Journal *of* Singularities
Volume 1 (2010), 146-168

received 14 October 2009
in revised form 9 November 2010
DOI: 10.5427/jsing.2010.1j

MONODROMY OF PLANE CURVES AND QUASI-ORDINARY SURFACES

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Abstract

We establish an explicit recursive formula for the vertical monodromies of an irreducible quasi-ordinary surface in $\mathbf{C}^3$. The calculation employs a local description of the singularity at the generic point of each singular component in terms of a "truncation" and a "derived" surface. These objects are also used to retrieve a formula for the (classical) horizontal monodromy in recursive terms.

Consider an irreducible germ of analytic surface S in $\mathbf{C}^3$, arranged so that the projection $\pi : (x, y, z) \mapsto (x, y)$ has its discriminant locus contained in the coordinate axes. This is the local picture of a *quasi-ordinary surface*. The theory of such surfaces (which we briefly recall in section 3) says that each sheet may be expressed in the following way:

$$\zeta = \sum c_{\lambda\mu} x^{\lambda} y^{\mu},$$

where the exponents range over certain non-negative rational numbers with a common denominator. Let d denote the number of sheets (equivalently the number of conjugates of ζ). One can write a function defining S by taking a product over all conjugates:

$$f(x, y, z) = \prod_{k=1}^{d} (z - \zeta_k).$$

In general the singular locus of such a surface is one-dimensional, with at most two components. In almost all instances, the x-axis is one such component. A transverse slice $x = C$ (where C is a small nonzero constant) cuts out a singular plane curve. The Milnor fiber of this curve undergoes a monodromy transformation when C loops around the origin; the action on its homology groups is called the *vertical monodromy*. In this article we show how to explicitly calculate this monodromy. Our formula is expressed recursively, by associating to our surface two related quasi-ordinary surfaces which we call its *truncation* S_1 and its *derived surface* S', and then expressing the vertical monodromy of S via the monodromies of S_1 and of S'.

As is well known, there is another fibration over a circle, called the *Milnor fibration*; here the action on homology is called the *horizontal monodromy*. In the course of working out our recursion for vertical monodromy, we have discovered what appears to be a new viewpoint about the horizontal monodromy, expressed in a similar recursion which again invokes the same two associated surfaces. In fact this recursion makes sense even outside the context of quasi-ordinary surfaces, and thus we have found a novel way to express the monodromy associated to the Milnor fibration of a singular plane curve. (There are known formulas for this monodromy, e.g. Theorem 2 of [3] and formula (6.1) of [4], as well as quasi-ordinary analogs presented in [7] and [14], but they are not framed in the same recursive manner.)

We begin by working out this situation, to motivate our later setup and to provide a model for the more elaborate calculation.

As a corollary to our formulas, we have found that from the vertical and horizontal monodromies (one pair for each component of the singular locus), together with the surface monodromy formula worked out in [14] and [7], one can recover the complete set of characteristic pairs of a quasi-ordinary surface. Since these data depend only on the embedded topology of the surface, we thus have a new proof of Gau's theorem [6] in the 2-dimensional case. As another application, we can employ a theorem of Steenbrink [17] (extended to the non-isolated case by M. Saito [15]) which relates the horizontal and vertical monodromies to the spectrum of the surface and to the spectrum of any member of the Yomdin series. Since the spectrum of an isolated singularity is computable in principle, we expect that the monodromies worked out here may be exploited to calculate the spectrum of a quasi-ordinary surface. We intend to explicate these two applications in subsequent papers. We have also begun, along with Mirel Caibăr and Manuel González Villa, to investigate whether our recursion has a motivic incarnation akin to that of [4]; we believe that it does.

We begin in section 1 with two "approximation lemmas" that allow us to replace one function by another when studying their associated fibrations. In section 2 we work out the monodromy of the Milnor fiber of a plane curve singularity. Everything in this section is well-known (although it is not usually presented in a recursive framework), and we present it merely as a prototype for our original contributions in subsequent sections. In section 3 we briefly recall the basic notions of quasi-ordinary surfaces and introduce the "transverse Milnor fiber." Section 4 formulates and proves our main results. In these results we assume that a certain characteristic exponent μ_1 does not vanish; our last (very brief) section discusses the case $\mu_1 = 0$.

We wish to thank Clement Caubel, Herb Clemens, Anatoly Libgober, and Joe Lipman for useful conversations regarding this project. We also thank the referee for several helpful suggestions for improving the article.

1. Approximation lemmas

In the proofs of our recursive formulas we use the following lemmas. For ease of reference, we give two separate formulations, but clearly the first lemma follows from the second.

Lemma 1.1. *Suppose that f and g are two holomorphic functions on a smooth compact analytic surface S with boundary. Suppose that they have the same divisor D, and that D_{red} is transverse to the boundary. Suppose that the unit $u = f/g$ always has positive real part. Then, for sufficiently small σ, the fibration over the circle $|\epsilon| = \sigma$ with fibers $f = \epsilon$ is smoothly isotopic to the fibration with fibers $g = \epsilon$.*

Lemma 1.2. *Over a circle $|x| = \rho$, let S be the total space of a continuous family of smooth compact analytic surfaces S_x with boundary. Suppose that f and g are two continuous functions such that, for each x, their restrictions f_x and g_x are holomorphic functions on S_x having the same divisor D_x. Suppose that each D_x is transverse to the boundary. Suppose that the unit $u = f/g$ always has positive real part. Then, for sufficiently small σ, the fibration over the torus $|x| = \rho, |\epsilon| = \sigma$ with fibers $f_x = \epsilon$ is isotopic to the fibration with fibers $g_x = \epsilon$.*

Proof. Let D be the union of the divisors D_x. We argue that in a punctured neighborhood of D, the interpolation $F_t = tf + (1-t)g$ (with $0 \le t \le 1$) has a non-vanishing gradient (as does its restriction to the boundary). Then by the Ehresmann fibration theorem, F_t provides a locally trivial fibration.

There is a neighborhood of D on which, away from D itself, the relative gradient ∇g does not vanish. Indeed, let V be the variety on which ∇g vanishes. Then g must be constant on each component of V, and each such component either misses D or is completely contained within it. Similarly, we claim that there is a (punctured) neighborhood of D on which ∇f is never a negative multiple of ∇g. To see this, consider the variety V on which the two gradients are linearly dependent; note that D is contained in V. Then the quotient $\lambda = \nabla f / \nabla g$ is a well-defined analytic function on V at least away from D. Suppose we have a map $\gamma : (C, p) \to V$ from a nonsingular curve germ, with $\gamma(p) \in D$. Then on C we have

$$\lambda = f'/g' = u + \frac{g}{g'} u'.$$

The quotient g/g' has a removable singularity at p and vanishes there. Thus we have $\lambda(p) = u(p)$. Since the curve C is arbitrary, this shows that λ is well-defined on D and agrees with u there. Thus there is a neighborhood of V in which the real part of λ cannot be negative; in the punctured neighborhood ∇F_t does not vanish.

Finally, since each D_x is transverse to the boundary, we can find a local trivialization of a neighborhood of $D_x \cap \partial S$ in ∂S, with fibers isomorphic to the complex disk. Then a similar argument as above applies to f and g restricted to the boundary. □

2. Plane curves

The material in this section is well-known. We present it to establish notations, to isolate certain technical details for later reference, and to elucidate our recursive point of view.

Consider a germ at the origin of an irreducible analytic plane curve defined by $f(y, z) = 0$; we will simply call it a "curve." (For basic notions and facts about singular plane curves see [5] or [18].) The *Milnor fiber* F is the set of points (y, z) obtained by the following process:

(1) requiring that $\|(y, z)\| \le \delta$, a sufficiently small radius,
(2) then requiring that $f(y, z) = \epsilon$, a number sufficiently close to zero.

The boundary of the Milnor fiber is a link in the sphere. Letting ϵ vary over a circle centered at 0 we obtain the *Milnor fibration* (which we will also call the *horizontal fibration*). Let $h_q : H_q(F; \mathbf{Q}) \to H_q(F; \mathbf{Q})$ be the monodromy operator. The graded characteristic function

$$\mathbf{H}(t) = \frac{\det(tI - h_0)}{\det(tI - h_1)}$$

is called the *horizontal monodromy.* (In the literature it is often called the *monodromy zeta function.*) Taking its degree computes the Euler characteristic χ of F.

Assuming that the curve is not the axis $y = 0$, there is a parametrization

$$y = t^d, \quad z = \sum_j c_j t^j,$$

where the exponents (taken all together) are relatively prime positive integers, and all coefficients are nonzero. The integer d (which we call the *degree*) is the number of sheets for the projection $\pi : (y, z) \mapsto y$, and over a slitted neighborhood of 0 we may parametrize each sheet by

$$\zeta = \sum_j c_j y^{j/d},$$

having chosen one of the d possible roots. We prefer to write this as follows:

$$\zeta = \sum c_\mu y^\mu, \tag{2.1}$$

where the sum is now over certain positive rational numbers with common denominator d (arranged in increasing order); this is called the *Puiseux series* of the curve. One can recover f by forming a product over all conjugates:

$$f(y, z) = \prod^d (z - \zeta).$$

(Note our notation for recording the number of conjugates.)

An exponent of the Puiseux series is called *essential* (or *characteristic*) if its denominator does not divide the common denominator of the previous exponents. In particular (by the convention that the least common multiple of the empty set is 1) all integer exponents are inessential, but the first noninteger exponent is essential. Clearly there are only finitely many essential exponents $\mu_1 < \mu_2 < \cdots < \mu_e$. The sum

$$\sum_{i=1}^{e} y^{\mu_i} \tag{2.2}$$

parametrizes the d sheets of a singular curve which we call the *prototype*.

Theorem 2.1. *A curve and its prototype have the same horizontal monodromy.*

(As an example, if there are no essential exponents then the curve is nonsingular at the origin, its prototype is $z = 0$, and the horizontal monodromy is $t - 1$.) This theorem is well-known; see for example [16]. We will prove Theorem 2.1 by induction on e, at the same time that we prove a set of recursive formulas. To this end, we define the *truncation* of a singular curve with prototype

$$\sum_{i=1}^{e} y^{\mu_i}$$

to be the curve with Puiseux series

$$\zeta_1 = y^{\mu_1} = y^{n/m}$$

(where the second equation defines the relatively prime integers m and n). Its *derived curve* is the curve with Puiseux series

$$\zeta' = \sum_{i=1}^{e-1} y^{\mu_i'},$$

with the new exponents computed by

$$\mu_i' = m(\mu_{i+1} - \mu_1 + n).$$

Example 2.2. Suppose we begin with the curve whose Puiseux series is

$$\zeta = y^{3/2} + y^{7/4} + y^{11/6}.$$

Then its truncation is parametrized by $\zeta_1 = y^{3/2}$, and its derived curve is parametrized by

$$\zeta' = y^{13/2} + y^{20/3}.$$

Repeating this process, we obtain truncation $\zeta_1' = y^{13/2}$ and second derived curve

$$\zeta'' = y^{79/3}.$$

Let d_1 and d' denote the degrees of the truncation and the derived curve, respectively. Similarly, let χ_1 and χ' denote the Euler characteristics of their Milnor fibers; let $\mathbf{H}_1$ and $\mathbf{H}'$ denote their horizontal monodromies.

Theorem 2.3. *The degree, Euler characteristic, and horizontal monodromy are determined by these formulas:*

(1) $d_1 = m$
(2) $d = d_1 d'$
(3) $\chi_1 = m + n - mn$
(4) $\chi = d'(\chi_1 - 1) + \chi'$
(5)
$$\mathbf{H}_1(t) = \frac{(t^m - 1)(t^n - 1)}{t^{mn} - 1}$$
(6)
$$\mathbf{H}(t) = \frac{\mathbf{H}_1(t^{d'}) \cdot \mathbf{H}'(t)}{t^{d'} - 1}$$

These formulas may be compared with the well-known (non-recursive) versions in the literature; see e.g. [18].

For the curve of Example 2.2, the first two formulas tell us that $d = 2d' = 4d'' = 12$. By formulas (3) and (4), the Euler characteristic of the Milnor fiber is

$$\chi = d'(\chi_1 - 1) + d''(\chi_1' - 1) + \chi'' = 6(-2) + 3(-12) + (-155) = -203.$$

By formulas (5) and (6), the horizontal monodromy is

$$\mathbf{H}(t) = \frac{\mathbf{H}_1(t^{d'})}{t^{d'} - 1} \cdot \frac{\mathbf{H}_1(t^{d''})}{t^{d''} - 1} \cdot \mathbf{H}''(t) = \frac{(t^{12} - 1)(t^{18} - 1)(t^{39} - 1)(t^{79} - 1)}{(t^{36} - 1)(t^{78} - 1)(t^{237} - 1)}.$$

Before embarking on the proof of Theorems 2.1 and 2.3, we describe its key idea, and elaborate it by working out the details of Example 2.2. As is well known, one may obtain an embedded resolution of a curve singularity by a resolution process whose steps are dictated by the Puiseux exponents, and from such a resolution one can compute the monodromy by invoking a formula of A'Campo [2]. Our proof does not use this full process of resolution, but just the first step of it: the toric transformation prescribed by the leading exponent. In the example, the toric transformation is given by

$$y = u^2 v$$
$$z = u^3 v^2.$$

Pulling back

$$f(y, z) = \prod^{12} \left(z - \left[y^{3/2} + y^{7/4} + y^{11/6} \right] \right)$$

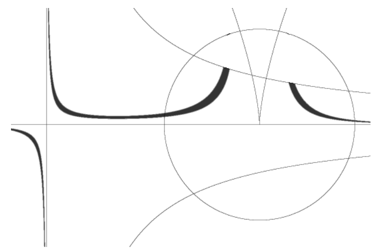

FIGURE 1. The Milnor fiber (the thickened curve) is divided into two pieces by the boundary of N (indicated by a circle). The rupture component is horizontal, and another exceptional divisor is shown vertically. The strict transform enters from above.

by this transformation and factoring, we see that

$$f = u^{36} v^{18} \prod^{12} \left(v^{1/2} - \left[1 + u^{1/2} v^{1/4} + u^{2/3} v^{1/3} \right] \right).$$

Thus there are two exceptional divisors of multiplicities 36 and 18; the former is called the *rupture component*. There is another exceptional divisor with multiplicity 12, not visible in the selected chart. Note that we have not achieved an embedded resolution, nor do we wish to do so; we are content to work with this "partial resolution." (Other authors have also used this idea of partial resolution, e.g. [8].)

The product of 12 conjugates defines the strict transform, and we note that it hits the rupture component at two different points, namely $(u, v) = (0, \pm 1)$. To focus attention at the point $(0, 1)$, we introduce two new variables y' and w. We let B denote a small ball $\|(y', w)\| \leq \delta'$ centered at the origin, and map it to a neighborhood N of $(u, v) = (0, 1)$ by letting $u = \frac{y'}{w+1}$ and $v = (w + 1)^2$. When pulled back via this map, just one of the two values $v^{1/2}$ becomes $w + 1$. Thus six of the 12 conjugates become units, and our function f is thus a unit times the following function:

$$(y')^{36} \prod^{6} \left(w - \left[(y')^{1/2} + (y')^{2/3} \right] \right). \tag{2.3}$$

Our Milnor fiber is thus divided into two pieces: the piece inside N and the outside piece; see Figure 1. Our decomposition is coarser than the usual decomposition of the Milnor fiber, as explained in [2]. Those pieces in the usual decomposition coming from the first sequence of blowups, i.e., dictated by the first characteristic exponent, constitute our outside piece, while the remaining pieces constitute our inside piece. As we show in our proof of Theorem 2.3, the outside piece consists of six copies of the Milnor fiber of the curve $z^2 = y^3$, i.e., the truncation.

To understand the inside piece, we observe that the configuration of curves defined by the vanishing of 2.3, consisting of the strict transform together with the rupture component, can be interpreted as the total transform of a new singular curve. The blowing down map is $(y', w) \mapsto (y', (y')^6 w)$, and the resulting curve has Puiseux series

$$\zeta' = (y')^{13/2} + (y')^{20/3}; \tag{2.4}$$

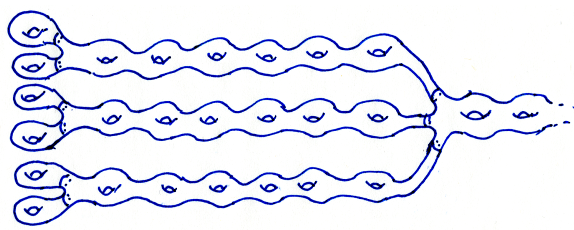

FIGURE 2. The Milnor fiber for Example 2.2 consists of six copies of the Milnor fiber for $z^2 = y^3$ attached to a single copy of the Milnor fiber of its derived curve. In turn, the Milnor fiber of the derived curve consists of three copies of the Milnor fiber for $z^2 = y^{13}$ attached to a single copy of the Milnor fiber of the second derived curve $z^3 = y^{79}$.

this is the derived curve. The blowing down map misses six small disks, and we observe that these disks are cyclically permuted by the monodromy. Figure 2 gives another picture of our decomposition, and indicates how the recursion will continue.

Proof. As indicated, we will simultaneously provide an inductive proof of Theorem 2.1 (inducting on the number of essential exponents) and a recursive proof of Theorem 2.3.

The Milnor fiber of the truncation, which is defined by $z^m - y^n = \epsilon$, is projected by π onto a neighborhood of 0 on the y-line, with total ramification above the nth roots of $-\epsilon$. This neighborhood can be retracted onto the union L of line segments from 0 to these points, in such a way that there is a compatible retraction of the Milnor fiber onto $\pi^{-1}L$, which is the complete bigraph on the n points $((-\epsilon)^{1/n}, 0)$ and the m points $(0, \epsilon^{1/m})$. As ϵ goes around a circle, each set of points is cyclically permuted. Since m and n are relatively prime, the mn edges of the graph are likewise cyclically permuted. Thus the odd-numbered formulas are confirmed.

To verify the recursive formulas and to handle the inductive step in the proof of Theorem 2.1, suppose we are given a curve with Puiseux series (2.1) and prototype (2.2). We first replace

$$\frac{z - \sum_{\mu \in \mathbf{Z}} c_\mu y^\mu}{c_{\mu_1}}.$$

by z. In the new coordinate system, the curve is defined by the vanishing of

$$f = \prod^{d} \left(z - \left[y^{n/m} + \sum_{\mu > n/m} c_\mu y^\mu \right] \right),$$

(where for simplicity the coefficients have been renamed). The truncation is defined by the vanishing of

$$f_1 = \prod^{m} (z - y^{n/m}) = z^m - y^n.$$

Note that m divides d, and that, as we vary the dth root of y, each value of $y^{1/m}$ occurs d/m times. Thus

$$\frac{f}{f_1^{d/m}} = \prod^{d} \left(1 - \frac{\sum_{\mu > n/m} c_\mu y^\mu}{z - y^{n/m}} \right). \tag{2.5}$$

One can obtain an embedded resolution of the truncation by a sequence of blowups dictated by its exponent $\mu_1 = n/m$ and the Euclidean algorithm. The total transform will consist of a chain of exceptional divisors occurring with certain multiplicities, together with a strict transform meeting just one such exceptional divisor, which we call the *rupture component*. Along this chain the function z^m/y^n has no indeterminacy, and in fact except along the rupture component its value is either 0 or ∞. In either case one immediately verifies that the value of (2.5) is 1.

To work in a chart containing the rupture component, we use substitutions dictated by the matrix

$$\begin{bmatrix} m & n \\ r & s \end{bmatrix},$$

where r and s are the smallest positive integers for which the determinant is 1, namely

$$\begin{aligned} y &= u^m v^r \\ z &= u^n v^s. \end{aligned}$$

We find that in this chart the total transform of the truncation is defined by the vanishing of

$$f_1 = u^{mn} v^{rn} (v - 1),$$

and its strict transform is defined by the vanishing of the last factor. Note that it meets the v-axis at the point $(u, v) = (0, 1)$. The total transform of the given curve is defined by the vanishing of

$$f = \prod^{d} \left(u^n v^s - \left[u^n v^{rn/m} + \sum_{\mu > n/m} c_\mu u^{m\mu} v^{r\mu} \right] \right)$$

which may be rewritten as

$$f = u^{nd} v^{rnd/m} \prod^{d} \left(v^{1/m} - \left[1 + \sum_{\mu > n/m} c_\mu u^{m\mu - n} v^{r(m\mu - n)/m} \right] \right). \tag{2.6}$$

The strict transform is defined by the vanishing of the last d factors, and again it meets the v-axis at $(0, 1)$ (as well as at $m - 1$ other points). Note that

$$\frac{f}{f_1^{d/m}} = \prod^{d} \left(1 - \frac{\sum_{\mu > n/m} c_\mu u^{m\mu - n} v^{r(m\mu - n)/m}}{v^{1/m} - 1} \right),$$

which is indeterminate at $(0, 1)$ but whose value elsewhere on the rupture component is 1.

Introducing two new variables y' and w, let B denote a small ball $\|(y', w)\| \leq \delta'$ centered at the origin, and map it to a neighborhood N of $(u, v) = (0, 1)$ by letting $u = \frac{y'}{(w+1)^r}$ and $v = (w + 1)^m$. Note that this map is nonsingular at the origin.

When pulled back via this map, just one of the values $v^{1/m}$ becomes $w+1$. Thus d/m of the factors at the end of (2.6) become

$$w - \sum_{\mu > n/m} c_\mu (y')^{m\mu - n},$$

whereas the remaining $d - d/m$ factors become units.

We can regard the Milnor fiber of our original curve as a subset of the surface obtained by the sequence of blowups. Let us assume that the choices of δ and ϵ made in defining the Milnor fiber are subsequent to the choice of δ'. We claim that by choosing δ sufficiently small we can guarantee that the strict transform of the original curve germ lies entirely within N. Indeed, we note that on the strict transform the following equation holds:

$$v^{1/m} = 1 + \sum_{\mu > n/m} c_\mu y^{\mu - n/m}$$

(for some choice of conjugate). Thus we can force v to be arbitrarily close to 1 by choosing δ sufficiently small, and since $u^m = y/v^r$ we can likewise force u arbitrarily close to 0. Then by appropriate choice of ϵ we can arrange that the Milnor fiber of our curve is transverse to the boundary of N, and that its boundary lies completely within N. Our Milnor fiber is thus divided into two pieces. (See Figure 1.)

Consider first the piece of the Milnor fiber lying outside of N. Having excluded the points of indeterminacy of $f/f_1^{d/m}$, we may apply the approximation lemma 1.1 to conclude that the monodromy of f is the same as the monodromy of $f_1^{d/m}$. The Milnor fiber has d/m connected components corresponding to all possible values of $\epsilon^{m/d}$, and each one is a copy of the Milnor fiber for f_1. Fixing one such value η, we see as above that the corresponding component can be contracted onto the complete bigraph on the n points $((-\eta)^{1/n}, 0)$ and the m points $(0, \eta^{1/m})$. As ϵ goes around a circle the values of $\epsilon^{m/d}$ are cyclically permuted; thus the components are likewise permuted. As ϵ goes around this circle d/m times, however, each η goes once around a circle. Thus the monodromy of this piece is $\mathbf{H}_1(t^{d/m})$.

Now consider the piece of the Milnor fiber lying inside N. Note that it has two sorts of boundary components: the components of the original link L and those components created by its intersection with the boundary sphere of N. To analyze it, we look at its inverse image in the ball B. By the approximation lemma 1.1, we may ignore all unit factors in f. Thus we may assume that the function defining this piece of the Milnor fiber is

$$(y')^{nd} \prod^{d/m} \left(w - \sum_{\mu > n/m} c_\mu (y')^{m\mu - n} \right).$$

The map $(y', w) \mapsto (y', (y')^{nm} w)$ takes this piece to the Milnor fiber of the curve with Puiseux series

$$\sum_{\mu > n/m} c_\mu (y')^{m\mu - n + nm}, \tag{2.7}$$

but it misses disks centered at the d/m points $(0, \epsilon^{m/d})$. Note that these disks are cyclically permuted by the monodromy. In (2.7) there are $e-1$ essential terms, whereas our original Puiseux series had e essential terms. By the inductive hypothesis, the monodromy of this curve is the same as that of its prototype, which has

Puiseux series

$$\sum_{i=2}^{e}(y')^{m(\mu_i-\mu_1+n)};$$

by reindexing we obtain the Puiseux series of the derived curve. Thus $d' = d/m$, confirming formula (2) of the theorem, and the monodromy of this piece of the Milnor fiber is

$$\frac{\mathbf{H}'(t)}{t^{d'}-1}.$$

Combining this with our conclusion about the monodromy of the first piece, we obtain formula (6). Finally we obtain formula (4) by computing the degree of both sides of (6). □

3. Quasi-ordinary surfaces

We now turn to quasi-ordinary surfaces, beginning with a compressed account of the essential facts and definitions. A reader seeking more information should consult [1, 4, 10, 11, 12].

We suppose that S is a germ at the origin of an irreducible analytic surface defined by the vanishing of a function $f(x, y, z)$. The quasi-ordinary condition means that we can arrange a projection $\pi : (x, y, z) \mapsto (x, y)$ so that $\pi|_S$ has discriminant locus contained in the coordinate axes $xy = 0$. In particular $\pi|_S$ is a finite covering space map over the complement of the axes, whose fundamental group is $\mathbf{Z}\times\mathbf{Z}$. It is known that S has many curve-like properties. Foremost among them is the existence of a fractional-exponent power series

$$\zeta(x, y) = \sum c_{\lambda\mu} x^{\lambda} y^{\mu} \tag{3.1}$$

which parametrizes S via $(x, y) \mapsto (x, y, \zeta(x, y))$, where we vary the conjugate of ζ so as to obtain the various sheets of the cover. The exponents can all be taken to have a common denominator, and we write only those terms in which $c_{\lambda\mu} \neq 0$. One can recover f by forming a product over all conjugates:

$$f(x, y, z) = \prod^{d}(z - \zeta(x, y)).$$

(Here d denotes the number of conjugates and thus the number of sheets.)

Define an ordering on pairs of exponents as follows: we say that $(\lambda, \mu) < (\lambda^*, \mu^*)$ if $\lambda \leq \lambda^*$, $\mu \leq \mu^*$, and they are not the same pair. The restriction on the discriminant locus implies that among the exponent pairs of (3.1) we may find a finite sequence of *characteristic pairs*

$$(\lambda_1, \mu_1) < (\lambda_2, \mu_2) < \cdots < (\lambda_e, \mu_e) \tag{3.2}$$

with these properties:

(1) $(0, 0) < (\lambda_1, \mu_1)$.
(2) Each (λ_i, μ_i) is not contained in the subgroup of $\mathbf{Q}\times\mathbf{Q}$ generated by $\mathbf{Z}\times\mathbf{Z}$ and by the previous characteristic pairs.
(3) If (λ, μ) is a noncharacteristic pair, then it is contained in the subgroup generated by those characteristic pairs for which $(\lambda_i, \mu_i) < (\lambda, \mu)$.

In our analysis we will assume that $\mu_1 \neq 0$. (Note that this covers the case of a *reduced* quasi-ordinary surface as defined in [12], viz., a surface for which $\lambda_1\mu_1 \neq 0$.) In this case one immediately verifies that the intersection of the surface with the plane $y = 0$ is the x-axis; except in trivial cases the x-axis is actually a component of the singular locus. For such a surface we define the *Milnor fiber of a transverse slice* to be the set of points (x, y, z) obtained by the following process:

(1) requiring that $\|(x, y, z)\| \leq \delta$, a sufficiently small radius,
(2) then requiring that x be a fixed number sufficiently close to (but different from) zero,
(3) then requiring that $f(x, y, z) = \epsilon$, a number sufficiently close to (but different from) zero.

Denote this transverse Milnor fiber by F and its Euler characteristic by χ. We should point out a subtlety in the definition: the tranverse slice (obtained by the first two steps but then staying on the surface $f = 0$) may be a plane curve with several branches. For example, the transverse slice of $z^2 = x^3y^2$ is a pair of lines, and thus its transverse Milnor fiber has two boundary components.

By keeping x fixed but letting ϵ vary over a circle centered at 0, we obtain the *horizontal fibration*. Keeping ϵ fixed but letting x vary over a circle centered at 0, we obtain the *vertical fibration*. Thus we have a fibration over a torus. Let $h_q : H_q(F; \mathbf{Q}) \to H_q(F; \mathbf{Q})$ and $v_q : H_q(F; \mathbf{Q}) \to H_q(F; \mathbf{Q})$ be the respective monodromy operators. We call the graded characteristic functions

$$\mathbf{H}(t) = \frac{\det(tI - h_0)}{\det(tI - h_1)} \qquad \text{and} \qquad \mathbf{V}(t) = \frac{\det(tI - v_0)}{\det(tI - v_1)}$$

the *horizontal monodromy* and *vertical monodromy*; in the literature they are often called *zeta functions*.

For a quasi-ordinary surface with $\mu_1 = 0$, the definitions of horizontal and vertical monodromy need to be formulated in a slightly different way. We discuss this case in the last section of the paper. In all circumstances our definitions agree with those of Kulikov [9], p. 137 (except in those cases where the surface is not singular along or above the x-axis, in which case our formulas yield trivial monodromy).

4. Recursive formulas for horizontal and vertical monodromy

Suppose we begin with a series (3.1) defining the germ at the origin of an irreducible quasi-ordinary surface S. As in the case of plane curves, we create a new series using just the characteristic pairs,

$$\sum_{i=1}^{e} x^{\lambda_i} y^{\mu_i}, \tag{4.1}$$

and call the corresponding surface the *prototype*.

Theorem 4.1. *A quasi-ordinary surface (with $\mu_1 \neq 0$) and its prototype have the same horizontal monodromy and the same vertical monodromy.*

We will establish this as in the case of plane curves: by induction on e, while simultaneously proving a set of recursive formulas. The case $e = 0$ is trivial, and henceforth we assume that $e > 0$. We define the *truncation* to be the surface S_1 determined by

$$\zeta_1 = x^{\lambda_1} y^{\mu_1} = x^{\frac{a}{mb}} y^{\frac{n}{m}},$$

where n and m are relatively prime, as are a and b.

As before, let r and s be the smallest nonnegative integers so that

$$\begin{bmatrix} m & n \\ r & s \end{bmatrix}$$

has determinant 1. The *derived surface* is the surface S' determined by

$$\zeta' = \sum_{i=1}^{e-1} x^{\lambda'_i} y^{\mu'_i},$$

where the new exponents are computed by these formulas:

$$\mu'_i = m(\mu_{i+1} - \mu_1 + mb\mu_1)$$
$$\lambda'_i = b(\lambda_{i+1} - \lambda_1 + mb\lambda_1 + r\mu'_i\lambda_1).$$

Example 4.2. For the quasi-ordinary surface with branch

$$\zeta = x^{1/2}y^{4/3} + x^{2/3}y^{4/3} + x^{11/12}y^{4/3},$$

the derived surface is determined by the branch

$$\zeta' = x^{163/3}y^{24} + x^{329/6}y^{24}.$$

For the truncation, let d_1, χ_1, $\mathbf{H}_1$, and $\mathbf{V}_1$ denote its degree, the Euler characteristic of its transverse Milnor fiber, and its horizontal and vertical monodromies. Let d', χ', $\mathbf{H}'$, and $\mathbf{V}'$ denote the same things for the derived surface. Let (n, a) denote the greatest common divisor.

Theorem 4.3. *For a quasi-ordinary surface germ (with $\mu_1 \neq 0$), its degree, the Euler characteristic of its transverse Milnor fiber, its horizontal monodromy, and its vertical monodromy are determined by the following formulas.*

(1) $d_1 = mb$
(2) $d = d_1 d'$
(3) $\chi_1 = mb + nb - mnb^2$
(4) $\chi = d'(\chi_1 - b) + b\chi' = d'\chi_1 + b(\chi' - d')$
(5)
$$\mathbf{H}_1(t) = \frac{(t^{mb} - 1)(t^{nb} - 1)}{(t^{mnb} - 1)^b}$$
(6)
$$\mathbf{H}(t) = \frac{\mathbf{H}_1(t^{d'})(\mathbf{H}'(t))^b}{(t^{d'} - 1)^b}$$
(7)
$$\mathbf{V}_1(t) = \frac{(t-1)^{mb}}{(t^{nb/(n,a)} - 1)^{(n,a)(mb-1)}}$$
(8)
$$\mathbf{V}(t) = \frac{(\mathbf{V}_1(t))^{d'}\mathbf{V}'(t^b)}{(t^b - 1)^{d'}}$$

Before embarking on the proof, we will illustrate its ideas by working out the details of Example 4.2, the surface with branch

$$\zeta = x^{1/2}y^{4/3} + x^{2/3}y^{4/3} + x^{11/12}y^{4/3}.$$

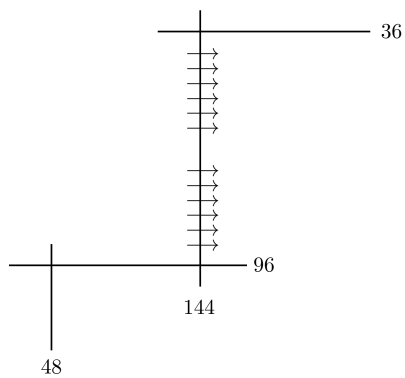

FIGURE 3. The resolution diagram for the transverse slice of the surface of Example 4.2, with multiplicities indicated. The rupture component meets the strict transform at 12 points.

Its intrinsic equation is a polynomial f of degree 36 in z (whose coefficients are functions of x and y):

$$f = \prod^{36} \left(z - \left[x^{1/2}y^{4/3} + x^{2/3}y^{4/3} + x^{11/12}y^{4/3} \right] \right).$$

As x moves on a circle of small radius ρ, each value of x determines a transverse slice of the surface. All of our constructions will be done equivariantly, i.e., by doing the same thing simultaneously to all transverse slices. First, in each transverse slice, we perform the series of blowups dictated by $\mu_1 = 4/3$ and the Euclidean algorithm: this in fact gives an embedded resolution of each transverse slice, with the resolution diagram shown in Figure 3. (This happens because $\mu_1 = \mu_2 = \mu_3$. In general this first set of blowups will only begin the resolution process, and the strict transform will continue to be singular.)

The exceptional divisor meeting the strict transform is called the *rupture component*, and to study it we examine the chart given by

$$y = u^3v^2$$
$$z = u^4v^3.$$

The pullback of f is a product of 36 conjugates:

$$f = \prod^{36} \left(u^4v^3 - \left[x^{1/2}u^4v^{8/3} + x^{2/3}u^4v^{8/3} + x^{11/12}u^4v^{8/3} \right] \right),$$

which we factor as follows

$$f = u^{144}v^{96}x^{18} \prod^{36} \left(\left(\frac{v}{x^{3/2}} \right)^{1/3} - \left[1 + x^{1/6} + x^{5/12} \right] \right). \tag{4.2}$$

Here the rupture component is the v-axis, and the strict transform meets it at the twelve points determined by the values

$$v = (1 + x^{1/6} + x^{5/12})x^{3/2}.$$

As shown in Figure 4, these twelve points are clustered around the two points where the torus knot $v^2 = x^3$ meets our transverse slice.

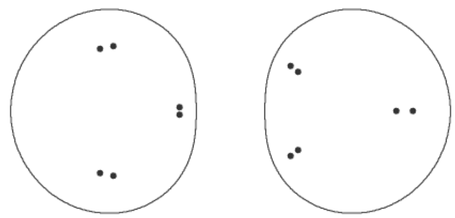

FIGURE 4. The strict transform of a transverse slice of the quasi-ordinary surface $\zeta = x^{1/2}y^{4/3} + x^{2/3}y^{4/3} + x^{11/12}y^{4/3}$ meets the rupture component in twelve points. The tubular neighborhood N meets the rupture component in two topological disks.

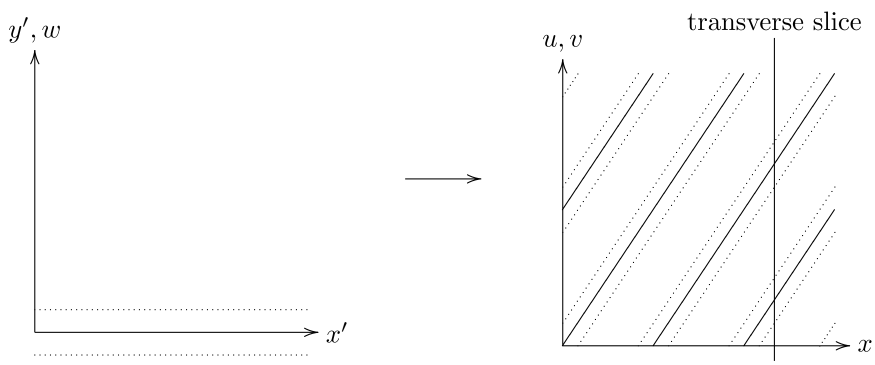

FIGURE 5. A tubular neighborhood B of the circle $\|x'\| = \rho^{1/b}$ is mapped onto a tubular neighborhood N of the torus knot $v^b = x^a$ (where $u = 0$, and x moves on the circle of radius ρ). Each transverse slice $x =$ constant meets N in b disjoint topological balls. In this example, $a = 3$ and $b = 2$.

Introducing three new variables x', y', and w, let B denote the product of the circle $\|x'\| = \rho^{1/2}$ and the 4-ball $\|(y', w)\| \leq \delta'$, where δ' is sufficiently small. We map B to a tubular neighborhood N of the torus knot as follows:

$$x = (x')^2$$
$$u = \frac{y'}{(w+1)^2\rho}$$
$$v = (w+1)^3(x')^3,$$

thus mapping the core circle of B to the knot. Figure 5 illustrates this map. In Figure 4, one sees that N meets the rupture component in two topological disks.

The Milnor fiber of the transverse slice is thus divided into two pieces: the piece lying within N, and the piece lying outside N. Our proof will show that the outside piece is unchanged if in (4.2) we replace f by

$$u^{144}v^{96}(v^{12} - x^{18}),$$

i.e., the pullback of $z^{36} - x^{18}y^{48}$. Thus this piece has six connected components, each of which is a copy of the transverse Milnor fiber of the truncation, the surface with branch

$$\zeta_1 = x^{1/2}y^{4/3}.$$

As for the inside piece, we will argue that it is the same as the transverse Milnor fiber of a new singular surface. When pulled back to B, thirty of the 36 factors at the end of (4.2) become units. To see this, first observe that we can force the value in square brackets to be arbitrarily close to 1 by choosing sufficiently small radii δ' and ρ. To obtain a non-unit, we must therefore pick the "principal value" of $x^{1/2}$ for which it equals x' and then similarly pick the appropriate cube root of $v/(x')^3$ so that

$$\left(\frac{v}{(x')^3}\right)^{1/3} = w + 1;$$

these choices can be made uniformly throughout B. Thus the inside piece is defined by the vanishing of

$$(x')^{324}(y')^{144}\prod^{6}\left(w - \left[(x')^{1/3} + (x')^{5/6}\right]\right)$$

The map $(x', y', w) \mapsto (x', y', (x')^{54}(y')^{24}w)$ takes this piece to the transverse Milnor fiber of the quasi-ordinary surface with branch

$$\zeta' = (x')^{163/3}(y')^{24} + (x')^{329/6}(y')^{24},$$

in accordance with our general formula. The image of the map misses six small disks centered at the points $(x', 0, \epsilon^{1/6})$.

Proof. As indicated, we will simultaneously provide an inductive proof of Theorem 4.1 (inducting on the number of characteristic pairs) and a recursive proof of Theorem 4.3.

Fixing a value of x, consider the transverse Milnor fiber of the truncation, defined by $z^{mb} - x^a y^{nb} = \epsilon$, and its image under the projection π. There is total ramification above the (nb)th roots of $(-\epsilon/x^a)$. We can retract a neighborhood of 0 onto the union L_x of line segments from 0 to these points, in such a way that there is a compatible retraction of the Milnor fiber onto $\pi^{-1}L_x$, which is the complete bigraph on the nb points

$$\left(\sqrt[nb]{-\epsilon/x^a}, 0\right) \tag{4.3}$$

and the mb points

$$\left(0, \sqrt[mb]{\epsilon}\right). \tag{4.4}$$

As ϵ goes around a circle, each set of points is cyclically permuted. Since m and n are relatively prime, the mnb^2 edges of the graph fall into b orbits of length mnb. This confirms formula (5). If ϵ is fixed but x varies, the retractions of the Milnor fibers fit together continuously. The points (4.4) are fixed but the points (4.3) fall into (n, a) orbits each of size $nb/(n, a)$. For the edges of the graph the orbits likewise have this size, and there are $(n, a)mb$ such orbits. This confirms formula (7). Formula (3) follows by taking the degree, and formula (1) is trivial.

To verify the recursive formulas and to handle the inductive step in the proof of Theorem 4.1, suppose we are given a curve with series (3.1) and prototype (4.1). We first replace

$$\frac{z - \sum_{(\lambda,\mu)\in\mathbf{Z}\times\mathbf{Z}} c_{\lambda\mu}x^{\lambda}y^{\mu}}{c_{\lambda_1\mu_1}}.$$

by z. In the new coordinate system, the surface is defined by the vanishing of

$$f = \prod^{d}\left(z - \left[x^{\frac{a}{mb}}y^{\frac{n}{m}} + \sum_{(\lambda,\mu)>\left(\frac{a}{mb},\frac{n}{m}\right)} c_{\lambda\mu}x^{\lambda}y^{\mu}\right]\right), \tag{4.5}$$

(where for simplicity the coefficients have been renamed). The truncation is defined by the vanishing of

$$f_1 = \prod^{mb}(z - x^{\frac{a}{mb}}y^{\frac{n}{m}}) = z^{mb} - x^a y^{nb}. \tag{4.6}$$

Dividing (4.5) by a power of (4.6), we claim that

$$\frac{f}{f_1^{d/(mb)}} = \prod^{d}\left(1 - \frac{\sum_{(\lambda,\mu)>\left(\frac{a}{mb},\frac{n}{m}\right)} c_{\lambda\mu}x^{\lambda}y^{\mu}}{z - x^{\frac{a}{mb}}y^{\frac{n}{m}}}\right). \tag{4.7}$$

To justify this we argue as follows. Let (x, y) be a point close to the origin but not lying on the x- or y-axis. Let d_x be the common denominator of all x-exponents appearing in (4.5); similarly let d_y be the common denominator of all y-exponents. Fix a value $\bar{x} = x^{1/d_x}$ and similarly a value $\bar{y} = y^{1/d_y}$. Then there is a map from the product of two groups of roots of unity:

$$\mu_{d_x} \times \mu_{d_y} \to \text{points on the surface projecting to } (x, y)$$

whose last coordinate is given by

$$(\alpha, \beta) \mapsto (\alpha\bar{x})^{ad_x/(mb)}(\beta\bar{y})^{nd_y/m} + \sum_{(\lambda,\mu)>\left(\frac{a}{mb},\frac{n}{m}\right)} c_{\lambda\mu}(\alpha\bar{x})^{\lambda d_x}(\beta\bar{y})^{\mu d_y}. \tag{4.8}$$

(Note that all exponents are integers.) This map factors through the quotient $(\mu_{d_x} \times \mu_{d_y})/K$, where K consists of all elements determining the same point as $(1, 1)$. This quotient group has order d. Similarly there is a map

$$(\alpha, \beta) \mapsto (\alpha\bar{x})^{ad_x/(mb)}(\beta\bar{y})^{nd_y/m}$$

onto the points of the truncation surface, with kernel K_1 and with quotient group $(\mu_{d_x} \times \mu_{d_y})/K_1$ of order mb. A fiber of the homomorphism

$$(\mu_{d_x} \times \mu_{d_y})/K \to (\mu_{d_x} \times \mu_{d_y})/K_1$$

(i.e, a coset of the kernel K_1/K) corresponds to all distinct series in (4.8) compatible with a specified first term. Since these fibers all have the same cardinality $d/(mb)$, the calculation leading to (4.7) is justified.

Now we suppose that x moves on the circle of radius ρ. All of our constructions will be done equivariantly, i.e., by doing the same thing simultaneously to all transverse slices. First, in each transverse slice, we perform the series of blowups dictated by $\mu_1 = n/m$ and the Euclidean algorithm. Doing this for the truncation, we obtain (for each transverse slice) a total transform consisting of certain exceptional divisors occurring with certain multiplicities, together with a strict transform meeting just one exceptional divisor, which we call the *rupture component*. Along this chain the function z^m/y^n has no indeterminacy, and in fact except along the rupture component its value is either 0 or ∞.

If all of the exponents μ appearing in (4.7) were strictly greater than n/m, then we could argue, as in the earlier proof of Theorem 2.3, that the value of (4.7) along a non-rupture exceptional divisor is 1. But since there may be a repetition of exponents (even in the characteristic pairs) we need to be more careful. If $z^m/y^n = 0$, then

$$\frac{f}{f_1^{d/(mb)}} = \prod^d \left(1 + \sum_{(\lambda,\mu) > \left(\frac{a}{mb}, \frac{n}{m}\right)} c_{\lambda\mu} x^{\lambda - a/(mb)} y^{\mu - n/m} \right),$$

and since y vanishes everywhere along the exceptional divisors we find that

$$\frac{f}{f_1^{d/(mb)}} = \prod^d \left(1 + \sum_{\lambda > \frac{a}{mb}} c_{\lambda\mu_1} x^{\lambda - a/(mb)} \right).$$

Note that by choosing x sufficiently close to 0 we can guarantee that this value has positive real part. If $z^m/y^n = \infty$, i.e. $y^n/z^m = 0$, then a similar calculation shows that the value of (4.7) is 1.

To work in a chart containing the rupture component, we use substitutions dictated by the matrix

$$\begin{bmatrix} m & n \\ r & s \end{bmatrix},$$

where r and s are the smallest positive integers for which the determinant is 1, namely

$$\begin{aligned} y &= u^m v^r \\ z &= u^n v^s. \end{aligned}$$

We find that in this chart the total transform of the truncation is defined by the vanishing of

$$f_1 = u^{mnb} v^{rnb} (v^b - x^a),$$

and its strict transform is defined by the vanishing of the last factor. Note that it meets the v-axis in b points, and that as x travels around a small circle these points trace out the torus knot $v^b = x^a$. The total transform of the given surface is defined by the vanishing of

$$f = \prod^d \left(u^n v^s - \left[x^{\frac{a}{mb}} u^n v^{rn/m} + \sum_{(\lambda,\mu) > \left(\frac{a}{mb}, \frac{n}{m}\right)} c_{\lambda\mu} x^{\lambda} u^{m\mu} v^{r\mu} \right] \right)$$

which may be rewritten as

(4.9)

$$\begin{aligned} f = {} & u^{nd} v^{rnd/m} x^{ad/(mb)} \\ & \prod^d \left(\left(\frac{v}{x^{a/b}}\right)^{1/m} - \left[1 + \sum_{(\lambda,\mu) > \left(\frac{a}{mb}, \frac{n}{m}\right)} c_{\lambda\mu} x^{\lambda - a/(mb)} u^{m\mu - n} v^{r(m\mu - n)/m} \right] \right). \end{aligned}$$

Again if all the values of μ appearing in (4.9) are strictly greater than n/m, then we can assert that the strict transform meets the v-axis in the same set of b points, but if there is a repetition of exponents then we find that the strict transform meets this axis at all points at which (for some choice of conjugate)

$$v^b = \left(1 + \sum_{\lambda > \frac{a}{mb}} c_{\lambda\mu_1} x^{\lambda - a/(mb)}\right)^{mb} x^a. \tag{4.10}$$

We also note that

$$\frac{f}{f_1^{d/(mb)}} = \prod^d \left(1 - \frac{\sum_{(\lambda,\mu) > \left(\frac{a}{mb}, \frac{n}{m}\right)} c_{\lambda\mu} x^{\lambda - a/(mb)} u^{m\mu - n} v^{r(m\mu - n)/m}}{\left(\frac{v}{x^{a/b}}\right)^{1/m} - 1}\right),$$

and that its restriction to the rupture component is

$$\prod^d \left(1 - \frac{\sum_{\lambda > \frac{a}{mb}} c_{\lambda\mu_1} x^{\lambda - a/(mb)}}{\left(\frac{v}{x^{a/b}}\right)^{1/m} - 1}\right). \tag{4.11}$$

Introducing three new variables x', y', and w, let B denote the product of the circle $\|x'\| = \rho^{1/b}$ and the 4-ball $\|(y', w)\| \le \delta'$. Map this product to a neighborhood N of the torus knot as follows:

$$x = (x')^b$$
$$u = \frac{y'}{(w+1)^r \rho^{ar/(mb)}}$$
$$v = (w+1)^m (x')^a$$

(See Figure 5.) Note that the circle $(y', w) = (0, 0)$ is mapped onto the knot. We claim that if δ' is sufficiently small then the map is injective (regardless of the value of ρ). Indeed, suppose that (x_1', y_1', w_1) and (x_2', y_2', w_2) are two points whose images agree. Then

$$\left(\frac{w_2 + 1}{w_1 + 1}\right)^m = \left(\frac{x_1'}{x_2'}\right)^a,$$

where the quantity on the right is a bth root of 1. If w_1 and w_2 are sufficiently close to 0 then this root must be 1 itself. Since a and b are relatively prime, this implies that $x_1'/x_2' = 1$. Since the map $w \mapsto (w+1)^m$ is injective near 0, we see that $w_1 = w_2$ and then that $y_1' = y_2'$.

Thus N is a tubular neighborhood of the torus knot: its intersection with each transverse plane consists of b disjoint topological disks, each of which encloses one of the points where the torus knot meets the plane.

We can regard each transverse Milnor fiber as a subset of the surface obtained from the transverse plane $x =$ constant by the sequence of blowups. Let us assume that the choices of δ, x, and ϵ which determine the transverse Milnor fiber are made subsequent to the choice of δ'. We claim that we can make these choices so as to guarantee that the strict transform of the surface lies entirely within N. Indeed, we note that on the strict transform

$$w = \sum_{(\lambda,\mu) > \left(\frac{a}{mb}, \frac{n}{m}\right)} c_{\lambda\mu} x^{\lambda - a/(mb)} y^{\mu - n/m},$$

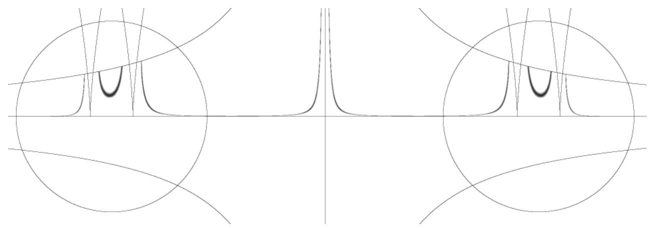

FIGURE 6. The transverse Milnor fiber is divided into two pieces by the boundary of N (indicated by two circles). The rupture component is horizontal, and another exceptional divisor is shown vertically. The strict transform enters from above.

where in each term at least one of the exponents is positive. Thus by choosing δ and $\|x\|$ sufficiently small we may force w arbitrarily close to 0. Now observe that

$$(y')^m = y\left(\frac{x'}{\rho^{1/b}}\right)^{-ar}$$

and that $\|x'/\rho^{1/b}\| = 1$. Thus we may also force $\|y'\|$ to be arbitrarily small. Note in particular that N will contain the points where the strict transform meets the v-axis (as determined by equation (4.10)); Figure 4 shows an example.

Looking at formula (4.11), we note that outside of N the value of $\left(\frac{v}{x^{a/b}}\right)^{1/m}$ along the rupture component is bounded away from 1, with the bound being independent of the choice of x; thus by choosing x sufficiently close to 0 we can guarantee that the value of (4.11) has positive real part. Finally by choosing ϵ sufficiently close to 0, we can guarantee that the Milnor fiber is transverse to the boundary of N and that its boundary lies entirely within N. Our transverse Milnor fiber is thus divided into two pieces. (See Figure 6.)

Consider first the piece of the Milnor fiber lying outside of N. By the approximation lemma 1.2, the monodromies of f and $f_1^{d/(mb)}$ are the same for this piece. The Milnor fiber has $d/(mb)$ connected components corresponding to all possible values of $\eta = \epsilon^{mb/d}$, and each one is a copy of the Milnor fiber for f_1. As ϵ goes around a circle, these copies are cyclically permuted. As ϵ goes around this circle $d/(mb)$ times, however, each η goes once around a circle. Thus the horizontal monodromy of this piece is $\mathbf{H}_1(t^{d/(mb)})$. But if ϵ is fixed and x varies, then each copy is individually acted upon by the vertical monodromy, so that the contribution from this piece is $(\mathbf{V}_1(t))^{d/(mb)}$.

Now consider the piece of the Milnor fiber lying inside N. Note that it has two sorts of boundary components: the components of the original link and those components created by its intersection with the boundary sphere of N. To analyze it, we look at its inverse image in B, which is contained in the b disjoint balls centered at the points $(x', y', w) = (x^{1/b}, 0, 0)$ (allowing all possible roots).

When pulled back to B, most of the d factors at the end of (4.9) become units. To see this, first observe that we can force the value in square brackets to be arbitrarily close to 1 by choosing sufficiently small radii δ' and ρ. To obtain a non-unit, we must therefore pick the "principal value" of $x^{1/b}$ for which it equals x' and then similarly pick the appropriate mth root of $v/(x')^a$ so that

$$\left(\frac{v}{(x')^a}\right)^{1/m} = w + 1;$$

note that these choices can be made uniformly throughout B. Thus $d/(mb)$ of the factors at the end of (4.9) become

$$w - \sum_{(\lambda,\mu)>\left(\frac{a}{mb},\frac{n}{m}\right)} c'_{\lambda\mu}(x')^{b\lambda-a/m+ar(m\mu-n)/m}(y')^{m\mu-n}$$

(where $c'_{\lambda\mu} = c_{\lambda\mu}\rho^{-ar(m\mu-n)/(mb)}$), whereas the remaining $d - d/(mb)$ factors become units. Each such unit takes its values in an arbitrarily small neighborhood of some $e-1$, where e is a nontrivial (mb)th root of unity. Thus by the approximation lemma 1.2, we may ignore all unit factors in f. Thus we may assume that the function defining this piece of the Milnor fiber is

$$(x')^{ads}(y')^{nd}\prod^{d/(mb)}\left(w - \sum_{(\lambda,\mu)>\left(\frac{a}{mb},\frac{n}{m}\right)} c'_{\lambda\mu}(x')^{b\lambda-a/m+ar(m\mu-n)/m}(y')^{m\mu-n}\right).$$

The map $(x',y',w)\mapsto(x',y',(x')^{asmb}(y')^{nmb}w)$ takes this piece to the transverse Milnor fiber of the quasi-ordinary surface with series

$$\sum_{(\lambda,\mu)>\left(\frac{a}{mb},\frac{n}{m}\right)} c'_{\lambda\mu}(x')^{b\lambda-a/m+ar(m\mu-n)/m+ambs}(y')^{m\mu-n+nmb}, \tag{4.12}$$

but it misses disks centered at the $d/(mb)$ points

$$(x',0,\epsilon^{d/(mb)}). \tag{4.13}$$

The horizontal monodromy permutes these disks. In (4.12) there are $e-1$ characteristic pairs, whereas our original series had e characteristic pairs. By the inductive hypothesis, the horizontal monodromy of this curve is the same as that of its prototype, which has series

$$\sum_{i=2}^{e}(x')^{b[\lambda_i-\lambda_1+mb\lambda_1+rm(\mu_i-\mu_1+mb\mu_1)\lambda_1]}(y')^{m(\mu_i-\mu_1+mb\mu_1)}.$$

(In calculating the first exponent we have used $ms = rn+1$.) By reindexing we obtain the series of the derived surface. (Note that all of the exponents on y' are positive; thus we are still in the hypothesized case.) Thus $d' = d/(mb)$, confirming formula (2) of the theorem. Since there are b copies of this situation (one for each bth root of x), the monodromy of this piece of the transverse Milnor fiber is

$$\left(\frac{\mathbf{H}'(t)}{t^{d'}-1}\right)^b.$$

Combining this with our conclusion about the monodromy of the first piece, we obtain formula (6). Then we obtain formula (4) by computing the degree of both sides of (6).

Turning to the vertical monodromy, we remark that it cyclically permutes the individual pieces of the Milnor fiber cut out by the b disjoint balls. Its bth power acts on each such piece by the vertical monodromy of the derived surface, in such a way that the disks of (4.13) are cyclically permuted. Thus the contribution to the vertical monodromy of our original surface is

$$\frac{\mathbf{V}'(T)}{(T-1)^{d'}}$$

where $T = t^b$. Combining this with our conclusion about the vertical monodromy of the first piece, we obtain formula (8). □

Here is another example. If we begin with the surface parametrized by

$$\zeta = x^{1/2}y^{3/2} + x^{1/2}y^{7/4} + x^{2/3}y^{11/6},$$

then its truncation and derived surface are parametrized by

$$\zeta_1 = x^{1/2}y^{3/2} \qquad \text{and} \qquad \zeta' = x^{17/4}y^{13/2} + x^{9/2}y^{20/3}.$$

Repeating the process, the new truncation and the second derived surface are parametrized by

$$\zeta_1' = x^{17/4}y^{13/2} \qquad \text{and} \qquad \zeta'' = x^{1438/3}y^{157/3}.$$

By repeated use of the first two formulas in Theorem 4.3, we find that the degree of the quasi-ordinary surface is

$$d = d_1 d_1' d'' = 2 \cdot 4 \cdot 3 = 24.$$

By formulas (3) and (4), the Euler characteristic of the transverse Milnor fiber is

$$\chi = d'(\chi_1 - b) + d''(\chi_1' - b') + b'\chi'' = 12(-1-1) + 3(-74-2) + 2(-311) = -874.$$

By formulas (5) and (6), the horizontal monodromy is

$$\begin{aligned}\mathbf{H}(t) &= \frac{\mathbf{H}_1(t^{d'})}{(t^{d'}-1)^b}\left[\frac{\mathbf{H}_1'(t^{d''})}{(t^{d''}-1)^{b'}}\right]^b [\mathbf{H}''(t)]^{bb'} \\ &= \frac{(t^{24}-1)(t^{36}-1)}{(t^{72}-1)(t^{12}-1)}\left[\frac{(t^{12}-1)(t^{78}-1)}{(t^{156}-1)^2(t^3-1)^2}\right]^1\left[\frac{(t^3-1)(t^{157}-1)}{t^{471}-1}\right]^2.\end{aligned} \tag{4.14}$$

By formulas (7) and (8), the vertical monodromy is

$$\begin{aligned}\mathbf{V}(t) &= \left[\frac{\mathbf{V}_1(t)}{t^b-1}\right]^{d'}\left[\frac{\mathbf{V}_1'(t^b)}{(t^{bb'}-1)}\right]^{d''} \cdot \mathbf{V}''(t^{bb'}) \\ &= \left[\frac{(t-1)^2}{(t^3-1)(t-1)}\right]^{12}\left[\frac{(t-1)^4}{(t^{26}-1)^3(t^2-1)}\right]^3 \cdot \frac{(t^2-1)^3}{(t^{314}-1)^2}.\end{aligned} \tag{4.15}$$

5. Quasi-ordinary surfaces for which $\mu_1 = 0$

Suppose that in (3.2) we have $\mu_i = 0$ for $1 \le i \le s < e$. Then the singular locus of S may contain a curve which does not lie in the x-y plane, namely the intersection of S with the plane $y = 0$. This curve projects to the x-axis, and if we restrict our attention to those points lying over a small circle we see an N-sheeted covering $C \to S^1$, where N is the least common denominator of $\{\lambda_i\}_{i=1}^s$. The transverse slice of S (as defined in section 3) will then be a curve with N singularities. For example, on the surface parametrized by $\zeta = x^{3/2} + x^2y^{3/2}$ the curve $z^2 = x^3$ is a component of the singular locus. A transverse slice is shown in Figure 7.

In this case, the correct definitions of the horizontal and vertical fibrations use Milnor fibers at the points of C. Such a Milnor fiber consists of those points within

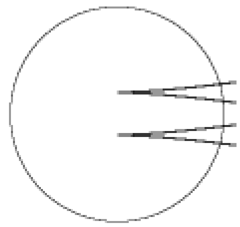

FIGURE 7. The real points of the transverse slice of the quasi-ordinary surface parametrized by $\zeta = x^{3/2} + x^2 y^{3/2}$. Here $N = 2$.

a transverse slice, within a sufficiently small neighborhood of the specified point of C, and satisfying $f = \epsilon$ (for sufficiently small ϵ). Each transverse slice will contain N such Milnor fibers, and they form the fibers of a fibration over $C \times S^1$ (the latter factor consisting of all ϵ on a small circle). One obtains the horizontal or vertical fibration by fixing (respectively) the point of C or the value of ϵ.

Lipman [12] (p. 65 ff.) shows that we can find a different quasi-ordinary surface S' with characteristic pairs $\{(\lambda_i', \mu_i') = (N\lambda_{i+s}, \mu_{i+s})\}$, $1 \leq i \leq e - s$, so that the horizontal and vertical fibrations of S (as just defined) are the same as those of S' (as defined in section 3). Thus the characteristic pairs $\{(\lambda_i, 0)\}_{i=1}^{s}$ are invisible in these monodromies, but they are precisely what is recovered by the topological zeta function of the two-dimensional singularity; see [14] and [13].

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