

$$T(S^3 \times S^2) \oplus \mathcal{E}^1 \longrightarrow S^3 \times S^2,$$

where $\mathcal{E}^1$ is the trivial real line bundle. This 6-dimensional vector bundle is the same as the restriction $T(S^3 \times \mathbb{R}^3)\Big|_{S^3\times S^2} = (p_1^* TS^3 \oplus p_2^* T\mathbb{R}^3)\Big|_{S^3\times S^2}$, where p_1 and p_2 are the projections of $S^3 \times \mathbb{R}^3$ onto the factors. The quaternionic multiplication on S^3 gives a trivialization of TS^3, i.e. an identification with $S^3 \times \mathbb{R}^3$. We need a trivialization of $T(TS^3)$. The standard spherical metric on S^3 gives a connection on the bundle $TS^3 \longrightarrow S^3$, that is a "horizontal" $\mathbb{R}^3 \subset T(TS^3)$ at any point. The trivialization of TS^3 gives a trivialization of both the horizontal and the vertical (tangent to the fibers) components in $T(TS^3)$. Restricting this to the sphere bundle $S(TS^3) = S^3 \times S^2$ we obtain the required trivialization of

$$T(TS^3)\Big|_{S^3\times S^2} = T(S^3 \times \mathbb{R}^3)\Big|_{S^3\times S^2} = T(S^3 \times S^2) \oplus \mathcal{E}^1.$$

Proof of Theorem 1. Having fixed a stable parallelization of M^5, any framed immersion $f : \ M^5 \longrightarrow \mathbb{R}^q$ gives a map $M^5 \longrightarrow SO_q$ that – by a slight abuse of notation – we will denote by df.

By the Smale–Hirsch immersion theory [S1, H] the map

$$\begin{array}{ccc} \text{Fr-Imm}\,(\overset{\circ}{M}, \mathbb{R}^q) & \longrightarrow & [M, SO_q] \\ \text{reg}\,[f] & \longrightarrow & [df] \end{array}$$

induces a bijection, where $[M, SO_q]$ denotes the homotopy classes of maps $M \longrightarrow SO_q$.

Since M^5 is simply connected there is a cell-decomposition having a single 0-cell, a single 5-cell, and no 1-dimensional, neither 4-dimensional cells.

Let $\overset{\circ}{M}$ be the punctured M^5: $\overset{\circ}{M} = M^5 \setminus D^5$. From the Puppe sequence of the pair $(\overset{\circ}{M}, \partial\overset{\circ}{M})$ (see [Hu]),

$$S^4 = \partial\overset{\circ}{M} \subset \overset{\circ}{M} \subset M \longrightarrow S^5,$$

it follows that the restriction map $[M^5, SO_q] \to [\overset{\circ}{M}, SO_q]$ is a bijection, since $\pi_4(SO) = 0$ and $\pi_5(SO_q) = 0$.

Now consider the Puppe sequence of the pair $(\overset{\circ}{M}, sk_2\, M)$. Note that $sk_2\, M$ is a bouquete of 2-spheres, while the quotient $\overset{\circ}{M}/sk_2\, M$ is homotopically equivalent to a bouquette of 3-spheres. Hence, a part of the Puppe sequence looks like this:

$$sk_2\, M \subset \overset{\circ}{M} \longrightarrow \vee S^3 \longrightarrow S(sk_2\, M) = \vee S^3$$

where $S(\ )$ means the suspension. Mapping the spaces of this Puppe sequence to SO_q, $q \geq 5$, we obtain the following exact sequence of groups (we omit q):

$$\big[sk_2\, \overset{\circ}{M},\, SO\big] \longleftarrow \big[\overset{\circ}{M},\, SO\big] \longleftarrow \big[\vee S^3,\, SO\big] \overset{\alpha}{\longleftarrow} \big[S(sk_2\, M), SO\big].$$

Here $\big[sk_2\, \overset{\circ}{M},\, SO\big] = 0$, because $\pi_2(SO) = 0$.

Since $\pi_3(SO) = \mathbb{Z}$ the group $[\vee S^3,\, SO]$ can be identified with the group of 3-dimensional cochains of M with integer coefficients, i.e. $[\vee S^3,\, SO] = C^3(M; \mathbb{Z})$.

Since there are no 4-dimensional cells this is also the group of 3-dimensional cocycles. The group $\big[S(sk_2\, M),\, SO\big]$ can be identified with the group of 2-dimensional cochains $C^2(M; \mathbb{Z})$.

Lemma. *The map α can be identified with the coboundary map*

$$\delta\colon C^2(M; \mathbb{Z}) \longrightarrow C^3(M; \mathbb{Z}).$$

Proof of this Lemma will be given in the Appendix.

Hence the cokernel of α, i.e. $\big[\overset{\circ}{M}, SO\big] = \text{Fr-Imm}\,(M, \mathbb{R}^q)$ can be identified with the cokernel of δ, i.e. with $H^3(M; \mathbb{Z})$. □

Remark 1. In the case when $M = S^3 \times S^2$ and $N \in S^2$ is a fixed point in S^2, for example the North pole, the inclusion $S^3 \hookrightarrow M$, $x \longrightarrow (x, N)$ gives an isomorphism

$$[M, SO] \longrightarrow [S^3, SO].$$

Hence, for $M = S^3 \times S^2$ two framed immersions $M^5 \longrightarrow \mathbb{R}^7$ (or $M^5 \longrightarrow S^7$) are regularly homotopic if their restrictions to $S^3 \times N$ are framed regularly homotopic (adding the two normal vectors of S^3 in M^5 to the framing).

Lemma 1. *The inclusion $j\,:\ SO_3 \hookrightarrow SO_q$ $(q \geq 5)$ induces in π_3 the multiplication by 2 (if we choose the generators in $\pi_3(SO_3) = \mathbb{Z}$ and in $\pi_3(SO_q) = \mathbb{Z}$ properly), i.e., for any $x \in \pi_3(SO_3) = \mathbb{Z}$ the image $j_*(x) \in \pi_3(SO) = \mathbb{Z}$ is $2x$.*

Proof. It is well-known that $\pi_3(SO_5) \approx \pi_3(SO_6) \approx \cdots \approx \pi_3(SO)$ and by Bott's result [B] $\pi_3(SO) \approx \mathbb{Z}$. Let us consider $V_2(\mathbb{R}^5) = SO_5/SO_3$. It is well-known that $\pi_3\big(V_2(\mathbb{R}^5)\big) = \mathbb{Z}_2$ (see for example [M-S]). It is also well-known that $\pi_3(SO_3) = \mathbb{Z}$.

Now the exact sequence of the fibration $SO_5 \longrightarrow V_2(\mathbb{R}^5)$ gives that the homomorphism $\pi_3(SO_3) \longrightarrow \pi_3(SO_5)$ induced by the inclusion is a multiplication by $+2$ (or -2, but choosing the generators properly it can be supposed that it is multiplication by $+2$). □

Remark 2. It is well-known that $\pi_3(SO_4) = \pi_3(S^3) \oplus \pi_3(SO_3)$ and the map $j_{4*}: \ \pi_3(SO_4) \longrightarrow \pi_3(SO_5)$ induced by the inclusion

$$j_4: \ SO_4 \hookrightarrow SO_5$$

is epimorphic.

It follows that j_{4*} maps $\pi_3(S^3) = \mathbb{Z}$ to the group $\mathbb{Z}_2 = \pi_3(SO_5)/j_{4*}\big(\pi_3(SO_3)\big)$ epimorphically.

From now on we shall denote by M the manifold $S^3 \times S^2$ (except in the Appendix). We shall write simply S^3 for the subset $S^3 \times N \subset S^3 \times S^2$, where $N \in S^2$.

Lemma 2. *For any class $2m \in 2\mathbb{Z} = \operatorname{im} j_{4*} \subset \mathbb{Z} = \pi_3(SO)$, there is a diffeomorphism $\alpha_m: \ M \longrightarrow M$ such that for any framed immersion $f: \ M \longrightarrow \mathbb{R}^7$ the difference of the regular homotopy classes of f and $f \circ \alpha_m$ is $2m$, i.e.*

$$\operatorname{reg}[f \circ \alpha_m] - \operatorname{reg}[f] \in \pi_3(SO) = \mathbb{Z}$$

is $2m$.

Proof. Let $\mu_m: \ S^3 \longrightarrow SO_3$ be a map representing the class $m \in \pi_3(SO_3)$ and define the diffeomorphism

$$\alpha_m: \ S^3 \times S^2 \longrightarrow S^3 \times S^2$$

by the formula

$$(x, y) \longmapsto (x, \mu_m(x)y).$$

We have the following diagram:

$$\begin{array}{ccc} \operatorname{reg}[f] \in \text{Fr-Imm}\,(M, \mathbb{R}^q) & \longrightarrow & \text{Fr-Imm}\,(S^3, \mathbb{R}^q) \ni \operatorname{reg}\left[f\Big|_{S^3}\right] \\ \downarrow\approx & & \downarrow\approx \\ [M, SO_q] & \longrightarrow & [S^3, SO_q] \\ df & \longmapsto & df\Big|_{S^3} \\ d(f \circ \alpha_m) & \longmapsto & d(f \circ \alpha_m)\Big|_{S^3} \end{array}$$

It shows that the regular homotopy class of the (framed) immersion f is detected by the homotopy class of $df\Big|_{S^3}$ in $\pi_3(SO)$, while the regular homotopy class of $f \circ \alpha_m$ is detected by the homotopy class of $d(f \circ \alpha_m)\Big|_{S^3}$.

So we have to compare the homotopy classes of maps

$$df\Big|_{S^3}: \ S^3 \longrightarrow SO_q \quad \text{and} \quad d(f \circ \alpha_m)\Big|_{S^3}: \ S^3 \longrightarrow SO_q.$$

By the chain rule one has:

$$d(f \circ \alpha_m)\Big|_{S^3} = df\Big|_{\alpha_m(S^3)} \cdot d\alpha_m\Big|_{S^3}.$$

The restriction map $\alpha_m\Big|_{S^3}: \ S^3 \longrightarrow S^3 \times S^2$ is homotopic to a map into $S^3 \vee S^2$, representing in the third homotopy group $\pi_3(S^3 \vee S^2) = \mathbb{Z} \oplus \mathbb{Z}$ the element $(1, *)$, where $*$ is an integer,

$* \in \pi_3(S^2) = \mathbb{Z}$ (at this point its value is not important, but later we shall show that it is m, see Lemma A). Since the map df maps $S^3 \times S^2$ into SO and $\pi_2(SO) = 0$, the map $df\Big|_{S^3 \vee S^2}$ can be extended to $S^3 \vee D^3 \cong S^3$.

Finally we have that $d(f \circ \alpha_m)\Big|_{S^3}$ is homotopic to the pointwise product of the maps $df\Big|_{S^3}$ and $d\alpha_m\Big|_{S^3}$.

But it is well-known that this gives the sum of the homotopy classes $\Big[df\Big|_{S^3}\Big] \in \pi_3(SO)$ and $\Big[d\alpha_m\Big|_{S^3}\Big] \in \pi_3(SO)$.

It remained to show the following

Sublemma. $\Big[d\alpha_m\Big|_{S^3}\Big] = 2m \in \pi_3(SO_q) = \mathbb{Z}$.

Proof. The differential $d\alpha_m$ acts on $T(S^3 \times \mathbb{R}^3)\Big|_{S^3 \times S^2} = p_1^* TS^3 \oplus p_2^* T\mathbb{R}^3\Big|_{S^3 \times S^2}$ as follows: by identity on $p_1^* TS^3$ and by $\mu_m(x)$ on $(x, y) \times \mathbb{R}^3$ for any $x \in S^3$, $y \in S^2$.

Hence, $d\alpha_m\Big|_{S^3}$ is $j \circ \mu_m$, where $j : SO_3 \hookrightarrow SO_q$ is the inclusion. Recall that the map $\mu_m : S^3 \longrightarrow SO_3$ was chosen so that its homotopy class $[\mu_m] \in \pi_3(SO_3)$ is $m \in \mathbb{Z} = \pi_3(SO_3)$. Since j_* is "the multiplication by 2" map it follows that $\Big[d\alpha_m\Big|_{S^3}\Big] = 2m$. □

This ends the proof of Lemma 2 too. □

Proposition. *Any self-diffeomorphism of $S^3 \times S^2$ changes the regular homotopy class of any immersion by adding an element of the subgroup in* $\operatorname{im} j_* = 2\mathbb{Z} \subset \mathbb{Z} = \pi_3(SO)$. *That is for any framed immersion $f : M \longrightarrow \mathbb{R}^q$ with (framed) regular homotopy class*

$$\operatorname{reg}[f] \in [M, SO] = \pi_3(SO)$$

and any diffeomorphism $\varphi : M \longrightarrow M$ the difference of regular homotopy classes

$$\operatorname{reg}[f] - \operatorname{reg}[f \circ \varphi]$$

belongs to the subgroup $\operatorname{im} j_* = 2\mathbb{Z}$ *in* $\mathbb{Z} = \pi_3(SO)$.

The proof will rely on the following two lemmas (Lemma A and Lemma B).

Definition. A self-diffeomorphism $\varphi : S^3 \times S^2 \longrightarrow S^3 \times S^2$ will be called *positive* if it induces on $H_3(S^3 \times S^2) = \mathbb{Z}$ the identity.

Lemma A. *For any positive self-diffeomorphism φ there exists a natural number $m \in \mathbb{Z}$ such that for $N \in S^2$ the restrictions $\varphi\Big|_{(S^3 \times N)}$ and $\alpha_m\Big|_{(S^3 \times N)}$ represent the same homotopy class in $\pi_3(M)$.*

Lemma B. *Let φ and ψ be self-diffeomorphisms of M such that the images of $S^3 \times N$ at φ and ψ represent the same element in $\pi_3(M)$. Then for any framed-immersion $f : M \longrightarrow \mathbb{R}^7$ the regular homotopy classes of $f \circ \varphi$ and $f \circ \psi$ coincide.*

Proof of Lemma B. Let us extend the self-diffeomorphisms φ and ψ to those of $M \times D^q$ by taking the product with the identity map of D^q, for some large q, and denote these self-diffeomorphisms of $M \times D^q$ by $\hat{\varphi}$ and $\hat{\psi}$. Similarly we shall denote by $\hat{f}$ the product of f with the standard inclusion $D^q \subset \mathbb{R}^q$.

By the Smale–Hirsch theory [S1, H] (or by the so-called Compression Theorem of Rourke–Sanderson [R-S]) the restriction induces a bijection

$$\text{Fr-Imm}\,(M, \mathbb{R}^7) \longleftarrow \text{Fr-Imm}\,(M \times D^q, \mathbb{R}^{7+q}).$$

Again the regular homotopy class of a framed immersion in

$$\text{Fr-Imm}\,(M, \mathbb{R}^{7+q}) = \text{Fr-Imm}\,(M \times D^q, \mathbb{R}^{7+q})$$

is uniquely defined by the restriction to $S^3 (= S^3 \times N)$.

The maps $\hat{\varphi}$ and $\hat{\psi}$ restricted to the sphere $S^3 \times N$ are framed isotopic. By Thom's isotopy lemma [T] there is an isotopy $\Psi_t :\ M \times D^q \longrightarrow M \times D^q$ such that $\Psi_0 = \hat{\varphi}$ and $\Psi_1 = \hat{\psi}$.

It follows that the induced maps $d\hat{\varphi} :\ M \longrightarrow SO$ and $d\hat{\psi} :\ M \longrightarrow SO$ are homotopic. Hence, the framed-regular homotopy classes of $\hat{f} \circ \hat{\varphi}$ and $\hat{f} \circ \hat{\psi}$ coincide. Then the compositions $f \circ \varphi$ and $f \circ \psi$ are also regularly homotopic. $\square$

Proof of Lemma A. Let m be the homotopy class of the composition

$$S^3 \overset{i_\varphi}{\hookrightarrow} S^3 \times S^2 \overset{p}{\longrightarrow} S^2,$$

where i_φ is the inclusion $x \mapsto \varphi(x, N)$ and p is the projection $S^3 \times S^2 \longrightarrow S^2$. We claim that the maps $\varphi' = p \circ \varphi\Big|_{(S^3 \times N)}$ and $\alpha'_m = p \circ \alpha_m\Big|_{(S^3 \times N)}$ are homotopic maps from S^3 to S^2. To show this it is enough to compute the Hopf invariants of these maps.

Let us consider first the case $m = 1$. We need to show that the Hopf invariant of α'_1 is equal to 1.

The map $\mu_1 :\ S^3 \longrightarrow SO_3$ representing the generator in $\pi_3(SO_3)$ can be provided by the standard double covering $S^3 \longrightarrow SO_3$. Then α_1 is the self-diffeomorphism of $S^3 \times S^2$

$$\alpha_1(x, y) = \big(x, \mu_1(x) y\big)$$

and α'_1 is the composition of the following three maps: the inclusion

$$S^3 \hookrightarrow S^3 \times S^2, \quad x \longmapsto (x, N);$$

the map α_1 and the projection $p :\ S^3 \times S^2 \longrightarrow S^2$.

In order to compute the Hopf invariant of $\alpha'_1 :\ S^3 \longrightarrow S^2$ first we need to compute the preimage of a regular value. Let us compute first the preimage of N in S^3, i.e., $(\alpha'_1)^{-1}(N)$. The map α'_1 can be further decomposed as the composition of $\mu_1 :\ S^3 \longrightarrow SO_3$ with the evaluation map $e :\ SO_3 \longrightarrow S^2$, $g \mapsto g(N)$, for $g \in SO_3$. The set $e^{-1}(N)$ is the subgroup $SO_2 \subset SO_3$, which consists of the rotations around the line $\overrightarrow{(N, -N)}$ (the stabilizer subgroup of N).

When we identify SO_3 with the ball D^3_π of radius π with identified antipodal points on the boundary S^2_π, then this subgroup SO_2 corresponds to the diameter $\overline{N, -N}$ with identified endpoints N and $-N$. The preimage of this diameter at $\mu_1 :\ S^3 \longrightarrow SO_3$ is a great circle. If we take any other point V in S^2, then $e^{-1}(V)$ is a coset of the previous subgroup SO_2. Then its preimage at μ_1 is also a great circle. Therefore the linking number of two such preimages is 1.

The map α'_m can be obtained from α'_1 by precomposing it with a degree m map $S^3 \longrightarrow S^3$. Hence the Hopf invariant of α'_m is m. $\square$

Parametrizations of the links L_d (or, equivalently, of the singularities X_d)

Let us denote by ζ the complex $\mathbb{C}^2$-bundle $T\mathbb{C}P^1 \oplus \varepsilon^1_C$ over $\mathbb{C}P^1 = S^2$, where $T\mathbb{C}P^1$ is the tangent bundle of $\mathbb{C}P^1$, and ε^1_C is the trivial complex line bundle. Note that the bundle ζ considered as a real $\mathbb{R}^4$-bundle is isomorphic to the trivial bundle. Hence its total space is diffeomorphic to $S^2 \times \mathbb{R}^4$. Let us denote by $E_0(\zeta)$ the complement of the zero section in the total space of the bundle ζ. We shall give below a *diffeomorphism* of this space $E_0(\zeta)$ onto $X_d \setminus 0$.

The existence of such a diffeomorphism will give a new proof of the result of [K-N] about the diffeomorphism type of L_d.

Proposition. *L_d is diffeomorphic to $S^3 \times S^2$.*

Proof. $X_d \setminus 0$ is diffeomorphic to $L_d \times \mathbb{R}^1$, and the space $E_0(\zeta)$ is diffeomorphic to $S^3 \times S^2 \times \mathbb{R}^1$. For simply connected 5-manifolds it is well-known, that two such manifolds are diffeomorphic if their products with the real line are diffeomorphic (see [Ba], Theorem 2.2). Hence L_d and $S^3 \times S^2$ are diffeomorphic. □

Next we give a concrete parametrization:

$$\varphi_d : \; E_0(\zeta) \longrightarrow X_d \setminus 0 = \big\{x, y, z, v \;\big|\; x^2 + y^2 + z^2 + v^{2d} = 0, \; |x| + |y| + |z| + |v| \neq 0\big\}.$$

The composition $i_d \circ \varphi_d$ (or its restriction to $\varphi_d^{-1}(S^7)$) will give a framed-immersion

$$S^3 \times S^2 \longrightarrow S^7,$$

and its regular homotopy class reg $[i_d \circ \varphi_d]$ will turn out to be the number

$$d \in \mathbb{Z} = \text{Fr-Imm}\,(S^3 \times S^2, S^7).$$

This will imply that the image-regular homotopy class of the link L_d in S^7 is $d \mod 2$ in $\mathbb{Z}_2 = I(S^3 \times S^2, S^7)$.

Proof of Theorem 3. For arbitrary manifolds N and Q the natural map

$$\text{Fr-Imm}\,(N, Q) \longrightarrow \text{Fr-Imm}\,(N, Q \times \mathbb{R}^1)$$

induces a bijection — by the Smale–Hirsch immersion theory (or by the Compression Theorem of Rourke–Sanderson). Hence Fr-Imm $(X_d \setminus 0 \subset \mathbb{C}^4 \setminus 0) =$ Fr-Imm $(S^3 \times S^2 \subset S^7)$. By a coordinate transformation of $\mathbb{C}^4$ we obtain the following equivalent equation defining X_d

$$X_d = \big\{x, y, z, v \;\big|\; xy - z(z + v^d) = 0\big\}.$$

The parametrization of $X_d \setminus 0$ is the following.

The inclusion

$$E_0(\zeta) \xrightarrow{\Psi} \mathbb{C}^4 = \{(x, y, z, v) \mid x, y, z, v \in \mathbb{C}\} \text{ with image } \operatorname{im}\Psi = X_d \setminus 0$$

will be described on two charts:

1) $\big((a : b), x, v\big)$, where $a, b, x, v \in \mathbb{C}$, $b \neq 0$, $(a : b) \in \mathbb{C}P^1$, and $\|x\| + \|v\| \neq 0$. Put $t = \frac{a}{b} \in \mathbb{C}$. The map Ψ on this chart will be given by the formula

$$\Psi : \; (t, x, v) \longrightarrow (x, t^2 x + t v^d, t x, v).$$

2) For $a \neq 0$ denote the quotient $\frac{b}{a}$ by t'. On the part of $E_0(\zeta)$ that projects to $\mathbb{C}P^1 \setminus (1 : 0)$ (that is diffeomorphic to $\mathbb{C}P^1 \setminus (1 : 0) \times (\mathbb{C}^2 \setminus 0)$) consider the coordinates (t', y, v) and define Ψ by the formula

$$\Psi : \; (t', y, v) \longrightarrow (t'^2 y - t' v^d, y, t' y - v^d, v).$$

The change of coordinates between the two coordinate charts of $E_0(\zeta)$ is

$$t' = t^{-1}, \; v = v, \; x = t'^2 y - t' v^d \quad \text{or equivalently}$$

$$y = t^2 x + t v^d.$$

In order to see that these local coordinates give indeed the bundle ζ over $\mathbb{C}P^1$ we can precompose the first local system with the map $(t, x, v) \mapsto (t, x - t v^d, v)$. (Note that this map can be connected to the identity by the diffeotopy $(t, x, v) \mapsto (t, x - s t v^d, v)$, $0 \le s \le 1$.) Then the change from the first coordinate system to the second one for $t \in S^1$ on the equator of $S^2 = \mathbb{C}P^1$ will be given by the map $(t, x, v) \mapsto (t, t^2 x, v)$, where $x, v \in \mathbb{C}$. Now it is clear that the obtained bundle is $\zeta = T\mathbb{C}P^1 \oplus \varepsilon_C^1$. (The map of the equator to $U(2)$ defining the bundle ζ gives in $\pi_1(U(2))$ the double of the generator, and its image in $\pi_1(SO_4) = Z_2$ is trivial. That is why the bundle ζ is trivial as a real bundle although it has first Chern class equal 2 as a complex bundle.) Note that Ψ maps the part of the first chart corresponding to the points $t = 0$, (i.e., the space $\mathbb{C}^2 = \{(0:1), x, v\}$) identically onto the coordinate space $\mathbb{C}^2_{x,v} = \{x, 0, 0, v\}$ of $\mathbb{C}^4$. The restriction of Ψ to this part determines the framed immersion of $X_d \setminus 0$ to $\mathbb{C}^4$. Hence, the immersion itself is very simple: just the inclusion of $\mathbb{C}^2 \setminus 0 \to \mathbb{C}^4$. But we need to consider also the framing. It is coming a) from the paramatrization Ψ and b) from the defining equation of X_d.

a) The parametrization gives the complex vector field

$$\frac{\partial \Psi}{\partial t}\Big|_{t=0} = (0, v^d, x, 0).$$

b) The defining equation $g(x,y,z,v) = xy - z(z + v^d) = 0$ at the points $(x, 0, 0, v)$ gives the complex vector field

$$\operatorname{grad} g(x, 0, 0, v) = (0, x, -v^d, 0).$$

These two complex vector fields have zero first and last complex coordinates (on the coordinate subspace $\mathbb{C}^2_{x,v} = \{x, 0, 0, v\}$). Hence, we shall write only their second and third coordinates: those are (v^d, x) and $(-x, v^d)$ respectively. These two complex vectors give four real vector fields if we add their i-images as well. Let us denote by a_1 and a_2 the real and imaginary coordinates of v^d: $v^d = a_1 + ia_2$. Similarly x_1 and x_2 are those of x, i.e., $x = x_1 + ix_2$. Then the four real vectors in $\mathbb{R}^4 = \mathbb{C}^2 = (0, y, z, 0)$ are:

$$\begin{aligned}
\mathbf{u}_1 &= (a_1, a_2, x_1, x_2)\\
\mathbf{u}_2 &= (a_2, -a_1, x_2, -x_1)\\
\mathbf{u}_3 &= (x_1, x_2, -a_1, -a_2)\\
\mathbf{u}_4 &= (-x_2, x_1, a_2, -a_1).
\end{aligned}$$

The map $(x, v) \in \mathbb{R}^4 \setminus 0 \longrightarrow (\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4)$ can be decomposed as a degree d branched covering $(x, v) \mapsto (x, v^d)$ and a map representing an element in $\pi_3(SO_4) = \pi_3(S^3) \oplus \pi_3(SO_3)$ of the form $(1, *)$ for some unknown element $*$ in $\pi_3(SO_3)$. (This is because the map

$$(x, v^d) = (x_1, x_2, a_1, a_2) \mapsto \mathbf{u}_1 = (a_1, a_2, x_1, x_2)$$

is almost the identity, it differs only by an even permutation of the coordinates.) Hence the composition represents an element of the form $(d, ?) \in \pi_3(S^3) \oplus \pi_3(SO_3)$, and its image in $\pi_3(SO)/j_{4*}(\pi_3(SO_3)) = \mathbb{Z}_2$ is $d \bmod 2$, see Remark 2. That finishes the proof of Theorem 3. □

Appendix

For any space Y let us denote by CY the cone over Y. Here we show that the map provided by the Puppe sequence

$$\alpha: \ \big[C(\overset{\circ}{M}) \cup C(sk_2 M), SO\big] \longrightarrow \big[\overset{\circ}{M} \cup C(sk_2 M), SO\big]$$

can be identified with the coboundary map in the cochain complex:

$$\delta: \ C^2(M; \mathbb{Z}) \longrightarrow C^3(M; \mathbb{Z}).$$

We have seen that the sources and targets of δ and α can be identified.

For simplicity let us consider the situation when $sk_2 M = S^2$ and M has a single 3-cell D^3, attached to this S^2 by a map θ of degree k. Then $\overset{\circ}{M} = S^2 \underset{\theta}{\cup} D^3$.

Let us denote the sets

$$\overset{\circ}{M} \cup C(sk_2 M) \quad \text{and} \quad C\overset{\circ}{M} \cup C(sk_2 M)$$

by A and B respectively.

Clearly we can choose *any* degree k map for θ in order to study the induced map α. Take for θ a branched k-fold cover of S^2 along S^0. Then the inclusion $A \subset B$ can be described homotopically as follows:

In $S^3 \times [0,1]$ contract an interval $* \times [0,1]$ for some $* \in S^3$ to a point. A will be identified with $S^3 \times \{0\}$. Further on $S^3 \times \{1\}$ identify the points that are mapped into the same point by the suspension of θ. The part of B coming from $S^3 \times \{1\}$ will be denoted by B_1. The space B_1 is a deformation retract of B.

Let us denote by r the retraction $B \longrightarrow B_1$. Clearly, its restriction $r|_A : A \longrightarrow B_1$ is a degree k map (it is actually the suspension of the branched covering θ). So the inclusion $A \subset B$ induces in the 3-dimensional homology group H_3 (or in π_3) a multiplication by k.

The proof of the special case (when in M there is a single 2-cell and a single 3-cell) is finished. The general case follows easily taking first the quotient of $sk_2 M$ by all but one 2-cell and considering any single 3-cell.

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