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SINGULARITIES OF AFFINE EQUIDISTANTS: PROJECTIONS AND CONTACTS

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Abstract. Using standard methods for studying singularities of projections and of contacts, we classify the stable singularities of affine λ-equidistants of n-dimensional closed submanifolds of $\mathbb{R}^q$, for $q \leq 2n$, whenever $(2n, q)$ is a pair of nice dimensions [12].

1. Introduction

When M is a smooth closed curve on the affine plane $\mathbb{R}^2$, the set of all midpoints of chords connecting pairs of points on M with parallel tangent vectors is called the *Wigner caustic* of M, or the *area evolute* of M, or still, the *affine* $1/2$-*equidistant* of M, denoted $E_{1/2}(M)$.

The $1/2$-equidistant is generalized to any λ-equidistant, denoted $E_\lambda(M)$, $\lambda \in \mathbb{R}$, by considering all chords connecting pairs of points of M with parallel tangent vectors and the set of all points of these chords which stand in the λ-proportion to their corresponding pair of points on M. In this case, when M is a curve on $\mathbb{R}^2$, the local classification of stable singularities of $E_\lambda(M)$ is well known [2, 5].

The definition of the affine λ-equidistant of M is generalized to the cases when M is an n-dimensional closed submanifold of $\mathbb{R}^q$, with $q \leq 2n$, by considering the set of all λ-points of chords connecting pairs of points on M whose direct sum of tangent spaces do not coincide with $\mathbb{R}^q$, the so-called *weakly parallel pairs* on M.

In addition to curves in $\mathbb{R}^2$, the possible stable singularities of $E_\lambda(M)$ have been previously studied in the general setting when M is a hypersurface [5, 6], or when M is a surface in $\mathbb{R}^4$ [7]. The cases of curves in $\mathbb{R}^2$ and surfaces in $\mathbb{R}^4$ have also been studied in the particular setting of Lagrangian submanifolds of affine symplectic spaces [3].

In this paper, we classify the possible stable singularities of $E_\lambda(M)$ in a quite more general circumstance, namely, when the double dimension of M, $2n$, and the dimension of the ambient affine space, q, form a pair of *nice dimensions* [12], see Theorem 5.3 below.

In order to obtain such a classification, we start in Section 2 by defining an affine λ-equidistant of $M^n \subset \mathbb{R}^q$ as the set of critical values of the λ-point map (projection)

$$\pi_\lambda : \mathbb{R}^q \times \mathbb{R}^q \to \mathbb{R}^q, (x^+, x^-) \mapsto \lambda x^+ + (1-\lambda)x^-$$

restricted to $M \times M$, thus locally a map

$$\tilde{\pi}_\lambda : \mathbb{R}^{2n} \to \mathbb{R}^q \ ,$$

see Definition 2.8, Remark 2.9 and equation (5.2), below. Then, we also present the characterization of affine equidistants by a contact map, extending previous construction for the Wigner caustic ([14, 7]).

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In Section 3 we review the standard $\mathcal{K}$-equivalence and the classification of $\mathcal{K}$-simple singularities [10, 12], Theorem 3.9 below. Then, in Section 4 we combine the study of singularities of projections and of contacts, in view of Theorem 4.6 below ([12, 11]), with emphasis on contact reduction to rank 0 map-germs, Proposition 4.14.

Our main result is obtained in Section 5. First, in Theorem 5.2 we apply the Multijet Transversality Theorem [8] to a $\mathcal{K}$-invariant stratification of the jet space. When $(2n, q)$ is a pair of nice dimensions, the relevant strata of this stratification are the $\mathcal{K}$-simple orbits in jet space. Then, we use the results of Section 4 in the context of affine equidistants: Proposition 5.4 and Corollary 5.5, as well as equations (5.8)-(5.12). The following table summarizes our main result, Theorem 5.6, which is presented more extensively as subsection 5.1. The normal forms for the $\mathcal{A}$-stable singularities of the map $\tilde{\pi}_\lambda$ follow the notation of [10] (see Theorem 3.9 below) for the $\mathcal{K}$-simple rank-0 contact map-germ

$$\theta_\lambda : (\mathbb{R}^k, 0) \to (\mathbb{R}^{k-(2n-q)}, 0) \ ,$$

where k is the degree of parallelism of the pair of points on M joined by the chord (cf. Definition 2.1 and Tables I, II, III in Theorem 3.9).

(n , q)	Stable $E_\lambda(M)$, $M^n \subset \mathbb{R}^q$	Restrictions
(1 , 2)	A_μ	$\mu \leq 2$
(2 , 3)	A_μ	$\mu \leq 3$
(2 , 4)	$A_\mu, C^{\pm}_{2,2}$	$\mu \leq 4$
(3 , 4)	$A_\mu, D^{\pm}_4$	$\mu \leq 4$
(3 , 5)	$A_\mu, D^{\pm}_4, D^{\pm}_5, S_5$	$\mu \leq 5$
(3 , 6)	$A_\mu, C^{\pm}_{\rho,\tau}, C_6$	$\mu \leq 6$, $2 \leq \rho \leq \tau$, $\rho + \tau \leq 6$
(4 , 5)	$A_\mu, D^{\pm}_4, D^{\pm}_5$	$\mu \leq 5$
(4 , 7)	$A_\mu, D^{\pm}_\nu, E_6, E_7, S_\beta, T_7, \tilde{T}_7$	$\mu \leq 7$, $4 \leq \nu \leq 7$, $5 \leq \beta \leq 7$
(4 , 8)	$A_\mu, C^{\pm}_{\rho,\tau}, C_6, C_8, F_7, F_8$	$\mu \leq 8$, $2 \leq \rho \leq \tau$, $\rho + \tau \leq 8$
(5 , 6)	$A_\mu, D^{\pm}_\nu, E_6$	$\mu \leq 6$, $4 \leq \nu \leq 6$

We note that the case $M^4 \subset \mathbb{R}^6$ is absent from the table of results. This is due to the fact that $(2n = 8, q = 6)$ is not a pair of nice dimensions (see Theorem 5.3 below). Similarly, $(2n, q > 6)$ is not a pair of nice dimensions, for all $n \geq 5$. Classification of stable singularities of $E_\lambda(M)$, in these cases, lies outside the scope of this paper.

As mentioned before, the cases in the table of results when

$$(n, q) \in \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 6)\}$$

correspond to hypersurfaces and have been previously studied in [5, 6], and the case $(n, q) = (2, 4)$ was partially studied in [7]. On the other hand, the results for the cases when

$$(n, q) \in \{(3, 5), (3, 6), (4, 7), (4, 8)\}$$

are entirely new.

We emphasize that, in all of the above, we are excluding the cases of *vanishing chords*, that is, when the λ-point of the chord connecting two points on M touches M because the pair of points on M lies in the diagonal of $M \times M$. Such "diagonal singularities" or *singularities on shell* for $E_\lambda(M)$ possess additional symmetries when $\lambda = 1/2$ and these have been studied for the cases of curves on the plane and surfaces in $\mathbb{R}^4$, both in the general setting [7] and in the more particular setting of Lagrangian submanifolds of affine symplectic space [4]. In this paper, we don't study such singularities on shell for $E_\lambda(M)$.

2. Affine equidistants

2.1. **Definition of affine equidistants.** Let M be a smooth closed n-dimensional submanifold of the affine space $\mathbb{R}^q$, with $q \leq 2n$. Let a, b be points of M and denote by

$$\tau_{a-b} : \mathbb{R}^q \ni x \mapsto x + (a-b) \in \mathbb{R}^q$$

the translation by the vector $(a-b)$.

Definition 2.1. A pair of points $a, b \in M$ $(a \neq b)$ is called a **weakly parallel** pair if

$$T_aM + \tau_{a-b}(T_bM) \neq \mathbb{R}^q.$$

$\text{codim}(T_aM + \tau_{a-b}(T_bM))$ in $T_a\mathbb{R}^q$ is called the **codimension of a weakly parallel pair** a, b. We denote it by $\text{codim}(a, b)$.

A weakly parallel pair $a, b \in M$ is called k**-parallel** if

$$\dim(T_aM \cap \tau_{b-a}(T_bM)) = k. \tag{2.1}$$

If $k = n$ the pair $a, b \in M$ is called **strongly parallel**, or just **parallel**. We also refer to k as the **degree of parallelism** of the pair (a, b) and denote it by $\deg(a, b)$. The degree of parallelism and the codimension of parallelism are related in the following way:

$$2n - \deg(a, b) = q - \text{codim}(a, b). \tag{2.2}$$

Definition 2.2. A **chord** passing through a pair a, b, is the line

$$l(a, b) = \{x \in \mathbb{R}^q | x = \lambda a + (1-\lambda)b, \lambda \in \mathbb{R}\}.$$

Definition 2.3. For a given λ, an **affine** λ**-equidistant** of M, $E_\lambda(M)$, is the set of all $x \in \mathbb{R}^q$ such that $x = \lambda a + (1-\lambda)b$, for all weakly parallel pairs $a, b \in M$. $E_\lambda(M)$ is also called a (affine) **momentary equidistant** of M. Whenever M is understood, we write E_λ for $E_\lambda(M)$.

Note that, for any λ, $E_\lambda(M) = E_{1-\lambda}(M)$ and in particular $E_0(M) = E_1(M) = M$. Thus, the case $\lambda = 1/2$ is special:

Definition 2.4. $E_{1/2}(M)$ is called the **Wigner caustic** of M [2, 14].

2.2. **Characterization of affine equidistants by projection.** Consider the product affine space: $\mathbb{R}^q \times \mathbb{R}^q$ with coordinates (x_+, x_-) and the tangent bundle to $\mathbb{R}^q$: $T\mathbb{R}^q = \mathbb{R}^q \times \mathbb{R}^q$ with coordinate system $(x, \dot{x})$ and standard projection $\pi : T\mathbb{R}^q \ni (x, \dot{x}) \to x \in \mathbb{R}^q$.

Definition 2.5. For $\lambda \in \mathbb{R}$, a λ**-chord transformation**

$$\Gamma_\lambda : \mathbb{R}^q \times \mathbb{R}^q \to T\mathbb{R}^q \ , \ (x^+, x^-) \mapsto (x, \dot{x})$$

is a linear diffeomorphism defined by the λ-*point equation*:

$$x = \lambda x^+ + (1-\lambda)x^- \ , \tag{2.3}$$

for the λ-point x, and a *chord equation*:

$$\dot{x} = x^+ - x^-. \tag{2.4}$$

Remark 2.6. For our purposes, the choice (2.4) for a chord equation is not unique, but is the simplest one. Among other possibilities, the choice $\dot{x} = \lambda x^+ - (1-\lambda)x^-$ is particularly well suited for the study of affine equidistants of *Lagrangian* submanifolds in symplectic space [3].

Now, let M be a smooth closed n-dimensional submanifold of the affine space $\mathbb{R}^q$ $(2n \geq q)$ and consider the product $M \times M \subset \mathbb{R}^q \times \mathbb{R}^q$. Let $\mathcal{M}_\lambda$ denote the image of $M \times M$ by a λ-chord transformation,

$$\mathcal{M}_\lambda = \Gamma_\lambda(M \times M) \ ,$$

which is a $2n$-dimensional smooth submanifold of $T\mathbb{R}^q$.

Then we have the following general characterization:

Theorem 2.7 ([3]). *The set of critical values of the standard projection $\pi : T\mathbb{R}^q \to \mathbb{R}^q$ restricted to $\mathcal{M}_\lambda$ is $E_\lambda(M)$.*

Definition 2.8. For $\lambda \in \mathbb{R}$, the **λ-point map** is the projection

$$\pi_\lambda : \mathbb{R}^q \times \mathbb{R}^q \to \mathbb{R}^q \ , \ (x^+, x^-) \mapsto x = \lambda x^+ + (1-\lambda)x^- \ .$$

Remark 2.9. Because $\pi_\lambda = \pi \circ \Gamma_\lambda$ we can rephrase Theorem 2.7: *the set of critical values of the projection π_λ restricted to $M \times M$ is $E_\lambda(M)$.*

2.3. **Characterization of affine equidistants by contact.** In the literature, if $M \subset \mathbb{R}^2$ is a smooth curve, the Wigner caustic $E_{1/2}(M)$ has been described in various ways. A particular description says that, if $\mathcal{R}_a : \mathbb{R}^2 \to \mathbb{R}^2$ denotes reflection through $a \in \mathbb{R}^2$, then $a \in E_{1/2}(M)$ when M and $\mathcal{R}_a(M)$ are not transversal [2, 14]. This description has also been used in [14] for the case of Lagrangian surfaces in symplectic $\mathbb{R}^4$ and, more recently [7], for the case of general surfaces in $\mathbb{R}^4$.

We now generalize this description for every λ-equidistant of submanifolds of more arbitrary dimensions.

Definition 2.10. For $\lambda \in \mathbb{R} \setminus \{0, 1\}$, a **$\lambda$-reflection** through $a \in \mathbb{R}^q$ is the map

$$\mathcal{R}_a^\lambda : \mathbb{R}^q \to \mathbb{R}^q \ , \ x \mapsto \mathcal{R}_a^\lambda(x) = \frac{1}{\lambda}a - \frac{1-\lambda}{\lambda}x \tag{2.5}$$

Remark 2.11. A λ-reflection through a is not a reflection in the strict sense because

$$\mathcal{R}_a^\lambda \circ \mathcal{R}_a^\lambda \neq id : \mathbb{R}^q \to \mathbb{R}^q,$$

instead,

$$\mathcal{R}_a^{1-\lambda} \circ \mathcal{R}_a^\lambda = id : \mathbb{R}^q \to \mathbb{R}^q \ ,$$

so that, if $a = a_\lambda = \lambda a^+ + (1-\lambda)a^-$ is the λ-point of $(a^+, a^-) \in \mathbb{R}^{2q}$,

$$\mathcal{R}_{a_\lambda}^\lambda(a^-) = a^+ \ , \ \mathcal{R}_{a_\lambda}^{1-\lambda}(a^+) = a^- \ .$$

Of course, for $\lambda = 1/2$, $\mathcal{R}_a^{1/2} \equiv \mathcal{R}_a$ is a reflection in the strict sense.

Now, let M be a smooth n-dimensional submanifold of $\mathbb{R}^q$, with $2n \geq q$, and let

$$a = a_\lambda = \lambda a^+ + (1-\lambda)a^-$$

be the λ-point of $(a^+, a^-) \in M \times M \subset \mathbb{R}^q \times \mathbb{R}^q$. Also, let M^+ be a germ of submanifold M around a^+ and M^- be a germ of submanifold M around a^-. We have:

Proposition 2.12. *The following statements are equivalent:*
(i) The λ-point a belongs to $E_\lambda(M)$.
(ii) M^+ and $\mathcal{R}_a^\lambda(M^-)$ are not transversal at a^+.
(iii) M^- and $\mathcal{R}_a^{1-\lambda}(M^+)$ are not transversal at a^-.

Remark 2.13. Furthermore, from Remark 2.9 we see that the study of the singularities of affine equidistants is the study of the singularities of π_λ. But this is the same as the study of the singularities at $a = 0$ of

$$(x^+, x^-) \to x^+ + \frac{1-\lambda}{\lambda} x^- = x^+ - \mathcal{R}_0^\lambda(x^-) \ .$$

In other words, *the study of the singularities of $E_\lambda(M) \ni 0$ can be proceeded via the study of the contact between M^+ and $\mathcal{R}_0^\lambda(M^-)$ or, equivalently, the contact between M^- and $\mathcal{R}_0^{1-\lambda}(M^+)$.*

3. $\mathcal{K}$-equivalence

We recall some basic definitions and results (for details, see [1]).

Henceforth, $\mathcal{E}_s$ denotes the local ring of smooth function-germs on $\mathbb{R}^s$, and $\mathfrak{m}_s$ its maximal ideal.

Definition 3.1. Map-germs $f, \widetilde{f} : (\mathbb{R}^s, y_0) \to (\mathbb{R}^t, 0)$ are $\mathcal{K}$**-equivalent** if there exists a diffeomorphism-germ $\phi : (\mathbb{R}^s, y_0) \to (\mathbb{R}^s, y_0)$ and a map-germ $A : (\mathbb{R}^s, y_0) \to GL(\mathbb{R}^t)$ such that $\tilde{f} = A \cdot (f \circ \phi)$.

Theorem 3.2 ([1])**.** *For the $\mathcal{K}$-equivalence of two map-germs it is necessary and sufficient that two ideals generated by the components of these map-germs may be mapped one to the other by an isomorphism of $\mathcal{E}_s$ induced by a diffeomorphism-germ of the source space $(\mathbb{R}^s, y_0)$.*

Definition 3.3. A map-germ $F : (\mathbb{R}^s \times \mathbb{R}^p, (y_0, z_0)) \to \mathbb{R}^t$ is a **deformation** of a map-germ $f : (\mathbb{R}^s, y_0) \to \mathbb{R}^t$ if $F|_{\mathbb{R}^s \times \{z_0\}} = f$, where p is the number of parameters of deformation F.

Definition 3.4. A diffeomorphism-germ $\Phi : (\mathbb{R}^s \times \mathbb{R}^p, (y_0, z_0)) \to (\mathbb{R}^s \times \mathbb{R}^p, (y_0, z_0))$ is called **fiber-preserving** if $\Phi(y, z) = (Y(y, z), Z(z))$ for a smooth map-germ

$$Y : (\mathbb{R}^s \times \mathbb{R}^p, (y_0, z_0)) \to (\mathbb{R}^s, y_0)$$

and a diffeomorphism-germ $Z : (\mathbb{R}^p, z_0) \to (\mathbb{R}^p, z_0)$. It means that Φ preserves the fibers of the projection $pr : (\mathbb{R}^s \times \mathbb{R}^p, (y_0, z_0)) \to (\mathbb{R}^p, z_0)$.

Definition 3.5. Deformations $F, \widetilde{F} : (\mathbb{R}^s \times \mathbb{R}^p, (y_0, z_0)) \to (\mathbb{R}^t, 0)$ of respective map-germs $f, \widetilde{f} : (\mathbb{R}^s, y_0) \to (\mathbb{R}^t, 0)$ are **fiber** $\mathcal{K}$**-equivalent** if there is a fiber-preserving diffeomorphism-germ $\Phi : (\mathbb{R}^s \times \mathbb{R}^p, (y_0, z_0)) \to (\mathbb{R}^s \times \mathbb{R}^p, (y_0, z_0))$, i.e. $\Phi(y, z) = (Y(y, z), Z(z))$, and a map-germ $\mathbb{A} : (\mathbb{R}^s \times \mathbb{R}^p, (y_0, z_0)) \to GL(\mathbb{R}^t)$ such that $\widetilde{F} = \mathbb{A} \cdot (F \circ \Phi)$.

Corollary 3.6. *For the fiber $\mathcal{K}$-equivalence of two deformations it is necessary and sufficient that the two ideals of $\mathcal{E}_{s+p}$ generated by the components of these deformations may be mapped one to the other by an isomorphism of $\mathcal{E}_{s+p}$ induced by a fiber-preserving diffeomorphism-germ of the source space $(\mathbb{R}^s \times \mathbb{R}^p, (y_0, z_0))$.*

Definition 3.7. The germ $f : (\mathbb{R}^s, 0) \to (\mathbb{R}^t, 0)$ is said to be $\mathcal{K}$**-simple** if its k-jet, for any k, has a neighborhood in the jet space $J^k_{0,0}(\mathbb{R}^s, \mathbb{R}^t)$ that intersects only a finite number of $\mathcal{K}$-equivalence classes (bounded by a constant independent of k).

Definition 3.8. The p-parameter **suspension** of the map-germ $f : (\mathbb{R}^s, 0) \to (\mathbb{R}^t, 0)$ is the map germ

$$F : (\mathbb{R}^s \times \mathbb{R}^p, 0) \ni (y, z) \mapsto (f(y), z) \in (\mathbb{R}^t \times \mathbb{R}^p, 0).$$

Theorem 3.9 ([10])**.** *$\mathcal{K}$-simple map-germs $(\mathbb{R}^s, 0) \to (\mathbb{R}^t, 0)$ with $s \geq t$ belong, up to $\mathcal{K}$-equivalence and suspension, to one of the following three lists in Tables 1-3:*

Notation	*Normal form*	*Restrictions*
A_μ	$y_1^{\mu+1} + Q_{s-1}$	$\mu \geq 1$
D_μ	$y_1^2 y_2 \pm y_2^{\mu-1} + Q_{s-2}$	$\mu \geq 4$
E_6	$y_1^3 + y_2^4 + Q_{s-2}$	-
E_7	$y_1^3 + y_1 y_2^3 + Q_{s-2}$	-
E_8	$y_1^3 + y_2^5 + Q_{s-2}$	-

TABLE 1. $\mathcal{K}$-simple germs $\mathbb{R}^s \to \mathbb{R}$. $Q_{s-i} = \pm y_{i+1}^2 \pm \cdots \pm y_s^2$.

Notation	*Normal form*	*Restrictions*
$C_{k,l}^{\pm}$	$(y_1 y_2, y_1^k \pm y_2^l)$	$l \geq k \geq 2$
$\widetilde{C}_{2k}$	$(y_1^2 + y_2^2, y_2^k)$	$k \geq 3$
F_{2m+1}	$(y_1^2 + y_2^3, y_2^m)$	$m \geq 3$
F_{2m+4}	$(y_1^2 + y_2^3, y_1 y_2^m)$	$m \geq 2$
G_{10}^*	(y_1^2, y_2^4)	-
$H_{m+5}^{\pm}$	$(y_1^2 \pm y_2^m, y_1 y_2^2)$	$m \geq 4$

TABLE 2. $\mathcal{K}$-simple germs $\mathbb{R}^2 \to \mathbb{R}^2$.

Notation	*Normal form*	*Restrictions*
S_μ	$(\pm y_1^2 \pm y_2^2 + y_3^{\mu-3}, y_2 y_3)$	$\mu \geq 5$
T_7	$(y_1^2 + y_2^3 + y_3^3, y_2 y_3)$	-
$\widetilde{T}_7$	$(y_1^2 + y_2^2, y_2^2 + y_3^2)$	-
T_8	$(y_1^2 + y_2^3 \pm y_3^4, y_2 y_3)$	-
T_9	$(y_1^2 + y_2^3 + y_3^5, y_2 y_3)$	-
U_7	$(y_1^2 + y_2 y_3, y_1 y_2 + y_3^3)$	-
U_8	$(y_1^2 + y_2 y_3 + y_3^3, y_1 y_2)$	-
U_9	$(y_1^2 + y_2 y_3, y_1 y_2 + y_3^4)$	-
W_8	$(y_1^2 + y_2^3, y_2^2 + y_1 y_3)$	-
W_9	$(y_1^2 + y_2 y_3^2, y_2^2 + y_1 y_3)$	-
Z_9	$(y_1^2 + y_3^3, y_2^2 + y_3^3)$	-
Z_{10}	$(y_1^2 + y_2 y_3^2, y_2^2 + y_3^3)$	-

TABLE 3. $\mathcal{K}$-simple germs $\mathbb{R}^3 \to \mathbb{R}^2$.

Definition 3.10. A deformation

$$F : (\mathbb{R}^s \times \mathbb{R}^p, (0,0)) \to (\mathbb{R}^t, 0)$$

of a map-germ $f : (\mathbb{R}^s, 0) \to (\mathbb{R}^t, 0)$ is **$\mathcal{K}$-versal** if any other deformation

$$\widetilde{F} : (\mathbb{R}^s \times \mathbb{R}^q, (0,0)) \to (\mathbb{R}^t, 0)$$

of f is of the form

$$\widetilde{F}(y,z) = \mathbb{A}(y,z) \cdot F(g(y,z), h(z)),$$

where $\mathbb{A} : \mathbb{R}^s \times \mathbb{R}^q \to GL(\mathbb{R}^t)$, $g : (\mathbb{R}^s \times \mathbb{R}^q, (0,0)) \to (\mathbb{R}^s, 0)$, $h : (\mathbb{R}^q, 0) \to (\mathbb{R}^p, 0)$ are map-germs such that $\mathbb{A}(0,0)$ is nondegerate matrix and $g(y,0) = y$.

Theorem 3.11 ([1]). *$\mathcal{K}$-versal deformations of $\mathcal{K}$-equivalent germs with the same number of parameters are fiber $\mathcal{K}$-equivalent.*

4. Singularities of projection and of contact

4.1. **Singularities of projection.** In view of Theorem 2.7, let M and $\widetilde{M}$ be smooth closed n-dimensional submanifolds of $\mathbb{R}^q$, $q \leq 2n$, and

$$\mathcal{M}_\lambda = \Gamma_\lambda(M \times M)\ ,\ \widetilde{\mathcal{M}}_\lambda = \Gamma_\lambda(\widetilde{M} \times \widetilde{M})\ ,$$

where Γ_λ is the λ-chord transformation.

For local classification of singularities, we introduce the definition:

Definition 4.1. $E_\lambda(M)$ and $E_\lambda(\widetilde{M})$ are **λ-chord equivalent** if there exists a fiber-preserving diffeomorphism-germ of $T\mathbb{R}^q$ that maps the germ of $\mathcal{M}_\lambda$ to the germ of $\widetilde{\mathcal{M}}_\lambda$ i.e. if the following diagram commutes (vertical arrows indicate diffeomorphism-germs):

$$\begin{array}{ccccc} M \times M & \stackrel{\Gamma_\lambda|_{M\times M}}{\longrightarrow} & T\mathbb{R}^q & \stackrel{\pi}{\longrightarrow} & \mathbb{R}^q \\ \downarrow & & \downarrow & & \downarrow \\ \widetilde{M} \times \widetilde{M} & \stackrel{\Gamma_\lambda|_{\widetilde{M}\times\widetilde{M}}}{\longrightarrow} & T\mathbb{R}^q & \stackrel{\pi}{\longrightarrow} & \mathbb{R}^q \end{array}$$

The λ-chord equivalence of E_λ is a special case of equivalence of projections studied by V. Goryunov ([9], [10]), as outlined below.

Definition 4.2. A **projection** of a (smooth) submanifold S from a total space E to the base B of the bundle $p : E \to B$ is a triple

$$S \stackrel{\iota}{\hookrightarrow} E \stackrel{p}{\to} B$$

where ι is an embedding. A projection is called a **projection "onto"** if the dimension of S is not less than the dimension of the base B.

Definition 4.3. Two projections $S_i \hookrightarrow E_i \to B_i$ for $i = 1, 2$ are **equivalent** if the following diagram commutes

$$\begin{array}{ccccc} S_1 & \stackrel{\iota_1}{\hookrightarrow} & E_1 & \stackrel{p_1}{\to} & B_1 \\ \downarrow & \iota_2 & \downarrow & p_2 & \downarrow \\ S_2 & \hookrightarrow & E_2 & \to & B_2 \end{array}$$

where vertical arrows indicate diffeomorphisms.

A projection of S onto B defines a family of subvarieties in the fibers of the bundle $p : E \to B$ parameterized by B: $S_b = S \cap p^{-1}(b)$ for any $b \in B$. A germ of the projection

$$(S, q_0) \hookrightarrow (E, e_0) \to (B, b_0)$$

can be considered in a natural way as a deformation of the subvariety S_{b_0}.

The germ of a bundle $E \to B$ can be identified with the germ of the trivial bundle

$$\mathbb{R}^s \times \mathbb{R}^p \to \mathbb{R}^p.$$

A germ of an embedded smooth submanifold S can be described by the germ of the variety of zeros of some mapping-germ $F : (\mathbb{R}^s \times \mathbb{R}^p, (y_0, z_0)) \to \mathbb{R}^t$. Then S_{z_0} can be identified with the germ of the variety of zeros of $F|_{\mathbb{R}^s\times\{z_0\}}$.

If deformations $F, \widetilde{F} : (\mathbb{R}^s \times \mathbb{R}^p, (y_0, z_0)) \to (\mathbb{R}^t, 0)$ of map-germs $f, \widetilde{f} : (\mathbb{R}^s, y_0) \to (\mathbb{R}^t, 0)$ (respectively) are fiber $\mathcal{K}$-equivalent then the following diagram commutes (Φ, Z indicate diffeomorphism-germs and pr indicate the projection):

$$\begin{array}{ccccc} F^{-1}(0) & \hookrightarrow & \mathbb{R}^s \times \mathbb{R}^p & \stackrel{pr}{\longrightarrow} & \mathbb{R}^p \\ \downarrow & & \downarrow \Phi & & \downarrow Z \\ \widetilde{F}^{-1}(0) & \hookrightarrow & \mathbb{R}^s \times \mathbb{R}^p & \stackrel{pr}{\longrightarrow} & \mathbb{R}^p \end{array}$$

If the ideal of function-germs vanishing on $F^{-1}(0)$ is generated by the components of F, then by Corollary 3.6 the inverse result is also true.

We remind that the group $\mathcal{A} = \mathrm{Diff}(\mathbb{R}^m, 0) \times \mathrm{Diff}(\mathbb{R}^p, 0)$ acts on map-germs $(\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$ by composition on source and target, with corresponding definitions for $\mathcal{A}$-equivalent and $\mathcal{A}$-simple (refer to Definitions 3.1 and 3.7 for the group $\mathcal{K}$). Then, from the above we have the following results:

Proposition 4.4 ([9, 10])**.** *F and $\widetilde{F}$ are fiber $\mathcal{K}$-equivalent if and only if the projections of $F^{-1}(0)$ and $\widetilde{F}^{-1}(0)$ onto $\mathbb{R}^p$ are $\mathcal{A}$-equivalent.*

Theorem 4.5 ([9])**.** *If the germ of a projection $(F^{-1}(0), (0, 0)) \hookrightarrow (\mathbb{R}^s \times \mathbb{R}^p, (0, 0)) \to (\mathbb{R}^p, 0)$ is $\mathcal{A}$-simple then $f = F|_{\mathbb{R}^s \times \{0\}}$ is $\mathcal{K}$-simple.*

Theorem 4.6 ([11, 12])**.** *The map-germ $F : \mathbb{R}^s \times \mathbb{R}^p \to \mathbb{R}^t$ is a $\mathcal{K}$-versal deformation of a rank-0 map-germ $f : \mathbb{R}^s \to \mathbb{R}^t$ of finite $\mathcal{K}$-codimension if and only if the projection-germ of $F^{-1}(0)$ onto $\mathbb{R}^p$ is $\mathcal{A}$-stable (infinitesimally stable).*

By Theorems 4.5 and 4.6, in order to classify stable singularities of projections one considers deformations of three classes of singularities: simple singularities of hypersurfaces (Table 1), simple singularities of curves in a 3-dimensional space (Table 3), simple singularities of a multiple point on a plane (Table 2). We are interested in projections "onto" when the projected submanifold $S = F^{-1}(0)$ is smooth and the dimension of the base B of the bundle is greater than 1.

In order to see in a more clear way how these three tables are applied to the classification of singularities of affine equidistants, we now turn to the contact viewpoint.

4.2. **Singularities of contact.** Let N_1, N_2 be germs at x of smooth n-dimensional submanifolds of the space $\mathbb{R}^q$, with $2n \geq q$. We describe N_1, N_2 in the following way:

- $N_1 = f^{-1}(0)$, where $f : (\mathbb{R}^q, x) \to (\mathbb{R}^{q-n}, 0)$ is a submersion-germ,
- $N_2 = g(\mathbb{R}^n)$, where $g : (\mathbb{R}^n, 0) \to (\mathbb{R}^q, x)$ is an embedding-germ.

Let $\tilde{N}_1, \tilde{N}_2$ be another pair of germs at $\tilde{x}$ of smooth n-dimensional submanifolds of the space $\mathbb{R}^q$, described in the same way.

Definition 4.7. The contact of N_1 and N_2 at x is of the same **contact-type** as the contact of $\tilde{N}_1$ and $\tilde{N}_2$ at $\tilde{x}$ if there exists a diffeomorphism-germ $\Phi : (\mathbb{R}^q, x) \to (\mathbb{R}^q, \tilde{x})$ such that $\Phi(N_1) = \tilde{N}_1$ and $\Phi(N_2) = \tilde{N}_2$. We denote the contact-type of N_1 and N_2 at x by $\mathcal{K}(N_1, N_2, x)$.

Definition 4.8. A **contact map** between submanifold-germs N_1, N_2 is the following map-germ $f \circ g : (\mathbb{R}^n, 0) \to (\mathbb{R}^{q-n}, 0)$.

Theorem 4.9 ([13])**.** *$\mathcal{K}(N_1, N_2, x) = \mathcal{K}(\tilde{N}_1, \tilde{N}_2, \tilde{x})$ if and only if the contact maps $f \circ g$ and $\tilde{f} \circ \tilde{g}$ are $\mathcal{K}$-equivalent.*

Remark 4.10. If N_1 and N_2 are transversal at x then it is obvious that the contact map $f \circ g : (\mathbb{R}^n, 0) \to (\mathbb{R}^{q-n}, 0)$ is a submersion-germ or a diffeomorphism-germ (when $q = 2n$).

The interesting cases are when N_1 and N_2 are not transversal at x_0

$$T_{x_0}N_1 + T_{x_0}N_2 \neq T_{x_0}\mathbb{R}^q.$$

Definition 4.11. We say that N_1 and N_2 are k-**tangent** at x_0 if

$$\dim(T_{x_0}N_1 \cap T_{x_0}N_2) = k\ .$$

If k is maximal, that is

$$k = n = \dim(T_{x_0}N_1) = \dim(T_{x_0}N_2)\ ,$$

we say that N_1 and N_2 are **tangent** at x_0.

Remark 4.12. In order to bring this definition into the context of affine equidistants, $E_\lambda(M)$, note that $N_1 = M^+$ and $N_2 = \mathcal{R}_0^\lambda(M^-)$ are k-**tangent** at 0 if and only if T_aM^+ and T_bM^- are k-**parallel**, where $\lambda a + (1-\lambda)b = 0 \in E_\lambda(M)$.

If N_1 and N_2 are k-tangent then we can describe germs of N_1 and N_2 at 0 in the following way:

$$N_1 = \{(y,z,u,v) \in \mathbb{R}^q : u = \phi(y,z),\ v = \psi(y,z)\}, \tag{4.1}$$

$$N_2 = \{(y,z,u,v) \in \mathbb{R}^q : z = \eta(y,v),\ u = \zeta(y,v)\}, \tag{4.2}$$

where $y = (y_1, \cdots, y_k)$, $z = (z_1, \cdots, z_{n-k})$, $u = (u_1, \cdots, u_{q+k-2n})$, $v = (v_1, \cdots, v_{n-k})$ and (y,z,u,v) is a coordinate system on the affine space $\mathbb{R}^q$,

$$\phi = (\phi_1, \cdots, \phi_{q+k-2n}), \qquad \psi = (\psi_1, \cdots, \psi_{n-k}),$$

$$\eta = (\eta_1, \cdots, \eta_{n-k}), \qquad \zeta = (\zeta_1, \cdots, \zeta_{q+k-2n}), \qquad \text{and} \qquad \phi_i, \psi_j, \eta_j, \zeta_i \in \mathcal{M}_q^2,$$

for $i = 1, \cdots, q+k-2n$ and $j = 1, \cdots, n-k$.

Then, the contact map $\kappa_{N_1,N_2} : (\mathbb{R}^n, 0) \to (\mathbb{R}^{q-n}, 0)$ is given by:

$$\kappa_{N_1,N_2}(y,z) = (z - \eta(y, \psi(y,z)), \phi(y,z) - \zeta(y, \psi(y,z))) \tag{4.3}$$

From the form of κ_{N_1,N_2} we easily obtain the following fact

Proposition 4.13. *If N_1 and N_2 are k-tangent at 0 then the corank of the contact map κ_{N_1,N_2} is k.*

We can interpret the contact between two k-tangent n-dimensional submanifolds N_1, N_2 of $\mathbb{R}^q$ as the contact between tangent k-dimensional submanifolds P_{N_1} and P_{N_2} of N_1 and N_2, respectively, in a smooth $q-2n+2k$-dimensional submanifold S of $\mathbb{R}^q$. These submanifolds are constructed in the following way:

Let H be a smooth $q+k-n$-dimensional submanifold-germ on $\mathbb{R}^q$ which contains N_1 and is transversal to N_2 at 0. Then $P_{N_2} = H \cap N_2$ is a smooth k-dimensional submanifold on N_2.

Let G be a smooth $q+k-n$-dimensional submanifold-germ on $\mathbb{R}^q$ which contains N_2 and is transversal to N_1 at 0. Then $P_{N_1} = G \cap N_1$ is a smooth k-dimensional submanifold on N_1.

P_{N_1} and P_{N_2} are tangent at 0 and they are contained in the smooth $q-2n+2k$-dimensional submanifold-germ $S = H \cap G$.

The contact between N_1 and N_2 at 0 can now be described as the contact between P_{N_1} and P_{N_2} at 0, which defines a rank-0 map

$$\kappa_{P_{N_1},P_{N_2}} : (\mathbb{R}^k, 0) \to (\mathbb{R}^{k-(2n-q)}, 0)\ . \tag{4.4}$$

Although in general P_{N_1} and P_{N_2} depend on the choices of H and G, the contact type of P_{N_1} and P_{N_2} does not depend on these choices. This means that if $\tilde{N}_1, \tilde{N}_2$ is another pair of germs at 0 of smooth n-dimensional submanifold of $\mathbb{R}^q$ then we have the following result.

Proposition 4.14. *$\mathcal{K}(N_1, N_2, 0) = \mathcal{K}(\tilde{N}_1, \tilde{N}_2, 0)$ if and only if*

$$\mathcal{K}(P_{N_1}, P_{N_2}, 0) = \mathcal{K}(P_{\tilde{N}_1}, P_{\tilde{N}_2}, 0).$$

Proof. It is easy to see that in general H can be described in the following way:

$$v = \psi(y,z) + A(y,z,u,v)(u - \phi(y,z)), \tag{4.5}$$

and G can be described in the following way:

$$z = \eta(y,v) + B(y,z,u,v)(u - \zeta(y,v)), \tag{4.6}$$

where $A = (a_{ij})_{i=1,\cdots,q+k-2n}^{j=1,\cdots,n-k}, B = (b_{ij})_{i=1,\cdots,q+k-2n}^{j=1,\cdots,n-k}$ and a_{ij}, b_{ij} are smooth function-germs on $\mathbb{R}^q$.

Thus $S = H \cap G$ is given by (4.5) and (4.6).

P_{N_1} is given by (4.5), (4.6), and $u = \phi(y,z)$ and P_{N_2} is given by (4.5), (4.6) and $u = \zeta(y,v)$.

On the other hand we can also describe N_1 by (4.5) and $u = \phi(y,z)$ and N_2 by (4.6) and $u = \zeta(y,v)$. Then it is easy to see that contact maps are the same after a suitable suspension. □

In view of Proposition 4.14, it is enough to classify the rank-0 map-germs of the form (4.4) with respect to the group $\mathcal{K}$.

5. Stable singularities of affine equidistants

Since our goal is to classify singularities of affine equidistants of n-dimensional submanifold M of $\mathbb{R}^q$, we substitute submanifold-germs N_1 and N_2 of the previous section by $N_1 = M^+$ and $N_2 = \mathcal{R}_0^\lambda(M^-)$, or equivalently by $N_1 = M^-$ and $N_2 = \mathcal{R}_0^{1-\lambda}(M^+)$, where M^+ and M^- are germs of $M \subset \mathbb{R}^q$ at points $a^+ \neq a^- \in M \subset \mathbb{R}^q$, such that $\lambda a^+ + (1-\lambda)a^- = 0$.

First, we state the following definition and theorem:

Definition 5.1. A mapping $\psi : N^m \to \mathbb{R}^q$ is *locally stable* at $p \in N^m$ if there exists a neighbourhood W_p of ψ in the space $C^\infty(N^m, \mathbb{R}^q)$ of C^∞-mappings from N^m into $\mathbb{R}^q$ with the Whitney C^∞-topology, and neighbourhoods U_p around p and V_p around $\psi(p)$ such that for all $\phi \in W_p$, it follows that $\phi : U_p \to V_p$ is $\mathcal{A}$- equivalent to $\psi : U_p \to V_p$, where $\mathcal{A} = \mathrm{Diff}(U_p) \times \mathrm{Diff}(V_p)$ (see [8]).

Theorem 5.2. *For a residual set of embeddings $\iota : M^n \to \mathbb{R}^q$ the map*

$$\pi_\lambda \circ (\iota \times \iota) : M \times M \setminus \Delta \to \mathbb{R}^q$$

is locally stable whenever the pair $(2n, q)$ is a pair of nice dimensions, where Δ is the diagonal in $M \times M$.

Proof. From the diagram of maps

$$M \times M \overset{\iota \times \iota}{\longrightarrow} \mathbb{R}^q \times \mathbb{R}^q \overset{\pi_\lambda}{\to} \mathbb{R}^q \ ,$$

we obtain the diagram of r-jet maps

$$M \times M \overset{j^r(\iota \times \iota)}{\longrightarrow} J^r(M \times M, \mathbb{R}^q \times \mathbb{R}^q) \overset{(\pi_\lambda)_*}{\to} J^r(M \times M, \mathbb{R}^q) \ .$$

A typical fiber of $J^r(M \times M, \mathbb{R}^q)$ is $J_0^r(M \times M, \mathbb{R}^q)$, the space of (degree $\leq r$)-polynomial map-germs $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^q$, vanishing at 0.

Let $\{W_1, \ldots, W_s\}$ be the finite set of all $\mathcal{K}$ simple orbits in $J^r(M\times M, \mathbb{R}^q)$; let $\{W_{s+1}, \ldots, W_t\}$ be a finite stratification of the complement of the union of simple orbits $W_1 \cup \ldots \cup W_s$. This stratification exists because these are semialgebraic sets. We denote by $\mathcal{S} = \{W_j\}_{1\le j\le t}$ the resulting stratification of $J^r(M \times M, \mathbb{R}^q)$. Because $(\pi_\lambda)_*$ is a submersion, $(\pi_\lambda)_*^{-1} W_j = W_j^*$ is a submanifold of $J^r(M\times M, \mathbb{R}^q\times\mathbb{R}^q)$, for all $j = 1, \ldots, t$, so that $\mathcal{S}^* = \{W_j^*\}_{1\le j\le t}$ is a stratification of this space.

Furthermore,

$$j^r(\iota\times\iota) \ \pitchfork\ \mathcal{S}^* \iff j^r(\pi_\lambda\circ(\iota\times\iota)) \ \pitchfork\ \mathcal{S}\ , \tag{5.1}$$

where transversality to $\mathcal{S}$ (respectively to $\mathcal{S}^*$) means transversality of $j^r(\iota \times \iota)$ (respectively $j^r(\pi_\lambda\circ(\iota\times\iota))$) to each stratum of the corresponding stratification.

On the other hand, under the natural identification

$$j^r(\iota\times\iota)|_{M\times M\setminus\Delta} \ \simeq\ {}_2j^r\iota \ \subset\ {}_2J^r(M,\mathbb{R}^q)\ ,$$

where ${}_2J^r(M,\mathbb{R}^q)$ is the space of double r-jets, we can apply the Multijet Transversality Theorem [8] to get that, for each W_j^* in ${}_2J^r(M,\mathbb{R}^q)$, the set of immersions

$$\mathcal{R}_{W_j} = \{\iota : M \to \mathbb{R}^q \,|{}_2j^r\iota \pitchfork W_j^*\}$$

is residual. Then, the set

$$\mathcal{R} = \cap_{j=1}^t \mathcal{R}_{W_j}$$

is also residual.

Now, it follows from equation (5.1) that $j^r(\pi_\lambda\circ(\iota\times\iota)) \pitchfork W_j$, for all $\iota\in\mathcal{R}$, for all $j = 1, \ldots, t$. When $(2n, q)$ is a pair of nice dimensions, this implies that $j^r(\pi_\lambda \circ (\iota \times \iota))$ is transversal to all $\mathcal{K}$ orbits in $J^r(M \times M, \mathbb{R}^q)$, which says that this mapping is locally stable (see [8, 12]). □

Theorem 5.3 ([12]). *The nice dimensions for pairs $(2n, q)$ are:*
(i) $n < q = 2n,\ n \le 4$
(ii) $n < q = 2n - 1,\ n \le 4$
(iii) $n < q = 2n - 2,\ n \le 3$
(iv) $n < q \le 2n - 3, q \le 6$

Thinking locally, denote two distinct germs of embedding $\iota : M^n \to \mathbb{R}^q$ by

$$\iota^+ : (\mathbb{R}^n, 0) \to (\mathbb{R}^q, a^+) \quad\text{and}\quad \iota^- : (\mathbb{R}^n, 0) \to (\mathbb{R}^q, a^-),$$

and by

$$\tilde\pi_\lambda = \pi_\lambda \circ (\iota^+ \times \iota^-) : (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^q, 0)\ , \tag{5.2}$$

the restriction of π_λ to $M^+ \times M^-$. Then, recalling the notation of (4.1)-(4.2), $\tilde\pi_\lambda$ is given by

$$\tilde\pi_\lambda : (y, z, \tilde y, v) \mapsto (\tilde\pi^1_\lambda(y, \tilde y), \tilde\pi^2_\lambda(z, \tilde y, v), \tilde\pi^3_\lambda(y, z, \tilde y, v), \tilde\pi^4_\lambda(y, z, v)) \tag{5.3}$$

where $y, \tilde y \in \mathbb{R}^k$, $z, v \in \mathbb{R}^{n-k}$, and

$$\tilde\pi^1_\lambda(y, \tilde y) = \lambda y + (1-\lambda)\tilde y, \tag{5.4}$$

$$\tilde\pi^2_\lambda(z, \tilde y, v) = \lambda z + (1-\lambda)\eta(\tilde y, v), \tag{5.5}$$

$$\tilde\pi^3_\lambda(y, z, \tilde y, v) = \lambda\phi(y, z) + (1-\lambda)\zeta(\tilde y, v), \tag{5.6}$$

$$\tilde\pi^4_\lambda(y, z, v) = \lambda\psi(y, z) + (1-\lambda)v. \tag{5.7}$$

Let

$$\kappa_\lambda : (\mathbb{R}^n, 0) \to (\mathbb{R}^{q-n}, 0)$$

denote the the contact-map between M^+ and $\mathcal{R}^\lambda_0(M^-)$. We have:

Proposition 5.4. *Local rings* $\dfrac{\mathcal{E}_{2n}}{\tilde{\pi}_\lambda^*(\mathfrak{m}_q)}$ *and* $\dfrac{\mathcal{E}_n}{\kappa_\lambda^*(\mathfrak{m}_{q-n})}$ *are isomorphic.*

Proof. From (5.3), we have that

$$\frac{\mathcal{E}_{2n}}{\tilde{\pi}_\lambda^*(\mathfrak{m}_q)} \simeq \frac{\mathcal{E}_{(y,z,\tilde{y},v)}}{\langle \tilde{\pi}_\lambda^1(y,\tilde{y}), \tilde{\pi}_\lambda^2(z,\tilde{y},v), \tilde{\pi}_\lambda^3(y,z,\tilde{y},v), \tilde{\pi}_\lambda^4(y,z,v)\rangle}$$

so that, using (5.4)-(5.7), this is isomorphic to

$$\frac{\mathcal{E}_{(y,z)}}{\langle z + \frac{(1-\lambda)}{\lambda}\eta(-\frac{\lambda}{(1-\lambda)}y, -\frac{\lambda}{(1-\lambda)}\psi(y,z)), \phi(y,z) + \frac{(1-\lambda)}{\lambda}\zeta(-\frac{\lambda}{(1-\lambda)}y, -\frac{\lambda}{(1-\lambda)}\psi(y,z))\rangle}$$

and, using (4.3) for $N_1 = M^+$ and $N_2 = \mathcal{R}_0^\lambda(M^-)$, we see that the above local ring is isomorphic to $\dfrac{\mathcal{E}_n}{\kappa_\lambda^*(\mathfrak{m}_{q-n})}$. □

On the other hand, we remind from Remark 4.12 that k is the degree of tangency of M^+ and $\mathcal{R}_0^\lambda(M^-)$ and therefore k is the degree of parallelism of $T_{a^+}M^+$ and $T_{a^-}M^-$, where

$$\lambda a^+ + (1-\lambda)a^- = 0 \in E_\lambda(M),$$

so that, denoting by

$$\theta_\lambda : (\mathbb{R}^k, 0) \to (\mathbb{R}^{k-(2n-q)}, 0)$$

the reduced (rank-0) contact map $\theta_\lambda = \kappa_{P_{N_1},P_{N_2}}$, for $N_1 = M^+$ and $N_2 = \mathcal{R}_0^\lambda(M^-)$, from Proposition 4.14 we have the following

Corollary 5.5. *The local rings* $\dfrac{\mathcal{E}_n}{\kappa_\lambda^*(\mathfrak{m}_{q-n})}$ *and* $\dfrac{\mathcal{E}_k}{\theta_\lambda^*(\mathfrak{m}_{k-(2n-q)})}$ *are isomorphic.*

Thus, by Theorems 4.6 and 5.2, Proposition 5.4 and Corollary 5.5, for the local classification of stable singularities of affine equidistants, we need to determine every rank-0 $\mathcal{K}$-simple map-germ

$$\theta_\lambda : (\mathbb{R}^k, 0) \to (\mathbb{R}^l, 0) \ , \tag{5.8}$$

that admits a $\mathcal{K}$-versal deformation $F_\lambda : \mathbb{R}^k \times \mathbb{R}^q \to \mathbb{R}^l$, so that

$$\tilde{\pi}_\lambda : (F_\lambda)^{-1}(0) = (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^q, 0) \tag{5.9}$$

is an $\mathcal{A}$-stable map. Here, $\theta_\lambda = \kappa_{P_{N_1},P_{N_2}}$, for $N_1 = M^+$ and $N_2 = \mathcal{R}_0^\lambda(M^-)$, and $\tilde{\pi}_\lambda$ is the restriction of π_λ to $M^+ \times M^-$, so that

$$l = k - (2n - q) \ \ , \ \ 1 \le k \le n \ \ , \ \ 2n \ge q > n \ \ , \tag{5.10}$$

for any pair $(2n, q)$ in the nice dimensions (Theorem 5.3).

In other words, we unfold the map-germ θ_λ with m parameters,

$$\tilde{\pi}_\lambda : (\mathbb{R}^m \times \mathbb{R}^k, 0) \to (\mathbb{R}^m \times \mathbb{R}^l, 0) \ \ , \ \ (w, y) \mapsto (w, u(w, y)) \ , \tag{5.11}$$

where $m = 2n - k$, so that $\tilde{\pi}_\lambda$ is $\mathcal{A}$-stable. Thus, in each case, we look for the rank-0 $\mathcal{K}$-simple map-germs θ_λ that can be unfolded with $m = 2n - k$ parameters so that its $\mathcal{K}_e$-codimension μ is such that

$$\mu \le l + m = q \ . \tag{5.12}$$

The list of $\mathcal{K}$-simple map-germs θ_λ is presented in Tables 1, 2 and 3, in section 2 above. Thus, for classifying the stable singularities of affine equidistants of smooth submanifolds $M^n \subset \mathbb{R}^q$, all we have to do is read those Tables with respect to the numbers k, l and μ, subject to conditions (5.10) and (5.12) for each pair $(2n, q)$ in the nice dimensions.

In this way, we arrive at our main result, as follows.

5.1. **All possible stable singularities in the nice dimensions.** First, remind the definition of k-parallelism, cf. (2.1). Then, we have:

Theorem 5.6. *Let $M^n \subset \mathbb{R}^q$ be a smooth closed submanifold of the affine space, such that $2n \geq q$ and $(2n, q)$ is a pair of nice dimensions, as listed in Theorem 5.3. Then, the possible stable singularities of the λ-affine equidistant $E_\lambda(M) \subset \mathbb{R}^q$ are listed case by case, as below.*

Curves:

In this case, we have curves in $\mathbb{R}^2$ and the rank-0 contact map is $\theta_\lambda : \mathbb{R} \to \mathbb{R}$, $\mu \leq 2$. From Table 1, the stable singularities of affine equidistants can be of type A_1 and A_2.

Surfaces:

(1) $M^2 \subset \mathbb{R}^3$.
2-parallelism. $\theta_\lambda : \mathbb{R}^2 \to \mathbb{R}$, $\mu \leq 3$.
$E_\lambda(M)$ with stable singularities of types A_1, A_2 and A_3.

(2) $M^2 \subset \mathbb{R}^4$.
(i) 1-parallelism. $\theta_\lambda : \mathbb{R} \to \mathbb{R}$, $\mu \leq 4$.
$E_\lambda(M)$ with stable singularities of types A_1, A_2, A_3 and A_4.
(ii) 2-parallelism. $\theta_\lambda : \mathbb{R}^2 \to \mathbb{R}^2$, $\mu \leq 4$.
$E_\lambda(M)$ with stable singularities of types $C^{\pm}_{2,2}$.

3-*manifolds*:

(1) $M^3 \subset \mathbb{R}^4$.
3-parallelism. $\theta_\lambda : \mathbb{R}^3 \to \mathbb{R}$, $\mu \leq 4$.
$E_\lambda(M)$ with stable singularities of types A_1, ..., A_4 and $D^{\pm}_4$.

(2) $M^3 \subset \mathbb{R}^5$.
(i) 2-parallelism. $\theta_\lambda : \mathbb{R}^2 \to \mathbb{R}$, $\mu \leq 5$.
$E_\lambda(M)$ with stable singularities of types A_1, ..., A_5, $D^{\pm}_4$, $D^{\pm}_5$.
(ii) 3-parallelism. $\theta_\lambda : \mathbb{R}^3 \to \mathbb{R}^2$, $\mu \leq 5$.
$E_\lambda(M)$ with stable singularites of types S_5.

(3) $M^3 \subset \mathbb{R}^6$.
(i) 1-parallelism. $\theta_\lambda : \mathbb{R} \to \mathbb{R}$, $\mu \leq 6$.
$E_\lambda(M)$ with stable singularities of types A_1, ..., A_6.
(ii) 2-parallelism. $\theta_\lambda : \mathbb{R}^2 \to \mathbb{R}^2$, $\mu \leq 6$.
$E_\lambda(M)$ with stable singularities of types $C^{\pm}_{2,2}$, $C^{\pm}_{2,3}$, $C^{\pm}_{2,4}$, $C^{\pm}_{3,3}$, C_6.
(iii) 3-parallelism. No stable singularities for $E_\lambda(M)$.

4-*manifolds*:

(1) $M^4 \subset \mathbb{R}^5$.
4-parallelism. $\theta_\lambda : \mathbb{R}^4 \to \mathbb{R}$, $\mu \leq 5$.
$E_\lambda(M)$ with stable singularities of types A_1, ..., A_5, $D^{\pm}_4$, $D^{\pm}_5$.

(2) $M^4 \subset \mathbb{R}^6$: The map $\tilde{\pi}_\lambda : \mathbb{R}^8 \to \mathbb{R}^6$ is not in nice dimensions.

(3) $M^4 \subset \mathbb{R}^7$.
(i) 2-parallelism. $\theta_\lambda : \mathbb{R}^2 \to \mathbb{R}$, $\mu \leq 7$.
$E_\lambda(M)$ with stable singularities A_1, ..., A_7, $D_4^\pm$, ..., $D_7^\pm$, E_6, E_7.
(ii) 3-parallelism. $\theta_\lambda : \mathbb{R}^3 \to \mathbb{R}^2$, $\mu \leq 7$.
$E_\lambda(M)$ with stable singularities of types S_5, S_6, S_7, T_7, $\tilde{T}_7$.
(iii) 4-parallelism. No stable singularities for $E_\lambda(M)$.

(4) $M^4 \subset \mathbb{R}^8$.
(i) 1-parallelism. $\theta_\lambda : \mathbb{R} \to \mathbb{R}$, $\mu \leq 8$.
$E_\lambda(M)$ with stable singularities of types A_1, ..., A_8.
(ii) 2-parallelism. $\theta_\lambda : \mathbb{R}^2 \to \mathbb{R}^2$, $\mu \leq 8$.
$E_\lambda(M)$ with stable singularities of types
$C_{2,2}^\pm$, $C_{2,3}^\pm$, $C_{2,4}^\pm$, $C_{2,5}^\pm$, $C_{2,6}^\pm$, $C_{3,3}^\pm$, $C_{3,4}^\pm$, $C_{3,5}^\pm$, $C_{4,4}^\pm$, C_6, C_8, F_7, F_8.
(iii) 3-parallelism, 4-parallelism. No stable singularities for $E_\lambda(M)$.

5-*manifolds*:

(1) $M^5 \subset \mathbb{R}^6$.
5-parallelism. $\theta_\lambda : \mathbb{R}^5 \to \mathbb{R}$, $\mu \leq 6$.
$E_\lambda(M)$ with stable singularities A_1, ..., A_6, $D_4^\pm$, $D_5^\pm$, $D_6^\pm$, E_6.

(2) For all other embeddings $M^5 \subset \mathbb{R}^q$, no map $\tilde{\pi}_\lambda$ in nice dimensions.

n-*manifolds*, $n \geq 6$: No map $\tilde{\pi}_\lambda$ in nice dimensions.

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