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# LIGHTLIKE HYPERSURFACES ALONG SPACELIKE SUBMANIFOLDS IN DE SITTER SPACE

SHYUICHI IZUMIYA AND TAKAMI SATO

Abstract. We consider the singularities of lightlike hypersurfaces along spacelike submanifolds with general codimension in de Sitter space. As an application of the theory of Legendrian singularities, we investigate the geometric meanings of the singularities of lightlike hypersurfaces from the viewpoint of the contact of spacelike submanifolds with de Sitter lightcones.

## 1. Introduction

One of the important objects in theoretical physics is the notion of lightlike hypersurfaces because they provide good models for different types of horizons [3, 5, 20, 23]. The lightlike hypersurfaces are constructed as ruled hypersurfaces along spacelike submanifolds whose rulings are the lightlike geodesics. A lightlike hypersurface is also called a *light sheet* in theoretical physics (cf., [2]), which plays a principal role in the quantum theory of gravity. In this paper, we consider the singularities of lightlike hypersurfaces along spacelike submanifolds in de Sitter space which is one of the Lorentz space forms. There are three kinds of Lorentz space forms: Lorentz-Minkowski space is a flat Lorentz space form, de Sitter space is a positively curved one, and anti-de Sitter space is a negatively curved one.

On the other hand, tools in the theory of singularities have proven to be useful in the description of geometrical properties of submanifolds immersed in different ambient spaces, from both the local and global viewpoint [6, 7, 9, 10, 11, 13, 16, 18]. The natural connection between geometry and singularities relies on the basic fact that the contacts of a submanifold with the models of the ambient space can be described by means of the analysis of the singularities of appropriate families of contact functions, or equivalently, of their associated Legendrian maps ([1, 21, 24]). When working in a Lorentz space form, the properties associated to the contacts of a given submanifold with lightcones have a special relevance. In [4, 8, 11, 17], a framework for the study of spacelike submanifolds with codimension two in Lorentz space forms was constructed, and a Lorentz invariant concerning their contacts with models related to lightlike hyperplanes was discovered. The geometry described in this framework is called the *lightlike geometry* of spacelike submanifolds with codimension two. By using the invariants of lightlike geometry, the singularities of lightlike hypersurfaces along spacelike submanifolds with codimension two in Lorentz-Minkowski space or de Sitter space were investigated in [10, 12, 16]. However, the situation is rather complicated for the general codimensional case. The main difference from the Euclidean space (or, Hyperbolic space) case is the fiber of the canal hypersurface of a spacelike submanifold is neither connected nor compact. In order to avoid the above difficulty, we arbitrarily choose a timelike future directed unit normal vector field along the spacelike submanifold, which always exists for an orientable submanifold (cf., [13, 14, 15]). Then we construct the unit spherical normal bundle relative to the above timeline unit normal vector field, which can be considered as a codimension two spacelike canal submanifold of the ambient Lorentz space form.

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Therefore, we can apply the idea of the lightlike geometry of spacelike submanifolds with codimension two in Lorentz space-forms. Recently, we have applied this framework and investigated the geometric meanings of the singularities of lightlike hypersurfaces along spacelike submanifolds in Lorentz-Minkowski space or anti-de Sitter space from the viewpoint of the theory of Legendrian singularities [14, 15]. In this paper, we consider spacelike submanifolds with general codimensions in de Sitter space applying an idea similar to [14, 15].

In §2 the basic notions of Lorentz-Minkowski space are described. We explain the differential geometry of spacelike submanifolds with general codimension in de Sitter space in §3. The notion of lightlike hypersurfaces is introduced in §4 and investigated the basic properties. In §5 we investigate the geometric meanings of the singularities of lightlike hypersurfaces in de Sitter space from the viewpoint of the theory of contact with de Sitter lightcones and the theory of Legendrian singularities. We review the classification result of Kasedou [17] on singularities of lightlike hypersurfaces along spacelike surfaces in de Sitter 4-space in §5.

## 2. Basic notions

In this section we prepare basic notions on Lorentz-Minkowski space. Let $\mathbb{R}^{n+1}$ be an $(n+1)$-dimensional cartesian space. For any vectors $\boldsymbol{x} = (x_0, x_1, \dots, x_n)$, $\boldsymbol{y} = (y_0, y_1, \dots, y_n) \in \mathbb{R}^{n+1}$, the *pseudo scalar product* of $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined by $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = -x_0y_0 + \sum_{i=1}^{n} x_iy_i$. The space $(\mathbb{R}^{n+1}, \langle , \rangle)$ is called *Lorentz-Minkowski* $(n+1)$*-space* and denoted by $\mathbb{R}^{n+1}_1$. We say that a vector $\boldsymbol{x}$ in $\mathbb{R}^{n+1}_1 \setminus \{\boldsymbol{0}\}$ is *spacelike*, *lightlike* or *timelike* if $\langle \boldsymbol{x}, \boldsymbol{x} \rangle > 0, = 0$ or $< 0$ respectively. The norm of the vector $\boldsymbol{x} \in \mathbb{R}^{n+1}_1$ is defined by $\|\boldsymbol{x}\| = \sqrt{|\langle \boldsymbol{x}, \boldsymbol{x} \rangle|}$. We define a *hyperplane with pseudo normal* $\boldsymbol{v}$ by $HP(\boldsymbol{v}, c) = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 \mid \langle \boldsymbol{x}, \boldsymbol{v} \rangle = c\}$, where $\boldsymbol{v} \in \mathbb{R}^{n+1}_1 \setminus \{\boldsymbol{0}\}$ and $c$ is a real number. We call $HP(\boldsymbol{v}, c)$ a *spacelike hyperplane*, a *timelike hyperplane* or a *lightlike hyperplane* if $\boldsymbol{v}$ is timelike, spacelike or lightlike respectively. We have the following three kinds of pseudo-spheres in $\mathbb{R}^{n+1}_1$: The *hyperbolic n-space* is defined by

$$H^n(-1) = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 | \ \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -1\},$$

the *de Sitter n-space* by

$$S^n_1 = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 | \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 1\}$$

and the (*open*) *lightcone* by

$$LC^* = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 \setminus \{\boldsymbol{0}\} | \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0\}.$$

We also define $LC_{\boldsymbol{\lambda}_0} = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 \ | \langle \boldsymbol{x} - \boldsymbol{\lambda}_0, \boldsymbol{x} - \boldsymbol{\lambda}_0 \rangle = 0\}$ which is called a *lightcone with the vertex* $\boldsymbol{\lambda}_0$.

For any $\boldsymbol{x}^1, \boldsymbol{x}^2, \dots, \boldsymbol{x}^n \in \mathbb{R}^{n+1}_1$, we define a vector $\boldsymbol{x}^1 \wedge \boldsymbol{x}^2 \wedge \cdots \wedge \boldsymbol{x}^n$ by

$$\boldsymbol{x}^1 \wedge \boldsymbol{x}^2 \wedge \cdots \wedge \boldsymbol{x}^n = \begin{vmatrix} -\boldsymbol{e}_0 & \boldsymbol{e}_1 & \cdots & \boldsymbol{e}_n \\ x^1_0 & x^1_1 & \cdots & x^1_n \\ x^2_0 & x^2_1 & \cdots & x^2_n \\ \vdots & \vdots & \cdots & \vdots \\ x^n_0 & x^n_1 & \cdots & x^n_n \end{vmatrix},$$

where $\boldsymbol{e}_0, \boldsymbol{e}_1, \dots, \boldsymbol{e}_n$ is the canonical basis of $\mathbb{R}^{n+1}_1$ and $\boldsymbol{x}^i = (x^i_0, x^i_1, \dots, x^i_n)$.

## 3. Differential geometry on spacelike submanifolds in de Sitter space

In [16] Kasedou has investigated differential geometry of spacelike submanifolds in de Sitter space from the viewpoint of contact with de Sitter hyperhorospheres. Here we construct another framework on differential geometry of spacelike submanifolds in de Sitter space. Let $\mathbb{R}^{n+1}_1$ be an oriented and time-oriented space. We choose $\boldsymbol{e}_0 = (1, 0, \dots, 0)$ as a future timelike vector field. We consider de Sitter $n$-space $S_1^n \subset \mathbb{R}_1^{n+1}$. Let $\boldsymbol{X} : U \longrightarrow S_1^n$ be a spacelike embedding of codimension $k$, where $U \subset \mathbb{R}^s$ $(s+k=n)$ is an open subset. We also write $M = \boldsymbol{X}(U)$ and identify $M$ and $U$ through the embedding $\boldsymbol{X}$ as usual. Since $M$ is a spacelike submanifold with codimension $k+1$ in $\mathbb{R}_1^{n+1}$, $N_p(M)$ is a $(k+1)$-dimensional Lorentzian subspace of $T_p\mathbb{R}_1^{n+1}$ (cf.,[22]). On the pseudo-normal space $N_p(M)$, we have two kinds of $k$-dimensional pseudo spheres:

$$\begin{aligned} N_p(M;-1) &= \{\boldsymbol{v} \in N_p(M) \mid \langle \boldsymbol{v}, \boldsymbol{v} \rangle = -1\ \} \\ N_p(M;1) &= \{\boldsymbol{v} \in N_p(M) \mid \langle \boldsymbol{v}, \boldsymbol{v} \rangle = 1\ \}, \end{aligned}$$

so that we have two unit pseudo-spherical normal bundles over $M$:

$$N(M;-1) = \bigcup_{p \in M} N_p(M;-1) \text{ and } N(M;1) = \bigcup_{p \in M} N_p(M;1).$$

Since $M = \boldsymbol{X}(U)$ is spacelike, $\boldsymbol{e}_0 \notin T_pM$. For any $\boldsymbol{v} \in T_p\mathbb{R}_1^{n+1}|M$, we have $\boldsymbol{v} = \boldsymbol{v}_1 + \boldsymbol{v}_2$, where $\boldsymbol{v}_1 \in T_pM$ and $\boldsymbol{v}_2 \in N_p(M)$. If $\boldsymbol{v}$ is timelike, then $\boldsymbol{v}_2$ is timelike. Let

$$\pi_{N(M)} : T\mathbb{R}_1^{n+1}|M \longrightarrow N(M)$$

be the canonical projection. Then $\pi_{N(M)}(\boldsymbol{e}_0)$ is a future directed timelike normal vector field along $M$. If we project $\pi_{N(M)}(\boldsymbol{e}_0)$ onto the normal space of $T_pM$ in $T_pS_1^n$, then we have a future directed unit timelike normal vector field in $TS_1^n$ along $M$ (even globally). We now arbitrarily choose a future directed unit timelike normal vector field $\boldsymbol{n}^T(u) \in N_p(M;-1) \cap T_pS_1^n$, where $p = \boldsymbol{X}(u)$. Therefore we have the pseudo-orthonormal compliment $(\langle \boldsymbol{n}^T(u) \rangle_{\mathbb{R}})^{\perp}$ in $N_p(M) \cap T_pS_1^n$ which is a $(k-1)$-dimensional subspace of $N_p(M)$. We define a $(k-2)$-dimensional spacelike unit sphere in $N_p(M)$ by $N_1^{dS}(M)_p[\boldsymbol{n}^T] = \{\boldsymbol{\xi} \in N_p(M;1) \mid \langle \boldsymbol{\xi}, \boldsymbol{n}^T(p) \rangle = \langle \boldsymbol{\xi}, \boldsymbol{X}(u) \rangle = 0\ \}$. Then we have a *spacelike unit* $(k-2)$*-spherical bundle over* $M$ *with respect to* $\boldsymbol{n}^T$ defined by

$$N_1^{dS}(M)[\boldsymbol{n}^T] = \bigcup_{p \in M} N_1^{dS}(M)_p[\boldsymbol{n}^T].$$

Since we have $T_{(p,\xi)}N_1^{dS}(M)[\boldsymbol{n}^T] = T_pM \times T_\xi N_1^{dS}(M)_p[\boldsymbol{n}^T]$, we have the canonical Riemannian metric on $N_1^{dS}(M)[\boldsymbol{n}^T]$ which is denoted by $(G_{ij}(p,\boldsymbol{\xi}))_{1 \leqslant i,j \leqslant n-2}$.

On the other hand, we define a map $\mathbb{LG}(\boldsymbol{n}^T) : N_1^{dS}(M)[\boldsymbol{n}^T] \longrightarrow LC^*$ by

$$\mathbb{LG}(\boldsymbol{n}^T)(u,\boldsymbol{\xi}) = \boldsymbol{n}^T(u) + \boldsymbol{\xi},$$

which we call the *de Sitter lightcone Gauss image* of $N_1^{dS}(M)[\boldsymbol{n}^T]$. This map leads us to the notions of curvatures. Let $T_{(p,\xi)}N_1^{dS}(M)[\boldsymbol{n}^T]$ be the tangent space of $N_1^{dS}(M)[\boldsymbol{n}^T]$ at $(p,\boldsymbol{\xi})$. Under the canonical identification

$$(\mathbb{LG}(\boldsymbol{n}^T)^* T\mathbb{R}_1^{n+1})_{(p,\boldsymbol{\xi})} = T_{(\boldsymbol{n}^T(p)+\boldsymbol{\xi})}\mathbb{R}_1^{n+1} \equiv T_p\mathbb{R}_1^{n+1},$$

we have

$$T_{(p,\boldsymbol{\xi})}N_1^{dS}(M)[\boldsymbol{n}^T] = T_pM \oplus T_\xi S^{k-2} \subset T_pM \oplus N_p(M) = T_p\mathbb{R}_1^{n+1},$$

where $T_\xi S^{k-2} \subset T_\xi N_p(M) \equiv N_p(M)$ and $p = \boldsymbol{X}(u)$. Let

$$\Pi^t : \mathbb{LG}(\boldsymbol{n}^T)^* T\mathbb{R}_1^{n+1} = TN_1(M)[\boldsymbol{n}^T] \oplus \mathbb{R}^{k+1} \longrightarrow TN_1^{dS}(M)[\boldsymbol{n}^T]$$

be the canonical projection. Then we have a linear transformation

$$S_\ell(\boldsymbol{n}^T)_{(p,\boldsymbol{\xi})} = -\Pi^t_{\mathbb{LG}(n^T)(p,\xi)} \circ d_{(p,\xi)}\mathbb{LG}(\boldsymbol{n}^T) : T_{(p,\xi)}N_1^{dS}(M)[\boldsymbol{n}^T] \longrightarrow T_{(p,\xi)}N_1^{dS}(M)[\boldsymbol{n}^T],$$

which is called the *de Sitter lightcone shape operator* of $N_1^{dS}(M)[\boldsymbol{n}^T]$ at $(p,\boldsymbol{\xi})$. Consider the eigenvalues of $S_\ell(\boldsymbol{n}^T)_{(p,\boldsymbol{\xi})}$, $(i = 1,\ldots,n-2)$. Then we write $\kappa_\ell(\boldsymbol{n}^T)_i(p,\boldsymbol{\xi})$, $(i = 1,\ldots,s)$ for the eigenvalues whose eigenvectors belong to $T_pM$ and $\kappa_\ell(\boldsymbol{n}^T)_i(p,\boldsymbol{\xi})$, $(i = s+1,\ldots n-2)$ for the eigenvalues whose eigenvectors belong to the tangent space of the fiber of $N_1^{dS}(M)[\boldsymbol{n}^T]$. By exactly the same arguments as those in [13, 15], we have $\kappa_\ell(\boldsymbol{n}^T)_i(p,\boldsymbol{\xi}) = -1$, $(i = s+1, \dots n-2)$. We call $\kappa_\ell(\boldsymbol{n}^T)_i(p,\boldsymbol{\xi})$, $(i = 1,\dots,s)$ the *de Sitter lightcone principal curvatures* of $M$ with respect to $(\boldsymbol{n}^T,\boldsymbol{\xi})$ at $p \in M$.

We deduce now the lightcone Weingarten formula. Since $\boldsymbol{X}$ is a spacelike embedding, we have a Riemannian metric (the *first fundamental form* ) on $M = \boldsymbol{X}(U)$ defined by

$$ds^2 = \sum_{i=1}^{s} g_{ij} du_i du_j,$$

where $g_{ij}(u) = \langle \boldsymbol{X}_{u_i}(u), \boldsymbol{X}_{u_j}(u)\rangle$ for any $u \in U$. Let $\boldsymbol{n}^S$ be a local section of $N_1^{dS}(M)[\boldsymbol{n}^T]$. Clearly, the vectors $\boldsymbol{n}^T(u) \pm \boldsymbol{n}^S(u)$ are lightlike. Here we choose $\boldsymbol{n}^T + \boldsymbol{n}^S$ as a lightlike normal vector field along $M$. We define a mapping $\mathbb{LG}(\boldsymbol{n}^T,\boldsymbol{n}^S) : U \longrightarrow LC^*$ by

$$\mathbb{LG}(\boldsymbol{n}^T,\boldsymbol{n}^S)(u) = \boldsymbol{n}^T(u) + \boldsymbol{n}^S(u).$$

We call it the *lightcone Gauss image* of $M = \boldsymbol{X}(U)$ with respect to $(\boldsymbol{n}^T,\boldsymbol{n}^S)$. Under the identification of $M$ and $U$ through $\boldsymbol{X}$, we have the linear mapping provided by the derivative of the lightcone Gauss image $\mathbb{LG}(\boldsymbol{n}^T,\boldsymbol{n}^S)$ at each point $p \in M$,

$$d_p\mathbb{LG}(\boldsymbol{n}^T,\boldsymbol{n}^S) : T_pM \longrightarrow T_p\mathbb{R}^{n+1}_1 = T_pM \oplus N_p(M).$$

Consider the orthogonal projection $\pi^t : T_pM \oplus N_p(M) \to T_p(M)$. We define

$$d_p\mathbb{LG}(\boldsymbol{n}^T,\boldsymbol{n}^S)^t = \pi^t \circ d_p(\boldsymbol{n}^T + \boldsymbol{n}^S).$$

We call the linear transformation $S_p(\boldsymbol{n}^T,\boldsymbol{n}^S) = -d_p\mathbb{LG}(\boldsymbol{n}^T,\boldsymbol{n}^S)^t$ the $(\boldsymbol{n}^T,\boldsymbol{n}^S)$*-shape operator* of $M = \boldsymbol{X}(U)$ at $p = \boldsymbol{X}(u)$. Let $\{\kappa_i(\boldsymbol{n}^T,\boldsymbol{n}^S)(p)\}_{i=1}^s$ be the eigenvalues of $S_p(\boldsymbol{n}^T,\boldsymbol{n}^S)$, which are called the *lightcone principal curvatures with respect to* $(\boldsymbol{n}^T,\boldsymbol{n}^S)$ at $p = \boldsymbol{X}(u)$. Then we have a *lightcone second fundamental invariant with respect to* $(\boldsymbol{n}^T,\boldsymbol{n}^S)$ defined by

$$h_{ij}(\boldsymbol{n}^T,\boldsymbol{n}^S)(u) = \langle -(\boldsymbol{n}^T + \boldsymbol{n}^S)_{u_i}(u), \boldsymbol{X}_{u_j}(u)\rangle$$

for any $u \in U$. By the similar arguments to those in the proof of [11, Proposition 3.2], we have the following proposition.

**Proposition 3.1.** *Let* $\{\boldsymbol{X},\boldsymbol{n}^T,\boldsymbol{n}^S_1,\dots,\boldsymbol{n}^S_{k-2}\}$ *be a pseudo-orthonormal frame of* $N(M)$ *with* $\boldsymbol{n}^S_{k-2} = \boldsymbol{n}^S$. *Then we have the following lightcone Weingarten formula* :

(a) $\mathbb{LG}(\boldsymbol{n}^T,\boldsymbol{n}^S)_{u_i} = \langle \boldsymbol{n}^T_{u_i},\boldsymbol{n}^S\rangle(\boldsymbol{n}^T+\boldsymbol{n}^S) + \sum_{\ell=1}^{k-3}\langle(\boldsymbol{n}^T+\boldsymbol{n}^S)_{u_i},\boldsymbol{n}^S_\ell\rangle\boldsymbol{n}^S_\ell - \sum_{j=1}^{s} h_i^j(\boldsymbol{n}^T,\boldsymbol{n}^S)\boldsymbol{X}_{u_j}$

(b) $\pi^t \circ \mathbb{LG}(\boldsymbol{n}^T,\boldsymbol{n}^S)_{u_i} = -\sum_{j=1}^{s} h_i^j(\boldsymbol{n}^T,\boldsymbol{n}^S)\boldsymbol{X}_{u_j}$.

*Here* $\left(h_i^j(\boldsymbol{n}^T,\boldsymbol{n}^S)\right) = \left(h_{ik}(\boldsymbol{n}^T,\boldsymbol{n}^S)\right)\left(g^{kj}\right)$ *and* $\left(g^{kj}\right) = \left(g_{kj}\right)^{-1}$.

Since $\langle -(\boldsymbol{n}^T+\boldsymbol{n}^S)(u), \boldsymbol{X}_{u_j}(u)\rangle = 0$, we have $h_{ij}(\boldsymbol{n}^T,\boldsymbol{n}^S)(u) = \langle \boldsymbol{n}^T(u)+\boldsymbol{n}^S(u), \boldsymbol{X}_{u_iu_j}(u)\rangle$. Therefore the lightcone second fundamental invariant at a point $p_0 = \boldsymbol{X}(u_0)$ depends only on the values $\boldsymbol{n}^T(u_0)+\boldsymbol{n}^S(u_0)$ and $\boldsymbol{X}_{u_iu_j}(u_0)$, respectively. Thus, the lightcone curvatures also depend only on $\boldsymbol{n}^T(u_0)+\boldsymbol{n}^S(u_0)$, $\boldsymbol{X}_{u_i}(u_0)$ and $\boldsymbol{X}_{u_iu_j}(u_0)$, independent of the derivation of the vector fields $\boldsymbol{n}^T$ and $\boldsymbol{n}^S$. We write $\kappa_i(\boldsymbol{n}^T_0,\boldsymbol{n}^S_0)(p_0)$ $(i = 1,\dots,s)$ as the lightcone principal curvatures at $p_0 = \boldsymbol{X}(u_0)$ with respect to $(\boldsymbol{n}^T_0,\boldsymbol{n}^S_0) = (\boldsymbol{n}^T(u_0),\boldsymbol{n}^S(u_0))$. So we write that

$$h_{ij}(\boldsymbol{n}^T,\boldsymbol{\xi})(u_0) = h_{ij}(\boldsymbol{n}^T,\boldsymbol{n}^S)(u_0)$$

and $\kappa_\ell(\boldsymbol{n}^T)_i(\boldsymbol{\xi},p_0) = \kappa_i(\boldsymbol{n}^T_0,\boldsymbol{n}^S_0)(p_0)$, where $\boldsymbol{\xi} = \boldsymbol{n}^S(u_0)$ for some local extension $\boldsymbol{n}^T(u)$ of $\boldsymbol{\xi}$. Let $\kappa_\ell(\boldsymbol{n}^T)_i(p,\boldsymbol{\xi})$ be the eigenvalues of $S_\ell(\boldsymbol{n}^T)_{(p,\boldsymbol{\xi})}$, $(i = 1,\dots,n-1)$. Here, we write

$$\kappa_\ell(\boldsymbol{n}^T)_i(p,\boldsymbol{\xi}),\ \ (i = 1,\dots,s)$$

for the eigenvalues belonging to the eigenvectors on $T_pM$ and

$$\kappa_\ell(\boldsymbol{n}^T)_i(p,\boldsymbol{\xi}),\ (i=s+1,\dots n-1)$$

for the eigenvalues belonging to the eigenvectors on the tangent space of the fiber of $N_1(M)[\boldsymbol{n}^T]$. Then we have the following proposition.

**Proposition 3.2.** *We choose a (local) pseudo-orthonormal frame* $\{\boldsymbol{X},\boldsymbol{n}^T,\boldsymbol{n}^S_1,\dots,\boldsymbol{n}^S_{k-2}\}$ *of* $N(M)$ *with* $\boldsymbol{n}^S_{k-2}=\boldsymbol{n}^S$*. For* $p_0=\boldsymbol{X}(u_0)$ *and* $\boldsymbol{\xi}_0=\boldsymbol{n}^S(u_0)$*, we have*

$$\kappa_\ell(\boldsymbol{n}^T)_i(p_0,\boldsymbol{\xi}_0)=\kappa_i(\boldsymbol{n}^T,\boldsymbol{n}^S)(u_0),\ (i=1,\dots,s)$$

*and* $\kappa_\ell(\boldsymbol{n}^T)_i(p_0,\boldsymbol{\xi}_0)=-1$, *(*$i=s+1,\dots n-1$*).*

*Proof.* Since $\{\boldsymbol{X},\boldsymbol{n}^T,\boldsymbol{n}^S_1,\dots,\boldsymbol{n}^S_{k-2}\}$ is a pseudo-orthonormal frame of $N(M)$, we have

$$\langle\boldsymbol{X}(u_0),\boldsymbol{\xi}_0\rangle=\langle\boldsymbol{n}^T(u_0),\boldsymbol{\xi}_0\rangle=\langle\boldsymbol{n}^S_i(u_0),\boldsymbol{\xi}_0\rangle=0.$$

Therefore, we have

$$T_{\boldsymbol{\xi}}S^{k-2}=\langle\boldsymbol{n}^S_1(u_0),\dots,\boldsymbol{n}^S_{k-2}(u_0)\rangle.$$

Using this orthonormal basis of $T_{\boldsymbol{\xi}_0}S^{k-2}$, the canonical Riemannian metric $G_{ij}(p_0,\boldsymbol{\xi}_0)$ is represented by

$$(G_{ij}(p_0,\boldsymbol{\xi}))=\begin{pmatrix} g_{ij}(p_0) & 0\\ 0 & I_{k-2}\end{pmatrix},$$

where $g_{ij}(p_0)=\langle\boldsymbol{X}_{u_i}(u_0),\boldsymbol{X}_{u_j}(u_0)\rangle$.

On the other hand, by Proposition 3.1, we have

$$-\sum_{j=1}^{s}h^j_i(\boldsymbol{n}^T,\boldsymbol{n}^S)(u_0)\boldsymbol{X}_{u_j}=\mathbb{LG}(\boldsymbol{n}^T,\boldsymbol{n}^S)_{u_i}(u_0)=d_{p_0}\mathbb{LG}(\boldsymbol{n}^T,\boldsymbol{n}^S)\left(\frac{\partial}{\partial u_i}\right),$$

so that we have

$$S_\ell(\boldsymbol{n}^T)_{(p_0,\boldsymbol{\xi}_0)}\left(\frac{\partial}{\partial u_i}\right)=\sum_{j=1}^{s}h^j_i(\boldsymbol{n}^T,\boldsymbol{n}^S)(u_0)\boldsymbol{X}_{u_j}.$$

Therefore, the representation matrix of $S_\ell(\boldsymbol{n}^T)_{(p_0,\boldsymbol{\xi}_0)}$ with respect to the basis

$$\{\boldsymbol{X}_{u_1}(u_0),\dots,\boldsymbol{X}_{u_s}(u_0),\boldsymbol{n}^S_1(u_0),\dots,\boldsymbol{n}^S_{k-2}(u_0)\}$$

of $T_{(p_0,\boldsymbol{\xi}_0)}(N^{dS}_1(M)[\boldsymbol{n}^T])$ is of the form

$$\begin{pmatrix} h^j_i(\boldsymbol{n}^T,\boldsymbol{n}^S)(u_0) & *\\ 0 & -I_{k-2}\end{pmatrix}.$$

It follows that the eigenvalues of this matrix are $\lambda_i=\kappa_i(\boldsymbol{n}^T,\boldsymbol{n}^S)(u_0)$, $(i=1,\dots,s)$ and $\lambda_i=-1$, $(i=s+1,\dots,n-1)$. This completes the proof. □

We call $\kappa_\ell(\boldsymbol{n}^T)_i(p,\boldsymbol{\xi})$, $(i=1,\dots,s)$ the *lightcone principal curvatures* of $M$ with respect to $(\boldsymbol{n}^T,\boldsymbol{\xi})$ at $p\in M$.

## 4. Lightlike hypersurfaces in de Sitter space

We define a hypersurface $\mathbb{LH}_M(\boldsymbol{n}^T): N_1^{dS}(M)[\boldsymbol{n}^T]\times\mathbb{R}\longrightarrow S_1^n$ by

$$\mathbb{LH}_M((p,\boldsymbol{\xi}),\mu)=\boldsymbol{X}(u)+\mu(\boldsymbol{n}^T+\boldsymbol{\xi})(u)=\boldsymbol{X}(u)+\mu\mathbb{LG}(\boldsymbol{n}^T)(u,\boldsymbol{\xi}),$$

where $p=\boldsymbol{X}(u)$, which is called the *de Sitter lightlike hypersurface* along $M$ relative to $\boldsymbol{n}^T$. We introduce the notion of height functions on spacelike submanifold, which is useful for the study of singularities of de Sitter lightlike hypersurfaces. We define a family of functions $H: M\times S_1^n\longrightarrow\mathbb{R}$ on a spacelike submanifold $M=\boldsymbol{X}(U)$ by

$$H(p,\boldsymbol{\lambda})=H(u,\boldsymbol{\lambda})=\langle\boldsymbol{X}(u),\boldsymbol{\lambda}\rangle-1,$$

where $p=\boldsymbol{X}(u)$. We call $H$ the *de Sitter height function* (briefly, *dS-height function*) on the spacelike submanifold $M$. For any fixed $\boldsymbol{\lambda}_0\in S_1^n$, we write $h_{\boldsymbol{\lambda}_0}(p)=H(p,\boldsymbol{\lambda}_0)$ and have the following proposition.

**Proposition 4.1.** *Suppose that $p_0=\boldsymbol{X}(u_0)\neq\boldsymbol{\lambda}_0$. Then we have the following*:

(1) *$h_{\boldsymbol{\lambda}_0}(p_0)=\partial h_{\boldsymbol{\lambda}_0}/\partial u_i(p_0)=0$, $(i=1,\dots,s)$ if and only if there exist $\boldsymbol{\xi}_0\in N_1^{dS}(M)_{p_0}[\boldsymbol{n}^T]$ and $\mu_0\in\mathbb{R}\setminus\{0\}$ such that*

$$\boldsymbol{\lambda}_0=\boldsymbol{X}(u_0)+\mu_0\mathbb{LG}(\boldsymbol{n}^T)(u_0,\boldsymbol{\xi}_0)=\mathbb{LH}_M(\boldsymbol{n}^T)((p_0,\boldsymbol{\xi}_0),\mu_0).$$

(2) *$h_{\boldsymbol{\lambda}_0}(p_0)=\partial h_{\boldsymbol{\lambda}_0}/\partial u_i(p_0)=\det\mathcal{H}(h_{\boldsymbol{\lambda}_0})(p_0)=0$ $(i=1,\dots,s)$ if and only if there exist $\boldsymbol{\xi}_0\in N_1^{dS}(M)_{p_0}[\boldsymbol{n}^T]$ and $\mu_0\in\mathbb{R}\setminus\{0\}$ such that*

$$\boldsymbol{\lambda}_0=\mathbb{LH}_M(\boldsymbol{n}^T)((p_0,\boldsymbol{\xi}_0),\mu_0)$$

*and $1/\mu$ is one of the non-zero lightcone principal curvatures $\kappa_\ell(\boldsymbol{n}^T)_i(p_0,\boldsymbol{\xi}_0),(i=1,\dots,s)$.*

(3) *With condition (2), $\operatorname{rank}\mathcal{H}(h_{\boldsymbol{\lambda}_0})(p_0)=0$ if and only if $p_0=\boldsymbol{X}(u_0)$ is a non-flat $(\boldsymbol{n}^T(u_0),\boldsymbol{\xi}_0)$-umbilical point.*

*Proof.* (1) We write that $p=\boldsymbol{X}(u)$. The condition $h_{\boldsymbol{\lambda}_0}(p)=\langle\boldsymbol{X}(u),\boldsymbol{\lambda}_0\rangle-1=0$ means that

$$\begin{aligned}\langle\boldsymbol{X}(u)-\boldsymbol{\lambda}_0,\boldsymbol{X}(u)-\boldsymbol{\lambda}_0\rangle &= \langle\boldsymbol{X}(u),\boldsymbol{X}(u)\rangle-2\langle\boldsymbol{X}(u),\boldsymbol{\lambda}_0\rangle+\langle\boldsymbol{\lambda}_0,\boldsymbol{\lambda}_0\rangle\\ &= -2(-1+\langle\boldsymbol{X}(u),\boldsymbol{\lambda}_0\rangle)=0,\end{aligned}$$

so that $\boldsymbol{X}(u)-\boldsymbol{\lambda}_0\in LC^*$. Since $\partial h_{\boldsymbol{\lambda}_0}/\partial u_i(p)=\langle\boldsymbol{X}_{u_i}(u),\boldsymbol{\lambda}_0\rangle$ and $\langle\boldsymbol{X}_{u_i},\boldsymbol{X}\rangle=0$, we have $\langle\boldsymbol{X}_{u_i}(u),\boldsymbol{\lambda}_0\rangle=-\langle\boldsymbol{X}_{u_i}(u),\boldsymbol{X}(u)-\boldsymbol{\lambda}_0\rangle$. Therefore, $\partial h_{\boldsymbol{\lambda}_0}/\partial u_i(p)=0$ if and only if

$$\boldsymbol{X}(u)-\boldsymbol{\lambda}_0\in N_pM.$$

On the other hand, the condition $h_{\boldsymbol{\lambda}_0}(p)=\langle\boldsymbol{X}(u),\boldsymbol{\lambda}_0\rangle-1=0$ implies that

$$\langle\boldsymbol{X}(u),\boldsymbol{X}(u)-\boldsymbol{\lambda}_0\rangle=0.$$

This means that $\boldsymbol{X}(u)-\boldsymbol{\lambda}_0\in T_pS_1^n$. Hence $h_{\boldsymbol{\lambda}_0}(p_0)=\partial h_{\boldsymbol{\lambda}_0}/\partial u_i((p_0)=0$ $(i=1,\dots,s)$ if and only if $\boldsymbol{X}(u_0)-\boldsymbol{\lambda}_0\in N_{p_0}M\cap LC^*\cap T_{p_0}S_1^n$. Let

$$\boldsymbol{v}=\boldsymbol{X}(u_0)-\boldsymbol{\lambda}_0\in N_{p_0}M\cap LC^*\cap T_{p_0}S_1^n.$$

If $\langle\boldsymbol{n}^T(u_0),\boldsymbol{v}\rangle=0$, then $\boldsymbol{n}^T(u_0)$ belongs to a lightlike hyperplane in the Lorentz space $T_{p_0}S_1^n$, so that $\boldsymbol{n}^T(u_0)$ is lightlike or spacelike. This contradicts the fact that $\boldsymbol{n}^T(u_0)$ is a timelike unit vector. Thus, $\langle\boldsymbol{n}^T(u_0),\boldsymbol{v}\rangle\neq0$. We set

$$\boldsymbol{\xi}_0=\frac{-1}{\langle\boldsymbol{n}^T(u_0),\boldsymbol{v}\rangle}\boldsymbol{v}-\boldsymbol{n}^T(u_0).$$

Then we have

$$\begin{aligned}\langle \boldsymbol{\xi}_0, \boldsymbol{\xi}_0\rangle &= -2\frac{-1}{\langle \boldsymbol{n}^T(u_0), \boldsymbol{v}\rangle}\langle \boldsymbol{n}^T(u_0), \boldsymbol{v}\rangle - 1 = 1\\ \langle \boldsymbol{\xi}_0, \boldsymbol{n}^T(u_0)\rangle &= \frac{-1}{\langle \boldsymbol{n}^T(u_0), \boldsymbol{v}\rangle}\langle \boldsymbol{n}^T(u_0), \boldsymbol{v}\rangle + 1 = 0,\end{aligned}$$

and $\langle \boldsymbol{\xi}_0, \boldsymbol{X}(u_0)\rangle = 0$. This means that $\boldsymbol{\xi}_0 \in N_1^{dS}(M)_{p_0}(M)[\boldsymbol{n}^T]$.

Since $-\boldsymbol{v} = \langle \boldsymbol{n}^T(u_0), \boldsymbol{v}\rangle(\boldsymbol{n}^T(u_0) + \boldsymbol{\xi}_0)$, we have $\boldsymbol{\lambda}_0 = \boldsymbol{X}(u_0) + \mu_0\mathbb{LG}(\boldsymbol{n}^T)(p_0, \boldsymbol{\xi}_0)$, where $p_0 = \boldsymbol{X}(u_0)$ and $\mu_0 = \langle \boldsymbol{n}^T(u_0), \boldsymbol{v}\rangle$. For the converse assertion, suppose that

$$\boldsymbol{\lambda}_0 = \boldsymbol{X}(u_0) + \mu_0\mathbb{LG}(\boldsymbol{n}^T)(p_0, \boldsymbol{\xi}_0).$$

Then $\boldsymbol{\lambda}_0 - \boldsymbol{X}(u_0) \in N_{p_0}(M) \cap LC^*$ and

$$\langle \boldsymbol{\lambda}_0 - \boldsymbol{X}(u_0), \boldsymbol{X}(u_0)\rangle = \langle \mu_0\mathbb{LG}(\boldsymbol{n}^T)(p_0, \boldsymbol{\xi}_0), \boldsymbol{X}(u_0)\rangle = 0.$$

Thus we have $\boldsymbol{\lambda}_0 - \boldsymbol{X}(u_0) \in N_{p_0}(M) \cap LC^* \cap T_{p_0}S^n_1$. By the previous arguments, these conditions are equivalent to the condition that $h_{\boldsymbol{\lambda}_0}(p_0) = \partial h_{\boldsymbol{\lambda}_0}/\partial u_i((p_0) = 0$ $(i = 1, \ldots, s)$.

(2) By a straightforward calculation, we have

$$\frac{\partial^2 h_{\boldsymbol{\lambda}_0}}{\partial u_i \partial u_j}(u) = \langle \boldsymbol{X}_{u_i u_j}, \boldsymbol{\lambda}_0\rangle.$$

Under the condition that $\boldsymbol{\lambda}_0 = \boldsymbol{X}(u_0) + \mu_0(\boldsymbol{n}^T(u_0) + \boldsymbol{\xi}_0)$, we have

$$\frac{\partial^2 h_{\boldsymbol{\lambda}_0}}{\partial u_i \partial u_j}(u_0) = \langle \boldsymbol{X}_{u_i u_j}(u_0), \boldsymbol{X}(u_0)\rangle + \mu_0\langle \boldsymbol{X}_{u_i u_j}(u_0), (\boldsymbol{n}^T(u_0) + \boldsymbol{\xi}_0)\rangle.$$

Since $\langle \boldsymbol{X}_{u_i}, \boldsymbol{X}\rangle = 0$, we have $\langle \boldsymbol{X}_{u_i u_j}, \boldsymbol{X}\rangle = -\langle \boldsymbol{X}_{u_i}, \boldsymbol{X}_{u_j}\rangle$. Thus, we have

$$\left(\frac{\partial^2 h_{\boldsymbol{\lambda}_0}}{\partial u_i \partial u_\ell}(u_0)\right)\left(g^{j\ell}(u_0)\right) = \left(\mu_0 h^j_i(\boldsymbol{n}^T, \boldsymbol{\xi}_0)(p_0) - \delta^j_i\right).$$

It follows that $\det\mathcal{H}(g)(p_0) = 0$ if and only if $1/\mu_0$ is an eigenvalue of $(h^i_j(\boldsymbol{n}^T, \boldsymbol{\xi}_0)(p_0))$, which is equal to one of the lightcone principal curvatures $\kappa_\ell(\boldsymbol{n}^T)_i(p_0, \boldsymbol{\xi}_0), (i = 1, \ldots, s)$.

(3) By the above calculation, $\operatorname{rank}\mathcal{H}(h_{\boldsymbol{\lambda}_0})(p_0) = 0$ if and only if $(h^i_j(\boldsymbol{n}^T)(p_0, \boldsymbol{\xi}_0)) = \frac{1}{\mu_0}(\delta^j_i)$, where $1/\mu_0 = \kappa_\ell(\boldsymbol{n}^T)_i(p_0, \boldsymbol{\xi}_0)$, $(i = 1, \ldots, s)$. This means that $p_0 = \boldsymbol{X}(u_0)$ is an $(\boldsymbol{n}^T(u_0), \boldsymbol{\xi}_0)$-umbilical point. □

In order to understand the geometric meanings of the assertions of Proposition 4.1, we briefly review the theory of Legendrian singularities For detailed expressions, see [1, 24]. Let $\pi : PT^*(\mathbb{R}^{n+1}) \longrightarrow \mathbb{R}^{n+1}$ be the projective cotangent bundle with its canonical contact structure. We next review the geometric properties of this bundle. Consider the tangent bundle $\tau : TPT^*(\mathbb{R}^{n+}) \to PT^*(\mathbb{R}^{n+1})$ and the differential map $d\pi : TPT^*(\mathbb{R}^{n+1}) \to T\mathbb{R}^{n+1}$ of $\pi$. For any $X \in TPT^*(\mathbb{R}^{n+1})$, there exists an element $\alpha \in T^*(\mathbb{R}^{n+1}_1$ such that $\tau(X) = [\alpha]$. For an element $V \in T_x(\mathbb{R}^{n+1})$, the property $\alpha(V) = 0$ does not depend on the choice of representative of the class $[\alpha]$. Thus we can define the canonical contact structure on $PT^*(\mathbb{R}^{n+1})$ by

$$K = \{X \in TPT^*(\mathbb{R}^{n+1}) \mid \tau(X)(d\pi(X)) = 0\}.$$

We have the trivialization $PT^*(\mathbb{R}^{n+1}) \cong \mathbb{R}^{n+1} \times P^n(\mathbb{R})^*$, and call

$$((v_0, v_1, \ldots, v_n), [\xi_0 : \xi_1 : \cdots : \xi_n])$$

*homogeneous coordinates* of $PT^*(\mathbb{R}^{n+1})$, where $[\xi_0 : \xi_1 : \cdots : \xi_n]$ are the homogeneous coordinates of the dual projective space $P^n(\mathbb{R})^*$. It is easy to show that $X \in K_{(x,[\xi])}$ if and only if

$$\sum_{i=0}^{n} \mu_i \xi_i = 0,$$

where $d\tilde{\pi}(X) = \sum_{i=0}^{n} \mu_i \partial/\partial v_i$. An immersion $i : L \to PT^*(\mathbb{R}^{n+1})$ is said to be *a Legendrian immersion* if $\dim L = n$ and $di_q(T_qL) \subset K_{i(q)}$ for any $q \in L$. The map $\pi \circ i$ is also called *the Legendrian map* of $i$ and the set $W(i) = \mathrm{image}\, \pi \circ i$, the *wave front set* of $i$. Moreover, $i$ (or, the image of $i$) is called the *Legendrian lift* of $W(i)$.

Let $F : (\mathbb{R}^k \times \mathbb{R}^{n+1}, \mathbf{0}) \longrightarrow (\mathbb{R}, \mathbf{0})$ be a function germ. We say that $F$ is *a Morse family of hypersurfaces* if the map germ

$$\Delta^* F = \left( F, \frac{\partial F}{\partial q_1}, \ldots, \frac{\partial F}{\partial q_k} \right) : (\mathbb{R}^k \times \mathbb{R}^{n+1}, \mathbf{0}) \longrightarrow (\mathbb{R} \times \mathbb{R}^k, \mathbf{0})$$

is submersive, where $(q, x) = (q_1, \ldots, q_k, x_0, \ldots, x_n) \in (\mathbb{R}^k \times \mathbb{R}^{n+1}, \mathbf{0})$. In this case we have a smooth $n$-dimensional submanifold

$$\Sigma_*(F) = \left\{ (q, x) \in (\mathbb{R}^k \times \mathbb{R}^{n+1}, \mathbf{0}) \;\middle|\; F(q, x) = \frac{\partial F}{\partial q_1}(q, x) = \cdots = \frac{\partial F}{\partial q_k}(q, x) = 0 \right\}$$

and the map germ $\mathscr{L}_F : (\Sigma_*(F), \mathbf{0}) \longrightarrow PT^*\mathbb{R}^{n+1}$ defined by

$$\mathscr{L}_F(q, x) = \left( x, \left[ \frac{\partial F}{\partial x_0}(q, x) : \cdots : \frac{\partial F}{\partial x_n}(q, x) \right] \right)$$

is a Legendrian immersion. We call $F$ *a generating family* of $\mathscr{L}_F(\Sigma_*(F))$, and the wave front set is given by $W(\mathscr{L}_F) = \pi_n(\Sigma_*(F))$, where $\pi_n : \mathbb{R}^k \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is the canonical projection. In the theory of unfoldings of function germs, the wave front set $W(\mathscr{L}_F)$ is called a *discriminant set* of $F$, which is also denoted by $\mathcal{D}_F$.

By the assertion (2) of Proposition 4.1, a singular point of the de Sitter lightlike hypersurface is a point $\boldsymbol{\lambda}_0 = \boldsymbol{X}(u_0) + \mu_0(\boldsymbol{n}^T + \boldsymbol{\xi}_0)(u_0)$ for $p_0 = \boldsymbol{X}(u_0)$ and $\mu_0 = 1/\kappa_\ell(\boldsymbol{n}^T)_i(p_0, \boldsymbol{\xi}_0)$, $i = 1, \ldots .s)$. Then we have the following corollary.

**Corollary 4.2.** *The critical value of* $\mathbb{LH}_M(\boldsymbol{n}^T)$ *is the point*

$$\boldsymbol{\lambda} = \boldsymbol{X}(u) + \frac{1}{\kappa_\ell(\boldsymbol{n}^T)_i(p, \boldsymbol{\xi})} \mathbb{LG}(\boldsymbol{n}^T)(u, \boldsymbol{\xi}),$$

*where* $p = \boldsymbol{X}(u)$ *and* $\kappa_\ell(\boldsymbol{n}^T)_i(p, \boldsymbol{\xi}) \neq 0$.

For a non-zero lightcone principal curvature $\kappa_\ell(\boldsymbol{n}^T)_i(p_0, \boldsymbol{\xi}_0) \neq 0$, we have an open subset $O_i \subset N_1^{dS}(M)[\boldsymbol{n}^T]$ such that $\kappa_\ell(\boldsymbol{n}^T)_i(p, \boldsymbol{\xi}) \neq 0$. Therefore, we have a non-zero lightcone principal curvature function $\kappa_\ell(\boldsymbol{n}^T)_i : O_i \longrightarrow \mathbb{R}$. We define a mapping $\mathbb{LF}_{\kappa_\ell(\boldsymbol{n}^T)_i} : O_i \longrightarrow AdS^{n+1}$ by

$$\mathbb{LF}_{\kappa_\ell(\boldsymbol{n}^T)_i}(p, \boldsymbol{\xi}) = \boldsymbol{X}(u) + \frac{1}{\kappa_\ell(\boldsymbol{n}^T)_i(p, \boldsymbol{\xi})} \mathbb{NG}(\boldsymbol{n}^T)(u, \boldsymbol{\xi}),$$

where $p = \boldsymbol{X}(u)$. We also define

$$\mathbb{LF}_M(\boldsymbol{n}^T) = \bigcup_{i=1}^{s} \left\{ \mathbb{LF}_{\kappa_\ell(\boldsymbol{n}^T)_i}(p, \boldsymbol{\xi}) \mid (p, \boldsymbol{\xi}) \in N_1^{dS}(M)[\boldsymbol{n}^T] \text{ s.t. } \kappa_\ell(\boldsymbol{n}^T)_i(p, \boldsymbol{\xi}) \neq 0 \right\}.$$

We call $\mathbb{LF}_M(\boldsymbol{n}^T)$ the *de Sitter lightlike focal set* of $M = \boldsymbol{X}(U)$ relative to $\boldsymbol{n}^T$, which is the critical value set of the de Sitter lightlike hypersurface $\mathbb{LH}_M(\boldsymbol{n}^T)(N_1^{dS}(M)[\boldsymbol{n}^T] \times \mathbb{R})$ along $M$ relative to $\boldsymbol{n}^T$.

By Proposition 4.1, the image of the lightlike hypersurface along $M$ relative to $\boldsymbol{n}^T$ is the discriminant set of the AdS-height function $H$ on $M$. Moreover, the focal set is the critical value set of the lightlike hypersurface along $M$ relative to $\boldsymbol{n}^T$. Since $H$ is independent of the choice of $\boldsymbol{n}^T$, we have shown the following corollary.

**Corollary 4.3.** *Let $\boldsymbol{n}^T$ and $\overline{\boldsymbol{n}}^T$ be future directed timelike unit normal fields along $M$. Then we have*

$$\mathbb{LH}_M(\boldsymbol{n}^T)(N_1(M)[\boldsymbol{n}^T]\times\mathbb{R}) = \mathbb{LH}_M(\overline{\boldsymbol{n}}^T)(N_1(M)[\overline{\boldsymbol{n}}^T]\times\mathbb{R}) \text{ and } \mathbb{LF}_M(\boldsymbol{n}^T) = \mathbb{LF}_M(\overline{\boldsymbol{n}}^T).$$

We have the following proposition.

**Proposition 4.4.** *For any point $(u,\boldsymbol{\lambda}) \in \Sigma_*(F) = \Delta^* H^{-1}(0)$, the germ of the dS-height function $H$ at $(u,\boldsymbol{\lambda})$ is a Morse family of hypersurfaces.*

*Proof.* We write

$$\boldsymbol{X}(u) = (X_0(u), X_1(u), \dots, X_n(u)) \text{ and } \boldsymbol{\lambda} = (\lambda_0, \lambda_1, \dots, \lambda_n).$$

We define an open subset $U_n^+ = \{\boldsymbol{\lambda} \in S_1^n \mid \lambda_n > 0\}$. For any $\boldsymbol{\lambda} \in U_n^+$, we have

$$\lambda_n = \sqrt{\lambda_0^2 - \sum_{i=1}^{n-1} \lambda_i^2 + 1}.$$

Thus, we have local coordinates on $S_1^n$ given by $(\lambda_0, \lambda_1, \dots, \lambda_{n-1})$ on $U_n^+$. By definition, we have

$$H(u,\boldsymbol{\lambda}) = -X_0(u)\lambda_0 + X_1(u)\lambda_1 + \cdots + X_{n-1}(u)\lambda_{n-1} + X_n(u)\sqrt{\lambda_0^2 - \sum_{i=1}^{n-1}\lambda_i^2 + 1} - 1.$$

We now prove that the mapping

$$\Delta^* H = \left(H, \frac{\partial H}{\partial u_1}, \dots, \frac{\partial H}{\partial u_s}\right)$$

is non-singular at $(u,\boldsymbol{\lambda}) \in \Sigma_*(F)$. Indeed, the Jacobian matrix of $\Delta^* H$ is given by

$$\begin{pmatrix} & X_n\dfrac{\lambda_0}{\lambda_n} - X_0 & -X_n\dfrac{\lambda 1}{\lambda_n} + X_1 & \cdots & -X_n\dfrac{\lambda_{n-1}}{\lambda_n} + X_{n-1} \\ \mathrm{A} & X_{nu_1}\dfrac{\lambda_0}{\lambda_n} - X_{0u_1} & -X_{nu_1}\dfrac{\lambda_1}{\lambda_n} + X_{1u_1} & \cdots & -X_{nu_1}\dfrac{\lambda_{n-1}}{\lambda_n} + X_{n-1u_1} \\ & \vdots & \vdots & \ddots & \vdots \\ & X_{nu_s}\dfrac{\lambda_0}{\lambda_n} - X_{0u_s} & -X_{nu_s}\dfrac{\lambda_1}{\lambda_n} + X_{1u_s} & \cdots & -X_{nu_s}\dfrac{\lambda_{n-1}}{\lambda_n} + X_{n-1u_s} \end{pmatrix},$$

where

$$\mathrm{A} = \begin{pmatrix} \langle \boldsymbol{X}_{u_1}, \boldsymbol{\lambda}\rangle & \cdots & \langle \boldsymbol{X}_{u_s}, \boldsymbol{\lambda}\rangle \\ \langle \boldsymbol{X}_{u_1u_1}, \boldsymbol{\lambda}\rangle & \cdots & \langle \boldsymbol{X}_{u_1u_s}, \boldsymbol{\lambda}\rangle \\ \vdots & \ddots & \vdots \\ \langle \boldsymbol{X}_{u_su_1}, \boldsymbol{\lambda}\rangle & \cdots & \langle \boldsymbol{X}_{u_su_s}, \boldsymbol{\lambda}\rangle \end{pmatrix}.$$

We now show that the rank of

$$
\mathrm{B}=\begin{pmatrix}
X_n\dfrac{\lambda_0}{\lambda_n}-X_0 & -X_n\dfrac{\lambda_1}{\lambda_n}+X_1 & \cdots & -X_n\dfrac{\lambda_{n-1}}{\lambda_n}+X_{n-1}\\
X_{nu_1}\dfrac{\lambda_0}{\lambda_n}-X_{0u_1} & -X_{nu_1}\dfrac{\lambda_1}{\lambda_n}+X_{1u_1} & \cdots & -X_{nu_1}\dfrac{\lambda_{n-1}}{\lambda_n}+X_{n-1u_1}\\
\vdots & \vdots & \ddots & \vdots\\
X_{nu_s}\dfrac{\lambda_0}{\lambda_n}-X_{0u_s} & -X_{nu_s}\dfrac{\lambda_1}{\lambda_n}+X_{1u_s} & \cdots & -X_{nu_s}\dfrac{\lambda_{n-1}}{\lambda_n}+X_{n-1u_s}
\end{pmatrix}
$$

is $s+1$ at $(u,\boldsymbol{\lambda})\in\Sigma_*(H)$. Since $(u,\boldsymbol{\lambda})\in\Sigma_*(H)$, we have

$$
\boldsymbol{\lambda}=\boldsymbol{X}(u)+\mu\left(\boldsymbol{n}^T(u)+\sum_{i=1}^{k-1}\xi_i\boldsymbol{n}_i(u)\right)
$$

with $\sum_{i=1}^{k-1}\xi_i^2=1$, where $\{\boldsymbol{X},\boldsymbol{n}^T,\boldsymbol{n}_1^S,\dots,\boldsymbol{n}_{k-1}^S\}$ is a pseudo-orthonormal (local) frame of $N(M)$. Without loss of generality, we assume that $\mu\neq 0$ and $\xi_{k-1}\neq 0$. We write

$$
\boldsymbol{n}^T(u)={}^t(n_0^T(u),\dots n_n^T(u)),\ \ \boldsymbol{n}_i^S(u)={}^t(n_0^i(u),\dots n_n^i(u)).
$$

It is enough to show that the rank of the matrix

$$
\mathrm{C}=\begin{pmatrix}
X_n\dfrac{\lambda_0}{\lambda_n}-X_0 & -X_n\dfrac{\lambda 1}{\lambda_n}+X_1 & \cdots & -X_n\dfrac{\lambda_{n-1}}{\lambda_n}+X_{n-1}\\
X_{nu_1}\dfrac{\lambda_0}{\lambda_n}-X_{0u_1} & -X_{nu_1}\dfrac{\lambda_1}{\lambda_n}+X_{1u_1} & \cdots & -X_{nu_1}\dfrac{\lambda_{n-1}}{\lambda_n}+X_{n-1u_1}\\
\vdots & \vdots & \ddots & \vdots\\
X_{nu_s}\dfrac{\lambda_0}{\lambda_n}-X_{0u_s} & -X_{nu_s}\dfrac{\lambda_1}{\lambda_n}+X_{1u_s} & \cdots & -X_{nu_s}\dfrac{\lambda_{n-1}}{\lambda_n}+X_{n-1u_s}\\
n_n^T\dfrac{\lambda_0}{\lambda_n}-n_0^T & -n_n^T\dfrac{\lambda_1}{\lambda_n}+n_1^T & \cdots & -n_n^T\dfrac{\lambda_{n-1}}{\lambda_n}+n_{n-1}^T\\
n_n^1\dfrac{\lambda_0}{\lambda_n}-n_0^1 & -n_n^1\dfrac{\lambda_1}{\lambda_n}+n_1^1 & \cdots & -n_n^1\dfrac{\lambda_{n-1}}{\lambda_n}+n_{n-1}^1\\
\vdots & \vdots & \ddots & \vdots\\
n_n^{k-2}\dfrac{\lambda_0}{\lambda_n}-n_0^{k-2} & -n_n^{k-2}\dfrac{\lambda_1}{\lambda_n}+n_1^{k-2} & \cdots & -n_{-1}^{k-2}\dfrac{\lambda_{n-1}}{\lambda_n}+n_{n-1}^{k-2}
\end{pmatrix}
$$

is $n$ at $(u,\boldsymbol{\lambda})\in\Sigma_*(H)$. We write

$$
\boldsymbol{a}_i={}^t(x_i(u),x_{iu_1}(u),\dots x_{iu_s}(u),n_i^T(u),n_i^1(u),\dots,n_i^{k-2}(u)).
$$

Then we have

$$
\mathrm{C}=\left(\boldsymbol{a}_n\frac{\lambda_0}{\lambda_n}-\boldsymbol{a}_0,-\boldsymbol{a}_n\frac{\lambda_1}{\lambda_n}+\boldsymbol{a}_1,\dots,-\boldsymbol{a}_n\frac{\lambda_{n-1}}{\lambda_n}+\boldsymbol{a}_{n-1}\right).
$$

It follows that

$$
\begin{aligned}
\det\mathrm{C}\ &=\frac{\lambda_0}{\lambda_n}(-1)^{n-1}\det(\boldsymbol{a}_1,\dots,\boldsymbol{a}_n)+\frac{\lambda_1}{\lambda_n}(-1)^{n-2}\det(\boldsymbol{a}_0,\boldsymbol{a}_2,\dots,\boldsymbol{a}_n)\\
&\quad+\cdots+(-1)^0\frac{\lambda_{n-1}}{\lambda_n}\det(\boldsymbol{a}_0,\boldsymbol{a}_1,\dots,\boldsymbol{a}_{n-2},\boldsymbol{a}_n)+(-1)^1\frac{\lambda_n}{\lambda_n}\det(\boldsymbol{a}_0,\boldsymbol{a}_1,\dots,\boldsymbol{a}_{n-1}).
\end{aligned}
$$

Moreover, we define $\delta_i=\det(\boldsymbol{a}_0,\boldsymbol{a}_1,\dots,\boldsymbol{a}_{i-1},\boldsymbol{a}_{i+1},\dots,\boldsymbol{a}_n)$ for $i=0,1,\dots,n$ and

$$
\boldsymbol{a}=(-(-1)^{n-1}\delta_0,(-1)^{n-2}\delta_1,\dots,(-1)^0\delta_{n-1},(-1)^1\delta_n).
$$

Then we have

$$
\boldsymbol{a}=(-1)^{n-1}\boldsymbol{X}\wedge\boldsymbol{X}_{u_1}\wedge\cdots\wedge\boldsymbol{X}_{u_s}\wedge\boldsymbol{n}^T\wedge\boldsymbol{n}_1\wedge\cdots\wedge\boldsymbol{n}_{k-2}.
$$

We remark that $\boldsymbol{a} \neq 0$ and $\boldsymbol{a} = \pm\|\boldsymbol{a}\|\boldsymbol{n}_{k-1}$. By the above calculation, we have

$$\begin{aligned}\det \mathrm{C} &= \left\langle \left(\frac{\lambda_0}{\lambda_n}, \frac{\lambda_1}{\lambda_n}, \dots, \frac{\lambda_n}{\lambda_n}\right), \boldsymbol{a} \right\rangle = \frac{1}{\lambda_n}\left\langle \boldsymbol{X}(u) + \mu\left(\boldsymbol{n}^T(u) + \sum_{i=1}^{k-1}\xi_i \boldsymbol{n}_i(u)\right), \boldsymbol{a}\right\rangle \\ &= \frac{1}{\lambda_n} \times \pm\mu\xi_{k-1}\|\boldsymbol{a}\| = \pm\frac{\mu\xi_{k-1}\|\boldsymbol{a}\|}{\lambda_n} \neq 0.\end{aligned}$$

Therefore the Jacobi matrix of $\Delta^* H$ is non-singular at $(u, \boldsymbol{\lambda}) \in \Sigma_*(F)$.

For other local coordinates of $S^n_1$, we can apply the same method for the proof as the above case. This completes the proof. □

Here we consider the open set $U_n^+$ again. Since $H$ is a Morse family of hypersurfaces, we have a Legendrian immersion

$$\mathscr{L}_H : \Sigma_*(H) \longrightarrow PT^*(S^n_1)|U^+_n$$

by the general theory of Legendrian singularities. By definition, we have

$$\frac{\partial H}{\partial \lambda_0}(u, \boldsymbol{\lambda}) = X_n(u)\frac{\lambda_0}{\lambda_n} - X_0(u),\ \frac{\partial H}{\partial \lambda_i}(u, \boldsymbol{\lambda}) = -X_n(u)\frac{\lambda_i}{\lambda_n} + X_i(u),\ (i = 1, \dots, n-1).$$

It follows that

$$\begin{aligned}&\left[\frac{\partial H}{\partial \lambda_0}(u, \boldsymbol{\lambda}) : \frac{\partial H}{\partial \lambda_1}(u, \boldsymbol{\lambda}) : \dots : \frac{\partial H}{\partial \lambda_{n-1}}(u, \boldsymbol{\lambda})\right] \\ &\qquad = [X_n(u)\lambda_0 - X_0(u)\lambda_n : X_1(u)\lambda_n - X_n(u)\lambda_1 : \dots : X_{n-1}(u)\lambda_n - X_n(u)\lambda_{n-1}].\end{aligned}$$

Therefore, we have

$$\mathscr{L}_H(u, \boldsymbol{\lambda}) = (\boldsymbol{\lambda}, [X_n(u)\lambda_0 - X_0(u)\lambda_n : X_1(u)\lambda_n - X_n(u)\lambda_1 : \dots : X_{n-1}(u)\lambda_n - X_n(u)\lambda_{n-1}]),$$

where

$$\Sigma_*(H) = \{(u, \boldsymbol{\lambda}) \mid \boldsymbol{\lambda} = \mathbb{LH}_M(\boldsymbol{n}^T)(p, \boldsymbol{\xi}, t)\ ((p, \boldsymbol{\xi}), t) \in N_1(M)[\boldsymbol{n}^T] \times \mathbb{R}\}.$$

We observe that $H$ is a generating family of the Legendrian immersion $\mathscr{L}_H$ whose wave front is $\mathbb{LH}_M(\boldsymbol{n}^T)(N_1(M)[\boldsymbol{n}^T] \times \mathbb{R})$. For other local coordinates of $S^n_1$, we have the similar results to the above case.

## 5. Contact with de Sitter lightcones

In this section, we consider the geometric meaning of the singularities of lightlike hypersurfaces in de Sitter space from the viewpoint of the theory of contact of submanifolds with model hypersurfaces in the view of Montaldi's theory. We review the theory of contact for submanifolds in [21]. Let $X_i$ and $Y_i$, $i = 1, 2$, be submanifolds of $\mathbb{R}^n$ with $\dim X_1 = \dim X_2$ and $\dim Y_1 = \dim Y_2$. We say that *the contact of* $X_1$ and $Y_1$ at $y_1$ is the same type as *the contact of* $X_2$ and $Y_2$ at $y_2$ if there is a diffeomorphism germ $\Phi : (\mathbb{R}^n, y_1) \longrightarrow (\mathbb{R}^n, y_2)$ such that $\Phi(X_1) = X_2$ and $\Phi(Y_1) = Y_2$. In this case we write $K(X_1, Y_1; y_1) = K(X_2, Y_2; y_2)$. Since this definition of contact is local, we can replace $\mathbb{R}^n$ by an arbitrary $n$-manifold. Montaldi gives in [21] the following characterization of contact by using $\mathcal{K}$-equivalence. We say that two function germs $h_i : (\mathbb{R}^s, \mathbf{0}) \longrightarrow (\mathbb{R}, 0)$ $(i = 1, 2)$ are *$\mathcal{K}$-equivalent* if there exist a diffeomorphism germ $\psi : (\mathbb{R}^s, \mathbf{0}) \longrightarrow (\mathbb{R}^s, \mathbf{0})$ and a function germ $\lambda : (\mathbb{R}^s, \mathbf{0}) \longrightarrow \mathbb{R}$ with $\lambda(\mathbf{0}) \neq 0$ such that $\lambda(x)h_1 \circ \psi(x) = h_2(x)$ for $x \in (\mathbb{R}^s, \mathbf{0})$.

**Theorem 5.1.** *Let $X_i$ and $Y_i$, $i = 1, 2$, be submanifolds of $\mathbb{R}^n$ for which $\dim X_1 = \dim X_2$ and $\dim Y_1 = \dim Y_2 = n - 1$. Let $g_i : (X_i, x_i) \longrightarrow (\mathbb{R}^n, y_i)$ be immersion germs and let $f_i : (\mathbb{R}^n, y_i) \longrightarrow (\mathbb{R}^p, 0)$ be submersion germs with $(Y_i, y_i) = (f_i^{-1}(0), y_i)$.*

*Then $K(X_1, Y_1; y_1) = K(X_2, Y_2; y_2)$ if and only if $f_1 \circ g_1$ and $f_2 \circ g_2$ are $\mathcal{K}$-equivalent.*

We remark that the assertion of the above theorem holds for submanifolds $Y_i$ with general codimension (cf., [21]).

Now, we return to the review of the theory of Legendrian singularities. We introduce a natural equivalence relation among Legendrian submanifold germs. Let

$$F, G : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, 0)$$

be Morse families of hypersurfaces. Then we say that $\mathscr{L}_F(\Sigma_*(F))$ and $\mathscr{L}_G(\Sigma_*(G))$ are *Legendrian equivalent* if there exists a contact diffeomorphism germ $H : (PT^*\mathbb{R}^n, z) \longrightarrow (PT^*\mathbb{R}^n, z')$ such that $H$ preserves fibers of $\pi$ and that $H(\mathscr{L}_F(\Sigma_*(F))) = \mathscr{L}_G(\Sigma_*(G))$, where $z = \mathscr{L}_F(\mathbf{0})$, $z' = \mathscr{L}_G(\mathbf{0})$. By using Legendrian equivalence, we can define the notion of Legendrian stability for Legendrian submanifold germs in the ordinary way (see, [1, Part III]). We can interpret Legendrian equivalence by using the notion of generating families. We denote by $\mathcal{E}_k$ the local ring of function germs $(\mathbb{R}^k, \mathbf{0}) \longrightarrow \mathbb{R}$ with the unique maximal ideal $\mathfrak{M}_k = \{h \in \mathcal{E}_k \mid h(\mathbf{0}) = 0\ \}$. Let $F, G : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, 0)$ be function germs. We say that $F$ and $G$ are *$P$-$\mathcal{K}$-equivalent* if there exists a diffeomorphism germ $\Psi : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0})$ of the form $\Psi(x, u) = (\psi_1(q, x), \psi_2(x))$ for $(q, x) \in (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0})$ such that $\Psi^*(\langle F \rangle_{\mathcal{E}_{k+n}}) = \langle G \rangle_{\mathcal{E}_{k+n}}$. Here $\Psi^* : \mathcal{E}_{k+n} \longrightarrow \mathcal{E}_{k+n}$ is the pull-back $\mathbb{R}$-algebra isomorphism defined by $\Psi^*(h) = h \circ \Psi$. We say that $F$ is an *infinitesimally $\mathcal{K}$-versal deformation of $f = F|\mathbb{R}^k \times \{\mathbf{0}\}$* if

$$\mathcal{E}_k = T_e(\mathcal{K})(f) + \left\langle \frac{\partial F}{\partial x_1}|\mathbb{R}^k \times \{\mathbf{0}\}, \dots, \frac{\partial F}{\partial x_n}|\mathbb{R}^k \times \{\mathbf{0}\} \right\rangle_{\mathbb{R}},$$

where $T_e(\mathcal{K})(f) = \left\langle \frac{\partial f}{\partial q_1}, \dots, \frac{\partial f}{\partial q_k}, f \right\rangle_{\mathcal{E}_k}$, (see [19].) The main result in the theory of Legendrian singularities ([1], §20.8 and [24], THEOREM 2) is the following:

**Theorem 5.2.** *Let $F, G : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, 0)$ be Morse families of hypersurfaces. Then we have the following assertions:*
*(1) $\mathscr{L}_F(\Sigma_*(F))$ and $\mathscr{L}_G(\Sigma_*(G))$ are Legendrian equivalent if and only if $F$ and $G$ are $P$-$\mathcal{K}$-equivalent,*
*(2) $\mathscr{L}_F(\Sigma_*(F))$ is Legendrian stable if and only if $F$ is an infinitesimally $\mathcal{K}$-versal deformation of $f = F|\mathbb{R}^k \times \{\mathbf{0}\}$.*

Since $F$ and $G$ are function germs on the common space germ $(\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0})$, we do not need the notion of stably $P$-$\mathcal{K}$-equivalence under this situation [24, page 27]. For any map germ $f : (\mathbb{R}^k, \mathbf{0}) \longrightarrow (\mathbb{R}^p, \mathbf{0})$, we define *the local ring of $f$* by $Q_r(f) = \mathcal{E}_k / (f^*(\mathfrak{M}_p)\mathcal{E}_k + \mathfrak{M}_k^{r+1})$. We have the following classification result of Legendrian stable germs (cf. [10, Proposition A.4]) which is the key for the purpose in this section.

**Proposition 5.3.** *Let $F, G : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, 0)$ be Morse families of hypersurfaces and $f = F|\mathbb{R}^k \times \{\mathbf{0}\}, g = G|\mathbb{R}^k \times \{\mathbf{0}\}$. Suppose that $\mathscr{L}_F(\Sigma_*(F))$ and $\mathscr{L}_G(\Sigma_*(G))$ are Legendrian stable. The the following conditions are equivalent:*
1. *$(W(\mathscr{L}_F), \mathbf{0})$ and $(W(\mathscr{L}_G), \mathbf{0})$ are diffeomorphic as set germs,*
2. *$(\mathscr{L}_F(\Sigma_*(F)), z)$ and $(\mathscr{L}_G(\Sigma_*(G)), z')$ are Legendrian equivalent,*
3. *$Q_{n+1}(f)$ and $Q_{n+1}(g)$ are isomorphic as $\mathbb{R}$-algebras.*

We have the following basic observations.

**Proposition 5.4.** *Let $M = \boldsymbol{X}(U)$ be a spacelike submanifold with*

$$\kappa_\ell(\boldsymbol{n}^T)_i(p, \boldsymbol{\xi}) \neq 0 \quad \text{for} \quad i = 1, \dots s.$$

*We consider $\boldsymbol{\lambda}_0 \in S^n_1$. Then $M \subset LC_{\boldsymbol{\lambda}_0} \cap S^n_1$ if and only if $\boldsymbol{\lambda}_0 = \mathbb{LF}_M(\boldsymbol{n}^T)$. In this case we have $\mathbb{LH}_M(\boldsymbol{n}^T)(N_1(M)[\boldsymbol{n}^T]) \subset LC_{\boldsymbol{\lambda}_0} \cap S^n_1$ and $M = \boldsymbol{X}(U)$ is totally lightcone umbilical.*

*Proof.* By Proposition 3.1, $\kappa_\ell(\boldsymbol{n}^T)_i(p,\boldsymbol{\xi}) \neq 0$ for $i = 1,\dots s$ if and only if

$$\{(\boldsymbol{n}^T+\boldsymbol{n}^S),(\boldsymbol{n}^T+\boldsymbol{n}^S)_{u_1},\dots,(\boldsymbol{n}^T+\boldsymbol{n}^S)_{u_s}\}$$

is linearly independent for $p_0 = \boldsymbol{X}(u_0) \in M$ and $\boldsymbol{\xi}_0 = \boldsymbol{n}^S(u_0)$, where $\boldsymbol{n}^S : U \longrightarrow N_1^{dS}(M)[\boldsymbol{n}^T]$ is a local section. By the proof of assertion (1) of Proposition 4.1, $M \subset LC_{\lambda_0} \cap S_1^n$ if and only if $h_{\boldsymbol{\lambda}_0}(u) = 0$ for any $u \in U$, where $h_{\boldsymbol{\lambda}_0}(u) = H(u,\boldsymbol{\lambda}_0)$ is the dS-height function on $M$. It also follows from Proposition 4.1 that there exists a smooth function $\eta : U \times N_1^{dS}(M)[\boldsymbol{n}^T] \longrightarrow \mathbb{R}$ and section $\boldsymbol{n}^S : U \longrightarrow N_1^{dS}(M)[\boldsymbol{n}^T]$ such that

$$\boldsymbol{X}(u) = \boldsymbol{\lambda}_0 + \eta(u,\boldsymbol{n}^S(u))(\boldsymbol{n}^T(u) \pm \boldsymbol{n}^S(u)).$$

In fact, we have $\eta(u,\boldsymbol{n}^S(u)) = -1/\kappa_\ell(\boldsymbol{n}^T)_i(p,\boldsymbol{\xi})$ $i = 1,\dots,s$, where $p = \boldsymbol{X}(u)$ and $\boldsymbol{\xi} = \boldsymbol{n}^S(u)$. It follows that $\kappa_\ell(\boldsymbol{n}^T)_i(p,\boldsymbol{\xi}) = \kappa_\ell(\boldsymbol{n}^T)_j(p,\boldsymbol{\xi})$, so that $M = \boldsymbol{X}(U)$ is totally lightcone umbilical. Therefore we have

$$\mathbb{LH}_M(\boldsymbol{n}^T)(u,\boldsymbol{n}^S(u),\mu) = \boldsymbol{\lambda}_0 + (\mu + \eta(u,\boldsymbol{n}^S(u))(\boldsymbol{n}^T(u) \pm \boldsymbol{n}^S(u)).$$

Hence we have $\mathbb{LH}_M(\boldsymbol{n}^T)(N_1(M)[\boldsymbol{n}^T]\times\mathbb{R}) \subset LC_{\lambda_0}$. By Corollary 4.2, the critical value set of $\mathbb{LH}_M(\boldsymbol{n}^T)(N_1(M)[\boldsymbol{n}^T]\times\mathbb{R})$ is the de Sitter lightlike focal set $\mathbb{LF}_M(\boldsymbol{n}^T)$. However, it is equal to $\lambda_0$ by the previous arguments.

For the converse assertion, suppose that $\boldsymbol{\lambda}_0 = \mathbb{LF}_M(\boldsymbol{n}^T)$. Then we have

$$\boldsymbol{\lambda}_0 = \boldsymbol{X}(u) + \frac{1}{\kappa_\ell(\boldsymbol{n}^T)_i(\boldsymbol{X}(u),\boldsymbol{\xi})}\mathbb{LG}(\boldsymbol{n}^T)(u,\boldsymbol{\xi}),$$

for any $i = 1,\dots,s$ and $(p,\boldsymbol{\xi}) \in N_1^{dS}(M)[\boldsymbol{n}^T]$, where $p = \boldsymbol{X}(u)$. Thus, we have

$$\kappa_\ell(\boldsymbol{n}^T)_i(\boldsymbol{X}(u),\boldsymbol{\xi}) = \kappa_\ell(\boldsymbol{n}^T)_j(\boldsymbol{X}(u),\boldsymbol{\xi})$$

for any $i,j = 1,\dots,s$, so that $M$ is totally lightcone umbilical. Since $\mathbb{LG}(\boldsymbol{n}^T)(u,\boldsymbol{\xi})$ is null, we have $\boldsymbol{X}(u) \in LC_{\boldsymbol{\lambda}_0}$. This completes the proof. □

According to the above proposition, $LC_{\boldsymbol{\lambda}_0} \cap S_1^n$ is regarded as a model lightlike hypersurface in $S_1^n$. We define

$$T(S_1^n)_{\boldsymbol{\lambda}_0} = \{\boldsymbol{x} \in \mathbb{R}_1^{n+1} \mid \boldsymbol{x} - \boldsymbol{\lambda}_0 \in T_{\boldsymbol{\lambda}_0}S_1^n\},$$

where $T_{\boldsymbol{\lambda}_0}S_1^n$ is the tangent space of $S_1^n$ at $\boldsymbol{\lambda}_0 \in S_1^n$. We call $T(S_1^n)_{\boldsymbol{\lambda}_0}$ a *tangent affine space* of $S_1^n$ at $\boldsymbol{\lambda}_0 \in S_1^n$. It is easy to show that

$$LC_{\boldsymbol{\lambda}_0} \cap S_1^n = T(S_1^n)_{\boldsymbol{\lambda}_0} \cap S_1^n.$$

We write $LC_{\boldsymbol{\lambda}_0}(S_1^n) = LC_{\boldsymbol{\lambda}_0} \cap S_1^n = T(S_1^n)_{\boldsymbol{\lambda}_0} \cap S_1^n$, which is called a *dS-lightcone* with the vertex $\boldsymbol{\lambda}_0 \in S_1^n$. Therefore, the model lightlike hypersurface is a dS-lightcone.

We consider the contact of spacelike submanifolds with dS-lightcones. Let

$$\mathcal{H} : S_1^n \times S_1^n \longrightarrow \mathbb{R}$$

be a function defined by $\mathcal{H}(\boldsymbol{x},\boldsymbol{\lambda}) = \langle \boldsymbol{x},\boldsymbol{\lambda}\rangle - 1$. Given $\boldsymbol{\lambda}_0 \in S_1^n$, we write $\mathfrak{h}_{\lambda_0}(\boldsymbol{x}) = \mathcal{H}(\boldsymbol{x},\boldsymbol{\lambda}_0)$, so that we have $\mathfrak{h}_{\lambda_0}^{-1}(0) = LC_{\boldsymbol{\lambda}_0}(S_1^n)$. For any $p_0 = \boldsymbol{X}(u_0) \in M$, $\mu_0 \in \mathbb{R}$ and $\boldsymbol{\xi}_0 \in N_1^{dS}(M)_p[\boldsymbol{n}^T]$, we consider the point $\boldsymbol{\lambda}_0 = \boldsymbol{X}(u_0) + \mu_0(\boldsymbol{n}^T(u_0) + \boldsymbol{\xi}_0)$. Then we have

$$\mathfrak{h}_{\lambda_0} \circ \boldsymbol{X}(u_0)) = \mathcal{H} \circ (\boldsymbol{X} \times 1_{AdS^{n+1}})(u_0,\boldsymbol{\lambda}_0) = H(p_0,\boldsymbol{\lambda}_0) = 0,$$

where $\mu_0 = 1/\kappa_\ell(\boldsymbol{n}^T)_i(p_0,\boldsymbol{\xi}_0)$, $i = 1,\dots,s$. We also have relations

$$\frac{\partial \mathfrak{h}_{\lambda_0} \circ \boldsymbol{X}}{\partial u_i}(u_0) = \frac{\partial H}{\partial u_i}(p_0,\boldsymbol{\lambda}_0) = 0,\ i = 1,\dots,s.$$

These imply that the dS-lightcone $\mathfrak{h}_{\lambda_0}^{-1}(0) = LC_{\boldsymbol{\lambda}_0}(S_1^n)$ is tangent to $M = \boldsymbol{X}(U)$ at $p_0 = \boldsymbol{X}(u_0)$. In this case, we call $LC_{\boldsymbol{\lambda}_0}(S_1^n)$ a *tangent dS-lightcone* of $M = \boldsymbol{X}(U)$ at $p_0 = \boldsymbol{X}(u_0)$, which is denoted by $TLC_{\boldsymbol{\lambda}_0}(M)_{p_0}$. Moreover, the tangent dS-lightcone $TLC_{\boldsymbol{\lambda}_0}(M)_{p_0}$ is called an *osculating dS-lightcone* if $\boldsymbol{\lambda}_0 = \mathbb{LF}_{\kappa_\ell(\boldsymbol{n}^T)_i(p_0,\boldsymbol{\xi}_0)}(u_0) \in \mathbb{LF}_M$, for one lightcone principal curvature $\kappa_\ell(\boldsymbol{n}^T)_i(p_0,\boldsymbol{\xi}_0)$. In this case, we call $\boldsymbol{\lambda}_0$ the *center of the lightcone principal curvature* $\kappa_\ell(\boldsymbol{n}^T)_i(p_0,\boldsymbol{\xi}_0)$. Therefore, we can interpret the lightlike focal set as the locus of the centers of the lightcone principal curvatures. This fact is analogous to the notion of the focal sets of submanifolds in Euclidean space.

We now describe the contacts of spacelike submanifolds in $S^n_1$ with dS-lightcones. We denote by $Q(\boldsymbol{X},u_0)$ the local ring of the function germ $\widetilde{h}_{\boldsymbol{\lambda}_0} : (U,u_0) \longrightarrow \mathbb{R}$, where $\boldsymbol{\lambda}_0 = \mathbb{LC}_M(u_0,\boldsymbol{\xi}_0,\mu_0)$. We remark that we can explicitly write the local ring as follows:

$$Q_{n+1}(\boldsymbol{X},u_0) = \frac{C^\infty_{u_0}(U)}{\langle\langle \boldsymbol{X}(u),\boldsymbol{\lambda}_0\rangle - 1\rangle_{C^\infty_{u_0}(U)} + \mathfrak{M}_{u_0}(U)^{n+2}},$$

where $C^\infty_{u_0}(U)$ is the local ring of function germs at $u_0$.

Let $\mathbb{LH}_{M_i}(\boldsymbol{n}_i^T) : (N_1(M_i)[\boldsymbol{n}_i^T]\times\mathbb{R},(p_i,\boldsymbol{\xi}_i,\mu_i)) \longrightarrow (S^n_1,\boldsymbol{\lambda}_i)$, $(i=1,2)$ be two lightlike hypersurface germs of spacelike submanifold germs $\boldsymbol{X}_i : (U,u^i) \longrightarrow (S^n_1,p_i)$. Let

$$H_i : (U\times S^n_1,(u^i,\boldsymbol{\lambda}_i)) \longrightarrow \mathbb{R}$$

be the dS-height function germ of $\boldsymbol{X}_i$. Then we have the following theorem:

**Theorem 5.5.** *Let $\boldsymbol{X}_i : (U,u^i) \longrightarrow (S^n_1,p_i)$, $i=1,2$, be spacelike submanifold germs such that the corresponding Legendrian submanifold germs $\mathscr{L}_{H_i}(\Sigma_*(H_i))$ are Legendrian stable. We write $\boldsymbol{X}_i(U) = M_i$. Then the following conditions are equivalent*:

(1) $(\mathbb{LH}_{M_1}(N_1(M_1)[\boldsymbol{n}_1^T]\times\mathbb{R}),\boldsymbol{\lambda}_1)$ *and* $(\mathbb{LH}_{M_2}(N_1(M_2)[\boldsymbol{n}_2^T]\times\mathbb{R}),\boldsymbol{\lambda}_2)$ *are diffeomorphic,*
(2) $(\mathscr{L}_{H_1}(\Sigma_*(H_1)),z_1)$ *and* $(\mathscr{L}_{H_2}(\Sigma_*(H_2)),z_2)$ *are Legendrian equivalent,*
(3) $H_1$ *and* $H_2$ *are $P$-$\mathcal{K}$-equivalent,*
(4) $h_{1,\lambda_1}$ *and* $h_{2,\lambda_2}$ *are $\mathcal{K}$-equivalent,*
(5) $K(M_1,TLC_{\boldsymbol{\lambda}_1}(M_1)_{p_1},p_1) = K(M_2,TLC_{\boldsymbol{\lambda}_2}(M_2)_{p_2},p_2)$.
(6) $Q_{n+1}(\boldsymbol{X}_1,u^1)$ *and* $Q_{n+1}(\boldsymbol{X}_2,u^2)$ *are isomorphic as $\mathbb{R}$-algebras.*

*Proof.* By Proposition 5.3, conditions (1), (2) and (6) are equivalent. These conditions are also equivalent to the condition that two generating families $H_1$ and $H_2$ are $P$-$\mathcal{K}$-equivalent by Theorem 5.2. If we denote $h_{i,\boldsymbol{\lambda}_i}(u) = H_i(u,\boldsymbol{\lambda}_i)$, then we have $h_{i,\boldsymbol{\lambda}_i}(u) = \mathfrak{h}_{\boldsymbol{\lambda}_i}\circ\boldsymbol{X}_i(u)$. By Theorem 5.1, $K(\boldsymbol{X}_1(U),LC_{\boldsymbol{\lambda}_1},p_1) = K(\boldsymbol{x}_2(U),LC\lambda_2,p_2)$ if and only if $\widetilde{h}_{1,\boldsymbol{\lambda}_1}$ and $\widetilde{h}_{2,\boldsymbol{\lambda}_2}$ are $\mathcal{K}$-equivalent. This means that (4) and (5) are equivalent. By definition, (3) implies (4). The uniqueness of the infinitesimally $\mathcal{K}$-versal deformation of $h_{i,\boldsymbol{\lambda}_i}$ (cf., [19]) leads that the condition (4) implies (3). This completes the proof. □

## 6. Spacelike submanifolds with codimension two

In [4], we previously investigated the singularities of lightlike surfaces along spacelike curves in $S^3_1$. As a consequence, we discovered a new invariant for spacelike curves which estimates the order of contact with de Sitter lightcones in $S^3_1$. After that, Kaseou [17] investigated the singularities of de Sitter lightlike hypersurfaces of spacelike submanifolds of codimension two in $S^n_1$. We remark that $N^{dS}(M)[\boldsymbol{n}^T]$ is a double covering of $M$ for codimension two spacelike submanifold $M$ in $S^n_1$. Then the de Sitter lightlike hypersurface is the image of the mapping $\mathbb{LH}^\pm_M(u,\mu) = \boldsymbol{X}(u)+\mu(\boldsymbol{n}^T\pm\boldsymbol{n}^S)(u)$, which coincides with the lightlike hypersurface along $M$ in [17]. Therefore, all results in the previous sections for de Sitter space are generalizations of the results in [17]. We now consider spacelike surfaces in $S^4_1$ here. Let $\boldsymbol{X} : U \longrightarrow S^4_1$ be a spacelike embedding from an open subset $U\subset\mathbb{R}^2$. In [17], it was shown that there is the following generic classification theorem. We say that two map germs $f, g : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^p, 0)$ are *$\mathcal{A}$-equivalent* if there exists diffeomorphism germs $\phi : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^n, 0)$ and $\psi : (\mathbb{R}^p, 0) \longrightarrow (\mathbb{R}^p, 0)$ such that $f \circ \phi = \psi \circ g$. Let $\mathrm{Emb}_{\mathrm{sp}}(U, S^4_1)$ be a space of spacelike embeddings from $U$ to $S^4_1$ with the Whitney $C^\infty$-topology.

**Theorem 6.1** ([17]). *There exists an open dense subset $\mathcal{O} \subset \mathrm{Emb}_{\mathrm{sp}}(U, S^4_1)$ such that for any $\boldsymbol{X} \in \mathcal{O}$, the germ of the corresponding lightlike hypersurfaces $\mathbb{LH}^{\pm}_M$ at any point $(u_0, \mu_0) \in U \times \mathbb{R}$ is $\mathcal{A}$-equivalent to one of the map germs $A_k$ $(1 \leq k \leq 4)$ or $D^{\pm}_4$ : where, $A_k$, $D^{\pm}_4$-map germs $f : (\mathbb{R}^3, 0) \longrightarrow (\mathbb{R}^4, 0)$ are given by*

$A_1$; $f(u_1, u_2, u_3) = (u_1, u_2, u_3, 0)$,
$A_2$; $f(u_1, u_2, u_3) = (3u_1^2, 2u_1^3, u_2, u_3)$,
$A_3$; $f(u_1, u_2, u_3) = (4u_1^3 + 2u_1u_2, 3u_1^4 + u_2u_1^2, u_2, u_3)$,
$A_4$; $f(u_1, u_2, u_3) = (5u_1^4 + 3u_2u_1^2 + 2u_1u_3, 4u_1^5 + 2u_2u_1^3 + u_3u_1^2, u_2, u_3)$,
$D_4^+$; $f(u_1, u_2, u_3) = (2(u_1^3 + u_2^3) + u_1u_2u_3, 3u_1^2 + u_2u_3, 3u_2^2 + u_1u_3, u_3)$,
$D_4^-$; $f(u_1, u_2, u_3) = \left(\left(\dfrac{u_1^3}{3} - u_1u_2^2\right) + (u_1^2 + u_2^2)u_3, u_2^2 - u_1^2 - 2u_1u_3, 2(u_1u_2 - u_2u_3), u_3\right)$.

As a corollary of the above theorem, we have the following generic local classification of AdS-lightlike focal sets along spacelike surfaces. We define $C(2,3,4) = \{(u_1^2, u_1^3, u_1^4) \mid u_1 \in \mathbb{R}\}$, which is called a $(2,3,4)$-*cusp*. We also define

$$C(BF) = \{(10u_1^3 + 3u_2u_1, 5u_1^4 + u_2u_1^2, 6u_1^5 + u_2u_1^3, u_2) \mid (u_1, u_2) \in \mathbb{R}^2\}.$$

We call $C(BF)$ a *C-butterfly* (i.e., the critical value set of the butterfly). Finally we define $C(2,3,4,5) = \{(u_1^2, u_1^3, u_1^4, u_1^5) \mid u_1 \in \mathbb{R}\}$, which is called a $(2,3,4,5)$-*cusp*.

**Corollary 6.2.** *There exists an open dense subset $\mathcal{O} \subset \mathrm{Emb}_{\mathrm{sp}}(U, S^4_1)$ such that for any $\boldsymbol{X} \in \mathcal{O}$, the germ of the corresponding dS-lightlike focal set $\mathbb{LF}^{\pm}_M$ at any point $(u_0, \mu_0) \in U \times \mathbb{R}$ is diffeomorphic to one of the following set germs at the origin in $\mathbb{R}^4$:*
$A_2$; $\{(0,0)\} \times \mathbb{R}^2$,
$A_3$; $C(2,3,4) \times \mathbb{R}$,
$A_4$; $C(BF)$,
$D_4^+$; $\{(2(u_1^3 + u_2^3) + u_1u_2u_3, 3u_1^2 + u_2u_3, 3u_2^2 + u_1u_3, u_3) \mid u_3^2 = 36u_1u_2\}$,
$D_4^-$; $\left\{\left(\left(\dfrac{u_1^3}{3} - u_1u_2^2\right) + (u_1^2 + u_2^2)u_3, u_2^2 - u_1^2 - 2u_1u_3, 2(u_1u_2 - u_2u_3), u_3\right) \;\middle|\; u_3^2 = u_1^2 + u_2^2\right\}$.

*Proof.* For $A_3$, we can calculate the Jacobi matrix of the normal form $f$ in Theorem 5.9:

$$J_f = \begin{pmatrix} 12u_1^2 + 2u_1 & 2u_1 & 0 \\ 12u_1^3 + 2u_1u_2 & u_1^2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

so that rank $J_f < 3$ if and only if $6u_1^2 + u_2 = 0$. Thus, the critical value set of $f$ is

$$C(f) = \{(-8u_1^3, -3u_1^4, -6u_1^2, u_3) \mid (u_1, u_3) \in \mathbb{R}^2\}.$$

It is $C(2,3,4) \times \mathbb{R}$. By a similar calculation, we can show that the germ of $A_4$ is diffeomorphic to $C(BF)$. For $D_4^+$, we can calculate the Jacobi matrix o the normal form $f$:

$$J_f = \begin{pmatrix} 6u_1^2 + u_2u_3 & 6u_2^2 + u_1u_3, u_1u_2 & 0 \\ 6u_1 & u_3 & u_2 \\ u_3 & 6u_2 & u_1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Therefore, rank $J_f < 3$ if and only if

$$\begin{vmatrix} 6u_1^2 + u_2u_3 & 6u_2^2 + u_1u_3, u_1u_2 \\ 6u_1 & u_3 \end{vmatrix} = \begin{vmatrix} 6u_1^2 + u_2u_3 & 6u_2^2 + u_1u_3, u_1u_2 \\ u_3 & 6u_2 \end{vmatrix} = \begin{vmatrix} 6u_1 & u_3 \\ u_3 & 6u_2 \end{vmatrix} = 0,$$

which is equivalent to the condition that $u_3^2 = 36u_1u_2$. For $D_4^-$, by a calculation similar to the above, we have the condition that $u_3^2 = u_1^2 + u_2^2$. This completes the proof. □

By using the above normal forms, we can investigate the detailed geometric properties of spacelike surface in $S_1^4$ corresponding to the singularities of dS-lightlike focal sets . However, we have limited space, so that we omit these discussions here.

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Shyuichi IZUMIYA, Department of Mathematics, Hokkaido University, Sapporo 060-0810,Japan
*E-mail address*: izumiya@math.sci.hokudai.ac.jp

Takami SATO, Shiseikan elementary school, Chuoku Minami 3 Nishi 7, Sapporo 060-0063, Japan
*E-mail address*: takami.s1218@gmail.com