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Algebraic Methods in Geometry:
Commutative and Homological Algebra in Foliations and Singularities;
School and conference in honor of Xavier Gómez-Mont on the occasion of his 60th birthday;
CIMAT, Guanajuato, México, Aug. 22-Sept. 2, 2011

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Preface

These are the proceedings of the School and Conference "Algebraic Methods in Geometry: Commutative and Homological Algebra in Foliations and Singularities", held at the Centro de Investigación en Matemáticas (CIMAT), in Guanajuato, México, during August 22nd to September 2nd of 2011. This meeting was in celebration of the 60th Birthday Anniversary of Xavier Gómez-Mont, and it was mainly devoted to the research topics covered by Gómez-Mont.

Xavier Gómez-Mont is a Mexican mathematician who finished his Ph.D. at Princeton University in 1978 sponsored by the Universidad Nacional Autónoma de México (UNAM). At Princeton, he worked with R. C. Gunning and he also profited from conversations with Bill Thurston, conversations that left a deep track in the –those days– young Gómez-Mont. After finishing his Ph.D., he joined the Institute of Mathematics at the UNAM, and then, in 1987, he moved to CIMAT, where he has been working since that time. During all these years he has been a world leading mathematician and a pillar of mathematics in México and Latin America.

The early works of Xavier Gómez-Mont were on holomorphic foliations, making significant contributions to laying down the foundations of the theory. Among other contributions to the subject, he was one of the first to study holomorphic foliations using tools and methods from Algebraic Geometry. He also proved striking theorems for codimension one holomorphic foliations in the spirit of Ahlfors' finiteness theorem and Sullivan's non-wandering domains theorem. In the 1990s, Gómez-Mont started working on indices of vector fields, and his research led to a remarkable index for holomorphic vector fields on singular varieties, called the homological index. This relies heavily on homological algebra and a certain Koszul complex. Part of that line of research is the "Gobelin": A certain double complex which is so fine and sophisticated that makes you think of a master piece of tapestry, hence the name "Gobelin". In Gómez-Mont's research one always finds algebra and geometry intertwined with dynamics, and this beautifully leads also to his work on the "foliated geodesic flow" and statistical methods to studying holomorphic foliations.

Each of the articles contained in this volume is an interesting piece of work on its own, and together they give a glimpse of important areas of mathematics where Gómez-Mont has made significant contributions.

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THE HOMOLOGICAL INDEX AND THE DE RHAM COMPLEX ON SINGULAR VARIETIES

A.G. ALEKSANDROV

Dedicated to Xavier Gómez-Mont on the occasion of his 60th birthday

Abstract. We discuss several methods of computation of the homological index originated in a paper by X. Gómez-Mont for vector fields given on singular complex varieties. Our approach takes into account basic properties of holomorphic and regular meromorphic differential forms and is applicable in different settings depending on concrete types of varieties. Among other things, we describe how to compute the index in the case of Cohen-Macaulay curves, graded normal surfaces and complete intersections by elementary calculations. For quasihomogeneous complete intersections with isolated singularities, an explicit formula for the index is obtained; it is a direct consequence of earlier results of the author. Indeed, in this case the computation of the homological index is reduced to the use of Newton's binomial formula only.

Introduction

The classical concept of topological index for vector fields with isolated singularities given on 2-dimensional manifolds goes back to H. Poincaré (1887). This notion was generalized to higher-dimensional case by H. Hopf who proved that the total index of a vector field on a closed smooth orientable manifold does not depend on the field and it is equal to the Euler-Poincaré characteristic of the manifold. Since then, many authors studied the index as a topological invariant in different contexts and various settings. However, being purely topological, the original definition of index essentially depends on concrete presentations of vector fields, on the topological and analytical structure of a manifold, on the existence of suitable metrics, etc. That is why when studying varieties with singularities the classical approach does not work perfectly for evident reasons.

A new algebraic concept of the homological index for vector fields on reduced pure-dimensional complex analytic spaces appeared in a work by X. Gómez-Mont [16]; it is easy and well adapted for use in the theory of singular varieties. His main idea is to consider an important algebraic and analytic invariant, the alternating sum of dimensions of the homology groups of the truncated De Rham complex of holomorphic differential forms whose differential is defined by the contraction $\iota_{\mathcal{V}}$ of differential forms along a vector field $\mathcal{V}$ given on a singular complex space. In other terms, this invariant is the Euler-Poincaré characteristic of the contracted De Rham complex. Then it is proved that under certain finiteness assumptions the homological index is equal to the classical *local* topological index of $\mathcal{V}$ *up to a constant* depending on the germ of singular space but not on the vector field. In its turn, the problem of computation of Euler-Poincaré characteristic is reduced to the analysis of the homology or hypercohomology of the contracted De Rham complex.

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Key words and phrases. holomorphic differential forms; contracted De Rham complex; regular meromorphic forms; torsion and cotorsion; generating functions; graded complete intersections; Lebelt resolutions.

Originally the homological index was computed explicitly for vector fields with isolated singularities tangent to a hypersurface embedded in a complex manifold with the use of standard resolvents and properties of spectral sequences [15]. In a paper by the author [6] another method for the calculation of the homological index is described; the main idea of his approach is to compute the homological index with the use of *meromorphic* differential forms defined on the ambient manifold and having logarithmic poles along the given hypersurface. Above all an auxiliary invariant, called the *logarithmic* index, is introduced and studied.

The next example, the case of complete intersections with isolated singularities was considered in [23], [10] with the help of quite a difficult technique and constructions of complicated resolvent or resolutions with detail analysis of spectral sequences, the use of computer algebra systems, and so on.

In the present paper we discuss several methods of computation of the homological index for some types of singular varieties. The key point of our approach lies in the fact that the homology of the contracted De Rham complex can be computed with the use of *meromorphic* differential forms. Among other things we show that in quite a general context the homological index can be naturally described in terms of the contracted complex of meromorphic differential forms or some subcomplexes. For example, in the case of *hypersurfaces* the complex $(\omega^\bullet_X, \iota_\mathcal{V})$ of regular meromorphic differential forms is closely related via the residue map with the contracted complex of *logarithmic* differential forms. In its turn, the latter complex is simply linked to the contracted De Rham complex $(\Omega^\bullet_X, \iota_\mathcal{V})$. Moreover, the Euler-Poincaré characteristics of these three complexes differ by constants which depend on singularities of the space and the vector field (see [6]). More generally, in the case of *normal* singularities it is useful to analyze the complex $(\omega^\bullet_X, \iota_\mathcal{V})$; in the case of *non-normal* singularities one can consider also the complex $(\Delta^\bullet_X, \iota_\mathcal{V})$ of *extendable* (to the normalization) meromorphic differential forms, and so on.

The paper is organized as follows. In the first sections we discuss some basic notions and definitions. Almost all of them are well-known in a more general setting; they are often exploited in many areas of singularity theory, analytic geometry, residue theory, etc. Our aim here is only to unify them and to consider related applications in some special situations. Then some simple methods of computation of the homological index are discussed; they are applied in different situations depending on concrete types of singular varieties. Thus, we show subsequently how to compute the index for Cohen-Macaulay curves, graded normal surfaces and complete intersections of arbitrary dimension. As a curious example, in the case of quasihomogeneous complete intersections with isolated singularities an explicit formula for the homological index is obtained; it is a direct consequence of earlier results by the author [1], [2]. In fact, in such a case the computation of the homological index is readily reduced to the use of Newton's binomial formula only.

I would like to thank all the organizers of Xavierfest at CIMAT Guanajuato for providing me with the opportunity of participating in this exciting meeting. I am also very indebted to Xavier Gómez-Mont for many stimulating conversations, discussions and suggestions concerning the subject of the paper.

1. The contracted De Rham complex

Let X be a complex (or real analytic) space or algebraic variety. Then the sheaves Ω^p_X, $p \geq 0$, of holomorphic (or analytic) differential p-forms on X are defined as follows.

Let $x \in X$ be a closed point. Choose one representative of the germ (X, x) embedded in an open neighborhood U of the origin in $\mathbb{C}^m$. Let $\mathcal{O}_U$ be the sheaf of analytic functions on U, and $\mathcal{I}$ a coherent sheaf of ideals with the zero-set X. Then $(X, \mathcal{O}_X)$ is a closed analytic subspace of U so that $\mathcal{O}_X = \mathcal{O}_U/\mathcal{I}|_X$. Assume that the ideal $\mathcal{I}$ is locally generated by a sequence of functions

$f_1, \ldots, f_k$ in $\mathcal{O}_U$, and set

$$\Omega^p_{X,x} = \Omega^p_U / \Big(\sum_{j=1}^{k} f_j \cdot \Omega^p_U + df_j \wedge \Omega^{p-1}_U\Big)\Big|_X.$$

By analogy with non-singular case elements of $\Omega^p_{X,x}$ are usually called germs of (regular) holomorphic forms of degree p on X. The differential d, acting on Ω^p_U, induces the differential on $\Omega^p_{X,x}$; it is denoted by the same symbol. As a result, the family of sheaves Ω^p_X, endowed with the differential d, forms an *increasing* complex $(\Omega^\bullet_X, d)$.

Remark. In the introduction to his famous article [19] A. Grothendieck wrote: "... we can consider the complex of sheaves $\Omega^\bullet_{X/k}$ of regular differentials on X, the differential operator being of course the exterior differentiation." Although this complex was firstly considered and studied already by Poincaré, he called it the *De Rham complex,* and its hypercohomology – the De Rham cohomology of X.

However, along with the structure of a complex, given by the De Rham differentiation, one can endow the family of sheaves of regular differential forms with a structure of a complex in other ways. For example, similarly to the classical theory of differentiation and integration one can associate with the exterior differentiation an important class of *inverse* operators as follows.

Let $\mathrm{Der}(X) = \mathrm{Hom}_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)$ be the sheaf of germs of holomorphic vector fields on X. Given an element $\mathcal{V} \in \mathrm{Der}(X) \cong \mathrm{Hom}_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)$, a canonical action of the interior multiplication (contraction) $\iota_\mathcal{V}$ along $\mathcal{V}$ on $\Omega^\bullet_X$ is well-defined. Since $\iota^2_\mathcal{V} = 0$, one obtains a *decreasing* complex $(\Omega^\bullet_X, \iota_\mathcal{V})$.

Remark. Apparently, J. Carrell and D. Liberman [13] investigated the complex $(\Omega^\bullet_X, \iota_\mathcal{V})$ for the first time; they proved in the cited work, that the Hodge numbers $h^{pq}(X)$ of a compact Kähler *manifold* X vanish as soon as the absolute value of the difference $|p-q|$ is greater than the dimension of the zero set of $\mathcal{V}$.

More generally, if X is a complex *manifold*, the sheaves of regular holomorphic forms Ω^p_X are locally *free*. In such a case, the complex $(\Omega^\bullet_X, \iota_\mathcal{V})$ is locally isomorphic to the classical Koszul complex. However, if X is a singular complex variety then the corresponding theory is considerably more difficult; the above complex belong to the class of the *generalized* Koszul complexes.

Following [6], in order to avoid an ambiguous terminology we call the complex $(\Omega^\bullet_X, \iota_\mathcal{V})$ the *contracted* De Rham complex of X.

2. Meromorphic differential forms

Let $Z \subset X$ be a closed subset, $j\colon X \setminus Z \hookrightarrow X$ be the natural inclusion, and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then there exists a standard exact sequence of $\mathcal{O}_X$-modules

$$0 \longrightarrow \mathcal{H}^0_Z(\mathcal{F}) \longrightarrow \mathcal{F} \longrightarrow j_*j^*\mathcal{F} \longrightarrow \mathcal{H}^1_Z(\mathcal{F}) \longrightarrow 0, \tag{1}$$

where $\mathcal{H}^*_Z(\bullet)$ is the local cohomology functor with supports in the closed subset $Z \subset X$, while j_* and j^* are functors of direct and inverse image, respectively, so that $j_*j^*\mathcal{F} \cong R^0j_*(\mathcal{F}|_{X\setminus Z})$. One says that $\mathcal{F}$ has support in Z, that is, $\mathrm{Supp}\,(\mathcal{F}) \subseteq Z$, if $\mathcal{H}^0_Z(\mathcal{F}) \longrightarrow \mathcal{F}$ is an isomorphism; $\mathcal{F}$ is said to have no Z-torsion, if $\mathcal{H}^0_Z(\mathcal{F}) = 0$. Usually, the latter local cohomology group is called Z-torsion of $\mathcal{F}$; it is denoted by $\mathrm{Tors}\,(\mathcal{F})$. Similarly $\mathcal{H}^1_Z(\mathcal{F})$ is called Z-cotorsion. For example, $j_*j^*\mathcal{F}$ itself has no Z-torsion. More generally, if $\mathcal{J} \subset \mathcal{O}_X$ is a coherent sheaf of ideals with the zero-locus $Z \subset X$, then for all $i \geq 0$ one has

$$\mathcal{H}^i_Z(\mathcal{F}) = \varinjlim_{\nu} \mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{J}^\nu, \mathcal{F}).$$

Given a morphism of quasi-coherent sheaves $\varrho\colon \mathcal{F} \to \mathcal{G}$, a family of natural morphisms

$$\mathcal{H}^i_Z(\varrho)\colon \mathcal{H}^i_Z(\mathcal{F}) \to \mathcal{H}^i_Z(\mathcal{G}),$$

$i \geq 0$, is well-defined. Suppose also that there is an *extension* $j_*(\varrho)$ of $\varrho|_{X\setminus Z}$ to X such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{H}^0_Z(\mathcal{F}) & \longrightarrow & \mathcal{F} & \longrightarrow & j_*j^*\mathcal{F} & \longrightarrow & \mathcal{H}^1_Z(\mathcal{F}) & \longrightarrow & 0 \\ & & \downarrow \mathcal{H}^0_Z(\varrho) & & \downarrow \varrho & & \downarrow j_*(\varrho) & & \downarrow \mathcal{H}^1_Z(\varrho) & & \\ 0 & \longrightarrow & \mathcal{H}^0_Z(\mathcal{G}) & \longrightarrow & \mathcal{G} & \longrightarrow & j_*j^*\mathcal{G} & \longrightarrow & \mathcal{H}^1_Z(\mathcal{G}) & \longrightarrow & 0. \end{array}$$

is commutative. Then for any complex $\mathcal{L}^\bullet = (\mathcal{L}^\bullet, \partial)$ of sheaves on X such that

$$j_*(\partial)^2 = j_*(\partial^2) = 0$$

the above diagram induces an exact sequence of complexes of $\mathcal{O}_X$-modules:

$$0 \longrightarrow \mathcal{H}^0_Z(\mathcal{L}^\bullet) \longrightarrow \mathcal{L}^\bullet \longrightarrow j_*j^*\mathcal{L}^\bullet \longrightarrow \mathcal{H}^1_Z(\mathcal{L}^\bullet) \longrightarrow 0, \tag{2}$$

since $\mathcal{H}^0_Z(\partial^2) = \mathcal{H}^0_Z(\partial)^2 = 0$, $\mathcal{H}^1_Z(\partial^2) = \mathcal{H}^1_Z(\partial)^2 = 0$.

Let us now take $\mathcal{F} = \Omega^p_X$, $p \geq 0$, and $Z = \operatorname{Sing} X$. Then $j_*j^*\Omega^p_X = j_*\Omega^p_X\big|_{X\setminus Z}$ consists of germs of meromorphic differential p-forms on X with singularities on Z.

Proposition 1. *With the same notations let $\mathcal{V}$ be an element of $\operatorname{Der}(X)$. Suppose that the restriction $j^*\mathcal{V} = \mathcal{V}|_{X\setminus Z}$ can be extended (in general, not necessarily uniquely) on X and denote the corresponding contraction, acting on $j_*j^*\Omega^\bullet_X$, by $j_*(\iota_\mathcal{V}) = \iota_{j_*\mathcal{V}}$. Then there exists an exact sequence of decreasing complexes*

$$0 \longrightarrow \mathcal{H}^0_Z(\Omega^\bullet_X) \longrightarrow \Omega^\bullet_X \longrightarrow j_*j^*\Omega^\bullet_X \longrightarrow \mathcal{H}^1_Z(\Omega^\bullet_X) \longrightarrow 0 \tag{3}$$

with differentials $\iota_\mathcal{V}$ and $\iota_{j_*\mathcal{V}}$.

For example, if X is a *normal* variety, then the desirable extension of $\mathcal{V}$ exists. Moreover, there exists also a similar exact sequence of *increasing* complexes with differentials induced by the usual De Rham differentiation d acting on $\Omega^\bullet_X$.

Remark. It should be underlined that, in general, for a coherent sheaf $\mathcal{F}$ the associated quasi-coherent sheaf $j_*j^*\mathcal{F}$ is *non-coherent*. However, there always exists a *coherent* subsheaf $\mathcal{G}_X \subset j_*j^*\mathcal{F}_X$ (as a rule, even not unique) such that $\mathcal{F}_U \cong \mathcal{G}_X|_U$.

The following useful assertion is an easy modification of the well-known statement due to M. Schlessinger (see [30]).

Lemma 1. *Let X be the germ of a complex space, $\mathcal{F}$ a coherent $\mathcal{O}_X$-module, and*

$$\mathcal{F}^\vee = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$$

its dual. Suppose that $Z \subset X$ is a closed subspace, and $\operatorname{depth}_Z X \geq 2$. *Then* $\operatorname{depth}_Z \mathcal{F}^\vee \geq 2$, *so that $\mathcal{H}^0_Z(\mathcal{F}^\vee) = \mathcal{H}^1_Z(\mathcal{F}^\vee) = 0$. Similarly, the condition* $\operatorname{depth}_Z X \geq 1$ *implies* $\operatorname{depth}_Z \mathcal{F}^\vee \geq 1$, *that is, $\mathcal{H}^0_Z(\mathcal{F}^\vee) = 0$.*

Proof. Taking the presentation $0 \to \mathcal{R} \to \mathcal{L} \to \mathcal{F} \to 0$, where $\mathcal{L}$ is a free $\mathcal{O}_X$-module, one gets the dual exact sequence

$$0 \to \mathcal{F}^\vee \to \mathcal{L}^\vee \to \mathcal{Q} \to 0$$

with $\mathcal{Q} \subseteq \mathcal{R}^\vee$. Since $\mathcal{L}$ is free and $\operatorname{depth}_Z \mathcal{O}_X \geq 2$, then $\mathcal{H}^0_Z(\mathcal{L}^\vee) = \mathcal{H}^1_Z(\mathcal{L}^\vee) = 0$. Hence, $\mathcal{H}^0_Z(\mathcal{F}^\vee) = 0$. Analogously, $\mathcal{H}^0_Z(\mathcal{R}^\vee) = 0$; this implies $\mathcal{H}^0_Z(\mathcal{Q}) = 0$. Finally, applying the functor of local cohomology $\mathcal{H}^*_Z(\bullet)$ to the dual exact sequence, one deduces $\mathcal{H}^1_Z(\mathcal{F}^\vee) = 0$. The remaining case $\operatorname{depth}_Z X \geq 1$ is analyzed in the same manner. QED.

Corollary 1. *Let X be a reduced complex space, $Z = \operatorname{Sing} X$. Then the $\mathcal{O}_X$-module* $\operatorname{Der}(X)$ *of vector fields on X has no Z-torsion.*

Proof. By definition, $\operatorname{Der}(X) \cong \operatorname{Hom}_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)$. Since X is reduced, then $\operatorname{codim}(Z, X) \geq 1$ and, hence, depth $_Z X \geq 1$. QED.

3. Regular meromorphic forms

Let M be a complex manifold, $\dim M = m$, and let $X \subset M$ be an analytical subset in a neighborhood of $x \in U \subset M$ defined by a sequence of functions $f_1, \dots, f_k \in \mathcal{O}_U$ as before. Throughout this section we assume that X is a *Cohen-Macaulay* space and $\dim X = n$. Then

$$\omega^n_X = \operatorname{Ext}^{m-n}_{\mathcal{O}_M}(\mathcal{O}_X, \Omega^m_M)$$

is called the Grothendieck *dualizing* module of X. It is well-known that the dualizing module has *no torsion*, $\operatorname{Tors} \omega^n_X = 0$.

Definition. The coherent sheaf of $\mathcal{O}_X$-modules ω^p_X, $p \geq 0$, is locally defined as the set of germs of meromorphic differential forms ω of degree p on X such that $\omega \wedge \eta \in \omega^n_X$ for any $\eta \in \Omega^{n-p}_X$. In other terms (see [8], [24], [22]),

$$\omega^p_X \cong \operatorname{Hom}_{\mathcal{O}_X}(\Omega^{n-p}_X, \omega^n_X) \cong \operatorname{Ext}^{m-n}_{\mathcal{O}_M}(\Omega^{n-p}_X, \Omega^m_M). \tag{4}$$

Elements of ω^p_X are called *regular meromorphic* differential forms of degree p on X. Some equivalent definitions of these sheaves in terms of Noether normalization and trace (see [24], [8]), in terms of residual currents (see [7]) and others are well-known.

Evidently, $\omega^p_X = 0$ for $p > n$ since $\Omega^{n-p}_X = 0$. Further, $\omega^p_X = 0$ for $p < 0$ because $\Omega^p_X \cong \operatorname{Tors} \Omega^p_X$ for $p > n$ and ω^n_X has no torsion. It is easy also to see that the De Rham differentiation d as well as the contraction $\iota_{\mathcal{V}}$ acting on $\Omega^\bullet_X$ are naturally extended to the family of modules ω^p_X, $0 \leq p \leq n$; they endow this family with structures of *increasing* complex $(\omega^\bullet_X, d)$ or *decreasing* complex $(\omega^\bullet_X, \iota^*_{\mathcal{V}})$, respectively. In particular, the contraction $\iota^*_{\mathcal{V}} \colon \omega^p_X \to \omega^{p-1}_X$ is naturally defined as a *dual* morphism to the action $\iota_{\mathcal{V}}$ on the complex $\Omega^\bullet_X$ in view of presentation (4).

Lemma 2 (see [8])**.** *Let $Z = \operatorname{Sing} X$, $j \colon X \setminus Z \longrightarrow X$. Then there exist natural inclusions $\omega^p_X \subseteq j_* j^* \Omega^p_X$ for all $p \geq 0$. Moreover, if $\operatorname{codim}(Z, X) \geq 1$, then ω^p_X has no Z-torsion; if $\operatorname{codim}(Z, X) \geq 2$, then ω^p_X has no cotorsion for all $p \geq 0$.*

Proposition 2. *If X is a normal space, that is, $c = \operatorname{codim}(\operatorname{Sing} X, X) \geq 2$, then $\omega^p_X \cong j_* j^* \Omega^p_X$ for $p \geq 0$. Thus, a meromorphic form is regular meromorphic on X if and only if it is holomorphic outside the singular subset Z of X. In particular, for $p \geq 0$ the sheaves $j_* j^* \Omega^p_X$ are coherent and exact sequence* (3) *of complexes transforms in the following way:*

$$0 \longrightarrow \mathcal{H}^0_Z(\Omega^\bullet_X) \longrightarrow \Omega^\bullet_X \longrightarrow \omega^\bullet_X \longrightarrow \mathcal{H}^1_Z(\Omega^\bullet_X) \longrightarrow 0.$$

Moreover, ω^p_X and the bidual sheaves $\Omega^{p\ \vee\vee}_X = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{H}om_{\mathcal{O}_X}(\Omega^p_X, \mathcal{O}_X), \mathcal{O}_X)$ of Zariski differential p-forms are isomorphic for all $0 \leq p < c$, respectively.

Remark. The sheaf ω^0_X contains all the germs of *weakly holomorphic* functions on X, or locally bounded meromorphic functions on X (see [8]); in other terms, it contains meromorphic germs whose preimages are holomorphic on the normalization of X [29]. Furthermore, the sheaves ω^{n-1}_X and $\operatorname{Der}(X)$ are naturally isomorphic.

Evidently, if $X \subset M$, then for any vector field $\mathcal{V} \in \operatorname{Der}(X)$ there exists a holomorphic vector field V given on the ambient manifold M such that $V|_X = \mathcal{V}$. For brevity, we often say that $\mathcal{V}$ has *isolated* singularities when its representative V does. In particular, this implies that such $\mathcal{V}$ has isolated singularities on X.

Proposition 3. *Let $\mathcal{V} \in \operatorname{Der}(X)$ be a vector field with isolated singularities. Then the $\iota_{\mathcal{V}}$-homology groups of the complex $\omega_X^\bullet$ are finite-dimensional vector spaces.*

Proof. Following [16], let us assume that $M = \mathbb{C}^m$ and the distinguished point

$$x_0 = 0 \in X \subset M$$

is an isolated singularity of the field $\mathcal{V}$, so that $\mathcal{V}(x_0) = 0$. Then $\mathcal{V}(x) \neq 0$ at any point x in a small enough punctured neighborhood of the point x_0. In a suitable neighborhood of x there exists a coordinate system $(t, z'_1, \ldots, z'_{m-1})$ such that $\mathcal{V} = \partial/\partial t$. Since $\mathcal{V}(\mathcal{I}) \subseteq (\mathcal{I})\mathcal{O}_{M,0}$, then $X \cong T \times X_0$, where a small disc in t is denoted by T and $X_0 \subseteq M_0 = \mathbb{C}^{m-1}$. Hence, for sheaves of holomorphic forms of degree $p \geq 0$ on X there are isomorphisms

$$\Omega^p_{X,0} \cong \big(\Omega^p_{X_0,0} \oplus \Omega^{p-1}_{X_0,0} \wedge dt\big) \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{C},0};$$

they are readily obtained from consideration of canonical projections $T \times X_0$ on the first and second cofactors and from definition of $\Omega^p_{X,0}$. A similar presentation exists for $j_*j^*\Omega^p_{X,0}$ as well as for $\omega^p_{X,0}$. Finally, on the p-th component of the complex $(\omega^\bullet_{X,0}, \iota^*_{\mathcal{V}})$ one has

$$\operatorname{Ker}(\iota^*_{\partial/\partial t}) \cong \operatorname{Im}(\iota^*_{\partial/\partial t}) \cong \big(\omega^p_{X_0,0} \oplus (0)\big) \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{C},0}$$

in virtue of duality. Thus, the corresponding homology groups vanish for all p. If $x_0 \in M \backslash X$, then one can easily get the same conclusion. This implies that $\iota^*_{\mathcal{V}}$-homology groups of the complex $\omega^\bullet_X$ may be non-trivial only at singular points of the field. The coherence of sheaves of regular meromorphic forms and their cohomology implies the statement. QED.

4. The homological index

Let $(\mathcal{L}^\bullet_X, \partial)$ be a (lower and upper) *bounded decreasing* complex of $\mathcal{O}_X$-modules with the differential of degree -1. Assume that all its homology groups $H_i(\mathcal{L}^\bullet_X, \partial)$ are modules of *finite* length, that is, $\ell(H_i(\mathcal{L}^\bullet_X, \partial)) < \infty$ for all $i \in \mathbb{Z}$. Then the Euler-Poincaré characteristic of the complex $(\mathcal{L}^\bullet_X, \partial)$ is defined as follows:

$$\chi(\mathcal{L}^\bullet_X, \partial) = \sum_{i \in \mathbb{Z}} (-1)^i \ell(H_i(\mathcal{L}^\bullet_X, \partial)).$$

Remark. If all the homology groups of $\mathcal{L}^\bullet$ are *finite-dimensional* vector spaces over the ground field $k = \mathbb{C}$, then

$$\chi(\mathcal{L}^\bullet_X, \partial) = \sum_{i \in \mathbb{Z}} (-1)^i \dim_k H_i(\mathcal{L}^\bullet_X, \partial).$$

Similarly, the Euler-Poincaré characteristic of the stalk of complex $\mathcal{L}^\bullet_{X,x}$ at any point $x \in X$ is well-defined.

Claim 1. *Under the finiteness assumption there exists the following relation:*

$$\chi(\Omega^\bullet_X, \iota_{\mathcal{V}}) = \chi(j_*j^*\Omega^\bullet_X, j_*\iota_{\mathcal{V}}) + \chi(\operatorname{Tors}\Omega^\bullet_X, \iota_{\mathcal{V}}) - \chi(\mathcal{H}^1_Z(\Omega^\bullet_X), \iota_{\mathcal{V}}),$$

Moreover, if X is normal then

$$\chi(\Omega^\bullet_X, \iota_{\mathcal{V}}) = \chi(\omega^\bullet_X, \iota^*_{\mathcal{V}}) + \chi(\operatorname{Tors}\Omega^\bullet_X, \iota_{\mathcal{V}}) - \chi(\mathcal{H}^1_Z(\Omega^\bullet_X), \iota_{\mathcal{V}}).$$

Proof. The exact sequence (3) yields two short exact sequences of complexes with differentials induced by the contractions along the vector field and its extension:

$$0 \longrightarrow \operatorname{Tors}\Omega^\bullet_X \longrightarrow \Omega^\bullet_X \longrightarrow \widetilde{\Omega}^\bullet_X \longrightarrow 0; \qquad 0 \longrightarrow \widetilde{\Omega}^\bullet_X \longrightarrow j_*j^*\Omega^\bullet_X \longrightarrow \mathcal{H}^1_Z(\Omega^\bullet_X) \longrightarrow 0,$$

where $\widetilde{\Omega}^\bullet_X = \Omega^\bullet_X/\operatorname{Tors}\Omega^\bullet_X$ is the quotient complex. Combining the associated long exact sequences of homologies, one obtains the first relation. Next, Proposition 2 implies

$$\chi(j_*j^*\Omega^\bullet_X) = \chi(\omega^\bullet_X);$$

it gives the second relation. QED.

Definition (see [16]). Let $\mathcal{V}$ be a holomorphic vector field given on the germ (X,x) of n-dimensional complex space and let $({}_{\sigma\le n}(\Omega^\bullet_{X,x}), \iota_{\mathcal{V}})$ be the *truncated* contracted De Rham complex of (X,x) :

$$0 \longrightarrow \Omega^n_{X,x} \xrightarrow{\iota_{\mathcal{V}}} \Omega^{n-1}_{X,x} \xrightarrow{\iota_{\mathcal{V}}} \Omega^{n-2}_{X,x} \to \cdots \to \Omega^1_{X,x} \xrightarrow{\iota_{\mathcal{V}}} \Omega^0_{X,x} \cong \mathcal{O}_{X,x} \longrightarrow 0.$$

The Euler-Poincaré characteristic of this complex is called the *homological* index of the vector field at x; it is denoted by $\operatorname{Ind}_{\mathrm{hom},x}(\mathcal{V})$. Thus,

$$\operatorname{Ind}_{\mathrm{hom},x}(\mathcal{V}) = \chi({}_{\sigma\le n}(\Omega^\bullet_{X,x}), \iota_{\mathcal{V}}).$$

Remark. In the standard terminology of homological algebra such kind of truncation is usually called the *"stupid"* or *"naïve"* truncation of level n.

Remark. The homological index was originally defined for vector fields on a *reduced pure-dimensional* complex analytic space with finite-dimensional homology groups $H_i(\Omega^\bullet_{X,x}, \iota_{\mathcal{V}})$; if such X and a vector field $\mathcal{V}$ both have isolated singularities at x then the homological index *coincides* with the *local topological (Poincaré-Hopf) index* up to a constant depending on the germ of singular space but not on the vector field (see [16]).

Definition. Let $\mathcal{V}$ be a holomorphic vector field on X and $\iota_{\mathcal{V}}$ the contraction along $\mathcal{V}$. If $(\mathcal{L}^\bullet_X, \partial)$ is equal to one of the described above complexes $(\Omega^\bullet_X, \iota_{\mathcal{V}})$, $(j_*j^*\Omega^\bullet_X, j_*\iota_{\mathcal{V}})$, $(\omega^\bullet_X, \iota^*_{\mathcal{V}})$, $\mathcal{H}^0_Z(\Omega^\bullet_X, \iota_{\mathcal{V}})$, $\mathcal{H}^1_Z(\Omega^\bullet_X, \iota_{\mathcal{V}})$, then we shall often call the corresponding Euler-Poincaré characteristic by holomorphic, meromorphic, regular meromorphic, torsion and cotorsion indices of the vector field $\mathcal{V}$ at $x \in X$, respectively.

It should be noted that the present paper is devoted in the main to the study of the homological index for varieties whose *cotorsion* index is equal to zero, that is, $\chi(\mathcal{H}^1_Z(\Omega^\bullet_X)) = 0$.

Proposition 4. *Assume that (X,x) is the germ of a complex space such that $\Omega^p_{X,x}$ are torsion modules for all $p > n$, that is, $\Omega^p_{X,x} = \operatorname{Tors}\Omega^p_{X,x}$. Then*

$$\operatorname{Ind}_{\mathrm{hom},x}(V) = \chi(\Omega^\bullet_{X,x}, \iota_{\mathcal{V}}) - \chi(\operatorname{Tors}({}_{\sigma>n}(\Omega^\bullet_{X,x})), \iota_{\mathcal{V}}).$$

Proof.It is enough to compare the Euler-Poincaré characteristics of the contracted $\iota_{\mathcal{V}}$-complexes in the exact sequence

$$0 \longrightarrow {}_{\sigma\le n}(\Omega^\bullet_X) \longrightarrow (\Omega^\bullet_X) \longrightarrow {}_{\sigma>n}(\Omega^\bullet_X) \longrightarrow 0,$$

and then to use an evident isomorphism ${}_{\sigma>n}(\Omega^\bullet_X) \cong \operatorname{Tors}({}_{\sigma>n}(\Omega^\bullet_X))$. QED.

Remark. It is well-known that reduced complete intersections with *non-isolated* singularities satisfy the conditions of Proposition 4 (see [17, Proposition 1.11]).

Corollary 2 (cf. [16], (1.4))**.** *Under the same assumptions suppose additionally that (X,x) is an n-dimensional isolated singularity of embedding dimension m. Then*

$$\operatorname{Ind}_{\mathrm{hom},x}(\mathcal{V}) = \chi({}_{\sigma\le n}(\Omega^\bullet_{X,x}), \iota_{\mathcal{V}}) = \chi(\Omega^\bullet_{X,x}, \iota_{\mathcal{V}}) - \sum_{p=n+1}^{m} (-1)^p \dim_k \operatorname{Tors}\Omega^p_{X,x}.$$

Proof. Since (X,x) is an isolated singularity, then torsion modules $\operatorname{Tors}\Omega^p_{X,x}$ are finite-dimensional vector spaces for all $p>n$. QED.

Corollary 3. *For an isolated complete intersection singularity of dimension $n\geq 1$ one has*

$$\operatorname{Ind}_{\hom,x}(\mathcal{V})=\chi(\omega^\bullet_{X,x},\iota^*_{\mathcal{V}})+(-1)^n\dim_k \operatorname{Tors}\Omega^n_{X,x}.$$

Proof. In fact, the cotorsion index is equal to zero, that is, $\chi(\mathcal{H}^1_{\{x\}}(\Omega^\bullet_X),\iota_{\mathcal{V}})=0$, because in our case there are only two non-trivial cotorsion modules of equal lengths (see [11]). It remains to use Claim 1 and Corollary 2. QED.

Claim 2. *Assume that (X,x) is a quasihomogeneous isolated complete intersection singularity of dimension $n\geq 1$. Then*

$$\dim_k \operatorname{Tors}\Omega^n_{X,x}=\sum_{p=n+1}^{m}(-1)^{p-n-1}\dim_k\Omega^p_{X,x},$$

where m is the embedding dimension of the singularity.

Proof. Let $\mathcal{V}$ be the Euler vector field. First observe that if $n=1$, then $\iota_{\mathcal{V}}(\operatorname{Tors}\Omega^1_{X,x})=0$ in $\mathcal{O}_{X,x}$. Indeed, if $\theta\in\operatorname{Tors}\Omega^1_{X,x}$ then there exists a non-zero divisor $u\in\mathcal{O}_{X,x}$ such that $u\theta\in\sum\mathcal{O}_{\mathbb{C}^m,o}\wedge df_j+(f_1,\dots,f_{m-1})\mathcal{O}_{\mathbb{C}^m,o}$. Then $\iota_{\mathcal{V}}(u\theta)=u\iota_{\mathcal{V}}(\theta)\in(f)\mathcal{O}_{\mathbb{C}^m,o}$, that is, one has $u\iota_{\mathcal{V}}(\theta)=0$ in $\mathcal{O}_{X,x}$ and, consequently, $\iota_{\mathcal{V}}(\theta)=0$ as was required. If $n\geq 2$ then $\operatorname{depth}_x\mathcal{O}_X\geq 2$ and the germ $X\setminus x$ is *connected* (see [18, Corollary 3.9]). In particular, the germ X is *irreducible* and its analytical algebra $\mathcal{O}_{X,x}$ has no zero-divisors, i.e. it is an integral domain. Further, $\operatorname{Tors}\Omega^p_X=0$ for all $0<p<n$, $\operatorname{Tors}\Omega^p_X$ are finite-dimensional for $p\geq n$ and $\Omega^p_X\cong\operatorname{Tors}\Omega^p_X$ for $p>n$ (see [17, Proposition 1.11]). Analogously to the above considerations for $n=1$ one gets $\iota_{\mathcal{V}}(\operatorname{Tors}\Omega^n_{X,x})=0$. Since (X,x) is a *contractible* singularity with respect to the Euler vector field then $\chi({}_{\sigma\geq n}(\operatorname{Tors}\Omega^\bullet_X),\iota_{\mathcal{V}})=0$. It remains to note that the Euler-Poincaré characteristic of a complex, whose terms are finite-dimensional vector spaces, is equal to the alternating sum of dimensions of all its terms (cf. [17, Lemma 5.6]). QED.

Remark. There exist also other cases when a slightly modified version of the Claim 2 is true (see, for example, [28, Satz 3]).

From the above observations it follows that the complex $(\omega^\bullet_{X,x},\iota^*_{\mathcal{V}})$ may have *non-trivial* homology groups H_i only for $i=0,1,\dots,n$, where n is the dimension of X. That is, in a certain sense the regular meromorphic index is defined in a more intrinsic manner than the homological one, since in the general case the former does not depend on the embedding dimension of the germ X, on the operation of truncation, etc.

5. Cohen-Macaulay curves

Let us consider in detail the case when $(X,\mathfrak{o})$ is the germ of a Cohen-Macaulay curve at the *distinguished* point $\mathfrak{o}\in X$ (for brevity, a singularity) and suppose that $Z=\operatorname{Sing}X=\{\mathfrak{o}\}$. Then $\omega^1_X\ncong j_*j^*\Omega^1_X$ and, moreover, the cotorsion of Ω^1_X, the right term $H^1_{\{\mathfrak{o}\}}(\Omega^1_X)$ of exact sequence (1), is *infinite-dimensional* over $\mathbb{C}$. Let $A=(A,\mathfrak{m})$ be the analytical algebra corresponding to the germ $(X,\mathfrak{o})$. Then A is a Cohen-Macaulay 1-dimensional local ring, $k\cong A/\mathfrak{m}$, $k=\mathbb{C}$ is the ground field and there exists an exact sequence

$$0\longrightarrow\operatorname{Tors}\Omega^1_A\longrightarrow\Omega^1_A\xrightarrow{c_A}\omega^1_A\longrightarrow{}^{\#}\omega^1_A\longrightarrow 0, \tag{5}$$

where the left and right terms are concentrated at the singularity so that they are *finite-dimensional* vector spaces.

In this case the *dualizing* (or, equivalently, *canonical*) module ω^1_A is contained properly in $\Omega^1_A \otimes_A F/A$, where F is the total ring of fractions of A, c_A is the canonical A-homomorphism of the fundamental class induced by the natural map $\Omega^1_A \to \Omega^1_A \otimes F/A$ (see [20], [25]). By definition, $\mathrm{Ker}(c_A) = \mathrm{Tors}\,\Omega^1_A \cong \mathcal{H}^0_{\mathfrak{m}}(\Omega^1_A)$ is the torsion submodule of Ω^1_A, and $\mathrm{Coker}(c_A) = {}^{\#}\omega^1_A$ is contained in $H^1_{\mathfrak{m}}(\Omega^1_A)$, it is called the ω_A-*cotorsion* of Ω^1_A.

Proposition 5. *Let A be a reduced 1-dimensional analytical algebra and $\mathcal{V} \in \mathrm{Der}_k(A)$ a k-differentiation of A. Then there exists the following commutative diagram with exact rows*

$$\begin{array}{ccccccccc} 0 \longrightarrow & \mathrm{Tors}\,\Omega^1_A & \longrightarrow & \Omega^1_A & \xrightarrow{c_A} & \omega^1_A & \longrightarrow {}^{\#}\omega^1_A & \longrightarrow 0 \\ & & & \downarrow\iota_{\mathcal{V}} & & \downarrow\iota^*_{\mathcal{V}} & \downarrow & \\ & 0 & \longrightarrow & \Omega^0_A \cong A & \longrightarrow & \omega^0_A & \longrightarrow {}^{\#}\omega^0_A & \longrightarrow 0, \end{array} \tag{6}$$

where $\omega^0_A = \mathrm{Hom}_A(\Omega^1_A, \omega^1_A)$, $A \hookrightarrow \omega^0_A$ is the canonical inclusion, ${}^{\#}\omega^0_A \cong \mathrm{Ext}^1_A({}^{\#}\omega^1_A, \omega^1_A)$, the middle left vertical arrow of diagram is the contraction $\iota_{\mathcal{V}}$, acting on Ω^1_A, while the middle term from right is induced by this contraction on $\omega^\bullet_A$ in a dual way.

Proof. Since A is, in fact, a Cohen-Macaulay local ring of Krull dimension 1, then

$$\mathrm{Hom}_A(k, \omega^1_A) = 0,$$

and, consequently, $\mathrm{Hom}_A(M, \omega^1_A) = 0$ for any A-module M of finite type with $\mathrm{Supp}\,M \subseteq \{\mathfrak{m}\}$. Applying the functor $\mathrm{Hom}_A(\bullet, \omega^1_A)$ to the following exact sequences subsequently

$$0 \to \mathrm{Tors}\,\Omega^1_A \to \Omega^1_A \to \widetilde{\Omega}^1_A \to 0, \qquad 0 \to \widetilde{\Omega}^1_A \xrightarrow{c_A} \omega^1_A \to {}^{\#}\omega^1_A \to 0, \tag{7}$$

one gets a natural isomorphism $\mathrm{Hom}_A(\widetilde{\Omega}^1_A, \omega^1_A) \cong \mathrm{Hom}_A(\Omega^1_A, \omega^1_A)$, and the first four terms of the long exact sequence

$$0 \to \mathrm{Hom}_A(\omega^1_A, \omega^1_A) \longrightarrow \mathrm{Hom}_A(\Omega^1_A, \omega^1_A) \longrightarrow \mathrm{Ext}^1_A({}^{\#}\omega^1_A, \omega^1_A) \longrightarrow \mathrm{Ext}^1_A(\omega^1_A, \omega^1_A)$$

since supports of $\mathrm{Tors}\,\Omega^1_A$ and ${}^{\#}\omega^1_A$ are contained in the singular point $\{\mathfrak{m}\}$. At last, from [20, 6.1 d)], it follows that $\mathrm{Ext}^1_A(\omega^1_A, \omega^1_A) = 0$, $\mathrm{Hom}_A(\omega^1_A, \omega^1_A) \cong A$, and there exists a canonical exact sequence

$$0 \longrightarrow A \longrightarrow \mathrm{Hom}_A(\Omega^1_A, \omega^1_A) \longrightarrow \mathrm{Ext}^1_A({}^{\#}\omega^1_A, \omega^1_A) \longrightarrow 0, \tag{8}$$

where the inclusion $A \longrightarrow \mathrm{Hom}_A(\Omega^1_A, \omega^1_A)$ is given by the correspondence $1_A \mapsto c_A$ (see details in [25, § 3]). In conclusion, the contraction $\iota_{\mathcal{V}} \colon \Omega^1_A \to \Omega^0_A \cong A$ induces the natural dual mapping $\iota^*_{\mathcal{V}} \colon \omega^1_A \to \omega^0_A$ in view of the definition of $\omega^\bullet_A$. It is not difficult to verify, that c_A is *compatible* with the contraction $\iota_{\mathcal{V}}$ and its extension $\iota^*_{\mathcal{V}}$, so that diagram (6), a combination of the latter exact sequence and (5), is, in fact, commutative. QED.

Corollary 4 ([22], (4.4))**.** *Under the same assumptions the lengths of ω-cotorsion modules are equal, that is, $\ell({}^{\#}\omega^1_A) = \ell({}^{\#}\omega^0_A)$, and the index of cotorsion complex is zero, $\chi({}^{\#}\omega^\bullet_A) = 0$.*

Remark. In a similar manner one can verify (see [22]) that ${}^{\#}\omega^1_A \cong \mathrm{Ext}^1_A({}^{\#}\omega^0_A, \omega^1_A)$. Really, applying $\mathrm{Hom}_A(\bullet, \omega^1_A)$ to the bottom row of the diagram (6), one gets

$$0 \to \mathrm{Hom}_A(\omega^0_A, \omega^1_A) \longrightarrow \omega^1_A \longrightarrow \mathrm{Ext}^1_A({}^{\#}\omega^0_A, \omega^1_A) \to 0.$$

By definition, the left module of this sequence is isomorphic to $\mathrm{Hom}_A(\mathrm{Hom}_A(\Omega^1_A, \omega^1_A), \omega^1_A)$, while the latter is isomorphic to $\mathrm{Hom}_A(\mathrm{Hom}_A(\widetilde{\Omega}^1_A, \omega^1_A), \omega^1_A) \cong \widetilde{\Omega}^1_A$. As a result, we obtain the second exact sequence (7).

Claim 3. *Let $\widetilde{A}$ be the normalization of a 1-dimensional singularity A in its total ring of fractions F, and let $\mathfrak{C}$ be the conductor of $\widetilde{A}$ in A. Then $\omega^1_{\widetilde{A}} \cong \Omega^1_{\widetilde{A}}$, $\omega^1_{\widetilde{A}} \cong \mathfrak{C} \cdot \omega^1_A$, and $\mathfrak{m} \cdot \omega^0_{\widetilde{A}} \subseteq \mathfrak{C} \cdot \omega^0_A$.*

Proof. The existence of both isomorphisms is well-known (see [25, (3.2)]), the inclusion is evident. QED.

Proposition 6. *Let $(A, \mathfrak{m})$ be a reduced 1-dimensional singularity, and let $\mathcal{V} \in \mathrm{Der}_k(A)$ be a vector field. Then $H_1(\omega^\bullet_A, \iota^*_\mathcal{V}) = 0$.*

Proof. Any vector field $\mathcal{V}$ on the curve singularity A can be extended to its normalization $\widetilde{A}$ (see [14, Lemma 2.33]); this extension is denoted by $\widetilde{\mathcal{V}} \in \mathrm{Der}_k(\widetilde{A})$. Since $\widetilde{\mathcal{V}}(\mathfrak{C}) \subseteq \mathfrak{C}$, then $\widetilde{\mathcal{V}}(\mathfrak{m}) \subseteq \mathfrak{m}$ (see [14]), and, consequently, $\mathrm{Ker}(\iota_{\widetilde{\mathcal{V}}}) = 0$. On the other hand, it is well-known that $\omega^1_{\widetilde{A}} \cong \omega^1_A$ and, making use of basic properties of Noether normalization and the definition of ω^1_A, we deduce that $\iota_{\widetilde{\mathcal{V}}}(\omega^1_{\widetilde{A}}) = \iota_{\widetilde{\mathcal{V}}}(\mathfrak{C} \cdot \omega^1_A) = \mathfrak{C} \cdot \iota^*_\mathcal{V}(\omega^1_A)$. Next, by definition, the conductor $\mathfrak{C}$ is the *maximal* element of the set of ideals of A which are also ideals of the *principal* ideal ring $\widetilde{A}$. It is not difficult to verify, that $\mathfrak{C} = (\theta)\widetilde{A}$, where $\theta \in \mathfrak{m}_A$ is a *non-zero divisor* (see [25, 3.1. b)]). Hence, $\mathrm{Ker}(\iota_{\widetilde{A}}) = \mathrm{Ker}(\iota^*_\mathcal{V})$; this completes the proof. QED.

Claim 4. *Under the assumptions of Proposition 6 suppose additionally that $\mathcal{V}$ has an isolated singularity on the germ $(X, \mathfrak{o})$. Then*

$$\dim_k H_0(\omega^\bullet_A, \iota^*_\mathcal{V}) = \dim_k H_0(\Omega^\bullet_A, \iota_\mathcal{V}) = \dim_k A/J_\mathfrak{o}\mathcal{V},$$

where $J_\mathfrak{o}\mathcal{V}$ is the ideal of A, generated by the coefficients of the vector field V in a suitable coordinate representation. In particular, we have

$$\chi(\omega^\bullet_A, \iota^*_\mathcal{V}) = \dim_k A/J_\mathfrak{o}\mathcal{V}.$$

Proof. The diagram (6) yields the following exact sequence

$$0 \to \mathrm{Ker}({}^\#\omega^1_A \to {}^\#\omega^0_A) \longrightarrow \mathrm{Coker}(\iota_\mathcal{V}) \longrightarrow \mathrm{Coker}(\iota^*_\mathcal{V}) \to \mathrm{Coker}({}^\#\omega^1_A \to {}^\#\omega^0_A) \to 0.$$

The difference of lengths of the left and right modules of this sequence *does not depend* on the vector field $\mathcal{V}$; it is equal to the difference of lengths of ${}^\#\omega^1_A$ and ${}^\#\omega^0_A$, which is zero (see [22, (4.4)]). Consequently, the lengths of the two middle modules are equal. Since both modules are concentrated at the singular point, they are vector spaces of the same dimension. It remains to note that $\mathrm{Coker}(\iota^*_\mathcal{V}) \cong H_0(\omega^\bullet_A, \iota^*_\mathcal{V})$, while $\mathrm{Coker}(\iota_\mathcal{V}) \cong H_0(\Omega^\bullet_A, \iota_\mathcal{V}) \cong A/J_\mathfrak{o}\mathcal{V}$. This completes the proof. QED.

Corollary 5. *Under the same assumptions, one has*

$$\chi(\Omega^\bullet_A) = \chi(\omega^\bullet_A) = \dim_k A/J_\mathfrak{o}\mathcal{V}, \quad \chi(_{\sigma\le 1}(\Omega^\bullet_A)) = \dim_k A/J_\mathfrak{o}\mathcal{V} - \dim_k(\mathrm{Tors}\,\Omega^1_A).$$

Proof. Since $H_1(\omega^\bullet_A, \iota^*_\mathcal{V}) = 0$, the diagram (6) implies $\mathrm{Ker}(\iota_\mathcal{V} \colon \widetilde{\Omega}^1_A \to A) = 0$. Consequently, $\chi(_{\sigma\le 1}(\Omega^\bullet_A)) = \chi(\omega^\bullet_A) - \dim_k(\mathrm{Tors}\,\Omega^1_A) = \chi(\Omega^\bullet_A) - \dim_k(\mathrm{Tors}\,\Omega^1_A)$. It remains to use Claim 4. QED.

Thus, the computation of homological index for reduced curves is reduced to the computation of the length of torsion module and dimension of the quotient algebra $A/J_\mathfrak{o}\mathcal{V}$.

Proposition 7. *Let A be the dual analytical algebra of the germ of a reduced Gorenstein 1-dimensional singularity and let $\mathcal{V} \in \mathrm{Der}_k(A)$ be a vector field with an isolated singularity. Then*

$$\mathrm{Ind}_{\mathrm{hom},\mathfrak{o}}(\mathcal{V}) = \dim_k A/J_\mathfrak{o}\mathcal{V} - \tau(A),$$

where $\tau(A)$ is the Tjurina number of the singularity A.

Proof. The local duality [18] implies an equality

$$\dim_k(\mathrm{Tors}\,\Omega^1_A) = \dim_k \mathrm{Ext}^1_A(\Omega^1_A, A).$$

Since A is *reduced* then the latter dimension is equal to $\dim_k T^1(A)$, the Tjurina number of the singularity A. QED.

Remark. In the general case the dimension of the torsion module can be computed in terms of Noether and Dedekind differentes with the use of a formula from [9]; many papers are devoted to the computation of this invariant for various types of curves and higher-dimensional singularities.

Remark. In fact, $A \subseteq \omega_A^0$, but, in general, $A \neq \omega_A^0$. To be more precise, let $\pi\colon \widetilde{X} \to X$ be the *normalization*. Then the sheaf ω_X^0 contains the direct image $\pi_*(\mathcal{O}_{\widetilde{X}})$, that is, all the germs of the *weakly holomorphic* functions on X, or, equivalently, the locally bounded meromorphic functions on X (see [8]). In other terms, it contains those meromorphic germs whose preimages are holomorphic on the normalization (cf. [29]).

6. Quasihomogeneous curves

First recall that if A is a 1-dimensional Gorenstein analytical k-algebra, then $\omega_A^1 \cong A(\eta)$, where $\eta \in \omega_A^1$ is a free generator of the dualizing module. Hence, the exact sequence (8) yields the following inclusion

$$A \longrightarrow \mathrm{Hom}_A(\Omega_A^1, \omega_A^1) \cong \mathrm{Der}_k(A).$$

The image of A does not depend on the generator η; it is denoted by D_A and its elements are called *trivial derivations* (or, equivalently, trivial differentiations) of A over k. Obviously, D_A is a *free* A-module of rank 1.

In the *compete intersection* case the module of trivial differentiations D_A has a canonical generator, the so-called *Hamiltonian* vector field. In this case the defining ideal $\mathcal{I}$ of the singularity is generated by a *regular* sequence of functions $f_1, \dots, f_{m-1} \in \mathcal{O}_U$ (in the notations of Section 1). Next, let Δ be the determinant of the $m \times m$-matrix arising from adjoining to the Jacobian matrix $\|\partial f_j/\partial z_i\|$ the extra row $(\partial/\partial z_1, \dots, \partial/\partial z_m)$, that is,

$$\Delta = \det \begin{pmatrix} \partial/\partial z_1 & \dots & \partial/\partial z_m \\ \partial f_1/\partial z_1 & \dots & \partial f_1/\partial z_m \\ \vdots & \vdots & \vdots \\ \partial f_{m-1}/\partial z_1 & \dots & \partial f_{m-1}/\partial z_m \end{pmatrix}.$$

Then the Hamiltonian vector field H is the cofactor expansion of the determinant Δ along the first row, so that $H(f_j) = 0$ for all $j = 1, \dots, m-1$.

By definition, a commutative ring A is called $\mathbb{Z}$-graded if it decomposes into a direct sum $A = \oplus_{\nu\in\mathbb{Z}} A_\nu$ of abelian groups A_ν such that $A_\nu A_\lambda \subseteq A_{\nu+\lambda}$ for all $\nu, \lambda \in \mathbb{Z}$. The elements of the group A_ν are said to be homogeneous of degree ν. In a similar way one can define graded modules, algebras, etc.

Now assume that for every $j = 1, \dots, m-1$ the defining function f_j of the singularity is *quasihomogeneous* of degree d_j with respect to the weights w_i, $i = 1, \dots, m$. In other terms, the type of homogeneity of the 1-dimensional complete intersection singularity is equal to $(d_1, \dots, d_{m-1}; w_1, \dots, w_m) \in \mathbb{Z}^{m-1} \times \mathbb{Z}^m$ (see [2]). Then the local analytical algebra A, as well as A-modules $\mathrm{Der}_k(A)$, Ω_A^p, ω_A^p, $p \geq 0$, and homology groups of the corresponding contracted complexes are endowed with a natural grading. Thus, the Poincaré *series* or *polynomials* of graded modules are well-defined. Moreover, in this case $\mathrm{Der}_k(A)/D_A$ is a *cyclic* module; it is generated by the Euler vector field (see [2, (6.1)], [25, Satz 2]).

It is clear that the weight of the differential form $\eta = dz_1 \wedge \dots \wedge dz_m / df_1 \wedge \dots \wedge df_{m-1}$ is equal to $-c$, where $c = \sum d_j - \sum w_i$. Hence, there exist natural isomorphisms $\omega_A^1 \cong A[-c]$,

$\omega_A^0 \cong \mathrm{Hom}_A(\Omega_A^1, \omega_A^1) \cong \mathrm{Der}_k(A)[-c]$, and the following identities for Poincaré series:

$$P(\omega_A^1; x) = x^{-c} P(A; x), \qquad P(\omega_A^0; x) = x^{-c} P(\mathrm{Der}_k(A); x).$$

Let us take $\mathcal{V} \in \mathrm{Der}_k(A)_v$ of weight $v \in \mathbb{Z}$. Then the middle column of the diagram (6) gives us an exact sequence:

$$0 \longrightarrow \iota_{\mathcal{V}}^*(\omega_A^1) \longrightarrow \omega_A^0 \longrightarrow H_0(\omega_A^\bullet) \longrightarrow 0.$$

By Proposition 6 one has $H_1(\omega_A^\bullet, \iota_{\mathcal{V}}^*) = 0$, that is, $\mathrm{Ker}(\omega_A^1 \xrightarrow{\iota_{\mathcal{V}}^*} \omega_A^0) = 0$, and, consequently,

$$P(H_0(\omega_A^\bullet, \iota_{\mathcal{V}}^*); x) = P(\omega_A^0; x) - x^v P(\omega_A^1; x) = x^{-c} P(\mathrm{Der}_k(A); x) - x^{v-c} P(A; x).$$

On the other hand, by [2, Proposition 6.1, Theorem 3.2], one has

$$P(\mathrm{Der}_k(A); x) = x^c + P(A; x), \quad P(A; x) = \prod(1 - x^{d_j}) / \prod(1 - x^{w_i}),$$

$$P(\mathrm{Tors}\,(\Omega_A^1); x) = P(H_{\mathfrak{m}}^0(\Omega_A^1); x) =$$

$$= 1 + P(A; x)\,\mathrm{res}_{t=0} t^{-2}(1+t)^{-1} \prod(1 + t x^{w_i}) / \prod(1 + t x^{d_j}) = 1 + P(A; x)(\textstyle\sum x^{w_i} - \sum x^{d_j} - 1).$$

Hence,

$$\begin{aligned} P(H_0(\omega_A^\bullet, \iota_{\mathcal{V}}^*); x) &= 1 + x^{-c} P(A; x) - x^{v-c} P(A; x), \\ \chi(\omega_A^\bullet) &= P(H_0(\omega_A^\bullet, \iota_{\mathcal{V}}^*); 1), \\ \mathrm{Ind}_{\mathrm{hom},\mathfrak{o}}(\mathcal{V}) = \chi({}_{\sigma \le 1}(\Omega_A^\bullet)) &= \chi(\omega_A^\bullet) - P(\mathrm{Tors}\,(\Omega_A^1); 1). \end{aligned}$$

Example 6.1. Let S_5 be the germ of a space curve, defined as the intersection of two quadrics in 3-dimensional space. The type of homogeneity is equal to $(2, 2; 1, 1, 1)$, $c = 1$. Hence,

$$P(A; x) = (1 - x^2)^2 / (1 - x)^3 = (1 + x)^2 / (1 - x),$$

$$P(\mathrm{Der}_k(A); x) = x + P(A; x) = (1 + 3x)/(1 - x),$$

$$P(H_0(\omega_A^\bullet, \iota_{\mathcal{V}}^*); x) = 1 + (x - x^{v+1})(1 + x)^2/(1 - x) = 1 + x(1 + x)^2(1 + \ldots + x^{v-1}), \ v \ge 1,$$

$$P(\mathrm{Tors}\,(\Omega_A^1); x) = 1 + (3x - 2x^2 - 1)(1 + x)^2/(1 - x) = (2x - 1)(1 + x)^2 = 3x^2 + 2x^3.$$

As a result we get

$$\chi(\omega_A^\bullet, \iota_{\mathcal{V}}^*) = 4v + 1, \quad \mathrm{Ind}_{\mathrm{hom},\mathfrak{o}}(\mathcal{V}) = \chi({}_{\sigma \le 1}(\Omega_A^\bullet), \iota_{\mathcal{V}}) = 4(v - 1), \ v \ge 1.$$

If $v = 0$, that is, $\mathcal{V}$ is the Euler vector field, then $\chi(\omega_A^\bullet, \iota_{\mathcal{V}}^*) = \chi(\Omega_A^\bullet, \iota_{\mathcal{V}}) = 1$, and

$$\mathrm{Ind}_{\mathrm{hom},\mathfrak{o}}(\mathcal{V}) = \chi({}_{\sigma \le 1}(\Omega_A^\bullet)) = -4.$$

Next, if $v = 1$, that is, $\mathcal{V}$ is a combination of the Euler and Hamiltonian vector fields, then $\chi(\omega_A^\bullet, \iota_{\mathcal{V}}^*) = \chi(\Omega_A^\bullet, \iota_{\mathcal{V}}) = 5$, and $\mathrm{Ind}_{\mathrm{hom},\mathfrak{o}}(\mathcal{V}) = \chi({}_{\sigma \le 1}(\Omega_A^\bullet)) = 0$, and so on.

Remark. This example was considered in a different style in [23, (4.4)], where the author recommends to use a computer algebra system for explicit computations.

Example 6.2. Quasihomogeneous and monomial curves. Evidently, every irreducible component of a quasihomogeneous curve has a *monomial* parametrization. For simplicity, let us consider the case of an irreducible curve, that is, X is a monomial curve. Let H be its *value semigroup* so that the local analytical algebra A of the germ X is generated by monomials t^h, $h \in H$, that is, $A \cong k\langle t^H \rangle$ in standard notations. Then one can compute explicitly dimensions of all graded components of the first cotangent cohomology $T^1(A) \cong \mathrm{Ext}_A^1(\Omega_A^1, A)$ in terms of the semigroup (see, for example, [12]).

In the Gorenstein case the local duality implies that $\dim_k \mathrm{Tors}\,\Omega_A^1 = \dim_k T^1(A) = \tau(A)$. Thus, one obtains also the dimension of the torsion module and, consequently, an explicit expression for the index in view of Proposition 7. It should be noted that for quasihomogeneous

Gorenstein curves one has $\dim_k({}^{\#}\omega_A^1) = \mu(A)$, that is, the dimension of the first ω-cotorsion module is equal to the Milnor number of the singularity (see [25, Satz 1]).

7. Normal two-dimensional singularities

Let us now discuss a simple generalization of Proposition 6 and Claim 4 to the higher dimensional case.

Proposition 8. *Let $(A, \mathfrak{m})$ be the local analytical algebra of a reduced isolated singularity of dimension $n \geq 1$ and let $\mathcal{V} \in \mathrm{Der}_k(A)$ be a vector field with an isolated singularity. Then $H_n(\omega_A^\bullet, \iota_\mathcal{V}^*) = 0$.*

Proof. Set $K_A^n = \mathrm{Ker}(\iota_\mathcal{V}^* \colon \omega_A^n \to \omega_A^{n-1})$. The module K_A^n is coherent; it is concentrated at the singular point of a *reduced* singularity A. Hence K_A^n is a finite-dimensional vector space over k. In other words, it is a torsion module, $K_A^n \cong H^0_\mathfrak{m}(K_A^n) \subseteq H^0_\mathfrak{m}(\omega_A^n) = \mathrm{Tors}\,\omega_A^n$. However, in view of basic properties of the dualizing module, ω_A^n has no torsion, that is, $K_A^n = 0$. QED.

Proposition 9. *Suppose that A is a normal Cohen-Macaulay singularity of dimension $n \geq 2$, and $\mathcal{V} \in \mathrm{Der}_k(A)$ is a vector field with an isolated singularity. Then there exists a natural isomorphism $H_0(\widetilde{\Omega}_A^\bullet, \iota_\mathcal{V}) \cong H_0(\omega_A^\bullet, \iota_\mathcal{V}^*)$.*

Proof. Since A is normal, then $\widetilde{\Omega}_A^0 \cong \Omega_A^0 \cong \omega_A^0 \cong A$. Similarly to (6), there exists the following commutative diagram with exact rows

$$\begin{array}{ccccccc}
0 & \longrightarrow & \widetilde{\Omega}_A^2 & \xrightarrow{c_A} & \omega_A^2 & \longrightarrow {}^{\#}\omega_A^2 & \longrightarrow 0 \\
 & & \downarrow\iota_\mathcal{V} & & \downarrow\iota_\mathcal{V}^* & \downarrow & \\
0 & \longrightarrow & \widetilde{\Omega}_A^1 & \xrightarrow{c_A} & \omega_A^1 & \longrightarrow {}^{\#}\omega_A^1 & \longrightarrow 0 \\
 & & \downarrow\iota_\mathcal{V} & & \downarrow\iota_\mathcal{V}^* & \downarrow & \\
0 & \longrightarrow & \Omega_A^0 \cong A & \longrightarrow & \omega_A^0 \cong A & \longrightarrow 0. &
\end{array} \tag{9}$$

The standard Ker-Coker exact sequence associated with the two lower rows looks like this

$$0 \to \mathrm{Ker}(\widetilde{\Omega}_A^1 \to A) \to \mathrm{Ker}(\omega_A^1 \to A) \to {}^{\#}\omega_A^1 \to H_0(\widetilde{\Omega}_A^\bullet, \iota_\mathcal{V}) \to H_0(\omega_A^\bullet, \iota_\mathcal{V}^*) \to 0. \tag{10}$$

Thus, it is enough to show that two right modules in (10) have the same dimension. To prove this, it is convenient to use an equivalent description of regular meromorphic forms in terms of Noether normalization and the trace map. To be more precise (see [22, (2.1)]), for an n-dimensional singularity A a meromorphic differential p-form $\omega \in \Omega_A^p \otimes_A F(A)$ is regular if and only if for any Noether normalization $Q = k\langle T_1, \dots, T_n\rangle \to A$ one has $\mathrm{Tr}_Q^A(\omega \wedge \eta) \in \Omega_Q^n$ for all holomorphic $(n-p)$-form $\eta \in \Omega_A^{n-p}$.

Now let $Q = k\langle T_1, \dots, T_n\rangle \to A$ be a Noether normalization of A, and let $\overline{\mathcal{V}}$ be an extension of the vector field $\mathcal{V}$ to $\Omega_Q^\bullet$. In fact, such (non-trivial) extension always exists because A is Cohen-Macaulay and one can choose suitable regular parameters $(T_1, \dots, T_n)$ from the jacobian ideal $J_\mathfrak{o}\mathcal{V} \subseteq \mathfrak{m}$, generated by $m \geq n+2$ *regular* elements by assumption (if A is quasihomogeneous then one can take homogeneous parameters). Thus, the two lower rows of the diagram (9) transform in the following commutative diagram

$$\begin{array}{ccccccc}
\Omega^1_Q & \longrightarrow & \Omega^1_A & \xrightarrow{c_A} & \omega^1_A & \xrightarrow{\mathrm{Tr}} & \Omega^1_Q \\
\downarrow\iota_{\overline{\mathcal{V}}} & & \downarrow\iota_{\mathcal{V}} & & \downarrow\iota^*_{\mathcal{V}} & & \downarrow\iota_{\overline{\mathcal{V}}} \\
\Omega^0_Q \cong Q & \longrightarrow & \Omega^0_A \cong A & \xrightarrow{c_A} & \omega^0_A & \xrightarrow{\mathrm{Tr}} & \Omega^0_Q \cong Q \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
Q/J_{\mathfrak{o}}\overline{\mathcal{V}} & \longrightarrow & \mathrm{Coker}(\iota_{\mathcal{V}}) & \longrightarrow & \mathrm{Coker}(\iota^*_{\mathcal{V}}) & \longrightarrow & Q/J_{\mathfrak{o}}\overline{\mathcal{V}}.
\end{array} \tag{11}$$

The left and right terms of the bottom row are finite-dimensional vector spaces. On the other hand, it is well-known that the composition of maps of this row is equal to the multiplication by $\mathrm{rank}_Q(A)$ (see [24], [22]). This completes the proof. QED.

Remark. For normal complete intersections of dimension $n \geq 3$ the proof of Proposition 9 is trivial, since $\Omega^p_A \cong \Omega^p_A{}^{\vee\vee}$ for all $0 \leq p < n-1$. In particular, Ω^1_A (as well as Ω^0_A) has no torsion and *cotorsion*, and the exact sequence (10) splits in two isomorphisms.

Claim 5. *Let A be a quasihomogeneous normal 2-dimensional singularity and let $\mathcal{V} \in \mathrm{Der}_k(A)$ be a vector field with an isolated singularity. Then*

$$\mathrm{Ind}_{\mathrm{hom},\mathfrak{o}}(\mathcal{V}) = \chi(\omega^\bullet_A, \iota^*_{\mathcal{V}}) - \dim_k(\mathrm{Tors}\,\Omega^1_A) + \dim_k(\mathrm{Tors}\,\Omega^2_A).$$

Proof. First observe that normal 2-dimensional singularities satisfy Serre's conditions R_1 and S_2. In particular, they are isolated and Cohen-Macaulay. Hence, ω^p_A, $p \geq 0$, are well-defined. Further, such singularities may have two non-trivial cotorsion modules only in dimension 1 and 2. Moreover, if A is *graded* then the cotorsion modules are isomorphic: ${}^{\#}\omega^2_A \cong {}^{\#}\omega^1_A$ (see [22, (4.8), Bem. (1)]). Making use of diagram (9), it remains to combine Claim 1 and Proposition 4. QED.

We are able to analyze the 2-dimensional case of an isolated complete intersection singularity similarly to Section 6. Let $(d_1, \ldots, d_{m-2}; w_1, \ldots, w_m)$ be the type of homogeneity of the singularity A, $c = \sum d_j - \sum w_i$. Then the weight of the differential form $\eta = dz_1 \wedge \ldots \wedge dz_m / df_1 \wedge \ldots \wedge df_{m-2}$ is equal to $-c$, and there are natural isomorphisms

$$\omega^2_A \cong A(\eta) \cong A[-c], \quad \omega^1_A \cong \mathrm{Hom}_A(\Omega^1_A, \omega^2_A) \cong \mathrm{Der}_k(A)[-c].$$

Since A is normal, then $\omega^1_A \cong \Omega^1_A{}^{\vee\vee}$ and $\omega^0_A \cong \Omega^0_A{}^{\vee\vee}$ by Proposition 2. In particular, $\omega^0_A \cong A$, and one has the following identities for Poincaré series:

$$P(\omega^2_A; x) = x^{-c}P(A;x), \quad P(\omega^1_A; x) = x^{-c}P(\mathrm{Der}_k(A);x), \quad P(\omega^0_A;x) = P(A;x).$$

Again, by [2, Proposition 6.1], one has $P(\mathrm{Der}_k(A);x) = P(A;x) + x^c P(A;x) - x^c$, and, consequently,

$$P(\omega^1_A;x) = x^{-c}P(A;x) + P(A;x) - 1.$$

Further, since $K^2_A = 0$, one has $\omega^2_A \cong \iota^*_{\mathcal{V}}(\omega^2_A)$. Hence,

$$P(\iota^*_{\mathcal{V}}(\omega^2_A);x) = x^{\nu}P(\omega^2_A;x) = x^{\nu-c}P(A;x).$$

The next step of computations is to consider the following exact sequence

$$0 \to K^1_A \longrightarrow \omega^1_A \xrightarrow{\iota^*_{\mathcal{V}}} A \longrightarrow A/\iota^*_{\mathcal{V}}(\omega^1_A) \to 0,$$

which implies the relations

$$P(K^1_A;x) = P(\omega^1_A;x) - x^{-\nu}P(A;x) + x^{-\nu}P(A/\iota^*_{\mathcal{V}}(\omega^1_A);x),$$

$$P(K^1_A;x) = x^{-c}P(A;x) + P(A;x) - 1 - x^{-\nu}P(A;x) + x^{-\nu}P(A/\iota^*_{\mathcal{V}}(\omega^1_A);x),$$

$$P(K^1_A/\iota^*_{\mathcal{V}}(\omega^2_A);x) = x^{-c}P(A;x) + P(A;x) - 1 - x^{-\nu}P(A;x) + x^{-\nu}P(A/\iota^*_{\mathcal{V}}(\omega^1_A);x) - x^{\nu-c}P(A;x),$$

$$P(H_1(\omega^\bullet_A, \iota^*_{\mathcal V}); x) = -1 + (1 + x^{-c} - x^{-\nu} - x^{\nu-c})P(A; x) + x^{-\nu}P(A/\iota^*_{\mathcal V}(\omega^1_A); x).$$

The last polynomial is equal to $x^{-\nu}P(A/J_{\mathfrak o}\mathcal V; x)$ by Proposition 9, while

$$P(A/J_{\mathfrak o}\mathcal V; x) = P(H_0(\Omega^\bullet_A, \iota_{\mathcal V}); x).$$

Example 7.1. The germ Q_7 of the intersection of two quadrics in 4-dimensional space. The type of homogeneity is equal to $(2, 2; 1, 1, 1, 1)$, $c = 0$. Hence, by [2, Proposition 6.1, Theorem 3.2],

$$P(A; x) = (1 - x^2)^2/(1 - x)^4 = (1 + x)^2/(1 - x)^2,$$
$$P(\mathrm{Der}_k(A); x) = 2P(A; x) - 1 = (1 + 6x + x^2)/(1 - x)^2,$$
$$P(H_1(\omega^\bullet_A, \iota^*_{\mathcal V}); x) = -1 + (2 - x^{-\nu} - x^{\nu})P(A; x) + x^{-\nu}P(A/J_{\mathfrak o}\mathcal V; x),$$
$$P(\mathrm{Tors}\,(\Omega^2_A); x) = P(H^0_{\mathfrak m}(\Omega^2_A); x) = -1 + P(A; x)\,\mathrm{res}_{t=0}t^{-3}(1 + t)^{-1}(1 + tx)^4)/(1 + tx^2) =$$
$$-1 + (1 + x)^2(1 - 2x + 3x^2) = 4x^3 + 3x^4,$$

so that $\dim_k \mathrm{Tors}\,(\Omega^2_A) = 7$.

If $v = 0$, then $P(A/J_{\mathfrak o}\mathcal V; 1) = P(H_0(\omega^\bullet_A, \iota^*_{\mathcal V}); 1) = 1$. Consequently, we obtain

$$P(H_1(\omega^\bullet_A, \iota^*_{\mathcal V}); x) = -1 + P(A/J_{\mathfrak o}\mathcal V; x) = 0,\ \chi(\omega^\bullet_A, \iota^*_{\mathcal V}) = 1,\ \mathrm{Ind}_{\mathrm{hom},\mathfrak o}(\mathcal V) = \chi(_{\sigma\le 2}(\Omega^\bullet_A, \iota_{\mathcal V})) = 8.$$

Next, if $v = 1$, then $P(A/J_{\mathfrak o}\mathcal V; x) = 1+4x+4x^2$. Indeed, there are 10 monomials of degree two in 4 variables and 6 generic relations between them which are given by 4 polynomial coefficients of the vector field and 2 defining equations. In addition, there are 4 variables of degree 1 and the field of constants. Thus,

$$P(H_1(\omega^\bullet_A, \iota^*_{\mathcal V}); x) = -1 + (2 - x^{-1} - x)(1 + x)^2/(1 - x)^2 + x^{-1}P(A/J_{\mathfrak o}\mathcal V; x)$$
$$= -1 - x^{-1}(1 + x)^2 + x^{-1} + 4 + 4x = -(x^{-1} + 3 + x) + x^{-1} + 4 + 4x = 1 + 3x.$$

As a result,

$$\chi(\omega^\bullet_A, \iota^*_{\mathcal V}) = 9-4 = 5, \quad \mathrm{Ind}_{\mathrm{hom},\mathfrak o}(\mathcal V) = \chi(_{\sigma\le 2}(\Omega^\bullet_A, \iota_{\mathcal V})) = \chi(\omega^\bullet_A, \iota^*_{\mathcal V})+\dim_k \mathrm{Tors}\,\Omega^2_A = 5+7 = 12.$$

Remark. This example was investigated in a different manner in [10, Ex.(4.3)], partially with the use of a computer algebra system of symbolic computations.

8. Holomorphic forms on complete intersections

Making use of explicit formulas for modules of holomorphic differential forms (see [2]), in this section the homological index is computed directly for quasihomogeneous isolated complete intersection singularities by another method. In addition, we verify the computational results for curves and surface singularities described above in a slightly different manner.

In the notations of Section 1 suppose that $X = (X, \mathfrak o)$ is the germ of a reduced complete intersection in a complex manifold M of dimension $m \ge 2$. Thus, the defining ideal $\mathcal I$ of $X \subset U$ is generated by a regular sequence of functions $f_1, \dots, f_k$ in $\mathcal O_U$, so that $\mathcal O_X = \mathcal O_U/\mathcal I|_X$, and $\dim_{\mathbb C} X = m - k = n \ge 1$.

Lemma 3 ([2], Lemma 3.2)**.** *Let $(A, \mathfrak m)$ be the dual analytical algebra of an isolated complete intersection singularity $(X, \mathfrak o)$ of dimension $n \ge 1$, $n = m - k \ge 1$, with the type of homogeneity $(d_1, \dots, d_k; w_1, \dots, w_m)$. Then for all $0 \le p \le n$*

$$P(\Omega^p_A; x) = P(A; x)\cdot\mathrm{res}_{t=0}t^{-p-1}\prod(1 + tx^{w_i})/\prod(1 + tx^{d_j}).$$

Proof. Let X' be an $(n+1)$-dimensional isolated complete intersection singularity and $X = f^{-1}(0)$, where $f\colon X' \to \mathbb{C}$, $f(\mathfrak{o}) = 0$, is a flat holomorphic map such that $f|_{X'-\{\mathfrak{o}\}}$ is regular. In other terms, the singularity X is the *hypersurface section* of X' defined by f (see [27]). Then the sequence of $\mathcal{O}_X$-modules

$$0 \to \Omega^p_X \xrightarrow{\wedge df} \Omega^{p+1}_{X'}/f\Omega^{p+1}_{X'} \longrightarrow \Omega^{p+1}_X \to 0$$

is exact for all $0 \le p \le n-1$ (see [17, Lemma 1.6]). In the quasihomogeneous case we obtain the following relations for Poincaré polynomials:

$$P(\Omega^{p+1}_X; x) = (1-x^d)P(\Omega^{p+1}_{X'}; x) - x^d P(\Omega^p_X; x),$$

where $d = \deg f = \deg(\wedge df)$.

First consider the case where the singularity $X = X_k$ can be defined by a regular sequence f_i, $i = 1, \dots, k$, such that for all $j = 1, 2, \dots, k$ every germ X_j determined by $f_1, \dots, f_j$ is the *hypersurface section* of X_{j-1} defined by f_j. For convenience of notations set $X_0 = (\mathbb{C}^m, 0)$. In this case we can apply a double induction on k and p similarly to [1]. The general case is reduced to the case of hypersurface sections by arguments in [27, 2.3]. QED.

Remark. In fact, for complete intersections with non-isolated singularities the identity of Lemma 3 is valid for all $0 \le p \le c$, where $c = \operatorname{codim}(\operatorname{Sing} X, X)$.

Proposition 10. *Let A be the local analytical algebra of a reduced n-dimensional isolated singularity, $n \ge 1$, and let $\mathcal{V} \in \operatorname{Der}_k(A)$ be a vector field with an isolated singularity. Then $H_n(\widetilde{\Omega}^\bullet_A, \iota_{\mathcal{V}}) = 0$.*

Proof. Since $\widetilde{\Omega}^n_A$ has no torsion, all the arguments of Proposition 8, applied for dualizing module ω^n_A, remain valid. Then we conclude that $\iota_{\mathcal{V}}\colon \widetilde{\Omega}^n_A \to \widetilde{\Omega}^{n-1}_A$ is injective and the n-th homology group vanishes as required. QED.

Corollary 6. *Under the same assumptions one has*

$$\operatorname{Ind}_{\mathrm{hom},\mathfrak{o}}(\mathcal{V}) = \chi({}_{\sigma\le n}(\Omega^\bullet_A), \iota_{\mathcal{V}}) = \chi(\widetilde{\Omega}^\bullet_A, \iota_{\mathcal{V}}) + (-1)^n \dim_k \operatorname{Tors}\Omega^n_A.$$

Example 8.1. Let us consider again the germ of space curve S_5 from Section 6 (see Example 6.1). By Lemma 3

$$P(\Omega^1_A; x) = (\textstyle\sum x^{w_i} - \sum x^{d_j})P(A; x) = (3x - 2x^2)P(A; x).$$

Then by [2, Theorem (3.2)] we have

$$P(\operatorname{Tors}(\Omega^1_A); x) = 1 + P(A; x)(\textstyle\sum x^{w_i} - \sum x^{d_j} - 1) = 3x^2 + 2x^3.$$

Hence,

$$P(\widetilde{\Omega}^1_A; x) = P(A; x) - 1, \quad P(\iota_{\mathcal{V}}(\widetilde{\Omega}^1_A); x) = x^v(P(A; x) - 1),$$
$$P(H_0(\widetilde{\Omega}^\bullet_A, \iota_{\mathcal{V}}); x) = P(A; x) - x^v P(A; x) + x^v =$$
$$= (1 - x^v)P(A; x) + x^v = x^v + (1+x)^2(1 + x + \dots x^{v-1}),\ v \ge 1.$$

As a result one gets

$$\chi(\Omega^\bullet_A, \iota_{\mathcal{V}}) = 4v + 1, \quad \operatorname{Ind}_{\mathrm{hom},\mathfrak{o}}(\mathcal{V}) = \chi({}_{\sigma\le 1}(\Omega^\bullet_A), \iota_{\mathcal{V}}) = 4(v-1),\ v \ge 1.$$

If $v = 0$, then $\chi(\Omega^\bullet_A, \iota_{\mathcal{V}}) = 1$, and $\operatorname{Ind}_{\mathrm{hom},\mathfrak{o}}(\mathcal{V}) = -4$, because $\dim_k \operatorname{Tors}\Omega^1_A = 5$. Further, if $v = 1$, then $\chi(\Omega^\bullet_A, \iota_{\mathcal{V}}) = 5$, and $\operatorname{Ind}_{\mathrm{hom},\mathfrak{o}}(\mathcal{V}) = 0$.

Remark. Combining observations of two approaches, in this example it is possible to compute the homogeneous structure of two modules of the ω_A-cotorson complex ${}^{\#}\omega^\bullet_A$ explicitly. As was shown before this complex has two non-trivial terms of the same dimension. In fact, they are cyclic A-modules and the module ${}^{\#}\omega_A^0$ is naturally isomorphic to the quotient of $\mathrm{Der}_k(A)$ by the submodule generated by the hamiltonian vector field. The bottom and top rows of diagram (6) imply the following two relations

$$P({}^{\#}\omega_A^0;x) = P(\omega_A^0;x) - P(A;x),$$

$$P({}^{\#}\omega_A^1;x) = P(\omega_A^1;x) - P(\Omega_A^1;x) + P(\mathrm{Tors}\,(\Omega_A^1);x)) = P(\omega_A^1;x) - P(\widetilde{\Omega}_A^1;x),$$

respectively. Hence,

$$P({}^{\#}\omega_A^0;x) = 1 + (x^{-1}-1)P(A;x) = 1 + x^{-1}(1+x)^2 = x^{-1} + 3 + x,$$

$$P({}^{\#}\omega_A^1;x) = x^{-1}(1+x)^2/(1-x) - (3x+x^2)/(1-x) = x^{-1}(1+2x-2x^2-x^3)/(1-x) = x^{-1}+3+x.$$

To simplify notations, in the sequel we will denote the contraction maps on the families $\Omega_A^\bullet$, $\omega_A^\bullet$ and ${}^{\#}\omega_A^\bullet$ by the same symbol $\iota_{\mathcal{V}}$.

Now we are able to describe the action of $\iota_{\mathcal{V}}$ on the cotorsion complex ${}^{\#}\omega_A^\bullet$ and to compute the homology groups. Obviously, if $v=0$ then this action is an isomorphism. If $v=1$, then

$$P(\mathrm{Ker}(\iota_{\mathcal{V}}\colon {}^{\#}\omega_A^1 \to {}^{\#}\omega_A^0);x) = 2+x; \quad P(\mathrm{Coker}(\iota_{\mathcal{V}}\colon {}^{\#}\omega_A^1 \to {}^{\#}\omega_A^0);x) = x^{-1}+2,$$

$\dim_k H_1({}^{\#}\omega_A^\bullet) = \dim_k H_0({}^{\#}\omega_A^\bullet) = 3$. If $v=2$, then

$$P(\mathrm{Ker}(\iota_{\mathcal{V}}\colon {}^{\#}\omega_A^1 \to {}^{\#}\omega_A^0);x) = 3+x; \quad P(\mathrm{Coker}(\iota_{\mathcal{V}}\colon {}^{\#}\omega_A^1 \to {}^{\#}\omega_A^0);x) = x^{-1}+3,$$

$\dim_k H_1({}^{\#}\omega_A^\bullet) = \dim_k H_0({}^{\#}\omega_A^\bullet) = 4$. For all $v \geq 3$ one has $\dim_k H_1({}^{\#}\omega_A^\bullet) = \dim_k H_0({}^{\#}\omega_A^\bullet) = 5$, that is, $\iota_{\mathcal{V}}$ is the zero map.

Example 8.2. In a similar manner for the surface singularity Q_7 from Section 7 one gets

$$P(\Omega_A^2;x) = (6x^2 - 8x^3 + 3x^4)P(A;x), \quad P(\Omega_A^1;x) = (4x-2x^2)P(A;x),$$

$$P(\widetilde{\Omega}_A^2;x) = P(\Omega_A^2;x) - P(\mathrm{Tors}\,(\Omega_A^2);x).$$

Further, there are two short exact sequences

$$0 \to \widetilde{\Omega}_A^2 \xrightarrow{\iota_{\mathcal{V}}} \Omega_A^1 \longrightarrow \Omega_A^1/\iota_{\mathcal{V}}(\Omega_A^2) \to 0, \quad 0 \to \mathrm{Ker}_A^1 \longrightarrow \Omega_A^1 \xrightarrow{\iota_{\mathcal{V}}} A/\iota_{\mathcal{V}}(\Omega_A^1) \to 0,$$

where $\mathrm{Ker}_A^1 = \mathrm{Ker}\,(\iota_{\mathcal{V}}\colon \Omega_A^1 \to A)$. They imply two relations

$$P(\iota_{\mathcal{V}}(\Omega_A^2);x) = x^v P(\widetilde{\Omega}_A^2;x), \quad P(\mathrm{Ker}_A^1;x) = P(\Omega_A^1;x) - x^{-v}P(A;x) + x^{-v}P(A/\iota_{\mathcal{V}}(\Omega_A^1);x),$$

respectively. As a result,

$$P(H_1(\widetilde{\Omega}_A^\bullet,\iota_{\mathcal{V}});x) = P(\mathrm{Ker}_A^1;x) - x^v P(\iota_{\mathcal{V}}(\widetilde{\Omega}_A^2);x) =$$
$$= P(\Omega_A^1;x) - x^{-v}P(A;x) + x^{-v}P(A/\iota_{\mathcal{V}}(\Omega_A^1);x) - x^v P(\Omega_A^2;x) + x^v P(\mathrm{Tors}\,(\Omega_A^2);x) =$$
$$= \{(4x-2x^2) - x^{-v} - x^v(6x^2-8x^3+3x^4)\}P(A;x) + x^{-v}P(A/J_{\mathfrak{o}}\mathcal{V};x) + x^v(4x^3+3x^4).$$

If $v=0$, then

$$P(H_1(\widetilde{\Omega}_A^\bullet,\iota_{\mathcal{V}});x) = \{(4x-2x^2) - 1 - (6x^2-8x^3+3x^4)\}P(A;x) + 1 + (4x^3+3x^4).$$

The expression in curly brackets is equal to $\{-2(1-x)^2 + (1-x)^3(1+3x)\}$. Hence,

$$P(H_1(\widetilde{\Omega}_A^\bullet,\iota_{\mathcal{V}});x) = \{-2 + (1-x)(1+3x)\}(1+x)^2 + 1 + (4x^3+3x^4) = 0,$$

that is,

$$\chi(\widetilde{\Omega}_A^\bullet,\iota_{\mathcal{V}}) = 1, \quad \mathrm{Ind}_{\mathrm{hom},\mathfrak{o}}(\mathcal{V}) = \chi(\sigma_{\leq 2}(\Omega_A^\bullet,\iota_{\mathcal{V}})) = 8.$$

Further, if $v=1$, then

$$P(H_1(\widetilde{\Omega}_A^\bullet,\iota_{\mathcal{V}});x) = \{(4x-2x^2) - x^{-1} - x(6x^2-8x^3+3x^4)\}P(A;x) + x^{-1}P(A/J_{\mathfrak{o}}\mathcal{V};x) + x(4x^3+3x^4).$$

The expression in curly brackets transforms in the following way:

$$x^{-1}\{-2(1-x)(1+x-x^2) + (1-x)(1+x+x^2+x^3-5x^4+3x^5)\} = -x^{-1}(1-x)^2(1+2x-x^2-2x^3+3x^4).$$

Hence,

$$P(H_1(\widetilde{\Omega}^\bullet_A,\iota_{\mathcal V});x) = -x^{-1}(1+x)^2(1+2x-x^2-2x^3+3x^4)+x^{-1}P(A/J_{\mathfrak o}\mathcal V;x)+x(4x^3+3x^4)$$
$$= x^{-1}(-1-4x-4x^2+2x^3+2x^4-4x^5-3x^6)+x(4x^3+3x^4)+x^{-1}P(A/J_{\mathfrak o}\mathcal V;x)$$
$$= -x^{-1}(1+4x+4x^2)+2x^2+2x^3+x^{-1}(1+4x+4x^2) = 2x^2+2x^3.$$

That is, $\dim_k H_1(\widetilde{\Omega}^\bullet_A,\iota_{\mathcal V}) = P(H_1(\widetilde{\Omega}^\bullet_A,\iota_{\mathcal V});1) = 4$. As a result,

$$\chi(\widetilde{\Omega}^\bullet_A,\iota_{\mathcal V}) = 9-4=5, \quad \mathrm{Ind}_{\mathrm{hom},\mathfrak o}(\mathcal V) = \chi({}_{\sigma\le 2}(\Omega^\bullet_A,\iota_{\mathcal V})) = 9-4+7=12.$$

In addition, since $\dim_k \Omega^4_A = 1$ and $\dim_k \Omega^3_A = 8$, one concludes $\chi(\Omega^\bullet_A,\iota_{\mathcal V}) = 5$.

For completeness, let us also analyze the homology of the ω_A-cotorson complex ${}^{\#}\omega^\bullet_A$. Two non-trivial terms of this complex has the same dimension, they are cyclic A-modules and the module ${}^{\#}\omega^1_A$ is naturally isomorphic to the quotient of $\mathrm{Der}_k(A)$ by the submodule generated by 4 hamiltonian vector fields. In fact, for 2-dimensional quasihomogeneous isolated complete intersection singularities these two cotorsion modules are isomorphic in view of the canonical isomorphisms ${}^{\#}\omega^1_A \cong H^1_{\mathfrak m}(\Omega^1_A)$ and ${}^{\#}\omega^2_A \cong H^1_{\mathfrak m}(\Omega^2_A) \cong H^1_{\mathfrak m}(\Omega^1_A)$. One can apply the general formulas for Poincaré polynomials of the local cohomology groups from [1] or [2, (3.2)]. However, it is also possible to compute the polynomials directly. To be more precise, similarly to the above example there exist two relations

$$P({}^{\#}\omega^1_A;x) = P(\omega^1_A;x) - P(\Omega^1_A;x),$$
$$P({}^{\#}\omega^2_A;x) = P(\omega^2_A;x) - P(\Omega^2_A;x) + P(\mathrm{Tors}\,(\Omega^2_A);x)).$$

Hence,

$$\begin{aligned} P({}^{\#}\omega^1_A;x) &= 2P(A;x)-1-P(\Omega^1_A;x) = P(A;x)(2-4x+2x^2)-1 = 1+4x+2x^2,\\ P({}^{\#}\omega^2_A;x) &= P(A;x)\{1-(6x^2-8x^3+3x^4)\}+(4x^3+3x^4)\\ &= (1+x)^2(1-x)(1+3x)+(4x^3+3x^4) = 1+4x+2x^2. \end{aligned}$$

Again, if $v=0$ then the homology groups of the cotorsion complex ${}^{\#}\omega^\bullet_A$ obviously vanish. If $v=1$, then

$$P(\mathrm{Ker}(\iota_{\mathcal V}\colon {}^{\#}\omega^2_A \to {}^{\#}\omega^1_A);x) = 2x+2x^2; \quad P(\mathrm{Coker}(\iota_{\mathcal V}\colon {}^{\#}\omega^1_A \to {}^{\#}\omega^0_A);x) = 1+3x,$$

and $\dim_k H_2({}^{\#}\omega^\bullet_A) = \dim_k H_1({}^{\#}\omega^\bullet_A) = 4$. If $v=2$, then

$$P(\mathrm{Ker}(\iota_{\mathcal V}\colon {}^{\#}\omega^1_A \to {}^{\#}\omega^0_A);x) = 4x+2x^2; \quad P(\mathrm{Coker}(\iota_{\mathcal V}\colon {}^{\#}\omega^1_A \to {}^{\#}\omega^0_A);x) = 1+4x+x^2$$

and $\dim_k H_2({}^{\#}\omega^\bullet_A) = \dim_k H_1({}^{\#}\omega^\bullet_A) = 6$.

For $v \ge 3$ one has $\dim_k H_2({}^{\#}\omega^\bullet_A) = \dim_k H_1({}^{\#}\omega^\bullet_A) = 7$, that is, the map $\iota_{\mathcal V}$ is identically *zero*.

9. The generating function and homology

Now we will study in some detail the case of quasihomogeneous complete intersections with isolated singularities of arbitrary dimension. The key idea of our approach is based on the fact that the Euler characteristic of a complex of finite-dimensional vector spaces is equal to the alternating sum of their dimensions. In the quasihomogeneous case the *graded components* of the modules of holomorphic differential forms play the role of such spaces. Our method of computation is based on a simplified procedure of "dévissage" applied in [2, (3.3)].

Definition. The generating function of the complex $(\Omega^\bullet_A,\iota_{\mathcal V})$ is defined as follows:

$$G_P((\Omega^\bullet_A,\iota_{\mathcal V});x,y) = \sum_{i\ge 0}(-1)^i P(H_i(\Omega^\bullet_A,\iota_{\mathcal V});x)\,y^i, \tag{12}$$

where $P(H_i(\Omega^\bullet_A, \iota_{\mathcal V}); x)$ are Poincaré polynomials of the corresponding homology groups which are *graded* vector spaces. Evidently, if all homology groups are finite-dimensional, then

$$\chi(\Omega^\bullet_A, \iota_{\mathcal V}) = G_P((\Omega^\bullet_A, \iota_{\mathcal V}); 1, 1).$$

Remark. Of course, similar generating functions are well-defined for all other complexes considered above; such functions can be considered as variants of χ_y-characteristic of Hirzebruch associated with the "twisted" De Rham cohomology (cf. [2, Introduction]).

Theorem 1. *In the notations of the previous section let $(A, \mathfrak m)$ be the local algebra of an n-dimensional isolated complete intersection singularity $(X, \mathfrak o)$ with the type of homogeneity $(d_1, \dots, d_k; w_1, \dots, w_m)$, $n = m - k \geq 1$. Suppose that the weight of $\mathcal V$ is equal to v. Then*

$$G_P(({}_{\sigma\leq n}(\Omega^\bullet_A), \iota_{\mathcal V}); x, x^v) =$$
$$(-1)^n x^{nv} P(A; x) \operatorname{res}_{t=0} t^{-n-1}(1 + tx^{-v})^{-1} \prod(1 + tx^{w_i}) / \prod(1 + tx^{d_j}).$$

In particular, $\operatorname{Ind}_{\mathrm{hom},\mathfrak o}(\mathcal V) = G_P(({}_{\sigma\leq n}(\Omega^\bullet_A), \iota_{\mathcal V}); 1, 1).$

Proof. Let us consider the truncated contracted De Rham complex of A :

$$({}_{\sigma\leq n}(\Omega^\bullet_A), \iota_{\mathcal V}) \colon \quad 0 \longrightarrow \Omega^n_A \xrightarrow{\iota_{\mathcal V}} \Omega^{n-1}_A \xrightarrow{\iota_{\mathcal V}} \Omega^{n-2}_A \to \dots \to \Omega^1_A \xrightarrow{\iota_{\mathcal V}} \Omega^0_A \cong A \longrightarrow 0.$$

According to Proposition 10, the kernel of the left contraction mapping is the torsion submodule, that is, $H_n({}_{\sigma\leq n}(\Omega^\bullet_A), \iota_{\mathcal V}) \cong \operatorname{Tors}\Omega^n_A$. Set $\operatorname{Ker}^i_A = \operatorname{Ker}(\iota_{\mathcal V} \colon \Omega^i_A \to \Omega^{i-1}_A)$. Then for all $1 \leq i < n$ there exist exact sequences

$$0 \longrightarrow \operatorname{Ker}^i_A \longrightarrow \Omega^i_A \xrightarrow{\iota_{\mathcal V}} \iota_{\mathcal V}(\Omega^i_A) \longrightarrow 0$$

and the following relations for Poincaré series:

$$P(\operatorname{Ker}^i_A; x) = P(\Omega^i_A; x) - x^{-v} P(\iota_{\mathcal V}(\Omega^i_A); x),$$
$$P(H_i(\Omega^\bullet_A, \iota_{\mathcal V}); x) = P(\Omega^i_A; x) - x^{-v} P(\iota_{\mathcal V}(\Omega^i_A); x) - P(\iota_{\mathcal V}(\Omega^{i+1}_A); x).$$

In addition, it is clear that

$$P(H_n({}_{\sigma\leq n}(\Omega^\bullet_A, \iota_{\mathcal V})); x) = P(\operatorname{Tors}(\Omega^n_A); x), \quad P(\iota_{\mathcal V}(\Omega^n_A); x) = x^v P(\Omega^n_A; x) - x^v P(\operatorname{Tors}(\Omega^n_A); x),$$
$$P(H_0(\Omega^\bullet_A, \iota_{\mathcal V}); x) = P(A; x) - P(\iota_{\mathcal V}(\Omega^1_A); x).$$

As a result, the n first terms for $i = 0, \dots, n-1$ of the generating function (12) with $y = x^v$ give us the following relations

$$\begin{aligned}
\textstyle\sum_{i=0}^{n-1}(-1)^i x^{iv} P(H_i(\Omega^\bullet_A, \iota_{\mathcal V}); x) &= \big(P(A; x) - P(\iota_{\mathcal V}(\Omega^1_A); x)\big) \\
&- \big(x^v P(\Omega^1_A; x) - P(\iota_{\mathcal V}(\Omega^1_A); x) - x^v P(\iota_{\mathcal V}(\Omega^2_A); x)\big) \\
&+ \big(x^{2v} P(\Omega^2_A; x) - x^v P(\iota_{\mathcal V}(\Omega^2_A); x) - x^{2v} P(\iota_{\mathcal V}(\Omega^3_A); x)\big) + \dots \\
&= \textstyle\sum_{i=0}^{n-1}(-1)^i x^{iv} P(\Omega^i_A; x) + (-1)^n x^{(n-1)v} P(\iota_{\mathcal V}(\Omega^n_A); x) \\
&= \textstyle\sum_{i=0}^{n}(-1)^i x^{iv} P(\Omega^i_A; x) + (-1)^{n+1} x^{nv} P(\operatorname{Tors}(\Omega^n_A); x).
\end{aligned}$$

Adding then the term of dimension n, one gets

$$G_P({}_{\sigma\leq n}(\Omega^\bullet_A, \iota_{\mathcal V}); x, x^v) = \sum_{i=0}^{n}(-1)^i x^{iv} P(H_i({}_{\sigma\leq n}(\Omega^\bullet_A), \iota_{\mathcal V}); x) = \sum_{i=0}^{n}(-1)^i x^{iv} P(\Omega^i_A; x).$$

Finally, making use of Lemma 3 and elementary transformations, we obtain the following identity

$$G_P(({}_{\sigma\leq n}(\Omega^\bullet_A), \iota_{\mathcal V}); x, x^v) = P(A; x) \sum_{p=0}^{n}(-1)^p x^{pv} \operatorname{res}_{t=0} t^{-p-1} \prod(1 + tx^{w_i}) / \prod(1 + tx^{d_j})$$

which implies the desired formula. QED.

It is possible to represent the obtained expression for the generating function G_P in an explicit form (cf. [2, (3.2)]). Let W_λ be the elementary symmetric polynomials in $y_1, \dots, y_m$ of *weight* $\lambda \geq 0$,

$$\prod_{i=1}^{m}(1 + y_i\zeta) = \sum_{\lambda=0}^{m} W_\lambda(y_1, \dots, y_m)\zeta^\lambda,$$

and let D_λ be the symmetric polynomials in $y_1, \dots, y_k$ of *degree* $\lambda \geq 0$,

$$\prod_{j=1}^{k}(1 + y_i\zeta)^{-1} = \sum_{\lambda=0}^{k}(-1)^\lambda D_\lambda(y_1, \dots, y_k)\zeta^\lambda.$$

Corollary 7. *Under the assumptions of Theorem 1 we have*

$$G_P({}_{\sigma\leq n}(\Omega^\bullet_A, \iota_{\mathcal{V}}); x, x^v) =$$
$$= \sum_{\lambda_1+\lambda_2+\lambda_3=n} (-1)^{\lambda_2} x^{(n-\lambda_1)v} W_{\lambda_2}(x^{w_1}, \dots, x^{w_m}) D_{\lambda_3}(x^{d_1}, \dots, x^{d_k}) P(A; x).$$

Making use of the expressions from the theorem and corollary above, for the surface germ Q_7 from Example 7.1 one gets immediately

$$G_P(({}_{\sigma\leq 2}(\Omega^\bullet_A), \iota_{\mathcal{V}}); x, x^v) = \big(1 - x^v(4x - 2x^2) + x^{2v}(6x^2 - 8x^3 + 3x^4)\big)P(A; x).$$

If $v = 1$, then

$$G_P(({}_{\sigma\leq 2}(\Omega^\bullet_A), \iota_{\mathcal{V}}); x, x) = 1 + 4x + 4x^2 - 2x^3 - 2x^4 + 4x^5 + 3x^6,$$

that is, $\operatorname{Ind}_{\hom,\mathfrak{o}}(\mathcal{V}) = \chi({}_{\sigma\leq 2}(\Omega^\bullet_A, \iota_{\mathcal{V}})) = 9 - 4 + 7 = 12$, as was required.

Comments. 1) The latter formula shows that the homological index is determined completely by the weights of variables, the defining equations and the weight of the vector field; it does not contain the Poincaré polynomials of the module of derivations, the torsion modules, the dualizing module and others except the polynomials of modules of holomorphic differential forms described in Lemma 3.

2) This formula is working correctly without the assumption that the vector field has an isolated singularity on the *ambient* manifold. Indeed, homology groups are finite-dimensional if the *grade* or *depth* of the ideal of A, generated by the coefficients of the vector field, is not less than the dimension of the singularity A. In this case the vector field $\mathcal{V}$ has isolated singularities on the singularity itself.

3) It should be also underlined that in contrast with the formulas for curves and surfaces obtained in Section 6 and Section 7, the homogeneous components of the generating function contain expressions for Poincaré series with shifted *non-canonical* grading since this function is a result of a weighted *convolution* of Poincaré series transcribed in the natural grading.

10. The Lebelt resolutions

Let us now discuss a direct method of computing the homological index in the case of normal complete intersections. This approach is partially based on the construction of a certain subcomplex of the generalized Koszul complex that gives a *free* resolution for exterior powers of a module whose homological dimension does not exceed 1 (see [26]).

Let A be the local analytical algebra of the germ of a reduced singularity of dimension $n \geq 1$ given by an ideal $I = (f_1, \dots, f_k) \subset P = k\langle z_1, \dots, z_m\rangle$, so that $A = P/I$. Then there is a standard exact sequence representing the module of Kähler differentials of k-algebra A as follows

$$0 \longrightarrow \textstyle\int I/I^2 \longrightarrow I/I^2 \xrightarrow{Df} \Omega^1_P \otimes_P A \longrightarrow \Omega^1_A \longrightarrow 0. \tag{13}$$

where $\int I$ is the *primitive ideal* of I, and $Df = \mathrm{Jac}(f)$ is the jacobian matrix associated with the sequence $(f_1, \ldots, f_k)$. By definition, the ideal $\int I \subset P$ consists of all $g \in I$, such that $\partial(g) \in I$ for all $\partial \in \mathrm{Der}_k(P)$ (see [31]).

It is well-known that for *complete intersection* germs one has $n = m - k$ and I/I^2 is a *free* A-module of rank k. Moreover, in the case of *reduced* complete intersections we have $\int I/I^2 = 0$ (see [31, §4, ex.(1)]). Hence, the homological dimension of the A-module Ω^1_A is not greater than one. Suppose additionally that A is an *isolated* singularity. According to [26, Folgerung 1(a)], for all A-modules $\Omega^p_A = \wedge^p \Omega^1_A$, $p = 0, 1, \ldots, n-1$, there are *free* resolutions

$$\mathcal{L}^k_p \colon 0 \longrightarrow \mathcal{L}^k_{p,p} \longrightarrow \ldots \longrightarrow \mathcal{L}^k_{p,r} \longrightarrow \ldots \longrightarrow \mathcal{L}^k_{p,1} \longrightarrow \mathcal{L}^k_{p,0} \xrightarrow{\varepsilon} \Omega^p_A \longrightarrow 0, \tag{14}$$

where

$$\mathcal{L}^k_{p,r} = S^r L_0 \otimes \wedge^{p-r} L_1, \quad r = 0, \ldots, p, \quad L_0 = I/I^2, \quad L_1 = \Omega^1_P \otimes_P A,$$

and $\varepsilon = \varepsilon^k_{p,0}$ is the quotient map of $\mathcal{L}^k_{p,0}$ to $\mathrm{Coker}\,(\mathcal{L}^k_{p,1} \to \mathcal{L}^k_{p,0}) \cong \Omega^p_A$ for all $p = 0, \ldots, n$. It is not difficult to see that

$$\mathcal{L}^k_{p,r} \cong \left(\wedge^{p-r} A^m \right)^{\nu(k,r)}, \quad \text{for all} \quad 0 \le r \le p,$$

where $\nu(k,r) = \binom{k-1+r}{r}$ is the number of homogeneous monomials of degree r in k variables.

Example 10.1. As an illustration let us apply this construction to the case of a *quasihomogeneous* isolated complete intersection singularity with type of homogeneity $(d_1, \ldots, d_k; w_1, \ldots, w_m)$. Then two free A-modules $L_0 \cong \prod A(-d_j)$ and $L_1 \cong \prod A(-w_i)$ of rank k and m, respectively, are endowed with the natural grading. In this grading the resolution $\mathcal{L}^k_p$ is an exact sequence of graded modules connecting by maps whose weights are equal to *zero* (cf. [2, (3.3)]). This property follows from the explicit expressions for differentials $\mathcal{D}^k_{p,r}$ of the Lebelt resolution (see [26]). Next, making use of sequence (14) and elementary transformations, one can deduce the formula of Lemma 3 (and vice versa).

Proposition 11. *Let A be the local analytical algebra of the germ of a normal isolated complete intersection singularity, $\mathcal{V} \in \mathrm{Der}_k(A)$ and $\iota_{\mathcal{V}} \colon \mathcal{L}^k_{p,r} \to \mathcal{L}^k_{p-1,r}$ a family of contraction maps for all admissible p and r. Then a bicomplex $\mathcal{L}^k_{\bullet\bullet}$ is well-defined and there exist natural isomorphisms of the homology groups*

$$H_i(\Omega^\bullet_A, \iota_{\mathcal{V}}) \cong H_i(\mathrm{Tot}(\mathcal{L}^k_{\bullet\bullet})), \quad i = 0, 1, \ldots, n-1,$$

where $\mathrm{Tot}(\mathcal{L}^k_{\bullet\bullet})$ is the total complex associated with the bicomplex.

Proof. It is not difficult to verify that all the differentials of the Lebelt resolutions commute with the contraction map. Hence, $\mathcal{L}^k_{\bullet\bullet}$ is a bicomplex of A-modules. All horizontal homologies of this bicomplex are trivial since its rows are given by the Lebelt resolutions of A-modules Ω^p_A, $p = 0, 1, \ldots, n-1$. The desired statement is a direct consequence of basic properties and standard relations between homology and hyperhomology functors. QED.

Claim 6. *If the vector field $\mathcal{V}$ has an isolated singularity then*

$$\dim_k H_n(\sigma_{\le n}(\Omega^\bullet_A), \iota_{\mathcal{V}}) = \dim_k \mathrm{Tors}\,\Omega^n_A = \dim_k {}^{\#}\omega^n_A = \dim_k \mathrm{Ext}^n_A(\Omega^n_A, A) = \dim_k A/F_0(Df),$$

where $F_0(Df)$ is the zeroth Fitting ideal generated by the maximal minors of the jacobian matrix Df.

Proof. The homology group $H_n(\sigma_{\le n}(\Omega^\bullet_A), \iota_{\mathcal{V}})$ and $\mathrm{Tors}\,\Omega^n_A$ are naturally isomorphic in view of Proposition 10. Further, for isolated complete intersection singularities the dimensions of torsion and cotorsion modules are equal (see [11]). On the other hand, the dimension of $\mathrm{Tors}\,\Omega^n_A$ is equal to $\dim_k \mathrm{Ext}^n_A(\Omega^n_A, A)$ (the latter is computed explicitly in the quasihomogeneous case in [2, (3.2)]), while the dimension of the cotorsion module ${}^{\#}\omega^n_A \cong H^1_{\mathfrak{m}}(\Omega^n_A)$ is nothing but the dimension of $A/F_0(Df)$. QED.

REMARK. From Claim 6 it follows that for an isolated singularity the dimensions of the *highest* homology groups of the complex $({}_{\sigma\le n}(\Omega^\bullet_A),\iota_{\mathcal V})$ and the torsion module $\operatorname{Tors}\Omega^n_A$ are equal; in the *quasihomogeneous* case this dimension coincides with the Milnor number. It is known at least n (generally speaking, *different*) expressions for Poincaré polynomials of the torsion module and, consequently, for the Milnor number of the singularity (see [1], [2]).

EXAMPLE 10.2. Let us show how one can compute directly homology groups with the use of the bicomplex $\mathcal{L}^k_{\bullet\bullet}$ in the 2-dimensional case. This bicomplex looks like this:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & A^{\binom{k+1}{2}} & \xrightarrow{\mathcal{D}^k_{2,1}} & (A^m)^k & \xrightarrow{\mathcal{D}^k_{2,1}} & \wedge^2 A^m & \xrightarrow{\varepsilon^k_{2,0}} & \Omega^2_A & \longrightarrow & 0 \\ & & & & \downarrow\iota_{\mathcal V} & & \downarrow\iota_{\mathcal V} & & \downarrow\iota_{\mathcal V} & & \\ & & 0 & \longrightarrow & A^k & \xrightarrow{\mathcal{D}^k_{1,1}} & A^m & \xrightarrow{\varepsilon^k_{1,0}} & \Omega^1_A & \longrightarrow & 0 \\ & & & & & & \downarrow\iota_{\mathcal V} & & \downarrow\iota_{\mathcal V} & & \\ & & & & 0 & \longrightarrow & A & \xrightarrow{\varepsilon^k_{0,0}} & \Omega^0_A\cong A & \longrightarrow & 0. \end{array} \tag{15}$$

By the definition above, $\varepsilon^k_{p,0}$ are natural quotient maps $\mathcal{L}^k_{p,0}\longrightarrow \operatorname{Coker}(\mathcal{D}^k_{p,1})\cong\Omega^p_A$ for all $p=0,1,2$. Next, the middle row is, in fact, equivalent to the truncated exact sequence (13) so that the differential $\mathcal{D}^k_{1,1}$ is given by the jacobian $(k\times m)$-matrix of the defining ideal, while $\mathcal{D}^k_{2,1}(y_1,\dots,y_k)=\operatorname{jac}(f_1)\wedge y_1+\ldots+\operatorname{jac}(f_k)\wedge y_k$, and so on (see [26, (2.12), (2.15)]).

In view of Proposition 11 the homology groups of the complex $\Omega^\bullet_A$ of dimensions $i=0,1$ can be computed as the corresponding homology groups of the total complex $\operatorname{Tot}(\mathcal{L}^k_{\bullet\bullet})$. As a result one has

$$H_0(\Omega^\bullet_A)\cong A/\iota_{\mathcal V}(A^m),\quad H_1(\Omega^\bullet_A)\cong \operatorname{Ker}(\iota_{\mathcal V}\colon A^m\to A)/\big(\iota_{\mathcal V}(\wedge^2A^m)+\mathcal{D}^k_{1,1}(A^k)\big).$$

REMARK. In this case all homology groups are *finite-dimensional*, and a simple script for calculations of the index can be readily implemented in any computer system of algebraic calculations similarly to [10], (4.4). It should be underlined that by contrast with the algorithm of [10] one needs expressions for the defining ideal of the singularity and for the coefficients of a vector field only. Explicit (highly non-trivial) expressions for the entries of the structure matrix $C=\| c_{ij}\|$ realizing the *tangency* relation $\mathcal{V}(f)=C(f)$ are no longer required since all calculations are carried out modulo the defining ideal in the local analytical algebra A.

EXAMPLE 10.3. In the case $n\ge 2$ the construction of the Lebelt resolutions implies that all the terms under the polygonal step-line of the lower left corner of the diagram (15) are zeros. Hence, one can described the homology groups of the next dimensions (until $n-1$ inclusively) analogously to the case considered in the Example 10.2, that is,

$$H_2(\Omega^\bullet_A)\cong \operatorname{Ker}(\iota_{\mathcal V}\colon\ \wedge^2 A^m\to A^m)/\big(\iota_{\mathcal V}(\wedge^3A^m)+\mathcal{D}^k_{2,1}((A^m)^k)\big).$$

The numerator in the presentation of $H_3(\Omega^\bullet_A)$ can be written down as follows

$$\operatorname{Ker}(\iota_{\mathcal V}\colon\ \wedge^3 A^m\to\wedge^2A^m)+\operatorname{Ker}(\iota_{\mathcal V}\colon (A^m)^k\to A^k),$$

while the corresponding denominator is equal to

$$\iota_{\mathcal V}(\wedge^4A^m)+\iota_{\mathcal V}((\wedge^2A^m)^k)+\mathcal{D}^k_{3,1}((\wedge^2A^m)^k),$$

and so on. As a result, for an *odd* integer $1\le\ell<n$ the numerator in the presentation of $H_\ell(\Omega^\bullet_A)$ is written down in the following way

$$\operatorname{Ker}(\iota_{\mathcal V}\colon\ \wedge^\ell A^m\to A^m)+\operatorname{Ker}(\iota_{\mathcal V}\colon\ \wedge^{\ell-2}(A^m)^k\to(A^m)^k)+\ldots,$$

while the corresponding denominator is equal to

$$\iota_{\mathcal{V}}(\wedge^{\ell+1}A^m) + \iota_{\mathcal{V}}(\wedge^{\ell-1}(A^m)^k) + \ldots + \mathcal{D}^k_{\ell,1}((\wedge^{\ell-1}A^m)^k).$$

Similar presentations also exist for all *even* $0 \le \ell < n$.

Remark. The above example can also be investigated with the use of considerations in the context of the theory of the *generalized* Koszul complex because the length of its homology groups of dimension $0 \le i \le n-2$ can be computed explicitly (see a detail review in [21]). To be more precise, these groups are expressed in terms of the symmetric polynomial algebra over the quotient $A/J_{\mathfrak{o}}\mathcal{V}$ similarly to [10]. The non-trivial homology groups of dimension $i = n-1$ are computed analogously to Example 10.2

Remark. This method shows also that a *necessary* condition under which the homology groups for $i = 0, 1, \ldots, n-1$ are finite-dimensional is the following: the *depth* of the ideal, generated by the coefficients of the vector field and defining equations, is not less than the dimension of the ambient manifold. In particular, for a Cohen-Macaulay singularity this means that the *grade* of the ideal $J_{\mathfrak{o}}\mathcal{V}$ is equal to the dimension of the singularity (cf. Comment 2 in Section 9).

11. Holomorphic and meromorphic indices

In conclusion we discuss some useful properties and relations between the indexes of complexes of holomorphic and regular meromorphic differential forms. Recall that for isolated complete intersection singularities there is an equality $\chi(\widetilde{\Omega}^{\bullet}_A, \iota_{\mathcal{V}}) = \chi(\omega^{\bullet}_A, \iota_{\mathcal{V}})$ (see Section 4).

Theorem 2. *For a normal isolated complete intersection singularity of dimension* $n \ge 2$ *there exist natural isomorphisms*

$$H_i(\Omega^{\bullet}_A, \iota_{\mathcal{V}}) \cong H_i(\omega^{\bullet}_A, \iota_{\mathcal{V}}), \quad 0 \le i \le n-2,$$

and the following exact sequence

$$0 \to H_n({}^{\#}\omega^{\bullet}_A) \to H_{n-1}(\Omega^{\bullet}_A, \iota_{\mathcal{V}}) \to H_{n-1}(\omega^{\bullet}_A, \iota_{\mathcal{V}}) \to H_{n-1}({}^{\#}\omega^{\bullet}_A) \to 0$$

which induces an equality $\dim_k H_{n-1}(\Omega^{\bullet}_A, \iota_{\mathcal{V}}) = \dim_k H_{n-1}(\omega^{\bullet}_A, \iota_{\mathcal{V}})$.

Proof. By Proposition 2 one has $\omega^p_A \cong {\Omega^p_A}^{\vee\vee} \cong \Omega^p_A$ for $0 \le p < n$. This gives us the required isomorphisms for all $0 \le i < n-2$.

On the other hand, in view of Proposition 8 one has $H_n(\widetilde{\Omega}^{\bullet}_A, \iota_{\mathcal{V}}) = H_n(\omega^{\bullet}_A, \iota_{\mathcal{V}}) = 0$. Hence the exact sequence of complexes

$$0 \longrightarrow \widetilde{\Omega}^{\bullet}_A \longrightarrow \omega^{\bullet}_A \longrightarrow {}^{\#}\omega^{\bullet}_A \longrightarrow 0$$

induces the long exact sequence of higher-dimensional homology groups

$$\begin{aligned} 0 \to H_n({}^{\#}\omega^{\bullet}_A) \to H_{n-1}(\Omega^{\bullet}_A, \iota_{\mathcal{V}}) \to H_{n-1}(\omega^{\bullet}_A, \iota_{\mathcal{V}}) \to \\ \to H_{n-1}({}^{\#}\omega^{\bullet}_A) \to H_{n-2}(\Omega^{\bullet}_A, \iota_{\mathcal{V}}) \to H_{n-2}(\omega^{\bullet}_A, \iota_{\mathcal{V}}) \to 0. \end{aligned} \tag{16}$$

Since the dimensions of the two non-trivial cotorsion modules are equal, this implies an equality

$$\dim H_{n-2}(\Omega^{\bullet}_A, \iota_{\mathcal{V}}) - \dim H_{n-1}(\Omega^{\bullet}_A, \iota_{\mathcal{V}}) = \dim H_{n-2}(\omega^{\bullet}_A, \iota_{\mathcal{V}}) - \dim H_{n-1}(\omega^{\bullet}_A, \iota_{\mathcal{V}}). \tag{17}$$

In addition, we get an inequality $\dim_k H_{n-2}(\Omega^{\bullet}_A, \iota_{\mathcal{V}}) \ge \dim_k H_{n-2}(\omega^{\bullet}_A, \iota_{\mathcal{V}})$ for evident reasons.

Similarly, the long Ker-Coker sequence, associated with the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widetilde{\Omega}^n_A & \xrightarrow{c_A} & \omega^n_A & \longrightarrow & {}^{\#}\omega^n_A & \longrightarrow & 0 \\ & & \downarrow \iota_{\mathcal{V}} & & \downarrow \iota_{\mathcal{V}} & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Ker}(\Omega^{n-1}_A \to \Omega^{n-2}) & \longrightarrow & \mathrm{Ker}(\omega^{n-1}_A \to \omega^{n-2}) & \longrightarrow & {}^{\#}\omega^{n-1}_A, & & \end{array}$$

gives us an exact sequence of finite-dimensional vector spaces

$$0 \longrightarrow H_n(^{\#}\omega^{\bullet}_A) \longrightarrow H_{n-1}(\Omega^{\bullet}_A, \iota_{\mathcal{V}}) \longrightarrow H_{n-1}(\omega^{\bullet}_A, \iota_{\mathcal{V}}) \longrightarrow H_{n-1}(^{\#}\omega^{\bullet}_A).$$

Again, since the dimensions of both cotorsion modules are equal, one obtains the inequality $\dim_k H_{n-1}(\Omega^{\bullet}_A, \iota_{\mathcal{V}}) \geq \dim_k H_{n-1}(\omega^{\bullet}_A, \iota_{\mathcal{V}})$.

At last, combining the above two inequalities with relation (17), one gets

$$\dim_k H_{n-2}(\Omega^{\bullet}_A, \iota_{\mathcal{V}}) = \dim_k H_{n-2}(\omega^{\bullet}_A, \iota_{\mathcal{V}}), \quad \dim_k H_{n-1}(\Omega^{\bullet}_A, \iota_{\mathcal{V}}) = \dim_k H_{n-1}(\omega^{\bullet}_A, \iota_{\mathcal{V}}).$$

In view of (16) the first equality implies an isomorphism $H_{n-2}(\Omega^{\bullet}_A, \iota_{\mathcal{V}}) \cong H_{n-2}(\omega^{\bullet}_A, \iota_{\mathcal{V}})$, and the exact sequence of the Theorem. QED.

Remark. For isolated complete intersection singularities we have already proved earlier that $H_n(\Omega^{\bullet}_A, \iota_{\mathcal{V}}) \cong \operatorname{Tors}\Omega^{\bullet}_A$, and $H_n(\widetilde{\Omega}^{\bullet}_A, \iota_{\mathcal{V}}) = H_n(\omega^{\bullet}_A, \iota_{\mathcal{V}}) = 0$. Further, for a vector field $\mathcal{V}$ of weight 1 on a surface Q_7-singularity computations in the above examples give us two Poincaré polynomials for first homology groups:

$$P(H_1(\Omega^{\bullet}_A, \iota_{\mathcal{V}}); x) = 2x^2 + 2x^3, \quad P(H_1(\omega^{\bullet}_A, \iota_{\mathcal{V}}); x) = 1 + 3x.$$

Hence, $H_1(\Omega^{\bullet}_A, \iota_{\mathcal{V}}) \ncong H_1(\omega^{\bullet}_A, \iota_{\mathcal{V}})$, although the dimensions of both homology groups are equal.

Remark. For a normal reduced complete intersection with *non-isolated singularities* the Lebelt resolutions exist for all Ω^p_A, $0 \leq p \leq c-1$, where c is the codimension of the singular subspace. Moreover, there are natural isomorphisms:

$$H_i(\Omega^{\bullet}_A, \iota_{\mathcal{V}}) \cong H_i(\omega^{\bullet}_A, \iota_{\mathcal{V}}), \quad 0 \leq i < c-1.$$

Claim 7. *Let D be the germ of a reduced normal hypersurface and $\mathcal{V}$ a vector field with isolated singularities. Then one has*

$$\chi(\Omega^{\bullet}_D, \iota_{\mathcal{V}}) = \chi(\omega^{\bullet}_D, \iota_{\mathcal{V}}).$$

Proof. If D is the germ of a hypersurface with an *isolated* singularity of dimension $n \geq 1$, then

$$H^{n-1}_{\{\mathfrak{o}\}}(\Omega^1_D) \cong H^0_{\{\mathfrak{o}\}}(\Omega^n_D) \cong \operatorname{Tors}\Omega^n_D, \quad \Omega^{n+1}_D \cong \operatorname{Tors}\Omega^{n+1}_D \cong H^1_{\{\mathfrak{o}\}}(\Omega^n_D)$$

(see [27, Pt. I]) and these modules are, in fact, τ-dimensional vector spaces, where τ is the *Tjurina* number of D. In the case of hypersurfaces with *non-isolated* singularities the statement follows from considerations in [6], where the logarithmic index is introduced and some relations between this index and the index of the contracted complex of regular meromorphic differential forms via the residue map are discussed. QED.

Remark. In fact, the logarithmic index is computed in the ambient space of a singularity in contrast with the regular meromorphic index and the residue map connects both realizations. In a more general context this idea leads to the study of *multi-logarithmic* differential forms [7], their residues and properties of the multi-logarithmic index.

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JACOBIAN MATES FOR NON-SINGULAR POLYNOMIAL MAPS IN $\mathbb{C}^n$ WITH ONE–DIMENSIONAL FIBERS

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A nuestro maestro Xavier Gómez-Mont, con gratitud

Abstract. Let $(F_2, \ldots, F_n) : \mathbb{C}^n \to \mathbb{C}^{n-1}$ be a non-singular polynomial map. We introduce a non-singular polynomial vector field X tangent to the foliation $\mathcal{F}$ having as leaves the fibers of the map $(F_2, \ldots, F_n)$. Suppose that the fibers of the map are irreducible in codimension ≥ 2, that the one forms of time associated to the vector field X are exact along the leaves, and that there is a finite set at the hyperplane at infinity containing all the points necessary to compactify the affine curves appearing as fibers of the map. Then, there is a polynomial F_1 (a Jacobian mate) such that the completed map $(F_1, F_2, \ldots, F_n)$ is a local biholomorphism. Our proof extends the integration method beyond the known case of planar curves (introduced by Ilyashenko [Ily69]).

1. Introduction and Statement of Results

The topological or analytical classification of non-singular polynomial foliations in $\mathbb{C}^n$ is a very hard problem, even in the lowest dimensional case $n = 2$. See [ACL98], [BT06], [Fer05], [NN02], [Tib07] and references therein.

We study the (holomorphic) polynomial foliations by curves $\mathcal{F}$ in $\mathbb{C}^n$ which can be obtained from the fibers of complex polynomials $F_2, \ldots, F_n \in \mathbb{C}[z_1, \ldots, z_n]$, chosen in such a way that

$$(1) \qquad \begin{cases} (F_2, \ldots, F_n) : \mathbb{C}^n \longrightarrow \mathbb{C}^{n-1} \qquad \text{and} \\ \\ dF_2 \wedge \cdots \wedge dF_n \qquad \text{does not vanish at any } z \in \mathbb{C}^n. \end{cases}$$

The fibers of the map in (1) are nonsingular, but possibly reducible, affine curves that we denote by $\{\mathcal{A}_c\}$. The leaves of $\mathcal{F}$ are the connected components (a unique one generically) of those affine curves. We say that $\mathcal{F}$ is a *non-singular polynomial foliation having* $n-1$ *first integrals.*

As a first step toward a general classification a natural problem is to study topologically or analytically this family of foliations.

An interesting subfamily is as follows. The map $(F_2, \ldots, F_n)$ has a *Jacobian mate* when there exists a polynomial $F_1 \in \mathbb{C}[z_1, \ldots, z_n]$ such that

$$(2) \qquad \begin{cases} F = (F_1, F_2, \ldots, F_n) : \mathbb{C}^n \longrightarrow \mathbb{C}^n \qquad \text{and} \\ \\ dF_1 \wedge dF_2 \wedge \cdots \wedge dF_n = \, dz_1 \wedge \cdots \wedge dz_n. \end{cases}$$

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Recall that the Jacobian Conjecture in $\mathbb{C}^n$ asserts the existence of the inverse map F^{-1} (which has to be also polynomial).

Given $\mathcal{F}$, where are the obstructions to the existence of F_1?

Note that the singularities of the extended foliation to projective space, still denoted by $\mathcal{F}$, are in the hyperplane at infinity of $\mathbb{C}^n$. In the classification problem one can study the singularities at infinity. Instead, our approach focus on the affine behavior and possible "jumps" in the geometry of the fibres $\{\mathcal{A}_c\}$. By a classical result of S. A. Broughton, see [Bro83], there exists an open Zariski set $U \subset \mathbb{C}^{n-1}$ such that the affine foliation $\mathcal{F}$ is a locally trivial fibration in $(F_2, \ldots, F_n)^{-1}(U)$.

Hence we must consider a priori the existence of atypical fibers (i.e. fibers outside U) of (1) and try to describe the behavior of $\mathcal{F}$. In particular, we point out that an example of (1) having atypical fibers and admitting a F_1, will provide a counterexample for the Jacobian Conjecture.

Another related problems with the existence of a Jacobian mate are the following. First, in the holomorphic category, on Stein manifolds the problem of the existence of F_1 is posed in [For03a] p. 146 and [For03b] p. 96, and it remains open (we thank Filippo Bracci for pointing this out to us). Second, the symmetric problem, i.e. given F_1 how to recognize the existence of $(F_2, \ldots, F_n)$ such that (2) is currently under study for $n \geq 3$, see [FR05] p. 3 or [Kal02].

The main tool that we introduce is a polynomial vector field X depending in an essential way of $\mathcal{F}$. Consider the Jacobian matrix of the map (1)

$$\left(\frac{\partial F_j}{\partial z_i}\right)_{2\leq j\leq n,\ 1\leq i\leq n},$$

and let $A_i(z_1, \ldots, z_n)$ be the determinant of the submatrix obtained after removing the i–th column, then

$$X := \sum_{i=1}^{n} (-1)^{i+1} A_i(z_1, \ldots, z_n)\frac{\partial}{\partial z_i}, \tag{3}$$

obviously X is nowhere zero. If there exists a Jacobian mate F_1, then

$$(F_1, \ldots, F_n)^* \frac{\partial}{\partial w_1} = X. \tag{4}$$

X restricted to any fiber $\mathcal{A}_c$, $c \in \mathbb{C}^{n-1}$, of the map (1), gives a tangent vector field on $\mathcal{A}_c$, that we will denote by X_c. It determines a unique holomorphic one form ω_c on $\mathcal{A}_c$, when we require $\omega_c(X_c) = 1$. Thus, each map $(F_2, \ldots, F_n)$ produces a collection of pairs

$$\{(\mathcal{A}_c,\ X_c) \mid c \in \mathbb{C}^{n-1}\}, \qquad \text{equivalently } \{(\mathcal{A}_c,\ \omega_c)\}. \tag{5}$$

In Section 2, we briefly develop this ideas to make the argument more transparent.

Remark 1. *1. The vector field X defines a singular holomorphic foliation $\mathcal{F}$ by curves in $\mathbb{CP}^n$, such that its singular locus is contained in the hyperplane at infinity $\mathbb{CP}^{n-1}_\infty$.*
2. The polynomial vector field X has $n-1$ polynomial first integrals on $\mathbb{C}^n$, and the leaves of the foliation defined by X in $\mathbb{C}^n$ are given by the curves $\{\mathcal{A}_c \mid c \in \mathbb{C}^{n-1}\}$.
3. The hyperplane $\mathbb{CP}^{n-1}_\infty$ is saturated by leaves of $\mathcal{F}$.

In addition

Remark 2. *Up to multiplication by a non-zero constant, X is the unique non vanishing polynomial vector field giving a trivialization for the tangent line bundle of the non-singular holomorphic foliation $\mathcal{F}$ on $\mathbb{C}^n$.*

Indeed, if a second polynomial vector field Y (providing a trivialization of the tangent line bundle to the foliation) exists, then $X = \lambda Y$, for λ an entire function on $\mathbb{C}^n$, nowhere zero. But λ is clearly polynomial, hence it is necessarily a non-zero constant. Moreover, X is independent on the choice of any polynomial F_1 satisfying (2): it only depends on $(F_2, \ldots, F_n)$. Hence, we can use X to explore the existence of F_1.

The main result about affirmative conditions for the existence of F_1, is the following

Theorem 1. *Let $(F_2, \ldots, F_n) : \mathbb{C}^n \longrightarrow \mathbb{C}^{n-1}$ be a polynomial map such that $dF_2 \wedge \cdots \wedge dF_n$ does not vanish at any point of $\mathbb{C}^n$. Consider X as in (3), and suppose furthermore that:*
(i) The reducible fibers $\{\mathcal{A}_c\} \subset \mathbb{C}^n$ determine an algebraic subset of codimension at least 2.
(ii) For every $c \in \mathbb{C}^{n-1}$, the pairs $(\mathcal{A}_c,\ \omega_c)$ satisfy that

$$\int_\gamma \omega_c = 0, \quad \textit{for every } [\gamma] \in H_1(\mathcal{A}_c, \mathbb{Z}).$$

(iii) There is a finite set $Y \subset \mathbb{CP}^n_\infty$ such that each affine curve $\mathcal{A}_c$ is completed in $\mathbb{CP}^n$ by adding points in Y.
Then, there is a polynomial F_1 such that

$$dF_1 \wedge dF_2 \wedge \cdots \wedge dF_n = dz_1 \wedge \cdots \wedge dz_n.$$

Note that the second hypothesis is clearly necessary for ω_c to be an exact one form on the fibers $\mathcal{A}_c$. Concerning the first, it is in fact necessary for the integration method that we use: Example 1 shows a function with a reducible fiber (of codimension one), with zero periods, and such that the function constructed by integration as a candidate for Jacobian mate has a pole on that fiber (see Remark 6).

The third hypothesis, obviously satisfied in the case $n = 2$, is automatically satisfied in case that the map F is surjective. In this case, as F has no critical points, all the fibers are one dimensional, and according to [Ga99] p. 158, they share the same cone at infinity, i.e. all the affine curves are completed by adding the same points at infinity (a finite set). Note that this cone at infinity is defined by the vanishing of the polynomials in the ideal generated by the terms of highest degree of the elements of the ideal generated by the components of the function F. This cone at infinity is contained in, but not necessary equal to, the singular set of the foliation $\mathcal{F}$ extended to projective space.

After proving our result by integration method (see below), we realized that in case F is surjective, it is a consequence of a Theorem of Ph. Bonnet (Theorem 1.5 in [Bon03]). Nevertheless, even in that case, as his approach is algebraic, and our proof extends the integration method beyond the case of planar curves previously known (starting with Ilyashenko [Ily69]), we consider that it can be of interest for the people working in the field. Moreover, with this technique as a fundamental tool, together with some considerations on the degree of the map F and computations of the index of X restricted to the fibers of $(F_2, \ldots, F_n)$ (see the end of this Introduction), we have also obtained some new results on negative conditions for the existence of a jacobian mate. They will be presented in a future work, including the solution in a particular case (see Example 1) of a problem posed by L. Dũng Tráng and C. Weber in [DW94].

1.1. **Method and Structure of the proof.** The proof of Theorem 1 is given in several steps below. Note that, to avoid confusion we use $\mathbb{C}^n_z$ and $\mathbb{C}^{n-1}_w$ to denote the domain and the target in map (1).
Step 1. We construct a polynomial one form of time ω for X on $\mathbb{C}^n_z$. By integration of ω along the irreducible fibers of F, see equation (8), we get a candidate function $\widetilde{F}_1$.
Step 2. We verify that the candidate function is holomorphic on the whole $\mathbb{C}^n_z$, see Proposition

1.
Step 3. We estimate the growth of $\widetilde{F}_1$. This is the hardest step. We will study the growth of $\widetilde{F}_1$ at infinity. We recognize the growth of $|\widetilde{F}_1(z)|$ in a suitable set of complex lines in $\mathbb{C}^n_z$. This requires bounds for: the norm of the end points of the integration paths in (8), see Lemma 1, the norm of the ramification points in the fibers $\{\mathcal{A}_c\}$, see Lemma 2, and the length of the integration paths in (8), see Lemma 3. Thus, $|\widetilde{F}_1(z)|$ has polynomial growth in suitable lines, see Lemma 4.
Step 4. In order to show that $\widetilde{F}_1(z)$ is a polynomial, we make an argument by contradiction, using a property of the growth of entire non-polynomial functions, see Lemma 5 and Proposition 2. We show explicitly that F_1 satisfies $dF_1 \wedge dF_2 \wedge \ldots \wedge dF_n = dz_1 \wedge \ldots \wedge dz_n$.

Concerning the proof of Theorem 1, we point out that the powerful method of integration of one forms ω along the algebraic leaves of a polynomial foliation $\{\mathcal{A}_c\}$ in $\mathbb{C}^2$ to find F_1, was introduced by Yu. Ilyashenko, in his foundational work on the second part of the Hilbert's 16–th problem [Ily69]; see also Yakovenko's article [Yak94], that inspired us when searching for the estimates in Step 3 above. The higher dimensional method of integration of rational one forms ω along the leaves of singular codimension–one foliations in higher dimensional affine and projective manifolds appeared in the work [Muc95] of the third author of this article. In our Theorem 1, the bounds for the integration of one forms along the leaves of an one–dimensional foliation on $\mathbb{C}^n$ is more difficult.

2. Meromorphic maps and vector fields on Riemann surfaces

Let $\mathbb{CP}^1 = \mathbb{C}_w \cup \{\infty\}$ be the projective line, having affine coordinate w. The vector field $\partial/\partial w$ induces a holomorphic vector field in $\mathbb{CP}^1$ having double zero at $\infty \in \mathbb{CP}^1$. Let $\mathcal{L}$ be a compact Riemann surface.

Remark 3. *Let $f : \mathcal{L} \to \mathbb{CP}^1$ be a non-constant meromorphic function. The non-identically zero meromorphic vector field*

$$\frac{\partial}{\partial f} := f^*\left(\frac{\partial}{\partial w}\right)$$

is well defined on $\mathcal{L}$. Moreover, f has canonically associated a meromorphic one form ω, such that the diagram commutes

$$\begin{array}{ccc} & \nearrow & \{\omega\} \\ \{f : \mathcal{L} \to \mathbb{CP}^1\} & & \updownarrow \\ & \searrow & \{X = \frac{\partial}{\partial f}\}, \end{array} \tag{6}$$

X and ω are non-identically zero.

In fact, given f, we define $\omega = df$. The one to one correspondence between meromorphic vector fields and meromorphic one forms is given by the equation $\omega(X) \equiv 1$. This ω is called the one form of time for X, since for $p_0,\ p \in \mathcal{L}$ we have

$$f(p) - f(p_0) = \int_{p_0}^{p} \omega = \left\{ \begin{array}{l} \text{complex time to travel from} \\ p_0 \text{ to } p \text{ under the local flow of } \frac{\partial}{\partial f}. \end{array} \right.$$

The diagram (6) comes from the theory of quadratic differentials, see [Muc02]. The correspondence from ω to f in (6) is elementary.

Remark 4. *A non-identically zero meromorphic one form ω, determines a univalued meromorphic function $f(p) = \int^p \omega$ if and only if the periods and residues of ω vanish, i.e.*

$$\int_\gamma \omega = 0 \qquad \textit{for each } [\gamma] \in H_1(\mathcal{L} - \{\textit{poles of } \omega\}, \mathbb{Z}).$$

In this case, all the arrows in (6) are bijections.

3. Proof of Theorem 1

Starting from the map $(F_2, \ldots, F_n)$ satisfying (1), we get the associated vector field X described by (3) in the Introduction.

3.1. **A candidate function.** For the construction of a polynomial one form of time ω, we show that the one form ω_c on $\mathcal{A}_c$ such that $\omega_c(X_c) = 1$ can be obtained as the restriction to the fiber $\mathcal{A}_c$ of a polynomial one form on $\mathbb{C}^n_z$.

Indeed, as X is never vanishing, recall equation (3), by Hilbert's Nullstellensatz we know that $1 \in (A_1, \ldots, A_n)$. Then, there are polynomials $a_1, \ldots, a_n \in \mathbb{C}[z_1, \ldots, z_n]$ such that

$$1 = a_1 A_1 + \cdots + a_n A_n.$$

These a_i are the coefficients of such an ω.

Observe that if $\{a'_i \mid i = 1, \cdots, n\}$ are polynomials giving another possible way of defining a one form ω' such that $\omega'|_{\mathcal{A}_c} = \omega_c$, then

$$(a_1 - a'_1)A_1 + \cdots + (a_n - a'_n)A_n = 0.$$

Hence, for ω on $\mathbb{C}^n_z$ (as above) and every path γ in $\mathcal{A}_c$ we have

$$\int_\gamma \omega_c = \int_\gamma \omega.$$

The third hypothesis in the statement of theorem asserts that there is a finite set

$$Y = \{\text{the points at infinity of the projective curves } \mathcal{P}_c \mid c \in \mathbb{C}^{n-1}_w\} \subset \mathbb{CP}^{n-1}_\infty,$$

so that we can choose a hyperplane H in $\mathbb{CP}^n$ such that $H \cap Y = \emptyset$. We can also assume that it is not contained in the union of the projective varieties given by the closures of the affine hypersurfaces defined by $A_i = 0$, $i = 1, \ldots, n$.

We consider the open set

$$\mathcal{R}^c = \mathbb{C}^n_z - \{\mathcal{A}_c \mid \text{reducible }\}.$$

Every point $z \in \mathcal{R}^c$ is in exactly one affine curve of the family $\{\mathcal{A}_c\}$ which we denote as $\mathcal{A}_{c(z)}$, where $c(z) := (c_2(z), \ldots, c_n(z)) := (F_2, \ldots, F_n)(z) \in \mathbb{C}^{n-1}_w$. The degree of the projective curve $\mathcal{P}_{c(z)}$ (the projectivization of $\mathcal{A}_{c(z)}$) is

$$d \leq d_2 \cdots d_n, \quad \text{where} \quad d_j = (\textit{degree } F_j).$$

We have in addition that $H \cap \mathcal{P}_{c(z)}$ consists of d points in $\mathbb{C}^n_z$, counted with multiplicities, for every $c(z) \in \mathbb{C}^{n-1}_w$. Therefore, we have

$$H \cap \mathcal{A}_{c(z)} = \{p_1(z), \ldots, p_d(z)\}.$$

By hypothesis $\mathcal{A}_{c(z)}$ is irreducible, having fixed some point $z \in \mathbb{C}^n_z$, we join z to the above points $p_\ell(z) \in H$, by using smooth paths $\gamma_\ell(z)$, $\ell = 1, \ldots, d$. inside the affine curve $\mathcal{A}_{c(z)}$. We observe that

$$f_c : \begin{array}{ccc} \mathcal{A}_{c(z)} & \longrightarrow & \mathbb{C} \\ z & \mapsto & \sum_{\ell=1}^d \int_{\gamma_\ell(z)} \omega \end{array} \tag{7}$$

is a well–defined holomorphic function, independently of the choices of paths, using the second hypothesis in Theorem 1. Moreover, f_c extends as a meromorphic function on the projective curve $\mathcal{P}_{c(z)}$ and there is a well–defined function

$$\widetilde{F}_1 : \begin{array}{ccc} \mathcal{R}^c \subset \mathbb{C}^n_z & \longrightarrow & \mathbb{C} \\ z & \mapsto & \sum_{\ell=1}^{d} \int_{\gamma_\ell(z)} \omega. \end{array} \tag{8}$$

3.2. **Holomorphicity of the candidate function.**

Proposition 1. *$\widetilde{F}_1(z)$ is holomorphic on the whole $\mathbb{C}^n_z$.*

Proof. Clearly $\widetilde{F}_1(z)$ is holomorphic along the irreducible curves $\mathcal{A}_{c(z)} \subset \mathcal{R}^c$. Now, we prove the holomorphicity of $\widetilde{F}_1$ at z_0 in the transverse directions to $\mathcal{F}$. We will distinguish two situations, depending on the number of points in the intersection of the leaf and the transversal H.

Case 1. Assume $\mathcal{A}_{c(z_0)} \cap H$ consists exactly of d different points. Thus, the leaf $\mathcal{A}_{c(z_0)}$ is transverse to H.

We may consider without loss of generality, that Σ^{n-1} is a tranversal to $\mathcal{F}$ at z_0, i.e. biholomorphic to some $(n-1)$–dimensional polydisk $\Delta^{n-1}(z_0, \epsilon)$, centered at z_0, embedded in $\mathbb{C}^n_z$ and transversal to $\mathcal{F}$. Now, let z be a point in the transverse directions $j = 2, \dots, n$, i.e. z is inside the polydisk Σ^{n-1}.

Fixed z_0, we consider the leaf $\mathcal{A}_{c(z_0)}$ and the smooth integration paths $\gamma_\ell(z_0)$, $\ell \in \{1, \dots, d\}$, inside the leaf $\mathcal{A}_{c(z_0)}$.

Since the foliation $\mathcal{F}$ is non-singular on $\mathcal{A}_{c(z_0)}$, a small variation of z in Σ^{n-1}, a transversal at z_0, induces a small (smooth) variation in the paths of integration in (8). In fact, the holonomy of the foliation $\mathcal{F}$ produces germs of biholomorphisms

$$hol(\gamma_\ell(z_0), \ \cdot\ ) : (\Sigma^{n-1}, z_0) \to (\Sigma^{n-1}_\ell, p_\ell(z_0)) \ , \ \ell \in \{1, \dots, d\},$$

where each $\Sigma^{n-1}_\ell \subset H$ is a local transversal to the foliation $\mathcal{F}$, given by a small $(n-1)$–dimensional polydisk in the hyperplane H centered at $p_\ell(z_0)$, the end point of the path $\gamma_\ell(z_0)$.

If z moves holomorphically in Σ^{n-1} around z_0, the respective end points of the paths $\gamma_\ell(z)$ move holomorphically in $\Sigma^{n-1}_\ell \subset H$, since the end points are given as the values of the biholomorphism $hol(\gamma_\ell(z_0), z) \in \Sigma^{n-1}_\ell$.

Summing up, the end points of the integration paths in (8) and the one form ω vary holomorphically with z. Thus, $\widetilde{F}_1$ is holomorphic in all directions around z_0, when $\mathcal{A}_{c(z_0)}$ is transverse to H.

Case 2. Assume $\mathcal{A}_{c(z_0)} \cap H$ consists of less than d different points. Thus, the leaf $\mathcal{A}_{c(z_0)}$ is tangent to H at some points.

The set $T = \{z \in \mathbb{C}^n_z \mid \mathcal{A}_{c(z)} \text{ is tangent to } H\}$ is a complex algebraic variety in $\mathbb{C}^n_z$ of codimension least one (T can be empty).

Recall that in (8) the intersection points $\mathcal{A}_{c(z_0)} \cap H$ are taken with multiplicities. It follows that $\widetilde{F}_1(z)$ extends continuously to T and is locally bounded at T. By the Riemann extension Theorem (see [FG02], p. 38), $\widetilde{F}_1(z)$ extends holomorphically to T, hence on $\mathcal{R}^c$.

Finally, the reducible fibers $\{\mathcal{A}_c\} \subset \mathbb{C}^n_z$ determine an algebraic subset of codimension at least 2. By the Second Riemann extension Theorem (see [FG02], p. 151), $\widetilde{F}_1(z)$ extends holomorphically over the points in reducible fibers, hence on the whole $\mathbb{C}^n_z$. □

3.3. **The candidate has polynomial growth.** We will prove that $\widetilde{F}_1(z)$ is a polynomial function. For this we study the growth of $|\widetilde{F}_1(z)|$, when $|z|$ goes to infinity along some lines, this in our goal in this subsection.

Let $\varrho = [m_1 : \ldots : m_n] \in \mathbb{CP}^{n-1}_\infty$ be a non-singular point of the foliation $\mathcal{F}$ in the hyperplane at infinity. We make z go to ϱ in a simple way. Let $(\{t \in \mathbb{C}\} \cup \{\infty\})$ be a projective line, consider the parametrized line

$$z(t) : (\mathbb{C} \cup \{\infty\}) \to \mathbb{CP}^n, \tag{9}$$

$$z(t) = \begin{cases} (m_1 t, \ldots, m_n t) & \text{for } t \in \mathbb{C}, \\ \varrho & \text{for } t = \infty. \end{cases}$$

In all what follows
1) z and $|z|$ will go to infinity as $z = z(t)$,
2) $(F_2, \ldots, F_n)(z(t)) := (c_2(t), \ldots, c_n(t)) := c(t) \in \mathbb{C}^{n-1}_w$,
3) $\mathcal{A}_{c(t)} := \mathcal{A}_{c(z(t))}$ and $H \cap \mathcal{A}_{c(t)} := \{p_1(t), \ldots, p_d(t)\}$.

In order to estimate the growth of

$$|\widetilde{F}_1(z(t))| \ = \left| \sum_{\ell=1}^{d} \int_{\gamma_\ell(z)} \omega \right|,$$

we will first construct integration paths

$$\gamma_\ell(z(t), s) : [0,1] \to \mathcal{A}_{c(t)}$$

inside the family of curves $\{\mathcal{A}_{c(t)}\}$ and bound their lengths in terms of $|t|$ (see Lemma 3). Note that we are using the notations

$$\gamma_\ell(z) = \gamma_\ell(z(t)) = \gamma_\ell(z(t), s)$$

simultaneously, the dependence on t will be continuous, and smooth on the real variable s. We will bound the growth of ω along the path, from a bound on $|\gamma_\ell(z(t), s)|$ for all the points in the trace of the paths, this is attained in the proof of Lemmas 1, 2.

The construction of the integration paths require formerly, the study of the projections of $\mathcal{A}_{c(t)}$ onto the coordinate axes.

Consider the natural projections $\Pi_i : \mathbb{C}^n_z \to \mathbb{C}_i$, $(z_1, \ldots, z_n) \mapsto z_i$, onto the i–th axis. Obviously, they induce functions for every fixed t,

$$\Pi_i : \mathcal{A}_{c(t)} \to \mathbb{C}_i \, ,$$

which are holomorphic branched coverings. Moreover in some special cases for $\mathcal{F}$ these functions can be constant.

Fixing t, and so the fiber $\mathcal{A}_{c(t)}$, and one direction of projection $i \in \{1, \ldots, n\}$ as above, we have two relevant sets of points and their corresponding associated disks in $\mathbb{C}_i$, having radii $r(i,t),\ R(i,t) > 0$ as follows:

The first collection of points and associated disks comes from the i–th projection of $z(t)$ and of the intersection points of $\mathcal{A}_{c(z)}$ with H

$$\{\Pi_i(z(t)), \Pi_i(p_1(t)), \ldots, \Pi_i(p_d(t))\} \subset \Delta(0, r(i,t)) \subset \mathbb{C}_i.$$

The second collection of points is determined by the ramification points of the function $\Pi_i : \mathcal{A}_{c(t)} \to \mathbb{C}_i$, $\{\rho_1(i,t), \ldots, \rho_\beta(i,t)\} \subset \mathcal{A}_{c(t)}$, and its projection to the i–th coordinate:

$$\{\Pi_i(\rho_i(i,t)), \ldots, \Pi_i(\rho_\beta(i,t))\} \subset \Delta(0, R(i,t)) \subset \mathbb{C}_i.$$

The number β depends on $\mathcal{A}_{c(t)}$ and Π_i, but we omit this dependence in the notation. By (b) of Corollary 1 below, β will be constant for large enough t.
So, fixed the i–th direction, our problem is *"for $z(t)$ going to fixed $\varrho \in \mathbb{CP}^{n-1}_\infty$, bound the growth of the radii $r(i,t)$, $R(i,t)$ for all sufficiently large $|t|$"*.

Now, we work in order to estimate of the radious $r(i,t)$.

Lemma 1. *Fixing $i \in \{1, \dots, n\}$, there exists $\xi \in \mathbb{N}$ such that $r(i,t) < |t|^{\xi}$ for large enough t. Moreover, this estimate holds for $z(t)$ going to infinity in the directions ϱ' in a small enough polydisk $\Delta^{n-1}(\varrho, \varepsilon) \subset \mathbb{CP}^{n-1}_{\infty}$ avoiding singularities of $\mathcal{F}$.*

Proof. The intersection points in $H \cap \mathcal{A}_{c(t)}$ are described by the system of algebraic equations in $(z_1, \dots, z_n)$,

$$F_2 - c_2(t) = 0,\ F_3 - c_3(t) = 0,\ \dots,\ F_n - c_n(t) = 0,\ H(z_1, \dots z_n) = 0.$$

For fixed t and i, we want to compute the values of z_i where these intersections appear. The elimination ideal of the above system (see [CLO07] Chapter 3, Section 2, in particular Theorem 3 p. 125), that by definition is

$$\langle F_2 - c_2(t), \dots, F_n - c_n(t), H \rangle \cap \mathbb{C}[z_i],$$

determines the required points.

The elimination procedure, described explicitly in [CLO07] p. 116–117, depends on the choice of a Groebner basis for the ideal of our system of equations (that always exists, see [CLO07] p. 77 Corollary 6). There is a polynomial

$$Q_i(z_i, t) = a_{i,0}(t) z_i^d + a_{i,1}(t) z_i^{d-1} + \cdots + a_{i,d-1}(t) z_i + a_{i,d}(t)$$

describing the position of $\{\Pi_i(p_1(t)), \dots, \Pi_i(p_d(t))\}$ in $\mathbb{C}_i$; here $\{a_{i,\alpha}(t)\}$ are polynomials in t, and d is the degree of the curves $\mathcal{A}_{c(t)}$.

The natural number $\xi(i) = \max_{\alpha}\{degree(a_{i,\alpha}(t))\}$ depends on the Groebner basis chosen. We can write

$$z_i^d + \frac{a_{i,1}(t)}{a_{i,0}(t)} z_i^{d-1} + \cdots + \frac{a_{i,d}(t)}{a_{i,0}(t)} = 0.$$

Recall that $a_{i,\alpha}(t)/a_{i,0}(t)$ are α–th elementary symmetric functions of the roots. The roots of $Q_i(z_i, t)$ grow at most as $\max_{\alpha}\{a_{i,\alpha}(t)/a_{i,0}(t)\}$, that is at most like $|t|^{\xi(i)}$, when t goes to infinity. So they are contained in a disk of radius $|t|^{\xi(i)}$.

The computation of the growth is similar for every $i \in \{1, \dots, n\}$. Let us define

$$\xi = (\max_i\{\xi(i)\}) + 1.$$

In addition, for the original point $z(t)$, the norm of the projection $|\Pi_i(z(t))|$ grows linearly, hence $\Pi_i(z(t)) \in \Delta(0, |t|^{\xi})$, for large enough t. The exponent ξ satisfies the assertion in the Lemma.

Finally, the bound is independent on the choice of ϱ' varying in a small enough polydisk $\Delta^{n-1}(\varrho, \epsilon)$, that is the second assertion in the Lemma. □

Now, we get the estimates for the radious $R(i,t)$ of the disks containing all the projections of the ramification points of Π_i restricted to $\mathcal{A}_{c(t)}$.

Lemma 2. *Fixing $i \in \{1, \dots, n\}$, there exists $\kappa \in \mathbb{N}$ such that $R(i,t) < |t|^{\kappa}$ for large enough t. Moreover, this estimate holds for $z(t)$ going to infinity in the directions ϱ' in a small enough polydisk $\Delta^{n-1}(\varrho, \varepsilon) \subset \mathbb{CP}^{n-1}_{\infty}$ avoiding singularities of $\mathcal{F}$.*

Proof. Observe that the ramification points of $\Pi_i : \mathcal{A}_{c(t)} \to \mathbb{C}_i$ come from the vanishing of the i–th coordinate of the vectors in the kernel of the differential of the map (1) at the points in $\mathcal{A}_{c(t)}$, which give the tangent space to $\mathcal{A}_{c(t)}$.

The condition above is given by the vanishing of the determinant of the matrix obtained by adding $(0, \dots, 1, \dots, 0)$, where the 1 is placed in the i–th column, as the last row to

$$\left(\frac{\partial F_j}{\partial z_i}\right)_{2 \le j \le n,\ 1 \le i \le n}.$$

This determinant is exactly $A_i(z_1, \ldots, z_n)$ in the definition of the vector field X, (3). Hence the tangencies of $\mathcal{A}_{c(t)}$ with the hyperplanes $\{z_i = const.\}$ in $\mathbb{C}^n_z$ are given by the following system of algebraic equations in $(z_1, \ldots, z_n)$

$$F_2 - c_2(t) = 0,\ F_3 - c_3(t) = 0,\ \ldots,\ F_n - c_n(t) = 0,\ A_i = 0.$$

For fixed t, we want to compute the i–th projection of the points where these tangencies appear. The elimination ideal of the above system

$$\langle F_2 - c_2(t), \ldots, F_n - c_n(t), A_i\rangle \cap \mathbb{C}[z_i],$$

determines the smallest algebraic variety containing the i–th projection of the ramification points of $\{\Pi_i(\rho_1(i,t)), \ldots, \Pi_i(\rho_\beta(i,t))\}$.

Using the elimination procedure and the existence of Groebner basis for the ideal as in the proof of Lemma 1, we know that there exists a polynomial

$$P_i(z_i, t) = b_{i,0}(t) z_i^\beta + \cdots + b_{i,\beta-1}(t) z_i + b_{i,\beta}(t)$$

whose roots give the projection of the ramification points above. The degree β is the number of ramification points of Π_i on $\mathcal{A}_{c(z)}$, and it is generically independent of i and t, for large enough t.

We can estimate the growth of the roots of $P_i(z_i, t)$ when t goes to infinity, as we did in the previous Lemma, so that we get a natural number $\kappa(i)$ (depending on the choice of the Grobner basis) such that they are contained in a disk of radius growing like $|t|^{\kappa(i)}$. Let us define

$$\kappa = (\max_i \{\kappa(i)\}) + 1;$$

this exponent provides the estimate in the Lemma. Finally, the bound is independent on the choice of ϱ' varying in a small enough polydisk $\Delta^{n-1}(\varrho, \epsilon)$, that is the second assertion in the Lemma. □

Summing up Lemmas 1 and 2, for the family of fibers $\mathcal{A}_{c(t)}$, we define the exponent $\varsigma := \max\{\xi, \kappa\}$. The n–dimensional polydisk $\Delta^n(0, |t|^\varsigma) \subset \mathbb{C}^n_z$, satisfies the following. The intersection

$$\mathcal{A}_{c(t)} \cap \Delta^n(0, |t|^\varsigma),$$

for large enough t, contains: the original point $z(t)$; the points $p_\ell(t)$, $\ell = 1, \ldots, d$ in $\mathcal{A}_{c(t)} \cap H$; and the ramification points $\rho_j(i,t)$, $j = 1, \ldots, \beta(i,t)$ of the functions $\Pi_i : \mathcal{A}_{c(t)} \to \mathbb{C}_i$, for all $i \in \{1, \ldots, n\}$.

Corollary 1. *There exists some t_0 such that for all $|t| > |t_0|$ the following facts hold.*
a) The intersection $\mathcal{A}_{c(t)} \cap \Delta^n(0, |t|^\varsigma)$ is a path connected Riemann surface.
b) The family of Riemann surfaces

$$\{\mathcal{A}_{c(t)} \cap \Delta^n(0, |t|^\varsigma) \mid |t| > |t_0|\}$$

is topologically trivial respect to t.

Proof. For a), note that there is always a direction such that the projection

$$\Pi_i : (\mathcal{A}_{c(t)} - \Delta^n(0, |t|^\varsigma)) \to \mathbb{C}_i$$

is a non-constant, unramified holomorphic covering. Without loss of generality, we can suppose $i = 1$. We remove from $\mathcal{A}_{c(t)}$ the preimages of the punctured closed disk $\{|z_1| \geq |t|^\varsigma\} \subset \mathbb{C}_1$. These preimages are disjoint punctured disks in $\mathcal{A}_{c(t)}$ (i.e. biholomorphic to $\Delta(0,1) - \{0\}$). Then, $\mathcal{A}_{c(t)} \cap \Delta^n(0, |t|^\varsigma)$ is path connected.

For b), the atypical fibers of each F_j, $j \in \{2, \ldots, n\}$, determine a finite number of hypersurfaces, see [Bro83]. The projective closure of each of them intersects the hyperplane at infinity in a hypersurface. If we choose the point at infinity ϱ, in (9) outside of these hypersurfaces in

$\mathbb{CP}^{n-1}_\infty$, then the family $\{\mathcal{A}_{c(t)}\}$ is locally trivial for suitable values of t. The assertion follows. □

Lemma 3. *There exists a point $\varrho \in \mathbb{CP}^{n-1}_\infty$, such that for $z(t)$ going to ϱ as in (9), we have a continuous family of smooth paths*

$$\left\{\gamma_\ell(z(t),\,\cdot) : [0,1] \to \mathcal{A}_{c(t)} \cap \Delta^n(0,|t|^\varsigma)\,,\ \ell = 1,\dots,d\,\right\}_{t\in U}\,,$$

where $U = \mathbb{C} - \overline{\Delta(0,M)}$, satisfying that for every ℓ, $\gamma_\ell(z(t),0) = z(t)$ and $\gamma_\ell(z(t),1) = p_\ell(t)$, as required for the paths in (8), with

$$(\textit{length of}\ \ \gamma_\ell(z(t),s)) < |t|^K,$$

for certain $K \in \mathbb{N}$.

Moreover, the above assertions are valid for $z(t)$ going to infinity in the directions ϱ' in a small enough polydisk $\Delta^{n-1}(\varrho,\varepsilon) \subset \mathbb{CP}^{n-1}_\infty$ avoiding singularities of $\mathcal{F}$.

Proof. Let us consider the polynomials A_i in the definition of the the vector field X, (3). Recall that $A_1,\dots,A_n$ do not vanish simultaneously. For a generic choice of $\varrho \in \mathbb{CP}^{n-1}_\infty$ and for large enough t, we have

$$A_i(z(t)) \neq 0,\ i \in \{1,\dots,n\}.$$

It follows that the initial points of the paths that we are searching for $\gamma_\ell(z(t),s)$ are not ramification points of Π_1, for large enough t. Observe that the projections $\Pi_i|_{\mathcal{A}_{c(t)}}$ are then ramified coverings over $\mathbb{C}_i$.

We also observe that for a generic choice of $\varrho \in \mathbb{CP}^{n-1}_\infty$ the paths $\gamma_\ell(z(t),s)$ will have no ramification points of Π_1 as end points $\{p_\ell(t)\}$ for large enough t. To see this, recall that the affine hyperplane determined by H is not contained in the hypersurface $A_i = 0$, for any i. Take a point $p \in H$ such that $A_i(p) \neq 0$, for every i. Clearly, the choice of p can be done in such a way that all the points in $F^{-1}(F(p)) \cap H$ satisfy the preceding condition.

Take the line through the origin in $\mathbb{C}^n$ determined by p, and let $\varrho \in \mathbb{CP}^{n-1}_\infty$ be the corresponding direction. For $z(t)$ going to infinity along this line, we define the algebraic affine surface

$$S^2 = \{\mathcal{A}_{c(t)} \mid t \in \mathbb{C}\}$$

given by the union of the fibers $\mathcal{A}_{c(t)}$ intersecting the line $\{z(t) \mid t \in \mathbb{C}\}$. In fact, $(F_2,\dots,F_n)(z(t)) : \mathbb{C} \to \mathbb{C}^{n-1}_w$ is a polynomial entire curve $\mathcal{C}$ and its closure $\overline{\mathcal{C}}$ is a rational projective curve in $\mathbb{CP}^{n-1}$. Consider $I(\mathcal{C}) = \langle g_1,\dots,g_\nu\rangle$ the affine ideal in $\mathbb{C}[w_2,\dots,w_n]$ describing $\mathcal{C}$ as an algebraic curve. The ideal $(F_2,\dots,F_n)^*I(\mathcal{C}) = \langle g_1\circ(F_2,\dots,F_n),\dots,g_\nu\circ(F_2,\dots,F_n)\rangle$ in $\mathbb{C}[z_1,\dots,z_n]$ determines S^2, showing that it is an algebraic surface.

By the conditions imposed in the choice of the direction ϱ along which $z(t)$ goes to infinity, we can assure that $\{A_1 = 0\} \cap H \cap S^2$ is at most a finite number of points. We get

$$\{A_1 = 0\} \cap H \cap \mathcal{A}_{c(t)} = \emptyset$$

for large enough t. It follows that the end points $\{p_\ell(t)\} = H \cap \mathcal{A}_{c(t)}$ are not ramification points for large enough t, as we asserted. Observe that this is still the case for ϱ' in a small enough polydisk $\Delta^{n-1}(\varrho,\varepsilon) \subset \mathbb{CP}^{n-1}_\infty$.

We take the polydisk $\Delta^n(0,|t|^\varsigma)$ in such a way that it contains $z(t)$, the points $p_1(t),\dots,p_d(t)$, and all the ramification points of the projection Π_1 in $\mathcal{A}_{c(t)}$ (see Lemmas 1 and 2). We will focus on the restricted map

$$\Pi_1 : \mathcal{A}_{c(t)} \cap \Delta^n(0,|t|^\varsigma) \to \Delta(0,|t|^\varsigma).$$

Recall that $\mathcal{A}_{c(t)} \cap \Delta^n(0,|t|^\varsigma)$ is path connected (Corollary 1, a). Choose a path from $z(t)$ to $p_\ell(t)$, and project it onto $\Delta(0,|t|^\varsigma)$. Now, choose a smooth path $\gamma_0(s) := z_1(s)$, $0 \le s \le 1$, in

$\Delta(0,|t|^\varsigma)$ joining $z_1(t)$ to $\Pi_1(p_\ell(t))$, homologous to this projected one, and so that it does not pass through the image of any ramification point of Π_1 on $\mathcal{A}_{c(t)}$. Lifting γ_0 to $\mathcal{A}_{c(t)}$ we have a smooth path

$$\gamma_\ell(z(t),s) = (z_1(s), z_2(z_1(s)), \ldots, z_n(z_1(s))),$$

joining $z(t)$ to $p_\ell(t)$ (we omit the dependence on ℓ and t, in the right term of above notation). Using the Implicit Function Theorem, we have

$$F_j(z_1, z_2(z_1), \ldots, z_n(z_1)) = c_j(t), \quad j = 2, \ldots, n$$

and taking derivatives we get a system of $n-1$ equations

$$\frac{\partial F_j}{\partial z_1} + \frac{\partial F_j}{\partial z_2} z_2' + \cdots \frac{\partial F_j}{\partial z_n} z_n' = 0, \quad j = 2, \ldots n$$

where we write $z_j' = \frac{\partial z_j}{\partial z_1}$. From the system above, we conclude that

$$z_j' = \frac{\widehat{A}_j}{A_1}, \tag{10}$$

where $\widehat{A}_j$ is the minor obtained after replacing the j–th column in the system by $(-\frac{\partial F_2}{\partial z_1}, \ldots, -\frac{\partial F_n}{\partial z_1})$. So we have that $\widehat{A}_j = (-1)^j A_j$, recall (3). If we now derive with respect to s the lifted path $\gamma_\ell(z(t), s)$, we get

$$\dot{\gamma}_\ell(z(t), s) = (\dot{z_1}, z_2'\,\dot{z_1}, \ldots, z_n'\,\dot{z_1})(s)$$

and

$$(\text{length of } \ \gamma_\ell(z(t),s)) = \int_0^1 |\dot{\gamma}_\ell(z(t),s)| ds =$$

$$= \int_0^1 \left(|\dot{z_1}| \sqrt{1 + |z_2'|^2 + \cdots + |z_n'|^2} \right) ds. \tag{11}$$

As $\{A_1 = 0\}$ and $S^2 \cap H$ are algebraic sets, there exists a number $K_0 \in \mathbb{N}$, such that each lifted path is chosen such that

$$|A_1(\gamma_\ell(z(t),s)| \geq \frac{1}{|t|^{K_0}}$$

going to infinity for all $0 \leq s \leq 1$ and large enough t. Note that this condition can be assured for all the directions ϱ' in a small enough polydisk $\Delta^{n-1}(\varrho, \varepsilon)$ at infinity.

We have from (10) and (11) that

$$(\text{length of } \ \gamma_\ell(z(t),s)) \leq \int_0^1 \left(|\dot{z_1}| \sqrt{1 + \frac{|\widehat{A}_2|^2 + \cdots + |\widehat{A}_n|^2}{\frac{1}{|t|^{K_0}}}} \right) ds.$$

As the determinants $|\widehat{A}_j|$ are products of polynomials of known degrees

$$\left(|\widehat{A}_2|^2 + \cdots + |\widehat{A}_n|^2 \right) \leq |t|^{K_1}$$

for certain $K_1 \in \mathbb{N}$ (for all $0 \leq s \leq 1$) which gives

$$(\text{length of } \ \gamma_\ell(z(t),s)) \leq (\text{length}\, \gamma_0) \cdot |t|^{K_0 + K_1}.$$

We finish by noting that a simple choice of the path γ_0 verifying all the conditions required above can be made inside the disk $\Delta(0,|t|^\varsigma)$, and in such a way that its length is less than twice the diameter of the disk. This ends the proof of Lemma 3, choosing $K > \varsigma + K_0 + K_1$. □

Remark 5. *The estimate for the length in Lemma 3 is inspired by Yakovenko, see* [Yak94]*, who dealt with the case* $n = 2$*. We tried to make the construction transparent by lifting smooth paths not passing through branching points, by means of the Implicit Function Theorem.*

Lemma 4. $|\widetilde{F}_1(z(t))|$ *grows polynomially if* $|z(t)|$ *goes linearly to infinity in the directions determined by* $\varrho' \in \Delta^{n-1}(\varrho, \varepsilon) \subset \mathbb{CP}^{n-1}_\infty$.

Proof. We fix one parametrized complex line $z = z(t)$ as in (9), going to a point at infinity in the polydisk determined in Lemma 3, and bound the growth of $\widetilde{F}_1(z(t))$. Recall that we have

$$|\widetilde{F}_1(z(t))| = \left| \sum_{\ell=1}^{d} \int_{\gamma_\ell(z(t))} \omega \right| \leq \sum_{i=1}^{n} \left| \sum_{\ell=1}^{d} \int_{\gamma_\ell(z(t))} a_i(\gamma_\ell(z(t))) dz_i \right|;$$

where $a_i(z_1, \ldots, z_n)$ are polynomials on $\mathbb{C}^n_z$ defining ω (see the begin of Subsection 3.1), and the notation $\gamma_\ell(z(t))$ omit the dependence on the real parameter s.

We bound the terms in the righthand side for each $a_i dz_i$ and each path $\gamma_\ell(z(t))$, where $i \in \{1, \ldots, n\}$ and $\ell \in \{1, \ldots, d\}$. Note that

$$\left| \int_{\gamma_\ell(z(t))} a_i(z_1, \ldots, z_n) dz_i \right| = \left| \int_0^1 a_i(\gamma_\ell(z(t))) dz_i(\gamma_\ell(z(t))) \right|.$$

Now we use the following bounds that were previously stated. Since $|\gamma_\ell(z(t))| < |t|^\varsigma$ and $a_i(z_1, \ldots, z_n)$ is a polynomial of degree $\delta(i)$ (this degree is not explicit, see Subsection 3.1), the norm $|a_i(\gamma_\ell(z(t)))|$ is bounded by $|t|^{\varsigma + \delta(i)}$. By Lemma 3, the lengths of the paths and their projections $dz_i(\gamma_\ell(z(t)))$ are bounded by $|t|^K$. Finally, if $\delta := \max_i\{\delta(i)\}$, then we can assert that

$$|\widetilde{F}_1(z(t))| < nd|z(t)|^{\varsigma + \delta + K},$$

for large enough t.

Moreover, all the bounds above remain true under variations of ϱ' in a small enough $(n-1)$–dimensional polydisk $\Delta^{n-1}(\varrho, \varepsilon) \subset \mathbb{CP}^{n-1}_\infty$, as asserted in Lemmas 1, 2, and 3. □

3.4. **The candidate is polynomial.** Now, in order to show that $\widetilde{F}_1(z)$ is polynomial we proceed by contradiction. The next result seems to be well known, however we could not find it explicitly in the literature.

Lemma 5. *Let* $\Lambda(z)$ *be an entire non-polynomial function in* $\mathbb{C}^n_z$*. The locus of points* $[m_1 : \ldots : m_n] \in \mathbb{CP}^{n-1}_\infty$ *such that* $|\Lambda(m_1 t, \ldots, m_n t)|$ *grows at most like* $|t|^\rho$*, for large enough* $\rho \in \mathbb{N}$*, is contained in an algebraic subvariety of codimension at least 1 in* $\mathbb{CP}^{n-1}_\infty$.

Proof. As usual, define the multi-index $\nu := (\nu_1, \ldots, \nu_n) \in (\mathbb{N} \cup \{0\})^n$ and its associated degree and monomial as

$$|\nu| := \nu_1 + \ldots + \nu_n, \quad z^\nu := z_1^{\nu_1} \cdots z_n^{\nu_n}.$$

The power series expansion of our entire function is

$$\Lambda(z) = \sum_{|\nu|=0}^{\infty} c_\nu z^\nu.$$

Consider the directions $m := [m_1 : \ldots : m_n]$ such that $|\Lambda(m_1 t, \ldots, m_n t)|$ grows less than $|t|^\rho$, for all sufficiently large $|t|$, where ρ is fixed. For these directions the higher order terms in the series must vanish, i.e.

$$\sum_{s \geq 1} \left(\sum_{|\nu| = \rho + s} c_\nu m^\nu \right) t^{\rho + s} = 0.$$

This equation must be true for sufficiently large $|t|$, in consequence it can be split in a numerable set of equations

$$\sum_{|\nu|=\rho+s} c_\nu m^\nu = 0, \ s \in \mathbb{N}.$$

For fixed s, the corresponding equation is homogeneous of degree $\rho+s$, in the variables $m_1, \ldots, m_n$ of $\mathbb{CP}^{n-1}_\infty$.

Λ is entire but it is not a polynomial, hence it has coefficients $c_\nu \neq 0$ for arbitrarily large $|\nu|$. Take such a ν_0 with $c_{\nu_0} \neq 0$. Then, the homogeneous equation of degree $|\nu_0|$ determines a non-trivial algebraic subvariety $T_{|\nu_0|} \subset \mathbb{CP}^{n-1}_\infty$.

For each ν with $|\nu| = \rho + s$, we have an algebraic subvariety $T_{|\nu|} \subset \mathbb{CP}^{n-1}_\infty$. The set of directions producing growth at most like $|t|^\rho$ is the intersection

$$\bigcap_{|\nu|\geq\rho+1} T_{|\nu|} \subset T_{|\nu_0|},$$

that is the desired algebraic variety. □

Proposition 2. *$\widetilde{F}_1(z)$ is polynomial.*

Proof. By Lemma 4 the restriction $\widetilde{F}_1(m_1 t, \ldots, m_n t)$ grows at most like a polynomial in $|t|$ for an open set $\Delta^{n-1}(\varrho, \varepsilon)$ of points in $\mathbb{CP}^{n-1}_\infty$. Assuming that $\widetilde{F}_1(z)$ is a non-polynomial entire function, we get a contradiction, since it must grow slowly in at most a proper algebraic subvariety of points in the hyperplane at infinity, by Lemma 5. Thus, $\widetilde{F}_1$ is a polynomial. □

Let us check the algebraic independence of $\widetilde{F}_1$ with respect to $F_2, \ldots, F_n$. Considering the holomorphic n–form

$$d\widetilde{F}_1 \wedge dF_2 \wedge \cdots \wedge dF_n = \phi(z_1, \ldots, z_n) dz_1 \wedge \cdots \wedge dz_n,$$

ϕ is a nowhere vanishing polynomial. Indeed, by contradiction, let $p \in \mathbb{C}^n_z$ be a point with $\phi(p) = 0$. This says that $d\widetilde{F}_1|_p$ is linearly dependent with $dF_2|_p, \ldots, dF_n|_p$. Then, for $d\widetilde{F}_1|_p$ induces the zero one form on the tangent line $T_p\mathcal{A}_c$ at p. This is a contradiction, since $d\widetilde{F}_1$ is the multiple $d \cdot \omega$ (here $d \geq 1$ is the degree of $\mathcal{A}_c$) and ω is non-zero in every $T_p\mathcal{A}_c$. Hence $\phi \in \mathbb{C}^*$. We define $F_1 := (1/\phi)\widetilde{F}_1$. *The proof of Theorem 1 is done.*

3.5. **Some examples.** For $n = 2$, we show polynomials F_2 satisfying the condition that dF_2 is nowhere zero, having in one case all the periods of ω zero, and with non-zero periods in the other.

Example 1. *A non-singular polynomial with zero periods*

$$F_2(z_1, z_2) = z_1 - z_1^2 z_2.$$

This is the polynomial described by S. A. Broughton, studied in [Bro83], [DW94] and [Dun08], but without considering the residues as we do here. It has irreducible typical fiber $\mathcal{A}_c = \{F_2(z_1, z_2) = c\}$, $c \neq 0$, biholomorphic to $\mathbb{C}^*$. When $\mathcal{A}_c$ is completed with its points at infinity, we have the rational curve: $z_0^2 z_1 - z_1^2 z_2 - c z_0^3 = 0$. It meets the infinity line $z_0 = 0$ at two points, $[0 : 1 : 0]$ and $[0 : 0 : 1]$. The first is a smooth point of the curve, and the second a singular one. Note that we can parametrize our projective curve as

$$\Upsilon[s : \zeta] = [\zeta^2 s : \zeta^3 : \zeta s^2 - c s^3] : \mathbb{CP}^1 \to \mathcal{A}_c,$$

the two distinguished points corresponding to $s = 0$ and to $\zeta = 0$.

In the affine neighbourhood given by $z_1 = 1$, we have that the curve can be parametrized as $\varphi(s) = (s, s^2 - cs^3)$, and so (for $s \neq 0$), we have $(1/s,\ s - cs^2)$ in the original $\mathbb{C}^2 = \{z_0 \equiv 1\}$. Hence, its derivative $(-1/s^2,\ 1 - 2cs)$ (that coincides with the restriction of

$$X = z_1^2 \frac{\partial}{\partial z_1} + (1 - 2z_1 z_2)\frac{\partial}{\partial z_2},$$

to the curve), says that we have the vector field $\varphi^* X = \partial/\partial s$ in $\mathbb{CP}^1$, which is regular at $s = 0$.

Concerning the point $[0 : 0 : 1]$, we have that the curve has a cusp. If $s \neq 0$, and for $\zeta \neq 0$, we note that the image point is $[1 : \zeta : (\zeta - c)/\zeta^2]$, that is the point of affine coordinates $\phi(\zeta) = (\zeta, (\zeta - c)/\zeta^2)$ in the original $\mathbb{C}^2 = \{z_0 \equiv 1\}$. The tangent vector to the affine curve so parametrized is $\phi' = (1, (-\zeta + 2c)/\zeta^3)$. Comparing with the restriction of the vector field X to it, we have that

$$X|_{\phi(\zeta)} = \phi_*(\zeta^2 \frac{\partial}{\partial \zeta}),$$

and the one form such that $\omega(X) = 1$, is written as $d\zeta/\zeta^2$ on the curve. Its period around the pole at $\zeta = 0$ vanishes.

Moreover, if we ignore for a moment the fact that the atypical fiber $\mathcal{A}_0 = \{z_1 = 0\} \cup \{1 - z_1 z_2 = 0\}$ has two irreducible components, and we try to construct $\widetilde{F}_1$ on $\mathbb{C}^2_z - \{F_2 = 0\}$, we get the next result.

Remark 6. *For $F_2(z_1, z_2) = z_1 - z_1^2 z_2$, the candidate function $\widetilde{F}_1$ has a pole in the atypical fiber $\mathcal{A}_0 = \{z_1 z_2 = 1\}$ of F_2.*

Indeed, a global one form of time is

$$\omega = 4z_2^2 dz_1 + (1 + 2z_1 z_2) dz_2,$$

in fact $\omega(X) \equiv 1$. Consider the line $H = \{z_1 - z_2 = 0\}$ transversal to the foliation defined by the fibers of F_2. For each point $z = (z_1, z_2)$, define $c = z_1 - z_1^2 z_2$ and consider the points

$$H \cap \mathcal{A}_c = \{p_1(c), p_2(c), p_3(c)\} = \{\phi(\zeta_1), \phi(\zeta_2), \phi(\zeta_3)\}.$$

For $c \neq 0$, they are determined in the domain of $\phi(\zeta) : \mathbb{C}^* \to \mathcal{A}_c$ by the three roots of the polynomial $\zeta^3 - \zeta + c = 0$. Note that $\phi(\zeta)$ depends on c, but we omit this fact in our notation. In particular since $c = z_1 - z_1^2 z_2$ we have that $\phi(z_1) = (z_1, z_2)$ holds. Following (8), there is a holomorphic function

$$\begin{array}{cccc} \widetilde{F}_1 : & \mathbb{C}^2_z - \mathcal{A}_0 & \longrightarrow & \mathbb{C} \\ & (z_1, z_2) & \mapsto & \sum_{\ell=1}^3 \int_{\gamma_\ell(z_1,z_2)} \omega = \sum_{\ell=1}^3 \int_{z_1}^{\zeta_\ell} \frac{d\zeta}{\zeta^2}. \end{array}$$

We want to study the behavior of $\widetilde{F}_1(z_1, z_2)$ near the atypical fiber $\mathcal{A}_0 := \{1 - z_1 z_2 = 0\}$. For example for $a \neq 0$, we fix $z_1 = a$ and compute

$$\lim_{(a,z_2)\to\mathcal{A}_0} |\widetilde{F}_1(a, z_2)| = \lim_{z_2 \to \frac{1}{a}} |\widetilde{F}_1(a, z_2)|.$$

Note that for $c = 0$,

$$H \cap \mathcal{A}_0 = \{(1, 1), (0, 0), (-1, -1)\} = \{\phi(1), \phi(0), \phi(-1)\}.$$

By using the continuity of the roots of $\zeta^3 - \zeta + c = 0$ as functions of the parameter $c = a - a^2 z_2$ near 0 (equivalently, for z_2 near $1/a$), we obtain that the values $\{\zeta_1(z_2),\ \zeta_2(z_2),\ \zeta_3(z_2)\}$ describing $H \cap \mathcal{A}_c$ remain near $\{1, 0, -1\}$ respectively. We get

$$\lim_{z_2 \to \frac{1}{a}} \left| \sum_{\ell=1}^{3} \int_{z_1}^{\zeta_\ell(z_2)} \frac{d\zeta}{\zeta^2} \right| = \left| \int_a^1 \frac{d\zeta}{\zeta^2} + \int_a^0 \frac{d\zeta}{\zeta^2} + \int_a^{-1} \frac{d\zeta}{\zeta^2} \right| = \infty.$$

In fact, in the righthand side the first and third integrals remain bounded when z goes to $1/a$. Hence, $|\widetilde{F}_1(a,z_2)|$ goes to infinity, when z_2 goes to $1/a$. $\widetilde{F}_1(z_1,z_2)$ is a rational function having a pole at the atypical fiber $\{1-z_1z_2=0\}$. □

Example 2. *A non-singular polynomial with non zero periods*

$$F_2(z_1,z_2)=z_1-z_1^4z_2^4.$$

This is also in the classification of polynomials with one critical value and no critical points in [Bod02]. The fiber over 0 is reducible, with a component which is topologically $\mathbb{C}$, and another one which is the Riemann sphere minus several points.

The level curve $\{F_2=c\}$ corresponds to an octic in $\mathbb{CP}^2$ of equation:

$$z_0^7z_1-z_1^4z_2^4-cz_0^8=0.$$

The curve meets the line at infinity $z_0=0$ at the two points $[0:1:0]$ and $[0:0:1]$. It is singular at the two and if we look at the affine $\mathbb{C}^2=\{z_1\equiv 1\}$ of the first, we have the affine curve $z_0^7(1-cz_0)-z_2^4=0$, that is singular (it has a cusp) at $(0,0)$, with tangent line $z_2=0$. Furthermore, the contact of this tangent with the curve is $\dim_{\mathbb{C}}\frac{\mathcal{O}_{\mathbb{C}^2_0}}{(z_2,z_0^7(1-cz_0)-z_2^4)}=8$.

On the other hand, if we look at the affine neighbourhood $\{z_2\equiv 1\}$ of the second point, we see that the affine curve is given by $z_0^7z_1-z_1^4-cz_0^8=0$. It is singular at $(0,0)$ and the tangent is $z_1=0$. The contact of the curve and the tangent is $\dim_{\mathbb{C}}\frac{\mathcal{O}_{\mathbb{C}^2_0}}{(z_1,z_0^7z_1-z_1^4-cz_0^8=0)}=7$.

Hence, in order to parametrize we can consider the conics that pass through $(0:1:0)$, $(0:0:1)$ and have as tangents at them the lines $z_2=0$ and $z_1=0$, respectively. The conics fulfilling the conditions are those written as

$$sz_1z_2+\zeta z_0^2,\quad [s:\zeta]\in\mathbb{CP}^1.$$

They meet the octic at 16 points, 15 prescribed by the base conditions, and the remaining one giving the parametrization for the curve. Thus, we have

$$\Upsilon[s:\zeta]=[cs^8+s^4\zeta^4:(cs^4+\zeta^4)^2:s^7\zeta]:\mathbb{CP}^1\to\mathcal{A}_c.$$

Note that $\Upsilon[0:1]=[0:1:0]$, while we have for points in $\mathbb{CP}^1$ (the roots of $cs^4+\zeta^4=0$) whose image is $[0:0:1]$, there are four branches of the projective curve through that point.

Proceeding as before, we study the periods of the form ω such that $\omega(X)=1$ on the level curve $\{F_2=c\}$. Note that, topologically, this is $\mathbb{CP}^1$ with five points removed. As the affine parametrization is $\varphi(\zeta)=(\zeta^4+c,\zeta/(\zeta^4+c))$, we have that

$$X_c:=X|_{\{F_2=c\}}=\varphi_*\left((\zeta^4+c)\frac{\partial}{\partial\zeta}\right),\quad\text{hence}\quad\omega_c(\zeta)=\frac{d\zeta}{\zeta^4+c}.$$

It is now easy to see that its periods around the finite poles are not zero.

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VECTOR FIELDS TANGENT TO FOLIATIONS AND BLOW-UPS

F. CANO AND C. ROCHE

Dedicated to Xavier Gómez-Mont on the occasion of his 60th birthday.

1. Introduction

In this note we consider germs of holomorphic vector fields at the origin of $(\mathbb{C}^3, 0)$

$$\xi = a(x,y,z)\frac{\partial}{\partial x} + b(x,y,z)\frac{\partial}{\partial y} + c(x,y,z)\frac{\partial}{\partial z}$$

having a formal invariant curve $\widehat{\Gamma}$ that is totally transcendental, that is $\widehat{\Gamma}$ is not contained in any germ of analytic hypersurface of $(\mathbb{C}^3, 0)$.

It is known (see [4, 6, 7]) that among such vector fields we find the only ones that cannot be desingularized by birational blow-ups in the sense that it is not possible to obtain *elementary singularities* (non nilpotent linear part).

On the other hand, not all germs of vector fields are tangent to a codimension one holomorphic foliation of $(\mathbb{C}^3, 0)$.

We present here a result relating the above two properties

Theorem 1. *Let ξ be a germ of vector field on $(\mathbb{C}^3, 0)$ having a totally transcendental formal invariant curve $\widehat{\Gamma}$ and let D be a normal crossings divisor of $(\mathbb{C}^3, 0)$. Denote by $\mathcal{L}$ the foliation by lines induced by ξ. Assume that there is a germ of codimension one holomorphic foliation $\mathcal{F}$ of $(\mathbb{C}^3, 0)$ such that ξ is tangent to $\mathcal{F}$. Then there is a finite sequence of local blow-ups*

$$(\mathbb{C}^3, 0) = (M_0, p_0) \overset{\pi_1}{\leftarrow} (M_1, p_1) \overset{\pi_2}{\leftarrow} \cdots \overset{\pi_n}{\leftarrow} (M_n, p_n) \tag{1}$$

with the following properties:

(1) *The center Y_{i-1} of π_i is a point or a germ of non-singular analytic curve invariant for the transformed foliation by lines $\mathcal{L}_{i-1}$ of $\mathcal{L}$. Moreover Y_{i-1} has normal crossings with the total transform D_{i-1} of D.*
(2) *The points p_i belong to the strict transform $\widehat{\Gamma}_i$ of $\widehat{\Gamma}$.*
(3) *The final transform $\mathcal{L}_n$ is generated by an elementary germ of vector field.*

As it has been noted by F. Sanz and F. Sancho, (in [3] one find a first reference to this example) there are examples of germs of vector fields ξ such that it is not possible to find a sequence as in Equation 1 with the above properties (1),(2) and (3). This is the starting point of the non-birational strategy of Panazzolo in [6]. The specific example is the following one

$$\xi_{\alpha,\beta,\lambda;x,y,z} = x\left(x\frac{\partial}{\partial x} - \alpha y\frac{\partial}{\partial y} - \beta z\frac{\partial}{\partial z}\right) + xz\frac{\partial}{\partial y} + (y - \lambda x)\frac{\partial}{\partial z},$$

where $\alpha, \beta \in \mathbb{R}_{\geq 0}$ and $\lambda \in \mathbb{R}_{>0}$. It is an obvious corollary of Theorem 1 that this vector field is not tangent to any codimension one foliation. Anyway, we start this note by giving a direct proof of this fact, based on geometrical arguments and on the behaviour of $\xi_{\alpha,\beta,\lambda;x,y,z}$ under blow-up. The proof of Theorem 1 comes just by remarking that the "bad" behaviour of the Sanz-Sancho vector fields does not occur when ξ is tangent to a codimension one foliation.

As a direct consequence of Theorem 1 we obtain that any germ of vector field tangent to a codimension one foliation can be desingularized.

2. The properties of Sanz-Sancho's example

We recall here the properties of the examples of Sanz-Sancho that allow to assure the non-existence of a desingularization sequence as in Theorem 1.

First of all, the singular locus of $\xi_{\alpha,\beta,\lambda;x,y,z}$ is exactly $x = y = 0$ and the divisor $x = 0$ is invariant.

Proposition 1. *Let $\pi : M \to (\mathbb{C}^3, 0)$ be the blow-up with center the origin of $\mathbb{C}^3$ and let ξ' be the transform of $\xi_{\alpha,\beta,\lambda;x,y,z}$ by π. Denote by $E = \pi^{-1}(0)$ the exceptional divisor and by H' the strict transform of $x = 0$ by π. Then*

1. *The exceptional divisor E is invariant by ξ'.*
2. *There is exactly one point $p' \in Sing(\xi') \cap E \setminus H'$ where ξ' has linear part of rank one.*
3. *The point p' is in the strict transform of the line $y - \lambda x = z - \lambda(\alpha + 1)x = 0$.*
4. *If we take local coordinates x', y', z' at p' given by $x' = x$, $y' = y/x - \lambda$ and $z' = z/x - \lambda(\alpha + 1)$, then the germ of ξ' at p' coincides with $\xi_{\alpha',\beta',\lambda';x',y',z'}$ where*
$$\alpha' = \alpha + 1,\ \beta' = \beta + 1,\ \lambda' = \lambda(\alpha + 1)(\beta + 1).$$
5. *The singular locus $Sing(\xi') \setminus H'$ outside H' corresponds to the projective straight line $L \subset E$ passing through p' with local coordinates $x' = y' = 0$.*

Proof. Consider coordinates x', y^*, z^* in the *first* chart of the blow-up, given by $x' = x$, $y^* = y/x$ and $z^* = z/x$. The transformed vector field ξ' is given by
$$\xi' = x' \{x'\partial/\partial x' - (\alpha + 1)y^*\partial/\partial y^* - (\beta + 1)z^*\partial/\partial z^*\} + x'z^*\partial/\partial y^* + (y^* - \lambda)\partial/\partial z^*.$$
We already see that $\mathrm{Sing}(\xi') \setminus H'$ is given by $x' = 0, y^* - \lambda = 0$. Put $y' = y^* - \lambda$ and $z' = z^* - \mu$, then
$$\begin{aligned}\xi' \quad = \quad & x' \{x'\partial/\partial x' - (\alpha + 1)(y' + \lambda)\partial/\partial y' - (\beta + 1)(z' + \mu)\partial/\partial z'\} + \\ & x'(z' + \mu)\partial/\partial y' + y'\partial/\partial z' = \\ & x' \{x'\partial/\partial x' - (\alpha + 1)y'\partial/\partial y' - (\beta + 1)z'\partial/\partial z'\} + \\ & x'(z' + \mu - \lambda(\alpha + 1))\partial/\partial y' + (y' - \mu(\beta + 1)x')\partial/\partial z'.\end{aligned}$$
The value $\mu = \lambda(\alpha + 1)$ gives the only point p' with linear part of rank one.

All the statements are now directly induced from the precedent computations. □

Now, let us recall a general fact on line foliations

Proposition 2. *Let $\widehat{\Gamma}$ be a formal curve for $(\mathbb{C}^3, 0)$. Let $\mathcal{L}$ be a foliation by lines of $(\mathbb{C}^3, 0)$ generated by a germ of vector field ξ. Let us consider the sequence of blow-ups corresponding to the infinitely near points of $\widehat{\Gamma}$*
$$\mathcal{S}_{\widehat{\Gamma}} :\ (\mathbb{C}^3, 0) = (M_0, q_0) \overset{\sigma_1}{\leftarrow} (M_1, q_1) \overset{\sigma_2}{\leftarrow} (M_2, q_2) \cdots \tag{2}$$
where the center of σ_i is q_{i-1} and q_i is in the strict transform $\widehat{\Gamma}_i$ of $\widehat{\Gamma}$. Then the following properties are equivalent

1. *$\widehat{\Gamma}$ is invariant by $\mathcal{L}$.*
2. *There is an index k_0 such that for all $k \geq k_0$ the point q_k is singular for the transform $\mathcal{L}_k$ of $\mathcal{L}$.*

Proof. See [1, 2] ☐

Let us start with $\xi_0 = \xi_{\alpha,\beta,\lambda;x,y,z}$. We blow-up to obtain the point p_1 and coordinates x_1, y_1, z_1 as in Proposition 1 where the transform ξ_1 of ξ is given by $\xi_1 = \xi_{\alpha_1,\beta_1,\lambda_1;x_1,y_1,z_1}$. We repeat the procedure indefinitely to obtain $p_0, p_1, p_2, \ldots$. These ones are the infinitely near points of a non singular formal curve $\widehat{\Gamma}$ transversal to $x = 0$. Moreover, by Proposition 2 the curve $\widehat{\Gamma}$ is invariant by ξ_0. In view of Proposition 1 we have that $\widehat{\Gamma}$ is parameterized by

$$y = \lambda x + \sum_{k=2}^{\infty} \lambda_{k-1} x^k;\ z = (\alpha + 1)\lambda x + \sum_{k=2}^{\infty} (\alpha_{k-1} + 1)\lambda_{k-1} x^k.$$

Remark 1. If we start with $\alpha = \beta = 0$, $\lambda = 1$, we get

$$y = x + \sum_{k=2}^{\infty} (k-1)!(k-1)! x^k;\ z = x + \sum_{k=2}^{\infty} k!(k-1)! x^k$$

that are obviously non convergent formal power series.

Let us give a general proof that $\widehat{\Gamma}$ is not contained in a germ of analytic surface $S \subset (\mathbb{C}^3, 0)$. We are going to do it by using elementary technics of blow-ups and transcendency. Let us work by contradiction by assuming that there is S containing $\widehat{\Gamma}$. First of all let us remark that $\widehat{\Gamma}$ is not a convergent germ of curve, otherwise its plane projection

$$y = \lambda x + \sum_{k=2}^{\infty} \lambda_{k-1} x^k$$

should be convergent. But this is not the case, since

$$\lambda_k = \lambda(\alpha + 1)(\beta + 1)(\alpha + 2)(\beta + 2) \cdots (\alpha + k)(\beta + k).$$

Next Lemma is a version of the transcendence argument known as "truc de Moussu" (see for instance [5]).

Lemma 1. *Let $\widehat{\Gamma}$ be a formal non convergent invariant curve of a germ of analytic vector field ξ of $(\mathbb{C}^3, 0)$ such that Sing(ξ) has codimension at least two. Assume that $\widehat{\Gamma}$ is contained in a germ of irreducible surface $(S, 0) \subset (\mathbb{C}^3, 0)$. Then $(S, 0)$ is invariant by ξ.*

Proof. The analytic set of the tangency locus between ξ and S contains $\widehat{\Gamma}$ but it cannot be equal to $\widehat{\Gamma}$. Thus it coincides with S. ☐

As a consequence of Lemma 1, we deduce that S is invariant by ξ_0. In particular the intersection $S \cap (x = 0)$ must be invariant by $\xi_0|_{(x=0)}$. Now, noting that

$$\xi_0|_{(x=0)} = y \frac{\partial}{\partial z}$$

we deduce that $S \cap (x = 0) = (x = y = 0)$.

By next Lemma 2 we reduce our problem to the case that S is non-singular and with normal crossings with $x = 0$.

Lemma 2. *Let $\widehat{\Gamma}$ be a non convergent formal curve for $(\mathbb{C}^3, 0)$ contained in a surface $S \subset (\mathbb{C}^3, 0)$. Consider the sequence of blow-ups corresponding to the infinitely near points of $\widehat{\Gamma}$*

$$\mathcal{S}_{\widehat{\Gamma}} :\ (\mathbb{C}^3, 0) = (M_0, q_0) \stackrel{\sigma_1}{\leftarrow} (M_1, q_1) \stackrel{\sigma_2}{\leftarrow} (M_2, q_2) \cdots$$

as in Equation 2. There is an index k_0 such that for all $k \geq k_0$ the strict transform S_k of the surface S is non-singular at q_k and has normal crossings with the exceptional divisor.

Proof. The proof is similar to the proof of Proposition 2. We do it for the sake of completeness. Up to a finite number of blow-ups, we can assume that $\widehat{\Gamma}$ is non singular and transversal to $x=0$. We can take formal coordinates $x,\hat{y},\hat{z}$ such that $\widehat{\Gamma}=(\hat{y}=\hat{z}=0)$. Let us express the blow-ups in that coordinates. The first one is given by

$$x=x';\ \hat{y}=x\hat{y}';\ \hat{z}=x\hat{z}'.$$

Now, let $f(x,\hat{y},\hat{z})=0$ be a formal equation of S. We know that $f=\hat{y}f'+\hat{z}f''$, moreover, $\widehat{\Gamma}$ is not in the singular locus of S since it is not convergent. Then, we have that

$$f'(x,0,0)=x^s\hat{u},\, f''(x,0,0)=x^t\hat{v}$$

where either $\hat{u}(0,0,0)\neq 0$ or $\hat{v}(0,0,0)\neq 0$. To fix ideas, assume that $\hat{u}(0,0,0)\neq 0$ and the origin is singular or has no normal crossings with $x=0$. After one blow-up we get $s'<s$ and this cannot be repeated indefinitely. □

Now, up to blow-up, we can assume that S in non singular at p, has normal crossings with $x=0$ and moreover $S\cap(x=0)=(x=y=0)$. This suggests to blow-up the line $x=y=0$. We explain the effect of performing this blow-up in next statement.

Proposition 3. *Let $\pi:M\to(\mathbb{C}^3,0)$ be the blow-up with center $x=y=0$ and let ξ' be the transform of $\xi_{\alpha,\beta,\lambda;x,y,z}$ by π. Denote by $E=\pi^{-1}(x=y=0)$ the exceptional divisor and by H' the strict transform of $x=0$ by π. Then*

1. *The exceptional divisor E is invariant by ξ'.*
2. *There is exactly one point $p'\in Sing(\xi')\cap\pi^{-1}(0)\setminus H'$ where ξ' has linear part of rank one. The point p' is in the strict transform of the plane $y-\lambda x=0$.*
3. *The singular locus $Sing(\xi')\setminus H'$ outside H' coincides with $\pi^{-1}(0)$.*
4. *If we take local coordinates x',y',z' at p' given by $x'=x$, $y'=z$ and $z'=y/x-\lambda$, then the germ of ξ' at p' coincides with $\xi_{\alpha',\beta',\lambda';x',y',z'}$ where*

$$\alpha'=\beta,\ \beta'=\alpha+1,\ \lambda'=\lambda(\alpha+1).$$

Proof. Consider coordinates x',y^*,z^* in the first chart of the blow-up, given by $x'=x$, $y^*=y/x$ and $z^*=z$. The transformed vector field ξ' is given in these coordinates by

$$\xi'=x'\left\{x'\partial/\partial x'-(\alpha+1)y^*\partial/\partial y^*-\beta z^*\partial/\partial z^*\right\}+z^*\partial/\partial y^*+x'(y^*-\lambda)\partial/\partial z^*.$$

We already see that $\mathrm{Sing}(\xi')\setminus H'$ is given by $x'=z^*=0$. Put $z'=y^*-\lambda$ and $y'=z^*$, then

$$\xi'=x'\left\{x'\partial/\partial x'-\beta y'\partial/\partial y'-(\alpha+1)z'\partial/\partial z'\right\}+x'z'\partial/\partial y'+(y'-\lambda(\alpha+1)x')\partial/\partial z'$$

All the statements are now directly induced from the precedent computations. □

Now Proposition 3 gives a contradiction with the existence of S. In fact, since S has normal crossings with $x=0$ and $x=y=0$ is contained in S, the strict transform S' of S by the blow-up π with center $x=y=0$ does not contain $\pi^{-1}(0)$. We can do the same argument as for S at the point p' to see that $S'\cap E'=\mathrm{Sing}(\xi')$, but $\mathrm{Sing}(\xi')=\pi^{-1}(0)$ (locally at p'). This is the desired contradiction.

Thus, we have proved that $\widehat{\Gamma}$ is totally transcendental.

Proposition 1 and Proposition 3 are the initial remarks of F. Sanz and F. Sancho to show that the vector fields $\xi_{\alpha,\beta,\lambda;x,y,z}$ cannot be desingularized by blow-ups with centers in the singular locus, since the only possibilities are the origin and the line $x=y=0$, and in both cases we repeat the situation. Anyway, in order to be complete, we need to show that there is no other analytic invariant curve that could be used as a center.

Corollary 1. *The singular locus $x=y=0$ is the only nonsingular germ of analytic curve invariant by $\xi_{\alpha,\beta,\lambda;x,y,z}$ and having normal crossings with the divisor $x=0$.*

Proof. Assume that γ is a nonsingular invariant curve having normal crossings with $x=0$ and different from $x=y=0$. The only invariant curve contained in $x=0$ is precisely $x=y=0$, hence γ must be transversal to $x=0$. By blowing-up the origin as in Proposition 1, we see that the strict transform of γ is transversal to the exceptional divisor in a point q' of the singular locus of ξ'. If $q'=p'$, we repeat the procedure. At one moment $q'\neq p'$, since otherwise γ and $\widehat{\Gamma}$ would have the same infinitely near points and thus $\gamma=\widehat{\Gamma}$ and this is not possible since $\widehat{\Gamma}$ is completely transcendental and γ is a germ of analytic curve. Now, assume that $q'\neq p'$. Actually it is enough to show that there is no invariant curve for $\xi_{\alpha,\beta,\lambda;x,y,z}$ in a point of coordinates $x=0, y=0, z=z_0\neq 0$ that is non singular and transversal to $x=0$. This is a consequence of Proposition 3 since blowing-up $x=y=0$, we see that there is no singular points over $(0,0,z_0)$ outside the strict transform of $x=0$. □

3. An example of vector field not tangent to a foliation

In this section we show that $\xi_{\alpha,\beta,\lambda;x,y,z}$ is not tangent to any codimension one foliation of $(\mathbb{C}^3,0)$.

Lemma 3. *Let η be a germ of vector field not collinear with $\xi_{\alpha,\beta,\lambda;x,y,z}$ and let $\mathcal{L}$ be the foliation by lines induced by η. Then*

(1) *$\widehat{\Gamma}$ is not an invariant curve of η.*
(2) *If we consider the sequence $\mathcal{S}_{\widehat{\Gamma}}$ of the infinitely near points of $\widehat{\Gamma}$ described in Equation 2, there is an index k_0 such that for all $k\geq k_0$ the transform $\mathcal{L}_k$ is generated by a non-singular vector field and the exceptional divisor is invariant.*

Proof. If $\widehat{\Gamma}$ is invariant for η, then it is contained in the set of collinearity of η and $\xi_{\alpha,\beta,\lambda;x,y,z}$, this is an analytic set that should be the whole space, because of the fact that $\widehat{\Gamma}$ is totally transcendental. The second part is a direct consequence of Proposition 2. □

Let us assume now that $\xi_{\alpha,\beta,\lambda;x,y,z}$ is tangent to a codimension one foliation $\mathcal{F}$. Then there is another germ of vector field η tangent to $\mathcal{F}$ and not collinear with $\xi_{\alpha,\beta,\lambda;x,y,z}$. Up to blowing-up points, and in order to find a contradiction, we can assume without loss of generality that η is non singular and tangent to $x=0$. Thus, the foliation $\mathcal{F}$ has dimensional type two, in the sense that it is trivialized by the flow of η, moreover it is singular, otherwise $\widehat{\Gamma}$ should be contained in a germ of hyper-surface. The singular locus $\mathrm{Sing}(\mathcal{F})$ is a curve invariant by η and $\xi_{\alpha,\beta,\lambda;x,y,z}$. The only possibility is then that

$$\mathrm{Sing}(\mathcal{F})=(x=y=0). \tag{3}$$

Now, we perform the blow-up with center $x=y=0$ to obtain transforms $\mathcal{F}'$, $\xi_{\alpha',\beta',\lambda';x',y',z'}$ and η' that we consider locally at the point p' described in Proposition 3. We take notations as in Proposition 3. By the same argument as before, and since η' is still a non singular vector field tangent to $\mathcal{F}'$, we have that

$$\mathrm{Sing}(\mathcal{F}')=(x'=y'=0).$$

But on the other hand, $\mathcal{F}$ has dimensional type two and thus the singular locus of $\mathcal{F}'$ must be etale over $\mathrm{Sing}(\mathcal{F})$ under the blow-up σ. This is not the case, since around p' we have that

$$\sigma(\mathrm{Sing}(\mathcal{F}'))=\{p\}.$$

This is the desired contradiction.

4. Vector fields tangent to a foliation

In this section we give a proof of Theorem 1. Take notations and hypothesis as in Theorem 1. We shall reason by contradiction by showing that if the vector field ξ cannot be desingularized, then it has the properties of Sanz-Sancho's examples that are contradictory with the fact of being tangent to a foliation.

We assume thus that ξ cannot be desingularized and that it is tangent to a foliation $\mathcal{F}$. We also consider the sequence $\mathcal{S}_{\widehat{\Gamma}}$ of infinitely near points of $\widehat{\Gamma}$ as in Equation 2

$$\mathcal{S}_{\widehat{\Gamma}}: \ (\mathbb{C}^3, 0) = (M_0, q_0) \overset{\sigma_1}{\leftarrow} (M_1, q_1) \overset{\sigma_2}{\leftarrow} (M_2, q_2) \cdots$$

We know that $\widehat{\Gamma}$ is desingularized by this sequence and thus there is k_0 such that for any $k \geq k_0$ the strict transform $\widehat{\Gamma}_k$ of $\widehat{\Gamma}$ is nonsingular and transversal to the exceptional divisor (this one is also non singular at p_k). We can assume without loss of generality that $k_0 = 0$ and that the exceptional divisor is given by $x = 0$. Now, we can parameterize $\widehat{\Gamma}$ by

$$y = \hat{\phi}(x); \quad z = \hat{\psi}(x).$$

Let us see how is transformed ξ under the sequence $\mathcal{S}_{\widehat{\Gamma}}$. For our purposes we can use the formal coordinates $x, \hat{y} = y - \hat{\phi}(x), \hat{z} = z - \hat{\psi}(x)$. Then all the blow-ups are given by a equation having the same shape, that is we have formal coordinates at q_k given inductively by

$$x_k = x, \ \hat{y}_k = \hat{y}_{k-1}/x, \ \hat{z}_k = \hat{z}_{k-1}/x,$$

starting by $\hat{y}_0 = \hat{y}, \ \hat{z}_0 = \hat{z}$. Let us write the vector field ξ (up to multiplying it by x if it is necessary to keep a logarithmic expression) as

$$\xi = \hat{a}(x, \hat{y}, \hat{z}) x \frac{\partial}{\partial x} + \hat{b}(x, \hat{y}, \hat{z}) \frac{\partial}{\partial \hat{y}} + \hat{c}(x, \hat{y}, \hat{z}) \frac{\partial}{\partial \hat{z}}.$$

Consider the invariant

$$r_0 = \min\{\nu_0(\hat{a}), \nu_0(\hat{b}) - 1, \nu_0(\hat{c}) - 1\},$$

where $\nu_0(f)$ is the order of f at the origin. Then the transformed line foliation $\mathcal{L}_k$ is given at q_k by

$$\xi_k = \hat{a}_k \left\{ x \frac{\partial}{\partial x} - k \hat{y}_k \frac{\partial}{\partial \hat{y}_k} - k \hat{z} \frac{\partial}{\partial \hat{z}_k} \right\} + \hat{b}_k \frac{\partial}{\partial \hat{y}_k} + \hat{c}_k \frac{\partial}{\partial \hat{z}_k}$$

where

$$\hat{a}_{k+1} = \hat{a}_k / x^{r_k}; \ \hat{b}_{k+1} = \hat{b}_k / x^{r_k + 1}; \ \hat{c}_{k+1} = \hat{c}_k / x^{r_k + 1},$$

and $r_k = \min\{\nu_{q_k}(\hat{a}_k), \nu_{q_k}(\hat{b}_k) - 1, \nu_{q_k}(\hat{c}_k) - 1\}$. The starting terms of this induction are evident. Let us note that $r_k \geq 0$ for all k since we are in a singular point of $\mathcal{L}_k$.

Now, we know that $\widehat{\Gamma} = (\hat{y} = \hat{z} = 0)$ is invariant and it is not in the singular locus of ξ (otherwise ξ should be identically zero, since $\widehat{\Gamma}$ is completely transcendental). In algebraic terms this is explained by saying that

$$\hat{a}(x, 0, 0) \neq 0; \quad \hat{b} = \hat{y} \hat{b}' + \hat{z} \hat{b}'', \ \hat{c} = \hat{y} \hat{c}' + \hat{z} \hat{c}''.$$

Write $\hat{a} = x^s \hat{u} + \hat{y} \hat{a}' + \hat{z} \hat{a}''$, with $\hat{u}(0,0,0) \neq 0$. Up to a finite number of steps, we obtain that x^s divides $\hat{a}$ and we can write

$$\hat{a} = x^s \hat{U}; \ \hat{U}(0,0,0) \neq 0.$$

Dividing by $\hat{U}$ we may assume that $\hat{a} = x^s$. Now, we conclude that $r_k = 0$ for $k >> 0$, otherwise s strictly decreases each time and once we obtain $s = 0$ we get an elementary singularity, contradiction with our hypothesis. So we assume without loss of generality that $s > 0$ and $r_k = 0$ for all $k \geq 0$. This implies that

$$\min\{\nu_0(\hat{b}), \nu_0(\hat{c})\} = 1.$$

Thus, up to one blow-up, we can write

$$\hat{b} = \alpha\hat{y} + \beta\hat{z} + \hat{y}x\tilde{b}' + \hat{z}x\tilde{b}'',\ \hat{c} = \gamma\hat{y} + \delta\hat{z} + \hat{y}x\tilde{c}' + \hat{z}x\tilde{c}'',$$

where $\alpha, \beta, \gamma, \delta$ are not all zero. Since the linear part must be nilpotent, up to a linear coordinate change in $\hat{y}, \hat{z}$ we may assume that

$$\hat{b} = \hat{y}x\tilde{b}' + \hat{z}x\tilde{b}'',\ \hat{c} = \hat{y} + \hat{y}x\tilde{c}' + \hat{z}x\tilde{c}'',$$

and hence ξ has the expression (we take $n \in \mathbb{Z}_{\geq} 0$)

$$\xi = x^s \left\{ x\frac{\partial}{\partial x} - n\hat{y}\frac{\partial}{\partial \hat{y}} - n\hat{z}\frac{\partial}{\partial \hat{z}} \right\} + \hat{y}\frac{\partial}{\partial \hat{z}} + x\left\{ (\hat{y}\tilde{b}' + \hat{z}\tilde{b}'')\frac{\partial}{\partial \hat{y}} + (\hat{y}\tilde{c}' + \hat{z}\tilde{c}'')\frac{\partial}{\partial \hat{z}} \right\}.$$

The singular locus $\mathrm{Sing}(\xi)$ is then $x = \hat{y} = 0$.

Recall that we assume ξ to be tangent to the codimension one foliation $\mathcal{F}$. By the same arguments as in the precedent Section 3, up to blow-up some infinitely near points of $\widehat{\Gamma}$, we may assume that $\mathcal{F}$ is if dimensional type two and $x = 0$ is invariant by $\mathcal{F}$. In particular it is also true that the singular locus $\mathrm{Sing}\mathcal{F}$ is $x = \hat{y} = 0$. Now, let us blow-up this singular locus and let us focus on the transform ξ' of ξ at the origin of the first chart (that corresponds to the strict transform of $\widehat{\Gamma}$). The local coordinates are given by $x = x, \hat{y} = xy', \hat{z} = z'$ and ξ' is given by

$$\begin{aligned}\xi' = \quad & x^s \left\{ x\tfrac{\partial}{\partial x} - (n+1)y'\tfrac{\partial}{\partial y'} - nz'\tfrac{\partial}{\partial z'} \right\} + xy'\tfrac{\partial}{\partial z'} + \\ & +(xy'\tilde{b}' + z'\tilde{b}'')\tfrac{\partial}{\partial y'} + x(y'x\tilde{c}' + z'\tilde{c}'')\tfrac{\partial}{\partial z'}.\end{aligned}$$

The new singular locus $\mathrm{Sing}(\xi')$ is $x = 0 = z'\tilde{b}''$. It contains $x = z' = 0$. But this is not possible since $x = \hat{z} = 0$ is not contained in the singular locus of the transform $\mathcal{F}'$ of $\mathcal{F}$, because $\mathcal{F}$ has dimensional type two, we have done a blow-up centered at the singular locus $x = \hat{y} = 0$ of $\mathcal{F}$ and $x = \hat{z} = 0$ projects under this blow-up to the origin and not to the whole singular locus of $\mathcal{F}$. This is the desired contradiction.

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FORMES LOGARITHMIQUES ET FEUILLETAGES NON DICRITIQUES

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jeune mathématicien à
l'humour inclassable

Abstract. Given an algebraic codimension 1 foliation $\mathcal{F}$ on the projective space $\mathbb{P}^n_{\mathbb{C}}$, under reasonable conditions on the nature of the singular set, one has that the degree of any invariant variety is at most $d+2$, where d is the degree of $\mathcal{F}$ (Carnicer [Car94], Cerveau-Lins-Neto [CLN91]). In this work we study the extreme case where the degree of the foliation attains its upper bound $d+2$, so completing results by Brunella [Bru97, CLN91].

Résumé. Pour un feuilletage algébrique $\mathcal{F}$ de codimension 1 sur l'espace projectif $\mathbb{P}^n_{\mathbb{C}}$, sous des conditions raisonnables portant sur la nature des singularités, le degré des hypersurfaces algébriques invariantes est majoré par $d+2$ où d est le degré de $\mathcal{F}$ (Carnicer [Car94], Cerveau-Lins-Neto [CLN91]). On s'intéresse ici au cas extrémal où le degré d'une telle hypersurface est précisément $d+2$ complétant en celà des résultats de Brunella [Bru97, CLN91].

1. Introduction

Soit Xune variété complexe ; si ω est une 1-forme différentielle méromorphe sur X on note $D = \mathrm{Pol}\,\omega$ son diviseur de pôles. On dit que ω est une forme logarithmique si ω et $d\omega$ sont à pôles simples le long de D. On sait qu'une 1-forme holomorphe sur une variété projective, et plus généralement sur une variété kählérienne, est fermée. C'est une conséquence de la formule de Stokes. Le résultat qui suit généralise ce fait ; il est du à P. Deligne :

Théorème 1.1 ([Del71]). *Soient X une variété projective complexe et ω une 1-forme logarithmique à pôles le long du diviseur D. Si les singularités de D sont des croisements ordinaires, la forme ω est fermée.*

On peut en fait alléger les hypothèses puisqu'une 1-forme méromorphe sur X est fermée dès qu'elle l'est en restriction à toute section hyperplane générale de dimension au moins 2. De sorte qu'il suffit de supposer que les singularités de D sont des croisements ordinaires en dehors d'un ensemble de codimension 3 de X. En un certain sens le Théorème 1.1 est de nature 2-dimensionnelle.

Comme l'ont remarqué plusieurs auteurs, ce résultat est directement lié au problème de l'estimation du degré des hypersurfaces invariantes des feuilletages de codimension 1 sur les variétés projectives. Les premiers résultats en ce sens sont dus à Carnicer [Car94] et Cerveau-Lins Neto [CLN91] dans le cadre des feuilletages algébriques du plan. Le théorème de Carnicer nécessite des hypothèses sur la nature des points singuliers du feuilletage :

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Théorème 1.2 ([Car94]). *Soit $\mathcal{F}$ un feuilletage de degré d, de codimension 1 sur l'espace projectif $\mathbb{P}^2_{\mathbb{C}}$. On suppose que $\mathcal{F}$ possède une courbe algébrique invariante S de degré m. Si les points singuliers de $\mathcal{F}$ situés sur S sont non dicritiques alors $m \leq d+2$.*

À l'inverse dans ce qui suit les hypothèses portent sur les points singuliers de S et non sur ceux de $\mathcal{F}$; rappelons qu'une hypersurface D d'une variété M est dite à **croisements ordinaires ou normaux** si en chaque point m elle est localement décrite par l'annulation d'un monome $x_1 \dots x_p = 0$ où $(x_1, \dots, x_n)$ est un système de coordonnées locales en m et $p = p(m) < n$.

Théorème 1.3 ([CLN91]). *Soit $\mathcal{F}$ un feuilletage de degré d du plan $\mathbb{P}^2_{\mathbb{C}}$ possédant une courbe algébrique invariante S de degré m. Si les singularités de S sont des croisements ordinaires alors $m \leq d+2$. Lorsque l'égalité est réalisée, $m = d+2$, le feuilletage $\mathcal{F}$ est défini par une forme fermée logarithmique.*

C'est la dernière partie de l'énoncé qui est directement reliée à celui de Deligne. Le Théorème 1.3 se généralise stricto sensu aux feuilletages de codimension 1 sur $\mathbb{P}^n_{\mathbb{C}}$ ayant une hypersurface invariante à croisements ordinaires. Mieux Brunella et Mendès établissent dans [BM00] un résultat plus général concernant les champs d'hyperplans (à priori non nécessairement intégrables) ayant encore une hypersurface à croisements normaux et ce sur les variétés projectives ayant $\mathbb{Z}$ comme groupe de Picard.

Dans cet article on précise le Théorème 1.2 dans le cas extrémal où l'inégalité est une égalité : $m = d+2$; pour celà on relie les concepts de non dicriticité et de formes logarithmiques (Propositions 2.1, 2.2, 2.4). Dans [Bru97, Proposition 10] Brunella présente un résultat similaire en utilisant des arguments d'indice de champs de vecteurs.

Théorème 1.4. *Soit $\mathcal{F}$ un feuilletage de degré d sur l'espace projectif $\mathbb{P}^2_{\mathbb{C}}$ possédant une courbe algébrique invariante S de degré précisément $d+2$. Si les points singuliers de $\mathcal{F}$ sur S sont non dicritiques, alors $\mathcal{F}$ est donné par une forme fermée logarithmique à pôles le long de S.*

On adapte ensuite cet énoncé aux dimensions supérieures et on donne quelques applications.

2. Formes logarithmiques et résolution des singularités

Soit $\mathcal{F}$ un germe de feuilletage singulier à l'origine de $\mathbb{C}^2$; on note $\omega = A\,dx + B\,dy$ un germe de 1-forme à singularité isolée en 0 définissant $\mathcal{F}$. Un tel ω est défini à unité multiplicative près. Par définition la **multiplicité algébrique** ou l'**ordre de $\mathcal{F}$ en** 0 est l'entier

$$\nu(\mathcal{F}) = \nu(\omega) = \inf(\nu(A), \nu(B))$$

où $\nu(A)$ et $\nu(B)$ désignent les ordres des fonctions holomorphes A et B en 0. Soit S un germe de courbe d'équation réduite $f = 0$ à l'origine de $\mathbb{C}^2$. On dit que S est une **séparatrice** ou une **courbe invariante** de $\mathcal{F}$ si $S \smallsetminus \{0\}$ est une feuille (au sens ordinaire) du feuilletage régulier $\mathcal{F}|_{\mathbb{C}^2 \smallsetminus \{0\}, 0}$. Ceci se traduit en termes algébriques par : la 2-forme $\omega \wedge df$ est divisible par f, *i.e.* s'annule identiquement sur S.

Le germe de feuilletage $\mathcal{F}$ est dit **non dicritique** s'il ne possède qu'un nombre fini de séparatrices. Il en possède d'ailleurs au moins une d'après un énoncé célèbre de Camacho et Sad [CS82]. Lorsque $\mathcal{F}$ possède une infinité de séparatrices il est donc dit **dicritique**. Cette notion de dicriticité s'interprète en termes de réduction des singularités. Soit $\pi \colon \widetilde{\mathbb{C}^2} \to (\mathbb{C}^2, 0)$ la réduction des singularités de $\mathcal{F}$; alors $\mathcal{F}$ est non dicritique si et seulement si chaque composante du diviseur exceptionnel $\pi^{-1}(0)$ est invariante par le feuilletage transformé strict $\pi^{-1}(\mathcal{F})$ de $\mathcal{F}$ par π. La notion de forme logarithmique se localise sans problème : le germe de 1-forme méromorphe Ω à l'origine de $\mathbb{C}^2$ est **logarithmique** si Ω et $d\Omega$ sont à pôles simples. L'énoncé qui suit est élémentaire :

Proposition 2.1. *Soient $\mathcal{F}$ un germe de feuilletage à l'origine de $\mathbb{C}^2$ défini par la 1-forme holomorphe ω et S une courbe invariante (pas nécessairement irréductible) de $\mathcal{F}$, d'équation réduite $f = 0$. Alors la forme méromorphe $\Omega = \omega/f$ est logarithmique.*

Démonstration. Puisque f est réduite Ω est à pôles simples. Maintenant $d\Big(\frac{\Omega}{f}\Big) = \frac{d\omega}{f} + \frac{\omega\wedge df}{f^2}$ est aussi à pôles simples puisque $\omega \wedge df$ est divisible par f. □

Le fait pour une 1-forme d'être logarithmique n'est pas "invariant" par éclatement. Par exemple la 1-forme $\Omega = \frac{x\,dy - y\,dx}{x^4+y^4}$ est logarithmique, mais si on éclate l'origine par la transformation quadratique

$$\sigma\colon (x,t) \to (x,tx)$$

alors $\sigma^*\Omega = \frac{dt}{x^2(1+t^4)}$ est à pôle double le long du diviseur exceptionnel $x = 0$. C'est en fait un avatar de la dicriticité du feuilletage radial associé à $x\,dy - y\,dx$.

Par contre dans le cas non dicritique on a la :

Proposition 2.2. *Soient $\mathcal{F}$ un germe de feuilletage non dicritique et S une courbe invariante par $\mathcal{F}$. Soient $f = 0$ une équation réduite de S et ω une 1-forme holomorphe définissant $\mathcal{F}$. Si σ est l'application d'éclatement de l'origine, alors la 1-forme méromorphe $\sigma^*\Big(\frac{\omega}{f}\Big)$ est logarithmique.*

Démonstration. Elle repose sur l'inégalité suivante [CLNS84] établie dans le cas non dicritique précisément : $\nu(f) \leq \nu(\omega) + 1$. La proposition est alors une simple vérification que l'on effectue par exemple dans la carte (x,t) où $\sigma(x,t) = (x,tx)$. On a :

$$\sigma^*\Big(\frac{\omega}{f}\Big) = \frac{x^{\nu(\omega)}\widetilde{\omega}}{x^{\nu(f)}\widetilde{f}}$$

avec $\widetilde{\omega}$ et $\widetilde{f}$ holomorphes. L'inégalité $\nu(f) \leq \nu(\omega)+1$ implique que $\sigma^*(\frac{\omega}{f})$est au pire à pôle simple le long de $x = 0$; le comportement de $\sigma^*\Big(\frac{\omega}{f}\Big)$ le long de $\widetilde{f} = 0$ est bien entendu le même que celui de ω le long de $f = 0$. Comme dans le cas non dicritique le diviseur exceptionnel $x = 0$ est invariant par le feuilletage $\sigma^*\mathcal{F}$ défini par $\widetilde{\omega}$, la 2-forme $d\Big(\sigma^*\big(\frac{\omega}{f}\big)\Big)$ est aussi à pôles au pire simples. □

Remarque 2.3. Il se peut, et c'est le cas si $\nu(f) \leq \nu(\omega)$, que $\sigma^*(\frac{\omega}{f})$ n'ait pas de pôle le long du diviseur $x = 0$.

Proposition 2.4. *Soit $\mathcal{F}$un germe de feuilletage non dicritique à l'origine de $\mathbb{C}^2$ donné par la 1-forme ω. Soit $S = (f = 0)$ une courbe invariante par $\mathcal{F}$, avec f réduite. Soit $\pi\colon \widetilde{\mathbb{C}}^2 \to (\mathbb{C}^2, 0)$ la résolution des singularités de $\mathcal{F}$. Alors la 1-forme $\pi^*\Big(\frac{\omega}{f}\Big)$ méromorphe sur $\widetilde{\mathbb{C}}^2$ est logarithmique.*

Démonstration. Comme l'application π est une composition finie d'éclatements et que la notion de non dicriticité compatible aux éclatements, c'est une application directe de la Proposition 2.2. □

3. Démonstration du théorème 1.4

Le feuilletage $\mathcal{F}$ est donné en coordonnées homogènes $(z_0 : z_1 : z_2)$ par une 1-forme

$$\omega = A_0 dz_0 + A_1 dz_1 + A_2 dz_2$$

où les A_i sont des polynômes homogènes de degré $d+1$, pgcd$\,(A_0, A_1, A_2) = 1$ satisfaisant l'identité d'Euler :

$$\sum z_i A_i = 0.$$

La courbe invariante S est elle donnée par un polynôme homogène réduit f de degré $d+2$. La 1-forme méromorphe $\frac{\omega}{f}$ est donc invariante par les homothéties $z \mapsto t.z$. En utilisant l'identité d'Euler on constate qu'elle définit une 1-forme méromorphe Ω sur $\mathbb{P}^2_{\mathbb{C}}$ à pôles simples le long de S. Comme S est invariante par $\mathcal{F}$ la forme Ω est donc logarithmique à pôles le long de S. On considère la réduction des singularités

$$\pi\colon \widetilde{\mathbb{P}^2_{\mathbb{C}}} \to \mathbb{P}^2_{\mathbb{C}}$$

du feuilletage $\mathcal{F}$. Comme en chaque point singulier $p \in S$ le feuilletage $\mathcal{F}_{,p}$ est non dicritique on peut appliquer la Proposition 2.4 ; ainsi $\pi^*\Omega$ est logarithmique sur $\widetilde{\mathbb{P}^2}_{\mathbb{C}}$. Son diviseur de pôles est contenu dans le transformé total $\pi^{-1}(S)$ de S par π (c'est l'union des diviseurs exceptionnels et de la transformée stricte de S). Comme $\pi^{-1}(S)$ est à croisements ordinaires, les pôles de $\pi^*(\Omega)$ le sont aussi et le théorème de Deligne affirme que $\pi^*\Omega$ est fermée ; par suite Ω aussi.

□

4. Applications et généralisation

Comme l'aura noté le lecteur on obtient de manière analogue et directe le :

Théorème 4.1. *Soient X une surface projective et ω une 1-forme logarithmique sur X. Si les singularités du feuilletage associé à ω situées sur le diviseur des pôles de ω sont non dicritiques, alors la forme ω est fermée.*

En fait le résultat précédent se généralise en toute dimension :

Théorème 4.2. *Soient $X \subset \mathbb{P}^n_{\mathbb{C}}$ une variété projective et ω une 1-forme logarithmique sur X. On suppose que dans une famille générique de sections linéaires de dimension $n - \dim X + 2$ les hypothèses du Théorème 4.1 sont réalisées. Alors la forme ω est fermée.*

Démonstration. Elle résulte du fait qu'une 1-forme méromorphe est fermée si et seulement si elle l'est dans une famille générique de sections comme ci-dessus. □

Dans l'esprit du Théorème 4.2 nous avons le :

Corollaire 4.3. *Soit $\mathcal{F}$ un feuilletage de codimension 1 sur l'espace $\mathbb{P}^n_{\mathbb{C}}, n \geq 2$. Si dans une section 2-plane générale $i : \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^n_{\mathbb{C}}$ le feuilletage restreint $i^*\mathcal{F}$ satisfait les hypothèses du Théorème 4.1 alors $\mathcal{F}$ est défini par une forme fermée logarithmique. En particulier $\mathcal{F}$ possède une hypersurface invariante de degré* $\deg \mathcal{F} + 2$.

Démonstration. Un feuilletage de $\mathbb{P}^n_{\mathbb{C}}$ défini dans une section plane générique par une 1-forme méromorphe fermée est aussi défini par une telle forme fermée. On trouve ce résultat par exemple dans [CM82]. □

5. Compléments

Considérons sur $\mathbb{P}^2_{\mathbb{C}}$ le feuilletage $\mathcal{F}$ donné en carte affine (z_1, z_2) par la 1-forme :

$$\omega = z_1 dz_2 - z_2 dz_1 + z_1 z_2 (z_2 - z_1) \left(\alpha \frac{dz_1}{z_1} + \beta \frac{dz_2}{z_2} + \gamma \frac{d(z_2 - z_1)}{z_2 - z_1} \right)$$

où les α, β, γ sont des constantes complexes.

C'est un feuilletage de degré 2, ayant une singularité dicritique en l'origine. Il possède les droites invariantes

$$z_1 = 0, \qquad z_2 = 0, \qquad z_1 = z_2$$

et la droite à l'infini (tout du moins lorsque $\alpha + \beta + \gamma \neq 0$) et par conséquent une séparatrice réduite de degré 4. On démontre facilement par des arguments holonomiques que pour α, β, γ

génériques $\mathcal{F}$ n'est pas donné par une 1-forme fermée. On note aussi que $\frac{\omega}{z_1 z_2 (z_2 - z_1)}$ définit sur $\mathbb{P}^2_{\mathbb{C}}$ une 1-forme logarithmique à pôles le long des 4 droites ci-dessus. Par contre si l'on éclate l'origine, la forme éclatée n'est pas logarithmique le long du diviseur exceptionnel.

Dans leur étude des feuilletages modulaires de Hilbert [MP05], Mendes et Pereira donnent l'exemple d'un feuilletage quadratique de $\mathbb{P}^2_{\mathbb{C}}$, non défini par une forme fermée, et possédant une courbe invariante irréductible de degré $S = \deg \mathcal{F} + 3$. Ce feuilletage est "transversalement projectif" et "non transversalement affine". Le lecteur intéressé pourra consulter l'article [LN02] de Lins Neto où l'auteur examine des familles de feuilletages de petit degré sur $\mathbb{P}^2_{\mathbb{C}}$ ayant des courbes invariantes de degré grand. Dans l'esprit de cet article on peut se demander si un feuilletage $\mathcal{F}$ de $\mathbb{P}^2_{\mathbb{C}}$ possèdant une courbe invariante de degré "très grand" relativement à celui de $\mathcal{F}$ est transversalement projectif.

Terminons par les remarques suivantes ; si un feuilletage $\mathcal{F}$ de $\mathbb{P}^2_{\mathbb{C}}$ a une courbe invariante de S de degré précisément $\deg \mathcal{F} + 2$ les singularités de $\mathcal{F}$ sur S étant non dicritiques alors S a au moins 3 composantes irréductibles.

Dans le même ordre d'idée soient $\mathcal{F}$ un feuilletage de degré d sur $\mathbb{P}^n_{\mathbb{C}}$ et H une hypersurface invariante de $\mathcal{F}$. Si les singularités de H sont de codimension supérieure où égale à 3 alors degré $H \leq d+1$. En effet si ω est une 1-forme homogène définissant $\mathcal{F}$ et h un polynôme homogène irréductible tel que $H = (h = 0)$ alors

$$\omega = a\,dh + h\eta$$

avec $a \in \mathcal{O}(\mathbb{C}^{n+1})$ et $\eta \in \Omega^1(\mathbb{C}^{n+1})$ homogènes ; c'est une conséquence du lemme de division de Rham-Saito. L'identité d'Euler $i_R\omega = 0$, où R désigne le champ radial $R = \sum_{i=0}^{n} z_i \frac{\partial}{\partial z_i}$, implique alors que :

$$h\Big((\deg h)a + i_R\eta\Big) = 0$$

en particulier $\deg a \geq 1$ et $\deg H \leq d+1$. Dans le cas extrémal où l'inégalité est une égalité, $\deg H = d+1$, on constate que $\eta = -\delta\, da$ où $\delta = \deg h$, de sorte que h/a^{δ} est une intégrale première rationnelle de $\mathcal{F}$. Notons que dans la carte affine $a = 1$ le feuilletage $\mathcal{F}$ a une intégrale première polynomiale. Remarquons que dans ce cas le feuilletage $\mathcal{F}$ a des singularités dicritiques le long de H.

Une pensée pour Marco Brunella qui s'est beaucoup intéressé à ce type de problèmes. Je tiens à remercier Julie Déserti pour son aide constante et désintéressée.

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VARIETIES OF COMPLEXES AND FOLIATIONS

FERNANDO CUKIERMAN

Dedicated to Xavier Gómez-Mont on his 60th Birthday.

ABSTRACT. Let $\mathcal{F}(r,d)$ denote the moduli space of algebraic foliations of codimension one and degree d in complex projective space of dimension r. We show that $\mathcal{F}(r,d)$ may be represented as a certain linear section of a variety of complexes. From this fact we obtain information on the irreducible components of $\mathcal{F}(r,d)$.

1. Basics on varieties of complexes.

1.1. Let K be a field and let $V_0,\dots,V_n$ be vector spaces over K of finite dimensions

$$d_i = \dim_K(V_i).$$

Consider sequences of linear functions

$$V_0 \xrightarrow{f_1} V_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} V_n \ ,$$

also written

$$f = (f_1,\dots,f_n) \in V = \prod_{i=1}^{n} \mathrm{Hom}_K(V_{i-1},V_i).$$

The variety of differential complexes is defined as

$$\mathcal{C} = \mathcal{C}(V_0,\dots,V_n) = \{f = (f_1,\dots,f_n) \in V / \ f_{i+1}\circ f_i = 0,\ i = 1,\dots,n-1\},$$

It is an affine variety in V, given as an intersection of quadrics. We intend to study the geometry of this variety (see also e.g., [3], [6]).

1.2. Since the defining equations $f_{i+1}\circ f_i = 0$ are bilinear, we may also consider, when it is convenient, the projective variety of complexes

$$P\mathcal{C} \subset \prod_{i=1}^{n} \mathbb{P}\mathrm{Hom}_K(V_{i-1},V_i),$$

as a subvariety of a product of projective spaces.

Denoting $V. = \oplus_{i=0}^{n} V_i$, each complex $f \in \mathcal{C}$ may be thought as a degree-one homomorphism of graded vector spaces $f : V. \to V.$ with $f^2 = 0$.

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1.3. For each $f \in \mathcal{C}$ and $i = 0, \ldots, n$ define

$$B_i = f_i(V_{i-1}) \subset Z_i = \ker\,(f_{i+1}) \subset V_i,$$

and

$$H_i = Z_i / B_i.$$

(we understand by convention that $B_0 = 0$)

From the exact sequences

$$0 \to B_i \to Z_i \to H_i \to 0,$$

$$0 \to Z_i \to V_i \to B_{i+1} \to 0,$$

we obtain for the dimensions

$$b_i = \dim_K(B_i), \quad z_i = \dim_K(Z_i), \quad h_i = \dim_K(H_i),$$

the relations

$$d_i = b_{i+1} + z_i = b_{i+1} + b_i + h_i,$$

where $i = 0, \ldots, n$ and $b_0 = b_{n+1} = 0$. Therefore,

Proposition 1. *a) The h_i and the b_j determine each other by the formulas:*

$$h_i = d_i - (b_{i+1} + b_i),$$

$$b_{j+1} = \chi_j(d) - \chi_j(h),$$

where for a sequence $e = (e_0, \ldots, e_n)$ and $0 \le j \le n$ we denote

$$\chi_j(e) = (-1)^j \sum_{i=0}^{j} (-1)^i e_i = e_j - e_{j-1} + e_{j-2} + \cdots + (-1)^j e_0,$$

the j-th Euler characteristic of e.

b) The inequalities $b_{i+1} + b_i \le d_i$ are satisfied for all i.

Proof. We write down the b_j in terms of the h_i: from

$$\sum_{i=0}^{j} (-1)^i d_i = \sum_{i=0}^{j} (-1)^i (b_{i+1} + b_i + h_i),$$

we obtain

$$b_{j+1} = (-1)^j (\sum_{i=0}^{j} (-1)^i d_i - \sum_{i=0}^{j} (-1)^i h_i),$$

as claimed.

□

Notice in particular that since $b_{n+1} = 0$, we have the usual relation

$$\sum_{i=0}^{n} (-1)^i d_i = \sum_{i=0}^{n} (-1)^i h_i.$$

1.4. Now we consider the subvarieties of $\mathcal{C}$ obtained by imposing rank conditions on the f_i.

Definition 2. *For each $r = (r_1, \ldots, r_n) \in \mathbb{N}^n$ define*

$$\mathcal{C}_r = \{f = (f_1, \ldots, f_n) \in \mathcal{C} /\ \operatorname{rank}(f_i) = r_i,\ i = 1, \ldots, n\}.$$

These are locally closed subvarieties of $\mathcal{C}$.

Proposition 3. *a) $\mathcal{C}_r \neq \emptyset$ if and only if $r_{i+1} + r_i \leq d_i$ for $0 \leq i \leq n$ (we use the convention $r_0 = r_{n+1} = 0$)*

b) In the conditions of a), $\mathcal{C}_r$ is smooth and irreducible, of dimension

$$\dim(\mathcal{C}_r) = \sum_{i=0}^{n} (d_i - r_i)(r_{i+1} + r_i) = \sum_{i=0}^{n} (d_i - r_i)(d_i - h_i) = \frac{1}{2} \sum_{i=0}^{n} (d_i^2 - h_i^2).$$

Proof. a) One implication follows from Proposition 1. Conversely, in the given conditions, we want to construct a complex with $\operatorname{rank}(f_i) = r_i$ for all i. Suppose we constructed

$$V_0 \xrightarrow{f_1} V_1 \xrightarrow{f_2} \ldots \xrightarrow{f_{n-1}} V_{n-1}\ .$$

We need to define $f_n : V_{n-1} \to V_n$ such that $f_n \circ f_{n-1} = 0$ and $\operatorname{rank}(f_n) = r_n$, that is, a map $V_{n-1}/B_{n-1} \to V_n$ of rank r_n. Such a map exists since $\dim(V_{n-1}/B_{n-1}) = d_{n-1} - r_{n-1} \geq r_n$.

b) Consider the projection (forgeting f_n)

$$\pi : \mathcal{C}(V_0, \ldots, V_n)_r \to \mathcal{C}(V_0, \ldots, V_{n-1})_{\bar{r}},$$

where $r = (r_1, \ldots, r_n)$ and $\bar{r} = (r_1, \ldots, r_{n-1})$. Any fiber $\pi^{-1}(f_1, \ldots, f_{n-1})$ is isomorphic to the subvariety in $\operatorname{Hom}(V_{n-1}/B_{n-1}, V_n)$ of maps of rank r_n; therefore, it is smooth and irreducible of dimension $r_n(d_{n-1} - r_{n-1} + d_n - r_n)$ (see [1]). The assertion follows by induction on n. The various expressions for $\dim(\mathcal{C}_r)$ follow by direct calculations.

Another proof of a): Given r such that $r_{i+1} + r_i \leq d_i$, put $h_i = d_i - (r_{i+1} + r_i) \geq 0$ and $z_i = d_i - r_{i+1} = h_i + r_i$. Choose linear subspaces $B_i \subset Z_i \subset V_i$ with $\dim(B_i) = r_i$ and $\dim(Z_i) = z_i$. Since $\dim(V_{i-1}/Z_{i-1}) = \dim(B_i)$, choose an isomorphism $\sigma_i : V_{i-1}/Z_{i-1} \to B_i$ for each i. Composing with the natural projection $V_{i-1} \to V_{i-1}/Z_{i-1}$ we obtain linear maps $V_{i-1} \to B_i$ with kernel Z_{i-1} and rank r_i, as wanted.

□

Remark 4. *In terms of dimension of homology, the condition in Proposition 3 a) translates as follows. Given $h = (h_0, \ldots, h_n) \in \mathbb{N}^{n+1}$, there exists a complex with dimension of homology equal to h if and only if $\chi_i(h) \leq \chi_i(d)$ for $i = 1, \ldots, n-1$ and $\chi_n(h) = \chi_n(d)$.*

Remark 5. *The group $G = \prod_{i=0}^{n}\ GL(V_i, K)$ acts on $V = \prod_{i=1}^{n} \operatorname{Hom}_K(V_{i-1}, V_i)$ via*

$$(g_0, g_1, \ldots, g_n) \cdot (f_1, f_2, \ldots, f_n) = (g_0 f_1 g_1^{-1}, g_1 f_2 g_2^{-1}, \ldots, g_{n-1} f_n g_n^{-1}).$$

This action clearly preserves the variety of complexes. It follows from the proof above that the action on each $\mathcal{C}_r$ is transitive. Hence, the non-empty $\mathcal{C}_r$ are the orbits of G acting on $\mathcal{C}(V_0, \ldots, V_n)$.

Definition 6. *For $r, s \in \mathbb{N}^n$ we write $s \leq r$ if $s_i \leq r_i$ for $i = 1, \ldots, n$.*

Corollary 7. *If $\mathcal{C}_r \neq \emptyset$ and $s \leq r$ then $\mathcal{C}_s \neq \emptyset$. Also,* $\dim(\mathcal{C}_s) > 0$ *if* $s \neq 0$.

Proof. The first assertion follows from Proposition 3 a), and the second from Proposition 3 b). □

Proposition 8. *With the notation above,*

$$\overline{\mathcal{C}}_r = \bigcup_{s\leq r} \mathcal{C}_s = \{f \in \mathcal{C}/\ \operatorname{rank}(f_i) \leq r_i,\ i = 1, \ldots, n\}.$$

Proof. Denote $X_r = \bigcup_{s\leq r} \mathcal{C}_s$. Since the second equality is clear, X_r is closed. It follows that $\overline{\mathcal{C}}_r \subset X_r$. To prove the equality, since $\mathcal{C}_r \subset X_r$ is open, it would be enough to show that X_r is irreducible. For this, consider $L = (L_1, \ldots, L_n)$ where $L_i \in \text{Grass}\ (r_i, V_i)$ and denote

$$X_L = \{f = (f_1, \ldots, f_n) \in \mathcal{C}/\ \text{ im } (f_i) \subset L_i \subset \text{ ker } (f_{i+1}),\ i = 1, \ldots, n\}.$$

Consider

$$\tilde{X}_r = \{(L, f)/\ f \in X_L\}\ \subset\ G \times \mathcal{C},$$

where $G = \prod_{i=0}^{n} \text{Grass}\ (r_i, V_i)$. The first projection $p_1 : \tilde{X}_r \to G$ has fibers

$$p_1^{-1}(L) = X_L \cong \text{Hom}(V_0, L_1) \times \text{Hom}(V_1/L_1, L_2) \times \cdots \times \text{Hom}(V_{n-1}/L_{n-1}, V_n),$$

which are vector spaces of constant dimension $\sum_{i=0}^{n}(d_i - r_i)r_{i+1}$. It follows that $\tilde{X}_r$ is irreducible, and hence $X_r = p_2(\tilde{X}_r)$ is also irreducible, as wanted.

□

Remark 9. *In the proof above we find again the formula*

$$\dim(X_r) = \dim(X_L) + \dim(G) = \sum_{i=0}^{n}(d_i - r_i)r_i + \sum_{i=0}^{n}(d_i - r_i)r_{i+1}.$$

Remark 10. *The fact that $p_1 : \tilde{X}_r \to G$ is a vector bundle implies that $\tilde{X}_r$ is smooth. On the other hand, since $p_2 : \tilde{X}_r \to X_r$ is birational (an isomorphism over the open set $\mathcal{C}_r$), it is a resolution of singularities.*

The following two corollaries are immediate consequences of Proposition 8.

Corollary 11. *$\mathcal{C}_s \subset \overline{\mathcal{C}}_r$ if and only if $s \leq r$.*

Corollary 12. *$\overline{\mathcal{C}}_r \cap \overline{\mathcal{C}}_s = \overline{\mathcal{C}}_t$ where $t_i = min\ (r_i, s_i)$ for all $i = 1, \ldots, n$.*

Definition 13. *For $d = (d_0, \ldots, d_n) \in \mathbb{N}^{n+1}$ let*

$$R = R(d) = \{(r_1, \ldots, r_n) \in \mathbb{N}^n/\ r_1 \leq d_0,\ r_{i+1} + r_i \leq d_i\ (1 \leq i \leq n-1),\ r_n \leq d_n\}.$$

We consider $\mathbb{N}^n$ ordered via $r \leq s$ if $r_i \leq s_i$ for all i; the finite set R has the induced order. Notice that R is finite since it is contained in the box $\{(r_1, \ldots, r_n) \in \mathbb{N}^n/\ 0 \leq r_i \leq d_i,\ i = 1, \ldots, n\}$.

Proposition 14. *With the notation above, the irreducible components of the variety of complexes $\mathcal{C} = \mathcal{C}(V_0, \ldots, V_n)$ are the $\overline{\mathcal{C}}_r$ with $r \in R(d_0, \ldots, d_n)$ a maximal element.*

Proof. From the previous Propositions, we have the equalities

$$\mathcal{C} = \bigcup_{r \in R} \mathcal{C}_r = \bigcup_{r \in R} \overline{\mathcal{C}}_r = \bigcup_{r \in R^+} \overline{\mathcal{C}}_r,$$

where R^+ denotes the set of maximal elements of R. The result follows because we know that each $\overline{\mathcal{C}}_r$ is irreducible and there are no inclusion relations among the $\overline{\mathcal{C}}_r$ for $r \in R^+$ (see Corollary 11).

□

1.5. **Morphisms of complexes. Tangent space of the variety of complexes.** Now we would like to compute the dimension of the tangent space of a variety of complexes at each point.

With the notation of 1.1 we consider complexes $f \in \mathcal{C}(V_0, \ldots, V_n)$ and $f' \in \mathcal{C}(V'_0, \ldots, V'_n)$ (the vector spaces V_i and V'_i are not necessarily the same, but the lenght n we may assume is the same). We denote

$$\mathrm{Hom}_{\mathcal{C}}(f, f'),$$

the set of morphisms of complexes from f to f', that is, collections of linear maps $g_i : V_i \to V'_i$ for $i = 0, \ldots, n$, such that $g_i \circ f_i = f'_i \circ g_{i-1}$ for $i = 1, \ldots, n$. It is a vector subspace of $\prod_{i=0}^n \mathrm{Hom}_K(V_i, V'_i)$, and we would like to calculate its dimension.

For this particular purpose and for its independent interest, we recall the following from [2] (§2 − 5. Complexes scindés):

For $f \in \mathcal{C}(V_0, \ldots, V_n)$, denote as in 1.1

$$B_i(f) = f_i(V_{i-1}) \subset Z_i(f) = \ker\ (f_{i+1}) \subset V_i.$$

Since we are working with vector spaces, we may choose linear subspaces $\bar{B}_i$ and $\bar{H}_i$ of V_i such that

$$V_i = Z_i(f) \oplus \bar{B}_i \quad \text{and} \quad Z_i(f) = B_i(f) \oplus \bar{H}_i.$$

Then $V_i = B_i(f) \oplus \bar{H}_i \oplus \bar{B}_i$ and clearly f_{i+1} takes $\bar{B}_i$ isomorphically onto $B_{i+1}(f)$. Notice also that

$$\dim(\bar{B}_i) = \dim(B_{i+1}(f)) = \mathrm{rank}(f_{i+1}) = r_{i+1}(f),$$

and

$$\dim(\bar{H}_i) = \dim(Z_i(f)/B_i(f)) = h_i(f).$$

Next, define the following complexes:

$\bar{H}(i)$ the complex of lenght zero consisting of the vector space $\bar{H}_i$ in degree i, the vector space zero in degrees $\neq i$, and all differentials equal to zero.

$\bar{B}(i)$ the complex of lenght one consisting of the vector space $\bar{B}_{i-1}$ in degree $i-1$, the vector space $B_i(f)$ in degree i, with the map $f_i : \bar{B}_{i-1} \to B_i(f)$, and zeroes everywhere else.

Proposition 15. *With the notation just introduced, $\bar{H}(i)$ and $\bar{B}(i)$ are subcomplexes of f and we have a direct sum decomposition of complexes:*

$$f = \bigoplus_{0 \le i \le n} \bar{H}(i) \ \oplus \ \bigoplus_{0 \le i \le n} \bar{B}(i).$$

Proof. Clear from the discussion above; see also [2], loc. cit. □

Now we are ready for the calculation of $\dim_K \mathrm{Hom}_{\mathcal{C}}(f, f')$.

Proposition 16. *With the previous notation, we have:*

$$\begin{aligned}\dim_K \mathrm{Hom}_{\mathcal{C}}(f,f') &= \sum_i h_i h'_i + h_i r'_i + r_i h'_{i-1} + r_i r'_i + r_i r'_{i-1} \\ &= \sum_i h_i(h'_i + r'_i) + r_i d'_{i-1}\end{aligned}$$

Proof. We may decompose f and f' as in Proposition 15:

$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}}(f,f') &= \mathrm{Hom}_{\mathcal{C}}(\oplus_i \bar{H}(i) \oplus \oplus_i \bar{B}(i), \oplus_i \bar{H}(i)' \oplus \oplus_i \bar{B}(i)') \\ &= \oplus_{i,j}\mathrm{Hom}_{\mathcal{C}}(\bar{H}(i),\bar{H}(j)') \ \oplus\ \oplus_{i,j}\mathrm{Hom}_{\mathcal{C}}(\bar{H}(i),\bar{B}(j)') \oplus \\ &\quad \oplus_{i,j}\mathrm{Hom}_{\mathcal{C}}(\bar{B}(i),\bar{H}(j)') \ \oplus\ \oplus_{i,j}\mathrm{Hom}_{\mathcal{C}}(\bar{B}(i),\bar{B}(j)')\end{aligned}$$

It is easy to check the following:

$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}}(\bar{H}(i),\bar{H}(j)') &= 0 \ \text{ for } i \neq j \\ \mathrm{Hom}_{\mathcal{C}}(\bar{H}(i),\bar{H}(i)') &= \mathrm{Hom}_K(\bar{H}_i,\bar{H}'_i)\end{aligned}$$

$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}}(\bar{H}(i),\bar{B}(j)') &= 0 \ \text{ for } i \neq j \\ \mathrm{Hom}_{\mathcal{C}}(\bar{H}(i),\bar{B}(i)') &= \mathrm{Hom}_K(\bar{H}_i,\bar{B}'_i)\end{aligned}$$

(the case $j = i+1$ requires special attention)

$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}}(\bar{B}(i),\bar{H}(j)') &= 0 \ \text{ for } i-1 \neq j \\ \mathrm{Hom}_{\mathcal{C}}(\bar{B}(i),\bar{H}(i-1)') &= \mathrm{Hom}_K(\bar{B}_{i-1},\bar{H}'_{i-1}) \cong \mathrm{Hom}_K(\bar{B}_i(f),\bar{H}'_{i-1})\end{aligned}$$

(the case $j = i$ requires special attention)

$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}}(\bar{B}(i),\bar{B}(i)') &\cong \mathrm{Hom}_K(B_i(f),B'_i(f)) \\ \mathrm{Hom}_{\mathcal{C}}(\bar{B}(i),\bar{B}(i-1)') &= \mathrm{Hom}_K(\bar{B}_{i-1},B'_{i-1}) \cong \mathrm{Hom}_K(B_i(f),B'_{i-1}) \\ \mathrm{Hom}_{\mathcal{C}}(\bar{B}(i),\bar{B}(j)') &= 0 \ \text{ otherwise}\end{aligned}$$

Taking dimensions we obtain the stated formula. □

Now we deduce the dimension of the tangent space to a variety of complexes at any point.

Proposition 17. *For $f \in \mathcal{C} = \mathcal{C}(V_0,\ldots,V_n)$ we have a canonical isomorphism*

$$T\mathcal{C}(f) = \mathrm{Hom}_{\mathcal{C}}(f, f(1)),$$

where $T\mathcal{C}(f)$ is the Zariski tangent space to $\mathcal{C}$ at the point f, and $f(1)$ denotes de shifted complex $f(1)_i = (-1)^i f_{i+1}$, $i = -1, 0, \ldots, n$.

Proof. Since $\mathcal{C}$ is an algebraic subvariety of the vector space $V = \prod_{i=1}^n \mathrm{Hom}_K(V_{i-1},V_i)$, an element of $T\mathcal{C}(f)$ is a $g = (g_1,\ldots,g_n) \in V$ such that $f + \epsilon g$ satisfies the equations defining $\mathcal{C}$ (i.e., a $K[\epsilon]$-valued point of $\mathcal{C}$), that is,

$$(f+\epsilon g)_{i+1} \circ (f+\epsilon g)_i = 0, \quad i = 1,\ldots,n-1 \ \ (\text{modulo } \epsilon^2),$$

which is equivalent to

$$f_{i+1} \circ g_i + g_{i+1} \circ f_i = 0, \quad i = 1, \dots, n-1,$$

and this means precisely that $g \in \mathrm{Hom}_{\mathcal{C}}(f, f(1))$. □

Corollary 18. *For $f \in \mathcal{C} = \mathcal{C}(V_0, \dots, V_n)$,*

$$\begin{aligned} \dim_K T\mathcal{C}(f) &= \sum_i h_i(h_{i+1} + r_{i+1}) + r_i d_i \\ &= \sum_i (d_i - r_i - r_{i+1})(d_{i+1} - r_{i+2}) + r_i d_i \end{aligned}$$

Proof. From Proposition 17 we know that $\dim_K T\mathcal{C}(f) = \dim_K \mathrm{Hom}_{\mathcal{C}}(f, f(1))$. Next we apply Proposition 16 with $f' = f(1)$, that is, replacing $d'_i = d_{i+1}$, $r'_i = r_{i+1}$, $h'_i = h_{i+1}$, to obtain the result.

□

1.6. **Varieties of exact complexes.** Now we apply the previous results to the case of exact complexes.

Let us fix $(d_0, \dots, d_n) \in \mathbb{N}^n$ so that

$$\chi_j(d) = (-1)^j \sum_{i=0}^{j} (-1)^i d_i \geq 0, \quad j = 1, \dots, n-1,$$

$$\chi_n(d) = (-1)^n \sum_{i=0}^{n} (-1)^i d_i = 0.$$

Denoting $\chi = \chi(d) = (\chi_1(d), \dots, \chi_n(d)) \in \mathbb{N}^n$, let us consider the variety $\mathcal{C}_\chi$ of complexes of rank χ as in Definition 2 . Since $\chi_i(d) + \chi_{i-1}(d) = d_i$ for all i, it follows from Proposition 3 that $\mathcal{C}_\chi$ is non-empty of dimension

$$\frac{1}{2} \sum_{i=0}^{n} d_i^2.$$

It follows from Proposition 1 that any complex $f \in \mathcal{C}_\chi$ is exact. Also, since $\chi \in R$ is clearly maximal, $\overline{\mathcal{C}}_\chi$ is an irreducible component of $\mathcal{C}$ (see Proposition 14). Let us denote

$$\mathcal{E} = \mathcal{E}(d_0, \dots, d_n) = \overline{\mathcal{C}}_\chi = \{f \in \mathcal{C} /\ \mathrm{rank}(f_i) \leq \chi_i, \quad i = 1, \dots, n\},$$

the closure of the variety $\mathcal{C}_\chi$ of exact complexes. Denote also, for $i = 1, \dots, n$

$$\chi^i = \chi - e_i = (\chi_1, \dots, \chi_{i-1}, \chi_i - 1, \chi_{i+1}, \dots, \chi_n),$$

and

$$\Delta_i = \overline{\mathcal{C}}_{\chi^i} = \{f \in \mathcal{C} /\ \mathrm{rank}(f) \leq \chi - e_i\},$$

the variety of complexes where the i-th matrix drops rank by one.

Proposition 19. *The codimension of Δ_i in $\mathcal{E}$ is equal to one, and*

$$\mathcal{E} = \mathcal{C}_\chi \cup \Delta_1 \cup \cdots \cup \Delta_n.$$

Proof. This follows from Proposition 8 and the fact that $s \in \mathbb{N}^n$ satisfies $s < \chi$ if and only if $s \leq \chi - e_i$ for some $i = 1, \dots, n$. □

2. Moduli space of foliations.

2.1. Let X denote a (smooth, complete) algebraic variety over the complex numbers, let L be a line bundle on X and let ω denote a global section of $\Omega^1_X \otimes L$ (a twisted differential 1-form). A simple local calculation shows that $\omega \wedge d\omega$ is a section of $\Omega^3_X \otimes L^{\otimes 2}$. We say that ω is integrable if it satisfies the Frobenius condition $\omega \wedge d\omega = 0$. We denote

$$\mathcal{F}(X, L) \subset \mathbb{P}H^0(X, \Omega^1_X \otimes L),$$

the projective classes of integrable 1-forms. The map

$$\varphi : H^0(X, \Omega^1_X \otimes L) \to H^0(X, \Omega^3_X \otimes L^{\otimes 2}),$$

such that $\varphi(\omega) = \omega \wedge d\omega$ is a homogeneous quadratic map between vector spaces and hence $\varphi^{-1}(0) = \mathcal{F}(X, L)$ is an algebraic variety defined by homogeneous quadratic equations.

Our purpose is to understand the geometry of $\mathcal{F}(X, L)$. In particular, we are interested in the problem of describing its irreducible components. For a survey on this problem see for example [7].

2.2. Let r and d be natural numbers. Consider a differential 1-form in $\mathbb{C}^{r+1}$

$$\omega = \sum_{i=0}^{r} a_i dx_i,$$

where the a_i are homogeneous polynomials of degree $d-1$ in variables $x_0, \ldots, x_r$, with complex coefficients. We say that ω has degree d (in particular the 1-forms dx_i have degree one). Denoting R the radial vector field, let us assume that

$$< \omega, R >= \sum_{i=0}^{r} a_i x_i = 0,$$

so that ω descends to the complex projective space $\mathbb{P}^r$ as a global section of the twisted sheaf of 1-forms $\Omega^1_{\mathbb{P}^r}(d)$. We denote

$$\mathcal{F}(r, d) = \mathcal{F}(\mathbb{P}^r, \mathcal{O}(d)),$$

parametrizing 1-forms of degree d on $\mathbb{P}^r$ that satisfy the Frobenius integrability condition.

3. Complexes associated to an integrable form.

Let us denote

$$H^0(\mathbb{P}^r, \Omega^k_{\mathbb{P}^r}(d)) = \Omega^k_r(d),$$

and

$$\Omega_r = \bigoplus_{d\in\mathbb{N}} \bigoplus_{0\leq k\leq r} \Omega^k_r(d),$$

with structure of bi-graded supercommutative associative algebra given by exterior product $\wedge$ of differential forms.

Definition 20. *Gelfand, Kapranov and Zelevinsky defined in [5] another product in Ω_r, the second multiplication $*$, as follows:*

$$\begin{aligned} \omega_1 * \omega_2 &= \frac{d_1}{d_1+d_2}\omega_1 \wedge d\omega_2 + (-1)^{(k_1+1)(k_2+1)}\frac{d_2}{d_1+d_2}\omega_2 \wedge d\omega_1, \\ &= \frac{d_1}{d_1+d_2}\omega_1 \wedge d\omega_2 + (-1)^{(k_1+1)}\frac{d_2}{d_1+d_2} d\omega_1 \wedge \omega_2, \end{aligned}$$

where $\omega_i \in \Omega^{k_i}_r(d_i)$ for $i = 1, 2$ and $d_1 + d_2 \neq 0$. In case $(d_1, d_2) = (0, 0)$ one defines $\omega_1 * \omega_2 = 0$.

It follows that $\omega_1 * \omega_2 = 0$ if $d_1 = 0$ or $d_2 = 0$.

Remark 21. *For $\omega_i \in \Omega^{k_i}_r(d_i)$ for $i = 1, 2$ as above,*

a) $\omega_1 * \omega_2$ belongs to $\Omega^{(k_1+k_2+1)}_r(d_1 + d_2)$.

b) $\omega_1 * \omega_2 = (-1)^{(k_1+1)(k_2+1)}\omega_2 * \omega_1$.

c) It follows from an easy direct calculation that $*$ is associative (see [5]).

d) For any $\omega \in \Omega^1_r(d)$ we have $\omega * \omega = \omega \wedge d\omega$. In particular, ω is integrable if and only if $\omega * \omega = 0$.

Definition 22. *For $\omega \in \Omega^k_r(d)$ we consider the operator δ_ω*

$$\delta_\omega : \Omega_r \to \Omega_r,$$

such that $\delta_\omega(\eta) = \omega * \eta$ for $\eta \in \Omega_r$.

Remark 23. *From Remark 21 a), if $\omega \in \Omega^{k_1}_r(d_1)$ then*

$$\delta_\omega(\Omega^{k_2}_r(d_2)) \subset \Omega^{(k_1+k_2+1)}_r(d_1 + d_2).$$

In particular, if $\omega \in \Omega^1_r(d_1)$,

$$\delta_\omega(\Omega^{k_2}_r(d_2)) \subset \Omega^{(k_2+2)}_r(d_1 + d_2).$$

Definition 24. *For $\omega \in \Omega^1_r(d)$ and $e \in \mathbb{Z}$ we define two differential graded vector spaces*

$$C^+_\omega(e) : \Omega^0_r(e) \to \Omega^2_r(e + d) \to \Omega^4_r(e + 2d) \to \cdots \to \Omega^{2k}_r(e + kd) \to \ldots,$$

$$C^-_\omega(e) : \Omega^1_r(e) \to \Omega^3_r(e + d) \to \Omega^5_r(e + 2d) \to \cdots \to \Omega^{2k+1}_r(e + kd) \to \ldots,$$

where all maps are δ_ω as in Remark 23.

Proposition 25. *Let $\omega \in \Omega^1_r(d)$, $e \in \mathbb{Z}$ and $k \in \mathbb{N}$ such that $k+2 \le r$. Then $\omega * \eta = 0$ for all $\eta \in \Omega^k_r(e)$ if and only if $\omega = 0$. In other words, the linear map*

$$\delta : \Omega^1_r(d) \to \mathrm{Hom}_K(\Omega^k_r(e), \Omega^{k+2}_r(e+d)),$$

sending $\omega \mapsto \delta_\omega$, is injective.

Proof. First remark that $\omega \wedge \eta = 0$ for all $\eta \in \Omega^k_r(e)$ (with $k+1 \le r$) easily implies $\omega = 0$. Now suppose $\omega * \eta = 0$, that is, $d\ \omega \wedge d\eta + e\ \eta \wedge d\omega = 0$, for all $\eta \in \Omega^k_r(e)$. Take

$$\eta = x_{i_1}^{e-k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

(here x_i denote affine coordinates and $1 < i_1 < \dots i_k < n$). Since $d\eta = 0$, we have

$$dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge d\omega = 0.$$

Hence $d\omega = 0$ by the first remark. Using the hypothesis again, we know $\omega \wedge d\eta = 0$ for all $\eta \in \Omega^k_r(e)$. Now take $\eta = x_{i_{k+1}}^{e-k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ (where $1 < i_1 < \cdots < i_{k+1} < n$). It follows that $dx_{i_1} \wedge \cdots \wedge dx_{i_{k+1}} \wedge \omega = 0$ and hence $\omega = 0$. □

Proposition 26. *$\omega \in \Omega^1_r(d)$ is integrable if and only if $\delta^2_\omega = 0$*

Proof. The associativity stated in Remark 21 c) implies that $\delta_{\omega_1} \circ \delta_{\omega_2} = \delta_{\omega_1 * \omega_2}$. In particular, $\delta^2_\omega = \delta_{\omega*\omega}$ and hence the claim follows from Remark 21 d) and Proposition 25. □

Remark 27. *It follows from Proposition 26 that $C^+_\omega(e)$ and $C^-_\omega(e)$ (Definition 24) are differential complexes (for any $e \in \mathbb{Z}$) if and only if ω is integrable.*

Remark 28. *To fix ideas we shall mostly discuss $C^-_\omega(e)$, but similar considerations apply to $C^+_\omega(e)$. If no confusion seems to arise we shall denote $C^-_\omega(e) = C_\omega(e)$.*

Theorem 29. *Fix $e \in \mathbb{Z}$. Let us consider the graded vector space*

$$\Omega_r(e) = \bigoplus_{0 \le k \le [\frac{r-1}{2}]} \Omega^{2k+1}_r(e+kd),$$

(direct sum of the spaces appearing in $C^-_\omega(e)$ above). Define the linear map

$$\delta(e) = \delta : \Omega^1_r(d) \to \prod_{k=1}^{[\frac{r-1}{2}]} \mathrm{Hom}_K(\Omega^{2k-1}_r(e+(k-1)d), \Omega^{2k+1}_r(e+kd)),$$

such that $\delta(\omega) = \delta_\omega$ for each $\omega \in \Omega^1_r(d)$, and its projectivization

$$\mathbb{P}\delta : \mathbb{P}\Omega^1_r(d) \to \prod_{k=1}^{[\frac{r-1}{2}]} \mathbb{P}\mathrm{Hom}_K(\Omega^{2k-1}_r(e+(k-1)d), \Omega^{2k+1}_r(e+kd)).$$

Denote $\mathcal{C} = \mathcal{C}(\Omega^1_r(e), \Omega^3_r(e+d), \Omega^5_r(e+2d), \dots, \Omega_r^{2[\frac{r-1}{2}]+1}(e+[\frac{r-1}{2}]d))$ the variety of complexes as in 1.1 and $\mathcal{F}(r,d)$ the variety of foliations as in 2.2. Then

$$\mathcal{F}(r,d) = (\mathbb{P}\delta)^{-1}(\mathcal{C}).$$

In other terms, $\mathbb{P}\delta(\mathcal{F}(r,d)) = L \cap \mathcal{C}$, that is, the variety of foliations $\mathcal{F}(r,d)$ corresponds via the linear injective map $\mathbb{P}\delta$ to the intersection of the variety of complexes with the linear space $L = \mathrm{im}(\mathbb{P}\delta)$.

Proof. The statement is a rephrasing of Remark 27. $\square$

Proposition 30. *Let us denote*

$$d_r^k(e) = \dim \Omega_r^k(e) = \binom{r-k+e}{r-k}\binom{d-1}{k},$$

(see [8]) and in particular

$$d_k = d_r^{2k+1}(e+kd) = \dim \Omega_r^{2k+1}(e+kd),\ 0 \leq k \leq [\frac{r-1}{2}].$$

For this $d = (d_0, d_1, \dots, d_{[\frac{r-1}{2}]})$ *we consider the finite ordered set* $R = R(d)$ *as in Proposition 14. Then each irreducible component of the variety of foliations* $\mathcal{F}(r,d)$ *is an irreducible component of the linear section* $(\mathbb{P}\delta)^{-1}(\overline{\mathcal{C}}_r)$ *for a unique* $r \in R^+$.

Proof. From Proposition 14, we have the decomposition into irreducible components

$$\mathcal{C} = \bigcup_{r \in R^+} \overline{\mathcal{C}}_r.$$

From Theorem 29 we obtain:

$$\mathcal{F}(r,d) = (\mathbb{P}\delta)^{-1}(\mathcal{C}) = \bigcup_{r \in R^+} (\mathbb{P}\delta)^{-1}(\overline{\mathcal{C}}_r).$$

This implies that each irreducible component X of $\mathcal{F}(r,d)$ is an irreducible component of $(\mathbb{P}\delta)^{-1}(\overline{\mathcal{C}}_r)$ for some $r \in R^+$. This element r is the sequence of ranks of δ_ω for a general $\omega \in X$, hence it is unique. $\square$

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REPRESENTATIONS OF SOME LATTICES INTO THE GROUP OF ANALYTIC DIFFEOMORPHISMS OF THE SPHERE $\mathbb{S}^2$

JULIE DÉSERTI

Abstract. In [11] it is proved that any morphism from a subgroup of finite index of $\mathrm{SL}(n,\mathbb{Z})$ to the group of analytic diffeomorphisms of $\mathbb{S}^2$ has a finite image as soon as $n \geq 5$. The case $n = 4$ is also claimed to follow along the same arguments; in fact this is not straightforward and that case indeed needs a modification of the argument. In this paper we recall the strategy for $n \geq 5$ and then focus on the case $n = 4$.

1. Introduction

After the works of Margulis ([15, 20]) on the linear representations of lattices of simple, real Lie groups with $\mathbb{R}$-rank larger than 1, some authors, like Zimmer, suggest to study the actions of lattices on compact manifolds ([22, 23, 24, 25]). One of the main conjectures of this program is the following: let us consider a connected, simple, real Lie group G, and let Γ be a lattice of G of $\mathbb{R}$-rank larger than 1. If there exists a morphism of infinite image from Γ to the group of diffeomorphisms of a compact manifold M, then the $\mathbb{R}$-rank of G is bounded by the dimension of M. There are a lot of contributions in that direction ([3, 4, 5, 8, 9, 10, 11, 12, 17, 18]). In this article we will focus on the embeddings of subgroups of finite index of $\mathrm{SL}(n,\mathbb{Z})$ into the group $\mathrm{Diff}^\omega(\mathbb{S}^2)$ of real analytic diffeomorphisms of $\mathbb{S}^2$ (*see* [11]).

The article is organized as follows. First of all we will recall the strategy of [11]: the study of the nilpotent subgroups of $\mathrm{Diff}^\omega(\mathbb{S}^2)$ implies that such subgroups are metabelian. But subgroups of finite index of $\mathrm{SL}(n,\mathbb{Z})$, for $n \geq 5$, contain nilpotent subgroups of length $n-1$ of finite index which are not metabelian; as a consequence Ghys gets the following statement.

Theorem A ([11])**.** *Let Γ be a subgroup of finite index of $\mathrm{SL}(n,\mathbb{Z})$. As soon as $n \geq 5$ there is no embedding of Γ into $\mathrm{Diff}^\omega(\mathbb{S}^2)$.*

To study nilpotent subgroups of $\mathrm{Diff}^\omega(\mathbb{S}^2)$ one has to study nilpotent subgroups of $\mathrm{Diff}^\omega_+(\mathbb{S}^1)$ (*see* §2), and then nilpotent subgroups of the group of formal diffeomorphisms of $\mathbb{C}^2$ (*see* §3). The last section is devoted to establish the following result.

Theorem B. *Let Γ be a subgroup of finite index of $\mathrm{SL}(n,\mathbb{Z})$. As soon as $n \geq 4$ there is no embedding of Γ into $\mathrm{Diff}^\omega(\mathbb{S}^2)$.*

The proof relies on the characterization, up to isomorphism, of nilpotent subalgebras of length 3 of the algebra of formal vector fields of $\mathbb{C}^2$ that vanish at the origin.

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2. Nilpotent subgroups of the group of analytic diffeomorphisms of $\mathbb{S}^1$

Let G be a group; let us set $\mathrm{G}^{(0)} = \mathrm{G}$ and $\mathrm{G}^{(i)} = [\mathrm{G}, \mathrm{G}^{(i-1)}]$ $\forall\, i \geq 1$. The group G is *nilpotent* if there exists an integer n such that $\mathrm{G}^{(n)} = \{\mathrm{id}\}$; the *length of nilpotence* of G is the smallest integer k such that $\mathrm{G}^{(k)} = \{\mathrm{id}\}$.

Set $\mathrm{G}_{(0)} = \mathrm{G}$ and $\mathrm{G}_{(i)} = [\mathrm{G}_{(i-1)}, \mathrm{G}_{(i-1)}]$ $\forall\, i \geq 1$. The group G is *solvable* if $\mathrm{G}_{(n)} = \{\mathrm{id}\}$ for some integer n; the *length of solvability* of G is the smallest integer k such that $\mathrm{G}_{(k)} = \{\mathrm{id}\}$.

We say that the group G (resp. algebra $\mathfrak{g}$) is *metabelian* if $[\mathrm{G}, \mathrm{G}]$ (resp. $[\mathfrak{g}, \mathfrak{g}]$) is abelian.

Proposition 2.1 ([11]). *Any nilpotent subgroup of* $\mathrm{Diff}^{\omega}_{+}(\mathbb{S}^1)$ *is abelian.*

Proof. Let G be a nilpotent subgroup of $\mathrm{Diff}^{\omega}_{+}(\mathbb{S}^1)$. Assume that G is not abelian; it thus contains a Heisenberg group

$$\langle f,\, g,\, h \,|\, [f,g] = h,\; [f,h] = [g,h] = \mathrm{id}\rangle.$$

The application "rotation number"

$$\mathrm{Diff}^{\omega}_{+}(\mathbb{S}^1) \to \mathbb{R}/\mathbb{Z}, \qquad\qquad \psi \mapsto \lim_{n\to+\infty} \frac{\psi^n(x) - x}{n}$$

is not a morphism but its restriction to a solvable subgroup is a morphism ([1]). Hence the rotation number of h is zero, and the set $\mathrm{Fix}(h)$ of fixed points of h is non-empty, and finite. Considering some iterates of f and g instead of f and g one can assume that f and g fix any point of $\mathrm{Fix}(h)$. The set of fixed points of a non-trivial element of $\langle f,\, g\rangle$ is finite and invariant by h so the action of $\langle f,\, g\rangle$ is free [1] on each component of $\mathbb{S}^1 \setminus \mathrm{Fix}(h)$. But the action of a free group on $\mathbb{R}$ is abelian: contradiction. □

3. Nilpotent subgroups of the group of formal diffeomorphisms of $\mathbb{C}^2$

Let us denote $\widehat{\mathrm{Diff}}(\mathbb{C}^2, 0)$ the group of formal diffeomorphisms of $\mathbb{C}^2$, *i.e.*, the formal completion of the group of germs of holomorphic diffeomorphisms at 0. Let Diff_i be the quotient of $\widehat{\mathrm{Diff}}(\mathbb{C}^2, 0)$ by the normal subgroups of formal diffeomorphisms tangent to the identity with multiplicity i; it can be viewed as the set of jets of diffeomorphisms at order i with the law of composition with truncation at order i. Note that Diff_i is a complex linear algebraic group. One can see $\widehat{\mathrm{Diff}}(\mathbb{C}^2, 0)$ as the projective limit of the Diff_i's: $\widehat{\mathrm{Diff}}(\mathbb{C}^2, 0) = \varprojlim \mathrm{Diff}_i$. Let us denote by $\widehat{\chi}(\mathbb{C}^2, 0)$ the algebra of formal vector fields in $\mathbb{C}^2$ vanishing at 0. One can define the set χ_i of the i-th jets of vector fields; one has $\varprojlim \chi_i = \widehat{\chi}(\mathbb{C}^2, 0)$.

Let $\widehat{\mathcal{O}}(\mathbb{C}^2)$ be the ring of formal series in two variables, and let $\widehat{K}(\mathbb{C}^2)$ be its fraction field; $\mathcal{O}_i$ is the set of elements of $\widehat{\mathcal{O}}(\mathbb{C}^2)$ truncated at order i.

The family $\big(\exp_i \colon \chi_i \to \mathrm{Diff}_i\big)_i$ is filtered, *i.e.*, compatible with the truncation. We then define the exponential application as follows: $\exp = \varprojlim \exp_i \colon \widehat{\chi}(\mathbb{C}^2, 0) \to \widehat{\mathrm{Diff}}(\mathbb{C}^2, 0)$.

As in the classical case, if X belongs to $\widehat{\chi}(\mathbb{C}^2, 0)$, then $\exp(X)$ can be seen as the "flow at time $t = 1$" of X. Indeed an element X_i of χ_i can be seen as a derivation of $\mathcal{O}_i$; so it can be written $S_i + N_i$ where S_i and N_i are two semi-simple (resp. nilpotent) derivations that commute. Passing to the limit, one gets $X = S + N$ where S is a semi-simple vector field, N a nilpotent one, and $[S, N] = \mathrm{id}$ (*see* [16]). A semi-simple vector field is a formal vector field conjugate to a diagonal linear vector field that is complete. A vector field is nilpotent if and only if its linear

1. The stabilizer of every point is trivial, *i.e.*, the action of a non-trivial element of $\langle f,\, g\rangle$ has no fixed point.

part is; let us remark that the usual flow φ_t of a nilpotent vector field is polynomial in t

$$\varphi_t(x) = \sum_I P_I(t)x^I, \qquad P_I \in (\mathbb{C}[t])^2$$

so $\varphi_1(x)$ is well-defined. As a consequence $\exp(tX) = \exp(tS)\exp(tN)$ is well-defined for $t = 1$. Note that the Jordan decomposition is purely formal: if X is holomorphic, then S and N are not necessary holomorphic.

Proposition 3.1 ([11])**.** *Any nilpotent subalgebra of* $\widehat{\chi}(\mathbb{C}^2, 0)$ *is metabelian.*

Proof. Let $\mathfrak{l}$ be a nilpotent subalgebra of $\widehat{\chi}(\mathbb{C}^2, 0)$, and let $Z(\mathfrak{l})$ be its center. Since

$$\widehat{\chi}(\mathbb{C}^2, 0) \otimes \widehat{K}(\mathbb{C}^2)$$

is a vector space of dimension 2 over $\widehat{K}(\mathbb{C}^2)$, one has the following alternatives:

- the dimension of the subspace generated by $Z(\mathfrak{l})$ in $\widehat{\chi}(\mathbb{C}^2, 0) \otimes \widehat{K}(\mathbb{C}^2)$ is 1;
- the dimension of the subspace generated by $Z(\mathfrak{l})$ in $\widehat{\chi}(\mathbb{C}^2, 0) \otimes \widehat{K}(\mathbb{C}^2)$ is 2.

Let us study these different cases.

Under the first assumption there exists an element X of $Z(\mathfrak{l})$ having the following property: any vector field of $Z(\mathfrak{l})$ can be written uX with u in $\widehat{K}(\mathbb{C}^2)$. Let us consider the subalgebra $\mathfrak{g}$ of $\mathfrak{l}$ given by

$$\mathfrak{g} = \big\{ \widetilde{X} \in \mathfrak{l} \,|\, \exists\, u \in \widehat{K}(\mathbb{C}^2),\ \widetilde{X} = uX \big\}.$$

Since X belongs to $Z(\mathfrak{l})$, the algebra $\mathfrak{g}$ is abelian; it is also an ideal of $\mathfrak{l}$. Let us assume that $\mathfrak{l}$ is not abelian: let Y be an element of $\mathfrak{l}$ whose projection on $\mathfrak{l}/\mathfrak{g}$ is non-trivial, and central. Any vector field of $\mathfrak{l}$ can be written as $uX + vY$ with u, v in $\widehat{K}(\mathbb{C}^2)$. As X belongs to $Z(\mathfrak{l})$, and Y is central modulo $\mathfrak{g}$ one has

$$X(u) = X(v) = Y(v) = 0.$$

The vector fields $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ being some linear combinations of X and Y with coefficients in $\widehat{K}(\mathbb{C}^2, 0)$, the partial derivatives of v are zero so v is a constant. Therefore $[\mathfrak{l}, \mathfrak{l}] \subset \mathfrak{g}$; but $\mathfrak{g}$ is abelian thus $\mathfrak{l}$ is metabelian.

In the second case $Z(\mathfrak{l})$ contains two elements X and Y which are linearly independent on $\widehat{K}(\mathbb{C}^2)$. Any vector field of $\mathfrak{l}$ can be written as $uX + vY$ with u and v in $\widehat{K}(\mathbb{C}^2)$. Since X and Y belong to $Z(\mathfrak{l})$ one has

$$X(u) = X(v) = Y(u) = Y(v) = 0.$$

As a consequence u and v are constant, *i.e.*, $\mathfrak{l} \subset \{uX + vY \,|\, u,\, v \in \mathbb{C}\}$; in particular $\mathfrak{l}$ is abelian. ☐

Proposition 3.2 ([11])**.** *Any nilpotent subgroup of* $\widehat{\mathrm{Diff}}(\mathbb{C}^2, 0)$ *is metabelian.*

Proof. Let G be a nilpotent subgroup of $\widehat{\mathrm{Diff}}(\mathbb{C}^2, 0)$ of length k. Let us denote by G_i the projection of G on Diff_i. The Zariski closure $\overline{\mathrm{G}_i}$ of G_i in Diff_i is an algebraic nilpotent subgroup of length k. It is sufficient to prove that $\overline{\mathrm{G}_i}$ is metabelian.

Since $\overline{\mathrm{G}_i}$ is a complex algebraic subgroup it is the direct product of the subgroup $\overline{\mathrm{G}_{i,u}}$ of its unipotent elements and the subgroup $\overline{\mathrm{G}_{i,s}}$ of its semi-simple elements (*see for example* [2]).

An element of Diff_i is unipotent if and only if its linear part, which belongs to $\mathrm{GL}(2, \mathbb{C})$, is; so $\overline{\mathrm{G}_{i,s}}$ projects injectively onto a nilpotent subgroup of $\mathrm{GL}(2, \mathbb{C})$. Therefore $\overline{\mathrm{G}_{i,s}}$ is abelian.

The group $\overline{\mathrm{G}_{i,u}}$ coincides with $\exp \mathfrak{l}_i$ where $\mathfrak{l}_i$ is a nilpotent Lie algebra of χ_i of length k. Passing to the limit one thus obtains the existence of a nilpotent subalgebra $\mathfrak{l}$ of $\widehat{\chi}(\mathbb{C}^2, 0)$ of length k such that $\exp(\mathfrak{l})$ projects onto $\overline{\mathrm{G}_{i,u}}$ for any i. According to Proposition 3.1 the subalgebra $\mathfrak{l}$, and thus $\overline{\mathrm{G}_{i,u}}$ are metabelian. ☐

4. Nilpotent subgroups of the group of analytic diffeomorphisms of $\mathbb{S}^2$

Proposition 4.1 ([11]). *Any nilpotent subgroup of* $\mathrm{Diff}^\omega(\mathbb{S}^2)$ *has a finite orbit.*

Proof. Let G be a nilpotent subgroup of $\mathrm{Diff}^\omega(\mathbb{S}^2)$; up to finite index one can assume that the elements of G preserve the orientation. Let ϕ be a non-trivial element of G that commutes with G. Let $\mathrm{Fix}(\phi)$ be the set of fixed points of ϕ; it is a non-empty analytic subspace of $\mathbb{S}^2$ invariant by G. If p is an isolated fixed point of ϕ, then the orbit of p under the action of G is finite. So it is sufficient to study the case where $\mathrm{Fix}(\phi)$ only contains curves; there are thus two possibilities:

- $\mathrm{Fix}(\phi)$ is a singular analytic curve whose set of singular points is a finite orbit for G;
- $\mathrm{Fix}(\phi)$ is a smooth analytic curve, not necessary connected. One of the connected component of $\mathbb{S}^2 \setminus \mathrm{Fix}(\phi)$ is a disk denoted by $\mathbb{D}$. Any subgroup Γ of finite index of G which contains ϕ fixes $\mathbb{D}$. Let us consider an element γ of Γ, and a fixed point m of γ that belongs to $\overline{\mathbb{D}}$. By construction ϕ has no fixed point in $\mathbb{D}$ so according to the Brouwer Theorem $(\phi^k(m))_k$ has a limit point on the boundary $\partial\mathbb{D}$ of $\overline{\mathbb{D}}$. Therefore γ has at least one fixed point on $\partial\mathbb{D}$. The group Γ thus acts on $\partial\mathbb{D}$, and any of its elements has a fixed point on $\mathbb{D}$. Then Γ has a fixed point on $\partial\mathbb{D}$ (Proposition 2.1).

□

Theorem 4.2 ([11]). *Any nilpotent subgroup of* $\mathrm{Diff}^\omega(\mathbb{S}^2)$ *is metabelian.*

Proof. Let G be a nilpotent subgroup of $\mathrm{Diff}^\omega(\mathbb{S}^2)$, and let Γ be a subgroup of finite index of G having a fixed point m (such a subgroup exists according to Proposition 4.1). One can embed Γ into $\widehat{\mathrm{Diff}}(\mathbb{R}^2,0)$, and so into $\widehat{\mathrm{Diff}}(\mathbb{C}^2,0)$, by considering the jets of infinite order of elements of Γ in m. According to Proposition 3.2 the group Γ is metabelian.

One can suppose that G is a finitely generated group.

Let us first assume that G has no element of finite order. Then G is a cocompact lattice of the nilpotent, simply-connected Lie group $\mathrm{G}\otimes\mathbb{R}$ (*see* [19]). The group G is metabelian if and only if $\mathrm{G}\otimes\mathbb{R}$ is; but Γ is metabelian so $\mathrm{G}\otimes\mathbb{R}$ also.

Finally let us consider the case where G contains at least one element of finite order. The set of such elements is a normal subgroup of G that thus intersects non-trivially the center $Z(\mathrm{G})$ of G. Let us consider a non-trivial element ϕ of $Z(\mathrm{G})$ which has finite order. Let us recall that a finite group of diffeomorphisms of the sphere is conjugate to a group of isometries. Denote by G^+ the subgroup of elements of G which preserve the orientation. It is thus sufficient to prove that G^+ is metabelian; indeed if ϕ does not preserve the orientation, then ϕ has order 2, and $\mathrm{G} = \mathbb{Z}/2\mathbb{Z}\times\mathrm{G}^+$. So let us assume that ϕ preserves the orientation; ϕ is conjugate to a direct isometry of $\mathbb{S}^2$, and has exactly two fixed points on the sphere. The group G has thus an invariant set of two elements. By considering germs in the neighborhood of these two points, one gets that G can be embedded into $2\cdot\mathrm{Diff}(\mathbb{R}^2,0)$ [2] and thus into $2\cdot\mathrm{Diff}(\mathbb{C}^2,0)$:

$$1 \longrightarrow \mathrm{Diff}(\mathbb{C}^2,0) \longrightarrow 2\cdot\mathrm{Diff}(\mathbb{C}^2,0) \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0.$$

Remark that $2\cdot\mathrm{Diff}(\mathbb{C}^2,0)$ is the projective limit of the algebraic groups $2\cdot\mathrm{Diff}_i$. One can conclude as in the proof of Proposition 3.2 except that the subgroup of the semi-simple elements of $2\cdot\mathrm{Diff}_i$ embeds now in $2\cdot\mathrm{GL}(2,\mathbb{C})$; it is metabelian because it contains an abelian subgroup of index 2. □

Let Γ be a subgroup of finite index of $\mathrm{SL}(n,\mathbb{Z})$ for $n\geq 5$. Since Γ contains nilpotent subgroups of finite index of length $n-1$ (for example the group of upper triangular unipotent matrices) which are not metabelian one gets the following statement.

2. Let G be a group and let q be a positive integer; $q\cdot\mathrm{G}$ denotes the semi-direct product of G^q by $\mathbb{Z}/q\mathbb{Z}$ under the action of the cyclic permutation of the factors.

Corollary 4.3 ([11])**.** *Let* Γ *be a subgroup of finite index of* $\mathrm{SL}(n,\mathbb{Z})$*; as soon as* $n\geq 5$ *there is no embedding of* Γ *into* $\mathrm{Diff}^\omega(\mathbb{S}^2)$*.*

5. Nilpotent subgroups of length 3 of the group of analytic diffeomorphisms of $\mathbb{S}^2$

Let us precise Proposition 3.1 for nilpotent subalgebras of length 3 of $\widehat{\chi}(\mathbb{C}^2,0)$. Let $\mathfrak{l}$ be such an algebra. The dimension of the subspace generated by $Z(\mathfrak{l})$ in $\widehat{\chi}(\mathbb{C}^2,0)\otimes\widehat{K}(\mathbb{C}^2)$ has dimension at most 1, for else $\mathfrak{l}$ would be abelian (Proposition 3.1) and this is impossible under our assumptions. So let us assume that the dimension of the subspace generated by $Z(\mathfrak{l})$ in $\widehat{\chi}(\mathbb{C}^2,0)\otimes\widehat{K}(\mathbb{C}^2)$ is 1. There exists an element X in $Z(\mathfrak{l})$ with the following property: any element of $Z(\mathfrak{l})$ can be written uX with u in $\widehat{K}(\mathbb{C}^2)$. Let $\mathfrak{g}$ denote the abelian ideal of $\mathfrak{l}$ defined by

$$\mathfrak{g}=\big\{\widetilde{X}\in\mathfrak{l}\,\big|\,\exists\, u\in\widehat{K}(\mathbb{C}^2),\,\widetilde{X}=uX\big\}.$$

By hypothesis $\mathfrak{l}$ is not abelian. Let Y be in $\mathfrak{l}$; assume that its projection onto $\mathfrak{l}/\mathfrak{g}$ is a non-trivial element of $Z(\mathfrak{l}/\mathfrak{g})$. Any vector field of $\mathfrak{l}$ can be written

$$uX+vY,\qquad u,\,v\in\widehat{K}(\mathbb{C}^2).$$

Since X, resp. Y belongs to $Z(\mathfrak{l})$ (resp. $Z(\mathfrak{l}/\mathfrak{g})$) and since the length of $\mathfrak{l}$ is 3, one has

$$X(u)=Y^3(u)=X(v)=Y(v)=0. \tag{5.1}$$

If X and Y are non-singular, one can choose formal coordinates x and y such that $X=\frac{\partial}{\partial x}$ and $Y=\frac{\partial}{\partial y}$. The previous conditions can be thus translated as follows: v is a constant and u is a polynomial in y of degree 2. We will see that we have a similar property without assumption on X and Y.

Lemma 5.1. *Let* X *and* Y *be two vector fields of* $\widehat{\chi}(\mathbb{C}^2,0)$ *that commute and are not colinear. One can assume that* $(X,Y)=\Big(\frac{\partial}{\partial\widetilde{x}},\frac{\partial}{\partial\widetilde{y}}\Big)$ *where* $\widetilde{x}$ *and* $\widetilde{y}$ *are two independent variables in a Liouvillian extension of* $\widehat{K}(\mathbb{C}^2,0)$*.*

Proof. Since X and Y are non-colinear, there exist two 1-forms α, β with coefficients in $\widehat{K}(\mathbb{C}^2)$ such that $\alpha(X)=1$, $\alpha(Y)=0$, $\beta(X)=0$, and $\beta(X)=1$. The vector fields X and Y commute if and only if α and β are closed (this statement of linear algebra is true for convergent meromorphic vector fields and is also true in the completion). The 1-form α is closed so according to [7] one has

$$\alpha=\sum_{i=1}^r\lambda_i\frac{d\widehat{\phi}_i}{\widehat{\phi}_i}+d\Big(\frac{\widehat{\psi}_1}{\widehat{\psi}_2}\Big)=d\Big(\sum_{i=1}^r\lambda_i\log\widehat{\phi}_i+\frac{\widehat{\psi}_1}{\widehat{\psi}_2}\Big)$$

where $\widehat{\psi}_1$, $\widehat{\psi}_2$, and the $\widehat{\phi}_i$ denote some formal series and the λ_i some complex numbers. One has a similar expression for β. So there exists a Liouvillian extension κ of $\widehat{K}(\mathbb{C}^2)$ having two elements $\widetilde{x}$ and $\widetilde{y}$ with $\alpha=d\widetilde{x}$ and $\beta=d\widetilde{y}$. One thus has $X(\widetilde{x})=1$, $X(\widetilde{y})=0$, $Y(\widetilde{x})=0$, and $Y(\widetilde{y})=1$. □

From (5.1) one gets: v is a constant, and u is a polynomial in $\widetilde{y}$ of degree 2; so one proves the following statement.

Proposition 5.2. *Let* $\mathfrak{l}$ *be a nilpotent subalgebra of* $\widehat{\chi}(\mathbb{C}^2,0)$ *of length* 3*. Then* $\mathfrak{l}$ *is isomorphic to a subalgebra of*

$$\mathfrak{n}=\Big\{P(\widetilde{y})\frac{\partial}{\partial\widetilde{x}}+\alpha\frac{\partial}{\partial\widetilde{y}}\;\Big|\;\alpha\in\mathbb{C},\,P\in\mathbb{C}[\widetilde{y}],\,\deg P=2\Big\}.$$

Remark 5.3. We use a real version of this statement whose proof is an adaptation of the previous one: a nilpotent subalgebra $\mathfrak{l}$ of length 3 of $\widehat{\chi}(\mathbb{R}^2,0)$ is isomorphic to a subalgebra of

$$\mathfrak{n}=\Big\{P(\widetilde{y})\frac{\partial}{\partial\tilde{x}}+\alpha\frac{\partial}{\partial\tilde{y}}\,\Big|\,\alpha\in\mathbb{R},\ P\in\mathbb{R}[\widetilde{y}],\ \deg P=2\Big\}.$$

Theorem 5.4. *Let Γ be a subgroup of finite index of $\mathrm{SL}(n,\mathbb{Z})$; as soon as $n\geq 4$ there is no embedding of Γ into $\mathrm{Diff}^\omega(\mathbb{S}^2)$.*

Proof. Let $\mathrm{U}(4,\mathbb{Z})$ (resp. $\mathrm{U}(4,\mathbb{R})$) be the subgroup of unipotent upper triangular matrices of $\mathrm{SL}(4,\mathbb{Z})$ (resp. $\mathrm{SL}(4,\mathbb{R})$); it is a nilpotent subgroup of length 3. Assume that there exists an embedding from a subgroup Γ of finite index of $\mathrm{SL}(4,\mathbb{Z})$ into $\mathrm{Diff}^\omega(\mathbb{S}^2)$. Up to finite index Γ contains $\mathrm{U}(4,\mathbb{Z})$. Let us set $\mathrm{H}=\rho(\mathrm{U}(4,\mathbb{Z}))$. Up to finite index H has a fixed point (Proposition 4.1). One can thus see H as a subgroup of $\mathrm{Diff}(\mathbb{R}^2,0)\subset\widehat{\mathrm{Diff}}(\mathbb{R}^2,0)$ up to finite index.

Let us denote by j^1 the morphism from $\widehat{\mathrm{Diff}}(\mathbb{R}^2,0)$ to Diff_i. Up to conjugation, $j^1(\rho(\mathrm{U}(4,\mathbb{Z})))$ is a subgroup of

$$\Big\{\begin{bmatrix}\lambda & t\\ 0 & \lambda\end{bmatrix}\,\Big|\,\lambda\in\mathbb{R}^*,\ t\in\mathbb{R}\Big\}.$$

Up to index 2 one can thus assume that $j^1\circ\rho$ takes values in the connected, simply-connected group T defined by

$$\mathrm{T}=\Big\{\begin{bmatrix}\lambda & t\\ 0 & \lambda\end{bmatrix}\,\Big|\,\lambda,\ t\in\mathbb{R},\ \lambda>0\Big\}.$$

Let us set

$$\mathrm{Diff}_i(\mathrm{T})=\big\{f\in\mathrm{Diff}_i\,|\,j^1(f)\in\mathrm{T}\big\};$$

the group $\mathrm{Diff}_i(\mathrm{T})$ is a connected, simply-connected, nilpotent and algebraic group. The morphism

$$\rho_i\colon\mathrm{U}(4,\mathbb{Z})\to\mathrm{Diff}_i$$

can be extended to a unique continuous morphism $\widetilde{\rho_i}\colon\mathrm{U}(4,\mathbb{R})\to\mathrm{Diff}_i(\mathrm{T})$ (*see* [13, 14]) so to an algebraic morphism [3]. Let us note that $\widetilde{\rho_i}(\mathrm{U}(4,\mathbb{Z}))$ is an algebraic subgroup of $\mathrm{Diff}_i(\mathrm{T})$ that contains $\rho_i(\mathrm{U}(4,\mathbb{Z}))$; in particular $\overline{\mathrm{H}_i}=\overline{\rho_i(\mathrm{U}(4,\mathbb{Z}))}\subset\widetilde{\rho_i}(\mathrm{U}(4,\mathbb{R}))$. By construction the family $(\mathrm{H}_i)_i$ is filtered; since the extension is unique, the family $(\widetilde{\rho_i})_i$ is also filtered. Therefore $\mathrm{K}=\varprojlim\overline{\mathrm{H}_i}$ is well-defined. Since ρ is injective, H is a nilpotent subgroup of length 3; as $\mathrm{H}\subset\mathrm{K}$ and as any $\overline{\mathrm{H}_i}$ is nilpotent of length at most 3 the group K is nilpotent of length at most 3. For i sufficiently large $\widetilde{\rho_i}(\mathrm{U}(4,\mathbb{R}))$ is nilpotent of length 3; this group is connected so its Lie algebra is also nilpotent of length 3. Therefore the image of

$$D\widetilde{\rho}:=\varprojlim D\widetilde{\rho_i}\colon\mathfrak{u}(4,\mathbb{R})\to\widehat{\chi}(\mathbb{R}^2,0)$$

is isomorphic to $\mathfrak{n}$ (Proposition 5.2). So there exists a surjective map ψ from $\mathfrak{u}(4,\mathbb{R})$ onto $\mathfrak{n}$. The kernel of ψ is an ideal of $\mathfrak{u}(4,\mathbb{R})$ of dimension 2; hence $\ker\psi=\langle\delta_{14},a\delta_{13}+b\delta_{24}\rangle$ where the δ_{ij} denote the Kronecker matrices. One concludes by noting that $\dim Z(\mathfrak{u}(4,\mathbb{R})/\ker\psi)=2$ whereas $\dim Z(\mathfrak{n})=1$. □

Corollary 5.5. *The image of a morphism from a subgroup of $\mathrm{SL}(n,\mathbb{Z})$ of finite index to $\mathrm{Diff}^\omega(\mathbb{S}^2)$ is finite as soon as $n\geq 4$.*

3. Let N_1 and N_2 be two connected, simply-connected, nilpotent and algebraic subgroups of $\mathbb{R}$; any continuous morphism from N_1 to N_2 is algebraic.

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RIEMANN-ROCH THEORY ON FINITE SETS

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Abstract. In [2] M. Baker and S. Norine developed a theory of divisors and linear systems on graphs, and proved a Riemann-Roch Theorem for these objects (conceived as integer-valued functions on the vertices). In [6] and [7] the authors generalized these concepts to real-valued functions, and proved a corresponding Riemann-Roch Theorem in that setting, showing that it implied the Baker-Norine result. In this article we prove a Riemann-Roch Theorem in a more general combinatorial setting that is not necessarily driven by the existence of a graph.

1. Introduction

Baker and Norine showed in [2] that a Riemann-Roch formula holds for an analogue of linear systems defined on the vertices of finite connected graphs. There, the image of the graph Laplacian induces an equivalence relation on the group of *divisors* of the graph, which are integer-valued functions defined on the set of vertices. This equivalence relation is the analogue of linear equivalence in the classical algebro-geometric setting. Gathmann and Kerber [5] later used the Baker-Norine result to prove a Riemann-Roch theorem for tropical curves.

We showed in [6] that the Baker-Norine result implies a generalization of the Riemann-Roch formula to edge-weighted graphs, where the edge weights can be R-valued, where R is an arbitrary subring of the reals; the equivalence relation induced by the image of the edge-weighted graph Laplacian applies equally well to divisors which are R-valued functions defined on the set of vertices. We prove in [7] our version of the R-valued Riemann-Roch theorem from first principles; this gave an independent proof of the Baker-Norine result as well. In [1], Amini and Caporoso develop a Riemann-Roch theory for vertex-weighted graphs over the integers; related work on computing the rank of these divisors can be found in [3] and [4].

The notion of linear equivalence above is induced by the appropriate graph Laplacian acting as a group in the space of divisors, which may be viewed as points in $\mathbb{Z}^n$ (the Baker-Norine case) or more generally R^n, where n is the number of vertices of the graph. A crucial role in this theory is played by a certain set of divisors $\mathcal{N}$, which is a union of orbits of the group action. In this paper, we propose a generalization of this Riemann-Roch formula where no graph structure need be present; however an appropriate set $\mathcal{N}$ must still be defined, having a specified symmetry property. The Baker-Norine proof is combinatorial and relies on properties of the finite graph, where our more general result presented in this paper only assumes this symmetry condition. In the graph case this symmetry condition holds, and therefore this is a generalization of the Baker-Norine result. Potentially our result may apply to many other discrete objects, with the additional generalization that the divisors are R-valued rather than restricted to integer values.

The setup we will use is as follows. Choose a subring R of the reals, and fix a positive integer n. Let V be the group of points in R^n under component-wise addition. If $x \in V$, we we will use the functional notation $x(i)$ to denote the the i-th component of x.

For any $x \in V$, define the *degree* of x as

$$\deg(x) = \sum_{i=1}^{n} x(i).$$

For any $d \in R$, define the subset $V_d \subset V$ to be

$$V_d = \{x \in V \mid \deg(x) = d\}.$$

Note that the subset V_0 is a subgroup; for any d, V_d is a coset of V_0 in V.

Fix the parameter $g \in R$, which we call the *genus*, and choose a set $\mathcal{N} \subset V_{g-1}$. For $x \in V$, define

$$\begin{aligned} x^+ &= \max(x, 0) \\ x^- &= \min(x, 0) \end{aligned}$$

where max and min are evaluated at each coordinate. It follows that

$$x = x^+ + x^- \quad \text{and} \quad x^+ = -(-x)^-.$$

We then define the *dimension* of $x \in V$ to be

$$\ell(x) = \min_{\nu \in \mathcal{N}} \{\deg((x - \nu)^+)\}.$$

This definition of the dimension agrees with the definition given by Baker and Norine in the special application to the graph setting, as we observed in [6].

We can now state our main result.

Theorem 1.1. *Let V be the additive group of points in R^n for a subring $R \subset \mathbb{R}$ and fix $g \in R$. Suppose $\kappa \in V_{2g-2}$, and $\mathcal{N} \subset V_{g-1}$, satisfying the symmetry condition*

$$\nu \in \mathcal{N} \Longleftrightarrow \kappa - \nu \in \mathcal{N}.$$

Then for every $x \in V$,

$$\ell(x) - \ell(\kappa - x) = \deg(x) - g + 1$$

We will give a proof of Theorem 1.1 in §2. In §3, we will give examples of κ and $\mathcal{N}$ (coming from the graph setting) which satisfy the conditions of Theorem 1.1, and show how this Riemann-Roch formulation is equivalent to that given in [7]. Finally in §4, we gives examples that do not arise from graphs.

2. Proof of Riemann-Roch Formula

The dimension of $x \in V$

$$\ell(x) = \min_{\nu \in \mathcal{N}} \{\deg((x - \nu)^+)\}$$

can be written as

$$\ell(x) = \min_{\nu \in \mathcal{N}} \left\{ \sum_{i=1}^{n} \max\{x(i) - \nu(i), 0\} \right\}.$$

If $x(i) \geq \nu(i)$ for each i, $\sum_{i=1}^{n} \max\{x(i) - \nu(i), 0\}$ is the *taxicab* distance from x to ν. Thus, $\ell(x)$ is the taxicab distance from x to the portion of the set $\mathcal{N}$ such that $x \geq \mathcal{N}$, where the inequality is evaluated at each component.

We will now proceed with the proof of the Riemann-Roch formula.

Proof. (Theorem 1.1)

Suppose that $\mathcal{N} \subset V_{g-1}$ and $\kappa \in V$ satisfy the symmetry condition. We can then write

$$\begin{aligned} \ell(\kappa - x) &= \min_{\nu \in \mathcal{N}} \{\deg((\kappa - x - \nu)^+)\} \\ &= \min_{\nu \in \mathcal{N}} \{\deg(((\kappa - \nu) - x)^+)\} \\ &= \min_{\mu \in \mathcal{N}} \{\deg((\mu - x)^+)\}. \end{aligned}$$

Using the identities $x = x^+ + x^-$ and $x^+ = -(x^-)$, we have

$$\begin{aligned}\min_{\mu\in N}\{\deg((\mu - x)^+)\} &= \min_{\mu\in\mathcal{N}}\{\deg((\mu - x)) - \deg((\mu - x)^-)\}\\ &= \min_{\mu\in\mathcal{N}}\{\deg((\mu - x)) + \deg((x - \mu)^+)\}.\end{aligned}$$

Since $\mu \in \mathcal{N}$ we know that $\deg(\mu) = g - 1$, thus $\deg(\mu - x) = g - 1 - \deg(x)$ and thus

$$\begin{aligned}\ell(\kappa - x) &= \deg((\mu - x)) + \min_{\mu\in\mathcal{N}}\{\deg((x - \mu)^+)\}\\ &= g - 1 - \deg(x) + \ell(x).\end{aligned}$$

□

Note that $\kappa \in V$ is the analogue to the canonical divisor in the classical Riemann-Roch formula.

3. Graph Examples

Let Γ be a finite, edge-weighted connected simple graph with n vertices. We will assume that Γ has no loops. Let $w_{ij} \in R$ with $w_{ij} \geq 0$ be the weight of the edge connecting vertices v_i and v_j. The no loops assumption is also applied to the edge weights so that $w_{ii} = 0$ for each i. We showed in [7] that such a graph satisfies an equivalent Riemann-Roch formula as in Theorem 1.1.

In this setting, the genus $g = 1 + \sum_{i<j} w_{ij} - n$, and the canonical element κ is defined by $\kappa(j) = \deg(v_j) - 2$. (Here $\deg(v)$ for a vertex v is the sum of the weights of the edges incident to v.) As shown in [7], the set $\mathcal{N} \subset V_{g-1}$ is generated by a set $\{\nu_1, \ldots, \nu_s\}$ as follows.

Fix a vertex v_k and let $(j_1, \ldots, j_n)$ be a permutation of $(1, \ldots, n)$ such that $j_1 = k$. There are then $(n-1)!$ such permutations; for each permutation, we compute a $\nu \in V_{g-1}$ defined by

$$\nu(j_l) = \begin{cases} -1 & \text{if } l = 1\\ -1 + \sum_{i=1}^{l-1} w_{j_i j_l} & \text{if } l > 1.\end{cases}$$

Each such ν may not be unique; set s to be the number of unique ν's and index this set $\{\nu_1, \ldots, \nu_s\}$. We then define the set $\mathcal{N}$ as

$$\mathcal{N} = \{x \in V \mid x \sim \nu_i \text{ for some } i = 1, \ldots s\}.$$

Here the equivalence relation $\sim$ is induced by the subgroup

$$H = < h_1, h_2, \ldots, h_{n-1} > \subset V_0$$

where each $h_i \in R^n$ is defined as

$$h_i(j) = \begin{cases} \deg(v_i) & \text{if } i = j\\ -w_{ij} & \text{if } i \neq j.\end{cases}$$

Note that H is the edge-weighted Laplacian of Γ.

As an example, consider a two vertex graph Γ with edge weight $w_{12} = p > 0$. Then $g = p - 1$ and $H =< (p, -p) >= \mathbb{Z}(p, -p)$. The set $\mathcal{N} \subset V_{g-1}$ is $\mathcal{N} = \{\nu \mid \nu \sim (p-1, -1)\}$ and $\kappa = (p-2, p-2)$. Figure 1 shows the divisors $x \in \mathbb{R}^2$ for this graph in the plane. The shaded region, which is bounded by the corner points in the set $\mathcal{N}$, represent points x with $\ell(x) = 0$.

We can show directly that $\mathcal{N}$ and κ for the two-vertex graph example satisfy the necessary condition for Theorem 1.1 to hold. If $\kappa - x \in \mathcal{N}$, then $\kappa - x = m(p, -p) + (p-1, -1)$ for some $m \in \mathbb{Z}$. Solving for x, we have

$$\begin{aligned} x &= (p-2, p-2) - m(p, -p) - (p-1, -1)\\ &= (-mp - 1, mp + p - 1)\\ &= (p-1, -1) - (m+1)(p, -p)\end{aligned}$$

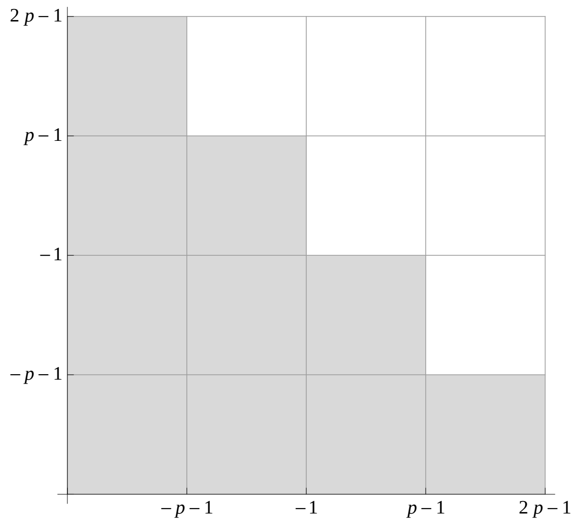

Figure 1. Divisors in $\mathbb{R}^2$ for a two-vertex graph with p edges. The shaded region represents points $x \in \mathbb{R}^2$ with $\ell(x) = 0$; for a general point $x \in \mathbb{R}^2$, $\ell(x)$ is the taxicab distance to the shaded region.

and thus $x \in \mathcal{N}$. Similarly, if $\nu \in \mathcal{N}$, it easily follows that $\kappa - \nu \in \mathcal{N}$.

Now consider a three vertex graph with edge weights $w_{12} = p$, $w_{13} = q$ and $w_{23} = r$. The set $\mathcal{N}$ can be generated by $\nu_1 = (-1, p-1, q+r-1)$ and $\nu_2 = (-1, p+r-1, q-1)$; H can be generated by $h_1 = (p+q, -p, -q)$ and $h_2 = (-p, p+r, -r)$. In Figure 2, the region representing points $x \in \mathbb{R}^3$ such that $\ell(x) = 0$ is shown for a three vertex graph with edge weights $p = 1$, $q = 3$ and $r = 4$.

4. Non-graph Examples

The main result of this paper would not be interesting if there were no examples $\mathcal{N}$ and κ that were not derived from graphs.

Theorem 4.1. *There exist $\kappa \in V$ and $\mathcal{N} \subset V_{g-1}$ such that Theorem 1.1 holds where $\mathcal{N}$ is not generated from a finite connected graph.*

Proof. Let $n = 2$ and choose $\mathcal{N} = \{\nu \in G \mid \nu \sim (2, -2)\}$ where $H = <(-4, 4)>$, with $\kappa = (0, 0)$ and $g = 1$. If H were generated from a two vertex graph, using the notation from the previous section we would have $p = 4$. This would require $\kappa = (2, 2)$ with $\mathcal{N}$ generated by $\nu_1 = (3, -1)$.

Since there is no integer m such that $\kappa = (0, 0) = (2, 2) + m(-4, 4)$ (and likewise there is no m such that $\nu_1 = (2, -2) = (3, -1) + m(-4, 4)$), H cannot be generated from a two vertex graph.

Now suppose that $\nu \in \mathcal{N}$. Then $\nu = (2, -2) + m(-4, 4)$ for some integer m, and

$$\begin{aligned} \kappa - \nu &= (0,0) - (2,-2) - m(-4,4) \\ &= (2,-2) - (m-1)(-4,4) \end{aligned}$$

thus $\kappa - \nu \in \mathcal{N}$.

Similarly, if $\kappa - \nu \in \mathcal{N}$, $\kappa - \nu = (2, -2) + m(-4, 4)$ for some integer m, and

$$\begin{aligned} \nu &= \kappa - (2,-2) - m(-4,4) \\ &= (4m-2, -4m+2) \\ &= (2,-2) - (m-1)(-4,4) \end{aligned}$$

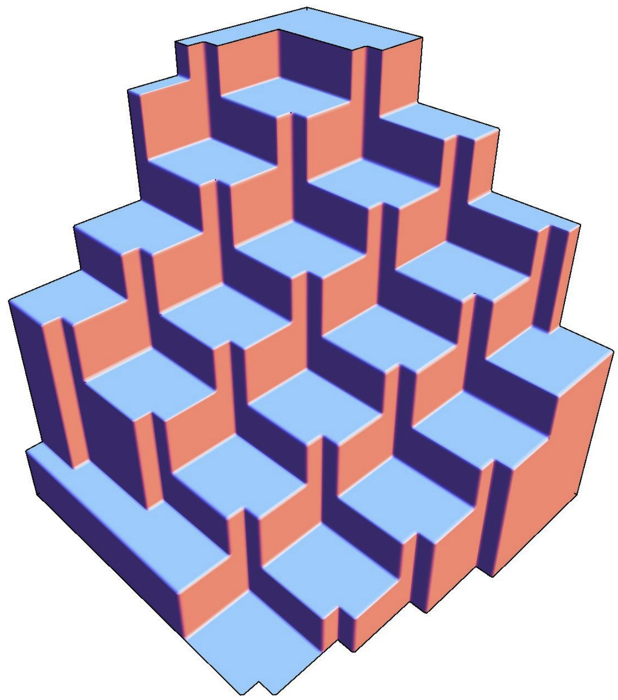

FIGURE 2. Divisors in $\mathbb{R}^3$ for a three-vertex graph edge weights $w_{12} = 1$, $w_{13} = 3$ and $w_{23} = 4$. The solid region represents points $x \in \mathbb{R}^3$ with $\ell(x) = 0$; for a general point $x \in \mathbb{R}^3$, $\ell(x)$ is the taxicab distance to the surface.

thus $\nu \in \mathcal{N}$ and H, κ, $\mathcal{N}$ satisfies Theorem 1.1. □

We include in Figure 3 a representation of the divisors $x \in \mathbb{R}^2$ with $\ell(x) = 0$ for the example used in the proof of Theorem 4.1. The plot is identical to that of a two vertex graph with $p = 4$ but is shifted by -1 in each direction.

It is also possible to produce non-graph examples by using more generators for $\mathcal{N}$. In Figure 4, the divisors $x \in \mathbb{R}^2$ with $\ell(x) = 0$ are shown where $\mathcal{N}$ is generated by two points $\nu_1 = (0, 4)$ and $\nu_2 = (1, 3)$, using $H =< (-3, 3) >$ and $\kappa = (0, 0)$.

As an application to discrete geometry, consider a set of points in $\mathcal{N} \subset \mathbb{R}^n$ along with $\kappa \in \mathbb{R}^n$ and $g \in \mathbb{R}$ which satisfy the conditions of Theorem 1.1. For each $\nu \in \mathcal{N}$, define

$$\mathcal{E}_\nu = \{x \in \mathbb{R}^n \mid x \leq \nu\}$$

and let

$$\mathcal{E} = \bigcup_{\nu \in \mathcal{N}} \mathcal{E}_\nu.$$

The set $\mathcal{E}$ consists of the points $x \in \mathbb{R}^n$ such that $\ell(x) = 0$; more generally, $\ell(x)$ is the taxicab distance from $x \in \mathbb{R}^n$ to $\mathcal{E}$. Theorem 1.1 then gives a lower bound $\ell(x) \geq \deg(x) - g + 1$ as well as an exact formula for $\ell(x)$ using the correction term $\ell(\kappa - x)$. It would be interesting to find and classify different discrete geometric structures, in addition to finite graphs, which admit such $\mathcal{N}$, g, and κ satisifying the symmetry condition of Theorem 1.1.

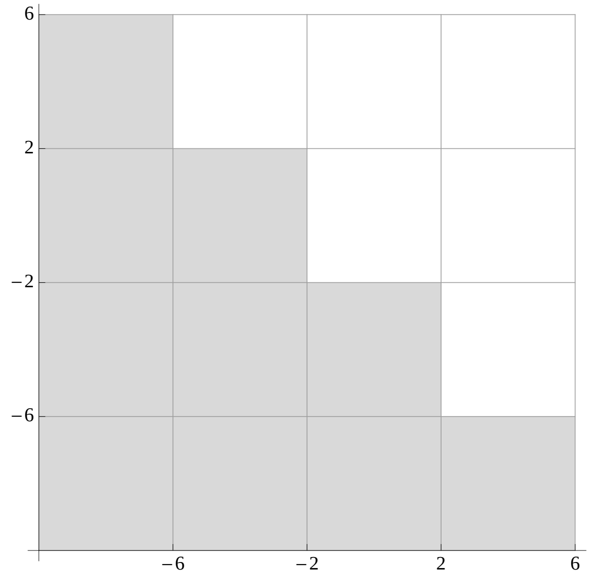

FIGURE 3. Divisors $x \in \mathbb{R}^2$ with $\ell(x) = 0$ for the non-graph example in the proof of Theorem 4.1. Note that this example is identical to the two vertex graph example in Figure 1 with $p = 4$, but shifted by $(-1, -1)$.

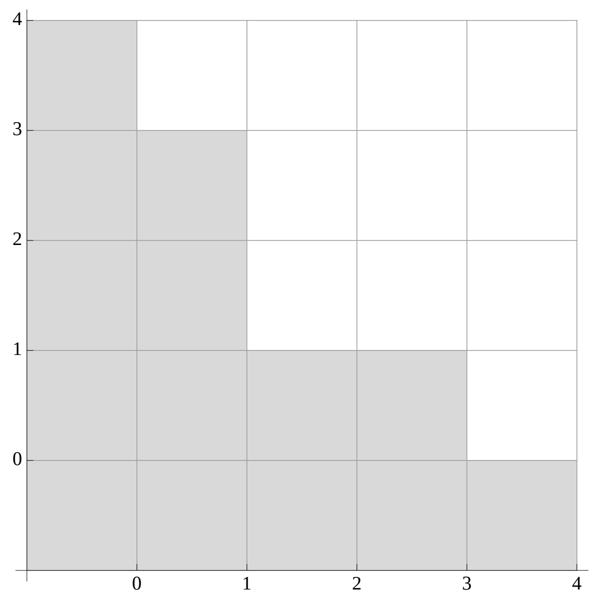

FIGURE 4. Divisors $x \in \mathbb{R}^2$ with $\ell(x) = 0$ for a non-graph example with $\mathcal{N}$ generated by $(0, 4)$ and $(1, 3)$, $H =< (-3, 3) >$ and $\kappa = (0, 0)$.

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FOLIATIONS WITH A MORSE CENTER

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Abstract. We say that a holomorphic foliation $\mathcal{F}$ on a complex surface M has a Morse center at $p \in M$ if $\mathcal{F}$ has a local first integral with a Morse singularity at p. Given a line bundle $\mathcal{L}$ on M, let $\mathbb{F}\text{ol}(M,\mathcal{L}) = \{$ foliations $\mathcal{F}$ on M such that $T^*(\mathcal{F}) = \mathcal{L}\}$ and $\mathbb{F}\text{ol}_C(M,\mathcal{L})$ be the closure of the set $\{\mathcal{F} \in \mathbb{F}\text{ol}(M,\mathcal{L}) \,|\, \mathcal{F}$ has a Morse center$\}$. In the first result of this paper we prove that $\mathbb{F}\text{ol}_C(M,\mathcal{L})$ is an algebraic subset of $\mathbb{F}\text{ol}(M,\mathcal{L})$. We apply this result to prove the persistence of more than one Morse center for some known examples, as for instance the logarithmic and pull-back foliations. As an application we give a simple proof that $\mathcal{R}(1, d+1)$ is an irreducible component of the space of foliations of degree d with a Morse center on $\mathbb{P}^2$, where $\mathcal{R}(m,n)$ denotes the space of foliations with a rational first integral of the form f^m/g^n with $m\, dg(f) = n\, dg(g)$.

1. Basic definitions and results

Given a complex surface M and a line bundle $\mathcal{L}$ on M we will denote by $\mathbb{F}\text{ol}(M,\mathcal{L})$ the set of holomorphic foliations on M with cotangent bundle $\mathcal{L}$ (cf. [Br]),

$$\mathbb{F}\text{ol}(M,\mathcal{L}) := \{\mathcal{F} \mid T^*_{\mathcal{F}} = \mathcal{L}\} = \mathbb{P}H^0(M, TM \oplus \mathcal{L}) \; .$$

Of course, we will assume that $\mathbb{F}\text{ol}(M,\mathcal{L}) \neq \emptyset$. In this case, if M is compact then $\mathbb{F}\text{ol}(M,\mathcal{L})$ is a finite dimensional projective space.

When $M = \mathbb{P}^2$ then the *degree* of a foliation $\mathcal{F}$, $dg(\mathcal{F})$, is the number of tangencies of a $\mathcal{F}$ with a generic straight line of $\mathbb{P}^2$. If $dg(\mathcal{F}) = d \geq 0$, then $\mathcal{F} \in \mathbb{F}\text{ol}(\mathbb{P}^2, \mathcal{O}(d-1))$ and we will denote $\mathbb{F}\text{ol}(\mathbb{P}^2, \mathcal{O}(d-1)) := \mathbb{F}\text{ol}(d)$.

Definition 1. We say that $p \in M$ is a *Morse center* of $\mathcal{F} \in \mathbb{F}\text{ol}(M,\mathcal{L})$ if p is an isolated singularity of $\mathcal{F}$ and the germ of $\mathcal{F}$ at p has a holomorphic first integral with a Morse singularity at p.

Remark 1.1. Definition 1 can be rephrased as follows: the germ of $\mathcal{F}$ at p is represented by some germ at p of holomorphic vector field X with an isolated singularity at p and there exists a germ $f \in \mathcal{O}_p$ such that $X(f) = 0$ and p is a Morse singularity of f. By Morse lemma, there exists a local holomorphic coordinate system $(x,y) \in \mathbb{C}^2$ such that $f(x,y) = x\,y$. If $X = P(x,y)\,\partial_x + Q(x,y)\,\partial_y$ in these coordinates, then $X(f) = 0$ implies

$$y\,P(x,y) + x\,Q(x,y) = 0 \quad \Longrightarrow \quad X = f(x,y)\,(x\,\partial_x - y\,\partial_y) \; ,$$

where $f(0,0) \neq 0$. In particular, the Baum-Bott index of $\mathcal{F}$ at p is zero and its characteristic values are both -1.

Recall that if $\mathcal{F}$ has a non-degenerate singularity at $q \in M$ and is represented by a vector field Y near q and the eigenvalues of $DY(q)$ are $\lambda_1, \lambda_2 \neq 0$ then the characteristic values are

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λ_1/λ_2 and λ_2/λ_1, whereas the Baum-Bott index is

$$BB(\mathcal{F},q) = \frac{tr(DY(q))^2}{det(DY(q))} = \frac{\lambda_2}{\lambda_1} + \frac{\lambda_1}{\lambda_2} + 2 \ .$$

We will denote by $\mathbb{F}\text{ol}_C(M,\mathcal{L})$ the closure in $\mathbb{F}\text{ol}(M,\mathcal{L})$ of the set

$$\{\mathcal{F} \in \mathbb{F}\text{ol}(M,\mathcal{L}) \,|\, \mathcal{F} \text{ has a Morse center}\}.$$

When $M = \mathbb{P}^2$, we will use the notations $\mathbb{F}\text{ol}(d) := \mathbb{F}\text{ol}(\mathbb{P}^2, \mathcal{O}(d-1))$ and

$$\mathbb{F}\text{ol}_C(d) := \mathbb{F}\text{ol}_C(\mathbb{P}^2, \mathcal{O}(d-1)), \ d \geq 0.$$

A well-known fact is that $\mathbb{F}\text{ol}_C(d)$ is an algebraic subset of $\mathbb{F}\text{ol}(d)$ (cf. [Mo1]). In our first result we generalize this fact.

Theorem 1. *Assume that M is compact and $\mathbb{F}ol(M,\mathcal{L}) \neq \emptyset$. Then $\mathbb{F}ol_C(M,\mathcal{L})$ is an algebraic subset of $\mathbb{F}ol(M,\mathcal{L})$.*

A natural problems that arises is the following :

Problem 1. *Classify the irreducible components of $\mathbb{F}ol_C(M,\mathcal{L})$.*

In the case of $\mathbb{P}^2$ three cases are known : $\mathbb{F}\text{ol}_C(0)$, $\mathbb{F}\text{ol}_C(1)$ and $\mathbb{F}\text{ol}_C(2)$. For instance, $\mathbb{F}\text{ol}_C(0) = \emptyset$ because any foliation of degree 0 is equivalent the foliation defined by the radial vector field $R = x\,\partial_x + y\,\partial_y$, which has no Morse centers (cf. [Br]).

On the other hand, any foliation $\mathcal{F}_o \in \mathbb{F}\text{ol}(1)$ can be defined by a holomorphic vector field on $\mathbb{P}^2$. Hence, if $\mathcal{F} \in \mathbb{F}\text{ol}(1)$ has a Morse center, then in some affine coordinate system it can be defined by the linear vector field $X = x\,\partial_x - y\,\partial_y$. It follows that $\mathbb{F}\text{ol}_C(1)$ is the closure of the orbit of $\mathcal{F}_o$ under the natural action of $Aut(\mathbb{P}^2)$ on $\mathbb{F}\text{ol}(1)$. In particular, this implies that $\mathbb{F}\text{ol}_C(1)$ is irreducible and $dim_{\mathbb{C}}(\mathbb{F}\text{ol}_C(1)) = 6$.

The case of $\mathbb{F}\text{ol}_C(2)$ is not so simple, but it is known that it has four irreducible components. Before describe them, let us consider a class of foliations with a Morse center.

Example 1 (Foliations defined by closed 1-forms)**.** Let ω be a closed meromorphic 1-form on the complex surface M, $\omega \not\equiv 0$. It is known that ω defines a foliation on M, which we will denote by $\mathcal{F}_\omega$ (cf. [Br]). In an open set $V \subset M \setminus |\omega|_\infty$, diffeomorphic to a polydisc, $\mathcal{F}$ can be defined by a holomorphic vector field X such that $\omega(X) = 0$. Since V is simply-connected, there exists $f \in \mathcal{O}(V)$ such that $\omega|_V = df$. In particular, f is a holomorphic first integral of $\mathcal{F}_\omega$ on V and if f has a Morse singularity $p \in V$ then p is a Morse center of $\mathcal{F}_\omega$. We will consider the following two cases :

(a). $\omega = dF$, where F is meromorphic on M. In this case F is a first integral of $\mathcal{F}_\omega$.
(b). ω is a *logarithmic* 1-form on M, that is $|\omega|_\infty \neq \emptyset$ and $(\omega)_\infty$ is reduced.

Let us consider $M = \mathbb{P}^2$ and $\Pi\colon \mathbb{C}^3 \setminus \{0\} \to \mathbb{P}^2$ be the canonical projection. If ω is a meromorphic closed 1-form in $\mathbb{P}^2$ then the 1-form $\Omega = \Pi^*(\omega)$ is closed and satisfies

$$i_R(\Omega) := \Omega(R) = 0,$$

where $R = x\,\partial_x + y\,\partial_y + z\,\partial_z$ is the radial vector field in $\mathbb{C}^3$. We can write $(\Omega)_\infty = F_1^{\ell_1}...F_k^{\ell_k}$, where $F_j \in \mathbb{C}[x,y,z]$ is a homogeneous of degree $d_j \in \mathbb{N}$ and $\ell_j \in \mathbb{N}$, $1 \leq j \leq k$. In this case, it can be proved that (cf. [Ce-Ma])

$$\Omega = \sum_{j=1}^{k} \lambda_j \, \frac{dF_j}{F_j} + d\left(\frac{G}{F_1^{\ell_1 - 1}...F_k^{\ell_k - 1}}\right) \ ,$$

where $\lambda_j = Res(\Omega, F_j = 0) = Res(\omega, \Pi(F_j = 0))$, $1 \leq j \leq k$, $dg(G) = \sum_{j=1}^{k}(\ell_j - 1)\, d_j$ and $\sum_{j=1}^{k} d_j\, \lambda_j = 0$. We will say that Ω *represents* $\mathcal{F}_\omega$ *in homogeneous coordinates*. Let us observe the following facts :

- If ω is exact then $\lambda_1 = ... = \lambda_k = 0$ and $\Omega = d(G/F)$, where $F = F_1^{\ell_1 - 1} ... F_k^{\ell_k - 1}$.
- If ω is a logarithmic form then $\ell_j = 1$, $\lambda_j \neq 0$, $1 \leq j \leq k$, and

$$\Omega = \sum_{j=1}^{k} \lambda_j \frac{dF_j}{F_j} \ . \tag{1}$$

Denote by $(\Omega)_0$ the divisor of zeroes of Ω, that is the codimension one part of

$$sing(\Omega) = \{q \in \mathbb{C}^3 \mid \Omega(q) = 0\}.$$

- If ω is logarithmic then $k \geq 2$ and

$$dg(\mathcal{F}_\omega) = d_1 + ... + d_k - 2 - dg((\Omega)_0) \ . \tag{2}$$

- If ω is logarithmic and $F_1, ..., F_k$ are generic polynomials of degrees $d_1, ..., d_k$, then $(\Omega)_0 = \emptyset$ and $\mathcal{F}_\omega \in \mathbb{F}\mathrm{ol}(d)$, where $d = d_1 + ... + d_k - 2$.

We will use the notation

$$\mathcal{L}(d_1, ..., d_k) = \overline{\{\mathcal{F}_\omega \mid \Pi^*(\omega) \text{ is like in } (1) \text{ and } dg(F_j) = d_j \ , \ 1 \leq j \leq k\}} \ .$$

Note that $\mathcal{L}(d_1, ..., d_k) \subset \mathbb{F}\mathrm{ol}(d)$.

Observe also that if $k = 2$ then the relation $\lambda_1\, d_1 + \lambda_2\, d_2 = 0$ implies that we can assume

$$\Omega = -d_2 \frac{dF_1}{F_1} + d_1 \frac{dF_2}{F_2} = \frac{d(F_2^{d_1}/F_1^{d_2})}{F_2^{d_1}/F_1^{d_2}} \implies F_2^{d_1}/F_1^{d_2}$$

is a first integral of $\mathcal{F}_\omega$. In this case we use also the notation $\mathcal{L}(d_1, d_2) = \mathcal{R}(d_1, d_2)$.

Denote by $\mathcal{P}_\ell \subset \mathbb{PC}[x, y, z]$ the projectivization of the set of homogeneous polynomials of degree ℓ. Given $D = (d_1, ..., d_k)$ then the set

$$\mathcal{P}(D, k) := \left\{ (F_1, ..., F_k, \lambda_1, ..., \lambda_k) \in \mathcal{P}_{d_1} \times ... \times \mathcal{P}_{d_k} \times \mathbb{C}^k \mid \sum_{j=1}^{k} \lambda_j\, d_j = 0 \right\}$$

parametrizes $\mathcal{L}(D) := \mathcal{L}(d_1, ..., d_k)$ as

$$(F, \Lambda) = (F_1, ..., F_k, \lambda_1, ..., \lambda_k) \in \mathcal{P}(D, k) \ \mapsto \ \mathcal{F}_{\Omega(F,\Lambda)} \ ,$$

where,

$$\Omega(F, \Lambda) := \sum_{j=1}^{k} \lambda_j \frac{dF_j}{F_j} \ .$$

We will denote by $\mathcal{F}(F, \Lambda)$ the foliation of $\mathbb{P}^2$ which is represented in homogeneous coordinates by the form $\Omega(F, \Lambda)$. With the above notations, in the next section we will sketch the proof of the following result:

Proposition 1. *There exists a Zariski open and dense set $Z \subset \mathcal{P}(D, k)$ such that*

(a). *If $(F, \Lambda) \in Z$ then all singularities of $\mathcal{F}(F, \Lambda)$ are non-degenerate. In particular, $\mathcal{F}(F, \Lambda) \in \mathbb{F}ol(d)$ with $d = d_1 + ... + d_k - 2$ and $\#(sing(\mathcal{F}(F, \lambda))) = d^2 + d + 1$.*

(b). *If* $(F,\Lambda) \in Z$ *then* $\mathcal{F}(F,\Lambda)$ *has* $N(d)$ *Morse centers, where*

$$N(d) = d^2 + d + 1 - \sum_{i<j} d_i\, d_j \ .$$

In particular, if $d \geq 2$ *then* $N(d) > 0$ *and* $\mathcal{L}(d_1, ..., d_k) \subset \mathbb{F}ol_C(d)$.

Example 2 (The exceptional component of $\mathbb{F}\mathrm{ol}_C(2)$)**.** Let $f(x,y,z) = x^3 - 3\,y\,x\,z$ and

$$g(x,y,z) = z^2 + y\,z - x^2/2.$$

Then the foliation $\mathcal{F}_o$ on $\mathbb{P}^2$ with first integral f^2/g^3 is of degree two. In fact, as the reader can check, z divides the form $\Omega = 2\,\frac{df}{f} - 3\,\frac{dg}{g}$ and so $dg(\mathcal{F}_o) = 3 + 2 - 2 - dg((\Omega)_0) = 2$ by (2). The foliation $\mathcal{F}_o$ has a Morse center at the point $[0:0:1] \in \mathbb{P}^2$.

In fact, it can be represented in the affine coordinate system $z = 1$ by the form

$$\omega = (y - x^2 + y^2)\,dx + x\,(1 - y/2)\,dy$$

or by the vector field $X = x\,(1 - y/2)\partial_x - (y - x^2 + y^2)\partial_y$ and so it has a non-degenerate singularity at $(0,0)$ with characteristic values -1. It is a Morse center because $\mathcal{F}_o$ has a first integral.

We denote by $\mathcal{E}(2)$ the orbit of $\mathcal{F}_o$ under the action of $Aut(\mathbb{P}^2)$:

$$\mathcal{E}(2) = \{T^*(\mathcal{F}_o) \mid T \in Aut(\mathbb{P}^2)\} \ .$$

Now, we can describe $\mathbb{F}\mathrm{ol}_C(2)$. The next result is a consequence of Dulac's classification of quadratic differential equations in $\mathbb{C}^2$ with a Morse center (cf. [Du]) and of a result of [Ce-LN].

Theorem 1.1. $\mathbb{F}\mathrm{ol}_C(2)$ has four irreducible components: $\mathcal{R}(1,3)$, $\mathcal{L}(1,1,2)$, $\mathcal{L}(1,1,1,1)$ and $\overline{\mathcal{E}(2)}$.

Now, let us state some known results about the components of $\mathbb{F}\mathrm{ol}_C(d)$, $d \geq 3$. Before state the first result, let us fix some notations. Recall that a foliation $\mathcal{F} \in \mathbb{F}\mathrm{ol}(d)$ can be defined in an affine coordinate system $(x,y) \in \mathbb{C}^2 \subset \mathbb{P}^2$ by a polynomial vector field $X = P\,\partial_x + Q\,\partial_y$, with

$$\begin{array}{l} P(x,y) = p(x,y) + x.\,g(x,y) \\ Q(x,y) = q(x,y) + y.\,g(x,y) \end{array} ,$$

where $max(dg(p), dg(q)) \leq d$ and g is homogeneous of degree d. Note that $g \equiv 0$ if, and only if, the line at infinity of the affine coordinates, denoted by L_∞, is $\mathcal{F}$-invariant. When $g \not\equiv 0$ then the intersection of the directions defined by $g(x,y) = 0$ and L_∞ are the tangent points of $\mathcal{F}$ with this line. In some contexts, like for instance in Ilyashenko's works, the line at infinity is invariant by the foliations. Motivated by this, we will use the following notation

$$\mathbb{F}\mathrm{ol}(d, L_\infty) = \{\mathcal{F} \in \mathbb{F}\mathrm{ol}(d) \,|\, L_\infty \text{ is } \mathcal{F}-\text{invariant}\}$$

and

$$\mathbb{F}\mathrm{ol}_C(d, L_\infty) = \mathbb{F}\mathrm{ol}_C(d) \cap \mathbb{F}\mathrm{ol}(d, L_\infty) \ .$$

The first result in the study of $\mathbb{F}\mathrm{ol}_C(d, L_\infty)$ was Dulac's theorem (cf. [Du]) in the case $d = 2$. The second one, due to Ilyashenko [Il], can be stated as follows

Theorem 1.2. $\mathcal{R}(1, d+1) \cap \mathbb{F}ol_C(d, L_\infty)$ *is an irreducible component of* $\mathbb{F}ol_C(d, L_\infty)$.

After that, J. Muciño in [Mu] proved the following :

Theorem 1.3. *If* $k \geq 3$ *then* $\mathcal{R}(k,k)$ *is an irreducible component of* $\mathbb{F}ol_C(2\,k-2)$.

Remark 1.2. We would like to observe that $\mathcal{R}(1,1) = \mathbb{F}\mathrm{ol}(0)$ and that $\mathcal{R}(2,2)$ is not an irreducible component of $\mathbb{F}\mathrm{ol}_C(2)$ because it is a proper subset of $\mathcal{L}(1,1,2)$.

The general case for foliations with a rational first integral was proved by H. Movasati in [Mo1].

Theorem 1.4. *If* $d_1 + d_2 \geq 5$ *then* $\mathcal{R}(d_1, d_2)$ *is an irreducible component of* $\mathbb{F}ol_C(d_1 + d_2 - 2)$.

The case of logarithmic foliations, in the context of foliations with L_∞ invariant, was considered also by H. Movasati. Given $k \geq 2$ and $D = (d_1, ..., d_k)$, set

$$\mathcal{L}_\infty(D) := \mathcal{L}(1, d_1, ..., d_k) \cap \mathbb{F}\mathrm{ol}(d, L_\infty),$$

where $d = d_1 + ... + d_k - 1$.

Theorem 1.5. *Given* $D = (d_1, ..., d_k)$ *with* $k \geq 2$ *and* $d = d_1 + ... + d_k - 1 \geq 2$, $\mathcal{L}_\infty(D)$ *is an irreducible component of* $\mathbb{F}ol_C(d, L_\infty)$.

In some of the above results, one of the tools of the proof is to prove that when we perturb the foliation in such a way that some of the centers persits then all others persist after the perturbation (see for instance [Il]). Motivated by this fact, we consider the following situation: let $\mathcal{L}$ be a line bundle on a compact surface M. Assume that $\mathbb{F}\mathrm{ol}_C(M, \mathcal{L}) \neq \emptyset$ and let $\mathcal{V}$ be an irreducible component of $\mathbb{F}\mathrm{ol}_C(M, \mathcal{L})$. Let $\mathcal{F}_o \in \mathcal{V}$ and p_o be a Morse center of $\mathcal{F}_o$. Since p_o is a non-degenerate singularity of $\mathcal{F}_o$, by applying the implicit function theorem, there exists a holomorphic map $\mathcal{F} \mapsto P(\mathcal{F})$, defined in some neighborhood U of $\mathcal{F}_o$, such that:

- $P(\mathcal{F}) \in (M, p_o)$ is a non-degenerate singularity of $\mathcal{F}$ for every $\mathcal{F} \in U$.

Definition 2. In the above situation, we say that p_o is a *persistent center in* $\mathcal{V}$ (briefly p.c. in $\mathcal{V}$) if $P(\mathcal{F})$ is a Morse center of $\mathcal{F}$ for every $\mathcal{F} \in \mathcal{V} \cap U$. We set

$$Npc(\mathcal{F}_o, \mathcal{V}) = \text{ number of persistent centers of } \mathcal{F}_o \text{ in } \mathcal{V}$$

and

$$Npc(\mathcal{V}) = max\,\{Npc(\mathcal{F}, \mathcal{V}) \mid \mathcal{F} \in \mathcal{V}\}\ .$$

We need another definition.

Definition 3. Let $\mathcal{F}_o \in \mathbb{F}\mathrm{ol}(M, \mathcal{L})$ and p_o be a non-degenerate singularity of $\mathcal{F}_o$. Let

$$\mathcal{G} \colon [0, 1] \to \mathbb{F}\mathrm{ol}(M, \mathcal{L})$$

be a continuous curve with $\mathcal{G}(0) = \mathcal{F}_o$. We say that p_o can be *continued* along $\mathcal{G}$ if there exists a curve $\gamma \colon [0, 1] \to M$ such that $\gamma(0) = p_o$ and $\gamma(t)$ is a non-degenerate singularity of $\mathcal{G}(t)$ for all $t \in [0, 1]$. We say also that γ is a *continuation* of p_o along $\mathcal{G}$.

Now we can state the following result:

Theorem 2. *Let* $\mathcal{V}$ *be an irreducible component of* $\mathbb{F}ol_C(M, \mathcal{L})$. *Fix* $\mathcal{F}_o \in \mathcal{V}$ *and let* p_o *be a p.c. in* $\mathcal{V}$ *of* $\mathcal{F}_o$. *Let* $\mathcal{G} \colon [0, 1] \to \mathcal{V}$ *be a continuous curve with* $\mathcal{G}(0) = \mathcal{F}_o$ *and assume that* p_o *can be continued along* $\mathcal{G}$ *by a curve* $\gamma \colon [0, 1] \to M$. *Then* $\gamma(1)$ *is a p.c. in* $\mathcal{V}$ *of* $\mathcal{G}(1)$.

A straightforward consequence is the following :

Corollary 1. *Let* $\mathcal{V}$ *be an irreducible component of* $\mathbb{F}ol_C(M, \mathcal{L})$ *and* $\mathcal{F}_o \in \mathcal{V}$ *be a foliation with* $k \geq 2$ *Morse centers, say* $p_1, ..., p_k$, *where* p_1 *is a p.c. in* $\mathcal{V}$. *Assume further that there exist continuous curves* $\mathcal{G}_j \colon [0, 1] \to \mathcal{V}$, $j = 2, ..., k$, *such that*

(a). $\mathcal{G}_j(0) = \mathcal{G}_j(1) = \mathcal{F}_o$, *for all* $j = 1, ..., k$.

(b). *For all* $j = 2, ..., k$, p_1 *admits a continuation* $\gamma_j \colon [0, 1] \to M$ *along* $\mathcal{G}_j$ *such that* $\gamma_j(1) = p_j$, $2 \leq j \leq k$.

Then $p_2, ..., p_k$ *are persistent centers of* $\mathcal{F}_o$ *in* $\mathcal{V}$ *and* $Npc(\mathcal{V}) \geq k$.

As an application of corollary 1 we will prove that all centers of a generic logarithmic foliation are persistent in the irreducible component. At this point we should say that Movasati's theorem 1.5 was only proved in the context of foliations with the line at infinity invariant. It is not known if $\mathcal{L}(d_1, ..., d_k)$ is an irreducible component of $\mathbb{F}\mathrm{ol}_C(d)$, $d = d_1 + ... + d_k - 2$. However, we have the following :

Corollary 2. *Let $k \geq 2$, $D = (d_1, ..., d_k)$, $d = d_1 + ... + d_k - 2 \geq 2$ and*

$$N(D) = d^2 + d + 1 - \sum_{i<j} d_i\, d_j.$$

Denote by $\mathcal{V}(D)$ the irreducible component of $\mathbb{F}ol_C(d)$ that contains $\mathcal{L}(D)$. Then the generic foliation $\mathcal{F} \in \mathcal{L}(D)$ has $N(D)$ Morse centers. Moreover, all these centers are persistent in $\mathcal{V}(D)$ and $Npc(\mathcal{V}(D)) = N(D)$.

As a consequence we will give a simple proof that $\mathcal{R}(1, d+1)$ is an irreducible component of $\mathbb{F}\mathrm{ol}_C(d)$ for all $d \geq 2$.

Corollary 3. *$\mathcal{R}(1, d+1)$ is an irreducible component of $\mathbb{F}ol_C(d)$ for all $d \geq 2$.*

Another class of foliation with Morse centers are the so called *pull-back foliations.*

Example 3 (Pull-back foliations)**.** Let M, N be compact surfaces, $\mathcal{G} \in \mathbb{F}\mathrm{ol}(M, \mathcal{L})$ and $\Psi\colon N \dashrightarrow M$ be a rational map of topological degree $k \geq 2$. We would like to remark that in some cases $\mathcal{F} := \Psi^*(\mathcal{G})$ has Morse centers. In fact, assume that:

(i). Ψ has a fold curve $\mathcal{C}$ and $\mathcal{D} := \Psi(\mathcal{C})$.

(ii). There are smooth points $p \in \mathcal{C}$ and $q = \psi(q) \in \mathcal{D}$ such that $q \notin sing(\mathcal{G})$, but $\mathcal{G}$ has a non-degenerate tangency with $\mathcal{D}$ at q.

In this case, p is a Morse center of $\mathcal{F}$. In fact, (i) and (ii) imply that there exist local coordinate systems, $(U, (x, y))$ at the source and $(V, (u, v))$ at the target, such that:

(iii). $\mathcal{C} \cap U = (y = 0)$, $\mathcal{D} \cap V = (v = 0)$, $p = (x = y = 0)$, $q = (u = v = 0)$ and $\Psi(x, y) = (x, y^2)$.

(iv). Condition (ii) implies that $\mathcal{G}$ has a local holomorphic first integral at q of the form $g(u, v) = v + u^2 + h.o.t.$.

It follows from (iv) that $f(x, y) := g(x, y^2) = x^2 + y^2 + h.o.t.$ is a local first integral of $\mathcal{F}$ at p of Morse type. Therefore, $\mathcal{F}$ has Morse centers.

In the case of $M = N = \mathbb{P}^2$ the map Ψ can be lifted by the projection $\Pi\colon \mathbb{C}^3 \setminus \{0\} \to \mathbb{P}^2$ to a polynomial map

$$\tilde{\Psi} = (F, G, H)\colon \mathbb{C}^3 \to \mathbb{C}^3 \ ,$$

with $\Pi \circ \tilde{\Psi} = \Psi \circ \Pi$, where F, G and H are homogeneous polynomials of the same degree, the *algebraic degree of* Ψ, which we denote $deg(\Psi)$.

Remark 1.3. If Ψ and $\mathcal{G}$ are generic, with $dg(\mathcal{G}) = d$ and $deg(\Psi) = k$, then

$$dg(\Psi^*(\mathcal{G})) = (d+2)\, k - 2.$$

Moreover, all singularities of $\Psi^*(\mathcal{G})$ are non-degenerate and $\Psi^*(\mathcal{G})$ has

$$N(d, k) := 3(k-1)(k(d+1) - 1)$$

Morse centers. Let us denote

$$PB(d, k) = \{\Psi^*(\mathcal{G}) \,|\, deg(\Psi) = k \ , \ dg(\mathcal{G}) = d\} \ .$$

Note that if $k > 1$ and $d \geq 0$ then

$$PB(d, k) \subset \mathbb{F}\mathrm{ol}_C((d+2)k - 2) \ ,$$

because $N(d,k) > 0$. We would like to remark that $PB(0,k) = \mathcal{R}(k,k)$ (because $\mathbb{F}\mathrm{ol}(0) = \mathcal{R}(1,1)$) and $PB(1,k) = \mathcal{L}(k,k,k)$ (because $\mathbb{F}\mathrm{ol}(1) = \mathcal{L}(1,1,1)$). Therefore, a natural question that arises is the following:

Problem 2. *Is $PB(d,k)$ an irreducible component of $\mathbb{F}ol_C((d+2)k-2)$ if $k \geq 2$ and $d \geq 2$?*

Another consequence of theorem 2 is the following :

Corollary 4. *Let $d,k \geq 2$ and $\mathcal{V}(d,k)$ be the irreducible component of $\mathbb{F}ol_C((d+2)k-2)$ which contains $PB(d,k)$. Then the generic foliation $\mathcal{F} \in PB(d,k)$ has $N(d,k) = 3(k-1)(k(d+1)-1)$ Morse centers. Moreover, all these centers are persistent in $\mathcal{V}(d,k)$ and $Npc(\mathcal{V}(d,k)) = N(d,k)$.*

We finish this section with an example.

Example 4 (An example with $\mathbb{F}\mathrm{ol}_C(M,\mathcal{L}) = \mathbb{F}\mathrm{ol}(M,\mathcal{L})$)**.** Let M be the rational surface obtained by blowing-up a point $p \in \mathbb{P}^2$. Denote by $\pi\colon (M,E) \to (\mathbb{P}^2,p)$ the blow-up map, where $E = \pi^{-1}(p)$ is the associated divisor. Given $\mathcal{G} \in \mathbb{F}\mathrm{ol}(d)$, where $d \geq 2$ and $p \notin sing(\mathcal{G})$, set $\mathcal{F}_\mathcal{G} := \pi^*(\mathcal{G})$. Since $p \notin sing(\mathcal{G})$ it is known that (cf. [Br]) :

- E is $\mathcal{F}_\mathcal{G}$ invariant.
- $\mathcal{F}_\mathcal{G}$ has an unique singularity $\hat{p}$ in E, which is a Morse center of $\mathcal{F}_\mathcal{G}$.
- $T^*_{\mathcal{F}_\mathcal{G}} = \Pi^*(T^*_\mathcal{G}) \oplus \mathcal{O}_M(E)$. In particular, $\mathcal{F}_\mathcal{G} \in \mathbb{F}\mathrm{ol}(M,\mathcal{L})$, where
$$\mathcal{L} = \Pi^*(\mathcal{O}(d-1)) \oplus \mathcal{O}_M(E).$$
- The map $\pi^*\colon \mathbb{F}\mathrm{ol}(d) \to \mathbb{F}\mathrm{ol}(M,\mathcal{L})$ is an isomorphism, because $\pi\colon M \to \mathbb{P}^2$ is birational.

This implies that $\mathbb{F}\mathrm{ol}_C(M,\mathcal{L}) = \mathbb{F}\mathrm{ol}(M,\mathcal{L})$, because the set $\{\mathcal{G} \in \mathbb{F}\mathrm{ol}(d) \mid p \notin sing(\mathcal{G})\}$ is a Zariski open and dense subset of $\mathbb{F}\mathrm{ol}(d)$.

2. Proofs

2.1. **Proof of Theorem 1.** Let M be a compact complex surface and $\mathcal{L}$ be a line bundle such that $\mathbb{F}\mathrm{ol}_C(M,\mathcal{L}) \neq \emptyset$. We will assume also that $\mathbb{F}\mathrm{ol}_C(M,\mathcal{L}) \subsetneqq \mathbb{F}\mathrm{ol}(M,\mathcal{L})$. Let us consider the analytic subset $\mathcal{S}(\mathcal{L})$ of $M \times \mathbb{F}\mathrm{ol}(M,\mathcal{L})$ defined by
$$\mathcal{S}(\mathcal{L}) = \{(p,\mathcal{F}) \in M \times \mathbb{F}\mathrm{ol}(\mathcal{L}), \mid p \text{ is a singularity of } \mathcal{F}\} \ .$$
We call $\mathcal{S}(\mathcal{L})$ the *total space of singularities* of $\mathbb{F}\mathrm{ol}(M,\mathcal{L})$. The set
$$\mathcal{S}_{dg}(\mathcal{L}) = \{(p,\mathcal{F}) \in \mathcal{S}(\mathcal{L}) \mid p \text{ is a degenerate singularity of } \mathcal{F}\}$$
will be called the total space of degenerate singularities of $\mathbb{F}\mathrm{ol}(M,\mathcal{L})$. Observe that $\mathcal{S}_{dg}(\mathcal{L})$ is an analytic subset of $\mathcal{S}(\mathcal{L})$. We leave the proof of this fact to the reader.

The *total space of Morse centers* of $\mathbb{F}\mathrm{ol}(M,\mathcal{L})$ is, by definition the set
$$\mathcal{S}_C(\mathcal{L}) := \{(p,\mathcal{F}) \in \mathcal{S}(\mathcal{L}) \mid p \text{ is a Morse center of } \mathcal{F}\} \ .$$
Note that, by definition $\mathcal{S}_C(\mathcal{L}) \subset \mathcal{S}(\mathcal{L}) \setminus \mathcal{S}_{dg}(\mathcal{L})$, because a Morse center is a non-degenerate singularity.

Remark 2.1. Denote by $P_2\colon \mathcal{S}(\mathcal{L}) \to \mathbb{F}\mathrm{ol}(M,\mathcal{L})$ the restriction of the second projection to $\mathcal{S}(\mathcal{L})$,
$$P_2(p,\mathcal{F}) = \Pi_2(p,\mathcal{F}) = \mathcal{F} \ , \ (p,\mathcal{F}) \in \mathcal{S}(\mathcal{L}) \ .$$
Note that $\mathbb{F}\mathrm{ol}_C(M,\mathcal{L}) = P_2\left(\overline{\mathcal{S}_C(\mathcal{L})}\right)$, where $\overline{\mathcal{S}_C(\mathcal{L})}$ denotes the closure of $\mathcal{S}_C(\mathcal{L})$. On the other hand, P_2 is finite to one in the subset of $\mathcal{S}(\mathcal{L})$ such that the singularities of $P_2(p,\mathcal{F}) = \mathcal{F}$ has only isolated singularities, because in this case the number of singularities of $\mathcal{F}$, counted with multiplicities, is given by (cf. [Br])
$$\mu(\mathcal{L}) = \mathcal{L}^2 + \mathcal{L}\,.\,K_M + C_2(M) \ ,$$

where K_M and $C_2(M)$ are the canonical bundle and the second Chern class of M, respectively. In particular, this implies that P_2 is proper. The idea is to prove that $\overline{\mathcal{S}_C(\mathcal{L})}$ is an analytic subset of $\mathcal{S}(\mathcal{L})$. This will imply, via the proper map theorem, that $P_2\left(\overline{\mathcal{S}_C(\mathcal{L})}\right) = \mathbb{F}\mathrm{ol}_C(M,\mathcal{L})$ is an analytic subset of $\mathbb{F}\mathrm{ol}(M,\mathcal{L})$, and therefore an algebraic subset, by Chow's theorem.

First of all, we will prove that $\mathcal{S}_C(\mathcal{L})$ is an analytic subset of $\mathcal{S}(\mathcal{L}) \setminus \mathcal{S}_{dg}(\mathcal{L})$. Fix $(p_o, \mathcal{F}_o) \in \mathcal{S}_C(\mathcal{L})$, so that p_o is a Morse center of $\mathcal{F}_o$. As we have seen in remark 1.1, p_o is a non-degenerate singularity of $\mathcal{F}_o$ and there exists a holomorphic chart $\psi = (x, y)\colon U \to D_1^2 \subset \mathbb{C}^2$ with $p_o \in U$, $D_r = \{z \in \mathbb{C} \,|\, |z| < r\}$, $\psi(p_o) = 0 \in D_1^2$ and $\mathcal{F}_o$ is represented in these coordinates by the vector field $X_o = x\,\partial_x - y\,\partial_y$. Since $\mathbb{F}\mathrm{ol}(M,\mathcal{L})$ is a finite dimensional projective space, say with $dim_{\mathbb{C}}(\mathbb{F}\mathrm{ol}(M,\mathcal{L})) = N$, there exists an affine neighborhood $\mathcal{U}$ of $\mathcal{F}_o$ in $\mathbb{F}\mathrm{ol}(M,\mathcal{L})$, holomorphic vector fields $X_1, ..., X_m$ on the polydisc U, such that any $\mathcal{F} \in \mathcal{U}$ is represented in U by the vector field

$$X_\alpha = X_o + \sum_{j=1}^{m} \alpha_j . X_j \ , \tag{3}$$

where $\alpha = (\alpha_1, ..., \alpha_m) \in \Delta$, Δ a polydisc of $\mathbb{C}^m$ with $0 \in \Delta$. The map $\alpha \in \Delta \mapsto X_\alpha$ parametrizes the set of foliations $\{\mathcal{F}|_U \,|\, \mathcal{F} \in \mathcal{U}\}$.

Since the characteristic values of $\mathcal{F}_o$ at p_o are both -1, by taking a smaller Δ if necessary, we can assume that

- for any $\alpha \in \Delta$, X_α has an unique non-degenerate singularity $p(\alpha) \in U \simeq D_1^2$, where $p(0) = p_o$ and the map $p\colon \Delta \to U$ is holomorphic. We can assume also that $\psi(p(\alpha)) \in D_{1/2}^2$ for any $\alpha \in \Delta$. This follows from the implicit function theorem.
- the characteristic values of X_α at $p(\alpha)$ are $\lambda(\alpha)$ and $\lambda(\alpha)^{-1}$, where $\lambda\colon \Delta \to \mathbb{C}$ is holomorphic.
- $\lambda(\alpha) \notin \mathbb{R}_+$ for any $\alpha \in \Delta$. This condition implies that X_α has exactly two analytic separatrices through $p(\alpha)$, which are smooth, for any $\alpha \in \Delta$ (cf. [Ma-Mo]).

Since $(y = 0) \subset U$ is a separatrix of $\mathcal{F}_o \simeq X_o$, by the theory of invariant manifolds we can assume also that (cf. [H-P-S])

- for any $\alpha \in \Delta$ the foliation defined by X_α has a separatrix $S(\alpha)$ through $p(\alpha)$ which is a graph of a holomorphic map $\phi_\alpha\colon D_1 \to D_1$, $S(\alpha) = \{(x, \phi_\alpha(x)) \,|\, x \in D_1\}$. Moreover, the map $\phi\colon D_1 \times \Delta \to D_1$ defined by $\phi(x, \alpha) = \phi_\alpha(x)$ is holomorphic.

Given $\alpha \in \Delta$, let us consider the leaf $L(\alpha) := S(\alpha) \setminus \{p(\alpha)\} \simeq \mathbb{D}^*$ of X_α. Set $\Sigma := (x = 1/2)$. Note that Σ cuts $S(\alpha)$ transversely at the point $\Sigma \cap S(\alpha) = (1/2, y_\alpha) = q_\alpha$, where $y_\alpha = \phi(1/2, \alpha)$. Since $\phi(p(\alpha)) \in D_{1/2}^2$ we have $p(\alpha) \notin \Sigma$. Moreover, the homotopy group of $L(\alpha)$ is generated by the closed curve $\delta_\alpha(\theta) = (e^{i\theta}/2, \phi(e^{i\theta}/2, \alpha))$, $\theta \in [0, 2\pi]$. Therefore, the holonomy group of $L(\alpha)$ can be considered as a sub-group of $Diff(\Sigma, q_\alpha)$ and is generated by the transformation $h_\alpha \in Diff(L(\alpha), q_\alpha)$ corresponding to δ_α. Its Taylor series in the coordinate y of Σ can be written as

$$h_\alpha(y) = y_\alpha + \sum_{j=1}^{\infty} a_j(\alpha)\,(y - y_\alpha)^j \ ,$$

where $a_1 \in \mathcal{O}^*(\Delta)$ and $a_j \in \mathcal{O}(\Delta)$ if $j \geq 2$. We would like to observe that $a_1(\alpha) = e^{2\pi i \lambda(\alpha)}$ for every $\alpha \in \Delta$ (cf. [Ma-Mo]).

Now, we use the following result due to Mattei and Moussu (cf. [Ma-Mo]) :

- X_α has a holomorphic first integral in a neighborhood of $p(\alpha)$ if, and only if, h_α has finite order. Moreover, the first integral is of Morse type if, and only if $\lambda(\alpha) = -1$ and $h_\alpha = id$, the identity transformation.

In particular, $p(\alpha)$ is a Morse center of some X_α if, and only if, $\lambda(\alpha) = -1$ and $a_j(\alpha) = 0$ for all $j \geq 2$.

Let $I \subset \mathcal{O}(U \times \Delta)$ be the ideal

$$I = \langle x - x(p(\alpha)), y - y(p(\alpha)), \lambda(\alpha) + 1, a_j(\alpha) \,|\, j \geq 2\rangle \tag{4}$$

and I_o be its germ at $(p_o, 0) \in U \times \Delta$. Since $U \times \Delta$ is finite dimensional, I_o is finitely generated. If we identify the neighborhood $\mathcal{U}$ of $\mathcal{F}_o$ with Δ, then $\mathcal{J}_o = \sqrt{I_o}$ defines the germ of $\mathcal{S}_C(\mathcal{L})$ at $(p_o, \mathcal{F}_o)$, by Mattei-Moussu theorem. This proves that $\mathcal{S}_C(\mathcal{L})$ is an analytic subset of $\mathcal{S}(\mathcal{L}) \setminus \mathcal{S}_{dg}(\mathcal{L})$.

Now, we will see that for any irreducible component $\mathcal{X}$ of $\mathcal{S}_C(\mathcal{L})$ there is an analytic subset $\mathcal{Y}$ of $\mathcal{S}(\mathcal{L})$ such that $\mathcal{X}$ is an open subset of $\mathcal{Y}$. This will imply that $\overline{\mathcal{X}}$ is an irreducible component of $\mathcal{Y}$, and so an analytic subset of $\mathcal{S}(\mathcal{L})$. Given a point $q_o = (p_o, \mathcal{F}_o) \in \mathcal{S}_C(\mathcal{L})$, we have seen that the ideal (see (4))

$$I(q_o) := \langle x - x(p(\alpha)), y - y(p(\alpha)), \lambda(\alpha) + 1, a_j(\alpha) \,|\, j \geq 2\rangle$$

defines the germ of $\mathcal{S}_C(\mathcal{L})$ at q_o. Given $m \in \mathbb{N}$ set

$$\begin{array}{rll} I_m(q_o) := \langle x - x(p(\alpha)), y - y(p(\alpha)), \lambda(\alpha) + 1, a_2(\alpha), ..., a_m(\alpha)\rangle & , \text{if } m \geq 2 \\ I_1(q_o) := \langle x - x(p(\alpha)), y - y(p(\alpha)), \lambda(\alpha) + 1\rangle & , \text{if } m = 1 \end{array}$$

In particular, if I_m is a representative of the ideal $I_m(q_o)$ and $(p, \mathcal{F}) \in I_m$, $m \geq 1$, then the $\mathcal{F}$ has two local separatrices through p, which are smooth, say Σ_1 and Σ_2. The holonomy of Σ_j is conjugated to some $f_j \in Diff(\mathbb{C}, 0)$ such that $j_0^m(f_j)(z) = z$, $j = 1, 2$, where j_0^m denotes the m^{th}-jet of f_j at 0.

Note that $I(q_o) = \bigcup_{m \geq 1} I_m(q_o)$ and that $I_m(q_o) \subset I_{m+1}(q_o)$ for all $m \geq 0$. Since $\mathcal{O}_{q_o}$ is a noetherian ring, there exists $N \in \mathbb{N} \cup \{0\}$ such that $I(q_o) = I_N(q_o)$, which means that $I_m(q_o) = I_N(q_o)$ for all $m \geq N$. Since we are assuming that $\mathbb{F}\text{ol}_C(M, \mathcal{L}) \neq \mathbb{F}\text{ol}(M, \mathcal{L})$, we must have $N \geq 1$. Define $N \colon \mathcal{S}_C(\mathcal{L}) \to \mathbb{N}$ by

$$N(q_o) = min\{N \in \mathbb{N} \,|\, I_m(q_o) = I_N(q_o) \ , \ \forall \ m \geq N\} \ .$$

Observe that the function $N \colon \mathcal{S}_C(\mathcal{L}) \to \mathbb{N}$ is upper semi-continuous. In fact, given $q_o \in \mathcal{S}_C(\mathcal{L})$ let $\mathcal{U}$ be a neighborhood of q_o such that the ideal $I_{N(q_o)}(q_o)$ has a representative in $\mathcal{U}$. It follows from the definition that $N(q) \leq N(q_o)$ for all $q \in \mathcal{U}$. This implies that, if $\mathcal{X} \subset \mathcal{S}(\mathcal{L}) \setminus \mathcal{S}_{dg}(\mathcal{L})$ is an irreducible component of $\mathcal{S}_C(\mathcal{L})$ then:

(i). $sup\{N(q) \mid q \in \mathcal{X}\} := N(\mathcal{X}) < +\infty$.

(ii). The subset $\mathcal{U}_{N(\mathcal{X})} := N^{-1}(N(\mathcal{X}))$ is an open and dense subset of $\mathcal{X}$.

Given $(p, \mathcal{F}) \in \mathcal{S}(\mathcal{L}) \setminus \mathcal{S}_{dg}(\mathcal{L})$ and a holomorphic vector field X that represents $\mathcal{F}$ in a neighborhood of p, we say that $\mathcal{F}$ *has trace zero at* p if $tr(DX(p)) = 0$. This condition does not depends on the vector field X representing $\mathcal{F}$ in a neighborhood of p. Define

$$\mathcal{Y}_1 := \{(p, \mathcal{F}) \in \mathcal{S}(\mathcal{L}) \setminus \mathcal{S}_{dg}(\mathcal{L}) \,|\, \mathcal{F} \text{ has trace zero at } p\} \ .$$

Note that if $(p, \mathcal{F}) \in \mathcal{Y}_1$ then

(iii). $\mathcal{F}$ has two smooth local separatrices through p.

(iv). The holonomy of both separatrices is tangent to the identity. Moreover, the order of tangency with the identity is the same for both separatrices.

Given $k \geq 2$ define $\mathcal{Y}_k := \{(p, \mathcal{F}) \in \mathcal{Y}_1 \mid$ the holonomy of a separatrix of $\mathcal{F}$ through p is conjugated to $f \in Diff(\mathbb{C}, 0)$ with $j_0^k(f)(z) = z\}$. It follows from the above arguments that:

(v). $\mathcal{Y}_k$ is an analytic subset of $\mathcal{S}(\mathcal{L}) \setminus \mathcal{S}_{dg}(\mathcal{L})$ for all $k \geq 1$.

(vi). The irreducible component $\mathcal{X}$ of $\mathcal{S}_C(\mathcal{L})$ coincides with one of the irreducible components of $\mathcal{Y}_{N(\mathcal{X})}$ (see (ii)).

By (vi) it is sufficient to prove the following:

Lemma 2.1. *$\overline{\mathcal{Y}}_k$ is an analytic subset of $\mathcal{S}(\mathcal{L})$ for all $k \geq 1$.*

Proof. The proof will be by induction on $k \geq 1$.

$\overline{\mathcal{Y}}_1$ is analytic. Given $(p_o, \mathcal{F}_o) \in \overline{\mathcal{Y}}_1$, take a parametrization $(X_\alpha)_{\alpha\in\Delta}$ of a neighborhood $\mathcal{U}$ of $\mathcal{F}_o$ in $\mathbb{F}\mathrm{ol}(M,\mathcal{L})$ as in (3), where X_α is a holomorphic vector field on a neighborhood U of p_o. Then $\overline{\mathcal{Y}}_1 \cap (U \times \mathcal{U})$ is defined by the analytic equations $X_\alpha(p) = 0$ and $tr(DX_\alpha(p)) = 0$.

If $k \geq 1$ then $\overline{\mathcal{Y}}_k$ analytic $\implies$ $\overline{\mathcal{Y}}_{k+1}$ analytic. Observe first that $\overline{\mathcal{Y}}_{k+1} \subset \overline{\mathcal{Y}}_k$, because $\mathcal{Y}_{k+1} \subset \mathcal{Y}_k$. Let $(p_o, \mathcal{F}_o) \in \mathcal{Y}_k$ and X be a holomorphic vector field representing $\mathcal{F}_o$ in a neighborhood of p_o. Fix a holomorphic coordinate system $(U, z = (x, y))$ with $p_o \in U$, $x(p_o) = y(p_o) = 0$. Write the Taylor series of X at $p_o = (0, 0)$, in this coordinate system, as

$$X(z) = \left(\sum_{|\sigma|\geq 1} a_\sigma z^\sigma\right) \partial_x + \left(\sum_{|\sigma|\geq 1} b_\sigma z^\sigma\right) \partial_y$$

where $\sigma = (m, n)$, $m, n \geq 0$, $|\sigma| = m + n$, $z^\sigma = x^m y^n$, and $a_\sigma, b_\sigma \in \mathbb{C}$. We will identify the ℓ^{th}-jet of X at 0, $j_0^\ell(X)$, with the point $(a_\sigma, b_\sigma \,|\, |\sigma| \leq L) \in \mathbb{C}^L$, where

$$L = L(\ell) = 2 \times \#\{(m,n) \,|\, 1 \leq m + n \leq \ell\}.$$

Claim 2.1. *If $(p_o, \mathcal{F}_o) \in \mathcal{Y}_k$ and X is as above, then there exists a polynomial P of $L(2k+1)$ variables such that $(p_o, \mathcal{F}_o) \in \mathcal{Y}_{k+1}$ if and only if $P(j_0^{2k+1}(X)) = 0$.*

Proof. Since $(p_o, \mathcal{F}_o) \in \mathcal{Y}_1$ the eigenvalues of $DX(p_o)$ are $a, -a \neq 0$ and we can assume that the linear part of X at $p_o = 0$ is $a\,X_1$, where $X_1 = x\,\partial_x - y\,\partial_y$. According to [M], X is formally equivalent to a formal vector field $\hat{X} = a\,X_1 + \hat{Y}$, where $DY(0) = 0$ and $[X_1, Y] = 0$. This implies that $\hat{X}$ can be written as below

$$\hat{X}(u,v) = a\,u\,(1 + \hat{F}(u\,v))\partial_u - a\,v\,(1 + \hat{G}(u\,v))\partial_v\ ,$$

where $\hat{F}$ and $\hat{G}$ are formal power series in one variable. On the other hand, the formal holonomy of the separatrix $(v = 0)$ can be obtained by integrating the formal differential equation

$$\frac{dV}{d\theta} = -i\,V\,\left(1 + H(r\,e^{i\theta}\,V)\right) \tag{5}$$

with initial condition $V(0) = v_o$, where $1+H(z)$ is the formal power series of $(1+\hat{G}(z))/(1+\hat{F}(z))$, $H = (\hat{G} - \hat{F})/(1 + \hat{F})$. Equation (5) is obtained by the restriction of the formal foliation given by $\hat{X}$ to the cilinder $\{(u,v) \,|\, u = r\,e^{i\theta}\,,\, \theta \in [0, 2\,\pi]\}$. The power series $\hat{f}(v_o) := V(2\,\pi, v_o)$ corresponds to the holonomy of the foliation in the section $(u = r)$. The formal diffeomorphism $\hat{f} \in \widehat{Diff}(\mathbb{C}, 0)$ is formally conjugated to the germ of holonomy $f \in Diff(\mathbb{C}, 0)$ of one of the two separatrices of the original vector field X (cf. [M]).

Equation (5) can be solved formally by series by writing the solution as

$$V(\theta, v_o) = \sum_{j=1}^{\infty} c_j(\theta)\, v_o^j$$

and substituting in (5). This gives:

$$\sum_{j\geq 1} c_j'(\theta)\, v_o^j \;=\; -i \sum_{j\geq 1} c_j(\theta)\, v_o^j\, \left(1 + H\left(r\,e^{i\,\theta} \sum_{j\geq 1} c_j(\theta)\, v_o^j\right)\right) \tag{6}$$

with initial conditions $c_1(0) = 1$ and $c_j(0) = 0$ if $j \geq 2$. If $H \not\equiv 0$ and the first non-zero jet of H is $j_0^\ell H(z) = h_\ell\, z^\ell$, $h_\ell \neq 0$, then (6) implies that

$$c_j' + i\, c_j = 0 \ , \text{ if } 1 \leq j \leq \ell\,, \text{ and } c_{\ell+1}' + i\, c_{\ell+1} = -i\, h_\ell\, r^\ell\, e^{i\ell\theta}\, c_1^{\ell+1} \implies$$

$$c_1(\theta) = e^{-i\theta} \ , \ c_j(\theta) = 0 \ , \text{ if } 2 \leq j \leq \ell \ , \text{ and } c_{\ell+1}(\theta) = -i\, h_\ell\, r^\ell\, \theta \ .$$

In particular, we get

$$j_0^\ell \hat{f}(v_o) = v_o - 2i\pi\, h_\ell\, v_o^{\ell+1} \implies \ell = k \ ,$$

because $(p_o, \mathcal{F}_o) \in \mathcal{Y}_k$. This proves also that $(p_o, \mathcal{F}_o) \in \mathcal{Y}_{k+1}$ if, and only if, $h_k = 0$.

Now, we use the known fact that there exists a germ of diffeomorphism $F \in Diff(\mathbb{C}^2, 0)$ such that (cf. [M]):

(I). $F(z) = z + G_2(z) + ... + G_{2k+1}(z)$, where $G_j(z)$ is homogeneous of degree j, $2 \leq j \leq 2k+1$, whoose coefficients are rational functions of the coefficients of $j_0^{2k+1}(X)$.

(II). $j_0^{2k+1}(F^*(j_0^{2k+1}(X))) = j_0^{2k+1}(F^*(X)) = j_0^{2k+1}(\hat{X}) =$
$= a\, u\, (1 + j_0^k(\hat{F})(u\, v))\partial_u - a\, v\, (1 + j_0^k(\hat{G})(u\, v))\partial_v$.

Since $F^*(X)(w) = DF(w)^{-1}.X \circ F(w)$, we get from (I) and (II) that the coefficients of $j_0^{2k+1}(\hat{X})$ are rational functions of the coefficients of $j_0^{2k+1}(X)$. Therefore, the coefficients of $j_0^k(\hat{F})$ and of $j_0^k(\hat{G})$ are rational functions of the coefficients of $j_0^{2k+1}(X)$. On the other hand, we have

(III). $h_k\, z^k = j_0^k(H(z)) = j_0^k(\hat{G}(z) - \hat{F}(z))/(1 + \hat{F}(z)) = j_0^k(\hat{G}(z) - \hat{F}(z))$

and this implies that h_k is a rational function of the coefficients of $j_0^{2k+1}(X)$, so that we can write $h_k = P(j_0^{2k+1}(X))/Q(j_0^{2k+1}(X))$, where P and Q are polynomials. In particular,

$$(p_o, \mathcal{F}_o) \in \mathcal{Y}_k \iff h_k = 0 \iff P(j_0^{2k+1}(X)) = 0 \ ,$$

which proves the claim. □

Let us finish the proof of lemma 2.1. Fix $q_o = (p_o, \mathcal{F}_o) \in \overline{\mathcal{Y}}_{k+1}$. Since $\overline{\mathcal{Y}}_{k+1} \subset \overline{\mathcal{Y}}_k$ and $\overline{\mathcal{Y}}_k$ is analytic, fix a neighborhood $U \times \mathcal{U}$ of q_o in $M \times \mathbb{F}\text{ol}(M, \mathcal{L})$ with the following properties:

(1). There exists a holomorphic chart $\phi = (x, y)\colon U \to \mathbb{C}^2$ such that $x(p_o) = y(p_o) = 0$ and $\phi(U)$ is a polydisk of $\mathbb{C}^2$.

(2). There exist holomorphic vector fields $X_0, X_1 ..., X_m$ on U such that the family

$$(X_\alpha := X_0 + \sum_{j=1}^{m} \alpha_j\, X_j)_{\alpha \in \Delta}$$

parametrizes the foliations in $\mathcal{U}$ (restricted to U), where $\Delta \subset \mathbb{C}^m$ is a polydisk. In this way, we can consider $U \times \mathcal{U}$ parametrized by (x, y, α).

(3). $\overline{\mathcal{Y}}_k \cap (U \times \mathcal{U})$ is defined by analytic equations $f_1(x, y, \alpha) = ... = f_n(x, y, \alpha) = 0$. Set $F = (f_1, ..., f_n)$.

According to claim 2.1 there exists a polynomial P in $\mathbb{C}^{L(2k+1)}$ such that if $(x, y, \alpha) \in \mathcal{Y}_k \cap (U \times \mathcal{U})$ then

$$(x, y, \alpha) \in \mathcal{Y}_{k+1} \cap (U \times \mathcal{U}) \iff P\left(j_{(x,y)}^{2k+1}\, X_\alpha\right) = 0 \ .$$

Since $(x, y, \alpha) \mapsto P\left(j_{(x,y)}^{2k+1}\, X_\alpha\right)$ extends analytically to $U \times \mathcal{U}$, this finishes the proof of lemma 2.1. □

2.2. **Proof of Theorem 2 and Corollary 1.** Let $\mathcal{V}$ be an irreducible component of $\mathbb{F}\mathrm{ol}_C(M,\mathcal{L})$, $\mathcal{F}_0 \in \mathcal{V}$ and p_0 be a p.c. of $\mathcal{F}_0$ in $\mathcal{V}$. Let us express this condition in terms of $\mathcal{S}(\mathcal{L})$. Since p_0 is a non-degenerate singularity of $\mathcal{F}_0$, by the implicit function theorem there exist neighborhoods $\mathcal{U}$ of $\mathcal{F}_0$ in $\mathbb{F}\mathrm{ol}(M,\mathcal{L})$, U of p_0 in M and a holomorphic map $P\colon \mathcal{U} \to U$ such that

(i) $\mathcal{U}$ is biholomorphic to a polydisc and $\mathcal{V} \cap \mathcal{U}$ is an analytic subset of $\mathcal{U}$.
(ii) $P(\mathcal{F}_0) = p_0$ and $sing(\mathcal{F}) \cap U = \{P(\mathcal{F})\}$, for all $\mathcal{F} \in \mathcal{U}$.
(iii) $P(\mathcal{F})$ is a non-degenerate singularity of $\mathcal{F}$, for all $\mathcal{F} \in \mathcal{U}$.

Lemma 2.2. *In the above situation define* $\Phi\colon \mathcal{U} \to \mathcal{S}(\mathcal{L})$ *by* $\Phi(\mathcal{F}) = (P(\mathcal{F}),\mathcal{F})$. *Then* $\Phi(\mathcal{V}\cap\mathcal{U}) \subset \mathcal{S}_C(\mathcal{L})$. *In particular, for any* $\mathcal{F} \in \mathcal{V}\cap\mathcal{U}$, $P(\mathcal{F})$ *is a p.c. of* $\mathcal{F}$ *in* $\mathcal{V}$.

Proof. In fact, since p_0 is a p.c. of $\mathcal{F}_0$ in $\mathcal{V}$, it follows from the definition of p.c. that there exists a neighborhood $\mathcal{U}_1 \subset \mathcal{U}$ of $\mathcal{F}_0$ such that if $\mathcal{F} \in \mathcal{V}\cap\mathcal{U}_1$ then $P(\mathcal{F})$ is a Morse center of $\mathcal{F}$. In particular, $\Phi(\mathcal{V}\cap\mathcal{U}_1) \subset \mathcal{S}_C(\mathcal{L})$. Since Φ is holomorphic, $\mathcal{V}\cap\mathcal{U}$ is an analytic subset of $\mathcal{U}$ and $\mathcal{S}_C(\mathcal{L})$ is an analytic subset of $\mathcal{S}(\mathcal{L})$, we get $\Phi(\mathcal{V}\cap\mathcal{U}) \subset \mathcal{S}_C(\mathcal{L})$. □

Lemma 2.2 implies the following: let $\mathcal{SV}$ be the irreducible component of $\mathcal{S}_C(\mathcal{L})$ containing $(p_o,\mathcal{F}_o)$ and $P_2 = \Pi_2|_{\mathcal{S}(\mathcal{L})}\colon \mathcal{S}(\mathcal{L}) \to \mathbb{F}\mathrm{ol}(\mathcal{L})$ be as in the proof of theorem 1. Then $P_2(\mathcal{SV}) = \mathcal{V}$ and $P_2|_{\mathcal{SV}}\colon \mathcal{SV} \to \mathcal{V}$ is a ramified covering.

In fact, since $\mathcal{SV}$ is irreducible and P_2 is finite to one and proper, the set $P_2(\mathcal{SV}) \subset \mathbb{F}\mathrm{ol}(M,\mathcal{L})$ is analytic and irreducible. On the other hand, lemma 2.2 implies that $\mathcal{V}\cap P_2(\mathcal{SV})$ contains $\mathcal{V}\cap\mathcal{U}$, which is an open set of $\mathcal{V}$ and of $P_2(\mathcal{SV})$. Hence, by irreducibility of $\mathcal{V}$ and $P_2(\mathcal{SV})$ we get $P_2(\mathcal{SV}) = \mathcal{V}$. This implies also that $P_2|_{\mathcal{SV}}\colon \mathcal{SV} \to \mathcal{V}$ is a ramified covering.

Now, let $\mathcal{G}\colon [0,1] \to \mathcal{V}$ be a continuous curve with $\mathcal{G}(0) = \mathcal{F}_o$ and such that p_o can be continued along $\mathcal{G}$ by a curve $\gamma\colon [0,1] \to M$. We want to prove that $\gamma(1)$ is a p.c. in $\mathcal{V}$ of $\mathcal{G}(1)$.

Define $\beta\colon [0,1] \to \mathcal{S}(\mathcal{L})$ by $\beta(t) = (\gamma(t),\mathcal{G}(t))$. Note that β is a lift of $\mathcal{G}\colon [0,1] \to \mathcal{V}$ by the covering $P_2\colon \mathcal{S}(\mathcal{L}) \to \mathbb{F}\mathrm{ol}(M,\mathcal{L})$: $P_2 \circ \beta = \mathcal{G}$. Since $\gamma(t)$ is a non-degenerate singularity of $\mathcal{G}(t)$ for all $t \in [0,1]$, lemma 2.2 implies that this lift is unique. It follows that $\beta[0,1] \subset \mathcal{SV}$, so that $\beta(1) \in \mathcal{SV}$. Since $P_2|_{\mathcal{SV}}\colon \mathcal{SV} \to \mathcal{V}$ is open the singularity $\gamma(1)$ must be a p.c. of $\mathcal{G}(1)$. □

2.3. **Proof of Proposition 1 and Corollary 2.** Fix $D = (d_1,...,d_k) \in \mathbb{N}^k$, $k \geq 2$. Let us sketch the proof that the set

$$Z_1 = \{(F,\Lambda) \in \mathcal{P}(D,k) \mid \text{all singularities of } \mathcal{F}(F,\Lambda) \text{ are non-degenerate}\}$$

is a Zariski open and dense subset of $\mathcal{P}(D,k)$. Recall that, if $F = (F_1,...,F_k)$ and $\Lambda = (\lambda_1,...,\lambda_k)$, where $\sum_j d_j\,\lambda_j = 0$, then $\mathcal{F}(F,\Lambda)$ is represented in homogeneous coordinates by the form

$$\Omega(F,\Lambda) = \sum_{j=1}^{k} \lambda_j \,\frac{dF_j}{F_j}\ .$$

Let $\mathbb{C}^2 \simeq E_0 \subset \mathbb{P}^2$ be the affine coordinate system given by $E_0 = \{(x,y,1) \in \mathbb{C}^3 \,|\, (x,y) \in \mathbb{C}^2\}$. If we set $f_j(x,y) := F_j(x,y,1)$ then $\mathcal{F}(F,\Lambda)$ is represented in E_0 by the polynomial 1-form $f_1...f_k\,.\,\omega(F,\Lambda)$, where

$$\omega(F,\Lambda) = \sum_{j=1}^{k} \lambda_j \,\frac{df_j}{f_j}$$

or by the vector field $X = X(F,\Lambda) = P(F,\Lambda)\,\partial_x + Q(F,\Lambda)\,\partial_y$, where

$$P(F,\Lambda) = f_1...f_k \sum_{j=1}^{k} \lambda_j \,\frac{\partial_y f_j}{f_j} \text{ and } Q(F,\Lambda) = -f_1...f_k \sum_{j=1}^{k} \lambda_j \,\frac{\partial_x f_j}{f_j}\ .$$

The singularities of $\mathcal{F}(F,\lambda)$ are non-degenerate if, and only if, the map $\Phi\colon \mathbb{C}^2 \to \mathbb{C}^2\times\mathbb{C}^2$ given by $\Phi(x,y) = (x,y,P(x,y),Q(x,y))$ is transverse to the zero section $\Sigma = \{(x,y,0,0)\,|\,(x,y)\in\mathbb{C}^2\}$. This implies already that the set Z_1 is Zariski open. Therefore, it is sufficient to prove that $Z_1 \neq \emptyset$. Let us sketch the proof of this fact.

Consider the analytic map $\mathbb{X}\colon \mathcal{P}(D,k)\times\mathbb{C}^2 \to \mathbb{C}^2\times\mathbb{C}^2$ defined by

$$\mathbb{X}(F,\Lambda,x,y) = (x,y,P(F,\Lambda)(x,y),Q(F,\Lambda)(x,y)) \ .$$

It is known from transversality theory that if $\mathbb{X}$ is transverse to Σ then the set

$$\{(F,\Lambda)\,|\,X(F,\Lambda)(x,y) := \mathbb{X}(F,\Lambda,x,y) \text{ is transverse to } \Sigma\}$$

has full measure. On the other hand, the reader can check that the map $\mathbb{X}$ is transverse to Σ. Therefore, Z_1 is Zariski open and dense.

Let us prove that there exists $Z \subset Z_1$, Zariski open and dense subset, such that for any $(F,\Lambda)\in Z$ then $\mathcal{F}(F,\Lambda)\in\mathbb{F}\text{ol}(d)$, $d = d_1+...+d_k-2$, and $\mathcal{F}(F,\Lambda)$ has at least

$$d^2+d+1-\sum_{i<j} d_i\,d_j$$

Morse centers. Recall that if $(F_1,...,F_k,\lambda_1,...,\lambda_k)\in\mathcal{P}(D,k)$ then

- $S_j := (F_j = 0)$ is $\mathcal{F}(F,\Lambda)$-invariant, $j\in\{1,...,k\}$. In particular, any singularity of the curve $S := \bigcup_j S_j$ is a singularity of $\mathcal{F}(F,\Lambda)$.

Let Z_2 be the Zariski open and dense subset of $\mathcal{P}(D,k)$ defined by $(F,\Lambda)\in Z_2$ if

(1). $S_1,...,S_k$ are smooth.
(2). for all $i<j$ the curves S_i and S_j are transverse, so that $\#(S_i\cap S_j) = d_i\,d_j$.
(3). if $i<j<\ell$ then $S_i\cap S_j\cap S_\ell = \emptyset$.
(4). $\lambda_j\neq 0$ for all $j=1,...,k$ for all $j=1,...,k$ and if $i<j$ then $\lambda_i\neq\lambda_j$.

If $(F,\Lambda)\in Z_2$ then :

(i). if $p\in S_j\setminus\bigcup_{i\neq j} S_i$ then $p\notin sing(\mathcal{F}(F,\Lambda))$.
(ii). if $i\neq j$ and $p\in S_i\cap S_j$ then p is a non-degenerate singularity of $\mathcal{F}(F,\Lambda)$ with characteristic values $-\lambda_i/\lambda_j$ and $-\lambda_j/\lambda_i$.

Properties (i) and (ii) are well known (cf. [LN-S]). In particular, if $(F,\Lambda)\in Z_2$ then $\mathcal{F}(F,\Lambda)$ has no Morse center on the curve S because the characteristic values of the singularities on S are different from -1, by (4). On the other hand, if $(F,\Lambda)\in Z_1\cap Z_2 := Z$ then the divisor of zeroes of $\Omega(F,\Lambda)$ is empty and so $\mathcal{F}(F,\Lambda)$ has degree $d = d_1+...+d_k-2$. Moreover, it follows from (1), (2) and (3) that $sing(\mathcal{F}(F,\Lambda))\cap S = \bigcup_{i<j} S_i\cap S_j$, and so

$$\#(sing(\mathcal{F}(F,\Lambda))\cap S) = \sum_{i<j} d_i\,d_j \quad\Longrightarrow$$

$$\#(\text{Morse centers of }\mathcal{F}(F,\Lambda)) = d^2+d+1-\sum_{i<j} d_i\,d_j := N(D) \ .$$

It remains to prove that all these Morse centers are persistent in $\mathcal{V}(D)$. Set

$$\mathcal{L}_Z = \{\mathcal{F}(F,\Lambda)\in\mathcal{L}(D)\ |\ (F,\Lambda)\in Z\}$$

and

$$\mathcal{S}_C\mathcal{L}_Z = \{(p,\mathcal{F})\in\mathcal{S}(d)\ |\ \mathcal{F}\in\mathcal{L}(D,Z) \text{ and } p \text{ is a Morse center of } \mathcal{F}\} \ ,$$

where $\mathcal{S}(d) = \{(p,\mathcal{F})\ |\ \mathcal{F}\in\mathbb{F}\text{ol}(d) \text{ and } p \text{ is a singularity of } \mathcal{F}\}$.

Remark 2.2. The map $P_Z := P_2|_{\mathcal{S}_C\mathcal{L}_Z} \colon \mathcal{S}_C\mathcal{L}_Z \to \mathcal{L}_Z$ is a covering map with $N(D)$ sheets.

In fact, since P_Z is a covering because Morse centers are non-degenerate singularities. On the other hand, the number of sheets is $N(D)$ because $\mathcal{F}(F,\Lambda)$ has $N(d)$ Morse centers for all $(F,\Lambda) \in Z$.

It follows from corollary 1 that it is enough to prove that $\mathcal{S}_C\mathcal{L}_Z$ is connected. In fact, fix $\mathcal{F}_o = \mathcal{F}(F_o,\Lambda_o) \in \mathcal{L}(D,Z)$ and let $p_1,...,p_{N(D)}$ be the Morse centers of $\mathcal{F}_o$. Note that $P_Z^{-1}(\mathcal{F}_o) = \{(p_j,\mathcal{F}_o)\,|\,j=1,...,N(D)\}$. On the other hand, it is clear that at least one of the Morse centers of $\mathcal{F}_o$, say p_1, is persistent in $\mathcal{V}(D)$. If we can prove that $\mathcal{S}_C\mathcal{L}_Z$ is connected then there exist continuous curves $\beta_j\colon [0,1] \to \mathcal{S}_C\mathcal{L}_Z$, $j=2,...,N(D)$, such that $\beta_j(0) = (p_1,\mathcal{F}_o)$ and $\beta_j(1) = (p_j,\mathcal{F}_o)$, $2 \le j \le N(D)$, and this implies, via corollary 1, that all centers of $\mathcal{F}_o$ are persistent in $\mathcal{V}(D)$.

Let us give an idea of the proof that $\mathcal{S}_C\mathcal{L}_Z$ is connected. Observe first that $\mathcal{L}_Z$ is connected. In particular, it is sufficient to prove that there is a fiber $P_Z^{-1}(\mathcal{F}_1)$ with the property that it is possible to connect any two points in this fiber by a curve in $\mathcal{S}_C\mathcal{L}_Z$. With this in mind, we consider a logarithmic foliation $\mathcal{G}$ on $\mathbb{P}^3$ defined in homogeneous coordinates by the form

$$\Omega = \sum_{j=1}^{k} \lambda_j \frac{dG_j}{G_j} \ , \tag{7}$$

where:

1. $G_j \in \mathbb{C}[z_0,...,z_3]$ is homogeneous of degree d_j, $1 \le j \le k$. We assume that the algebraic set S_j of $\mathbb{P}^3$ defined by $G_j = 0$ is smooth, $1 \le j \le k$.
2. if $i \ne j$ then S_i and S_j are tranverse.
3. if $k \ge 3$ and $1 \le i < j < \ell \le k$ then $dG_i(p) \wedge dG_j(p) \wedge dG_\ell(p) \ne 0$ for any $p \in \mathbb{C}^4 \setminus \{0\}$ with $G_i(p) = G_j(p) = G_\ell(p) = 0$.
4. if $k \ge 4$ and $1 \le i < j < \ell < m \le k$ then $(G_i = G_j = G_\ell = G_m = 0) = \{0\}$.
5. $\lambda_j \ne 0$ for all $j = 1,...,k$ and $\lambda_i \ne \lambda_j$ for all $1 \le i < j \le k$.

Given a linearly embedded plane $\mathbb{P}^2 \simeq \Sigma \subset \mathbb{P}^3$ then we can define a logarithmic foliation on $\mathbb{P}^2$ by the restriction $\Omega|_\Sigma$. In fact we will consider a more general situation as below:

Remark 2.3. Let $\mathcal{H}$ be a codimension one holomorphic foliation of $\mathbb{P}^3$. We say that a 2-plane $\mathbb{P}^2 \simeq \Sigma \subset \mathbb{P}^3$ is in general position with respect (notation g.p.w.r.) to $\mathcal{H}$ if

- Σ is not $\mathcal{H}$-invariant.
- outside $\Sigma \cap sing(\mathcal{H})$ the tangencies of $\mathcal{H}$ with Σ are isolated points in Σ.

Note that the set of 2-planes in g.p.w.r. to $\mathcal{H}$ is a Zariski open and dense subset of $\check{\mathbb{P}}^3$, the dual of $\mathbb{P}^3$ (cf. [C-LN-S]).

Given a 2-plane $\mathbb{P}^2 \simeq \Sigma \subset \mathbb{P}^3$ in g.p.w.r. to $\mathcal{H}$ then the restriction $\mathcal{H}|_\Sigma$ is defined as $i^*(\mathcal{H})$, where $i\colon \Sigma \to \mathbb{P}^3$ is a linear embedding. Note that the singular set of $\mathcal{H}|_\Sigma$ can be written as

$$sing(\mathcal{H}|_\Sigma) = T(\mathcal{H},\Sigma) \cup (\Sigma \cap sing(\mathcal{H}))$$

where $T(\mathcal{H},\Sigma)$ denotes the set of points $q \in \mathbb{P}^3 \setminus sing(\mathcal{H})$ such that Σ is tangent at q to the leaf of $\mathcal{H}$ through q. Since $q \notin sing(\mathcal{H})$, $\mathcal{H}$ has a holomorphic first integral in a neighborhood of q, say $f\colon U \to \mathbb{C}$. In particular, $g := f|_{\Sigma\cap U}$ is a holomorphic first integral of $\mathcal{H}_\Sigma$ in a neighborhood of q in Σ. Since Σ is tangent to $\mathcal{H}$ at q, q is a singular point of g. We say that the tangency is non-degenerate at $q \in T(\mathcal{H},\Sigma)$ if q is a Morse singularity of g and so a Morse center of $\mathcal{H}|_\Sigma$. Otherwise, we say that the tangency is degenerate.

Now, we introduce the Gauss map of $\mathcal{H}$, $G\colon \mathbb{P}^3 \setminus sing(\mathcal{H}) \to \check{\mathbb{P}}^3$, defined by

$$G(q) = \text{2-plane tangent at } q \text{ to } \mathcal{H} \ .$$

Note that G can be considered as a rational map $G\colon \mathbb{P}^3 \dashrightarrow \check{\mathbb{P}}^3$. Given $q \in \mathbb{P}^3 \setminus sing(\mathcal{H})$ such that $G(q)$ is in g.p.w.r. to $\mathcal{H}$ we set $\mathcal{H}(q) := \mathcal{H}|_{G(q)}$. Note that q is a singular point of $\mathcal{H}(q)$. Set

$$M(\mathcal{H}) = \{q \in \mathbb{P}^3 \setminus sing(\mathcal{H}) \mid q \text{ is a Morse center of } \mathcal{H}(q)\}\ ,$$

$$S(\mathcal{H}) = \{\Sigma \mid \Sigma \subset \mathbb{P}^3 \text{ is a 2-plane and } \forall\ q \in T(\mathcal{H},\Sigma) \text{ then } q \text{ is a Morse center of } \mathcal{H}|_\Sigma\}$$

and

$$MS(\mathcal{H}) := \{q \in \mathbb{P}^3 \setminus sing(\mathcal{H}) \mid G(q) \in S(\mathcal{H})\}\ .$$

The following result was proved in [Ce-LN]:

Theorem 2.1. *Let $\mathcal{H}$ be a holomorphic codimension one foliation on $\mathbb{P}^3$. If $M(\mathcal{H}) = \emptyset$ then all leaves of $\mathcal{H}$ are ruled surfaces and*

(a). *either $\mathcal{H} = \Phi^*(\mathcal{G})$, where $\Phi\colon \mathbb{P}^3 \to \mathbb{P}^2$ is a linear map (a linear pull-back),*

(b). *or $\mathcal{H}$ has rational first integral ϕ that can be written in some homogeneous coordinate system as*

$$\phi(x,y,z,w) = \frac{z\,P(x,y) + Q(x,y)}{w\,P(x,y) + R(x,y)}\ ,$$

where P, Q, R are homogeneous polynomials with $deg(Q) = deg(R) = deg(P) + 1$.

As a consequence, we get the following:

Corollary 2.1. *If $\mathcal{H}$ is not as in (a) or (b) of theorem 2.1 then $MS(\mathcal{H})$ is a Zariski open and dense subset of $\mathbb{P}^3$. In particular, $MS(\mathcal{H})$ is connected.*

Sketch of the proof. By analycity of $\mathcal{H}$ it can be proved that $Y := \mathbb{P}^3 \setminus M(\mathcal{H})$ is an algebraic subset of $\mathbb{P}^3$. In particular, $M(\mathcal{H})$ is a Zariski open subset of $\mathbb{P}^3$. Since $\mathcal{H}$ is not as (a) or (b) of theorem 2.1 we get $M(\mathcal{H}) \neq \emptyset$ and so Y is proper and the set $Z := Y \setminus sing(\mathcal{H})$ has dimension ≤ 2. Since the Gauss map G is rational the set $W := \overline{G(Z)}$ is algebraic of dimension ≤ 2. In particular, G is dominant and $G^{-1}(W)$ is a proper algebraic subset of $\mathbb{P}^3$. It follows that $U := \mathbb{P}^3 \setminus G^{-1}(W)$ is a Zariski open and dense subset of $\mathbb{P}^3$. Now, it follows from the definition that $U = MS(\mathcal{H})$, which proves the result. □

Let us finish the proof that $\mathcal{S}_C\mathcal{L}_Z$ is connected. The reader can check that the logarithmic foliation $\mathcal{G}$ on $\mathbb{P}^3$ defined by (7) with the properties 1,...,5, is not like in (a) or (b) of theorem 2.1. As a consequence, $MS(\mathcal{G}) \subset \mathbb{P}^3$ is open dense and connected. Fix $p_0 \in MS(\mathcal{G})$ and let $\mathcal{F}_o := \mathcal{G}|_{G(p_0)}$, where G denotes the Gauss map of $\mathcal{G}$. Note that with condition 5 then the set of Morse centers of $\mathcal{F}_o$ coincides with $T(\mathcal{G}, G(p_0))$. In particular, p_o is a Morse center of $\mathcal{F}_o$ and $\mathcal{F}_o \in \mathcal{L}_Z(D)$. Fix another Morse center p_1 of $\mathcal{F}_o$. Since $MS(\mathcal{G})$ is connected let $\gamma\colon [0,1] \to MS(\mathcal{G})$ be a curve with $\gamma(0) = p_0$ and $\gamma(1) = p_1$. Let $\mathcal{I}\colon [0,1] \times \mathbb{P}^2 \to \mathbb{P}^3$ be a continuous map such that for any $t \in [0,1]$ the map $\mathcal{I}_t\colon \mathbb{P}^2 \to \mathbb{P}^3$ is a linear embedding with $\mathcal{I}_t(\mathbb{P}^2) = G(\gamma(t))$. This defines a continuous curve $\Gamma\colon [0,1] \to \mathcal{L}_Z$ by $\Gamma(t) = \mathcal{I}^*(\mathcal{G}|_{G(\gamma(t))})$ with the property that $\Gamma(0) = \Gamma(1) = \mathcal{I}_0^*(\mathcal{F}_o)$ and $\mathcal{I}_0^{-1}(p_o)$ can be continued along Γ by the curve $\delta =: [0,1] \to \mathbb{P}^2$ defined by $\delta(t) = \mathcal{I}_t^{-1}(\gamma(t))$, so that the hypothesis of corollary 1 is verified. This finishes the proof of corollary 2. □

2.4. **Proof of Corollary 3.** A foliation $\mathcal{F}_o \in \mathcal{R}(1, d+1)$ has a rational first integral written in homogeneous coordinates as F_o/L_o^{d+1}, where F_o is homogeneous of degree $d+1$ and L_o is linear. By corollary 2 if F_o and L_o are generic then $\mathcal{F}_o$ has degree d and $N(1, d+1) = d^2$ Morse centers. Let $\mathcal{V}(1, d+1)$ be the irreducible component of $\mathbb{F}\mathrm{ol}_C(d)$ containing $\mathcal{R}(1, d+1)$. By corollary 2 the d^2 Morse centers of $\mathcal{F}_o$ are persistent in $\mathcal{V}(1, d+1)$, so that there is a neighborhood $\mathcal{U}$ of $\mathcal{F}_o$ in $\mathbb{F}\mathrm{ol}(d)$ such that any foliation $\mathcal{F} \in \mathcal{V}(1, d+1) \cap \mathcal{U}$ has d^2 Morse centers. It is enough to prove that $\mathcal{V}(1, d+1) \cap \mathcal{U} \subset \mathcal{R}(1, d+1)$. The proof of this fact is based on the following:

Lemma 2.3. *Let $\mathcal{F} \in \mathbb{F}ol(d)$ be such that $\mathcal{F}$ has d^2 non-degenerate singularities with Baum-Bott index zero. Then $\mathcal{F} \in \mathcal{R}(1, d+1)$.*

Proof. Let $p_1, ..., p_{d^2}$ be the non-degenerate singularities of $\mathcal{F}$ with Baum-Bott index zero. Let us prove first that $\mathcal{F}$ has an invariant straight line ℓ such that $p_j \notin \ell$, $1 \leq j \leq d^2$. Fix an affine coordinate system $(x, y) \in \mathbb{C}^2 \subset \mathbb{P}^2$ such that the line at infinity is not $\mathcal{F}$-invariant and $p_1, ..., p_{d^2} \in \mathbb{C}^2$. In this case, $\mathcal{F}$ is induced in $\mathbb{C}^2$ by a vector field X of the form,

$$X = (a + xg)\frac{\partial}{\partial x} + (b + yg)\frac{\partial}{\partial y} \ ,$$

where a, b are polynomials with $\deg(a), \deg(b) \leq d$ and g is a non-identically zero degree d homogeneous polynomial.

Let I be the ideal generated by $a + xg$ and $\operatorname{div}(X)$, where

$$\operatorname{div}(X) = \frac{\partial(a + xg)}{\partial x} + \frac{\partial(b + yg)}{\partial y} = \frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} + (d + 2)g \ .$$

By Bezout's Theorem we have that $V(I) = \{p \in \mathbb{P}^2 \,|\, f(p) = 0\,, \forall f \in I\}$ has degree $\deg(\operatorname{div}(X))\deg(a + xg) = d(d + 1)$, i.e., $V(I)$ has $d^2 + d$ points(counted with multiplicity): d of these points are at infinity; they correspond to the intersection of the curve $\{g = 0\}$(which is a union of lines) with the line at infinity; the other d^2 correspond to the singularities of X in $\mathbb{C}^2$ where $div(X) = 0$, i.e., with Baum-Bott index zero.

Since $b + yg$ vanishes on all points of $V(I)$ it must belong to I. Keeping in mind that $\deg(b + yg) = \deg(a + xg) = \deg(\operatorname{div}(X)) + 1$ we see that there exists $\ell_1, \ell_2 \in \mathbb{C}[x, y]$ such that $\deg(\ell_1) = \deg(\ell_2) = 1$ and

$$X(\ell_1) = \ell_2 \cdot \operatorname{div}(X)$$

Note that the left-hand side of the above equation vanishes at all singularities of X. We can suppose, without loss of generality, that all the singularities of $\mathcal{F}$ are contained in $\mathbb{C}^2$. Thus all the singularities of $\mathcal{F}$ with Baum-Bott index distinct from zero are in ℓ_2. Comparing the homogeneous terms of degree $d + 1$ of the equation one obtains that

$$g\left(\frac{\partial \ell_1}{\partial x}\,x + \frac{\partial \ell_1}{\partial y}\,y\right) = (d+2)g\left(\frac{\partial \ell_2}{\partial x}\,x + \frac{\partial \ell_2}{\partial y}\,y\right).$$

Thus $\ell_1 - (d + 2)\ell_2 \in \mathbb{C}$, and consequently

$$\frac{X(\ell_2)}{\ell_2} = \frac{1}{d+2} \cdot \operatorname{div}(X),$$

proving that ℓ_2 is invariant.

From now on we will suppose that $\mathcal{F}$ has an invariant line and will choose an affine coordinate system where the line at infinity is invariant and

$$X = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y},$$

with $\deg(a) = \deg(b) = d$. We claim that $\operatorname{div}(X) = 0$. Suppose not and let I be the ideal generated by $\operatorname{div}(X)$ and a. $V(I)$ in this case has degree $d(d-1)$ and has to vanish at d^2 points what is clearly impossible unless $\operatorname{div}(X) = 0$.

Now, note that $\operatorname{div}(X) = 0$ is equivalent $d\omega = 0$, where $\omega = bdx - ady$, which implies $\omega = df$ for some polynomial f of degree $d + 1$, i.e., f is a first integral of $\mathcal{F}|_{\mathbb{C}^2}$. This implies that $\mathcal{F} \in \mathcal{R}(1, d+1)$. □

2.5. **Proof of Corollary 4.** We identify a holomorphic map $\Phi\colon \mathbb{P}^2 \to \mathbb{P}^2$, via the projection $\Pi\colon \mathbb{C}^3 \setminus \{0\} \to \mathbb{P}^2$, with its lifiting $\tilde{\Phi} = (F_0, F_1, F_2)\colon \mathbb{C}^3 \to \mathbb{C}^3$, where $F_0, F_1, F_2 \in \mathbb{C}[x_0, x_1, x_2]$ are homogeneous polynomials of the same degree such that $(F_0 = F_1 = F_2 = 0) = \{0\}$. The algebraic degree of Φ is the common degree of F_0, F_1 and F_2. We denote the set of holomorphic maps of algebraic degree k by $\mathcal{H}(k)$. Note that $\mathcal{H}(k)$ can be identified with a Zariski open and dende subset of a projective space of polynomials. Given $\Phi = (F_0, F_1, F_2) \in \mathcal{H}(k)$ we define its Jacobian, $J(\Phi)$, by

$$dF_0 \wedge dF_1 \wedge dF_2 = J(\Phi)\,.\,dx_0 \wedge dx_1 \wedge dx_2 \ .$$

Observe that the singular set of Φ is $S(\Phi) := \Pi(J(\Phi) = 0) \subset \mathbb{P}^2$. If $J(\Phi) \not\equiv 0$ then $S(\Phi)$ defines a divisor of degree $3(k-1)$ in $\mathbb{P}^2$.

Let us consider the 1-forms Ω_{ij} on $\mathbb{C}^3$, $0 \le i < j \le 2$, defined by

$$\Omega_{ij} := F_i\, dF_j - F_j\, dF_i \ .$$

It can be proved that the subset of $\mathbb{P}^2$ where Φ has rank 0 (that is $D\Phi(p) = 0$) is defined in homogeneous coordinates by

$$Z(\Phi) := \{p \in \mathbb{C}^3 \setminus \{0\} \mid \Omega_{ij}(p) = 0 \ , \ 0 \le i < j \le 2\} \ .$$

We observe $Z(\Phi) \subset S(\Phi)$. Moreover, if $dF_i \wedge dF_j \not\equiv 0$ then the set $Z_{ij} := \Pi(\mathcal{O}_{ij} = 0)$ is finite and contains $4k^2 - 6k + 3$ points counted with multiplicities.

As the reader can check, this implies that the following subset of $\mathcal{H}(k)$ is Zariski open and dense:

$$W(k) = \{\Phi = [F_0 : F_1 : F_2] \mid J(\Phi) \text{ is irreducible and } Z(\Phi) = \emptyset\} \ .$$

If $\Phi \in W(k)$ then:

- $S(\Phi)$ is a smooth curve of $\mathbb{P}^2$.
- $rank(D\Phi(p)) \ge 1$ for all $p \in \mathbb{P}^2$ and $rank(D\Phi(p)) = 1 \iff p \in S(\Phi)$. In particular, $dim(ker(D\Phi(p))) = 1$ and $dim(Im(D\Phi(p)) = 1$ for all $p \in S(\Phi)$.
- $C(\Phi) := \{p \in S(\Phi) \,|\, ker(D\Phi(p)) = T_pS(\Phi)\}$ is a finite subset of $S(\Phi)$. Note that $\Phi(S(\Phi))$ is an irreducible singular curve of $\mathbb{P}^2$ and if $p \in C(\Phi)$ then $\Phi(p)$ is a singularity of $\Phi(S(\Phi))$ of cuspidal type.
- if $p \in S(\Phi) \setminus C(\Phi)$ then p is a fold singularity, that is there exists a holomorphic chart $\phi = (x, y)\colon U \to \mathbb{C}^2$, $p \in U$, such that $\phi(p) = 0$ and $\Phi(x, y) = (x, y^2)$.

Now, fix $\Phi \in W(k)$ and assume that $\mathcal{G} \in \mathbb{F}\mathrm{ol}(d)$ satisfies the following conditions:

- all singularities of $\mathcal{G}$ are non-degenerate and $\mathcal{G}$ has no Morse center.
- $\Phi(S(\Phi)) \cap sing(\mathcal{G}) = \emptyset$.
- if $p \in C(\Phi)$ then $Im(D\Phi(p))$ is transverse to the leaf of $\mathcal{G}$ through p.
- the tangencies of $\mathcal{G}$ with $\Phi(S(\Phi) \setminus C(\Phi))$ are non-degenerate.

It can be proved that the set $Z(\Phi)$ of foliations in $\mathbb{F}\mathrm{ol}(d)$ that satisfy the above conditions is a Zariski open and dense subset. We leave the proof to the reader.

On the other hand, if $\Phi = [F_0 : F_1 : F_2]$ and $\mathcal{G}$ are as above then $\mathcal{G}$ is represented in homogeneous coordinates by a polynomial 1-form $\omega = P\,dx + Q\,dy + R\,dz$, where P, Q and R are homogeneous polynomials of degree $d+1$ and $x\,P + y\,Q + z\,R \equiv 0$. It follows that $\Phi^*(\mathcal{G})$ is represented in homogeneous coordinates by the form

$$P(F_0, F_1, F_2)\, dF_0 + Q(F_0, F_1, F_2)\, dF_1 + R(F_0, F_1, F_2)\, dF_2$$

whose coeffitients are homogeneous of degree $(d+1)\,k + k - 1$. This implies that $\Phi^*(\mathcal{G})$ has degree $\ell := (d+1)\,k + k - 2 = (d+2)\,k - 2$. On the other hand, the map Φ has topological degree k^2 and if we set $X = \Phi^{-1}(\Phi(S(\Phi)))$ then the map

$$\Phi|_{\mathbb{P}^2 \setminus X}\colon \mathbb{P}^2 \setminus X \to \mathbb{P}^2 \setminus \Phi(S(\Phi))$$

is a regular covering with k^2 sheets. In particular, for any point $p \notin \Phi(S(\Phi))$ we have $\#(\Phi^{-1}(p) = k^2$. Now, $\mathcal{G}$ has d^2+d+1 non-degenerate singularities and $sing(\mathcal{G}) \cap \Phi(S(\Phi)) = \emptyset$, so that $\Phi^{-1}(sing(\mathcal{G}))$ contains exactly $k^2\,(d^2+d+1)$ singularities of $\Phi^*(\mathcal{G})$ which are not Morse centers and are non-degenerate , because $rank(D\Phi(q)) = 2$ for all $q \notin X$. Since the tangencies of $\mathcal{G}$ with $\Phi(S(\phi) \setminus C(\Phi))$ are non-degenerate, the remaining singularities of $\Phi^*(\mathcal{G})$ are Morse centers. Finally, the total number of singularities of $\Phi^*(\mathcal{G})$ is $\ell^2+\ell+1$, so that the number of Morse centers is

$$\ell^2+\ell+1-k^2\,(d^2+d+1) = 3(k-1)(k(d+1)-1) = N(d,k)\ .$$

It remains to prove that all these centers are persistent in the irreducible component of $\mathbb{F}\mathrm{ol}_C(\ell)$ that contains $PB(d,k)$. It is sufficient to find an example $\Phi^*(\mathcal{F}_o) \in PB(d,k)$ with $N = N(d,k)$ centers, say $p_1, ..., p_N$, such that for every $1 \le i < j \le N$ there exist curves $\mathcal{G}\colon [0,1] \to PB(d,k)$ and $\gamma\colon [0,1] \to \mathbb{P}^2$ with $\mathcal{G}(0) = \mathcal{G}(1) = \Phi^*(\mathcal{F}_o)$, $\gamma(0) = p_i$, $\gamma(1) = p_j$ and such that $\gamma(t)$ is a Morse center for $\mathcal{G}(t)$ for all $t \in [0,1]$. The idea is the same of the proof of corollary 2: to extend a foliation $\Phi^*(\mathcal{G})$ on $\mathbb{P}^2 \subset \mathbb{P}^3$, with exactly $N(d,k)$ Morse centers, to a foliation $\mathcal{H}$ on $\mathbb{P}^3$ with the following properties:

- $\mathcal{H} = \Psi^*(\mathcal{G})$ where $\Psi\colon \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ is a rational extension of Φ.
- if we denote by $G\colon \mathbb{P}^3 \setminus sing(\mathcal{H}) \to \check{\mathbb{P}}^3$ the Gauss map associated to $\mathcal{H}$ then the subset

  $$MS(\mathcal{H}) := \left\{p \in \mathbb{P}^3 \setminus sing(\mathcal{H}) \mid \mathcal{H}|_{G(p)} \text{ has exactly } N(d,k) \text{ Morse centers}\right\}$$

  is a Zariski open and dense subset of $\mathbb{P}^3$. In particular, it is connected.

This is not very difficult to do and we leave the proof of the existence of this extension to the reader. Finally, we consider a curve $\gamma\colon [0,1] \to MS(\mathcal{H})$ such that $\gamma(0) = p_i$ and $\gamma(1) = p_j$ and set $\mathcal{G}(t) = \mathcal{H}|_{G(\gamma(t))}$, as in the proof of corollary 2. This finish the proof of corollary 4. □

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ON SMOOTH DEFORMATIONS OF FOLIATIONS WITH SINGULARITIES

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Abstract. We study smooth deformations of codimension one foliations with Morse and Bott-Morse singularities of center-type. We show that in dimensions ≥ 3, every small smooth deformation by foliations of a Morse function with only center type singularities is a deformation by Morse functions. We also show that this statement is false in dimension 2. In the same vein we show that if $\mathcal{F}$ is a foliation with Bott-Morse singularities on a manifold M, all of center type, and if we assume there is a component $N \subset \operatorname{sing}(\mathcal{F})$ of codimension $m \geq 3$ such that $H^1(N, \mathbb{R}) = 0$, then every small smooth deformation $\{\mathcal{F}_t\}$ of $\mathcal{F}$ is compact, stable and given by a Bott-Morse function $f_t \colon M \to [0, 1]$ with only two critical values at 0 and 1. Furthermore, each such foliation $\{\mathcal{F}_t\}$ is topologically equivalent to $\mathcal{F}$. Hence, Bott-Morse foliations with only center-type singularities and having a component $N \subset \operatorname{sing}(\mathcal{F})$ of codimension $m \geq 3$ such that $H^1(N, \mathbb{R}) = 0$, are structurally stable under smooth deformations. These statements are false in general if we drop the codimension $m \geq 3$ condition.

1. Introduction and results

An important problem in geometry and dynamics is studying the stability of singular foliations under deformations. This is classical for 1-dimensional foliations defined by (real or complex) vector fields. For higher dimensional foliations, we need to impose some additional structure on the foliations and/or on the type of singularities, in order to be able to say something about them.

For instance, in the interesting article [4], the authors give extensions of Reeb's Stability Theorem to singular holomorphic foliations of codimension 1 having a meromorphic first integral and defined on projective manifolds $\mathcal{M}$ with $H^1(\mathcal{M}, \mathcal{C}) = 0$. In doing so, the authors study foliations defined by Lefschetz pencils, defined by a general meromorphic function, and prove a stability theorem for these. A key ingredient in the proof of that theorem is looking at the behavior of the foliation near a Kupka component of its singular set.

Let us recall that given any integrable polynomial homogeneous 1-form ω on $\mathbb{C}^{n+1}$ with singular set of codimension ≥ 2, we define the *Kupka singular set of* ω as

$$K(\omega) = \{p \in \mathbb{C}^{n+1} \backslash 0 \mid \omega(p) = 0, d\omega(p) \neq 0\}.$$

The *Kupka singular set of* the corresponding foliation $\mathcal{F} = \mathcal{F}(\omega)$ in $\mathbb{C}P(n)$ is $K(\mathcal{F}) = \pi(K(\omega))$ where π is the projectivization map.

We know from [3, 4, 6, 9] that if $n \geq 3$, then the Kupka set is a locally closed codimension 2 smooth submanifold of $\mathbb{C}P(n)$ which has a local product structure: Given a connected component $K \subset K(\mathcal{F})$ there exist a holomorphic 1-form η, called the transversal type of K, defined on a neighborhood of $0 \in \mathbb{C}^2$ and vanishing only at 0, a covering $\{U_\alpha\}$ of a neighborhood of K in $\mathbb{C}P(n)$ and a family of holomorphic submersions $\varphi_\alpha \colon U_\alpha \to \mathbb{C}^2$ satisfying that $\varphi_\alpha^{-1}(0) = K \cap U_\alpha$ and $\varphi_\alpha^* \eta$ defines $\mathcal{F}$ in U_α. Furthermore, $K(\mathcal{F})$ is persistent under small perturbations of $\mathcal{F}$, namely, fixed any $p \in K(\mathcal{F})$ with defining 1-form $\varphi^*\eta$ as above, and for any foliation $\mathcal{F}'$ sufficiently close to $\mathcal{F}$, there is a holomorphic 1-form η' close to η and a submersion φ' close to φ, such that $\mathcal{F}'$ is defined by $(\varphi')^*\eta'$ near the point p.

In this work we study a different but somehow similar setting. Here we look at the class of codimension one real foliations in smooth manifolds, such that at each point the foliation is locally defined by a Bott-Morse function of center type. The singular set consists of a disjoint union of submanifolds and one has for these, all the properties mentioned above for the Kupka set. We also prove that just as in the case of the Kupka set, all these properties are preserved under appropriate deformations of the foliation.

Before describing with more care what we do, let us recall that probably the most important foundational result in the theory of foliations is the celebrated Local Stability Theorem of Reeb (see for instance [3, 10]): *A compact leaf of a foliation having finite holonomy group is stable, i.e., it admits a fundamental system of invariant neighborhoods where each leaf is compact with finite holonomy group.* This is followed in importance by Reeb's Global Stability Theorem: *If $\mathcal{F}$ is codimension one foliation, of class $C^r, r \geq 1$, on a closed connected manifold M and $\mathcal{F}$ has a compact leaf with finite fundamental group, then all leaves of $\mathcal{F}$ are compact with finite fundamental group. Moreover, if $\mathcal{F}$ is transversely orientable then the leaves of $\mathcal{F}$ have trivial holonomy group and they are the fibers of a locally trivial fibration $M \to S^1$.* In fact, according to Thurston ([14]), the same conclusion holds if $\mathcal{F}$ is transversely orientable and exhibits a compact leaf L with zero first Betti number $H^1(L, \mathbb{R}) = 0$.

Some interesting questions arise when we consider small perturbations of a given foliation. For instance the classical Tischler's fibration theorem ([15]) states that a codimension one foliation induced by a nonsingular closed one-form on a compact manifold, can be approximated by compact foliations induced by closed one-forms, and hence the manifold fibers over the circle. The basic idea is to perturb the closed one-form into a closed one-form with rational periods. On the other hand, it is not true that every compact foliation can be approximated by noncompact foliations, even if the compact foliation is defined by a closed one-form. This was already considered by Reeb, who proved the following classical result concerning stability for perturbations, which strengthens his Local Stability Theorem:

Theorem (Reeb), [3]: *Let $\mathrm{Fol}_k^r(M)$ be the space of codimension $k \geq 1$ foliations of class C^r on M, $2 \leq r \leq \omega$, endowed with the C^0-topology. Let $\mathcal{F}$ be an element in $\mathrm{Fol}_k^r(M)$ with a compact leaf L having finite fundamental group. Then for each neighborhood W of L in M and for each point $q \in L$, there exist an open neighborhood V of q in W and a neighborhood $\mathcal{V}$ of $\mathcal{F}$ in $\mathrm{Fol}_k^r(M)$, such that for each foliation $\mathcal{G} \in \mathcal{V}$, the saturated of V by $\mathcal{G}$ is contained in W and it is a union of compact leaves of $\mathcal{G}$, each leaf being a finite covering of L.*

Using arguments as in Thurston's version of Reeb's Global Stability Theorem, Langevin and Rosenberg gave in [7] a generalization of the preceding result: *Equip the space $\mathrm{Fol}_k^r(M)$ with the C^1 topology and assume $\mathcal{F} \in \mathrm{Fol}_k^r(M)$ has a compact leaf L such that $H^1(L, \mathbb{R}) = 0$ and $\mathrm{Hom}(\pi_1(L), \mathrm{GL}(k, \mathbb{R})) = \mathrm{Id}$. Then we have the same conclusions as in Reeb's theorem of stability for perturbations. Moreover, the compact leaves of $\mathcal{G} \in \mathcal{V}$ close enough to L have trivial holonomy and they are diffeomorphic to L.* In fact when $k = 1$ it is enough to assume $H^1(L, \mathbb{R}) = 0$.

On the other hand, foliations with singularities play a significant role in several areas of mathematics. It is thus natural to search for stability theorems for singular foliations in the spirit of the preceding results, and that was the motivation for this article.

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Our starting point is Reeb's Sphere Recognition theorem (see [11]): If a compact manifold M of dimension $m \geq 3$ admits a foliation with Morse singularities all of center type, then all leaves are compact and diffeomorphic to spheres S^{m-1}, M is homeomorphic to the sphere S^m and the foliation is given by the level surfaces of a Morse function having only two centers as singular set. Such a foliation will be called a *Morse-Reeb fibration*.

Our first result is:

Theorem 1. *In dimension $m \geq 3$ Morse-Reeb fibrations on spheres are stable under small smooth deformations. Moreover, every small enough smooth deformation by foliations of a Morse function with only center type singularities in dimension ≥ 3, is a deformation by Morse functions.*

It is well-known that Morse singularities are stable under deformations as functions. The point here is proving the persistence of center type Morse singularities under smooth deformations as foliations. We give an example showing that this condition fails in dimension 2.

The next step we envisage in this article is considering a compact connected manifold M of dimension $m \geq 2$ and a codimension one smooth (*i.e.*, of class C^∞) foliation $\mathcal{F}$ on M with Bott-Morse singularities in the sense of [12, 13]. This means that its singular set, $\operatorname{sing}(\mathcal{F})$, is a union of a finite number of disjoint compact connected submanifolds, $\operatorname{sing}(\mathcal{F}) = \bigcup_{j=1}^{t} N_j$, each of codimension ≥ 2, and for each $p \in N_j \subset \operatorname{sing}(\mathcal{F})$ there exists a neighborhood V of p in M where the foliation is defined by a Bott-Morse function. That is, there is a diffeomorphism $\varphi\colon V \to D \times B$, where $D \subset \mathbb{R}^{n_j}$, $n_j = \dim N_j$, and $B \subset \mathbb{R}^{m-n_j}$ are open balls centered at the origin, such that φ takes $\mathcal{F}|_V$ into the product foliation $D \times \mathcal{G}$, where $\mathcal{G} = \mathcal{G}(N_j)$ is the foliation on B given by a Morse function with a singularity at the origin.

Given a point $p \in N_j$, we write $\varphi(p) = (x(p), y(p))$, so that the discs $\Sigma_p = \varphi^{-1}(x(p) \times B)$ are transverse to $\mathcal{F}$ outside $\operatorname{sing}(\mathcal{F})$ and the restriction $\mathcal{F}|_{\Sigma_p}$ is an ordinary Morse singularity, whose Morse index $r = r(N_j)$ does not depend on the point p in the component N_j. The restriction $\mathcal{G}(N_j) = \mathcal{F}|_{\Sigma_p}$ is the *transverse type* of $\mathcal{F}$ along N_j; it is a codimension one foliation in the disc Σ_p with an ordinary *Morse singularity* at $\{p\} = N_j \cap \Sigma_p$. A component $N \subset \operatorname{sing}(\mathcal{F})$ is of *center type* (or just *a center*) if the transverse type $\mathcal{G}(N) = \mathcal{F}|_{\Sigma_q}$ of $\mathcal{F}$ along N is a center, *i.e.*, its Morse index is either 0 or $r = \dim \Sigma_q$.

Such a foliation $\mathcal{F}$ is *transversally orientable* if there exists a vector field X on M, possibly with singularities at $\operatorname{sing}(\mathcal{F})$, such that X is transverse to $\mathcal{F}$ outside $\operatorname{sing}(\mathcal{F})$. Throughout this paper, all foliations are assumed to be transversely oriented.

Recall that in the classical framework of nonsingular foliations, a compact leaf is *stable* if it admits a fundamental system of invariant neighborhoods such that on each neighborhood the leaves are compact. In codimension one, Reeb's local stability theorem implies that this is equivalent to finiteness of the holonomy group of the leaf.

One has the similar notion of stability for a center type component $N \subset \operatorname{sing}(\mathcal{F})$ of the singular set of a foliation with Bott-Morse singularities: N is *stable* if it admits a fundamental system of invariant neighborhoods such that on each neighborhood the leaves are compact. The foliation is *stable* if all its leaves are compact and stable and all components of the singular set are of center type and stable.

In [12] the authors prove a natural version of Reeb's global stability theorem in this setting: *Let $\mathcal{F}$ be a foliation with Bott-Morse singularities on a closed oriented manifold M of dimension $m \geq 3$ having only center type components in* $\operatorname{sing}(\mathcal{F})$*. Assume that $\mathcal{F}$ has some compact leaf L_o with finite fundamental group, or there is a codimension ≥ 3 component N of* $\operatorname{sing}(\mathcal{F})$ *with finite fundamental group. Then all leaves of $\mathcal{F}$ are compact, stable, with finite fundamental group. If,* *moreover, $\mathcal{F}$ is transversely orientable, then* $\operatorname{sing}(\mathcal{F})$ *has exactly two components and there is a differentiable Bott-Morse function* $f\colon M \to [0,1]$ *whose critical values are* $\{0,1\}$ *and such that* $f\big|_{M\setminus \operatorname{sing}(\mathcal{F})}\colon M \setminus \operatorname{sing}(\mathcal{F}) \to (0,1)$ *is a fiber bundle with fibers the leaves of* $\mathcal{F}$*.* According to [8] the same conclusion holds if we assume that we have a compact leaf or a codimension ≥ 3 center type component $N \subset \operatorname{sing}(\mathcal{F})$ with first Betti number zero.

In this article we prove the following stability theorem:

Theorem 2. *Let M be a compact oriented connected manifold and $\mathcal{F}$ a foliation with Bott-Morse singularities on M all of center type. Assume there is a component $N \subset \operatorname{sing}(\mathcal{F})$ of codimension $\ell \geq 3$ and such that $H^1(N,\mathbb{R}) = 0$. Given a smooth deformation $\{\mathcal{F}_t\}$, $t \in [0,\epsilon)$ of $\mathcal{F}$ there is $0 < \epsilon_1 < \epsilon$ such that if $0 \leq t \leq \epsilon_1$ then $\mathcal{F}_t$ is compact, stable and given by a Bott-Morse function $f_t\colon M \to [0,1]$ with critical values at 0 and 1.*

Just as for Theorem 1, Example **??** below shows that Theorem 2 is sharp in the sense that one cannot drop the codimension ≥ 3 condition. These two theorems are similar, with the additional condition of the existence of a smooth deformation, to the fact that the class of Morse functions is an open subset in the C^1-topology.

As a corollary of the proof of Theorem 2 we have:

Corollary 1. *Let $\mathcal{F}$ be a foliation with Bott-Morse singularities on a manifold M. Assume there is a center type component $N \subset \operatorname{sing}(\mathcal{F})$ of codimension $\ell \geq 3$ such that $H^1(N,\mathbb{R}) = 0$. Given a smooth deformation $\{\mathcal{F}_t\}$, $t \in [0,\epsilon]$, of $\mathcal{F}$ there is $0 < \epsilon_1 < \epsilon$ such that if $0 \leq t \leq \epsilon_1$ then $\mathcal{F}_t$ also exhibits an stable center type component $N_t \subset \operatorname{sing}(\mathcal{F}_t)$ which is close and isotopic to N, and therefore it is stable.*

Also from the proof of Theorem 2 and from Theorems A, B and C in [12] we have the following weak structural stability:

Corollary 2. *In the situation of Theorem 2 the foliations $\mathcal{F}_t$ are topologically conjugate to $\mathcal{F}$ for t small enough.*

2. An example

Given a foliation $\mathcal{F}$ on M with singular set $\operatorname{sing}(\mathcal{F}) \subset M$, by a C^∞ *deformation* of $\mathcal{F}$ we mean a family $\{\mathcal{F}_t\}_{t\in[0,\epsilon)}$ of foliations $\mathcal{F}_t$ on M, with $\mathcal{F}_0 = \mathcal{F}$ and which is smooth in the sense that for each point $p \in M$, there are an open set $p \in U \subset M$ and a smooth family of differential one-forms $\Omega_t(x) := \Omega(x,t)$ in $U \times [0,\epsilon)$ such that for each t the one-form Ω_t is integrable and defines $\mathcal{F}_t$ in U.

Before proving Theorems 1 and 2, let us show that these results are sharp in the sense that the codimension ≥ 3 condition cannot be dropped. Notice that in [8] examples are given showing that the conditions on the component $N \subset \operatorname{sing}(\mathcal{F})$ cannot be dropped without destroying the stability of the foliation $\mathcal{F}_0$.

Let $\Omega = d(x^2+y^2)$ and $\Omega_\lambda = xdy - \lambda ydx$ in affine coordinates $(x,y) \in \mathbb{R}^2$, where $\lambda \in \mathbb{R}$ is not zero. Put

$$\Omega_t := \Omega + t\Omega_\lambda = (2x - t\lambda y)dx + (2y + tx)dy.$$

Then $\operatorname{sing}(\Omega_t) = \{(0,0)\}$. For

$$X_t := (2y + tx)\frac{\partial}{\partial x} + (t\lambda y - 2x)\frac{\partial}{\partial y}$$

we have $\Omega_t \cdot X_t = 0$. Thence

$$DX_t(0,0) = \begin{pmatrix} t & 2 \\ -2 & t\lambda \end{pmatrix}.$$

The eigenvalues of X_t at $(0,0)$ are the α given by

$$0 = \mathrm{Det}(DX_t(0,0) - \alpha I) = (t-\alpha)(t\lambda - \alpha) + 4\,.$$

Thus we have

$$\alpha = \frac{(1+\lambda)t \pm \sqrt{t^2(1+\lambda)^2 - 4(4+t^2\lambda)}}{2}\,.$$

For $t = 0$ we have

$$\alpha = \pm 2\sqrt{-1}\,.$$

For $t \approx 0$ but $t \neq 0$ we have $\alpha = a + b\sqrt{-1} \in \mathbb{C}$ where $b \approx 2$ and $0 \neq a \approx 0$ provided that $\lambda \neq -1$. In this case the quotient of eigenvalues of X_t at the origin is of the form

$$\frac{a + b\sqrt{-1}}{a - b\sqrt{-1}} = \frac{a^2 - b^2 + 2\sqrt{-1}ab}{a^2 + b^2} \notin \mathbb{R}$$

and therefore X_t has a hyperbolic singularity at the origin. In particular, thanks to the dynamics of such a singularity, *the leaves of* Ω_t *are not closed and the foliation* $\Omega_t = 0$ *exhibits no continuous first integral in a neighborhood of the origin* $(0,0) \in \mathbb{R}^2$.

Now, by gluing two copies of the 2-disk D^2 we obtain the 2-sphere S^2. Endowing each copy of D^2 with a foliation given by $\Omega_t = 0$ we obtain a deformation $\mathcal{F}_t$ of the foliation $\mathcal{F}_0$ by parallels, $\mathcal{F}_0$ is of Morse type with singularities only at the North and South poles, both of center type. The foliation $\mathcal{F}_t$ (obtained indeed as an extension of the foliation in $\mathbb{R}^2$ given by $\Omega_t = 0$) exhibits singularities at the North and South poles either, but these are not of Morse type as seen above. By taking products with a closed manifold N we obtain a foliation $\tilde{\mathcal{F}}_0$ with singularities of Bott-Morse type, all of center type, which is deformed into foliations which are *not* of Bott-Morse type. We can of course take N such that $H^1(N,\mathbb{R}) = 0$, thus showing that the codimension ≥ 3 condition on the singular component $N \subset \mathrm{sing}(\mathcal{F})$ in Theorem 2 cannot be dropped.

3. Deformations of Morse singularities by foliations

Let us consider a differential one-form $\Omega = \sum\limits_{j=1}^{m} f_j\, dx_j$ in coordinates $(x_1,...,x_n) \in U \subset \mathbb{R}^m$ in an open subset.

Definition 1. The *gradient* vector field of Ω is defined as $\mathrm{grad}(\Omega) := \sum\limits_{j=1}^{m} f_j \frac{\partial}{\partial x_j}$.

This is a differentiable vector field which, away from the (singular) zero-set, is orthogonal to the distribution $\mathrm{Ker}(\Omega)$. Also $\mathrm{sing}(\mathrm{grad}(\Omega)) = \mathrm{sing}(\Omega)$.

Theorem 3. *Let $\mathcal{F}_t$ be a smooth deformation of $\mathcal{F}$ in an open neighborhood U of the origin $0 \in \mathbb{R}^m$. Assume $\mathcal{F}$ has a Morse singularity of center type at the origin and either $m \geq 3$, or else $m = 2$ and the leaves of Ω_t are compact for t small enough. Then there exist $\epsilon > 0$, a neighborhood $V \subset \mathbb{R}^m$ of 0 and a smooth function $\xi\colon [0,\epsilon) \to V$ such that:*

(i) $\xi(0) = 0$*;*
(ii) *For $t < \epsilon$ we have $\mathrm{sing}(\mathcal{F}_t) \cap V = \{\xi(t)\}$ and the leaves of $\mathcal{F}_t$ close enough to $\xi(t)$ are compact and diffeomorphic to the sphere S^{m-1};*
(iii) *Moreover, for each such t, $\xi(t)$ is a center type Morse singularity of $\mathcal{F}_t$: there is a smooth map $\rho_t\colon V \to \mathbb{R}$ with $\rho_t(\xi(t)) = 0$, which is a first integral for $\mathcal{F}_t$ in V and has a nondegenerate critical point at $\xi(t)$ of center-type.*

First part of the proof of Theorem 3. Let $\{\Omega_t\}_{t\in[0,\epsilon)}$ be a smooth family of integrable one-forms in the neighborhood U of the origin such that $\mathcal{F}_t$ is defined by the one-form Ω_t and $\mathcal{F}_0 = \mathcal{F}$. Since $\mathcal{F}$ has a center-type singularity at 0, there is a neighborhood $W \subset U$ of the origin where we can choose local coordinates $(x_1, ..., x_m) \in W$ such that Ω is of the form:

$$\Omega = gd(\sum_{j=1}^{m} x_j^2) = \sum_{j=1}^{m} 2gx_j dx_j \, .$$

We set $\Omega_t(x_1, ..., x_m) = \sum\limits_{j=1}^{m} a_j(t,x)dx_j$, then each a_j is smooth. Define a smooth map

$$F\colon [0,\epsilon) \times W \to \mathbb{R}^m$$

by $F(t,x) = (a_1(t,x), ..., a_m(t,x))$. Then we have

$$\frac{\partial}{\partial(x_1, ..., x_m)}\Big|_{t=0} F(t,x) = D(a_1(0,x), ..., a_m(0,x)).$$

This last is a diagonal matrix and its determinant is $(2g(0))^m \neq 0$. Since $F(0,0) = 0$, by the Implicit Function theorem, if $\epsilon > 0$ is small enough, there is a smooth map $\xi\colon [0,\epsilon) \to W$ such that $\xi(0) = 0$, $F(t,\xi(t)) = 0$ and $\text{sing}(\Omega_t) \cap W = \{\xi(t)\}$. Moreover, the partial derivative $\frac{\partial F}{\partial(x_1,...,x_m)}(t,\xi(t))$ is non-singular, so that Ω_t has a nondegenerate singularity at $\xi(t)$. In order to prove that $\xi(t)$ is an stable singularity (i.e, a singularity surrounded by compact leaves with finite holonomy) of Ω_t, we proceed as follows. The leaves of $\mathcal{F}$ in W are spheres of dimension $m-1 \geq 2$. Choose a small neighborhood $V \subset W$ of the origin, invariant by $\mathcal{F}$. Fix a leaf $L_0 \in \mathcal{F}$ such that L_0 bounds a region (a ball) contained in V. By Reeb's stability for perturbations theorem if $\epsilon > 0$ is small enough then $\mathcal{F}_t$ exhibits a compact leaf L_t close to L_0, contained in W. Denote by $R(L_t) \subset W$ the region (diffeomorphic to a closed ball), containing the origin and therefore the singularity $\xi(t) \in \text{sing}(\mathcal{F}_t)$, bounded by the leaf L_t. By Reeb's complete stability theorem all leaves of $\mathcal{F}_t$ in $R(L_t)$ are compact diffeomorphic to L_t. This proves (i) and (ii) in Theorem 3. □

Now we consider the family of vector fields $X_t := -\operatorname{grad}(\Omega_t)$ in W (cf. Definition 1). Then X_t is a smooth deformation of the vector field $X_0 = -\operatorname{grad}(\Omega_0) = -2g\vec{R}$ where $\vec{R}$ is the radial vector field. Using what we have seen above we have:

Lemma 1. *Assume that the dimension $m \geq 3$ is odd. Then for t small enough the vector field X_t exhibits a smooth separatrix through its unique singularity $\xi(t)$ close to the origin.*

Proof. Concerning the existence of separatrices, we may indeed assume that $g = \frac{1}{2}$ and $X_0 = -\vec{R}$. Denote by $\xi(t)$ the singular point of X_t close to the origin $0 = \xi(0)$ in $\mathbb{R}^m$. Then the derivative $DX_t(\xi(t))$ is a perturbation of the derivative $DX_0(0) = \text{Id} \in \text{GL}(m,\mathbb{R})$. This implies that its characteristic equation $P_t(\lambda) = \text{Det}(DX_t - \lambda\,\text{Id}) = 0$ is a perturbation of the characteristic equation $P_0(\lambda) = \text{Det}(DX_0 - \lambda I) = (1-\lambda)^m = 0$. By continuity, for t small enough, the eigenvalues of DX_t at $\xi(t)$ have positive real part, in particular X_t has a hyperbolic singularity at $\xi(t)$ (see Hartman [5]). Since by hypothesis m is odd, there is at least one real eigenvalue and therefore (by the classical Hartman-Grobman theorem [5]) we have at least one smooth unstable separatrix through the singular point $\xi(t)$. □

Using now the fact that X_t is transverse to the leaves of $\mathcal{F}_t$ which are compact manifolds filling up a neighborhood of the singularity $\xi(t)$, we obtain the following fact:

Lemma 2. *The vector field X_t exhibits a smooth separatrix Γ_t through the singularity $\xi(t)$.*

Proof. If m is odd then we apply Lemma 1. Assume now that $m \geq 4$ is even. The one-form Ω_t defines a compact foliation $\mathcal{F}_t$ with a non-degenerate singularity at $\xi(t)$, with $\Omega_0 = g_0\, d(\sum\limits_{j=1}^{m} x_j^2)$ and $\xi(0) = 0$. We may assume that for each leaf L_t of $\mathcal{F}_t$, the vector field X_t points inwards the region $R(L_t)$, bounded by L_t, that contains the singularity $\xi(t)$. Since the regions $R(L_t)$ form a fundamental system of neighborhoods of $\xi(t)$ we conclude that the singularity $\xi(t)$ is asymptotically stable with respect to X_t.

Claim 1. *The spectrum* $\operatorname{Spec}(DX_t(\xi(t))) \subset \mathbb{C}$ *of* X_t *at* $\xi(t)$ *exhibits some real eigenvalue.*

Proof. Write $X_t = (a_1^t, b_1^t, ..., a_n^t, b_n^t)$ where $n = m/2$. Put $Y_t := X_t^{\perp} = (-b_1^t, a_1^t, ..., -b_n^t, a_n^t)$. Then Y_t is orthogonal to X_t and therefore its orbits are tangent to the leaves of Ω_t. In particular, the orbits of Y_t are contained in compact manifolds. The nonsingular orbits of Y_t do not accumulate at the singularity $\xi(t)$. Suppose by contradiction that the characteristic polynomial P_t of $DX_t(\xi(t))$ is of the form $P_t(\lambda) = \prod\limits_{j=1}^{m/2} (\lambda^2 + a_j\lambda + b_j)$ in irreducible polynomials over $\mathbb{R}[\lambda]$. We have several cases to consider.

If there are no multiple eigenvalues then we can write $DX_t(\xi(t))$ as a diagonal matrix of $m/2$ blocks B_j of the form

$$B_j = \begin{pmatrix} \alpha_j & \beta_j \\ -\beta_j & \alpha_j \end{pmatrix}$$

where $\beta_j \neq 0$. Assume for simplicity that $DX_t(\xi(t) = B_j$ is one block. We write $X_t = (a_1^t, b_1^t)$ and $Y_t = X_t^{\perp} = (-b_1^t, a_1^t)$. Then the same linear coordinates that give $DX_t(\xi(t)) = B_j$ give $DY_y(\xi(t)) = B_j^{\perp}$ which is defined as

$$B_j^{\perp} = \begin{pmatrix} \beta_j & -\alpha_j \\ \alpha_j & \beta_j \end{pmatrix}$$

The linear system

$$\dot{x} = DY_t(\xi(t)) \cdot x = B_j^{\perp} \cdot x$$

has its solutions given explicitly in terms of $\exp(\beta_j t)\cos(\alpha_j t)x_j$ and $\exp(\beta_j t)\sin(\alpha_j t)x_j$ so that, thanks to the terms in $\exp(\beta_j t)$ we conclude that the solutions of $\dot{x} = DY_t(\xi(t))x$ cannot be contained in a compact manifold surrounding the singularity $\xi(t)$, they must instead accumulate at the singular point $\xi(t)$. By Hartman-Grobman theorem this same statement holds for the solutions of Y_t. This excludes this "diagonalizable" case.

Now we consider the case where $DX_t(\xi(t))$ is a matrix with blocks of the form

$$\begin{pmatrix} B_j & O \\ I_2 & B_j \end{pmatrix}$$

where

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Therefore we have

$$DY_t(\xi(t)) = \begin{pmatrix} B_j^{\perp} & O \\ J_2 & B_j^{\perp} \end{pmatrix},$$

where

$$J_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Again we conclude that the linear system

$$\dot{x} = Y_t(x) = B_j^{\perp} \cdot x$$

has solutions that cannot be contained in a compact manifold surrounding the singularity $\xi(t)$. They must instead accumulate at the singular point $\xi(t)$. By the classical Hartman-Grobman linearization theorem this gives a final contradiction and proves the claim in this case. The cases of even dimension ≥ 6 are similar and can be proved in the same way.

□

Let us now finish the proof of Lemma 2. Because the spectrum of $DX_t(\xi(t))$ contains some real eigenvalue, it contains some negative eigenvalue and then X_t exhibits some smooth stable separatrix Γ_t through the singularity $\xi(t)$ thanks to the Stable manifold theorem. This proves the lemma in the even dimensional case. □

End of the proof of Theorem 3. The trace of Γ_t is (diffeomorphic to) an interval $[0,1]$ with the origin corresponding to the singularity $\xi(t)$, and transverse to each leaf of $\mathcal{F}_t$ in $R(L_t)$. We may take any smooth function $\rho_t\colon [0,1] \to \mathbb{R}$ such that $\rho_t(0) = 0$ and extend ρ_t to $R(L_t)$ as constant through the leaves of $\mathcal{F}_t$ in $R(L_t)$. Now, if we choose $\rho_t\big|_{\Gamma_t}$ such that it has an order two zero at the origin then we claim that the extension $\rho_t\colon R(L_t) \to \mathbb{R}$ has a nondegenerate singularity at $\xi(t)$. Indeed, since ρ_t is a first integral for Ω_t we have $\Omega_t \wedge d\rho_t = 0$, and since Ω_t has a nondegenerate singularity at $\xi(t)$ we can write $d\rho_t = h_t.\Omega_t$ for some smooth function h_t. In coordinates we have $d\rho_t = h_t \cdot \sum\limits_{j=1}^{m} a_j(t,x)dx_j$ so that $\frac{\partial \rho_t}{\partial x_j} = h_t \cdot a_j(x,t), \forall j = 1,...,m$. Since $a_j(t,\xi(t)) = 0$ we conclude that $\frac{\partial^2 \rho_t}{\partial x_i \partial x_j} = h_t(\xi(t)) \cdot \frac{\partial a_j(t,\xi(t))}{\partial x_i}$. If $h_t(\xi(t)) = 0$ then $D^2\rho_t(\xi(t)) = 0$ what is a contradiction to our original choice of ρ_t as having an order two zero at the origin. Therefore, $h_t(\xi(t)) \neq 0$ and the Hessian of ρ_t at $\xi(t)$ is nonsingular. This implies that ρ_t has a nondegenerate Morse type singularity at $\xi(t)$ and, since the leaves of $\mathcal{F}_t$ in $R(L_t)$ are compact, this singularity is a center. □

4. Integrable deformations of non-isolated singularities

As for the non-isolated case we have the following version of the first part of Theorem 3.

Lemma 3. *Let $\mathcal{F}$ be a foliation on M having a Bott-Morse component $N \subset \operatorname{sing}(\mathcal{F})$ of center type and* $\operatorname{codim} N = \ell \geq 3$. *Let now $\mathcal{F}_t$ be a C^∞ deformation of $\mathcal{F} = \mathcal{F}_0$, where $t \in [0,\epsilon)$. There are a neighborhood W of N in M and $0 < \epsilon_1 < \epsilon$ such that if $t \leq \epsilon_1$ then:*

(1) $\operatorname{sing}(\mathcal{F}_t) \cap W = N_t$ *is a compact nondegenerate component, diffeomorphic to N_0.*
(2) N_t is isotopic to N_0.
(3) $N_t \subset \operatorname{sing}(\mathcal{F}_t)$ is a Bott-Morse component of center type.

Proof. The same ideas as in the proof of Theorem 3 apply here. Indeed, let N be a codimension ℓ component of the singular set of $\mathcal{F}$. Given a point $p \in N$ there is a neighborhood U of p in M diffeomorphic to the product $D^\ell \times D^{m-\ell}$ of discs $D^\ell \subset \mathbb{R}^\ell$ and $D^{m-\ell} \subset \mathbb{R}^{m-\ell}$, such that the restriction $\mathcal{F}\big|_U$ is equivalent to the product foliation $D^{m-\ell} \times \mathcal{F}_1$, where $\mathcal{F}_1$ is a foliation on the disc D^ℓ with an isolated Morse type singularity of center type at the origin $0 \in D^\ell$. At each disc $\mathcal{D}_q := \{q\} \times D^\ell$, for any point $q \in D^{m-\ell}$, $\mathcal{F}$ induces an ordinary Morse singularity of center type, isomorphic to $\mathcal{F}_1$. Given any smooth deformation $\mathcal{F}_t$ of $\mathcal{F} = \mathcal{F}_0$, for small t the foliation $\mathcal{F}_t$ is transverse to the discs $\mathcal{D}_q$ and induces by restriction a smooth deformation $\mathcal{F}_t\big|_{\mathcal{D}_q}$ of $\mathcal{F}\big|_{\mathcal{D}_q}$. Since $\dim \mathcal{D}_q = \ell \geq 3$, by Theorem 3 above there is a smooth function $\xi_q(t)$ of the parameter t such that $\xi_q(t)$ is the only singularity of $\mathcal{F}\big|_{\mathcal{D}_q}$. Moreover, this singularity is of Morse center type.

Finally, the map ξ_q depends also smoothly on the point q as it follows from the Implicit function theorem, where we consider $q \in D^{m-\ell}$ as a parameter on which the coefficients of the map F (which is just the map having as coordinate functions the coefficients of the form $\Omega(t,x) = \Omega_t(x)$, in the proof of Theorem 3) depend smoothly. This shows that there is a neighborhood W of N in U such that for t small enough $\operatorname{sing}(\mathcal{F}_t) \cap W = N_t$ is center type Bott-Morse component, mapped as the graph of a smooth map $\xi(q,t)$ taking values on the transverse disc $\mathcal{D}_q$. By uniqueness these maps glue and this shows that for a suitable neighborhood V of N in M and for small t, the singular set $\operatorname{sing}(\mathcal{F}_t) \cap V$ is a nondegenerate component N_t, diffeomorphic (indeed isotopic) to $N_0 = N$. This shows (1) and (2) in the lemma. Assume now that $\ell = \operatorname{codim} N$ is odd. Then the above arguments show, as in the proof of Theorem 3, that for t small enough we may choose local defining functions ρ_t for $\mathcal{F}_t$ around the points of N_t such that each ρ_t has a center type Bott-Morse singularity at the points in N_t. This proves (3). □

Definition 2. Given an integrable one-form Ω in a manifold M we say that Ω *has nondegenerate singularities* if its singular set $\operatorname{sing}(\Omega)$ is a disjoint union of closed submanifolds $N \subset M$ such that for each point $p \in N \subset \operatorname{sing}(\Omega)$ there are local coordinates $(x_1, ..., x_\ell, x_{\ell+1}, ..., x_m)$ for M, centered at p, such that $N : (x_{\ell+1} = ... = x_m = 0)$ and writing $\Omega = \sum\limits_{j=1}^{m} a_j(x) dx_j$ we have $\operatorname{Det}\big(\frac{\partial a_i}{\partial x_j}\big)_{i,j=\ell+1}^{m}(0) \neq 0$.

5. Proof of the results

Let us now prove our main results.

Proof of Theorem 1. Let $\mathcal{F}_t$ be a smooth deformation of a Morse-Reeb fibration on a compact manifold M of dimension $m \geq 3$. We claim that for t small enough the foliation $\mathcal{F}_t$ is a Morse-Reeb fibration. Indeed, by Theorem 3 , for t small enough, the foliation $\mathcal{F}_t$ is a foliation with nondegenerate singularities of center type, and the leaves close to the singularities are spheres. By Reeb's theorem in [11] $\mathcal{F}_t$ is a Morse-Reeb fibration. □

The proof of Theorem 2 relies on the following local stability result, similar to Thurston's version of Reeb local stability (Corollary 1 in [14]).

Proposition 1 (Proposition 1 in [8])**.** *Let $\mathcal{F}$ be a transversely orientable codimension one foliation with Bott-Morse singularities on a manifold M. Assume that $N \subset \operatorname{sing}(\mathcal{F})$ is a center type component with $H^1(N;\mathbb{R}) = 0$. Then N is stable. Indeed, there is a fundamental system of saturated neighborhoods W of N in M such that each leaf $L \subset W$ is compact with $H^1(L;\mathbb{R}) = 0$. Moreover, the holonomy of the component N is trivial and there is a Bott-Morse function $f\colon W \to \mathbb{R}$, defined in an invariant neighborhood W of N, which defines $\mathcal{F}$ in W.*

Proof of Theorem 2. Consider a deformation $\mathcal{F}_t$, $t \in [0,\epsilon)$, of the Bott-Morse foliation $\mathcal{F}$ having a component $N \subset \operatorname{sing}(\mathcal{F})$ with $H^1(N,\mathbb{R}) = 0$. By Proposition 1 we may apply Lemma 3 and conclude that if $\epsilon > 0$ is small enough then the singular set of $\mathcal{F}_t$ exhibits a center type component N_t isotopic to $N = N_0$. In particular, for $t < \epsilon$ small enough the foliation $\mathcal{F}_t$ is a Bott-Morse foliation having all singularities of center type and some component $N_t \subset \operatorname{sing}(\mathcal{F}_t)$ such that $H^1(N_t,\mathbb{R}) = 0$. Then, according to Theorem 1 in [8] there is a Bott-Morse function $f_t\colon M \to \mathbb{R}$ that defines $\mathcal{F}_t$. □

Proof of Corollary 2. For t small enough in Theorem 2 the foliation $\mathcal{F}_t$ is a Bott-Morse foliation with only center type singularities. Moreover, there is a Bott-Morse function $f_t\colon M \to \mathbb{R}$ that defines $\mathcal{F}_t$. From Theorem A in [12] the singular set $\operatorname{sing}(\mathcal{F}_t)$ has only two components say

N_1^t, N_2^t and $f_t\big|_{M\setminus(N_1^t\cup N_2^t)}$ is a fibre bundle over $(0,1)$ with fibers the leaves of $\mathcal{F}_t$. Moreover, by Lemma 3, each component N_j^t is isotopic (and therefore homeomorphic) to $N_j^0 = N_j \subset \mathrm{sing}(\mathcal{F})$. Finally, from Reeb stability theorem, all leaves of $\mathcal{F}_t$ are diffeomorphic to a (typical) leaf $L^t \in \mathcal{F}_t$ and each leaf L_t is homeomorphic to the (typical) leaf $L^0 \in \mathcal{F}$. Thus the bundles $\mathcal{F}_t\big|_{M\setminus(N_1^t\cup N_2^t)}$ and $\mathcal{F}\big|_{M\setminus(N_1\cup N_2)}$ are topologically equivalent. This and the product type of $\mathcal{F}_t$ around the singularities N_j^t give the topological equivalence between $\mathcal{F}_t$ and $\mathcal{F}$. □

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INDEX OF SINGULARITIES OF REAL VECTOR FIELDS ON SINGULAR HYPERSURFACES

PAVAO MARDEŠIĆ

...with affection and respect,
for all the pleasure of working with Xavier

ABSTRACT. Gómez-Mont, Seade and Verjovsky introduced an index, now called GSV-index, generalizing the Poincaré-Hopf index to complex vector fields tangent to singular hypersurfaces. The GSV-index extends to the real case.

This is a survey paper on the joint research with Gómez-Mont and Giraldo about calculating the GSV-index $\mathrm{Ind}_{V_\pm,0}(X)$ of a real vector field X tangent to a singular hypersurface $V = f^{-1}(0)$. The index $\mathrm{Ind}_{V_\pm,0}(X)$ is calculated as a combination of several terms. Each term is given as a signature of some bilinear form on a local algebra associated to f and X. Main ingredients in the proof are Gómez-Mont's formula for calculating the GSV-index on *singular complex* hypersurfaces and the formula of Eisenbud, Levine and Khimshiashvili for calculating the Poincaré-Hopf index of a singularity of a *real* vector field in $\mathbb{R}^{n+1}$.

1. Introduction

This paper is a survey of the joint work with Xavier Gómez-Mont and Luis Giraldo spread over some 15 years. We give a formula for calculating the index of singularities of real vector fields on singular hypersurfaces. Some partial results are published in [8], [10], [11], [12].

In [13], Gómez-Mont, Seade and Verjovsky studied vector fields tangent to a complex hypersurface with isolated singularity. They introduced a notion of index, now called GSV-index at a common singularity of the vector field and the hypersurface (see also [1]). It is a kind of relative version of the Poincaré-Hopf index at a singularity. A natural question is how can one calculate this index. Complex case was studied first. It was solved by Gómez-Mont in his seminal paper [6]. Gómez-Mont's formula expresses the GSV index via dimensions of certain local algebras. The GSV index can be generalized to the real case. More precisely, depending on the side of the singular hypersurface, there are two GSV indices. Real case, is more difficult than the complex case since in the real case a simple singularity can carry the index $+1$ or -1, whereas in the complex case all simple singularities count as $+1$.

In the absolute real case Eisenbud, Levine and Khimshiashvili expressed the Poincaré-Hopf index of a vector field in terms of the signature of a bilinear form.

Our result in the relative real case expresses the GSV-index of a real vector field on a singular variety as a sum of certain terms. Each term is a signature of a non-degenerate bilinear form on some local algebra.

Our proof has two essential ingredients: on one hand Gomez-Mónt's result in the singular complex case and on the other hand the Eisenbud, Levine, Khimshiashvili's result in the real absolute case.

1.1. **Real absolute case.** Let us recall first the definition of the Poincaré-Hopf index of a singularity of a real vector field in $\mathbb{R}^{n+1}$. Let

$$X = \sum_{i=0}^{n} X^i \frac{\partial}{\partial x_i} \tag{1}$$

be a smooth vector field in $\mathbb{R}^{n+1}$ having an isolated singularity at the origin $X_0 = 0$. One can identify the vector field X with a mapping $X : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}^{n+1}, 0)$. Taking a small sphere S^n around the origin, the vector field X induces a map $N = \frac{X}{||X||} : S^n \to \mathbb{S}^n$, where $\mathbb{S}^n$ is the unitary sphere in $\mathbb{R}^{n+1}$. The Poincaré-Hopf index $\mathrm{Ind}_0(X)$ of the vector field X at the origin is defined as the degree of N. That is, $\mathrm{Ind}(X, 0)$ is the number of pre-images of generic points taken with orientation.

Example 1. Let X be the vector field $X(x, y) = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ in $\mathbb{R}^2$ having a node at the origin and let Y be the vector field $Y(x, y) = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial x}$ having a saddle at the origin.

Then $\mathrm{Ind}_0(X) = 1$ and $\mathrm{Ind}_0(Y) = -1$.

1.2. **Complex absolute case.** Consider the complex n-dimensional space $\mathbb{C}^n$, with complex coordinates $x_1, \ldots, x_n$ and a complex vector field X of the form $X = \sum_{i=0}^{n} X^i \frac{\partial}{\partial x_i}$. We can identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$. With this identification a holomorphic vector field on $\mathbb{C}^n$ becomes a smooth real vector field on $\mathbb{R}^{2n}$ and one can apply the previous definition of the Poincaré-Hopf index $\mathrm{Ind}_0(X)$ to a singularity of a holomorphic vector field. Note that not every smooth real vector field on $\mathbb{R}^{2n}$ comes from a holomorphic vector field on $\mathbb{C}^n$. By holomorphy, a holomorphic vector field seen as a map preserves orientation. Hence the index of a singularity of a holomorphic vector field is necessarily positive.

Example 2. Let $n = 1$ and let $X = x\frac{\partial}{\partial x}$ and $Y = x^2\frac{\partial}{\partial x}$ be vector fields in $\mathbb{C}$. Then $\mathrm{Ind}_0(X) = 1$ and $\mathrm{Ind}_0(Y) = 2$.

In the complex case, the Poincaré-Hopf index is simply the multiplicity. One counts how many points are hidden at the singularity at the origin.

2. Definition of the GSV-index in the complex and real case

2.1. **Smooth points.** Let now $f : (\mathbb{R}^{n+1}, p) \to (\mathbb{R}, 0)$ be a germ of an analytic function. Then $V = f^{-1}(0)$ is a germ of a hypersurface at p. We say that a vector field defined in a neighborhood of $p \in V$ is a vector field tangent to V, if there exists an analytic function h such that

$$X(f) = fh. \tag{2}$$

The function h is sometimes called the *cofactor* of X. Assume first that $p \in V$ is a regular point of f. Then the variety V is smooth in a neighborhood of p. Let $x = (x_1, \ldots, x_n)$ be a chart of V in a neighborhood of p. We assume moreover that the orientation of $\nabla f, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$ is positive. The chart $x = (x_1, \ldots, x_n)$ transports the vector field X to $\mathbb{R}^n$. One then applies the usual definition of the Poincaré-Hopf index. Thus we define the *relative Poincaré-Hopf index* $\mathrm{Ind}_{V,p}(X)$ of a vector field tangent to a hypersurface, relative to the surface. It is easy to verify that the definition is independent of the choices.

If $f : (\mathbb{C}^{n+1}, p) \to (\mathbb{C}, 0)$ is a germ of holomorphic function instead, $p \in \mathbb{C}^{n+1}$ is a regular point of f, $V = f^{-1}(f(p)) \subset \mathbb{C}^{n+1}$ is a complex hypersurface, and X a holomorphic vector field tangent to V, one transports as previously the vector field to $\mathbb{C}^n$ and defines the *relative Poincaré-Hopf index* $\mathrm{Ind}_{V,p}(X)$ in the complex case. Note that in the relative complex case, just as in the absolute complex case, the relative index is always positive.

2.2. **Singular points, GSV-index in the complex case.** Let as previously,

$$f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$$

be a germ of a holomorphic function. Assume now that $p \in \mathbb{C}^{n+1}$ is an isolated singularity of f. Then $V = f^{-1}(0) \subset \mathbb{C}^{n+1}$ is a complex hypersurface with isolated singularity at p. Let X be a holomorphic vector field defined in a neighborhood of $p \in \mathbb{C}^{n+1}$ tangent to V. That is, relation (2) holds. In [13], Gómez-Mont, Seade and Verjovsky defined what is now called the GSV-index of a vector field tangent to a singular variety at the singularity $\mathrm{Ind}_{V,0}(X)$.

In order to formulate the definition, let us first recall that the holomorphic function $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ having an isolated singularity at the origin defines a Milnor fibration: $f : B \setminus \{0\} \to \mathbb{C}^*$, where $B \subset \mathbb{C}^{n+1}$ is a small ball around the origin. Denote $V_\varepsilon = f^{-1}(\varepsilon)$. For $\varepsilon \neq 0$ small, close enough to zero, all fibers $V_\varepsilon \cap B$ are isotopic. Note that the vector field X is not necessarily tangent to the fibers $V_\varepsilon \cap B$, for $\varepsilon \neq 0$. We modify X slightly, giving a C^∞ vector field X_ε tangent to a fiber $V_\varepsilon \cap B$, for $\varepsilon \neq 0$ close to zero. We assume moreover that the restriction of the vector field X_ε on $\partial(V_\varepsilon \cap B)$ is isotopic to the restriction of the vector field X to $\partial(V \cap B)$ see [15] and [1].

The GSV-index can be defined by the formula

$$\mathrm{Ind}_{V,0}(X) = \sum_{p_i(\varepsilon) \in V_\varepsilon \cap B} \mathrm{Ind}_{V_\varepsilon, p_i(\varepsilon)}(X_\varepsilon). \tag{3}$$

It follows from the Poincaré-Hopf theorem that the definition is independent of all choices. Indeed, the Poincaré-Hopf theorem says that the right-hand side of (3) is the Euler characteristic $\chi(V_\varepsilon \cap B)$ up to some correction term given by the behavior of any vector field X_ε on $\partial(V_\varepsilon \cap B)$. Note that by the Milnor fibration theorem all regular fibers $V_\varepsilon \cap B$, $\varepsilon \neq 0$, have the same Euler characteristic. Moreover, the behavior of any vector field in X_ε on $\partial(V_\varepsilon \cap B)$ is the same as the behavior of X on $\partial(V \cap B)$. Hence the correction term is independent of the choices.

For an equivalent topological definition using residues see Suwa [17].

Proposition 1. [1] *Up to a constant $K(V)$ independent of the vector field X, the GSV-index* $\mathrm{Ind}_{V,0}(X)$ *is characterized by the two following conditions:*

- **(i):** *At smooth points $p \in V$, the GSV-index coincides with the relative Poincaré-Hopf index* $\mathrm{Ind}_{V,p}(X)$.
- **(ii):** *The GSV-index satisfies the* law of conservation of number*: For any holomorphic vector field X' tangent to V sufficiently close to X the following law of conservation of number holds:*

$$\mathrm{Ind}_{V,0}(X) = \sum_{p_i \in V} \mathrm{Ind}_{V,p_i(\varepsilon)}(X'). \tag{4}$$

Here p_i are singularities of X' belonging to V, which are close to 0.

The constant can be determined by calculating the GSV-index $\mathrm{Ind}_{V,0}(X)$, for any vector field tangent to V.

2.3. **GSV-index in the real case.** Let now $f : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}, 0)$ be a germ of a real analytic function. In this case, there is no Milnor fibration, or more precisely there are two Milnor fibrations: one for strictly positive small values of ε and one for small strictly negative values of ε. The Euler characteristic of all fibers $V_\varepsilon \cap B$, for ε small of the same sign are the same, but can be different for ε positive or ε negative. (Think of $f : \mathbb{R}^3 \to \mathbb{R}$ given by $f(x, y, z) = x^2 + y^2 - z^2$.) As in the complex case, in the real case one now defines the GSV-index. More precisely, one defines two GSV indices $\mathrm{Ind}_{V^\pm,0}(X)$, taking V_ε, for ε positive or negative respectively.

3. Calculating the GSV-index on complex hypersurfaces

A formula for calculating the GSV-index in the complex case was given by Gómez-Mont in [6]. Let us first define the principal ingredients. Let $\mathcal{O}_{\mathbb{C}^{n+1},0}$ be the algebra of germs of holomorphic functions at the origin. Let $f \in \mathcal{O}_{\mathbb{C}^{n+1},0}$ be given, with $f(0) = 0$. Let $f_i = \frac{\partial f}{\partial z_i}$, $i = 0, \dots, n$, be the partial derivatives of f. Assume that 0 is an isolated singularity of f. This means that the algebra

$$\mathbb{A}^{\mathbb{C}} = \frac{\mathcal{O}_{\mathbb{C}^{n+1},0}}{(f_0, \dots, f_n)} \tag{5}$$

is finite dimensional. Here $\mathcal{O}_{\mathbb{C}^{n+1,0}}$ is the algebra of germs at 0 of holomorphic functions. The dimension $\mu = \dim(\mathbb{A}^{\mathbb{C}})$ is the Milnor number of the singularity. Let X be a germ of holomorphic vector field at $0 \in \mathbb{C}^{n+1}$ given by (1). Assume that 0 is an isolated singularity of X. This means that the algebra

$$\mathbb{B}^{\mathbb{C}} = \frac{\mathcal{O}_{\mathbb{C}^{n+1},0}}{(X^0, \dots, X^n)} \tag{6}$$

is finite dimensional. Its dimension $\dim(\mathbb{B}^{\mathbb{C}})$ is the Poincaré-Hopf index $\mathrm{Ind}_0(X)$ of the vector field X in the ambient space.

Let $V = f^{-1}(0)$ be the hypersurface defined by f and assume that X is tangent to V. That is, (2) holds for some holomorphic function h.

Theorem 1. [6] *The GSV-index of a holomorphic vector field X tangent to a complex hypersurface V at an isolated singularity 0 is given by.*

$$\mathrm{Ind}_{V,0}(X) = \begin{cases} \dim \frac{\mathbb{B}^{\mathbb{C}}}{(f)} - \dim \frac{\mathbb{A}^{\mathbb{C}}}{(f)}, & \text{if (n+1) even,} \\ \dim \mathbb{B}^{\mathbb{C}} - \dim \frac{\mathbb{B}^{\mathbb{C}}}{(h)} + \dim \frac{\mathbb{A}^{\mathbb{C}}}{(f)}, & \text{if (n+1) odd.} \end{cases} \tag{7}$$

We give the idea of proof of Theorem 1. As recalled in Proposition 1, the GSV index is defined up to a constant by condition (i) and (ii) in Proposition 1. In [6] Gómez-Mont considers the *Koszul complex*:

$$0 \to \Omega^{n-1}_{V,0} \to \Omega^{n-1}_{V,0} \to \cdots \to \Omega^1_{V,0} \to \mathcal{O}_{V,0} \to 0, \tag{8}$$

where

$$\Omega^i_{V,0} = \frac{\Omega_{\mathbb{C}^{n+1},0}}{f\Omega_{\mathbb{C}^{n+1},0} + df \wedge \Omega_{\mathbb{C}^{i-1}_{n+1,0}}}. \tag{9}$$

is the space of *relatively exact forms* on V and the arrows in (8) are given by contraction of forms by the vector field X. Gómez-Mont defines the *homological index* $\mathrm{Ind}^{hom}_{V,0}$ as the Euler characteristic of the complex (8):

$$\mathrm{Ind}^{hom}_{V,0} = \sum_{i=0}^{n-1} (-1)^i \dim H_i(K) \tag{10}$$

where $H_i(K)$, $i = 0, \dots, n-1$, are the i-th homology groups of the Koszul complex (8). It is easy to see that at smooth points the homological index coincides with the relative Poincaré-Hopf index. In [7], Giraldo and Gómez-Mont show that the homological index verifies the law of conservation (ii) of Proposition 1. Hence, the homological index coincides with the GSV-index up to a constant $K(V)$. The homological index has the advantage that it can be calculated using projective resolutions of a double complex. The horizontal complexes in the double complex are obtained as a *mapping cone* induced by multiplication by the cofactor h in (2) in the Koszul complex in the ambient space. Vertical complexes are obtained as the mapping cone induced by multiplication by f in the *de Rham complex* in the ambient space. To show that the homological index $\mathrm{Ind}^{hom}_{V,0}$ coincides with the GSV-index $\mathrm{Ind}_{V,0}$, it is sufficient for each f to calculate both indices on a vector field X associated to f. If the dimension of the ambient space $(n+1)$ is even, a natural candidate is the Hamiltonian vector field

$$X_f = \sum_{i=1}^{(n+1)/2} [f_{2i}\frac{\partial}{\partial x_{2i-1}} - f_{2i-1}\frac{\partial}{\partial x_{2i}}]. \tag{11}$$

If $(n+1)$ is odd, Gómez-Mont uses the vector field

$$Y_f = f\frac{\partial}{\partial x_0} + \sum_{i=1}^{(n+1)/2} [f_{2i}\frac{\partial}{\partial x_{2i-1}} - f_{2i-1}\frac{\partial}{\partial x_{2i}}] \tag{12}$$

in generic coordinates x_i.

4. Calculating the Poincaré-Hopf index of vector fields in $\mathbb{R}^{n+1}$

When studying the Poincaré-Hopf index in the real case, one has to take into account orientation and not just multiplicity. This is done using some bilinear forms. We recall in this section the results of Eisenbud, Levine [4] and Khimshiashvili [14] who solve this problem for real vector fields in the ambient space $\mathbb{R}^{n+1}$. This, in addition to Gómez-Mont's formula for calculating the GSV-index on complex hypersurfaces, are the two main ingredients in our study.

Let

$$\mathbb{B} = \frac{\mathcal{A}_{\mathbb{R}^{n+1},0}}{(X^0,\ldots,X^n)}, \tag{13}$$

where $\mathcal{A}_{\mathbb{R}^{n+1},0}$ is the algebra of germs at 0 of analytic functions in $\mathbb{R}^{n+1}$. Let X, given by (1), be a germ of analytic vector field with an algebraically isolated singularity. That is, the singularity when considered over the complex domain remains isolated. Then the algebra $\mathbb{B}$ is finite dimensional. Let $J = \det(\frac{\partial X^i}{\partial x_j}) \in \mathcal{A}_{\mathbb{R}^{n+1},0}$ be the Jacobian of the map defined by the vector field X. It can be shown that the class $[J] \in \mathbb{B}$ of J in $\mathbb{B}$ is non-zero. In [4] and [14] Eisenbud, Levine and Khimshiashvili define a nondegenerate bilinear form $<,>_{\mathbb{B},J}$ as follows.

$$\mathbb{B}\times\mathbb{B}\xrightarrow{\cdot}\mathbb{B}\xrightarrow{L}\mathbb{R}. \tag{14}$$

Here the first arrow is simply multiplication in the algebra $\mathbb{B}$ and L is any linear mapping such that $L([J]) > 0$. Of course, the bilinear form depends on the choice of L. However its signature $\mathrm{sgn}(\mathbb{B}, J) = \mathrm{sgn}(<,>_{\mathbb{B},J})$ does not. More precisely Eisenbud, Levine, Khimshiashvili show

Theorem 2. *Let X be a germ at 0 of a real analytic vector field on $\mathbb{R}^{n+1}$ having an algebraically isolated singularity at the origin. Then the Poincaré-Hopf index $\mathrm{Ind}_{\mathbb{R}^{n+1},0}(X)$ of the vector filed X at the origin is given by*

$$\mathrm{Ind}_{\mathbb{R}^{n+1},0}(X) = \mathrm{sgn}(\mathbb{B}, J). \tag{15}$$

In order to prove the theorem, one has to prove that the signature $\mathrm{sgn}(\mathbb{B}, J)$ coincides with the Poincaré-Hopf index for simple singularities and verifies the law of conservation of number. The first claim is easily verified. The key-point of the proof of the law of conservation of number is the claim that the bilinear form $<,>_{\mathbb{B},J}$ is nondegenerate.

Once one knows that the form is nondegenerate, the law of conservation of number will follow. Indeed, let X' be a small real deformation of the vector field X. As the bilinear form is nondegenerate, its signature does not change by a small deformation. The local algebra $\mathbb{B}$ will decompose into a multilocal algebra $\mathbb{B}(X')$ of the same dimension concentrated in some real point and complex conjugated pairs of points. One verifies that the contribution to the signature of the pairs of complex conjugated points is zero. From the preservation of signature, there follows the law of conservation of number once one knows that the bilinear form is nondegenerate.

The nondegeneracy of the form $<,>_{\mathbb{B},J}$ is a more general feature. It follows from the fact that J generates the *socle* of the algebra $\mathbb{B}$. By definition a socle in an algebra is the *minimal* nonzero ideal of the algebra.

In general, let $\mathbb{B}$ be a real algebra. Assume that the socle of $\mathbb{B}$ is one-dimensional generated by $J \in \mathbb{B}$. We can define a bilinear form $<,>_{\mathbb{B},J}$ as above. Following the proof of Eisenbud-Levine in [4] one verifies that the form $<,>_{\mathbb{B},J}$ is nondegenerate. Its signature does not depend on the choice of the linear map L such that $L(J) > 0$.

Example 3. Consider for instance $\mathbb{B} = \frac{\mathcal{A}_{\mathbb{R}^2,0}}{(x^2,y^3)}$. Then the socle is one-dimensional generated by $J = xy^2$. The bilinear form $<,>_{\mathbb{B},J}$ is a nondegenerate form on the six dimensional space $\mathbb{B}$.

If $\mathbb{B} = \frac{\mathcal{A}_{\mathbb{R}^2,0}}{(x^2,xy^2,y^3)}$, then the socle is generated by xy and y^2. It is not one-dimensional and one cannot define a nondegenerate bilinear form as above.

5. Bilinear Forms on Local Algebras

Let $\mathbb{B} = \frac{\mathcal{A}_{\mathbb{R}^{n+1},0}}{(X^0,\dots,X^n)}$ be a finite dimensional complete intersection algebra. This assures that its socle is one-dimensional generated by the Jacobian $J = \det(\frac{\partial X^i}{\partial x_j})$.

In [10], we observed that the Eisenbud-Levine, Khimshiashvili signature generalizes. Let $h \in \mathbb{B}$ be arbitrary. Denote $\mathrm{Ann}(h) = \{g \in \mathbb{B} : gh = 0\}$ the *annihilator ideal* of h. For $\mathbb{B}$ as above, the algebra $\frac{\mathbb{B}}{\mathrm{Ann}(h)}$ has a one-dimensional socle generated by the element $\frac{J}{h} \in \frac{\mathbb{B}}{\mathrm{Ann}(h)}$. The assumption that (J) is minimal guarantees that J can be divided by h. We define the bilinear form $<,>_{\mathbb{B},h,J}$ on $\frac{\mathbb{B}}{\mathrm{Ann}(h)}$ by $<b,b'>_{\mathbb{B},h,J} = L(bb'h)$, where $L : \mathbb{B} \to \mathbb{R}$ is a linear mapping such that $L(J) > 0$. In other words $<b,b'>_{\mathbb{B},h,J} = L_h(bb')$, where $L_h(\frac{J}{h}) > 0$ is a linear mapping. Note that in general the element $\frac{J}{h}$ is not well defined in $\mathbb{B}$. However, the ambiguity is lifted in the quotient space $\frac{\mathbb{B}}{\mathrm{Ann}(h)}$.

We put

$$\mathrm{sgn}(\mathbb{B},h,J) = \mathrm{sgn} <,>_{\mathbb{B},h,J} = \mathrm{sgn}(\frac{\mathbb{B}}{\mathrm{Ann}(h)}, \frac{J}{h}). \tag{16}$$

5.1. Signatures associated to a singular point of a hypersurface.

Let now

$$f : (\mathbb{R}^{n+1},0) \to (\mathbb{R},0)$$

be a germ of analytic function having an algebraically isolated singularity at the origin. Let $f_i = \frac{\partial}{\partial x_i}$ be the partial derivatives of f. Consider the local algebra $\mathbb{A} = \frac{\mathcal{A}_{\mathbb{R}^{n+1},0}}{(f_0,\dots,f_n)}$. It is a finite complete intersection algebra. Its socle is one-dimensional generated by the Hessian

$$\mathrm{Hess}(f) = \det(\frac{\partial^2 f}{\partial x_i \partial x_j}).$$

Define a flag of ideals in $\mathbb{A}$

$$K_m = \mathrm{Ann}_{\mathbb{A}}(f) \cap (f^{m-1}), \quad m \geq 1. \tag{17}$$

Note that

$$0 \subset K_{\ell+1} \subset \cdots \subset K_1 \subset K_0 = \mathbb{A}. \tag{18}$$

Define a family of bilinear forms $<,>_{f,m}: K_m \times K_m \to \mathbb{R}$ by

$$<a, a'>_{f,m} = <\frac{a}{f^{m-1}}, a'>, \quad m = 0, \dots, \ell+1, \tag{19}$$

where $<,>_{\mathbb{A},Hess(f)}$ is the bilinear form defined in (14) for some linear map L with

$$L(\mathrm{Hess}(f)) > 0.$$

In particular $<a, a'>_{f,0} = <fa, a'>_{\mathbb{A},\mathrm{Hess}(f)}$. The form $<,>_{f,0}$ degenerates on $\mathrm{Ann}_{\mathbb{A}}(f)$, but on K_0/K_1 defines a nondegenerate form. We have $<a, a'>_{f,1} = <a, a'>_{\mathbb{A},\mathrm{Hess}(f)}$. This form degenerates on $K_2 = \mathrm{Ann}_{\mathbb{A}}(f) \cap (f)$ etc. In [12], we define

$$\sigma_i = \mathrm{sgn} <,>_{f,i}, \quad i = 0 \dots, \ell. \tag{20}$$

The signatures σ_i are intrinsically associated to the singularity 0 of f.

6. Main Result

The following theorem resumes our results [10], [11], [8], [12] about the calculation of the GSV-index of singularities of real vector fields on hypersurfaces:

Theorem 3. *Let $f : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}, 0)$ be a germ of analytic function with algebraically isolated singularity at the origin. Let X be an analytic vector field in $\mathbb{R}^{n+1}$ having an algebraically isolated singularity at the origin. Assume that X is tangent to $V = f^{-1}(0)$. That is $X(f) = hf$, for some analytic function h. Then*

(i): *if $(n+1)$ is even,*

$$\mathrm{Ind}_{V^+,0}(X) = \mathrm{Ind}_{V^-,0}(X) = \mathrm{sgn}(\mathbb{B}, h(X), J(X)) - \mathrm{sgn}(\mathbb{A}, h(X), \mathrm{Hess}(f)). \tag{21}$$

(ii): *if $(n+1)$ is odd,*

$$\mathrm{Ind}_{V^\pm,0}(X) = \mathrm{sgn}(\mathbb{B}, h(X), J(X)) + K_\pm, \tag{22}$$

where

$$K_+ = \sum_{i \geq 1} \sigma_i, \quad K_- = \sum_{i \geq 1} (-1)^i \sigma_i. \tag{23}$$

7. Proof of the Main theorem

We give here the main ingredients of the proof of Theorem 3. The GSV-index is determined by three properties:

(i): Value at smooth points
(ii): The law of conservation of number
(iii): Constants $K_\pm$ depending only on the orientation (side) $V_\pm$ of the variety $V = f^{-1}(0)$ and not on the vector field.

One verifies easily that at smooth points of V, the formula is valid. Indeed, from the tangency condition there follows $(f) \subset \mathrm{Ann}(h)$. In smooth points the converse is also true. Hence $\mathrm{Ann}(h) = (f)$. Next, working in a local chart at smooth points one shows that $\mathrm{sgn} <,>_{\mathbb{B},h,J}$ gives the relative Poincaré-Hopf index of the vector field. Then, one has to show that our formulas (21) and (22) verify the law of conservation of number. Some parts are easier in the even case and some other are easier in the odd case.

7.1. $(n+1)$ **odd case.** The law of conservation of number is easy for $(n+1)$ odd. Indeed, in this case the complex index, up to a constant depending only of f, is

$$\dim \mathbb{B}^{\mathbb{C}} - \dim \frac{\mathbb{B}^{\mathbb{C}}}{(h)} = \dim \frac{\mathbb{B}^{\mathbb{C}}}{\mathrm{Ann}(h)}$$

(see Theorem 1). On the other hand on $\frac{\mathbb{B}}{\mathrm{Ann}(h)}$ there is the non-degenerate form $<,>_{\mathbb{B},h,J}$. Make a small deformation X' of X, tangent to V. The corresponding local algebra $\mathbb{B}$ or rather its complexification decomposes into a multilocal algebra concentrated in several points corresponding to singular points of X'. The dimension of the multilocal algebra is equal to the sum of the dimensions at points in which it is concentrated. On the other hand, by Theorem 1 of Gomez-Mont, the dimension $\dim \frac{\mathbb{B}^{\mathbb{C}}(X')}{\mathrm{Ann}(h)}$ verifies the law of conservation of number. Hence, the dimension of the multilocal algebra obtained after deformation X' of X is equal to the dimension of the local algebra $\dim \frac{\mathbb{B}^{\mathbb{C}}}{\mathrm{Ann}(h)}$ before the deformation. This permits to extend continuously the bilinear form $<,>_{h,J}$ from the algebra $\frac{\mathbb{B}}{\mathrm{Ann}(h)}$ to its deformation. By nondegeneracy of the form $<,>_{h,J}$, its signature is unchanged by a small deformation. This gives the law of conservation of number for the signature of $<,>_{h,J}$ when adding the signatures for all (real or complex) singular points of X' appearing after deformation. Note that from the tangency condition (2), it follows that $(f) \subset \mathrm{Ann}(h)$, so only points in $V = f^{-1}(0)$ can contribute to the signature $\mathrm{sgn} <,>_{\mathbb{B}(X'),h,J}$ after deformation. At the end, let us note that complex zeros of X' come in pairs. One verifies that the contribution to the signature of each pair is equal to zero. Hence only real singular points of X' belonging to V contribute. The law of conservation of number (in the real case) for the formula $\mathrm{sgn}(\mathbb{B}, h, J)$ follows.

The final step in proving the formula in the case $(n+1)$ odd is to adjust the constant $\mathrm{sgn}(\mathbb{A}, \mathrm{Hess}(f)) + K_{\pm}$. This is difficult in the odd case. We will come back to it in subsection 7.4.

7.2. $(n+1)$ **even case.** In the $(n+1)$ even case Theorem 1 says that in the complex case, up to a constant, the index is given by $\dim \frac{\mathbb{B}^{\mathbb{C}}}{(f)}$. There is no natural bilinear form on $\frac{\mathbb{B}^{\mathbb{C}}}{(f)}$. We consider a non-degenerate bilinear form on $\frac{\mathbb{B}^{\mathbb{C}}}{\mathrm{Ann}(h)}$. We stratify the space of bilinear vector fields by the dimension of the ideal (h) in the algebra $\mathbb{A}$. The signature $\mathrm{sgn}(h(X), J(X))$ verifies the law of conservation of number in restriction to each stratum. We show that when changing the stratum the jump in $\mathrm{sgn}(\mathbb{B}, h, J)$ is equal to the jump in $\mathrm{sgn}(\mathbb{A}, h, \mathrm{Hess}(f))$. The two jumps hence compensate in the index formula (21). In order to show the equality of the jumps it is sufficient to study the place where all strata meet i.e. the stratum of highest codimension. One has the highest codimension for the Hamiltonian vector field X_f given by (11), when $h = 0$. Note that in this case the two algebras $\mathbb{A}$ and $\mathbb{B}$ coincide and $J(X) = \mathrm{Hess}(f)$.

In this case it is very easy to determine the constant (independent of the vector field) adjusting the signature formula with index. For that purpose, one studies the Hamiltonian vector field X_f. Note that the Hamiltonian vector field X_f is tangent to all fibers $V_\varepsilon = f^{-1}(\varepsilon)$. Moreover, it has the same behavior on the boundary $V_\varepsilon \cap B$, for $\varepsilon \neq 0$ as on $V \cap B$. The Hamiltonian vector field X_f has no zeros on $V_\varepsilon = f^{-1}(\varepsilon)$, for $\varepsilon \neq 0$. Hence $\mathrm{Ind}_{V_\pm}(X_f) = 0$. On the other hand $\mathrm{sgn}(\mathbb{B}, h(X), J(X)) - \mathrm{sgn}(\mathbb{A}, h, Hess) = 0$, as $\mathbb{A} = \mathbb{B}$ and $J = \mathrm{Hess}$. It follows that no correction term has to be added to $\mathrm{sgn}(\mathbb{B}, h(X), J(X)) - \mathrm{sgn}(\mathbb{A}, h, Hess)$ in order to obtain the formula for $\mathrm{Ind}_{V_\pm,0}(X)$.

7.3. **Why is** $\mathrm{Ind}_{V_+,0}(X) = \mathrm{Ind}_{V_-,0}(X)$ **in the** $(n+1)$ **even case and not in the odd case?** We explain here why $\mathrm{Ind}_{V_+,0}(X) = \mathrm{Ind}_{V_-,0}(X)$ in the $(n+1)$ even case and not in the odd case. Note first that the index of a vector field in the ambient space is an even function if the dimension $(n+1)$ of the ambient space is even and is an odd function if $(n+1)$ is odd. We next use Morse theory. Consider the vector field ∇f. By Morse theory, the Euler characteristic χ verifes:

$$\begin{aligned} \chi(V_+) &= 1 + \operatorname{Ind}(\nabla f) \\ \chi(V_-) &= 1 + \operatorname{Ind}(-\nabla f). \end{aligned} \tag{24}$$

Here $\chi(V_+)$ is the Euler characteristic of $V_\varepsilon \cap B$, for $\varepsilon > 0$ small. The value $\chi(V_-)$ is defined analogously.

If $(n+1)$ is even, then $\operatorname{Ind}(\nabla f) = \operatorname{Ind}(-\nabla f)$, so $\chi(V_+) = \chi(V_-)$ and

$$\operatorname{Ind}_{V_+,0}(X) = -\operatorname{Ind}_{V_-,0}(X).$$

If $(n+1)$ is odd, then $\operatorname{Ind}(-\nabla f) = -\operatorname{Ind}(\nabla f)$, so $\chi(V^+) - 1 = -(\chi(V_-) - 1)$ and

$$\operatorname{Ind}_{V_-,0}(X) = 2 - \operatorname{Ind}_{V_+,0}(X).$$

7.4. **Adjusting the constant K in the $(n+1)$ odd case.** In order to complete the sketch of proof of the main theorem, we have to explain how do we calculate the constant $K_\pm$ appearing in the $(n+1)$ odd case (22).

As shown previously, the two signature terms in (22) calculate the GSV-index up to a constant independent of the vector field. In order to determine the constant, for each $V = f^{-1}(0)$, one has to take a vector field tangent to V, having an algebraically isolated singularity at the origin. Contrary to the situation in the $(n+1)$ even case, in the odd case, there is no such natural vector field. As in [6], we use the family of vector fields

$$X_t = (f-t)\frac{\partial}{\partial x_0} + \sum_{i=1}^{(n+1)/2} [f_{2i}\frac{\partial}{\partial x_{2i-1}} - f_{2i-1}\frac{\partial}{\partial x_{2i}}] \tag{25}$$

in generic coordinates. The local algebra is $\mathbb{B} = \mathbb{B}(X_0) = \frac{\mathcal{A}_{\mathbb{R}^{n+1},0}}{(f,f_1,f_2,\ldots,f_n)}$. Note that X_t is tangent to $V_t = f^{-1}(t)$, for any t. More precisely, $X_t(f) = f_0 f$, so $h = f_0$ is the cofactor of X_t. Hence, by definition

$$\operatorname{Ind}_{V_+}(X,0) = \sum_{p_t \in V_t \cap B} \operatorname{Ind}_{V_t,p_t}(X_t). \tag{26}$$

But, these indices are calculated using the multilocal algebra $\mathbb{B}_t$ and the relative Jacobian $\frac{J(X_t)}{f_0}$. That is, the index is given by the signature of the bilinear form $<,>_{\mathbb{B}_t}$, for $t \neq 0$ small. For the index $\operatorname{Ind}_{V_+,0}(X_0)$, we have to take it positive and for $\operatorname{Ind}_{V_-,0}(X_0)$ it is negative. The problem is that this form degenerates on $\operatorname{Ann}_{\mathbb{B}_t}(f_0)$, for $t = 0$.

We prove in [12] a general result for algebras $\mathbb{A} = \mathbb{A}(f)$ and $\mathbb{B} = \mathbb{B}(X)$ associated to a vector field X tangent to $V = f^{-1}(0)$ i.e. verifying (2):

Lemma 1. *There exists a natural isomorphism between the algebras* $\operatorname{Ann}_{\mathbb{B}}(h)$ *and* $\operatorname{Ann}_{\mathbb{A}}(f)$.

Proof. The isomorphism is given by the mapping $g \mapsto k$ if $gh = fk$. □

Lemma 1 permits to transport all higher order signature vanishing in $\operatorname{Ann}_{\mathbb{B}_0}$ to a natural algebra $\mathbb{A}$. We apply it to our vector field X_t. When looking at the signature of the form $<,>_{\mathbb{B}_t}$, we have one part which does not degenerate. It is the part in $\frac{\mathbb{B}}{\operatorname{Ann}_{\mathbb{B}}(f_0)}$. The bilinear form $<,>_{\mathbb{B}_t}$ degenerates at different orders on different parts of $\operatorname{Ann}_{\mathbb{B}_t}(f_0)$. By Lemma 1, we transport the bilinear form $<,>_{\mathbb{B}_t}$ to a bilinear form in the coordinate independent algebra $\operatorname{Ann}_{\mathbb{A}}(f)$. Note that in $\mathbb{B}_t$, we have $f = t$, so degeneration of $<,>_{\mathbb{B}_t}$ at different orders of t corresponds to multiplication by f in the algebra $\operatorname{Ann}_{\mathbb{A}}(f)$. For more details see [12].

8. Open problems

8.1. Geometric interpretation of the signatures σ_i. Filtration of contributions to the Euler characteristic of the generic fiber. In Theorem 3 appear higher order signatures σ_i defined in (20). These signatures are associated to the singularity f alone. We would like to give a geometric interpretation of these numbers. We believe that they correspond to parts of the Euler characteristic of the generic fiber, filtered by the speed of arrival at the singular fiber.

Let us be more precise. In [20] Teissier studies polar varieties in the complex case (see also [18], [19]). He considers a germ of a function $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ having an isolated critical point at the origin. He considers a Morsification $f_s = f - sx_0$ of f in generic coordinates $(x_0, \dots, x_n)$. Its critical points are given by

$$f_0 - s = f_1 = \cdots = f_n = 0. \tag{27}$$

Let Γ be the curve given by $f_1 = \cdots = f_n = 0$. The curve Γ is called *polar curve*. In general it has several branches $\Gamma = \cup_{q=1}^{\ell} \Gamma_q$. By Morsification, the critical point 0 of f decomposes in several critical points arriving along the polar curve to the origin. For each value of $s \neq 0$, the critical points of f_s belong to $f_0^{-1}(s) \cap \Gamma$. Each critical point corresponds to a vanishing cycle contributing to $H_n(V_{t_0})$. In [20], Teissier observed that, after Morsification, critical points arrive at different speed at the origin. More precisely, each component Γ_q of the polar curve Γ is parametrized as

$$\begin{array}{rcl} x_0(t_q) & = & t_q^{m_q} + \cdots \\ \cdots & \cdots & \cdots \\ x_n(t_q) & = & \lambda_n t_q^{k_{q,n}} + \cdots \end{array} \tag{28}$$

where $m_q \leq k_{q,i}$. In [20], Teissier calculates the exponent m_q. One can use x_0 (or the corresponding critical value) as a measure for the speed of approach of a critical point in the Morsification. One can filter the n-th group of homology of the generic fiber $H_n(f^{-1}(t))$ i.e. the space of vanishing cycles, by the speed of arrival of the corresponding critical points. We believe that this filtration is given by the filtration (18) or rather its complex counterpart. The dimensions

$$0 = \dim \frac{\mathbb{A}}{K_0} \leq \dim \frac{\mathbb{A}}{K_1} \leq \dim \frac{\mathbb{A}}{K_{\ell+1}} \leq \dim \mathbb{A} \tag{29}$$

would measure the dimension of the space of vanishing cycles arriving at a certain minimal speed.

The signatures σ_i would be the real counterpart. The signature σ_0 is a signature of a bilinear form on $\mathbb{A}$. It measures the Euler characteristic $\chi(V_t)$. We believe that the signatures σ_i that we introduced measure the filtered part of the Euler characteristic of the generic fiber $\chi(V_t)$, the filtration being done by taking only the part of the topology of the fiber arriving at a certain minimal speed. We hope to be able to address this problem in a continuation of our research.

8.2. Generalization to higher codimension. In [2] Bothmer, Ebeling and Gómez-Mont generalized Gómez-Mont's formula (Theorem 1) to a formula for the index of a vector field on an isolated complete intersection singularity in the complex case. A natural problem would be to extend the result to the real case. Here, as in our Theorem 3, one would certainly have to define some bilinear forms on the spaces studied in [2].

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TOPOLOGY OF SINGULAR HOLOMORPHIC FOLIATIONS ALONG A COMPACT DIVISOR

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Abstract. We consider a singular holomorphic foliation $\underline{\mathcal{F}}$ defined near a compact curve $\underline{\mathcal{C}}$ of a complex surface. Under some hypothesis on $(\underline{\mathcal{F}}, \underline{\mathcal{C}})$ we prove that there exists a system of tubular neighborhoods U of a curve $\underline{\mathcal{D}}$ containing $\underline{\mathcal{C}}$ such that every leaf L of $\underline{\mathcal{F}}_{|(U\setminus\underline{\mathcal{D}})}$ is incompressible in $U \setminus \underline{\mathcal{D}}$. We also construct a representation of the fundamental group of the complementary of $\underline{\mathcal{D}}$ into a suitable automorphism group, which allows to state the topological classification of the germ of $(\underline{\mathcal{F}}, \underline{\mathcal{D}})$, under the additional but generic dynamical hypothesis of transverse rigidity. In particular, we show that every topological conjugation between such germs of holomorphic foliations can be deformed to extend to the exceptional divisor of their reductions of singularities.

Dedicated to Xavier Gómez-Mont on the occasion of his 60h birthday

1. Introduction and main results

We consider a smooth complex surface $\underline{M}$ endowed with a holomorphic foliation $\underline{\mathcal{F}}$ having isolated singularities and a compact connected holomorphic curve $\underline{\mathcal{C}}$. To treat in a unified way the local setting we will also allow the case that $\underline{\mathcal{C}}$ reduces to an isolated singular point. There are two main results in this paper under some hypothesis concerning the pair $(\underline{\mathcal{F}}, \underline{\mathcal{C}})$, which we will precise in the sequel:

(A) The existence of a fundamental system of neighborhoods of $\underline{\mathcal{C}}$ where the leaves of $\underline{\mathcal{F}}$ are incompressible in the complementary of an "adapted" curve $\underline{\mathcal{D}}$ containing $\underline{\mathcal{C}}$. Recall that a subset A of a topological space V is **incompressible** in V if the natural inclusion $A \subset V$ induces a monomorphism at the fundamental group level for every choice of the base point in A.

(B) The construction of a representation of the fundamental group of the complementary of $\underline{\mathcal{D}}$ into a suitable automorphism group, which allows us to state the topological classification of the germ $(\underline{\mathcal{F}}, \underline{\mathcal{D}})$ of $\underline{\mathcal{F}}$ along $\underline{\mathcal{D}}$. When the curve $\underline{\mathcal{C}}$ is smooth and invariant by $\underline{\mathcal{F}}$, this object is directly equivalent to the classical holonomy representation of $\pi_1(\underline{\mathcal{C}})$ into the automorphisms of a transverse section.

A particular situation of this context occurs when the pair $(\underline{M}, \underline{\mathcal{C}})$ is a resolution of a surface singularity (S, O), see Example 1.6. In the general setting it is well known that there exists a composition $E : M \to \underline{M}$ of blow-ups such that the curve $\mathcal{C} := E^{-1}(\underline{\mathcal{C}})$ and the foliation $\mathcal{F} := E^*\underline{\mathcal{F}}$ satisfy the following properties:

- $\mathcal{C}$ has normal crossings and all its irreducible components $\mathcal{C}_i$, $i \in \mathfrak{I}$ are smooth,
- two different irreducible components of $\mathcal{C}$ are disjoint or intersect in a unique point,

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- $\mathcal{F}$ is reduced in the sense of [4], i.e. each singular point of $\mathcal{F}$ has Camacho-Sad index in $\mathbb{C} \setminus \mathbb{Q}_{>0}$ and each component $\mathcal{C}_i$ is either $\mathcal{F}$-invariant or $\mathcal{C}_i \cap \mathrm{Sing}(\mathcal{F}) = \emptyset$ and $\mathcal{F}$ is totally transverse to $\mathcal{C}_i$.

All the notions that we introduce in the sequel are germified along $\mathcal{D}$ or $\underline{\mathcal{D}}$. By definition the **isolated separatrix set** of $\mathcal{F}$ is the set $\mathcal{S}$ constituted by invariant curves by $\mathcal{F}$, which are not contained in $\mathcal{C}$ and which intersect some $\mathcal{F}$-invariant irreducible component of $\mathcal{C}$. The image of the components of $\mathcal{S}$ by E are called the **isolated separatrices** of $\underline{\mathcal{F}}$.

Let $\mathcal{G}_{\mathcal{C}}$ be the **dual graph** associated to $(M, \mathcal{C})$ having one vertex $\mathbf{s}_i$ for each irreducible component $\mathcal{C}_i$ of $\mathcal{C}$ and one edge when two irreducible components of $\mathcal{C}$ intersect. We also introduce a double weighting (g_i, ν_i) in each vertex $\mathbf{s}_i$, by giving the genus $g_i = g(\mathcal{C}_i)$ and minus the self-intersection $\nu_i = -\mathcal{C}_i \cdot \mathcal{C}_i$. It is well known that we can topologically recover a tubular neighborhood of $\mathcal{C}$ in M by a plumbing procedure from the data given by the dual graph with weights $\mathcal{G}_{\mathcal{C}}$, see Section 2.1.

In the sequel we will need to consider a (not necessarily compact) holomorphic curve $\mathcal{D} \subset M$ containing $\mathcal{C}$. We define the **valence** with respect to $\mathcal{D}$ of an irreducible component D of $\mathcal{D}$ as the number $v(D)$ of singular points of $\mathcal{D}$ lying on D. A **dead branch** of $\mathcal{D}$ is a connected maximal union of irreducible components of $\mathcal{C}$ of genus 0 with valence 2 with respect to $\mathcal{D}$ except for one of them whose valence must be 1.

Making an additional iterative blowing down process if necessary, without loss of generality we can also assume that

- there is no exceptional (i.e. having self-intersection -1) $\mathcal{F}$-invariant rational component of $\mathcal{C}$ of valence ≤ 2 with respect to $\mathcal{D}$.

Notice that an irreducible component D (not necessarily compact) of $\mathcal{D}$ may be transverse to $\mathcal{F}$. In that case we will say that D is a **dicritical** component of $\mathcal{F}$.

In order to state our first main result we must introduce some new notions. Denote by $\mathcal{G}_{\mathcal{D}}$ the dual graph associated to the divisor $\mathcal{D}$.

- A **breaking element** of $\mathcal{G}_{\mathcal{D}}$ is every vertex corresponding to a dicritical component of $\mathcal{F}$ and every edge corresponding to a linearizable singularity of $\mathcal{F}$.
- The **break graph** associated to $(\mathcal{F}, \mathcal{D})$ is the graph obtained from $\mathcal{G}_{\mathcal{D}}$ by removing all the breaking elements and the edges whose one of its endpoints is a breaking vertex.
- An **initial component** of $(\mathcal{F}, \mathcal{D})$ is a $\mathcal{F}$-invariant irreducible component C of $\mathcal{C}$ such that one of the following situations holds:
  - (a) $g(C) = 0$, there is a non-linearizable singular point p_0 of $\mathcal{F}$ on C and every point $p \in \mathrm{Sing}(\mathcal{D}) \cap C$, $p \neq p_0$, belongs to some dead branch;
  - (b) $g(C) > 0$ and the holonomy of the boundary of every embedded conformal disk in C containing $\mathrm{Sing}(\mathcal{D}) \cap C$ is not linearizable.

We introduce two hypothesis on the pair $(\mathcal{F}, \mathcal{C})$. The first one is of local nature and it concerns only the singularities of $\mathcal{F}$. The second one is global and it also concerns the topology of $\mathcal{C}$.

(L) *The reduced foliation $\mathcal{F}$ has no saddle-nodes and each singularity $s \in \mathrm{Sing}(\mathcal{F})$ having Camacho-Sad index $\lambda_s \notin \mathbb{Q}$ is linearizable.*

(G) *Each connected component of the break graph associated to $(\mathcal{F}, \mathcal{C})$ is a tree, which contains at most one vertex corresponding to an initial component C of $\mathcal{C}$ of genus $g(C) > 0$.*

Notice that Condition (L) is generic in the following sense: let $\mathcal{B} \subset \mathbb{C}$ be the set of Brjuno numbers, namely those complex numbers λ verifying that the germ of every singular foliation defined by a 1-form of type $(u + \cdots)dv - (\lambda v + \cdots)du$ is always linearizable. It is well known that $\mathbb{C} \setminus \mathbb{R} \subset \mathcal{B}$ and that $\mathbb{R}^- \setminus \mathcal{B}$ has zero measure.

If $\lambda \in \mathbb{R}_{>0}$ then the singularity is a **node**. Because the reducedness of $\mathcal{F}$ we have that λ is necessarily irrational. If such a singular point s belongs to the strict transform of a (necessarily isolated) separatrix Z of $\underline{\mathcal{F}}$ we say that Z is a **nodal separatrix** of $\underline{\mathcal{F}}$ and s a **nodal singularity**. The topological specificity of such singularity is the existence, in any small neighborhood of s, of a saturated closed set whose complement is an open disconnected neighborhood of the two punctured local separatrices of the node. We call **nodal separator** such a saturated closed set. A nodal separator of $\underline{M}$ is the image by E of a nodal separator in M.

If D is a dicritical component of $\mathcal{F}$ then for each singular point $s \in \mathrm{Sing}(\mathcal{D}) \cap D$ we consider a conformal closed disk $D_s \subset D$ containing s in its interior such that their pairwise intersections are empty. A **dicritical separator** associated to D is a tubular neighborhood of the closure of $D \setminus \bigcup_{s \in \mathrm{Sing}(\mathcal{D}) \cap D} D_s$ which is the total space of a holomorphically trivial disk fibration whose fibers are contained in the leaves of $\mathcal{F}$. A dicritical separator of $\underline{M}$ is the image by E of a dicritical separator of M.

On the other hand, Condition (G) is not generic and we do not know if the incompressibility of the leaves of $\mathcal{F}$ in the complementary of some $\mathcal{D} \supset \mathcal{C}$ holds when it is not fulfilled. Even in the case that Condition (G) holds, the first choice $\mathcal{D} = \mathcal{C}$ does not work for instance by considering the case that $\mathcal{C}$ is the exceptional divisor of the reduction of singularities of a germ of foliation $\underline{\mathcal{F}}$ in $\underline{M} = (\mathbb{C}^2, 0)$ because $M \setminus \mathcal{C} \cong \mathbb{C}^2 \setminus \{0\}$ is simply connected. The next natural choice consists to add to $\mathcal{C}$ the isolated separatrices $\mathcal{S}$ of $\mathcal{F}$ but this is not enough as the following example shows.

Exemple 1.1. Consider the dicritical foliation $\underline{\mathcal{F}}$ in $(\mathbb{C}^2, 0)$ defined by the rational first integral $\underline{f}(x, y) = \frac{y^2 - x^3}{x^2}$ whose isolated separatrix set is the cusp $S = \{y^2 - x^3 = 0\}$. Let $\underline{M}$ be a Milnor ball for S. The composition $E : M \to \underline{M}$ of blow-ups considered in the introduction for this case corresponds to the minimal desingularization of S. The exceptional divisor $\mathcal{C} = E^{-1}(0)$ has three irreducible components D_1, D_2, D_3 which we numerate according to the order that they appear in the blowing up process. Thus $D_1^2 = -3$, $D_2^2 = -2$ and $D_3^2 = -1$. The strict transform $\mathcal{S}$ of S only meets D_3. It turns out that D_1 and D_2 are two dead branches composed by a single irreducible component attached to D_3. Moreover D_1 a dicritical component. In fact, it is totally transverse to $\mathcal{F} = E^*\underline{\mathcal{F}}$. Thus, $\mathcal{C} \cup \mathcal{S}$ do not satisfy Condition (c) in Definition 1.2. On the other hand, it is well-known that if a, b and c are meridian loops around D_1, D_2 and D_3 respectively, with common origin, then $\pi_1(\underline{M} \setminus S, \cdot) = \langle a, b, c \,|\, a^3 = b^2 = c \rangle$. We shall see that there exist non-incompressible leaves of $\mathcal{F}$ inside $M \setminus (\mathcal{C} \cup \mathcal{S})$. Indeed, looking at the situation after the first blowing-up, we immediately see that there are two types of leaves of $\mathcal{F}$: those that are near to the isolated separatrix set $\mathcal{S}$, which are disks minus two points and the others which are diffeomorphic to $\mathbb{D}^*$. If L is a leaf of the first kind then $\pi_1(L) = \langle \alpha_+, \alpha_- | - \rangle$ is a free group of rank 2. We claim that we can choose the generators so that the morphism $\imath : \pi_1(L) \to \pi_1(M \setminus (\mathcal{C} \cup \mathcal{S}))$ induced by the inclusion is given by $\imath(\alpha_+) = a$ and $\imath(\alpha_-) = b^{-1}ab$. It follows that $\imath(\alpha_+^3 \alpha_-^{-3}) = a^3 b^{-1} a^{-3} b = 1$ and consequently L is not incompressible in $M \setminus (\mathcal{C} \cup \mathcal{S})$. In order to prove the claim we consider the coordinate system (t, x) on $M \setminus (\mathcal{C} \cup \mathcal{S})$ induced by the first blowing-up, defined by $E(t, x) = (x, tx) = (x, y)$. We have $f(x, t) := (E^*\underline{f})(x, t) = t^2 - x$ and the restriction of f to $U_\varepsilon := \{|f| = \varepsilon\}$, $0 < \varepsilon \ll 1$, is a locally trivial C^∞-fibration over the standard circle $\mathbb{S}^1_\varepsilon$ of radius ε, whose fiber over ε is $F_\varepsilon := \mathbb{C} \setminus \{\pm\sqrt{\varepsilon}\}$. Since the pull-back of $U_\varepsilon \xrightarrow{f} \mathbb{S}^1_\varepsilon$ by the exponential map $\exp : [0, 2\pi] \to \mathbb{S}^1_\varepsilon$, $\exp(\theta) = \varepsilon e^{i\theta}$, is trivial, we obtain a trivializing map $\tau : F_\varepsilon \times [0, 2\pi] \to U_\varepsilon$ sending (z, θ) into $(t, x) = (z e^{i\frac{\theta}{2}}, (z^2 - \varepsilon) e^{i\theta})$. We consider the path $\beta : s \mapsto (z, \theta) = (0, s + \pi)$, $s \in [0, 2\pi]$, projecting by τ into the loop $(t, x) = (0, -\varepsilon e^{is})$ which is a meridian of D_2. Hence, we can take the generator $b \in \pi_1(M \setminus (\mathcal{C} \cup \mathcal{S}))$ as the homotopy class of β. Let $z(s)$ be a simple loop in F_ε based on $z=0$ having index $+1$ around $+\sqrt{\varepsilon}$ and index 0 around $-\sqrt{\varepsilon}$. We define $\alpha_-(s)=(z(s),0)$ and $\alpha_+(s)=(z(s),2\pi)$. It is clear that α_+ is homotopic to $\beta\alpha_-\beta^{-1}$ in $F_\varepsilon\times[0,2\pi]$. Hence its respective projections by τ are also homotopic loops in U_ε. Notice that $\tau(\alpha_-(s))=(z(s),z^2(s)-\varepsilon)$ and $\tau(\alpha_+(s))=(-z(s),z^2(s)-\varepsilon)$ are meridians around D_1 so that we can choose the generator $a\in\pi_1(M\setminus(\mathcal{C}\cup\mathcal{S}))\cong\pi_1(U_\varepsilon)$ as the homotopy class of $\tau(\alpha_+)$. The fundamental group of the leaf L passing through the point $(t,x)=(0,-\varepsilon)$ is $\pi_1(L)=\langle\alpha_+,\alpha_-\,|\,-\rangle$ and the images of its generators by $\imath$ are given by $\imath(\alpha_+)=a$ and $\imath(\alpha_-)=b^{-1}ab$.

However, if we define $\mathcal{D}:=\mathcal{S}\cup T\cup\mathcal{C}$, where T is the strict transform of $\{x=0\}$, we can directly see that all the leaves are incompressible in $M\setminus\mathcal{D}$. Indeed T meets D_1 transversely, then

$$\pi_1(M\setminus\mathcal{D},\cdot)=\langle a,b,c\,|\,b^2=c,\ [c,a]=1\rangle=\langle a,b\,|\,[a,b^2]=1\rangle$$

and the elements a and $b^{-1}ab$ are without relation in this group. □

Thus, we must make some additional "holes" in $M\setminus(\mathcal{C}\cup\mathcal{S})$ in order to obtain a bigger fundamental group which could contain the fundamental group of each leaf. This will be done by considering a new divisor $\mathcal{D}\supset\mathcal{C}\cup\mathcal{S}$ obtained by adding some small curves transverse to $\mathcal{C}$ satisfying the following technical properties.

Definition 1.2. *We say that a (generally not compact) divisor $\mathcal{D}\subset M$ is **adapted** to $(\mathcal{F},\mathcal{C})$ if the following conditions hold:*

(a) the adherence of $\mathcal{D}\setminus\mathcal{C}$ is a finite union of conformal disks transverse to $\mathcal{C}$ at regular points of $\mathcal{C}$ and $\mathcal{D}\setminus\mathcal{C}$ does not contain any singular point of $\mathcal{F}$;
(b) the isolated separatrix set $\mathcal{S}$ is contained in $\mathcal{D}$;
(c) for every irreducible components C and D of $\mathcal{C}$ we have $C\cap D=\emptyset$ provided that C is contained in a dead branch and D is dicritical;
(d) if $\mathcal{D}=\mathcal{C}$, then it contains at least two irreducible components which do not belong to any dead branch;
(e) each connected component of the break graph associated to $(\mathcal{F},\mathcal{D})$ contains at most one vertex corresponding to an initial component of $(\mathcal{F},\mathcal{D})$

Adding to $\mathcal{C}\cup\mathcal{S}$ one non-isolated separatrix over each dicritical component of $\mathcal{C}$ having valence 1 and one transverse curve over certain initial components of genus zero, we obtain a divisor $\mathcal{D}$ adapted to $(\mathcal{F},\mathcal{C})$ provided it fulfills Condition (G):

Proposition 1.3. *If $(\mathcal{F},\mathcal{C})$ satisfies Condition (G) then there always exists a divisor $\mathcal{D}$ adapted to $(\mathcal{F},\mathcal{C})$.*

In the case that $\mathcal{C}$ is the exceptional divisor of the reduction of a germ $\underline{\mathcal{F}}$ at $(\mathbb{C}^2,0)$, in the statement of Corollary A we will precise the "minimal" divisor adapted to $(\mathcal{F},\mathcal{C})$.

For $A\subset B\subset\underline{M}$ we denote by $\mathrm{Sat}_{\underline{\mathcal{F}}}(A,B)$ the union of all the leaves of $\mathcal{F}_{|B}$ passing through some point of A. We fix a plumbing tubular neighborhood W of $\mathcal{C}$ in M (see Section 2.1). The first main result of this paper is the following.

Theorem A. *Let $\mathcal{D}$ be a divisor adapted to $(\mathcal{F},\mathcal{C})$. Assume that $(\mathcal{F},\mathcal{D})$ satisfies the assumptions (L) and (G) stated below. Then there exists a fundamental system $(U_n)_{n\in\mathbb{N}}$, $U_{n+1}\subset U_n$, of open neighborhoods of $\underline{\mathcal{D}}:=E(\mathcal{D})$ in $\underline{M}$ and there exists a smooth holomorphic curve $\Upsilon\subset M$ transverse to $\mathcal{F}$ having a finite number of connected components, such that for each $n\in\mathbb{N}$ the open sets $U_n^*:=U_n\setminus\underline{\mathcal{D}}$ and $V^*:=E(W)\setminus\underline{\mathcal{D}}$ satisfy the following properties:*

(i) the inclusions $U_{n+1}^*\subset U_n^*\subset V^*$ induce isomorphisms of their fundamental groups,
(ii) every leaf of $\underline{\mathcal{F}}_{|U_n^*}$ is incompressible in U_n^*,

(iii) *each connected component of $Y_n^* := E(\Upsilon) \cap U_n^*$ is a punctured topological disk which is incompressible in U_n^* and $\mathrm{Sat}_{\underline{\mathcal{F}}}(Y_n^*, U_n)$ is the complementary in U_n^* of a finite union of nodal and dicritical separators,*

(iv) *there does not exist any path lying on a leaf of $\mathcal{F}_{|U_n^*}$ with distinct endpoints on Y_n^* which is homotopic in U_n^* to a path lying on Y_n^*,*

(v) *the leaf space of the foliation induced by $\mathcal{F}$ in the universal covering space of U_n^* is a not necessarily Hausdorff one-dimensional complex manifold.*

Remark 1.4. It will follow from the proof that a curve Υ satisfying the properties of Theorem A can be constructed in the following way. We choose a vertex in each connected component of the break graph of $(\mathcal{F}, \mathcal{D})$, a regular point in the corresponding irreducible component of $\mathcal{D}$ and we take transversal disks through these points as branches of Υ. Hence, the irreducible components of this curve are in one-to-one correspondence with the connected components of the break graph.

The following corollary of Theorem A completes the main result of [10].

Corollary A. *Let $\mathcal{F}$ be a germ of singular holomorphic foliation in $(\mathbb{C}^2, 0)$ which is a generalized curve such that all its singularities after reduction whose Camacho-Sad index is not rational are linearizable. Then, there exists an open ball $\mathbb{B}$ centered at 0, an analytic curve $\mathcal{Z}$ closed in $\mathbb{B}$ containing all the isolated separatrices of $\mathcal{F}$, a fundamental system $(U_n)_{n\in\mathbb{N}}$ of neighborhoods of $\mathcal{Z}$ in $\mathbb{B}$ and a curve $\Upsilon \subset \mathbb{B}$, transverse to $\mathcal{F}$ outside the origin, such that the open sets $U_n^* := U_n \setminus \mathcal{Z}$ and $V^* := \mathbb{B} \setminus \mathcal{Z}$ satisfy Properties (i)-(v) of Theorem A. Moreover, if $\mathcal{F}$ is not dicritical then we can take $\mathcal{Z}$ as the set of all the separatrices of $\mathcal{F}$. Otherwise, we can take $\mathcal{Z}$ as the set of all the isolated separatrices of $\mathcal{F}$ jointly with one non-isolated separatrix of $\mathcal{F}$ for each dicritical component containing a unique singular point of the exceptional divisor of the reduction of $\mathcal{F}$.*

Remark 1.5. We point out some issues of each requirement of adapted divisor in Definition 1.2:

(a) As we have already pointed out, roughly speaking, $W \setminus \mathcal{D}$ is obtained from $W \setminus \mathcal{C}$ making some holes in order to that $\pi_1(W \setminus \mathcal{D})$ is big enough to contain the fundamental group of each leaf.

(b) As it was stated by R. Thom in the seventies, the separatrix set can be viewed as the organization center of the topology of the foliation around a singular point. Hence it is natural to study the topological embedding of the leaves in the complement of it. In the dicritical case there is an infinite number of separatrices, so the first natural candidate curve to eliminate from the ambient space is the isolated separatrix set.

(c) If $\mathfrak{m} \subset \mathcal{C}$ is an invariant dead branch of $\mathcal{D}$ then on a neighborhood of $\mathfrak{m}$, the leaves sufficiently close to $\mathfrak{m}$ are disks or rational curves. If moreover $\mathfrak{m}$ attaches to a dicritical component D, then Condition (c) of Definition 1.2 is not satisfied. Near D the leaves L far away from $\mathfrak{m}$ are punctured disks with infinite cyclic fundamental group $\mathbb{Z}c$, but we can deform the loop $c \subset L$ in the ambient space so that it is conjugated to a loop in a simply connected leaf close to $\mathfrak{m}$. Hence, in this case we never have the incompressibility of all the leaves. On the other hand, there exist counter-examples to the incompressibility of the leaves if we admit some dicritical component contained in a dead branch, as we have already seen in Example 1.1, where we have treated in detail the simplest non-trivial dicritical foliation in $(\mathbb{C}^2, 0)$ showing this behavior.

(d) The radial vector field is a trivial counter-example for the incompressibility of its leaves if Condition (d) of Definition 1.2 is not satisfied. On the other hand, if $\mathcal{C}$ is an $\mathcal{F}$-invariant divisor whose dual graph is a tree and $(\mathcal{F}, \mathcal{D})$ do not satisfy Condition (d) of Definition 1.2, then the intersection matrix $(\mathcal{C}_i \cdot \mathcal{C}_j)$ can not be negative definite. Indeed, the main result of [1] implies the existence of separatrices (which are necessarily isolated because $\mathcal{C}$ is $\mathcal{F}$-invariant) in that case. Hence $\mathcal{D} \supset \mathcal{S} \cup \mathcal{C} \supsetneq \mathcal{C}$. Consequently, such divisors do not come from foliations on surface singularities. However, it would be interesting to study the topology of the leaves in this context. The simplest situation occurs when $\mathcal{D} = \mathcal{C}$ is a chain. Since the restriction of the leaves to a neighborhood of a component of valence 1 and 2 are disks and annuli respectively, we deduce that the global leaves in the chain situation are topologically spheres, hence simply connected.

(e) Condition (e) of Definition 1.2 is of technical nature and it comes from the method of construction developed in [10] which is used in this work.

Exemple 1.6. Let (S, O) the surface singularity

$$\{z^2 = (x^2 + y^2)(x^2 + y^7)\} \subset (\mathbb{C}^3, 0)$$

considered in [1]. The desingularization $(\underline{M}, \underline{\mathcal{C}})$ of (S, O) is described by a triangular graph whose vertex represent rational curves having self-intersections -2, -2 and -3, cf. [8]. After [17], the fundamental group G of $S \setminus \{O\} \cong \underline{M} \setminus \underline{\mathcal{C}}$ can be presented as

$$G = \langle a, b, c \,|\, cac^{-1} = a^{-3}b^5,\ cbc^{-1} = a^{-5}b^8,\ [a, b] = 1\rangle$$

and it is solvable. By the synthesis theorem of [9] there exists a singular holomorphic foliation $\mathcal{F}$ on (S, O) such that after desingularization defines a singular foliation $\underline{\mathcal{F}}$ on $\underline{M}$ whose singularities are reduced and correspond to the three intersection points of the precedent rational curves. By applying the index theorem of [3] we deduce that the Camacho-Sad index of these singularities belong to the list $\{-\frac{11}{10} \pm \frac{\sqrt{21}}{10}, -\frac{9}{10} \pm \frac{\sqrt{21}}{10}, -\frac{3}{2} \pm \frac{\sqrt{21}}{6}\}$. From Siegel and Liouville theorems we deduce that all three singularities are linearizable. By applying Theorem A we obtain that the fundamental group of each leaf of $\mathcal{F}$ is solvable because it is a subgroup of G. Therefore all the leaves of $\mathcal{F}$ are disks or annuli. □

The precedent arguments show a more general result.

Corollary 1.7. *Let (S, O) be a surface singularity such that the fundamental group of $S \setminus \{O\}$ is solvable. If $\mathcal{F}$ is a singular holomorphic foliation on (S, O) without local separatrices then all the leaves of $\mathcal{F}$ are disks and annuli.*

Proof. The classification of configurations with solvable fundamental group given by [17] and the hypothesis about the non-existence of local separatrices force all the Camacho-Sad indices to be algebraic numbers, hence of Brjuno type. □

To deal with the second objective of the paper, the topological classification, we fix the topological type of $\underline{\mathcal{C}}$ as embedded divisor in $\underline{M}$, a divisor $\underline{\mathcal{D}}$ adapted to $(\underline{\mathcal{F}}, \underline{\mathcal{C}})$, a fundamental system $(U_n)_{n\in\mathbb{N}}$ of neighborhoods of $\underline{\mathcal{C}}$ fulfilling conditions (i)-(v) of Theorem A and a universal covering $q : \widetilde{U}_0 \to U_0 \setminus \underline{\mathcal{D}}$. In the sequel we will use the following notations:

$$\text{if} \quad A \subset U_0 \quad \text{then} \quad A^* = A \setminus \underline{\mathcal{D}} \quad \text{and} \quad \widetilde{A} = q^{-1}(A^*).$$

Thanks to Property (i) in Theorem A, we can take the restriction of q to $\widetilde{U}_n$ as universal covering of U_n^*. The deck transformation groups of all these coverings will be identified to $\Gamma := \mathrm{Aut}_q(\widetilde{U}_0)$. We denote by $\mathcal{Q}_n$ the leaf space of the foliation induced by $\underline{\mathcal{F}}$ in $\widetilde{U}_n$. Clearly the holomorphic natural maps $\mathcal{Q}_{n+1} \to \mathcal{Q}_n$ form an inverse system denoted by $\mathcal{Q}^{\underline{\mathcal{F}}}$. As we already pointed out in [12, §3] in the local setting, each deck transformation factorizes through $\mathcal{Q}_n$ and allows us to consider the notion of monodromy. To this end, we denote by $\underleftarrow{\mathcal{A}}$ the category of inverse systems of objects in some category $\mathcal{A}$. We refer to [5, 12] for a precise description of the morphisms in $\underleftarrow{\mathcal{A}}$.

Definition 1.8. *The monodromy representation of* $\underline{\mathcal{F}}$ *along* $\underline{\mathcal{D}}$ *is the natural morphism of groups*

$$\mathfrak{m}^{\underline{\mathcal{F}}} : \Gamma \to \mathrm{Aut}_{\underleftarrow{\mathrm{An}}}(\mathcal{Q}^{\underline{\mathcal{F}}}),$$

where $\underleftarrow{\mathrm{An}}$ *is the category of pro-objects associated to the category* An *of analytic spaces.*

Consider now another foliation $\underline{\mathcal{F}}'$ defined in a neighborhood of a curve $\underline{\mathcal{C}}'$ embedded in a surface $\underline{M}'$ and a divisor $\underline{\mathcal{D}}'$ adapted to $(\underline{\mathcal{F}}', \underline{\mathcal{C}}')$. In order to state our second main result, we need to adapt to our new context some additional notions that we have already considered in [12]:

- We say that a topological conjugation between the germs $(\underline{\mathcal{F}}, \underline{\mathcal{D}})$ and $(\underline{\mathcal{F}}', \underline{\mathcal{D}}')$ is **$\mathcal{S}$-transversely holomorphic** if it is transversely holomorphic outside some nodal and dicritical separators. We have the same notion for conjugations between the germs $(\mathcal{F}, \mathcal{D})$ and $(\mathcal{F}', \mathcal{D}')$. Notice that if there are no dicritical components nor nodal singularities then a $\mathcal{S}$-transversely holomorphic conjugation is just a transversely holomorphic conjugation.

- A **$\mathcal{S}$-conjugation** between the monodromies $\mathfrak{m}^{\underline{\mathcal{F}}}$ and $\mathfrak{m}^{\underline{\mathcal{F}}'}$ consists of $(\varphi, \tilde{\varphi}, h)$ where

  $$h : \mathcal{Q}^{\underline{\mathcal{F}}} \to \mathcal{Q}^{\underline{\mathcal{F}}'}$$

  is an isomorphism in the category $\underleftarrow{\mathrm{Top}}$, which is holomorphic outside the subset corresponding to the leaves of some nodal and dicritical separators (we will say that h is a **$\mathcal{S}$-$\underleftarrow{\mathrm{An}}$ isomorphism**), $\varphi : (\underline{U}, \underline{\mathcal{D}}) \to (\underline{U}', \underline{\mathcal{D}}')$ is a germ of homeomorphism defined in some neighborhoods of $\underline{\mathcal{D}}$ and $\underline{\mathcal{D}}'$ and $\tilde{\varphi}$ is a lifting of φ to the universal coverings of $\underline{U} \setminus \underline{\mathcal{D}}$ and $\underline{U}' \setminus \underline{\mathcal{D}}'$ such that the following diagram commutes

  $$\begin{array}{ccccc} \Gamma & \xrightarrow{\mathfrak{m}^{\underline{\mathcal{F}}}} & \mathrm{Aut}_{\underleftarrow{\mathrm{An}}}(\mathcal{Q}^{\underline{\mathcal{F}}}) & \subset & \mathrm{Aut}_{\underleftarrow{\mathrm{Top}}}(\mathcal{Q}^{\underline{\mathcal{F}}}) \\ \tilde{\varphi}_* \downarrow & & & & \downarrow h_* \\ \Gamma' & \xrightarrow{\mathfrak{m}^{\underline{\mathcal{F}}'}} & \mathrm{Aut}_{\underleftarrow{\mathrm{An}}}(\mathcal{Q}^{\underline{\mathcal{F}}'}) & \subset & \mathrm{Aut}_{\underleftarrow{\mathrm{Top}}}(\mathcal{Q}^{\underline{\mathcal{F}}'}). \end{array}$$

  In addition, we say that $(\varphi, \tilde{\varphi}, h)$ is **realized over** germs of subsets $\Sigma \subset \underline{M}$ and $\Sigma' \subset \underline{M}'$, if $\varphi(\Sigma) = \Sigma'$ and the following diagram commutes:

  $$\begin{array}{ccc} \widetilde{\Sigma} & \xrightarrow{\widetilde{\varphi}_{|\widetilde{\Sigma}}} & \widetilde{\Sigma}' \\ \downarrow & & \downarrow \\ \mathcal{Q}^{\underline{\mathcal{F}}} & \xrightarrow{h} & \mathcal{Q}^{\underline{\mathcal{F}}'}, \end{array}$$

  where the vertical arrows are the natural morphisms. These notions also apply to $(\mathcal{F}, \mathcal{D})$ and $(\mathcal{F}', \mathcal{D}')$.

- We define the **cut divisor** $\mathcal{D}^{\mathrm{cut}}$ as the disjoint union of the closure of each connected component of the complementary in $\mathcal{D}$ of nodal singular points and dicritical components of $(\mathcal{F}, \mathcal{D})$. Notice that the dual graph of $\mathcal{D}^{\mathrm{cut}}$ is not the break graph of $(\mathcal{F}, \mathcal{D})$. These notions are independent.

- A **$\mathcal{S}$-collection of transversals** for $\underline{\mathcal{F}}$ and $\underline{\mathcal{D}}$ is a finite collection $\Sigma = \{(\Sigma_i, p_i)\}_i$, where each (Σ_i, p_i) is the image by $E : M \to \underline{M}$ of the germ of a regular curve transverse to $\mathcal{F}$ at a regular point $p_i \in \mathcal{D} \setminus \mathrm{Sing}(\mathcal{D})$ not belonging to the exceptional divisor $\mathcal{E}$ of E, the whole collection satisfying that for each connected component $\mathcal{D}^{\mathrm{cut}}_\alpha \subset \mathcal{D}^{\mathrm{cut}}$ of the cut divisor there exists $i \in \{1, \ldots, m\}$ such that $p_i \in \mathcal{D}^{\mathrm{cut}}_\alpha$. The existence of a such collection follows from the below lemma whose proof is just adapted from that of the Strong Camacho-Sad Separatrix Theorem given in [14].

Lemma 1.9. *There is no irreducible component of $\mathcal{D}^{\mathrm{cut}}$ contained in the exceptional divisor $\mathcal{E}$ of E.*

Proof. By contradiction, let $\mathcal{D}^{\mathrm{cut}}_\alpha$ be a component of $\mathcal{D}^{\mathrm{cut}}$ contained in $\mathcal{E}$ and denote by $\mathcal{T}$ its dual graph. As in [14, Section 3] the vertices $\mathbf{s}_i$ of $\mathcal{T}$ are weighted by the self-intersection of the corresponding component D_i multiplied by -1 and to each edge $\mathbf{a}_{ij}$ (joining $\mathbf{s}_i$ and $\mathbf{s}_j$) is associated the pair $(\wp_{ij}, \wp_{ji})$, where $-\wp_{ij}$ is equal to the real part Camacho-Sad index $\mathrm{CS}(\mathcal{F}, D_i, s_{ij})$ and $\{s_{ij}\} := D_i \cap D_j$. At the singular points s of $\mathcal{D}$ lying in the regular part of $\mathcal{D}^{cut}_\alpha$ the Camacho-Sad index of $\mathcal{F}$ are not negative real number. Indeed, it is zero if s is the attaching point of a dicritical component and it is positive if s is a nodal singularity. Then the index formulae give the inequalities $\sum_j \wp_{ij} \geq D_i \cdot D_i$ and, using the terminology introduced in [14], $\mathcal{T}$ is a fair quasi-proper tree. We also have the inequalities $\wp_{ij}\wp_{ji} \leq 1$ and $\mathcal{T}$ is well-balanced. This cannot occur because of Lemma 2.1 of [14], which asserts the no existence of well balanced fair proper tree, is extended to quasi-proper trees in [14, Section 4]. □

Remark 1.10. The method developed in [14] immediately give a lower bound for the number of isolated separatrices for dicritical foliations in terms of the number of nodal singularities and dicritical components.

- We say that a foliation $\underline{\mathcal{F}}$ is **$\mathcal{S}$-transversely rigid** if every topological conjugation between $\underline{\mathcal{F}}$ and another foliation $\underline{\mathcal{F}}'$ is necessarily $\mathcal{S}$-transversely holomorphic. There are many situations in which we have this property. For instance, an extended version of the Transverse Rigidity Theorem of [15] already used in [12] asserts that the following condition implies the $\mathcal{S}$-transversal rigidity:

  (R) *Each connected component of the cut divisor contains an irreducible component with non-solvable holonomy group.*

- We call **$\underline{\mathcal{D}}$-extended divisor** every curve $\underline{\mathcal{D}}^+ \supset \underline{\mathcal{D}}$ such that $\overline{\underline{\mathcal{D}}^+ \setminus \underline{\mathcal{D}}}$ consists in the union of pairs of non-isolated separatrices, one pair for each dicritical separator of $\underline{\mathcal{F}}$.

- A germ of homeomorphism $\varphi : (M, \mathcal{D}) \to (M', \mathcal{D}')$ is **excellent** if it satisfies the following properties:
  (a) outside some neighborhoods of the singular locus of $\mathcal{D}$ and $\mathcal{D}'$, φ conjugates the smooth disk fibrations π_i and π_i' given by Lemma 2.1;
  (b) φ is holomorphic in a neighborhood of the singular set of $\mathcal{D}$.

Theorem B. *Let $\mathcal{D}$ (resp. $\mathcal{D}'$) be a divisor adapted to $(\mathcal{F}, \mathcal{C})$ (resp. $(\mathcal{F}', \mathcal{C}')$). Assume that $(\mathcal{F}, \mathcal{D})$ and $(\mathcal{F}', \mathcal{D}')$ satisfy the assumptions (L) and (G). Then the following statements are equivalent:*

(1) $(\underline{\mathcal{F}}, \underline{\mathcal{D}})$ and $(\underline{\mathcal{F}}', \underline{\mathcal{D}}')$ are $\mathcal{S}$-transversely holomorphic conjugated;
(2) $(\mathcal{F}, \mathcal{D})$ and $(\mathcal{F}', \mathcal{D}')$ are $\mathcal{S}$-transversely holomorphic conjugated by an excellent homeomorphism;
(3) there exists a $\mathcal{S}$-conjugation $(\underline{\varphi}, \underline{\widetilde{\varphi}}, h)$ between the monodromies representations of $\underline{\mathcal{F}}$ along $\underline{\mathcal{D}}$ and $\underline{\mathcal{F}}'$ along $\underline{\mathcal{D}}'$, which is realized over $\mathcal{S}$-collections of transversals, such that:
  (a) there exist a $\underline{\mathcal{D}}$-extended divisor $\underline{\mathcal{D}}^+$ such that $\underline{\varphi}(\underline{\mathcal{D}}^+)$ is a $\underline{\mathcal{D}}'$-extended divisor; in addition, for each irreducible component D of $\underline{\mathcal{D}}$ we have that D is $\underline{\mathcal{F}}$-invariant if and only if $\underline{\varphi}(D)$ is $\underline{\mathcal{F}}'$-invariant;

  (b) for each singular point s of $\underline{\mathcal{F}}$ and each invariant local irreducible component of $\underline{\mathcal{D}}$ at s we have the equality of Camacho-Sad indices $\mathrm{CS}(\underline{\mathcal{F}}, D, s) = \mathrm{CS}(\underline{\mathcal{F}}', \underline{\varphi}(D), \underline{\varphi}(s))$;

(4) there exists a $\mathcal{S}$-conjugation $(\varphi, \widetilde{\varphi}, h)$ between the monodromies representations of $\mathcal{F}$ along $\mathcal{D}$ and $\mathcal{F}'$ along $\mathcal{D}'$, which is realized over $\mathcal{S}$-collections of transversals, such that:
  (a) for each irreducible component D of $\mathcal{D}$ we have that D is $\mathcal{F}$-invariant if and only if $\varphi(D)$ is $\mathcal{F}'$-invariant;
  (b) for each invariant local irreducible component $D \subset \mathcal{D}$ at a point $s \in D \cap \mathrm{Sing}(\mathcal{D})$ we have $\mathrm{CS}(\mathcal{F}, D, s) = \mathrm{CS}(\mathcal{F}', \varphi(D), \varphi(s))$;
  (c) φ is excellent.

Moreover, if $\underline{\mathcal{F}}$ satisfies Condition (R) (more generally if $\underline{\mathcal{F}}$ is $\mathcal{S}$-transversely rigid) then the precedent properties (1)-(4) are also equivalent to:

(1') $(\underline{\mathcal{F}}, \underline{\mathcal{D}})$ and $(\underline{\mathcal{F}}', \underline{\mathcal{D}}')$ are topologically conjugated;
(2') $(\mathcal{F}, \mathcal{D})$ and $(\mathcal{F}', \mathcal{D}')$ are topologically conjugated by an excellent homeomorphism.

Remark 1.11. The proof of Theorem B shows in fact that the conjugations in (1) and (2) (or (1') and (2')) are homotopic in the complementary of the corresponding divisors.

Corollary B. *Let $\mathcal{F}$ be a germ of singular holomorphic foliation in $(\mathbb{C}^2, 0)$ which is a generalized curve such that all its singularities after reduction whose Camacho-Sad index is not rational are linearizable. Assume that $\mathcal{F}$ satisfies Condition (R) below. Let $\mathcal{F}'$ be another germ of singular holomorphic foliation in $(\mathbb{C}^2, 0)$.*

Then for every topological conjugation germ $\varphi : (\mathbb{B}, \mathcal{F}) \to (\mathbb{B}', \mathcal{F}')$ there exists a new topological conjugation germ $\widehat{\varphi} : (\pi^{-1}(\mathbb{B}), \pi^*\mathcal{F}) \to (\pi'^{-1}(\mathbb{B}'), \pi'^*\mathcal{F}')$ defined after the reduction processes π and π' of singularities of $\mathcal{F}$ and $\mathcal{F}'$ such that

  (1) $\widehat{\varphi}$ is holomorphic at a neighborhood of $\mathrm{Sing}(\pi^*\mathcal{F})$,
  (2) there exist germs of invariant curves $\mathcal{Z} \subset \mathbb{B}$ and $\mathcal{Z}' \subset \mathbb{B}'$ satisfying conclusions of Corollary A such that $\varphi(\mathcal{Z}) = \mathcal{Z}'$, $\widehat{\varphi}(\pi^{-1}(\mathcal{Z})) = \pi'^{-1}(\mathcal{Z}')$ and such that the restrictions $\varphi : \mathbb{B} \setminus \mathcal{Z} \to \mathbb{B}' \setminus \mathcal{Z}'$ and $\widehat{\varphi} : \pi^{-1}(\mathbb{B} \setminus \mathcal{Z}) \to \pi'^{-1}(\mathbb{B}' \setminus \mathcal{Z}')$ are homotopic.

In particular, the analytic type of the singularities of $\pi^*\mathcal{F}$ and its projective holonomy representations are topological invariants of the germ of $\mathcal{F}$ at 0.

Theorem B with $\underline{\mathcal{D}}$ reduced to a point and Corollary B generalize Theorems I and II of [12] to the case of dicritical foliations. Moreover, the topological conjugations considered in [12] are assumed to send nodal separatrices into nodal separatrices preserving its corresponding Camacho-Sad indices. In this paper we have used the following result of R. Rosas [16, Proposition 11] which allows us to eliminate this constraint and to extend our results to general topological conjugations.

Theorem 1.12. *Every topological conjugation Φ between two germs $\mathcal{F}$ and $\mathcal{F}'$ of holomorphic foliations in $(\mathbb{C}^2, 0)$ maps nodal separatrices into nodal separatrices preserving its corresponding Camacho-Sad indices.*

The idea of the proof is the following.

(a) Let Z be a nodal separatrix of $\mathcal{F}$. Any tubular neighborhood of $Z \setminus \{0\}$ retracts into a 2-torus T whose first homology group is endowed with a natural $\mathbb{Z}$-basis given by monomial coordinates after reduction of singularities of the foliation, cf. [12, Definition 6.1.2].
(b) Up to a foliated isotopy we can assume that Φ preserves the 2-tori T and T' corresponding to Z and $Z' := \Phi(Z)$. It is possible to prove that $\Phi_* : H_1(T, \mathbb{Z}) \to H_1(T', \mathbb{Z})$ conjugates its corresponding canonical basis, see [16, Theorem 10] and [12, Theorem 6.2.1].

(c) We can canonically identify T and T' with the standard 2-torus and $\mathcal{F}_{|T}$ and $\mathcal{F}'_{|T'}$ with 1-dimensional linear irrational foliations. It remains to see that the slopes of two linear foliations on the torus are equal once we assume that they are topologically conjugated by a homeomorphism homotopic to the identity.

2. Localisation

2.1. **Plumbing.** The following result is well known in the literature, cf. for instance [13, 7, 17, 19]:

Lemma 2.1. *There exist an open tubular neighborhood W of $\mathcal{C}$ in M and a decomposition $W = \bigcup_{i\in\mathfrak{I}} W_i$ satisfying the following conditions:*

(i) each W_i is a tubular neighborhood of an irreducible component $\mathcal{C}_i$ of $\mathcal{C}$;

(ii) each W_i admits a smooth disk fibration $\pi_i : W_i \to \mathcal{C}_i$ over $\mathcal{C}_i$ whose Euler number $-\nu_i$ is the self-intersection of $\mathcal{C}_i$; moreover each nonempty intersection $\mathcal{C}_j \cap W_i$, $i \neq j$, is a fiber of π_i;

(iii) there exists a differentiable function $h : W \to \mathbb{R}^+$ which is a submersion on $W \setminus \mathcal{C}$, such that $h^{-1}(0) = \mathcal{C}$ and $\{h^{-1}([0,\varepsilon))\}_{\varepsilon>0}$ is a fundamental system of neighborhoods of $\mathcal{C}$, which do not meet the boundary of $\overline{W}$ in M;

(iv) there exists a simplicial map $\pi : W \to \mathcal{C}$ having connected fibres whose restriction to $W_i \setminus \bigcup\limits_{j\neq i} W_j$ coincides with π_i, $i \in \mathfrak{I}$.

Furthermore, we can endow W with a riemannian metric so that the flow of the gradient vector field of h preserves the level hypersurfaces $h = \varepsilon$. In particular, all the neighborhoods $h^{-1}([0,\varepsilon))$ are homeomorphic. Moreover, we can topologically recover W by making the plumbing procedure described in [13, 7] of the fibrations $\pi_i : W_i \to \mathcal{C}_i$ obtained from the data given by the dual graph with weights $\mathcal{G}$.

Remark 2.2. We point out some considerations.

(a) If additionally the intersection matrix $(\mathcal{C}_i \cdot \mathcal{C}_j)_{i,j}$ is definite negative then, after Grauert's theorem, there exists a complex structure on the plumbing W such that the quotient $W/\mathcal{C}$ becomes a complex surface with an isolated singularity.

(b) The existence of the simplicial map $\pi : W \to \mathcal{C}$ having connected fibres implies the existence of a epimorphism

$$\pi_1(\partial\overline{W}) \to \pi_1(\mathcal{C}) \cong \pi_1(\mathcal{G}) * \pi_1(\mathcal{C}_1) * \cdots * \pi_1(\mathcal{C}_n),$$

where $\mathcal{G}$ is the dual graph associated to $(W, \mathcal{C})$ and $\mathcal{C}_1, \ldots, \mathcal{C}_n$ are the irreducible components of $\mathcal{C}$, cf. [17].

(c) We can assume that the fibrations $\pi_i : W_i \to \mathcal{C}_i$ are holomorphic in a neighborhood of $\mathrm{Sing}(\mathcal{C}) \cap \mathcal{C}_i$. Moreover, if $\mathcal{C}_i$ is a dicritical component of $(\mathcal{F}, \mathcal{C})$ then we can assume that the fibers of π_i are the leaves of the restriction $\mathcal{F}_{|W_i}$.

2.2. **Boundary assembly.** Let V be a smooth manifold endowed with a regular foliation $\mathcal{F}$ of class C^1 and let A be an arbitrary subset of V. By definition, a leaf of $\mathcal{F}_{|A}$ is a connected component of $L \cap A$, where L is a leaf of $\mathcal{F}$. For every $A \subset V$ we define the **boundary** of A as $\partial A := A \setminus \overset{\circ}{A}$, where $\overset{\circ}{A}$ is the interior of A. The definitions and results of this section are borrowed from [10].

Definition 2.3. *If $A \subset B \subset V$ we will say that A is **1-$\mathcal{F}$-connected** in B (denoted by $A \underset{\mathcal{F}}{\hookrightarrow} B$) if for every leaf L of $\mathcal{F}_{|B}$ and for all paths $a : [0,1] \to A$ and $b : [0,1] \to L$ with the same endpoints*

m_0, m_1, which are homotopic (with fixed endpoints) in B, there exists a path $c : [0,1] \to A \cap L$ with endpoints m_0, m_1, which is homotopic to a inside A and to b inside L.

Definition 2.4. *Let $(V_i)_{i\in I}$ a finite or numerable collection of submanifolds (with boundary) of V of the same dimension that V. We will say that V_i is a **$\mathcal{F}$-adapted block** if it satisfy the following properties:*

(B1) ∂V_i is incompressible in V_i,
(B2) ∂V_i is a transversely orientable submanifold of V transverse to $\mathcal{F}$,
(B3) ∂V_i is 1-$\mathcal{F}$-connected in V_i,
(B4) every leaf of $\mathcal{F}_{|V_i}$ is incompressible in V_i.

*We will say that V is a **boundary assembly** of the blocks V_j if for all $i \in I$ Condition (B1) and the following property hold:*

(B5) for all different $i, j \in I$ either $V_i \cap V_j = \emptyset$ or $V_i \cap V_j$ is a connected component or ∂V_i and a connected component of ∂V_j.

*We will say that V is a **foliated boundary assembly** if each block V_i is $\mathcal{F}$-adapted and if V is a boundary assembly of V_j.*

Theorem 2.5 (Localisation)**.** *If V is a foliated boundary assembly of V_i then every leaf of $\mathcal{F}$ is incompressible in V and for every $I' \subset I$, the union $V' = \bigcup_{i\in I'} V_i$ is incompressible and 1-$\mathcal{F}$-connected in V.*

Remark 2.6. If $V = \bigcup_{i\in I} V_i$ and each block V_i satisfy Condition (B5) in previous Definition 2.4, then we define its dual graph $\mathcal{G}_V$ by putting one vertex for each element of I and one edge between vertex i and j for each common boundary component of V_i and V_j. We can give an explicit presentation of the fundamental group of V uniquely from $\pi_1(\mathcal{G}_V)$ and the morphisms $\pi_1(V_i \cap V_j) \to \pi_1(V_i)$ thanks to the following generalization of the classical Seifert-Van Kampen theorem ($r = 0$).

Proposition 2.7. *Let A be a connected simplicial complex with connected sub-complex A_0 and A_1 such that $A = A_0 \cup A_1$ and $A_0 \cap A_1 = B_0 \sqcup \cdots \sqcup B_r$, where each B_i is a connected sub-complex of A_j for each $i = 0, \ldots, r$ and $j = 0, 1$. Let $\varphi_{ij} : \pi_1(B_i) \to \pi_1(A_j)$ be the morphisms induces by the natural inclusions $B_i \subset A_j$. Then $\pi_1(A)$ is isomorphic to the quotient*

$$(\pi_1(A_0) * \pi_1(A_1) * \mathbb{Z}(u_0) * \cdots * \mathbb{Z}(u_r))/K\,,$$

where K is the normal subgroup generated by the relations $u_0 = 1$ and

$$\varphi_{i,0}(b_i) = u_i^{-1}\varphi_{i,1}(b_i)u_i, \quad \forall b_i \in \pi_1(B_i), \quad i = 0, \ldots, r.$$

Proof. See the proof of Proposition 2.1. of [17] for the case $r = 1$. The case $r > 1$ is completely analogous. □

2.3. **Decomposition of $\mathcal{D}$ and boundary assembly of Milnor tubes.** We consider the function $h : W \to \mathbb{R}^+$ given by Lemma 2.1 with $h^{-1}(0) = \mathcal{C}$. If $f : W \to \mathbb{C}$ is a reduced holomorphic equation of $\overline{\mathcal{D} \setminus \mathcal{C}}$ then we consider the product $H := h \cdot |f|$ and we define the **open** 4**-Milnor tube** of **height** $\eta > 0$ associated to $\mathcal{D}$ as $\mathcal{T}_\eta := H^{-1}([0, \eta))$. We also denote

$$\mathcal{T}_\eta^* := \mathcal{T}_\eta \setminus \mathcal{D} = H^{-1}((0, \eta))$$

and we remark that if $\eta > 0$ is small enough then the **closed** 3**-Milnor tube** $\mathcal{M}_\eta$, defined as the adherence of $H^{-1}(\eta)$ in $\overline{W}$, is transverse to $\partial\overline{W}$. The set of open 4-Milnor tubes associated to $\mathcal{D}$ is a fundamental system of neighborhoods of $\mathcal{D} \subset W$. In [19] it is shown that there exists a vector field ξ such that $\xi(H) > 0$, by gluing suitable local models with a partition of the unity.

The flow of ξ allows to define homeomorphisms between the open 4-Milnor tubes of different height, provided they are small enough.

For each irreducible component D of $\mathcal{D}$ we also consider the disk fibrations $\pi_D : W_D \to D$ given by Lemma 2.1 if $D \subset \mathcal{C}$ and trivial ones if $D \subset \overline{\mathcal{D} \setminus \mathcal{C}}$. After Point (c) of Remark 2.2 we can choose the tubular neighborhoods W_D and the fibrations π_D in such a way that for each singular point $s \in \mathrm{Sing}(\mathcal{D})$ the following properties hold:

(a) If $\{s\} = D \cap D'$ then $W_s := W_D \cap W_{D'}$ admits holomorphic local coordinates

$$(x_s, y_s) : W_s \xrightarrow{\sim} \mathbb{D}_2 \times \mathbb{D}_2$$

such that the germ of $\mathcal{F}$ at s is given by a 1-form of the following type:
- $x_s dy_x - \lambda_s y_s dx_s$ with $\lambda_s \in \mathbb{C}$, if s is a linearizable singularity;
- $x_s dy_s - (\lambda_s y_s + x_s y_s(\cdots))dx_s$ with $\lambda_s \in \mathbb{Q}_{<0}$, if s is a resonant singularity;
- dx_s (resp. dy_s) if D (resp. D') is a dicritical component of $(\mathcal{F}, \mathcal{D})$.

(b) $D \cap W_s = \{y_s = 0\}$, $D' \cap W_s = \{x_s = 0\}$ and the restrictions of π_D and $\pi_{D'}$ to $W_s \cap \{|x_s| < \frac{3}{2}\}$ and $W_s \cap \{|y_s| < \frac{3}{2}\}$ coincide with $(x_s, y_s) \mapsto x_s$ and $(x_s, y_s) \mapsto y_s$ respectively.

For each irreducible component D of $\mathcal{D}$ we denote $\Sigma_D := \mathrm{Sing}(\mathcal{D}) \cap D$ and

$$D_s := D \cap \{|x_s| \le 1,\ |y_s| \le 1\} \quad \text{for} \quad s \in \Sigma_D\,. \tag{1}$$

For each irreducible $\mathcal{F}$-invariant component D of $\mathcal{D}$ of genus $g(D) > 0$ we fix a smooth real analytic curve Γ_D which is the boundary of a closed conformal disk D_{Γ_D} containing Σ_D such that the holonomy of Γ_D is linearizable, provided that D is not an initial component, see the introduction. Notice that $\Sigma_D \neq \emptyset$ because of Condition (e) in Definition 1.2. If D contains a unique singular point s of $\mathcal{D}$ then we shall take $\Gamma_D = \partial D_s$ and $D_{\Gamma_D} = D_s$. Otherwise we can assume that every two closed disks D_s and $D_{s'}$, $s, s' \in \Sigma_D$, are disjoint and contained in the open disk $D_{\Gamma_D} \setminus \partial D_{\Gamma_D}$ when $g(D) > 0$. We also denote

$$D^* := \overline{D \setminus \bigcup_{s \in \Sigma_D} D_s}, \quad \text{if} \quad g(D) = 0 \tag{2}$$

and

$$D^* := \overline{D_{\Gamma_D} \setminus \bigcup_{s \in \Sigma_D} D_s}, \quad \text{if} \quad g(D) > 0. \tag{3}$$

Consider the union $\mathfrak{J} \subset \mathcal{D}$ of all the Jordan curves of the form Γ_D with $g(D) > 0$ and all the curves ∂D_s with $s \in D \cap \mathrm{Sing}(\mathcal{D})$. Let $\mathfrak{A}$ be the set of **elementary blocks of** $\mathcal{D}$ defined as the adherence of the connected components of $\mathcal{D} \setminus \mathfrak{J}$. There exists an **uniformity height** $\eta_1 > 0$ such that for all $\eta \in (0, \eta_1]$ the set $\{\mathcal{T}_\eta(A)\}_A$ composed by the adherence of the connected components of

$$\mathcal{T}_\eta \setminus \bigcup_{D \subset \mathcal{D}} \pi_D^{-1}(\mathfrak{J} \cap D)$$

is in one to one correspondence with $\mathfrak{A}$. More precisely, for each $A \in \mathfrak{A}$ there is a unique connected component of $\mathcal{T}_\eta \setminus \bigcup_{D \subset \mathcal{D}} \pi_D^{-1}(\mathfrak{J} \cap D)$ containing $A \subset \mathcal{D}$ and whose adherence we denote by $\mathcal{T}_\eta(A)$. Notice that for each elementary block $A \subset \mathcal{D}$ we can construct a vector field ξ_A whose flow induces deformation retracts between $(\mathcal{T}_\eta^*(A), \partial\mathcal{T}_\eta^*(A))$ and $(\mathcal{T}_{\eta_1}^*(A), \partial\mathcal{T}_{\eta_1}^*(A))$ for all $\eta \in (0, \eta_1]$, see Theorem 5.1.5 and Proposition 9.3.2 of [19]. If $B = \cup_i A_i \subset \mathcal{D}$ is an arbitrary union of elementary blocks of $\mathcal{D}$ we also adopt the following convenient notation

$$\mathcal{T}_\eta(B) := \bigcup_i \mathcal{T}_\eta(A_i) \quad \text{and} \quad \mathcal{T}_\eta^*(B) := \mathcal{T}_\eta(B) \setminus \mathcal{D}\,. \tag{4}$$

Definition 2.8. *We will say that an inclusion $\imath : A \subset B$ between two subspaces of a topological space is* ***rigid*** *if $\imath_* : \pi_1(A,p) \xrightarrow{\sim} \pi_1(B,p)$ is an isomorphism for all $p \in A$. We will say that $\imath$ is ∂-**rigid** if $\partial A \subset \partial B$ and the two inclusions $A \subset B$ and $\partial A \subset \partial B$ are rigid. Recall that $\partial A = A \setminus \mathring{A}$.*

Proposition 2.9. *Consider a subset $\mathcal{B} \subset \mathcal{T}^*_{\eta_1}$. If for each elementary block A of $\mathcal{D}$ the inclusion $\mathcal{B} \cap \mathcal{T}^*_{\eta_1}(A) \subset \mathcal{T}^*_{\eta_1}(A)$ is ∂-rigid, then the inclusion $\mathcal{B} \subset \mathcal{T}^*_{\eta_1}$ is also rigid. In particular the inclusion $\mathcal{T}^*_\eta \subset \mathcal{T}^*_{\eta_1}$ is rigid for all $\eta \in (0, \eta_1]$.*

Proof. This assertion follow immediately from Remark 2.6 and the following (trivial) result. □

Lemma 2.10. *Let $A \subset B \subset C$ be topological spaces. If two of the three inclusions $A \subset B$, $B \subset C$ and $A \subset C$ are (∂-)rigid then the third one is also (∂-)rigid.*

Notice that the collection $\{\mathcal{T}^*_\eta(A)\}_{A \in \mathfrak{A}}$ does not define a boundary assembly of $\mathcal{T}^*_\eta$ because Condition (B1) in Definition 2.4 is not always verified. More precisely, if C is an irreducible component of $\mathcal{C}$ having genus 0 and valence 1 then the boundary of $\mathcal{T}^*_\eta(C)$ is not incompressible. This situation leads us to consider bigger blocks of $\mathcal{D}$ as we have already done in [10].

Definition 2.11. *The* ***fundamental blocks*** *of $\mathcal{D}$ are the unions of elementary blocks of $\mathcal{D}$ described below:*

(a) For each $\mathcal{F}$-invariant irreducible component D of $\mathcal{D}$ not contained in a dead branch, we consider the ***aggregate block*** *defined as*

$$\mathfrak{m}_D \cup D^* \cup \left(\bigcup_{s \in \mathfrak{m}_D \cap D} D_s \right),$$

where $\mathfrak{m}_D$ is the union of all the dead branches meeting D, D^* is defined by Equations (2) or (3), and D_s is given by (1).

(b) For each singularity $s \in \mathrm{Sing}(\mathcal{D})$ belonging to different irreducible components D and D' of $\mathcal{D}$, we consider the ***singularity block*** *$D_s \cup D'_s$ provided that s do not belong to any dead branch.*

(c) For each $\mathcal{F}$-invariant irreducible component $D \subset \mathcal{C}$ of genus $g(D) > 0$, we consider the ***genus block*** *$\overline{D \setminus D_{\Gamma_D}}$.*

(d) For each dicritical irreducible component D of $\mathcal{F}$, we consider the ***dicritical block***

$$D \cup \bigcup_{(s,D') \in \mathfrak{K}_D} D'_s,$$

where $\mathfrak{K}_D$ is the set of pairs (s, D') constituted by a singular point s of $\mathcal{D}$ lying on D and the irreducible component $D' \neq D$ of $\mathcal{D}$ meeting D at s.

An ***initial block*** *of $\mathcal{D}$ is either an aggregate block containing a single singular point of $\mathcal{D}$ which do not belong to any dead branch, or a genus block associated to an initial component D of $\mathcal{D}$ of genus $g(D) > 0$ such that the holonomy of Γ_D is not linearizable. A* ***breaking block*** *of $\mathcal{D}$ is either a singularity block associated to a linearizable singular point or a dicritical block.*

Proposition 2.12. *For every $\eta \in (0, \eta_1]$, $\mathcal{T}^*_\eta$ is a boundary assembly of the blocks $\{\mathcal{T}^*_\eta(B)\}_{B \in \mathfrak{B}}$ defined by (4), where $\mathfrak{B}$ is the set of fundamental blocks of $\mathcal{D}$.*

The proof of this proposition will be based on explicit descriptions of the fundamental groups of $\mathcal{T}^*_\eta(B)$, by generators and relations. But before we must give some preliminary information about the topology of tubular neighborhoods of dead branches. Let $\mathfrak{m} = \bigcup_{i=1}^{\ell} D_i$ be a $\mathcal{F}$-invariant dead branch with $v(D_1)=1$, $v(D_i)=2$ for $i=2,\ldots,\ell$. For each $j=1,\ldots,\ell$ the intersection matrix of $\bigcup_{i=1}^{j} D_i$ is

$$A_j=\begin{pmatrix} e_1 & 1 & 0 & \cdots & 0 \\ 1 & e_2 & 1 & \ddots & \vdots \\ 0 & 1 & e_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 1 & e_j \end{pmatrix}$$

whose determinant is denoted $\delta_j=\det(A_j)$. Assume that the attaching component C of $\mathfrak{m}$ is also $\mathcal{F}$-invariant according to Condition (c) in Definition 1.2 and that $\mathfrak{m}$ can not be blow-down according to the initial assumptions stated in the introduction. Let $d_i\in\pi_1(\mathcal{T}^*_\eta(\mathfrak{m}))$ be a meridian of D_i and let $c\in\pi_1(\mathcal{T}^*_\eta(\mathfrak{m}))$ be a meridian of C.

Lemma 2.13. *There exist coprime positive integers $p\geq 2$ and $q\geq 1$ such that $d_\ell^p=c^q$.*

Proof of Lemma 2.13. By assumption $C\cup\mathfrak{m}$ is $\mathcal{F}$-invariant and the singularities of $\mathcal{F}$ are reduced and they are not saddle-nodes. Then classically the Camacho-Sad indices

$$\lambda_i=\mathrm{CS}(\mathcal{F},D_i,s_i)\,,\qquad i=1,\ldots,\ell$$

are rational strictly negative numbers. By Camacho-Sad formula follows that $\lambda_i=e_i-\frac{1}{\lambda_{i-1}}$. On the other hand, by developing the determinant of A_{i+1} by the last row, we have the equality $\delta_{i+1}=e_{i+1}\delta_i-\delta_{i-1}$. We claim that $\lambda_i=\frac{\delta_i}{\delta_{i-1}}$, for $i=2,\ldots,\ell$. Indeed, this is trivially the case for $i=2$ and the inductive step $i\Rightarrow i+1$:

$$\lambda_{i+1}=e_{i+1}-\frac{1}{\lambda_i}=e_{i+1}-\frac{\delta_{i-1}}{\delta_i}=\frac{e_{i+1}\delta_i-\delta_{i-1}}{\delta_i}=\frac{\delta_{i+1}}{\delta_i}$$

completes the proof of the claim. Since $\lambda_i<0$ for all $i=1,\ldots,\ell$ and $\delta_1=e_1<0$, it follows that $(-1)^i\delta_i>0$ and, by Silvester's criterion, the matrix A_ℓ is definite negative. We take $p=(-1)^\ell\delta_\ell\geq 1$ and $q=(-1)^{\ell-1}\delta_{\ell-1}\geq 1$. By Grauert's criterion, $\mathfrak{m}$ can be blow-down if and only if A_ℓ is definite negative and $\delta_\ell=\pm 1$. Hence $p\geq 2$ by the assumption on $\mathfrak{m}$. Moreover we have

$$\gcd(p,q)=\gcd(\delta_\ell,\delta_{\ell-1})=\gcd(\delta_{\ell-1},\delta_{\ell-2})=\cdots=\gcd(\delta_2,\delta_1)=\gcd(e_1,-1).$$

Hence $\gcd(p,q)=1$. It only remains to prove that $d_\ell^p=c^q$. This equality follows directly from the fact that $(A_\ell^{-1})_{\ell\ell}=\frac{\delta_{\ell-1}}{\delta_\ell}$ and from the relation $A_\ell v+w=0$ in $H_1(\mathcal{T}^*_\eta(\mathfrak{m}),\mathbb{Z})$, where $v=([d_1],\cdots,[d_\ell])^t$ and $w=(0,\ldots,0,[c])^t$. This relation being a matrix reformulation of the Camacho-sad index formulae along the components of $\mathfrak{m}$. □

The following result is well-known in combinatorial group theory.

Lemma 2.14. *If $p_i\geq 2$ and $q_i\geq 1$ are coprime integers then every element γ of the group Γ presented by*

$$\langle c,d_1,\ldots,d_m\,|\,[c,d_i]=1,\,d_i^{p_i}=c^{q_i},i=1,\ldots,m\rangle$$

can be written in a unique way as $\gamma=u_1\cdots u_r c^s$ with $u_i=d_{j_i}^{\varepsilon_i}$, $0\leq\varepsilon_i<p_i$ and $s\in\mathbb{Z}$.

Proof of Proposition (2.12). By construction the family $\{\mathcal{T}^*_\eta(B)\}_{B\in\mathfrak{B}}$ satisfy property (B5) in Definition 2.4. In order to check Condition (B1) for each block $\mathcal{T}^*_\eta(B)$, we will distinguish four cases according to the type of $B\in\mathfrak{B}$:

(a) If B is the aggregated block associated to an $\mathcal{F}$-invariant irreducible component D of $\mathcal{D}$ then, after [17], we obtain a presentation of $\pi_1(\mathcal{T}_\eta^*(B))$ by considering the generators $a_1, \ldots, a_g, b_1, \ldots, b_g, c, d_1, \ldots, d_v$ and the relations

$$[c, *] = 1, \quad c^\nu \cdot \prod_{i=1}^{g}[a_i, b_i] \cdot \prod_{j=1}^{v} d_j = 1, \quad d_k^{p_k} = c^{q_k},\ k = 1, \ldots, m \leq v, \tag{5}$$

where g, v and ν are respectively the genus, the valence and the self-intersection of D, m is the number of dead branches contained in B and p_k, q_k are the positive coprime integers given by Lemma 2.13. Each connected component of the boundary of $\mathcal{T}_\eta^*(B)$ is a torus whose fundamental group is $\langle c, d_j | [c, d_j] = 1\rangle$, $j = m+1, \ldots, v$. The incompressibility of the boundary is equivalent to the following implication

$$(j > m \text{ and } d_j^\alpha c^\beta = 1 \text{ in } \pi_1(\mathcal{T}_\eta^*(B))) \Longrightarrow \alpha = \beta = 0, \tag{6}$$

which is trivially true if $m = v$. Hence, in the sequel we will assume that $m \leq v - 1$. Notice that $\pi_1(\mathcal{T}_\eta^*(B)) = \Gamma *_C G$ where Γ is the group considered in Lemma 2.14, G is defined by

$$G := \langle a_1, \ldots, a_g, b_1, \ldots, b_g, c, d_{m+1}, \ldots, d_{v-1} |\, [c, *] = 1\rangle \cong \mathbb{Z}^{*2g+v-m-1} \oplus \mathbb{Z}$$

and $C = \langle c | -\rangle \cong \mathbb{Z}$. Trivially C injects in G. On the other hand, because $p_i \geq 2$ for $i = 1, \ldots, m$, C also injects into Γ. Seifert-Van Kampen Theorem implies that G can also considered as a subgroup of $\pi_1(\mathcal{T}_\eta^*(B))$. Thus, for $j \leq v-1$ implication (6) can be considered in the subgroup G, where it is trivially true. It only remains to treat the case of $j = v$. But $d_v^\alpha c^\beta$ is equal to $\left(\prod_{i=1}^{g}[a_i, b_i]\prod_{j=1}^{v-1} d_j\right)^{-\alpha} c^{\beta-\nu\alpha}$ and this expression can not be simplified using the relations (5), if $g > 0$ or $v - m \geq 2$ provided $(\alpha, \beta) \neq (0, 0)$. It remains to consider the situation $g = 0$ and $v - m = 1$. In this case $G = C$ and the element of Γ given by

$$d_v^\alpha c^\beta = \underbrace{(d_1 \cdots d_m) \cdots (d_1 \cdots d_m)}_{-\alpha} c^{\beta-\nu\alpha}$$

is written in the unique reduced form stated in Lemma 2.14. Consequently it is trivial if and only if $\alpha = \beta = 0$.

(b) If B is a singularity block then $\mathcal{T}_\eta^*(B) \cong T \times [0, 1]$ and $\partial\mathcal{T}_\eta^*(B) \cong T \times \{0, 1\}$, so that each connected component of its boundary is incompressible.

(c) If B is a genus block ($g > 0$) then

$$\pi_1(\mathcal{T}_\eta^*(B)) \cong \langle a_1, \ldots, a_g, b_1, \ldots, b_g, c \,|\, [c, *] = 1\rangle$$

contains $\pi_1(\partial\mathcal{T}_\eta^*(B)) = \langle \prod_{i=1}^{g}[a_i, b_i], c \,|\, [c, *] = 1\rangle$.

(d) If B is the dicritical block associated to some dicritical component D of $\mathcal{D}$ of genus $g \geq 0$ and valence $v \geq 1$ then D is not adjacent to any dead branch of $\mathcal{D}$ by Condition (c) in Definition 1.2 and consequently $\pi_1(\mathcal{T}_\eta^*(B))$ is the group G considered in case (a) taking $m = 0$. Each connected component of $\partial\mathcal{T}_\eta^*(B)$ is a torus whose fundamental group $\langle d_j, c \,|\, [c, d_j] = 1\rangle$ injects into G.

□

2.4. **Existence of adapted blocks.** In order to control the topology of the foliated blocks that we will construct in Section 3 we must consider the notions of size and roughness of a suspension type subset introduced in [10]. First of all we recall the notion of suspension type subset. Let P be a regular point of $\mathcal{F}$ lying on an irreducible component D of $\mathcal{D}$, let Δ be a subset contained in the fibre $\pi_D^{-1}(P)$ and let μ be a path contained in D with origin P.

Definition 2.15. *The **suspension** of* Δ *over* μ *along the fibration* π_D *is the union*

$$\mathbb{V}_{\Delta,\mu} := \bigcup_{m\in\Delta} |\mu_m|,$$

where μ_m *denotes the path of origin* m *lying on the leaf of* $\mathcal{F}$ *passing through* m *which lifts the path* μ *via* π_D*, i.e.* $\pi_D \circ \mu_m = \mu$ *and* $\mu_m(0) = m$*.*

This notion is well defined provided Δ is small enough. In [10] we have also introduced the notion of **roughness** $\mathfrak{e}(\xi)$ of an oriented curve $\xi \subset \mathbb{C}^*$. Here we will say that $\Omega \subset \mathbb{C}$ is of infinite roughness if $\Theta = \overline{\Omega} \setminus \overset{\circ}{\Omega}$ is not a piecewise smooth curve. Otherwise we will define the **roughness** of Ω as $\mathfrak{e}(\Omega) = \inf\{(\Theta^+), (\Theta^-)\}$, where Θ^+ and Θ^- are two curves of opposite orientations parameterizing Θ. The finiteness of the roughness is equivalent to the starlike property with respect to the origin.

Since every open Riemann surface is Stein, each fibration $\pi_D : W_D \to D$ is analytically trivial over every open set $D' \subsetneq D$. Fix on $W' := \pi_D^{-1}(D')$ a trivializing coordinate $z_{D'} : W' \to \mathbb{C}$, i.e. $(z_{D'}, \pi_D)$ is a biholomorphism from W' onto the product of the unit disk of $\mathbb{C}$ times D'; we define the **roughness** of a subset E of W' with respect to $z_{D'}$ as

$$\mathfrak{e}_{z_{D'}}(E) := \sup\{\mathfrak{e}(z_{D'}(E \cap \pi_D^{-1}(m))),\, m \in D\} \in \mathbb{R}^+ \cup \{\infty\}.$$

We also define the **size** of E with respect to $z_{D'}$ as

$$\|E\|_{z_{D'}} := \max\{|z_{D'}(m)|,\, m \in E\},$$

and we denote $\mathfrak{c}(\cdot) = \max\{\mathfrak{e}_{z_{D'}}(\cdot), \|\cdot\|_{z_{D'}}\}$ called control function.

Now we present an existence theorem of $\mathcal{F}$-adapted blocks having controlled size and roughness, which we will prove in next section. In Section 4 we shall prove Theorem A by gluing inductively these $\mathcal{F}$-adapted blocks and using Localization Theorem 2.5. We keep the notation $\mathcal{T}_\eta^*(A)$ for the blocks of the boundary assembly given in Proposition 2.12.

Theorem 2.16. *Fix* $\varepsilon > 0$ *and* $\eta \in (0, \eta_1]$*.*

(I) Let A *be an initial fundamental block of* $\mathcal{D}$*. Then there exists a holomorphic regular curve* $\Upsilon_A \subset \mathcal{T}_{\eta_1}^*$ *transverse to* $\mathcal{F}$ *and there exists a subset* $\mathcal{B}_\eta(A)$ *of* $\mathcal{T}_\eta^*(A)$ *satisfying the following conditions:*

(1) for $\eta' > 0$ *small enough the inclusion* $\mathcal{T}_{\eta'}^*(A) \subset \mathcal{B}_\eta(A)$ *is* ∂*-rigid;*

(2) $\mathcal{B}_\eta(A)$ *is a* $\mathcal{F}$*-adapted block;*

(3) the connected components $\mathcal{V}_1, \ldots, \mathcal{V}_{\mathfrak{n}_A}$ *of* $\partial\mathcal{B}_\eta(A)$ *are of suspension type over the connected components of* ∂A*;*

(4) $\mathfrak{c}(\mathcal{V}_j) \leq \varepsilon$ *for each* $j = 1, \ldots, \mathfrak{n}_A$*;*

(5) the intersection $\Upsilon_{A,\eta} := \Upsilon_A \cap \mathcal{B}_\eta(A)$ *is incompressible in* $\mathcal{B}_\eta(A)$ *and it satisfies* $\mathrm{Sat}_{\mathcal{F}}(\Upsilon_{A,\eta}, \mathcal{B}_\eta(A)) = \mathcal{B}_\eta(A)$ *and* $\Upsilon_{A,\eta} \underset{\mathcal{F}}{\hookrightarrow} \mathcal{B}_\eta(A)$*.*

(II) Let A *be a fundamental block of* $\mathcal{D}$ *which is not an initial or breaking block. Then there exists a holomorphic regular curve* $\Upsilon_A \subset \mathcal{T}_{\eta_1}^*$ *transverse to* $\mathcal{F}$*, there exist a constant* $C_A > 0$ *and a function* $\rho_A : \mathbb{R}^+ \to \mathbb{R}^+$ *with* $\lim_{c\to 0} \rho_A(c) = 0$*, such that for every suspension type subset* $\mathcal{V} \subset \mathcal{T}_\eta^*(A)$ *over a connected component of* ∂A *satisfying* $\mathfrak{c}(\mathcal{V}) \leq C_A$*, there exists a subset* $\mathcal{B}_\eta(A)$ *of* $\mathcal{T}_\eta^*(A)$ *satisfying Properties (1), (2), (3) and (5) of Part (I) as well as*

(3') $\mathcal{V}_1 \underset{\mathcal{F}}{\hookrightarrow} \mathcal{V}$*;*

(4') $\mathfrak{c}(\mathcal{V}_j) \leq \rho_A(\mathfrak{c}(\mathcal{V}))$ *for each* $j = 1, \ldots, \mathfrak{n}_A$*.*

(III) Let A be a breaking block of $\mathcal{D}$. Then for every choice of suspension type subsets

$$\mathcal{V}_1, \ldots, \mathcal{V}_{\mathfrak{n}_A} \subset \mathcal{T}^*_\eta(A)$$

over the connected components of ∂A such that the inclusion $\bigcup_{i=1}^{\mathfrak{n}_A} \mathcal{V}_i \subset \partial\mathcal{T}^*_\eta(A)$ be rigid, there exists a subset $\mathcal{B}_\eta(A)$ of $\mathcal{T}^*_\eta(A)$ satisfying Properties (1) and (2) of Part (I), such that the connected components $\mathcal{V}'_1, \ldots, \mathcal{V}'_{\mathfrak{n}_A}$ of $\partial\mathcal{B}_\eta(A)$ are of suspension type and they satisfy
(3") $\mathcal{V}'_j \underset{\mathcal{F}}{\hookrightarrow} \mathcal{V}_j$
(4") $\mathfrak{c}(\mathcal{V}'_j) \leq \varepsilon$
for each $j = 1, \ldots, \mathfrak{n}_A$.

We will prove this theorem in the following section.

3. Construction of foliated adapted blocks

Theorem 2.16 is proved in [10, Theorem 3.2.1] when the fundamental block A is an aggregated block or a singularity block. Thus, it suffices to consider the cases of genus blocks and dicritical blocks which we treat separately in sections 3.1 and 3.2 respectively.

3.1. **Genus type foliated adapted block.** In the sequel we will assume that the genus of D is $g > 0$. In order to simplify the notations in this section we will denote

$$\Gamma := \Gamma_D, \qquad D_\Gamma := D_{\Gamma_D}, \qquad \text{and} \qquad D' := \overline{D \setminus D_{\Gamma_D}}.$$

3.1.1. *Preliminary constructions.* We fix a **normal form** for D given by

- a closed regular polygon $\mathcal{P} \subset \mathbb{C}$ of $4g$ sides of length 1 centered at the origin;
- arc-length parameterizations $a_1, b_1, a'_1, b'_1, \ldots, a_g, b_g, a'_g, b'_g$ of the adjacent sides of $\mathcal{P}$ positively oriented according to $\partial\mathcal{P}$ such that $a_1(0) \in \mathbb{R}^+$;
- a continuous map $\Psi : \mathcal{P} \to D$ such that the restriction of Ψ to each side $|a_j|$ or $|b_j|$ is a smooth immersion and the compositions $\alpha_j := \Psi \circ a_j$, $\beta_j := \Psi \circ b_j$, $j = 1, \ldots, g$, are simple loops having a same origin m_Λ which only meets each other in that point; moreover $\alpha_j^{-1} = \Psi \circ a'_j$ and $\beta_j^{-1} = \Psi \circ b'_j$ for $j = 1, \ldots, g$;
- Ψ has an extension to an open neighborhood of $\mathcal{P}$ into $\mathbb{C}$ which is a local homeomorphism and its restriction to $\mathcal{P} \setminus \partial\mathcal{P}$ is a homeomorphism onto $D \setminus \Lambda$, where

$$\Lambda = \bigcup_{j=1}^{g} |\alpha_j| \cup |\beta_j|$$

  is a wedge of $2g$ circles.

We fix an open disk $\mathbb{D}_\epsilon \subset \mathcal{P}$ centered at the origin of radius $\epsilon < \cos\left(\frac{\pi}{2g}\right)$. Up to modifying slightly Ψ we can assume that $\Psi(\overline{\mathbb{D}_\epsilon}) = \overline{D_\Gamma}$ so that the loop $\theta : [0,1] \to \Gamma = \partial D'$ given by $\theta(s) = \Psi(\epsilon e^{2i\pi s})$ is a simple parametrization of Γ. We consider the pull-back $\widehat{\pi} : \widehat{W}_{\mathcal{P}'} \to \mathcal{P}'$ by the restriction of Ψ to $\mathcal{P}' := \mathcal{P} \setminus \mathbb{D}_\epsilon$ of the fibration $\pi_D : W_{D'} \to D'$, where $W_{D'} := \pi_D^{-1}(D') \setminus D$. Thus, $\widehat{\pi}$ is a continuous $\mathbb{D}^*$-fibration which is globally trivial. Let $\widehat{\Psi} : \widehat{W}_{\mathcal{P}'} \to W_{D'}$ the continuous map which make commutative the cartesian diagram

$$\begin{array}{ccc} \widehat{W}_{\mathcal{P}'} & \xrightarrow{\widehat{\Psi}} & W_{D'} \\ \widehat{\pi} \downarrow & \square & \downarrow \pi_D \\ \mathcal{P}' & \xrightarrow{\Psi} & D'\,. \end{array}$$

Clearly $\widehat{\Psi}$ is a local homeomorphism whose restriction to $\widehat{\pi}^{-1}(\mathcal{P}' \setminus \partial\mathcal{P})$ is a homeomorphism onto $\pi_D^{-1}(D' \setminus \Lambda) \setminus D$. The foliation $\mathcal{F}_{|W_{D'}}$ lifts to a regular foliation $\widehat{\mathcal{F}}$ on $\widehat{W}_{\mathcal{P}'}$ transverse to the fibres of $\widehat{\pi}$.

We fix a conformal pointed disk $T \subset \pi_D^{-1}(m_\Lambda)$ whose size and roughness is bounded by a constant $C_D > 0$ small enough so that all the constructions we shall done in the sequel lead us to sets having finite size and roughness. By construction there exists $\widehat{T} \subset \widehat{\pi}^{-1}(\widehat{m})$ such that $T = \widehat{\Psi}(\widehat{T})$, where $\widehat{m}$ is the vertex of $\mathcal{P}$ lying on $\mathbb{R}^+$. The image by $\widehat{\Psi}$ of the suspension $\mathbb{V}_{\widehat{T},\widehat{\mu}}$ of $\widehat{T}$ via $\widehat{\pi}$ over the loop $\widehat{\mu} := a_1 \vee b_1 \vee a_1' \vee b_1' \vee \cdots \vee a_g \vee b_g \vee a_g' \vee b_g'$ can be considered as the suspension of T via π_D over the loop

$$\mu := \Psi \circ \widehat{\mu} = \alpha_1 \vee \beta_1 \vee \alpha_1^{-1} \vee \beta_1^{-1} \vee \cdots \vee \alpha_g \vee \beta_g \vee \alpha_g^{-1} \vee \beta_g^{-1}. \tag{7}$$

The subset $\underline{\mathcal{B}} := \widehat{\Psi}^{-1}(\widehat{\Psi}(\mathbb{V}_{\widehat{T},\widehat{\mu}}))$ of $\widehat{\pi}^{-1}(\partial\mathcal{P}')$ is not necessarily a multisuspension set in the sense of [10, Definition 4.2.1] because over each vertex of $\mathcal{P}$ this set is the union of $4g$ pointed disks, two of them are contained in the adherence of $\underline{\mathcal{B}} \setminus \widehat{\pi}^{-1}(\mathcal{S}_{\mathcal{P}})$, where $\mathcal{S}_{\mathcal{P}}$ is the vertex set of $\mathcal{P}$, but the other two could not satisfy this condition. We put

$$\widehat{\mathcal{B}}_{\partial\mathcal{P}} := \overline{\underline{\mathcal{B}} \setminus \widehat{\pi}^{-1}(\mathcal{S}_{\mathcal{P}})}, \qquad \mathcal{B}_\Lambda := \widehat{\Psi}(\widehat{\mathcal{B}}_{\partial\mathcal{P}}).$$

The following diagram is commutative but not necessarily cartesian

$$\begin{array}{ccc} \widehat{\mathcal{B}}_{\partial\mathcal{P}} & \xrightarrow{\widehat{\Psi}} & \mathcal{B}_\Lambda \\ \widehat{\pi}\downarrow & \circlearrowleft & \downarrow \pi_D \\ \partial\mathcal{P} & \xrightarrow{\Psi} & \Lambda\,. \end{array}$$

Now we will precise the geometry of $\mathcal{B}_\Lambda$. Let us denote by h_{α_j} (resp. h_{β_j}) the holonomy transformations of $\mathcal{F}$ along the loops α_j (resp. β_j), represented over the transverse section $\pi_D^{-1}(m_\Lambda)$. If $C_D > 0$ is small enough then the following pointed $4g$ disks are well defined

$$T_0 := T\,, \quad T_{4j+1} = h_{\alpha_j}(T_j)\,, \quad T_{4j+2} = h_{\beta_j}(T_{4j+1})\,,$$
$$T_{4j+3} = h_{\alpha_j}^{-1}(T_{4j+2})\,, \quad T_{4j+4} = h_{\beta_j}^{-1}(T_{4j+3})\,,$$

$j = 1, \ldots, g$. We have a decomposition

$$\mathcal{B}_\Lambda = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \cdots \mathcal{B}_{2g}\,,$$

of $\mathcal{B}_\Lambda$ in $2g$ pieces of suspension type

$$\mathcal{B}_{2j-1} := \mathbb{V}_{T_{4j-4},\,\alpha_j} \cup \mathbb{V}_{T_{4j-2},\,\alpha_j^{-1}} = \mathbb{V}_{T_{4j-4} \cup T_{4j-1},\,\alpha_j}\,,$$
$$\mathcal{B}_{2j} := \mathbb{V}_{T_{4j-3},\,\beta_j} \cup \mathbb{V}_{T_{4j-1},\,\beta_j^{-1}} = \mathbb{V}_{T_{4j-3} \cup T_{4j},\,\beta_j}\,,$$

$j = 1, \ldots, g$, with finite roughness. Moreover, $\pi_D(\mathcal{B}_i \cap \mathcal{B}_j) = \{m_\Lambda\}$. From this description follows:

($*$) *If $\lambda : [0,1] \to L$ is a simple parametrization of a leaf L of $\mathcal{F}_{|\mathcal{B}_k}$, $k = 1, \ldots, 2g$, then there exists a unique path $\widehat{\lambda} : [0,1] \to \widehat{\mathcal{B}}_{\partial\mathcal{P}}$ such that $\widehat{\Psi} \circ \widehat{\lambda} = \lambda$ and the orientations of $\widehat{\pi} \circ \widehat{\lambda}$ and $\partial\mathcal{P}$ coincide.*

Lemma 3.1. *If χ is a path lying on a leaf L of $\mathcal{B}_\Lambda$ such that $\pi \circ \chi = \mu^\nu$ then there exists a unique path $\widehat{\chi} : [0,1] \to \widehat{\mathcal{B}}_{\partial\mathcal{P}}$ lying on a leaf of $\widehat{\mathcal{F}}$ such that $\widehat{\Psi} \circ \widehat{\chi} = \chi$.*

Proof. We decompose $\chi = \chi_1 \vee \cdots \vee \chi_n$ with $|\chi_j| \subset \mathcal{B}_{k_j}$. By property ($*$) each χ_j possesses a unique lift $\widehat{\chi}_j$ with the same orientation as $\partial\mathcal{P}$. We must prove that all these lifts glue in a unique continuous path $\widehat{\chi}$. Fix $j \in \{1, \ldots, n\}$ and notice that the point $\chi_j(1)$ possesses exactly $4g$ pre-images by $\widehat{\Psi}$, one over each vertex of $\mathcal{P}$. To prove that $\widehat{\chi}_j(1) = \widehat{\chi}_{j+1}(0)$ it suffices to see that $\widehat{\pi}\circ\widehat{\chi}_j(1)=\widehat{\pi}\circ\widehat{\chi}_{j+1}(0)$. But $\widehat{\pi}\circ\widehat{\chi}_j$ is the unique lift of $\pi\circ\chi_j$ with the same orientation as $\partial\mathcal{P}$ and theses lifts glue because $\pi\circ\chi$ lifts, by hypothesis. $\square$

Remark 3.2. Since $C_D>0$ is small enough so that the roughness of $\mathcal{B}_j$ is finite, we have that for every $\eta'>0$ small enough $\mathcal{B}_\Lambda$ retracts onto $\mathcal{T}^*_{\eta'}(\Lambda)$, which has the homotopy type of a product of a circle by the wedge of circles Λ. More precisely, for all $m_0\in\bigcup_{k=1}^{4g}T_k$ the map

$$\chi:\pi_1(\mathcal{B}_\Lambda,m_0)\to\pi_1(\Lambda,m_\Lambda)\oplus\mathbb{Z}\,,\quad [\lambda]_{\mathcal{B}_\Lambda}\mapsto\left([\pi_D\circ\lambda]_\Lambda,\ \frac{1}{2i\pi}\int_\lambda\frac{dz}{z}\right)\,, \tag{8}$$

is an isomorphism.

Lemma 3.3. *There exist a neighborhood $\widehat{\mathcal{B}}$ of $\mathcal{P}'$ in $\widehat{W}_{D'}$ and two retractions by deformation $r:D'\to\Lambda$ and $R:\mathcal{B}_{D'}:=\widehat{\Psi}(\widehat{\mathcal{B}})\to\mathcal{B}_\Lambda$ such that*

(i) $\widehat{\mathcal{B}}\cap\widehat{\pi}^{-1}(\partial\mathcal{P})=\widehat{\mathcal{B}}_{\partial\mathcal{P}}$, $\mathcal{B}_{D'}\cap\pi_D^{-1}(\Lambda)=\mathcal{B}_\Lambda$ *and the following diagram is commutative:*

$$\begin{array}{ccc}\widehat{\mathcal{B}} & \xrightarrow{\widehat{\Psi}} & \mathcal{B}_{D'}\\ \widehat{\pi}\downarrow & \circlearrowleft & \downarrow\pi_D\\ \mathcal{P}' & \xrightarrow{\Psi} & D'\,.\end{array}$$

(ii) *R and r commute with the fibration π_D, i.e.* $\pi_D\circ R=r\circ\pi_D$*;*

(iii) *for every leaf L of $\mathcal{F}_{|\mathcal{B}_{D'}}$, the restriction $R_{|L}$ is a retraction by deformation of L onto $L\cap\mathcal{B}_\Lambda$;*

(iv) *every path γ on $\mathcal{B}_{D'}$ with endpoints lying on the fibre $\pi_D^{-1}(m_\Gamma)$ of a point $m_\Gamma\in\Gamma$ is homotopic inside $\mathcal{B}_{D'}$ to a path contained in $\mathcal{B}_\Gamma:=\mathcal{B}_{D'}\cap\pi_D^{-1}(\Gamma)$ if and only if the element $[\pi_D\circ R\circ\gamma]_\Lambda$ of $\pi_1(\Lambda,m_\Lambda)$ belongs to the subgroup generated by the loop μ defined in Equation (7);*

(v) *a path γ lying on a leaf L of $\mathcal{F}_{\mathcal{B}_{D'}}$ with endpoints on $\pi_D^{-1}(m_\Gamma)$ is homotopic inside L to a path lying on $\mathcal{B}_\Gamma\cap L$ if $[\pi_D\circ R\circ\gamma]_\Lambda$ belongs to the subgroup $\langle\mu\rangle$ of $\pi_1(\Lambda,m_\Lambda)$.*

Proof. Let $\Phi(t,z)$ the flow of the radial vector field $\mathcal{R}=z\frac{\partial}{\partial z}$ on $\mathbb{C}$. If $z\in\mathcal{P}'$ we define $\zeta(z)=\inf\{t\in\mathbb{R}_{>0}\,|\,\Phi(t,z)\notin\mathcal{P}'\}$. The map

$$\widehat{h}:\mathcal{P}'\times[0,1]\longrightarrow\mathcal{P}'\,,\qquad \widehat{h}(z,t):=\Phi\left(t\,\varsigma(z),\,z\right)$$

is a homotopy defining a retraction by deformation $\widehat{r}:=h(\cdot,1):\mathcal{P}'\to\partial\mathcal{P}$. Its restriction to $\partial\overline{\mathbb{D}}_\epsilon\times[0,1]$ is a homeomorphism sending each segment $\{z\}\times[0,1]$ onto the intersection of the half line $\mathbb{R}_{\geq0}\cdot z$ with $\mathcal{P}'$. We define $r:=\widehat{\Psi}\circ\widehat{r}\circ(\Psi_{|\Gamma})^{-1}$. The vector field $\mathcal{R}$ lifts (via $\widehat{\pi}$) to a unique vector field $\widehat{\mathcal{R}}$ tangent to the foliation $\widehat{\mathcal{F}}$. Let $\widehat{\Phi}(t,m)$ be its flow and denote $\varsigma'(z):=\inf\{t\in\mathbb{R}_{>0}\mid\Phi(-t,z)\in\mathbb{D}_\epsilon\}$. The map $(m,t)\mapsto\widehat{\Phi}(-t\varsigma'(\widehat{\pi}(m)),m)$ define a homeomorphism of $\widehat{\mathcal{B}}_{\partial\mathcal{P}'}\times[0,1]$ onto a neighborhood $\widehat{\mathcal{B}}$ of $\mathcal{P}'$ in $\widehat{W}_{D'}$. Consider now the homotopy

$$\widehat{H}:\widehat{\mathcal{B}}\times[0,1]\longrightarrow\widehat{\mathcal{B}}\,,\quad \widehat{H}(m,t):=\widehat{\Phi}(t\varsigma(\widehat{\pi}(m)),m)\,,\qquad \widehat{\pi}\circ\widehat{H}=\widehat{h}\circ\widehat{\pi}\,,$$

which lifts $\widetilde{h}$, and the homotopy

$$\widehat{H}':\widehat{\mathcal{B}}\times[0,1]\longrightarrow\widehat{\mathcal{B}}\,,\quad \widehat{H}'(m,t):=\widehat{\Phi}(-t\varsigma'(\widehat{\pi}(m)),m)\,,\qquad \widehat{\pi}\circ\widehat{H}'=\widehat{h}'\circ\widehat{\pi}\,,$$

which lifts the homotopy

$$\widehat{h}':\partial\overline{\mathbb{D}}_\epsilon\times[0,1]\longrightarrow\mathcal{P}'\,,\qquad \widehat{h}'(z,t):=\Phi\left(-t\,\varsigma'(z),\,z\right)\,.$$

Clearly the restrictions

$$\widehat{H}'_{|\widehat{\mathcal{B}}_{\partial\mathcal{P}}\times[0,1]} : \widehat{\mathcal{B}}_{\partial\mathcal{P}} \times [0,1] \xrightarrow{\sim} \widehat{\mathcal{B}} \text{ and } \widehat{H}_{|\widehat{\mathcal{B}}_{\partial\mathbb{D}_\epsilon}\times[0,1]} : \widehat{\mathcal{B}}_{\partial\mathbb{D}_\epsilon} \times [0,1] \xrightarrow{\sim} \widehat{\mathcal{B}},$$

are homeomorphisms which conjugate the product foliations $\widehat{\mathcal{F}}_{|\widehat{\mathcal{B}}_{\partial\mathcal{P}}} \times [0,1]$ and $\widehat{\mathcal{F}}_{|\widehat{\mathcal{B}}_{\partial\mathbb{D}_\epsilon}} \times [0,1]$ to the foliation $\widehat{\mathcal{F}}$, where $\widehat{\mathcal{B}}_{\partial\mathbb{D}_\epsilon} = \widehat{\Psi}^{-1}(\mathcal{B}_\Gamma) \subset \widehat{\pi}^{-1}(\partial\mathbb{D}_\epsilon)$. The maps

$$\widehat{R} := \widehat{H}(\cdot,1) : \widehat{\mathcal{B}} \longrightarrow \widehat{\mathcal{B}}_{\partial\mathcal{P}} \quad \text{and} \quad \widehat{R}' := \widehat{H}'(\cdot,1) : \widehat{\mathcal{B}} \longrightarrow \widehat{\mathcal{B}}_{\partial\mathbb{D}_\epsilon}$$

are retractions by deformation inducing retractions by deformation on each leaf of $\widehat{\mathcal{F}}$ lifting respectively $\widehat{r}$ and

$$\widehat{r}' := \widehat{h}'(\cdot,1) : \mathcal{P}' \longrightarrow \partial\mathbb{D}_\epsilon .$$

Since the restriction of $\widehat{\Psi}$ to $\widehat{\mathcal{B}} \setminus \widehat{\mathcal{B}}_{\partial\mathbb{D}_\epsilon}$ is a homeomorphism onto $\mathcal{B}_{D'} \setminus \mathcal{B}_\Lambda$, the map $\widehat{\Psi}\circ\widehat{R}\circ\widehat{\Psi}^{-1}$: $\mathcal{B}_{D'} \setminus \mathcal{B}_\Lambda \to \mathcal{B}_\Lambda$ is well defined and it extends to a map $R : \mathcal{B}_{D'} \to \mathcal{B}_\Lambda$ by being the identity on $\mathcal{B}_\Lambda$. Indeed, the restriction of $\widehat{\Psi}$ to each subset $\widehat{\mathcal{B}}_k := \widehat{\mathcal{B}} \cap \widehat{\pi}^{-1}\big(\{\frac{2\pi k}{4g} < \arg(z) < \frac{2\pi(k+1)}{4g}\}\big)$ and $\widehat{\mathcal{B}}'_k := \widehat{\mathcal{B}} \cap \widehat{\pi}_D^{-1}\big(\{\arg(z) = \frac{2\pi k}{4g}\}\big)$, $k = 0,\ldots,4g-1$, is a homeomorphism onto their image. Moreover $\widehat{R}(\widehat{\mathcal{B}}_k) = \widehat{\mathcal{B}}_k \cap \widehat{\mathcal{B}}_{\partial\mathcal{P}}$, $\widehat{R}(\widehat{\mathcal{B}}'_k) = \widehat{\mathcal{B}}'_k \cap \widehat{\mathcal{B}}_{\partial\mathcal{P}}$. Therefore the restriction of $\widehat{\Psi}\circ\widehat{R}\circ\widehat{\Psi}^{-1}$ to each subset $\widehat{\Psi}(\widehat{\mathcal{B}}_k)$, $\widehat{\Psi}(\widehat{\mathcal{B}}'_k)$ is well-defined and continuous. All these restrictions coincide with the identity map on $\mathcal{B}_\Lambda$ because $\widehat{R} = \mathrm{Id}$ on $\widehat{\Psi}^{-1}(\mathcal{B}_\Lambda) = \widehat{\mathcal{B}}_{\partial\mathcal{P}}$. Thus $R : \mathcal{B}_{D'} \to \mathcal{B}_\Lambda$ is a retraction by deformation satisfying Properties (ii) and (iii) of the lemma. We shall see now that R also satisfies Property (iv).

Let $\gamma : [0,1] \to \mathcal{B}_{D'}$, $\gamma(0), \gamma(1) \in \pi_D^{-1}(m_\Gamma)$, $m_\Gamma \in \Gamma$, be a path homotopic to another path γ_1 lying on $\mathcal{B}_\Gamma$. It follows from (8) that we can take for $\pi_D\circ\gamma_1$ a power $\breve{\Gamma}^\nu$ of the simple parametrization $\breve{\Gamma}(t) := \Psi(\epsilon e^{2i\pi t})$, $t \in [0,1]$ of Γ which satisfies $r\circ\breve{\Gamma} = \mu$. Hence $\pi_D \circ R\circ\gamma = r\circ\pi_D\circ\gamma = \mu^\nu$.

Conversely, let $\gamma : [0,1] \to \mathcal{B}_{D'}$, $\gamma(0), \gamma(1) \in \pi_D^{-1}(m_\Gamma)$, be a path such that $\pi_D\circ R\circ\gamma = r\circ\pi_D\circ\gamma$ is homotopic to μ^ν. We consider a path ξ contained in $\mathcal{B}_\Gamma$ having the same endpoints as γ and such that $\pi_D\circ\xi \sim \breve{\Gamma}$. The loop $\delta := \gamma\vee\xi^{-\nu}$ satisfy $[\pi_D\circ R\circ\delta]_\Lambda = 0$. Consequently $R\circ\delta$ is homotopic in $\mathcal{B}_\Lambda$ to a loop lying on the fibre $\pi_D^{-1}(m_\Lambda)$. Since R is a retraction by deformation commuting to the projection π_D, we obtain that δ is homotopic inside $\mathcal{B}_{D'}$ to a loop δ_1 lying on $\pi_D^{-1}(m_\Gamma)$. Hence γ is homotopic inside $\mathcal{B}_{D'}$ to the path $\delta_1\vee\xi^{-\nu}$ which is contained in $\mathcal{B}_\Gamma$.

Now, we shall prove Property (v) from the following assertion:

($\star$) Let δ be a path lying on a leaf L of $\mathcal{F}_{|\mathcal{B}_\Lambda}$ such that $\pi_D\circ\delta$ is homotopic to μ^ν. Then there exists a path χ homotopic to δ inside L such that $\pi_D\circ\chi = \mu^\nu$.

We can apply this property to the path $\delta := R\circ\gamma$ because $\pi_D\circ R\circ\gamma$ is homotopic to μ^ν for some $\nu\in\mathbb{Z}$ by hypothesis. Thus we obtain a path χ homotopic to $R\circ\gamma$ inside $L\cap\mathcal{B}_\Lambda$ such that $\pi_D\circ\chi = \mu^\nu$. By applying Lemma 3.1 to it we get a continuous $\widehat{\Psi}$-lift $\widehat{\chi}$ lying on the leaf $\widehat{L} = \widehat{\Psi}^{-1}(L)$ of $\widehat{\mathcal{F}}$. On the other hand, by using the foliated retraction R we construct two paths ξ_0, $\xi_1 : [0,1] \to L$ such that $\xi_0(0) = \gamma(0)$, $\xi_0(1) = R\circ\gamma(0)$, $\xi_1(0) = R\circ\gamma(1)$, $\xi_1(1) = \gamma(1)$ and

$$\gamma \sim_L \xi_0\vee(R\circ\gamma)\vee\xi_1 \sim_L \xi_0\vee\chi\vee\xi_1 = \widehat{\Psi}\circ(\widehat{\xi}_0\vee\widehat{\chi}\vee\widehat{\xi}_1)$$

for the unique continuous $\widehat{\Psi}$-lifts $\widehat{\xi}_0$, $\widehat{\xi}_1$ of ξ_0 and ξ_1 respectively. Moreover $|\xi_0|$ and $|\xi_1|$ are contained in orbits of $\widehat{\mathcal{R}}$ and clearly $\widehat{\xi}_0\vee\widehat{\chi}\vee\widehat{\xi}_1$ is homotopic in $\widehat{L}$ to $\widehat{R}'\circ\widehat{\chi}$. Then γ is homotopic in L to $\widehat{\Psi}\circ\widehat{\chi}$ which is contained in $\mathcal{B}_\Gamma\cap L$.

In order to prove Assertion ($\star$) we consider a path $\delta : [0,1] \to L$ satisfying the hypothesis of this assertion. Without loss of generality we can assume that δ is smooth and transverse to the fibre $\pi_D^{-1}(m_\Lambda)$. We get a subdivision $t_1 = 0 < t_2 < \cdots < t_{q'+1} = 1$ of the interval $[0,1]$, such that each curve $\delta([t_j,t_{j+1}])$ is contained in a single block $\mathcal{B}_{\tau(j)}$. The endpoints $m_j := \delta(t_j)$ and $m_{j+1} := \delta(t_{j+1})$ of the path $\delta_j := \delta_{|[t_j,t_{j+1}]}$ project by π_D onto the point m_Λ and the image of δ_j is the closed segment L_j of the leaf L (of real dimension one) delimited by the points m_j and m_{j+1}. The projection of the path δ_j by π_D is a loop based on m_Λ with image $|\alpha_{\tau(j)}|$ or $|\beta_{\tau(j)}|$ depending on the parity of $\tau(j)$. Moreover, the equality $\pi_D \circ \delta_j(t) = m_\Lambda$ only holds for $t = t_j$ and $t = t_{j+1}$. Thus, if we assume that $\pi_D \circ \delta_j$ is not null-homotopic then there exists a homotopy inside $|\pi_D \circ \delta_j|$ between $\pi_D \circ \delta_j$ and one of the loops $\alpha_{\tau(j)}$, $\alpha_{\tau(j)}^{-1}$, $\beta_{\tau(j)}$ or $\beta_{\tau(j)}^{-1}$. Clearly this homotopy lifts to a homotopy inside L_j between δ_j and a new path ρ_j. Finally, there exists $q \leq q'$ such that δ is homotopic inside L to the path

$$\rho := \rho_1 \vee \cdots \vee \rho_q\,, \quad \pi_D \circ \rho_j =: \mu_{\tau(j)} \in \{\, \alpha_{\tau(j)}\,, \alpha_{\tau(j)}^{-1}\,, \beta_{\tau(j)}\,, \beta_{\tau(j)}^{-1}\,\}\,.$$

Let us consider the word

$$M(\delta) := \mu_{\tau(1)}\, \mu_{\tau(2)} \cdots \mu_{\tau(q)}$$

composed by the signs of the alphabet

$$\mathcal{A} := \{\alpha_1, \alpha_1^{-1}, \ldots, \alpha_g, \alpha_g^{-1}, \beta_1, \beta_1^{-1}, \ldots, \beta_g, \beta_g^{-1}\}\,.$$

By a sequence of moves of type

$$u_1 \cdots u_k\, v\, v^{-1}\, u_{k+1} \cdots u_N \;\to\; u_1 \cdots u_k\, u_{k+1} \cdots u_N\,, \qquad u_j,\; v \in \mathcal{A}\,, \tag{9}$$

$$u_1 \cdots u_k\, v\, v^{-1}\, u_{k+1} \cdots u_N \;\leftarrow\; u_1 \cdots u_k\, u_{k+1} \cdots u_N\,, \qquad u_j,\; v \in \mathcal{A}\,,$$

we can transform the word $M(\delta)$ in a unique word $M(\delta)^{\mathrm{red}}$, called the **reduced word** associated to $M(\delta)$, which do not contain any sub-word of length two of type $v\,v^{-1}$, $v \in \mathcal{A}$. The uniqueness of $M(\delta)^{\mathrm{red}}$ follows from the solution of the word problem in a free group, cf. [6]. More precisely, every element of the free group

$$\pi_1(\Lambda, m_\Lambda) \xrightarrow{\sim} \langle \dot\alpha_1, \ldots \dot\alpha_g, \dot\beta_1, \ldots \dot\beta_g \mid - \rangle\,,$$

can be written in a unique way in the form $\dot M := \dot u_1 \vee \cdots \vee \dot u_p$, where $M := u_1 \cdots u_p$, $u_j \in \mathcal{A}$ is a reduced word. Now we use the hypothesis that $\pi_D \circ \delta$ is homotopic inside Λ to a loop of type μ^ν. We have equality of reduced words:

$$M(\delta)^{\mathrm{red}} = \overline{\mu}^{\nu}\,, \quad \overline{\mu} := \alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1} \cdots \alpha_g\beta_g\alpha_g^{-1}\beta_g^{-1}\,.$$

Notice that we pass from $M(\delta)$ to $M(\delta)^{\mathrm{red}}$ by a sequence of suppression moves of type (9)

$$M_0 = M(\delta) \to \cdots \to M_q = M(\delta)^{\mathrm{red}}\,, \quad M_k = u_{k,\,1} \cdots u_{k,\,n_k}\,, \quad u_{k,\,j} \in \mathcal{A}\,.$$

To finish the proof it suffices to remark that

- to any suppression move we can associate a homotopy inside Λ between the loops

  $$\underline{M}_k := u_{k,\,1} \vee \cdots \vee u_{k,\,n_k}$$

  and $\underline{M}_{k+1} := u_{k+1,\,1} \vee \cdots \vee u_{k+1,\,n_{k+1}}$,
- if one of the paths ρ_j composing ρ satisfies $\mu_{\tau(j+1)} = \mu_{\tau(j+1)}^{-1}$, then there exists a homotopy inside L between $\rho_j \vee \rho_{j+1} \vee \rho_{j+2}$ and a path $\widetilde\rho_{j+2}$ such that $\pi_D \circ \widetilde\rho_{j+2} = \mu_{\tau(j)}$.

□

3.1.2. *Checking the properties of Theorem 2.16.* First of all, we will precise the construction of the foliated block $\mathcal{B}_\eta(A)$ associated to the genus fundamental block $A = D'$. In the construction made in the precedent section we take a conformal disk $T \subset \pi_D^{-1}(m_\Lambda)$ of size small enough so that it is contained in the open 4-Milnor tube $\mathcal{T}_\eta$. Since $\mathcal{B}_\Gamma$ is a subset of type multi-suspension we can apply to it the *rabotage* procedure described in [10, Definition 4.3.5] in order to obtain a subset $\mathcal{R}_\Gamma$ of $\mathcal{B}_\Gamma$ of suspension type such that the inclusion $\mathcal{R}_\Gamma \subset \mathcal{B}_\Gamma$ is rigid and verifies $\mathcal{R}_\Gamma \underset{\mathcal{F}}{\hookrightarrow} \mathcal{B}_\Gamma$. Then we define

$$\mathcal{B}_\eta(A) := (\mathcal{B}_{D'} \setminus \mathcal{B}_\Gamma) \cup \mathcal{R}_\Gamma.$$

Clearly the inclusion $\mathcal{B}_\eta(A) \subset \mathcal{B}_{D'}$ is ∂-rigid.

We begin checking the properties of the part (I) in Theorem 2.16. In order to see the point (1) we consider the following commutative diagram induced by the natural inclusions:

$$\begin{array}{ccc} \pi_1(\mathcal{T}^*_{\eta'}(\Lambda)) & \longrightarrow & \pi_1(\mathcal{T}_{\eta'}(D')) \\ \downarrow & & \downarrow \\ \pi_1(\mathcal{B}_\Lambda) & \longrightarrow & \pi_1(\mathcal{B}_{D'}) \end{array}$$

Remark 3.2 implies that the first vertical arrow is an isomorphism. The bottom horizontal arrow is also an isomorphism because the map R in Lemma 3.3 is a deformation retract. By lifting conveniently the retraction r to $\mathcal{T}_{\eta'}(D')$ we see that the top horizontal arrow is also an isomorphism. The fourth arrow is also an isomorphism. Consequently the inclusion $\mathcal{T}_{\eta'}(D') \subset \mathcal{B}_{D'}$ is rigid. The inclusion $\mathcal{T}_{\eta'}(D') \subset \mathcal{B}_\eta(A)$ is also rigid. The fact that it is also ∂-rigid follows immediately from the construction.

In order to show (2) we must prove Properties (B1)-(B4) of Definition 2.4:

(B1) $\partial\mathcal{B}_\eta(A)$ is incompressible in $\mathcal{B}_\eta(A)$ because the inclusion $\partial\mathcal{T}^*_{\eta'}(D') \subset \partial\mathcal{B}_\eta(A)$ is rigid and $\partial\mathcal{T}^*_{\eta'}(D')$ is incompressible in $\mathcal{T}^*_{\eta'}(D')$ thanks to Proposition 2.12.

(B2) The boundary $\partial\mathcal{B}_\eta(A)$ is transverse to $\mathcal{F}$ because it is of suspension type.

(B3) Since $\partial\mathcal{B}_\eta(A)$ has been obtained by the *rabotage* procedure from a multi-suspension type subset $\mathcal{B}_\Gamma$, in order to prove the 1-$\mathcal{F}$-connectedness of $\partial\mathcal{B}_\eta(A)$ inside $\mathcal{B}_\eta(A)$ it suffices to show that $\mathcal{B}_\Gamma \underset{\mathcal{F}}{\hookrightarrow} \mathcal{B}_{D'}$ because $\partial\mathcal{B}_\eta(A) \underset{\mathcal{F}}{\hookrightarrow} \mathcal{B}_\Gamma$. In order to prove this, we consider a leaf L of $\mathcal{B}_{D'}$ and two paths $a : [0,1] \to \mathcal{B}_\Gamma$ and $b : [0,1] \to L$ which are homotopic in $\mathcal{B}_{D'}$ By point (3) of Lemma 3.3 we deduce that $[\pi_D \circ R \circ b] \in \langle \mu \rangle \subset \pi_1(\Lambda, m_\Lambda)$. By applying point (4) of Lemma 3.3 we obtain a new path $c : [0,1] \to L \cap \mathcal{B}_\Gamma$ which is homotopic to b inside L. By transitivity, c is homotopic to a in $\mathcal{B}_{D'}$. Since $|a|$ and $|b|$ are contained in $\mathcal{B}_\Gamma$ which is incompressible in $\mathcal{B}_{D'}$ we conclude that a is homotopic to c in $\mathcal{B}_\Gamma$.

(B4) After point (2) of Lemma 3.3 we know that every leaf L of $\mathcal{B}_\eta(A)$ is a deformation retract of $L \cap \mathcal{B}_\Lambda$, which outside of the fibre $\pi_D^{-1}(m_\Lambda)$ is a suspension type subset. We deduce that every leaf $L \cap \mathcal{B}_\Lambda$ of $\mathcal{F}_{|\mathcal{B}_\Lambda}$ is incompressible.

Properties (3) and (4) of Part (I) are trivial because in this case $\mathfrak{n}_A = 1$. To see (5) we define first $\Upsilon_{D'}$ as the holonomic transport of $\pi_D^{-1}(m_\Lambda) \cap \mathcal{T}^*_{\eta_1}(D')$ along the oriented segment joining the point m_Λ to m_Γ. It is clear that $\Upsilon_{D'} \cap \mathcal{B}_\eta(A)$ is incompressible in $\mathcal{B}_\eta(A)$ and that $Sat_{\mathcal{F}_{|\mathcal{B}_\eta(A)}}(\Upsilon_{D'}, \mathcal{B}_\eta(A)) = \mathcal{B}_\eta(A)$. On the other hand, $\Upsilon_{D'} \cap \mathcal{B}_\eta(A) \underset{\mathcal{F}}{\hookrightarrow} \partial\mathcal{B}_\eta(A) \underset{\mathcal{F}}{\hookrightarrow} \mathcal{B}_\eta(A)$ because $\partial\mathcal{B}_\eta(A)$ is of suspension type.

To prove Part (II) of Theorem 2.16 we recall that if $A = D'$ is not an initial block then the holonomy transformation h_Γ associated to Γ is linearizable. Therefore, there exists a conformal disk $\Sigma \subset \pi_D^{-1}(m_\Gamma)$ such that $h_\Gamma(\Sigma) \subset \Sigma$ or $h_\Gamma^{-1}(\Sigma) \subset \Sigma$. We define $\mathcal{V}_1 = \mathbb{V}_{\Sigma,\Gamma}$ and we begin the precedent construction with the conformal disk $T \subset \pi_D^{-1}(m_\Lambda)$ obtained by holonomic transport of $\mathcal{V}_1 \cap \pi_D^{-1}(m_\Gamma)$ along the segment joining m_Γ to m_Λ, choosing m_Λ as the breaking point of $\mathcal{V}_1$. Indeed, from this choice the precedent construction shows that $\mathcal{V}_1$ is of suspension type and $\partial\mathcal{B}_\eta(A) = \mathcal{V}_1 \underset{\mathcal{F}}{\hookrightarrow} \mathcal{V}$. Thus, we have proved point (3'). Since $\mathfrak{n}_A = 1$ we can take $\rho_A(c) = c$ to obtain trivially (4').

Finally, by definition a genus block is not a breaking block, so Part (III) do not apply in this case.

3.2. **Dicritical type foliated adapted block.** We fix a fundamental block $A \subset \mathcal{D}$ associated to a dicritical irreducible component D of $\mathcal{F}$ of genus g and valence $\mathfrak{n}_A \geq 1$, given by Definition 2.11. Condition (c) in Definition 1.2 implies that there are no dead branches adjacent to D.

Each connected component of ∂A is the boundary of a closed disk $D_{s_i}^{(i)}$ contained in an adjacent component $D^{(i)}$ of $\mathcal{D}$ and $D \cap D^{(i)} = \{s_i\}$, $i = 1, \ldots, \mathfrak{n}_A$. Let $\mathcal{V}_i$ be the given suspension sets over $\partial D_{s_i}^{(i)}$. Since the holonomy of $\partial D_{s_i}^{(i)}$ is the identity we can choose a saturated subset $\mathcal{V}_i' \subset \mathcal{V}_i$ having $\mathfrak{c}(\mathcal{V}_i') \leq \varepsilon$, i.e. satisfying Condition (4"). The saturation condition of $\mathcal{V}_i'$ inside $\mathcal{V}_i$ implies that each $\mathcal{V}_i'$ is of suspension type and satisfies Property (3").

Next we define $\mathcal{B}_{\mathcal{V}_i'}$ as the saturation of $\mathcal{V}_i' \subset \mathcal{V}_i$ by $\mathcal{F}$ inside $\pi_{D^{(i)}}^{-1}(D_{s_i}^{(i)})$, where $\pi_{D^{(i)}}$ is the Hopf fibration over the component D_i. We put $\mathcal{B}_{\mathcal{V}'} := \bigcup_{i=1}^{\mathfrak{n}_A} \mathcal{B}_{\mathcal{V}_i'}$ and we finally define

$$\mathcal{B}_\eta(A) := \left(\pi_D^{-1}\left(D \setminus \mathcal{B}_{\mathcal{V}'}\right) \cap \mathcal{T}_\eta^*(A)\right) \cup \left(\mathcal{B}_{\mathcal{V}'} \setminus D\right).$$

Recall that we have choose Hopf fibration π_D to be constant along the leaves of $\mathcal{F}|_{W_D}$, see Point (c) of Remark 2.2.

In order to prove Part (III) of Theorem 2.16 it suffices to show Assertions (1) and (2) of Part (I) because $\partial\mathcal{B}_\eta(A) = \bigcup_{i=1}^{\mathfrak{n}_A} \mathcal{V}_i'$ is automatically satisfied by construction. It is clear that $\overline{\mathcal{B}}_\eta(A)$ is a tubular neighborhood of A so that $\mathcal{B}_\eta(A)$ contains $\mathcal{T}_{\eta'}^*(A)$ for $\eta' > 0$ small enough. This inclusion is ∂-rigid because the inclusions $\partial\mathcal{B}_\eta(A) \subset \partial\mathcal{T}_\eta^*(A)$ and $\partial\mathcal{T}_{\eta'}^*(A) \subset \partial\mathcal{T}_\eta^*(A)$ are rigid and, on the other hand, we can easily see that $\mathcal{T}_{\eta'}(A) \subset \mathcal{B}_\eta(A)$ is a retract by deformation and consequently this last inclusion is also rigid.

To prove that $\mathcal{B}_\eta(A)$ is a $\mathcal{F}$-adapted block it suffices to observe the following assertions concerning properties (B1)-(B4) of Definition 2.4:

(B1) By using Proposition 2.12,

$$\pi_1(\partial\mathcal{B}_\eta(A)) \cong \pi_1(\partial\mathcal{T}_\eta^*(A)) \hookrightarrow \pi_1(\mathcal{T}_\eta^*(A)) \cong \pi_1(\mathcal{B}_\eta(A))$$

after the ∂-rigidity of $\mathcal{B}_\eta(A) \subset \mathcal{T}_\eta^*(A)$ that we have seen before.

(B2) The boundary $\partial\mathcal{B}_\eta(A) = \bigcup_{i=1}^{\mathfrak{n}_A} \mathcal{V}_i'$ is a suspension type subset over ∂A and consequently it is transverse to $\mathcal{F}$.

(B3) Each connected component of $\bigcup_{L \in \mathcal{F}} (L \cap \partial\mathcal{B}_\eta(A))$ is diffeomorphic to the product $\mathbb{D}^* \times \mathbb{D}^*$ endowed with the horizontal foliation. Consequently, we have that $\partial\mathcal{B}_\eta(A) \underset{\mathcal{F}}{\hookrightarrow} \mathcal{B}_\eta(A)$.

(B4) Every leaf L of $\mathcal{F}_{|\mathcal{B}_\eta(A)}$ is diffeomorphic to $\mathbb{D}^*$ and a generator of $\pi_1(L)$ is sent to the element c contained in the center of the group $\pi_1(\mathcal{B}_\eta(A))$ which is isomorphic to the direct sum of $\mathbb{Z}c$ and a free group of rank $2g + \mathfrak{n}_A - 1$.

4. Proofs of the main results

4.1. **Proof of Theorem A.** Recall that the break graph associated to $(\mathcal{F}, \mathcal{D})$ was obtained by considering the complement of the breaking elements $\mathcal{R}$ inside $\mathcal{G}_{\mathcal{D}}$, see Introduction. We consider the graph $\check{\mathcal{G}}$ obtained by eliminating the part of the break graph associated to $(\mathcal{F}, \mathcal{D})$ corresponding to the dead branches of $\mathcal{D}$. After Condition (e) in Definition 1.2 and Hypothesis (G) on $(\mathcal{F}, \mathcal{D})$, each connected component Λ of $\check{\mathcal{G}}$ is a tree with at most one vertex corresponding to an initial component $C \subset \mathcal{C}$. We apply Part (I) of Theorem 2.16 to the initial block A_C associated to C. If Λ does not contain any initial element then we begin the construction from a fundamental block A associated to an arbitrary element of Λ by applying Part (II). To do that we choose some suspension type initial boundary $\mathcal{V}$ with $\mathfrak{c}(\mathcal{V})$ small enough. Since the fundamental blocks $A \neq A_C$ corresponding to elements of Λ are not initial blocks we can apply to them by adjacency order Part (II) of Theorem 2.16 from the suspension type boundary obtained in the precedent step. Since Λ is finite this procedure stops. In this way we obtain a $\mathcal{F}$-adapted block for each fundamental block of $\mathcal{D}$ except for the breaking blocks of $\mathcal{D}$. The size and roughness of the boundary of the $\mathcal{F}$-adapted block obtained at each step of this inductive process is controlled by those of the block constructed in the precedent step. If we choose the size and roughness sufficiently small at the beginning then we have finite roughness at each step of the induction, see [10, §3.2] for more details. We make the boundary assembly of these $\mathcal{F}$-adapted blocks obtaining a connected subset $\mathcal{B}_\eta(\Lambda)$ of $\mathcal{T}^*_\eta(\Lambda)$ for each connected component Λ of $\check{\mathcal{G}}$.

In order to make the boundary assembly of all these sets $\mathcal{B}_\eta(\Lambda)$ we need also to consider $\mathcal{F}$-adapted blocks associated to the breaking elements $\rho \in \mathcal{R}$ adjacent to two connected components Λ and Λ' of $\check{\mathcal{G}}$, which we construct from the suspension type boundaries of $\mathcal{B}_\eta(\Lambda)$ and $\mathcal{B}_\eta(\Lambda')$ by using Part (III) of Theorem 2.16. Notice that the case $\Lambda = \Lambda'$ is not excluded. In fact, this situation could happen when $\partial\mathcal{B}_\eta(\Lambda)$ is not connected.

In this way we obtain a foliated boundary assembly

$$\mathcal{B}_\eta = \bigcup_{\Lambda \subset \check{\mathcal{G}}} \mathcal{B}_\eta(\Lambda) \cup \bigcup_{\rho \in \mathcal{R}} \mathcal{B}_\eta(\rho) \subset \mathcal{T}^*_\eta.$$

We take $U_1 = E(\mathcal{B}_{\eta_1}) \cup \mathcal{D}$. There exists $\eta_2 > 0$ such that $\mathcal{T}^*_{\eta_2} \subset \mathcal{B}_{\eta_1}$ and we define

$$U_2 = E(\mathcal{B}_{\eta_2}) \cup \mathcal{D} \subset U_1.$$

By induction we construct a decreasing a sequence (η_n) tending to zero such that

$$U_n := E(\mathcal{B}_{\eta_n}) \cup \mathcal{D} \subset U_{n-1}.$$

Put $\Upsilon := \sqcup_A \Upsilon_A$, where A varies in the set of fundamental blocks of $\mathcal{D}$ which are not breaking blocks. To finish it suffices to remark the validity of the following assertions:

(i) The inclusion $U^*_{n+1} \subset U^*_n$ is rigid by Remark 2.10, Corollary 2.9 and Property (1) of Theorem 2.16.

(ii) Every leaf L of $\underline{\mathcal{F}}_{|U^*_n}$ is incompressible after Property (2) of Theorem 2.16 by using Localization Theorem 2.5.

(iii) Thanks to Property (5) of Theorem 2.16 each irreducible component of Y^*_n is incompressible in the corresponding $\mathcal{F}$-adapted block, which is incompressible in U^*_n by Localization Theorem 2.5. Hence Y^*_n is incompressible in U^*_n. Let Ω be the union of all the $\mathcal{F}$-adapted blocks associated to non-breaking fundamental blocks of $\mathcal{D}$. Thanks to Property (5) in Theorem 2.16 we have $\mathrm{Sat}_{\mathcal{F}}(\Upsilon \cap \mathcal{B}_\eta, \Omega) = \Omega$. Clearly, the connected components of $\partial\Omega$ are exactly the connected components of the boundary of all $\mathcal{F}$-adapted blocks associated to breaking fundamental blocks of $\mathcal{D}$. Finally, for each $\mathcal{F}$-adapted block B associated to a fundamental breaking block A of $\mathcal{D}$ we have that $B \setminus \mathrm{Sat}_{\mathcal{F}}(\partial B, B)$ is a nodal or dicritical separator according to whether A is a dicritical block or a singularity block (necessarily associated to a nodal singularity).

(iv) Property (iv) of Theorem A is equivalent to the relation $Y_n^* \underset{\underline{\mathcal{F}}}{\hookrightarrow} U_n^*$. This follows from $\sqcup_A \Upsilon_A \underset{\mathcal{F}}{\hookrightarrow} \mathcal{B}_\eta$ because $\Upsilon_A \underset{\mathcal{F}}{\hookrightarrow} \mathcal{B}_\eta(A)$ by Theorem 2.16, $\mathcal{B}_\eta(A) \underset{\mathcal{F}}{\hookrightarrow} \mathcal{B}_\eta$ by Localization Theorem 2.5 and the transitivity of the relation $\underset{\mathcal{F}}{\hookrightarrow}$.

(v) Let U_n be one of the open sets that we have constructed. We still denote by $\widetilde{\mathcal{F}}_{U_n}$ the the pull-back by the universal covering $q_{U_n} : \widetilde{U_n} \to U_n^*$ of the foliation $\underline{\mathcal{F}}$ restricted to U_n^* and we denote $\widetilde{\mathcal{Q}}_{U_n}$ its leaf space. It is easy to see that the open subset of $\widetilde{\mathcal{Q}}_{U_n}$ corresponding to leaves of $\widetilde{\mathcal{F}}_{U_n}$ projecting onto an open fixed separator has a natural structure of Hausdorff one-dimensional complex manifold. To obtain a complete holomorphic atlas on $\widetilde{\mathcal{Q}}_{U_n}$ we proceed as follows. From the fact that Y_n^* is incompressible and 1-$\underline{\mathcal{F}}$-connected in U_n^* follows that each connected component $\widetilde{Y}_\alpha \cong \mathbb{D}$ of $q_{U_n}^{-1}(Y_n^*)$ intersects every leaf of $\widetilde{\mathcal{F}}_{U_n}$ in at most one point. Consequently, the open canonical maps $\tau_\alpha : \widetilde{Y}_\alpha \to \widetilde{Q}_{U_n}$, sending each point $p \in \widetilde{Y}_\alpha$ to the leaf L_p of $\widetilde{\mathcal{F}}_{U_n}$ passing through p, are injective. Hence the inverse maps τ_α^{-1} are holomorphic charts on $\widetilde{\mathcal{Q}}_{U_n}$. We achieve the proof by noting that $U_n \setminus Sat_{\underline{\mathcal{F}}}(Y_n^*, U_n^*)$ is a disjoint finite union of nodal and dicritical separators and that the transition functions induce the holonomy pseudo-group of $\underline{\mathcal{F}}$; hence they are holomorphic.

4.2. **Proof of Corollary A.** We must check that the total transform of $\mathcal{Z}$ by the minimal reduction map π of $\mathcal{F}$ is an adapted divisor of $(\pi^*\mathcal{F}, \pi^{-1}(0))$. Conditions (a) and (b) of Definition 1.2 are obviously fulfilled. Condition (d) can not occur by the existence of local separatrices.

To prove Condition (c) notice that on a neighborhood of a dead branch with branching point lying on a dicritical component all the leaves are compact. This situation can not happen because it does not exist compact analytic curves in $\mathbb{C}^2$.

To prove Condition (e) we will use the well-known fact that the total divisor of the desingularisation of a germ of curve $(X, 0)$ contains at most one irreducible component of the exceptional divisor adjacent to at least two dead branches. We take for X the union of the isolated separatrices of $\mathcal{F}$ and two non-isolated separatrices for each dicritical component of $\pi^*\mathcal{F}$. We can easily check that the minimal desingularisation morphism of X coincide with π, see [2, Theorem 2]. Then there exists at most one initial component of $(\pi^*\mathcal{F}, \pi^{-1}(0))$.

4.3. **Proof of Theorem B.** As we have already point out in the introduction, the equivalences $(1) \Leftrightarrow (1')$ and $(2) \Leftrightarrow (2')$ follow from the main result of [15] thanks to Condition (R). Since the implication $(2) \Rightarrow (1)$ is obvious it only remains to prove implications $(1) \Rightarrow (3) \Rightarrow (4) \Rightarrow (2)$. To do that we will make a strong use of the notions and statements introduced in [12].

(4) ⇒ (2): Conditions (a) and (b) in (4) imply that if $D \subset \mathcal{D}$ is a dicritical component of $(\mathcal{F}, \mathcal{D})$ then $\varphi(D)$ is a dicritical component of $(\mathcal{F}', \mathcal{D}')$ and if $s \in \mathcal{D}$ is a nodal singularity of $\mathcal{F}$ then $\varphi(s)$ is a nodal singularity of $\mathcal{F}'$. Consequently φ sends connected components of the cut divisor $\mathcal{D}^{\mathrm{cut}}$ defined in the introduction into connected components of $\mathcal{D}'^{\mathrm{cut}}$. On the other hand, by assumption $(\psi, \widetilde{\psi}, h) := (\varphi_{|\Sigma}, \widetilde{\varphi}_{|\widetilde{\Sigma}}, h)$ is a realization of the $\mathcal{S}$-conjugation $(\varphi, \widetilde{\varphi}, h)$ between the monodromies $\mathfrak{m}^{\underline{\mathcal{F}}}$ and $\mathfrak{m}^{\underline{\mathcal{F}}'}$ over $\mathcal{S}$-collections of transversals (Σ, Σ') in the sense of [12, Definition 3.6.1]. Moreover, by definition it satisfies trivially the additional condition

$$\widetilde{\psi}_\bullet = \widetilde{\varphi}_\bullet : \pi_0(\widetilde{\Sigma}) \to \pi_0(\widetilde{\Sigma}')$$

required in the Extension Lemma of [12, Lemma 8.3.2], whose proof is also valid for genus blocks. If $\mathcal{D}^{\mathrm{cut}}$ is not a tree we choose singularity blocks B_α such that $\mathcal{D}^{\mathrm{tree}} := \mathcal{D}^{\mathrm{cut}} \setminus \bigcup B_\alpha$ does not contain any cycle of components. Let B'_α be the singularity block of $\mathcal{D}'^{\text{cut}}$ corresponding to B_α by φ and put $\mathcal{D}'^{\text{tree}} := \mathcal{D}'^{\text{cut}} \setminus \bigcup B'_\alpha$. By applying iteratively the Extension Lemma beginning by $(\psi, \widetilde{\psi}, h)$ we obtain a realization $(\psi^0, \widetilde{\psi}^0, h)$ of $(\varphi, \widetilde{\varphi}, h)$ over a union W of foliated adapted blocks covering $\mathcal{D}^{\text{tree}}$ and $\mathcal{D}'^{\text{tree}}$.

Now we fix transversal disks Υ, Θ to the local separatrices associated to the singularity block B_α contained in the boundary of W. Extension Lemma implies that ψ^0 is excellent and that ψ^0 and φ coincide over $\mathcal{D}$. Consequently, $\Upsilon' := \psi^0(\Upsilon) = \varphi(\Upsilon)$ and $\Theta' := \psi^0(\Theta) = \varphi(\Theta)$ are transversal disks to the local separatrices associated to the singularity block B'_α. Let $(\psi^1, \widetilde{\psi}^1, h)$ be the restriction of the realization $(\psi^0, \widetilde{\psi}^0, h)$ to Υ.

Applying again Extension Lemma to the realization $(\psi^0_{|\Theta}, \widetilde{\psi}^0_{|\widetilde{\Theta}}, h)$ for the block B_α we obtain a new realization whose restriction $(\psi^2, \widetilde{\psi}^2, h)$ to Υ satisfies $\psi^2(\Upsilon) = \Upsilon'$,

$$\widetilde{\psi}^1_\bullet = \widetilde{\varphi}_\bullet = \widetilde{\psi}^2_\bullet : \pi_0(\widetilde{\Upsilon}) \to \pi_0(\widetilde{\Upsilon}'),$$

and the commutativity of the following diagrams

$$\begin{array}{ccc} \widetilde{\Upsilon}_\alpha & \hookrightarrow & \mathcal{Q}^{\underline{\mathcal{F}}} \\ \widetilde{\psi}^i \downarrow & & \downarrow h \\ \widetilde{\Upsilon}'_{\widetilde{\varphi}_\bullet(\alpha)} & \hookrightarrow & \mathcal{Q}^{\underline{\mathcal{F}}'} \end{array}$$

for all $\alpha \in \pi_0(\widetilde{\Upsilon})$ and $i = 1, 2$. Since the horizontal arrows of these diagrams are monomorphisms we deduce that $(\psi^1, \widetilde{\psi}^1, h) = (\psi^2, \widetilde{\psi}^2, h)$. Consequently, we can glue these realizations to obtain a new realization $(\Psi, \widetilde{\Psi}, h)$ defined in a union of adapted foliated blocks covering $\mathcal{D}^{\text{cut}}$.

Finally, it only remains to extend Ψ to the dicritical components and the nodal singularities in order to obtain a global realization of $(\varphi, \widetilde{\varphi}, h)$ which will be the desired $\mathcal{S}$-transversely holomorphic conjugation between $(\mathcal{F}, \mathcal{D})$ and $(\mathcal{F}', \mathcal{D}')$. In fact, the extension to nodal singularities has been described in [12, §8.5].

Now we fix dicritical components $D \subset \mathcal{D}$ and $D' := \varphi(D) \subset \mathcal{D}'$. On neighborhoods of these components the foliations $\mathcal{F}$ and $\mathcal{F}'$ are disk fibrations. Because D and D' have the same self-intersection number, we can identify two tubular neighborhoods of D and D' endowed with the restriction of the foliations $\mathcal{F}$ and $\mathcal{F}'$ with a tubular neighborhood of the zero section of the normal bundle of D in M endowed with the natural fibration. Thus, we can consider the realization to be extended as a map from a disjoint union K of closed disks contained in D to $\mathrm{Aut}_0(\mathbb{D}, 0)$. We can extend it to a union K' of bigger disks containing K, being a constant automorphism of the fibres over $\partial K'$ and consequently to the whole dicritical component D using the connectedness of $\mathrm{Aut}_0(\mathbb{D}, 0)$.

(1) ⇒ (3): Let $g : (\underline{U}, \underline{\mathcal{D}}) \overset{\sim}{\to} (\underline{U'}, \underline{\mathcal{D}'})$ be a $\mathcal{S}$-transversely holomorphic conjugation between $(\underline{\mathcal{F}}, \underline{\mathcal{D}})$ and $(\underline{\mathcal{F}'}, \underline{\mathcal{D}'})$ and $\widetilde{g} : \widetilde{U} \to \widetilde{U}'$ a lifting to the universal coverings of $\underline{U} \setminus \underline{\mathcal{D}}$ and $\underline{U}' \setminus \underline{\mathcal{D}}'$. By [12, Remark 3.6.2] there exists a $\mathcal{S}$-$\overleftarrow{\underline{\mathrm{An}}}$ isomorphism $h : \mathcal{Q}^{\underline{\mathcal{F}}} \to \mathcal{Q}^{\underline{\mathcal{F}}'}$ such that $(g, \widetilde{g}, h)$ is a $\mathcal{S}$-conjugation between the monodromies $\mathfrak{m}^{\underline{\mathcal{F}}}$ and $\mathfrak{m}^{\underline{\mathcal{F}}'}$. Consider Σ a $\mathcal{S}$-collection of transversals for $\underline{\mathcal{F}}$ and $\underline{\mathcal{D}}^+$ a $\underline{\mathcal{D}}$-extended divisor. Using [12, Proposition 3.6.4] and by composing g by a suitable $\underline{\mathcal{F}}'$-isotopy Θ_t, having support on a neighborhood W' of $g(\Sigma)$, we obtain an homeomorphism $\underline{\varphi} := \Theta_1 \circ g$ such that $\Sigma' := \underline{\varphi}(\Sigma)$ is a $\mathcal{S}$-collection of transversals for $\underline{\mathcal{F}}'$ and $\underline{\mathcal{D}}'$. Now we choose $\underline{\mathcal{D}}'^+$ as $\underline{\varphi}(\underline{\mathcal{D}}^+)$. On the universal covering $\widetilde{U}$ we also consider the lifting $\widetilde{\underline{\varphi}}$ of $\underline{\varphi}$ which coincides with $\widetilde{g}$ on the complementary of $\widetilde{W}'$. Again by the same proposition, we see that $(\underline{\varphi}, \widetilde{\underline{\varphi}}, h)$ is a $\mathcal{S}$-conjugation of the monodromies realized over the $\mathcal{S}$-collections of transversals Σ and Σ'. It remains to check properties (a) and (b) of Point (3). First remark that $\underline{\varphi}$ maps isolated separatrices of $\underline{\mathcal{F}}$ into isolated separatrices of $\underline{\mathcal{F}}'$ because we have the following topological characterization:

S is a non-isolated separatrix if and only if there is a family $\{S_j\}_{j\in\mathbb{N}}$ of pairwise disjoint separatrices such that every $i,j\in\mathbb{N}$ we have that S_i is topologically conjugated to S and $S_i\cup S_j$ is topologically conjugated to $S\cup S_i$.

We deduce that $\underline{\mathcal{D}}'^{+}$ is a $\underline{\mathcal{D}}'$-extended divisor. The last assertion of Condition (a) is trivially satisfied by the topological conjugation $\underline{\varphi}$. In (b) equality of Camacho-Sad indices follows from Theorem 1.12 of R. Rosas if D is a nodal separatrix of $\underline{\mathcal{F}}$. Otherwise, $\underline{\varphi}$ is transversely holomorphic in a neighborhood of D and the desired equality is proved in [12, §7.2].

(3) ⇒ (4): We apply the following result which will be proved later.

Lemma 4.1. *Under the hypothesis of Point (3) there exists a germ of homeomorphism*

$$\varphi:(M,\mathcal{D})\to(M',\mathcal{D}')$$

sending the strict transform of $\underline{\mathcal{D}}^{+}$ into the strict transform of $\underline{\mathcal{D}}'^{+}$ and a there is a lift $\widetilde{\varphi}$ of φ to the universal coverings of the complementaries of $\mathcal{D}$ and $\mathcal{D}'$ satisfying the following properties:

- *(i) at each singular point of $\underline{\mathcal{D}}^{+}$ the actions of φ and $\underline{\varphi}$ on the set of local irreducible components of $\underline{\mathcal{D}}^{+}$ coincide;*
- *(ii) $\varphi_{|\Sigma}=\underline{\varphi}_{|\Sigma}$ and $\widetilde{\varphi}_{\widetilde{\Sigma}}=\underline{\widetilde{\varphi}}_{|\widetilde{\Sigma}}$;*
- *(iii) $\widetilde{\varphi}_{*}=\underline{\widetilde{\varphi}}_{*}:\Gamma\to\Gamma'$;*
- *(iv) φ is excellent.*

Properties (ii) et (iii) trivially imply that $(\varphi,\widetilde{\varphi},h)$ is a $\mathcal{S}$-conjugation between the monodromies $\mathfrak{m}^{\mathcal{F}}$ and $\mathfrak{m}^{\mathcal{F}'}$ realized over the $\mathcal{S}$-collections of transversals Σ and Σ'. From property (i) easily follows Condition (a) of Point (4) because the strict transforms of $\overline{\underline{\mathcal{D}}^{+}\setminus\underline{\mathcal{D}}}$ and $\overline{\underline{\mathcal{D}}'^{+}\setminus\underline{\mathcal{D}}}$ allows to identify the dicritical components of $(\mathcal{F},\mathcal{D})$ and $(\mathcal{F}',\mathcal{D}')$. Condition (b) of (4) follows from Condition (b) of (3) for local separatrices $D\subset\mathcal{D}$ which are not contained in the exceptional divisor $\mathcal{E}$ of $E:M\to\underline{M}$. Since the dual graph of $\mathcal{E}$ is a disjoint union of trees we can apply the same argument of [12, §7.3] to the $\mathcal{F}$-invariant part of $\mathcal{D}$ in order to obtain the equalities of all Camacho-Sad indices corresponding by φ from those of the local separatrices of $\underline{\mathcal{F}}$ and $\underline{\mathcal{F}}'$. Finally (iv) gives (c) in Point (4).

Proof of Lemma 4.1. Following the notations of Section 2.3, for $0<\eta\ll\eta'\ll\varepsilon\ll 1$ we consider an open 4-Milnor tube $\mathcal{T}_\eta$ (resp. $\mathcal{T}'_{\eta'}$) associated to the divisor $\mathcal{D}^{+}:=E^{-1}(\underline{\mathcal{D}}^{+})$ (resp. $\mathcal{D}'^{+}:=E'^{-1}(\underline{\mathcal{D}}'^{+})$) and we denote by $\mathcal{T}$ (resp. $\mathcal{T}'$) the image by E (resp. E') of its closure in the neighborhood $\overline{W}$ (resp. $\overline{W}'$) considered in Lemma 2.1. It is worth to notice that the boundary of $\mathcal{T}$ is constituted by the closed 3-Milnor tube $\mathcal{M}=E(\mathcal{M}_\eta)$ and a finite union of solid tori whose boundaries are the connected components of $\partial\mathcal{M}$. The same property holds for $\mathcal{T}'$ and $\mathcal{M}'$. In the neighborhood of each singular point s of $\underline{\mathcal{D}}^{+}$ (resp. $\underline{\mathcal{D}}'^{+}$) we consider an euclidian metric given by holomorphic coordinates. The boundaries of the closed balls $B(s,r)$ centered at s with radius r are transverse to $\mathcal{M}$ if $0<r\le 2\varepsilon$. We define a collar piece of $\mathcal{T}$ or $\mathcal{M}$ as the intersection of $\overline{B(s,2\varepsilon)\setminus B(s,\varepsilon)}$ with $\mathcal{T}$ or $\mathcal{M}$. The connected components of the adherence of the complementary of the collar pieces of $\mathcal{T}$ or $\mathcal{M}$ are called essential pieces of $\mathcal{T}$ or $\mathcal{M}$. A continuous map between $\mathcal{M}$ and $\mathcal{M}'$ or $\mathcal{T}$ and $\mathcal{T}'$ will be called piece-adapted if the image of a piece is contained in a piece and the image of the boundary of a piece is contained in the boundary of a piece.

First step. Without loss of generality we can assume that $\underline{\varphi}(\mathcal{T}) \subset \mathcal{T}'$ and that any essential piece $\mathcal{T}_s$ of $\mathcal{T}$ containing a singular point s of $\underline{\mathcal{D}}^+$ is mapped into an essential piece $\mathcal{T}'_{s'}$ of $\mathcal{T}$ containing also a singular point s' of $\underline{\mathcal{D}}'^+$. Using the local conical structures of the divisors at their singular points and the retraction $\mathcal{T}^* := \mathcal{T} \setminus \underline{\mathcal{D}}^+ \to \mathcal{M}$ defined by the vector field ξ considered in Section 2.3, we can adapt the constructions of [11, Section 4.1] and a variant of [11, Lemma 4.6] to obtain a piece-adapted continuous map $\psi_T : \mathcal{T}^* \to \mathcal{M}' \subset \mathcal{T}'^*$ such that

(a) ψ_T is homotopic to $\underline{\varphi}_{|\mathcal{T}^*}$ as maps from $\mathcal{T}^*$ into $\mathcal{T}'^*$ by a homotopy preserving all essential pieces associated to the singularities;
(b) the restriction of ψ_T to each connected component of a piece of $\mathcal{M}$ is a homeomorphism onto a connected component of a piece of $\mathcal{M}'$ which respect the circle fibrations considered in Lemma 2.1.

We define ψ as the restriction of ψ_T to $\mathcal{M}$.

Second step. For any essential piece $\mathcal{M}_\alpha$ of $\mathcal{M}$, the part of the proof of the main result of [11] corresponding to Sections 4.2 to 4.4 gives us a homotopy, which preserves the boundaries, between the continuous map $\psi_{|\mathcal{M}_\alpha}$ and a homeomorphism $\psi_\alpha : \mathcal{M}_\alpha \to \mathcal{M}'_\alpha = \psi(\mathcal{M}_\alpha)$ such that

(a) ψ_α extends to a homeomorphism $\Psi_\alpha : \mathcal{T}_\alpha \to \mathcal{T}'_\alpha$ between the corresponding pieces of $\mathcal{T}$ and $\mathcal{T}'$ containing $\mathcal{M}_\alpha$ and $\mathcal{M}'_\alpha$;
(b) Ψ_α is excellent in the sense of [11, Definition 2.5]; in particular, the restriction of Ψ_α to $\partial\mathcal{T}_\alpha$ conjugates the disk fibrations considered in Lemma 2.1.

Third step. Using the product structure, it is straightforward to construct homotopies on the collar pieces of $\mathcal{M}$ gluing the previous homotopies defined in the essential pieces of $\mathcal{M}$. In this way we obtain a piece-adapted continuous map $\psi' : \mathcal{M} \to \mathcal{M}'$ whose restriction to each essential $\mathcal{M}_\alpha$ coincides with the homeomorphism ψ_α but whose restriction to any collar piece is not necessarily a homeomorphism. However, up to deforming ψ' by suitable homotopies with support on the collar pieces provided by [18, Theorem 6.1] we can assume that ψ' is a piece-adapted global homeomorphism.

It remains to extend ψ' to an excellent homeomorphism Ψ between $\mathcal{T}$ and $\mathcal{T}'$ possessing a lifting $\widetilde{\Psi}$ to the universal coverings of $\mathcal{T} \setminus \underline{\mathcal{D}}$ and $\mathcal{T}' \setminus \underline{\mathcal{D}}'$ fulfilling properties of Lemma 4.1. On each essential piece $\mathcal{T}_\alpha$ we define Ψ as Ψ_α constructed in second step. Since the restriction of Ψ_α to the boundary of the essential pieces conjugates the disk fibrations, we can apply the techniques given in [11, Sections 4.4.2 and 4.4.4] to obtain the desired extension Ψ. In addition, it is not difficult to modify Ψ by an excellent isotopy in order to have $\Psi_{|\Sigma} = \underline{\varphi}_{|\Sigma} : \Sigma \to \Sigma'$. Classically there exists a lifting $\widetilde{\Psi}$ to the universal coverings $\widetilde{\mathcal{T}}$ and $\widetilde{\mathcal{T}}'$ of $\mathcal{T} \setminus \underline{\mathcal{D}}$ and $\mathcal{T}' \setminus \underline{\mathcal{D}}'$ such that $\widetilde{\Psi}_* = \underline{\widetilde{\varphi}}_* : \Gamma \to \Gamma'$.

Moreover, since the restriction of $\underline{\varphi}$ and Ψ to each singular piece $\mathcal{M}_\alpha$ are related by a homotopy localized in $\mathcal{M}_\alpha$ it follows that

$$\underline{\widetilde{\varphi}}_\bullet = \widetilde{\Psi}_\bullet : \pi_0(\widetilde{\mathcal{T}}_\alpha) \to \pi_0(\widetilde{\mathcal{T}}'_\alpha).$$

Thanks to this last equality we can apply the procedure described in [12, Section 8.4] in order to modify $(\Psi, \widetilde{\Psi})$ by Dehn twists to obtain a new pair $(\varphi, \widetilde{\varphi})$ which satisfy the same properties (iii) and (iv) and fulfills also the equality

$$\underline{\widetilde{\varphi}}_\bullet = \widetilde{\varphi}_\bullet : \pi_0(\widetilde{\Sigma}) \to \pi_0(\widetilde{\Sigma}').$$

Up to making an additional Dehn twist if necessary we obtain that $\widetilde{\varphi}_{|\Sigma} = \underline{\widetilde{\varphi}}_{|\widetilde{\Sigma}}$, showing Property (ii).

Since Property (i) follows from our construction, the proof of the lemma is achieved. □

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WEBS AND SINGULARITIES

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Dedicated to Xavier Gómez-Mont on his 60th birthday

ABSTRACT. We investigate the singular web structure of first-order PDEs from the viewpoint of singularity theory. Most of the results given have already appeared in papers by others, as well as the author [28, 29, 30, 31, 32] in various different terminologies. The new results are the construction of mini-versal webs from the deformation of isolated singularities, and their classification. We prove also the existence of the resonance curve for generic 3-webs with cuspidal singular locus. We introduce also Klein-Halphen webs with polyhedral symmetry and Fermat webs, and we investigate their properties.

1. VERSAL WEB

Let f be a holomorphic function germ with an isolated singularity at $o \in \mathbb{C}^{n+1}$, i.e., $f(o) = 0$ and $V(\frac{\partial f}{\partial x_0}, \cdots, \frac{\partial f}{\partial x_n}) = \{o\}$, and let us consider the ideal quotients

$$A = \frac{\mathcal{O}_{\mathbb{C}^{n+1},o}}{\langle f, \ \frac{\partial f}{\partial x_1}, \ \cdots \ \frac{\partial f}{\partial x_n}\rangle \mathcal{O}_{\mathbb{C}^{n+1},o}}, \qquad B = \frac{\mathcal{O}_{\mathbb{C}^{n+1},o}}{\langle \frac{\partial f}{\partial x_0}\rangle \mathcal{O}_{\mathbb{C},o} + \langle f, \ \frac{\partial f}{\partial x_1}, \ \cdots \ \frac{\partial f}{\partial x_n}\rangle \mathcal{O}_{\mathbb{C}^{n+1},o}},$$

where $x_0, \ldots, x_n$ are Cartesian coordinates of $\mathbb{C}^{n+1}$ and $\mathcal{O}_{\mathbb{C},o}$ denotes the local ring of germs of holomorphic functions of x_0 at $0 \in \mathbb{C}$. Clearly, if A is finite over $\mathbb{C}$, then B is also finite over $\mathbb{C}$. On the variety $V' = V(f, \frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n})$, the coordinate x_0 vanishes identically as f has an isolated singularity. If B is finite over $\mathbb{C}$, then A is finite over $\mathcal{O}_{\mathbb{C},o}$; hence V' is isolated and in particular A is finite over $\mathbb{C}$. The dimension of A over $\mathbb{C}$ is called the G_0-*codimension* of f in this paper, and differs slightly from the codimension of the G_0-*equivalence* class of f defined in §4. In this note, a τ-*web structure* (locally a configuration of τ codimension-one foliations) is introduced on A, $\tau = \dim_{\mathbb{C}} A$, and its various properties are investigated.

The quotient A was studied by Teissier [36] from the viewpoint of polar variety, and by Gomez Mont from the viewpoint of Euler obstruction, and also used to compute the dimension of the space of logarithmic vector fields of $f = 0$ (see e.g. [34]). The deformation theory with respect to B was investigated by Goryunov [15], and reviewed in detail in the book [5]. A link of the theory of projection of hypersurface singularities with respect to some intermediate quotient and the classification of first-order ordinary differential equations (ODEs) was investigated by Izumiya, Takahashi et al. (see e.g., [21]). From the viewpoint of the *equivalence problem* for ODEs, the web structure of families of solutions was initiated by Cartan and his followers [11, 14]. This short note is devoted to recollecting various links among these old and new subjects from the viewpoint of web geometry, and also emphasizing the role of versal web structure in the classification problem.

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Let $\{g_1, \ldots, g_\tau\}$ be a $\mathbb{C}$-basis of A, $\tau = \dim_{\mathbb{C}} A$. Consider the linear deformation $V_t : f_t = 0 \subset \mathbb{C}^{n+1}$, $f_t = f + t_1 g_1 + \cdots + t_\tau g_\tau$ and the divergent diagram

$$\mathbb{C} \xleftarrow{\lambda = x_0} V = \cup V_t \xrightarrow{\ t\ } \mathbb{C}^\tau .$$

The surface V_t is smooth for a generic t, on which λ has τ critical points of Morse-type. The above diagram may be regarded as a deformation of the projection of hypersurface singularities with parameter $t \in \mathbb{C}^\tau$,

$$\mathbb{C} \xleftarrow{\lambda = x_0} V_t \subset \mathbb{C}^{n+1} .$$

Let D denote the τ-valued function on $\mathbb{C}^\tau$ that assigns these critical values to a t. The *mini-versal* τ*-web* $W_{\{f_t\}}$ on $\mathbb{C}^\tau$ is defined to be the codimension-one "foliation" by the level hypersurfaces of D. Specifically, $W_{\{f_t\}}$ is locally a configuration of τ foliations, i.e., a τ-web, as D is τ-valued.

It was shown [28] that the web thus constructed is non-singular at a generic point; in other words, the τ-tuple of critical values (*critical-value-map*) $D' = (d_1, \ldots, d_\tau)$ is a local diffeomorphism of $\mathbb{C}^\tau$. This is a consequence of the versality in Theorem 4.1 and the existence of a deformation of f_t with an additional parameter $t' \in \mathbb{C}^{\tau'}$ for which the critical-value-map, defined on the extended parameter space $\mathbb{C}^{\tau+\tau'}$, is submersive at the generic point. By the symmetry quotient, D' induces a map (*classifying map*) $\tilde{D} : \mathbb{C}^\tau \to \mathbb{C}^\tau / S_\tau = \mathbb{C}^\tau$, branched over the discriminant set in the quotient. Thus, by the argument given by Looijenga [25], if the classifying map is proper and finite-to-one, the non-singular locus, i.e., the set of those t for which the critical values are all distinct and D' is a local diffeomorphism, possesses the $K(\pi, 1)$-property (see also [28]). Interestingly, this property was first proved for simple (G-simple) function germs in the weak equivalence relation with respect to B in [15]. The above τ-web is defined also for an arbitrary non-linear deformation of $f = 0$ in the same manner, though a deformation can be linearized if it is *versal*, i.e., $\partial f_t / \partial t$ generates A.

Another feature of the versal web is that all "leaves" are diffeomorphic to a discriminant of a Thom-Mather stable map germ [28]. In other words, a versal web is a complex one-parameter family of discriminant hypersurfaces. A versal deformation with the smallest number (i.e. τ) of parameters is called a *mini-versal* deformation (see also §3).

Brunella [9] investigated the various "real" one-parameter subfamilies of complex one-parameter families of hypersurfaces from the viewpoint of singular Levi-flat surfaces, and called them "*tissus microlocaux*".

2. An example

Let $n = 1$ and $f = x_1^2 + x_0^3$ for simplicity. Then the Jacobian quotients A, B are generated by $1, x_0, x_0^2$ and by $1, x_0$ respectively over $\mathbb{C}$. Our mini-versal linear deformation (with respect to A) with parameter $t = (t_1, t_2, t_3) \in \mathbb{C}^3$ is

$$\mathbb{C}_{x_0} \xleftarrow{\ x_0\ } V = \{x_1^2 + x_0^3 + t_1 x_0^2 + t_2 x_0 + t_3 = 0\} \xrightarrow{\ \pi\ } \mathbb{C}_t^3.$$

The restriction of x_0 to a fiber over a generic $t \in \mathbb{C}_t^3$

$$x_0 : \mathbb{C}_{x_0} \leftarrow V_t = \{f_t = x_1^2 + x_0^3 + t_1 x_0^2 + t_2 x_0 + t_3 = 0\} \subset \mathbb{C}^5_{x_0 x_1 t_1 t_2 t_3}$$

has three Morse-type singularities. The critical values of the restriction are the solutions in x_0 of $x_0^3 + t_1 x_0^2 + t_2 x_0 + t_3 = 0$. Thus the classifying map $\tilde{D} : \mathbb{C}_t^3 \to \mathbb{C}^3_{XYZ} = \mathbb{C}^3_{xyz}/S_3$ is the diffeomorphism $(X, Y, Z) = (\Sigma_1, \Sigma_2, \Sigma_3) = (-t_1, t_2, -t_3)$, where $\Sigma_1(x, y, z) = x + y + z$, $\Sigma_2(x, y, z) = xy + yz + zx$, and $\Sigma_3(x, y, z) = xyz$. The coordinate foliations by x, y, z on $\mathbb{C}^3_{xyz}$ induce the singular 3-web on the quotient $\mathbb{C}^3_{XYZ}$, which induces the mini-versal web $W_{\{f_t\}}$ on $\mathbb{C}_t^3$ via $\tilde{D}$.

The leaves (*solutions*) satisfy the following implicit first-order PDE

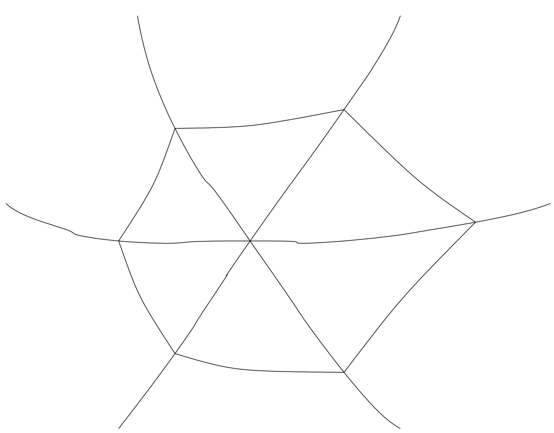
$$\begin{cases} p^3 + (t_1^2 - 2t_2)q^2 + (t_2^2 - 2t_1t_3)p - t_3^2 &= 0 \\ q^3 - t_1q^2 + t_2q - t_3 &= 0 \end{cases},$$

where $p = \frac{\partial t_3}{\partial t_1}$ and $q = \frac{\partial t_3}{\partial t_2}$. The figure on the left in Figure 2 is the cross-section of $W_{\{f_t\}}$ by $T = \mathbb{C}^2_{t_2t_3} : t_1 = 0$. The leaves on T are the solutions of the Clairaut equation

$$q^3 + t_2q - t_3 = 0,$$

in the coordinates t_2, t_3 (c.f. the equation E_1 in §7). This construction of the 3-web by quotient is generalized in §6, 7 and 8. The right figure in Figure 1 is a cross-section of the mini-versal web by a generic $T' : t_1 = s(t_2, t_3)$, where the hexagonal structure is violated so that the closed hexagon with concurrent 3 diagonal curves (Brianchon hexagon, the figure on the left in Figure 1) can not be embedded in a small shape. By this non-embeddability, the *hexagonal web* (for which all hexagons are closed, see §5,) is distinguished from the other apparently similar but non-hexagonal webs. The s is called the *function moduli* following the Russian school. Basically, the same cuspidal web structure was first investigated by Arnold [3, 5], and also independently by Carneiro [10] and Dufour [13] from the viewpoint of web geometry. By a suitable leaf preserving diffeomorphism of the mini-versal web on $\mathbb{C}^3_t$, the hypersurface T' can be transformed so that $s = 0$ on the cuspidal singular locus. The diffeo-type of induced web structure on the plane is then determined by the equivalence class of such an s by the weighted $\mathbb{C}^*$-automorphisms respecting the cusp, which is formally in one-to-one correspondence with an equivalence class of a 2-form on the plane by the same $\mathbb{C}^*$-automorphisms via the web curvature 2-form introduced in §5 [29].

In general, the codimension-one τ-webs with regular first integrals on $\mathbb{C}^s$, $s \leq \tau$, (see the next section for the definition) are described by codimension-$(\tau - s)$ sections, or pullbacks by maps from the s-space, of a mini-versal τ-web on $\mathbb{C}^\tau$ by the versality theorem (Theorem 4.1).



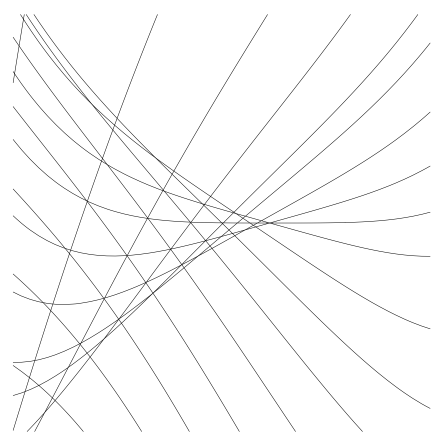


FIGURE 1. Left: a closed hexagon in a 3-web; Right: a non-hexagonal Clairaut 3-web

3. LEGENDRE CONSTRUCTION

Let $S^\tau \subset J^1(\mathbb{C}_q^{\tau-1}, \mathbb{C}_{t_\tau}) = \mathbb{C}_q^{\tau-1} \times \mathbb{C}_p^{\tau-1} \times \mathbb{C}_{t_\tau}$, $q = (q_1, \ldots, q_{\tau-1})$, be a germ of a submanifold of dimension τ in the first jet space such that the projection $\pi = \pi_q \times \pi_{t_\tau}$ to the base space (*observation space*) $\mathbb{C}_q^{\tau-1}\times\mathbb{C}_{t_\tau}$ is d-to-one. Assume the canonical contact form $\theta = dt_\tau - \sum p_i dq_i$ of the jet space is integrable on S. Then the push-forward $\pi_*\theta|_S$ defines a d-web on the base space, which we denote by W_S and call S the *skeleton* of W_S. A *first integral* of W_S is a function λ on the skeleton S such that $d\lambda\wedge\theta$ vanishes identically. A *solution* in $\mathbb{C}_q^{\tau-1}\times\mathbb{C}_{t_\tau}$ is an image of the projection of a level surface $\lambda = c$, which we denote by S_c. The family of these solutions $\{S_c\}$ constitutes the web W_S. If λ is non-singular, then $\alpha\ d\lambda = \theta$ with a holomorphic function α on S, thus

$$dt_\tau - \left(\alpha d\lambda + \sum p_i dq_i\right) = 0 \quad \text{ on } S.$$

This equation describes the canonical contact form on the extended jet space $J^1(\mathbb{C}_{x_0}\times\mathbb{C}_q^{\tau-1},\mathbb{C}_{t_\tau})$ vanishes identically on the image of

$$(\lambda,\pi_q,\alpha,\pi_p,\pi_{t_\tau}) : S^\tau \to (\mathbb{C}_{x_0}\times\mathbb{C}_q^{\tau-1})\times(\mathbb{C}_{p_0}\times\mathbb{C}_p^{\tau-1})\times\mathbb{C}_{t_\tau} = J^1(\mathbb{C}_{x_0}\times\mathbb{C}_q^{\tau-1},\mathbb{C}_{t_\tau}),$$

where π_p denotes the p-coordinates on $S\subset J^1(\mathbb{C}_q^{\tau-1},\mathbb{C}_{t_\tau})$. Thus the image is a *Legendre sub manifold.*

By a well-known result attributed to Hörmander and Arnold [4], the image of the above inclusion is represented by a Nash blow-up by the tangent hyperplane of a discriminant

$$D(F)\subset\mathbb{C}_{x_0}\times\mathbb{C}_{\tilde t}^{\tau-1}\times\mathbb{C}_{t_\tau}$$

of an unfolding of a function on a $\mathbb{C}_x^n$, $F = (t', f_{t'}) : \mathbb{C}_{t'}^\tau\times\mathbb{C}_x^n\to\mathbb{C}_{t'}^\tau\times\mathbb{C}$ by the slopes $p_1,\dots,p_\tau$ of the critical values of $f_{t'}$ with respect to the parameter $t' = (x_0,\tilde t) = (x_0,t_1,\dots,t_{\tau-1})\in\mathbb{C}^\tau$. Thus we may suppose

$$(\lambda,\pi_q,\alpha,\pi_p,\pi_{t_\tau}) = (x_0,t_1,\dots,t_{\tau-1},\frac{\partial f_{t'}}{\partial x_0},\frac{\partial f_{t'}}{\partial t_1},\dots,\frac{\partial f_{t'}}{\partial t_{\tau-1}},f_{t'})$$

identifying S and the critical point set $\Sigma(F)$ of F. (The unfolding is possibly nonlinear. The critical locus $\Sigma(F)$ is smooth.) By this identification, the solution S_c is the image of

$$\tilde F = (t_1,\dots,t_{\tau-1},f_{t'}) : \Sigma(F)\cap\{x_0 = c\} = \Sigma(F|x_0 = c)\to\mathbb{C}_{\tilde t}^{\tau-1}\times\mathbb{C}_{t_\tau},$$

which is the discriminant locus of the restriction of $\tilde F$ to $x_0 = c$. Let us consider the diagram

$$\mathbb{C}\xleftarrow{\ x_0\ }\mathbb{C}^{\tau+n}\xrightarrow{\tilde F = (t_1,\dots,t_{\tau-1},f_{t'})}\mathbb{C}_{\tilde t}^{\tau-1}\times\mathbb{C}_{t_\tau}\ .$$

Here we have the following *Legendre duality*

$$c\in D(x_0|\tilde F^{-1}(t_1\dots,t_\tau))\Longleftrightarrow(t_1\dots,t_\tau)\in S_c = D(\tilde F|x_0 = c)\ .$$

Thus the solution web W_S on the right space $\mathbb{C}_{\tilde t}^{\tau-1}\times\mathbb{C}_{t_\tau} = \mathbb{C}_q^{\tau-1}\times\mathbb{C}_{t_\tau}$ in the above divergent diagram is the *Legendre transformation* of the codimension-1 foliation of the left $\mathbb{C}$ by points.

Let

$$f_t = f_{(*,t_1,\dots,t_{\tau-1})}(*)\ -\ t_\tau\in\mathcal{O}_{\mathbb{C}_{x_0}\times\mathbb{C}_x^n,o}\ ,$$

where $t = (t_1,\dots,t_{\tau-1},t_\tau)\in\mathbb{C}_{\tilde t}^{\tau-1}\times\mathbb{C}_{t_\tau}$. Then $W_S = W_{\{f_t\}}$. The following theorem is stated in [28] and also found in the paper by Hayakawa, et al [17] in an apparently different form.

Theorem 3.1. *Let $S\subset J^1(\mathbb{C}^{\tau-1},\mathbb{C})$ be a germ of a submanifold at a (o,p,o) such that the projection to $\mathbb{C}^{\tau-1}\times\mathbb{C}$ is d-to-1 and the fiber over the origin is (o,p,o). Assume there exists a regular (nonsingular) first integral on S. Then there exists a family of functions $f_t\in\mathcal{O}_{\mathbb{C}^{n+1},o}$, $n>0$, with parameter $t\in\mathbb{C}^\tau$ such that $W_S = W_{\{f_t\}}$.*

The deformation $\{f_t\}$ above constructed is unique up to *stable equivalence* [5], which is called the *generating function* of the web with a first integral W_S. If $\{f_t\}$ is versal, as will be described in the next section, the restriction $F_c = \tilde{F} : \{x_0 = c\} \to \mathbb{C}^\tau$ is stable in the sense of Thom-Mather theory, and the solutions $S_c, c \in \mathbb{C}$, are all diffeomorphic—trivial by an ambient isotopy. The family $\{F_c\}$ is also *stable as a family* [28].

4. Versality

Let us consider the group G of triples (ϕ, ψ, h), where ϕ and ψ are respectively germs of diffeomorphisms of $\mathbb{C}^{n+1}, o$ and the x_0-line $\mathbb{C}, o$ compatible via the projection of $\mathbb{C}^{n+1}$ onto the first x_0-factor $\mathbb{C}$, and h is a function germ on $\mathbb{C}^{n+1}, o$ with $h(o) \neq 0$. The *product* $\circ$ on G is defined by

$$(\phi, \psi, h) \circ (\phi', \psi', h') = (\phi \circ \phi', \psi \circ \psi', \phi'^* h \times h').$$

Let $G_0 \subset G$ be the normal subgroup of triples (ϕ, id, h). These groups act on the local ring $\mathcal{O}_{\mathbb{C}^{n+1},o}$ of function germs defined at $o \in \mathbb{C}^{n+1}$ from the right by $f \cdot (\phi, \psi, h) = h \cdot \phi^* f$.

Two function germs f, g at $o \in \mathbb{C}^{n+1}$ are *G-equivalent* (respectively *G_0-equivalent*) if they lay on a common G-(resp. G_0-)orbit. Note that function germs non-vanishing at o are all G_0-equivalent, and hence G-equivalent. These notions are variants of the contact equivalence stated in terms of fibered diffeomorphisms of $\mathbb{C}^{n+1}$ over $\mathbb{C}$ and found in various articles (see cf. [15, 35]). For instance, G-equivalence is called *parameterized contact equivalence* in some papers. However the deformation theory was developed mostly involving an intermediate equivalence relation, for instance, the $\mathbb{R}^+$-equivalence, which has resulted in some confusion in terminology. Some incorrect conclusions were drawn, for instance, the "classification" of webs with two functional moduli in [35]. This matter will be explained in Example 1.

The present author investigated the theory in the latter equivalence relation and arrived at a natural versality notion suitable for classifying singular τ-webs with first integrals [28]. It is worth recalling the theory developed and restating it in a common language in terms of deformation of singularities. Theorem 4.3 was stated in [28] in an alternative form; its proof is for the first given in the present paper.

G_0-orbits are contained in G-orbits by definition. Because $G_0 \subset G$ is a normal subgroup, all G-orbits are foliated by G_0-orbits (possibly of codimension 0) in a finite jet level, and a triple (ϕ, ψ, h) in G sends a G_0-orbit to a G_0-orbit, thus it leaves each G-orbit invariant respecting the foliation by G_0-orbits. Here we can build two deformation theories with respect to G_0- and G-equivalence relations. If an f is G_0-simple, i.e., there exist only finitely many G_0-orbits on a sufficiently small neighborhood of f, then it is also G-simple and the G_0-orbit locally coincides with its G-orbit, hence the two theories coincide.

The above actions induce those of the Lie groups of their k-jets, denoted G_0^k, G^k, on the k-jet space of function germs $J^k(\mathbb{C}^{n+1}, o) = \mathcal{O}_{\mathbb{C}^{n+1},o}/m^{k+1}_{\mathbb{C}^{n+1},o}$, where $m_{\mathbb{C}^{n+1},o} \subset \mathcal{O}_{\mathbb{C}^{n+1},o}$ denotes the maximal ideal consisting of function germs vanishing at the origin. Of course, the tangent space of the G_0^k-orbit $\mathcal{O}_{G_0^k}(J^k f(o))$ as well as the G^k-orbit $\mathcal{O}_{G^k}(J^k f(o))$ at a k-jet $J^k f(0)$ is spanned by the tangent lines of the action of one-parameter subgroups generated by vector fields in a suitable form. The respective normal spaces of these orbits are presented as ideal quotients

$$N_{J^k f(0)} \mathcal{O}_{G_0^k}(J^k f(o)) = \frac{\mathcal{O}_{\mathbb{C}^{n+1},o}}{\langle f \rangle_{\mathcal{O}_{\mathbb{C}^{n+1},o}} + \langle \frac{\partial f}{\partial x_1}, \cdots \frac{\partial f}{\partial x_n} \rangle_{m_{\mathbb{C}^{n+1},o}} + m^{k+1}_{\mathbb{C}^{n+1},o}},$$

$$N_{J^k f(o)} \mathcal{O}_{G^k}(J^k f(o)) = \frac{\mathcal{O}_{\mathbb{C}^{n+1},o}}{\langle f \rangle_{\mathcal{O}_{\mathbb{C}^{n+1},o}} + \langle \frac{\partial f}{\partial x_0} \rangle_{m_{\mathbb{C},o}} + \langle \frac{\partial f}{\partial x_1}, \cdots \frac{\partial f}{\partial x_n} \rangle_{m_{\mathbb{C}^{n+1},o}} + m^{k+1}_{\mathbb{C}^{n+1},o}}.$$

Assume $\mathbb{C}$-dimensions of the Jacobian quotients A, B are finite. Then the ideals in the denominators are decreasing and stabilized as $k \to \infty$, hence the $\mathbb{C}$-dimensions of the quotients converge to certain limits, which are invariant under the G-equivalence relation. We say a deformation $f_t \in \mathcal{O}_{\mathbb{C}^{n+1},o}$ with parameter $t \in \mathbb{C}^r$ (possibly nonlinear in t) is *versal* (more precisely G_0-*versal*) if the differential $\delta : \mathbb{C}^r \to A$, $\delta(t) = \sum t_i \frac{\partial f_t}{\partial t_i}|_{t=o} \in A$ is surjective, and *mini-versal* if δ is an isomorphism.

It is not difficult to see $f = \{f_t\}$ is versal if and only if

$$\frac{\partial f_0}{\partial x_i}, \quad i = 1, \ldots, n, \qquad \frac{\partial f_t}{\partial t_j}|_{t=0}, \quad j = 1, \ldots, r$$

generate the normal space of the $\mathcal{O}_{G_0^k}$-orbit over $\mathbb{C}$ for any sufficiently large k; in other words, the k-jet section $J^k f : \mathbb{C}^{n+r} \to J^k(\mathbb{C}^{n+1}, o)$ defined by

$$J^k f(a, t) = \text{“}k\text{-jet of } f_t(x + (0, a)) \text{ at } x = o \in \mathbb{C}^{n+1}\text{”}$$

is transverse to the G_0^k-orbit at $o \in \mathbb{C}^{n+r}$.

We say a d-web W_S with a smooth skeleton S and a non-singular first integral is *versal* (respectively *mini-versal*) if its generating function $\{f_t\}$, constructed in the previous section, is versal (resp. mini-versal) in the above sense.

Given a generic map μ of $\mathbb{C}^\sigma$ to a parameter space $\mathbb{C}^\tau$ of a deformation $\{f_t\}, t \in \mathbb{C}^r$, the pullback deformation $\{f_{\mu(s)}\}$, $s \in \mathbb{C}^\sigma$, defines a singular d-web on $\mathbb{C}^\sigma$. This does not require the transversality of μ to the leaves of the web $W_{\{f_t\}}$. We denote the pullback web by $\mu^* W_{\{f_t\}}$ or $W_{\mu^*\{f_t\}} = W_{\{f_{\mu(s)}\}}$ and call μ the *classifying map*. The pullback $\mu^* W_{\{f_t\}}$ has the natural first integral induced from $W_{\{f_t\}}$, which is tautologically defined assigning the critical values of $f_{\mu(s)}$ to $s \in \mathbb{C}^\sigma$ and lifting it to the skeleton in a natural manner.

Theorem 4.1. *A germ of a smooth codimension-one τ-web on $\mathbb{C}^\sigma$ with a smooth skeleton, a regular first integral, and a finite G_0-codimension is a pullback of a mini-versal τ-web on $\mathbb{C}^\tau$ via a map germ $\mu : \mathbb{C}^\sigma, o \to \mathbb{C}^\tau, o$.*

This theorem follows immediately from the next theorem and its corollary in the deformation theory.

Theorem 4.2. *Let $f \in \mathcal{O}_{\mathbb{C}^{n+1},o}$ be an isolated singularity and assume f has a finite G_0-codimension, or equivalently, $\dim_{\mathbb{C}} A$ is finite. Let $f_t, g_t, t \in \mathbb{C}^\tau$, be versal deformations of an equal $f = f_o = g_o$. Then there exist a germ of diffeomorphism χ of $\mathbb{C}^\tau, o$ and a family of diffeomorphisms ϕ_t of $\mathbb{C}^{n+1}, o$ leaving the first coordinate x_0 invariant such that*

$$\phi_t(\{g_t = 0\}) = \{f_{\chi(t)} = 0\}, \quad t \in \mathbb{C}^\tau.$$

In particular, $\chi^* W_{\{f_t\}} = W_{\{g_t\}}$.

The theorem states in particular the mini-versal web $W_{\{f_t\}}$ is determined by f_o up to diffeomorphism. The proof is routine in Thom-Mather theory, being based on first constructing the one-parameter family of versal deformations joining $\{f_t\}$ to $\{g_t\}$, and second the trivialization of the family. A detailed proof is found in [28] in a different terminology.

Corollary 4.1. *Let $f \in \mathcal{O}_{\mathbb{C}^{n+1},o}$ be an isolated singularity with a finite G_0-codimension, and let $\dim_{\mathbb{C}} A = \tau$. Let $f_t, t \in \mathbb{C}^\tau$, $g_s, s \in \mathbb{C}^\sigma$ be deformations of $f_o = g_o = f$. Assume $\{f_t\}$ is versal. Then there exists a map germ $\mu : \mathbb{C}^\sigma, o \to \mathbb{C}^\tau, o$ such that*

$$W_{\{g_s\}} = \mu^* W_{\{f_t\}}.$$

Proof. Let $\{g_{rs}\}$ be a versal deformation with $g_{os} = g_s$ with an additional parameter $r \in \mathbb{C}^\rho$, and put $f_{ut} = f_t$ for $u \in \mathbb{C}^{\rho+\sigma-\tau}$. Here we suppose $\rho + \sigma - \tau \geq 0$ choosing a large ρ. By Theorem 4.2, these versal deformation with an equal dimension of parameters $\{f_{ut}\}$, $\{g_{rs}\}$ are equivalent, thus there exists a diffeomorphism χ of $\mathbb{C}^{\rho+\sigma}, 0$ such that $W_{\{g_{rs}\}} = \chi^* W_{\{f_{ut}\}}$. Let $i : \mathbb{C}^\sigma \to \mathbb{C}^{\rho+\sigma}$ be the natural embedding and $\pi : \mathbb{C}^{\rho+\tau} \to \mathbb{C}^\tau$ be the natural projection. Then $W_{\{g_s\}} = i^* W_{\{g_{rs}\}}$ and $W_{\{f_{ut}\}} = \pi^* W_{\{f_t\}}$. Therefore $W_{\{g_s\}} = (\pi \circ \chi \circ i)^* W_{\{f_t\}}$. □

Example 1. *Let us consider the versal web $W_{\{f_t\}}$ constructed in §2. The graph of the critical values of x_0 on $f_{t_1t_2t_3} = x_1^2 + x_0^3 + t_1x_0^2 + t_2x_0 + t_3 = 0 \subset \mathbb{C}^2_{x_0x_1}$ is the set*

$$\{x_0^3 + t_1x_0^2 + t_2x_0 + t_3 = 0\} \subset \mathbb{C}^4_{x_0t_1t_2t_3}$$

and the versal web on $\mathbb{C}^3_{t_1t_2t_3}$ is given by the family of x_0-level sets. A 3-web $W_{\{g_{uv}\}}$ on the plane $\mathbb{C}^2_{uv}$ with $g_o = f_o = x_1^2 + x_0^3$ is of the form $\mu^* W_{\{f_t\}}$ induced from the versal web by Theorem 4.1. A generic embedding $\mu : \mathbb{C}^2_{uv} \to \mathbb{C}^3_{t_1t_2t_3}$ can be presented as $\mu(u,v) = (\alpha(u,v), u, v)$. Thus the critical-value-graph of the induced web $W_{\{g_{uv}\}} = \mu^* W_{\{f_t\}}$ is presented as

$$\mathbb{C} \xleftarrow{x_0} \{x_0^3 + \alpha(u,v)x_0^2 + ux_0 + v = 0\} \subset \mathbb{C}^3_{x_0uv} \xrightarrow{\pi} \mathbb{C}^2_{uv}.$$

Here the graph in the middle is smooth with coordinates (u, x_0); the first integral is fixed to x_0, while the second projection π varies as the embedding μ varies. If we normalize the projection π to the Whitney cusp map, we obtain the normal form of 3-webs with the cuspidal singular locus due to Carneiro [10] and Dufour [13]. Similarly, a generic 4-web on $\mathbb{C}^2_{uv}$ with $g_o = x_1^2 + x_0^4$ is given by

$$\mathbb{C} \xleftarrow{x_0} \{x_0^4 + \alpha(u,v)x_0^3 + \beta(u,v)x_0^2 + ux_0 + v = 0\} \subset \mathbb{C}^3_{x_0uv} \xrightarrow{\pi} \mathbb{C}^2_{uv}.$$

These α and β are function moduli of the web structures. This normal form conflicts with a classification result in [35], and waits for a better explanation.

By Theorem 4.2, a deformation of a versal web is trivial, i.e., equivalent to a trivial family of the versal web. In particular, a versal web is a cylinder of a mini-versal web. Thus the complement of the discriminant locus of a versal web possesses the $K(\pi, 1)$-property, where the fundamental group π is the braid subgroup of τ strings given by the τ critical values of the first integral x_0 on $V_t : f_t = 0$.

The following theorem is remarkable as it reduces the classification of versal webs to that of functions on varieties by G-equivalence, which is weaker than G_0-equivalence.

Theorem 4.3. *Let $f, g \in \mathcal{O}_{\mathbb{C}^{n+1},o}$ be G-equivalent. Then f, g have an equal G_0-codimension and their mini-versal webs $W_{\{f_t\}}$ and $W_{\{g_t\}}$ are diffeomorphic.*

Proof. Assume f, g are equivalent by a triple (ϕ, ψ, h): $h\phi^* f = g$. By a straight forward calculation,

$$\frac{\partial g}{\partial x_i} = \frac{\partial h\phi^* f}{\partial x_i} = \frac{\partial h}{\partial x_i}\phi^* f + h \sum_{j=1,\dots,n} \frac{\partial \phi_j}{\partial x_i} \phi^* \frac{\partial f}{\partial x_j}.$$

This states that ϕ^* sends the $\mathcal{O}_{\mathbb{C}^{n+1},o}$-submodule $< f > + J_0 f$ to $< g > + J_0 g$, where $< g > + J_0 g$ stands for the denominator of the quotient A for g. If $\{s_1, \dots, s_\tau\}$ is a $\mathbb{C}$-basis of the Jacobian quotient A of f, its pullback $\{\phi^* s_1, \dots, \phi^* s_\tau\}$ is a $\mathbb{C}$-basis of the quotient for g. Thus

$$g_t = g + \sum t_i \phi^* s_i = h\phi^* f + \sum t_i \phi^* s_i = \phi^*(h' f + \sum t_i s_i)$$

is a mini-versal deformation of g, where h' is a unit. If we write $f'_t = h' f + \sum t_i s_i$, then the solution $S_c \in W_{\{g_t\}}$ coincides with the solution $S_{\psi(c)} \in W_{\{f'_t\}}$. Thus $W_{\{g_t\}} = W_{\{f'_t\}}$. By a similar calculation, we have also $< h' f > + J_0 h' f = < f > + J_0 f$ as h' is a unit. Thus $\{f'_t\}$ is versal, and $\{f''_t = h' f + \sum t_i h' s_i\}$ is also versal as h' is a unit. Thus $W_{\{f'_t\}}$ and $W_{\{f''_t\}}$ are diffeomorphic by Theorem 4.2. As ${f''}_t = h' f_t$, we obtain $W_{\{{f''}_t\}} = W_{\{f_t\}}$. This completes the proof. □

A result of Matsuoka [26] asserts that functions on a variety are classified by associated homomorphisms of $\mathbb{C}$-algebras.

5. Affine Connection of 3-webs on the plane

A *complex first-order ODE of one valuable*, local in $p = dy/dx$, is

$$f(x, y, p) = 0 \tag{$*$}$$

where f is a germ of complex analytic function at a $(0, 0, p_0)$. In suitable coordinates one may assume $p_0 = 0$. The *skeleton* $S \in J^1(\mathbb{C}, \mathbb{C}) = \mathbb{C}_x \times \mathbb{C}_y \times \mathbb{C}_p$ is defined by $f = 0$. From now on we do not assume S to be smooth.

To obtain the solutions of $(*)$, we assume the equation is locally solved in p as

$$p = f_i(x, y)\ ,\ \ i = 1, \dots, d$$

on a domain nearby the origin with implicit functions f_i. The solutions of each explicit differential equation as above form a germ of curvilinear foliations, hence the entire family of solutions of $(*)$ form a configuration of d foliations, i.e., a d-web. We recall some classic results obtained by Cartan and Blaschke (for details, see e.g. [11, 7, 8, 12]).

One of the basic ideas to extract geometric invariants from a web is to extend the Bott connection (parallel translation of normal vectors along leaves) of its constituent foliations (if possible) to an equal affine connection ∇ of the xy-plane. For $d = 3$, such a connection exists and called the *Chern connection*. This connection is defined on the complement of the discriminant of the equation (in p), and extends meromorphically to the discriminant [4]. The singularity of the connection depends subtly on that of the equation in general. Hence one may expect to classify the equations in terms of connection. Indeed, the transverse sections of the mini-versal 3-webs $W_{\{f_t\}}$, $f_o = x_1^2 + x_0^3$, in §2 are classified by their curvature 2-forms [29].

To introduce such a common affine connection, let us consider

$$\omega_i = U_i\ (dy - f_i\ dx), \quad i = 1, \dots, d$$

with units $U_i \neq 0$. In the simplest non-trivial case $d = 3$, one may impose the *normalization condition*

$$\omega_1 + \omega_2 + \omega_3 = 0\ .$$

Then it is seen that there exists a unique θ such that

$$d \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \begin{bmatrix} \theta & 0 & 0 \\ 0 & \theta & 0 \\ 0 & 0 & \theta \end{bmatrix} \wedge \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} + (T = 0) \tag{$**$}$$

The i-th row of the equation

$$d\omega_i = \theta \wedge \omega_i$$

is just the integrability condition for ω_i. Omitting the i-th row for any i, the equation $(**)$ is regarded as the structure equation for an affine connection without torsion. By the above normalization condition, the resulting connection is independent of the choice of i and U_1, U_2, and U_3. The affine connection thus defined is called the *Chern connection* of the 3-web of ω_1, ω_2, and ω_3; it has connection form $\Theta = \theta I$ and curvature form

$$\Omega = d\Theta + \Theta \wedge \Theta = d\theta\ I$$

where I stands for the 2×2 identity matrix. The curvature form is independent of the choice of co-frame because it is a similarity matrix. The $d\theta$ is called the *web curvature* by Blaschke [8]. It is not difficult to see that if $\omega_1 = -f_x dx, \omega_2 = -f_y dy$ and $\omega_3 = df$, then

$$d\theta = \frac{\partial^2}{\partial x \partial y} \log \frac{f_x}{f_y} dx \wedge dy.$$

The skeleton S is locally identified with the xy-plane via the natural projection. The above method is generalized to define an affine connection on (the smooth part of) S, which is an extension of the Bott connection of the foliation given by the contact form for any $d \geq 3$.

The following proposition was proved by Lins Neto and the author [24].

Proposition 5.1. *Assume the natural projection of a germ of skeleton $f(x, y, p) = 0$ to xy-plane has multiplicity d. Then f is equivalent to the polynomial equation of degree d in p,*

$$p^d + B_2\ p^{d-2} + B_3 p^{d-3} + \cdots + B_d = 0$$

where B_i are germs of analytic functions of x, y.

This is seen simply by changing the coordinate y with a first integral of the mean slope equation $y' = -B_1/d$ for a polynomial $f(x, y, p) = p^d + B_1 p^{d-1} + \cdots + B_d$. The general case is reduced to the polynomial case by the Weierstrass preparation theorem.

For $d = 3$, our normal form is

$$p^3 + B\ p + C = 0. \tag{$***$}$$

Mignard[7] calculated the curvature form of 3-webs given by ODEs without this normalization and using computer produced a large formula. Henaut [6] provided insight into the web curvature form from the D-module theory. The following curvature form for the above normal form was presented by Lins Neto and the author in [24, 32].

Theorem 5.1. The web curvature form $d\theta$ of the normal form $(***)$ is

$$\frac{1}{6}\ (\log \Delta)_{xy}\ dx \wedge dy + d \left\{ \frac{(6BB_y C - 4B^2 C_y)\ dx + (6BC_x - 9B_x C)\ dy}{\Delta} \right\},$$

where $\Delta = 4B^3 + 27C^2$ is the discriminant of the cubic polynomial in p.

From the theorem we immediately obtain

Theorem 5.2. (Resonance Curve Theorem (Lins Neto, Nakai) [24]**)** Assume $d = 3$, $B = C = 0$ at the origin and the germ of discriminant Δ at the origin is diffeomorphic to the $(2, 3)$ cusp; assume also the skeleton is smooth and transverse to the canonical contact element $dy - p dx = 0$ in the first jet space. Then the curvature form vanishes on a union of two transverse non-singular curves passing through the origin; one is tangent to the discriminant at the origin and the other is transverse. The statement remains valid also in the real smooth case.

Proof. The first assumption implies that (B, C) is a local diffeomorphism of $\mathbb{C}^2$ and the second assumption tells $C_x \neq 0$ at the origin. By straightforward calculation one sees that the curvature multiplied by Δ^2 has a trivial linear part, the second-degree part is non-degenerate, and its zero splits into the tangent line of the cusp and another transverse line. □

In the real case, the resonance curve is also real; moreover, the theorem provides a law of positivity and negativity of the curvature in the sectorial areas between the cusp and the component of the resonance curve passing through inside. In the 3-web on the left in Fig 4, the curvature is negative and positive on the respective upper and lower sectorial domains inside the cusp separated by the resonance curve (see also [29] for the web curvature nearby singular locus).

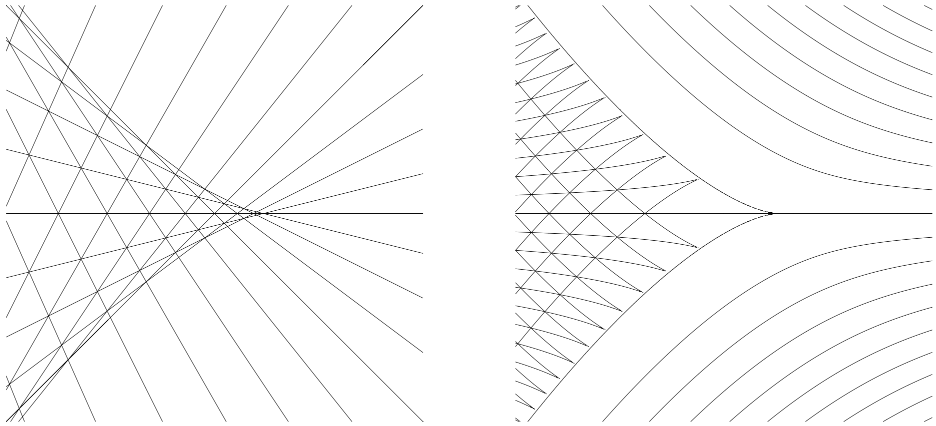


FIGURE 2. Left:Cusp-Clairaut 3-web: Exponent 1; Right:Rectangular 3-web: Exponent $\frac{1}{2}$

For $d \geq 4$, there is no canonical affine connection associated to a d-web, unless the cross ratio of the tangent lines of d leaves is constant. Henaut [18, 19, 20] defines a curvature from the viewpoint of Abelian relations. Recently Henaut proved his curvature form is the sum of web curvatures of all subtracted 3-subwebs in [20].

6. Flat Differential Equations

A 3-web is *hexagonal* if it is locally flat, in other words, the web curvature form vanishes identically. The next fact is classically known.

Theorem 6.1. (Linearization Theorem [1,2,4]) *A non-singular flat 3-web is locally diffeomorphic to the linear 3-web defined by*

$$dx\ ,\ dy\ ,\ -(dx+dy).$$

From the intuitive geometric point of view, it is interesting to classify singular hexagonal 3-webs on the plane. The following theorem was announced by Lins Neto and the author in [31, 32] without proof, which has now been given by Agafonov [1].

Theorem 6.2. *Assume the solution web of the local first-order ODE* $(*)$ *in* §5 *is a hexagonal 3-web and the discriminant locus in* p *is diffeomorphic to the* $(2,3)$*-cusp. Then the equation* $(*)$ *is equivalent to one of the following two equations by transformation of the* xy*-plane.*

(Cusp-Clairaut) $$p^3 + xp - y = 0,$$

(Rectangular) $$p^3 + \frac{1}{4}xp + \frac{1}{8}y = 0.$$

These equations have smooth skeletons and their projections onto the xy-plane are the Whitney cusp map. The portraits of the solution hexagonal 3-webs of these equations are drawn in Figure 2. The reader may appreciate the affine (linear) structure on the complement of the discriminant. Curiously these hexagonal webs appear in geometric optics: The figure on the left is apparently the most symmetric 3-web by ray lines tangent to the cuspidal caustics, and the figure on the right is give by the contour lines of the differences in phases (critical values) $d_1 - d_2$, $d_2 - d_3$, and $d_3 - d_1$ in the Pearcey web in §10, where d_1, d_2, and d_3 are the critical values of the potential function $\frac{1}{4}p^4 + \frac{1}{2}xp^2 + yp$.

Agafonov [2] explains these singular hexagonal 3-webs from the viewpoint of Frobenius manifolds.

7. Klein-Halphen webs and Fermat webs

In this section we introduce some other singular hexagonal 3-webs on the plane. The *coordinate 3-web* on an analytic surface $V \subset \mathbb{C}^3$ is defined by the coordinate 1-forms dx, dy, dz. The coordinate web is hexagonal on the generalized Brieskorn variety

$$V_{\alpha,\beta,\gamma} : X^\alpha + Y^\beta + Z^\gamma = 0 \ , \quad \alpha, \beta, \gamma \in \mathbb{Q}^*$$

as $d\theta = \frac{\partial^2}{\partial x \partial y} \log \frac{z_x}{z_y} dx \wedge dy$ vanishes identically on the variety. Clearly a pullback of this coordinate 3-web by any non-degenerate map germ $\phi : \mathbb{C}^2, o \to V_{\alpha,\beta,\gamma}, o$ is hexagonal, but it is highly singular at the preimage of the origin in general.

For positive integers α, β, γ, Halphen [16], Klein [22] and Lins Neto [23] proved that the germ $V_{\alpha,\beta,\gamma}, o$ admits a finite-to-one dominating map germ ϕ from $\mathbb{C}^2, o$ if and only if

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} > 1 \ .$$

Moreover, for such a triple (α, β, γ) with $\alpha, \beta, \gamma \geq 2$, i.e., one of $(2, 2, n)$, $(2, 3, 3)$, $(2, 3, 4)$, $(2, 3, 5)$, the variety is a quotient of $\mathbb{C}^2$ by a certain subgroup of the binary polyhedral group in $SU(2)$ corresponding to the triple, i.e., a cyclic group of order n, binary Klein quartic group, binary alternating group of order 24 or the whole binary octahedral group, respectively. Thus the hexagonal coordinate 3-web on the variety lifts to a singular hexagonal 3-web on $\mathbb{C}^2$ with symmetry of the subgroup (see c.f. [22] for the explicit form of the quotient map). It is seen that the web thus constructed possesses also the natural $\mathbb{C}^*$-symmetry and moreover the symmetry of the whole binary polyhedral group corresponding to the triple. The complete quotients of $\mathbb{C}^2$ by these binary polyhedral groups are known as *duVal singularities* of type D_n, E_6, E_7, and E_8. Therefore every duVal singularity of type ADE, including type A_n, admits a hexagonal singular web structure, which carries affine structure off the singular locus. (The lift of the coordinate 3-web on $V_{2,3,3}$ possesses also the symmetry of binary octahedral group if we allow transposition of the constituent coordinate foliations of the web. Therefore the E_7 singularity admits two distinct singular affine structures induced from $V_{2,3,3}$ and $V_{2,3,4}$.) We call these webs on the source $\mathbb{C}^2$ lifted from $V_{\alpha,\beta,\gamma}$, *Klein-Halphen* webs. This construction may be generalized to certain other values of α, β, γ.

Let us consider the generalized homogeneous *Fermat surface*

$$V_\alpha : X^\alpha + Y^\alpha + Z^\alpha = 0 \ ,$$

where $\alpha \neq 0$ is a rational number. This is the only symmetric germ of an analytic subspace, up to diffeomorphisms of Cartesian form $\phi(X) \times \phi(Y) \times \phi(Z)$, such that the coordinate 3-web is hexagonal. The coordinate web on V_α induces a hexagonal web W_α on its S_3-quotient $\tilde{V}_\alpha$. The coordinate functions X, Y, Z induce a 3-valued function on the quotient, which is the "defining level function" of the quotient 3-web. The natural $\mathbb{C}^*$-action on V_α induces that on $\tilde{V}_\alpha$, and then $\tilde{V}_{p/q}$ is the $\mathbb{Z}_q$-quotient of $\tilde{V}_p$, and the $\mathbb{Z}_p$-quotient of $\tilde{V}_p$ is $\tilde{V}_1$.

For an exponent α in a certain class, there exists a finite covering map $P : \mathbb{C}^2, o \to \tilde{V}_\alpha, o$ branched at o. Among the positive integers, such a covering map exists only for $\alpha = 1, 2, 3, 4$ and 5: $\tilde{V}_4$, $\tilde{V}_5$ are diffeomorphic to the A_1 singularity and A_4 singularity, respectively, which are the quotients of $\mathbb{C}^2$ by cyclic groups of orders 2 and 5. For such an α (not necessarily a positive integer in general) the quotient coordinate 3-web on $\tilde{V}_\alpha, o$ lifts to a hexagonal 3-web on the source $\mathbb{C}^2$, and its first integral is induced from the coordinate functions X, Y, Z. We call the web thus constructed *Fermat web* if it exists, and denote by W_α.

Here we present the first-order differential equations defining the Fermat webs for certain exponents α:

E_1 **:** $p^3 + xp - y = 0$ (Cusp-Clairaut),
$E_{\frac{1}{2}}$ **:** $p^3 + 4xyp - 8y^2 = 0,$
$E_{\frac{1}{3}}$ **:** $27x^2p^3 + 3xp - y = 0,$
$E_{\frac{1}{6}}$ **:** $27x^2p^3 + 12xyp - 8y^2 = 0,$

where $p = dy/dx$. The portrait of $W_{\frac{1}{2}}$ is the figure on the left in Figure 4. The equations thus obtained are not local in p in general, as is seen in the small list above, and do not fall within the classification scheme given by Theorem 3.1.

The equation E_1 has the natural $\mathbb{C}^*$-symmetry induced from that on the plane

$$V_1 : \Sigma_1 = X + Y + Z = 0$$

in XYZ-space. For instance, the action by -1 induces the involution $(x, y) \to (x, -y)$ on the xy-plane, $x = \Sigma_2 = XY + YZ + ZX, y = \Sigma_3 = XYZ$, and the action of the cubic root of unity ω gives the symmetry of order 3, $(x, y) \to (\omega^{-1}x, y)$. The quotients of E_1 by these symmetries of order 2, 3, and their generating group of order 6 are respectively $E_{\frac{1}{2}}, E_{\frac{1}{3}}$ and $E_{\frac{1}{6}}$.

For $\alpha = -1$, we obtain the following Clairaut equation

E_{-1} **:** $p^3 + x^2p^2 - 2xyp + y^2 = 0,$

where $x = \Sigma_1, y = \Sigma_3$ and $p = dy/dx$. The explicit form of the differential equations for the other α can be calculated by computer but they are too big and unsuitable to present here. The other cases will be published elsewhere.

8. Abelian relation of first integral and exponent

Assume the Fermat web W_α exists for a rational α. By construction, the coordinates X, Y, Z on the Fermat surface V_α induce a single valued first integral λ on the skeleton of E_α, and it enjoys the relation

$$\text{Trace } \lambda^\alpha = \sum \lambda^\alpha = 0$$

on the xy-plane, where the sum in the middle is taken over the fiber of projection of the skeleton onto the xy-plane choosing suitable branches of λ^α. For instance, $\lambda = p$ for the equation E_1 in the previous section, and the relation Trace $p = 0$ is apparent by the presentation of the equation. We call the relation an *Abelian relation* of the first integral λ.

Consider a germ of 3-web at the origin in the xy-plane defined by a first-order ODE $f(x, y, p) = 0$ as in §5, which is not necessarily local in p. The Trace λ^α is well defined in a similar manner to the above on a punctured neighborhood of the origin for a first integral λ on the skeleton $S : f = 0$. Then the Abelian relation Trace $\lambda^\alpha = 0$ implies the web is hexagonal.

Assume another Abelian relation Trace $\mu^\beta = 0$ holds. As the space of Abelian relations for triples of defining 1-forms of a germ of non-singular plane 3-web is of dimension at most 1 (see [7, 8, 20] for the detail), it follows $d\lambda^\alpha = c\, d\mu^\beta$ on S with a constant $c \neq 0$, from which $\lambda^\alpha = c\mu^\beta$. Assume S is smooth at a lift $\tilde{C}$ of a solution $C \subset \mathbb{C}^2$ containing the origin in its closure, and λ, μ vanishe on $\tilde{C}$ in order 1. Then comparing the orders of vanishing of both sides of $\lambda^\alpha = c\mu^\beta$ at $\tilde{C}$, we obtain $\alpha = \beta$. Moreover a first integral with an Abelian relation of exponent α is unique up to multiplication by a constant. The exponent α is uniquely determined by the 3-web if there exists only one solution C with the above property. We call α the *exponent* of the hexagonal 3-web with first integerl. (Remark that the first integral λ^n fulfills the Abelian relation of exponent

α/n for any positive integer n, but it vanishes at $\tilde{C}$ in order n. We define the exponent to be ∞ if the Abelian relation does not exist.)

Theorem 8.1. (Universality of Fermat web) *Assume a germ of a hexagonal 3-web on the plane admits an irreducible holomorphic first integral λ with an Abelian relation of exponent α as above, and assume the Fermat web W_α exists. Assume also $\lambda \neq 0$ at a point on each fiber of the projection of skeleton over $(x, y) \neq o$ near the origin o. Then the web and the first integral λ are induced from the Fermat web W_α via a holomorphic map of the plane.*

Proof. Let $\Delta \subset \mathbb{C}^2$ denote the subset of those points where the projection of $\pi : S \to \mathbb{C}^2$ are not regular at the fiber of π or λ is not regular at the fiber. Define $\mu : \mathbb{C}^2 \setminus \Delta \to \mathbb{C}^3$ by $\mu = (\Sigma_1, \Sigma_2, \Sigma_3)$ with the symmetric polynomials Σ_i of degree i of the values of λ at the fiber of π. It is bounded on a neighborhood of the origin, so extends holomorphically to a map of $\mathbb{C}^2$ to the quotient Fermat surface $\tilde{V}_\alpha$, and the web W_S is the pullback of the quotient web on $\tilde{V}_\alpha$. By assumption, the extension μ has the fiber $\mu^{-1}(o) = o \in \mathbb{C}^2$. Therefore the pullback of the branched covering $P : \mathbb{C}^2 \to V_\alpha$ by μ decomposes into a union of non-singular surfaces meeting at the origin. Let s be a section of the pullback, $\tilde{\mu}'$ the natural bundle map covering μ, and set $\tilde{\mu} = \tilde{\mu}' \circ s$. Then $P \circ \tilde{\mu} = \mu$ and $W_S = \tilde{\mu}^* W_\alpha$. By the definition of μ, the lift $\tilde{\mu}$ respects the first integrals of the leaves passing through those points in the source and target. Therefore λ is induced from the first integral of W_α. $\square$

For instance, Rectangular web in Figure 2 has exponent $\frac{1}{2}$ as is seen by the argument for Dual-Cusp-Clairaut web in the end of the next §9, thus it is a pullback of the Fermat web $W_{\frac{1}{2}}$. We remark that Klein-Halphen webs may possess a similar universal property.

In general, a singular d-web admits at most $(d-1)(d-2)/2$ linearly independent Abelian relations of first integrals by the virtue of the theory of Abelian relation of integrable forms (see c.f. [20]). The following Clairaut equation defines a 4-web, and the first integral $p = dy/dx$ enjoys the maximal number (i.e. 3) of Abelian relations with the various exponents

$$p^4 + xp - y = 0, \quad \text{Trace } p = 0, \ \text{Trace } p^2 = 0, \ \text{Trace } p^5 = 0.$$

This suggests generalizing our construction of Fermat webs to Brieskorn varieties of higher codimensions.

9. Dual 3-web

The hexagonal 3-web structure can be also produced by rotation of leaves as follows. The *dual* L^* of a configuration $L = L_1 \cup L_2 \cup L_3$ of lines in the plane passing through the origin is the unique 3-line configuration (different from L) invariant under the linear symmetry group of L. The *dual 3-web* W^* of a 3-web W is defined by integrating the dual 3-line configuration of the tangent 3-line fields of W.

Theorem 9.1. *The bi-duality holds: $W^{**} = W$, and W, W^* share the same Chern connection.*

Corollary 9.1. *A 3-web W is flat if and only if its dual W^* is flat.*

The dual equations of (Cusp-Clairaut), (Rectangular) in Theorem 6.2 are respectively

$$\text{(Dual-Cusp-Clairaut)} \qquad y'^3 + \frac{2x^2}{3y}\, y'^2 - x\, y' + \frac{2x^3 + 27y^2}{27y} = 0,$$

$$\text{(Dual-Rectangular)} \qquad y'^3 - \frac{x^2}{3y}\, y'^2 - \frac{x}{4}\, y' - \frac{2x^3 + 27y^2}{216y} = 0.$$

These equations have the exponents $\frac{1}{2}$ and 1 respectively. The portraits of the solution webs of these equations are drawn in Figure 3. Dual-Cusp-Clairaut web is the S_3-quotient of the plane

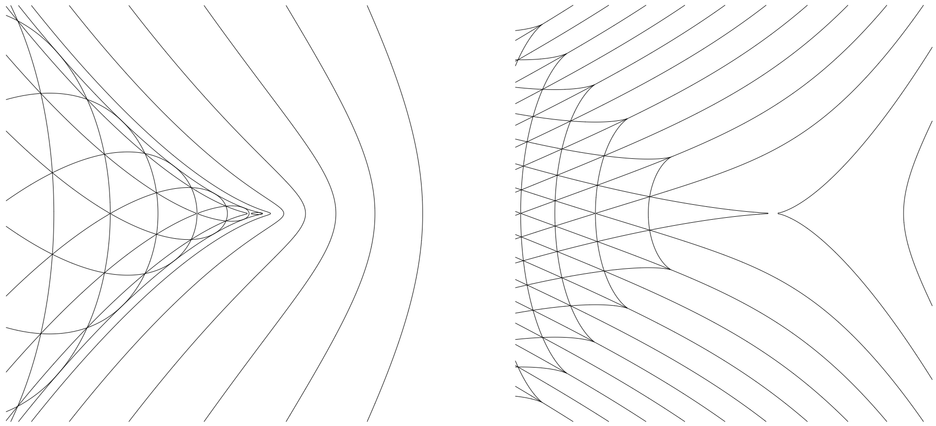


FIGURE 3. Left: Dual-Cusp-Clairaut web: Exponent $\frac{1}{2}$; Right: Dual-Rectangular web: Exponent 1

$\alpha+\beta+\gamma=0$ in the $\alpha\beta\gamma$-space foliated by the level lines of $\alpha-\beta$, and its skeleton is diffeomorphic to the quotient of the plane by the involution transposing α and β. Thus $\lambda=(\alpha-\beta)^2$, well defined on the skeleton, is a first integral of the equation. The Abelian relation of λ of exponent $\frac{1}{2}$ follows from the obvious relation $(\alpha-\beta)+(\beta-\gamma)+(\gamma-\alpha)=0$. Fermat web $W_{\frac{1}{2}}$ is also interesting because it is self-dual, i.e., identical to its dual 3-web. Generalization of the duality to flat webs in higher dimensions would be interesting.

10. Pearcey web and the stationary phase method

Let us consider the Pearcey integral

$$\int_{-\infty}^{\infty} \exp\ \sqrt{-1}\{\frac{1}{4}x^4+\frac{1}{2}t_1x^2+t_2x+h(t_1,t_2)\}\ dx.$$

In the theory of geometric optics, the potential function is supposed to be the distance from a point in the observation plane $(t_1,t_2)\in\mathbb{C}^2$ to a wave front [37]. Huygens principle suggests the intensity of light at the (t_1,t_2) near caustics is well approximated by the absolute value of the integral[6, 37, 33]. This model is associated with the generating function

$$f_t(x_0,x_1)=\frac{1}{4}x_1^4+\frac{1}{2}t_1x_1^2+t_2x_1+h(t_1,t_2)-x_0, \quad t=(t_1,t_2)\in\mathbb{C}^2,$$

which is non-versal in the manner in §1, 4 and embedded into the versal family with an additional parameter

$$F_T(x_0,x_1)=\frac{1}{4}x_1^4+\frac{1}{2}t_1x_1^2+t_2x_1+t_3-x_0, \quad T=(t_1,t_2,t_3)\in\mathbb{C}^3.$$

The stationary phase method reveals that the integral is approximated by the functional value of the integrand at the critical points of the potential function in the exponent. This leads us to the geometry of the family of wavefronts: Phase = constant, which is just our 3-web $W_{\{f_t\}}$ introduced in §1 and illustrated on the right in Figure 4. We call this web of wave fronts, *Pearcey 3-web*.

Clearly the intensity is independent of the constant term h. The contour map of the intensity is given on the left in Figure 5. In the paper of Pearcey [33], the contour map of the phase is also given for the h identically 0. On the right in Figure 5, we present it with a h suitable for the natural wave propagation. The skeleton is smooth if $h_{t_1}(0,0) \neq 0$. It is interesting to compare the figures on the right in Figures 2, 3, and the Pearcey 3-web in Figure 4 with the contour maps of the respective intensity and phase of the Pearcey integral in Figure 5.

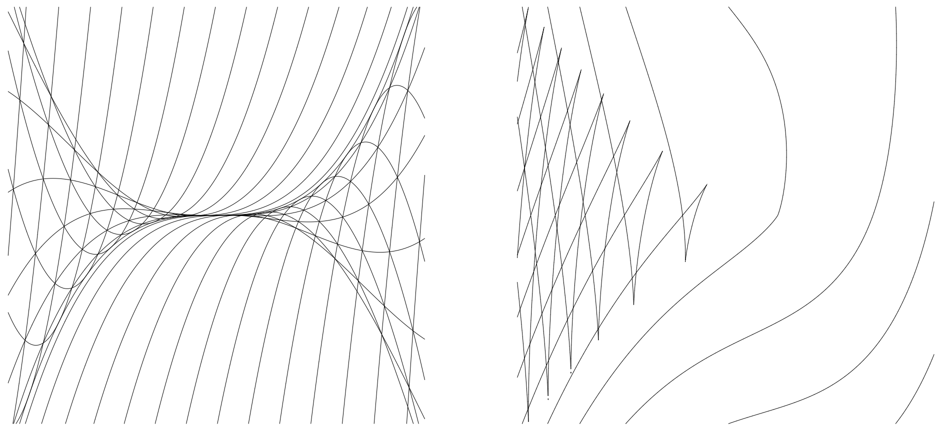


FIGURE 4. Left: Fermat web $W_{\frac{1}{2}}$; Right: non-hexagonal Pearcey 3-web

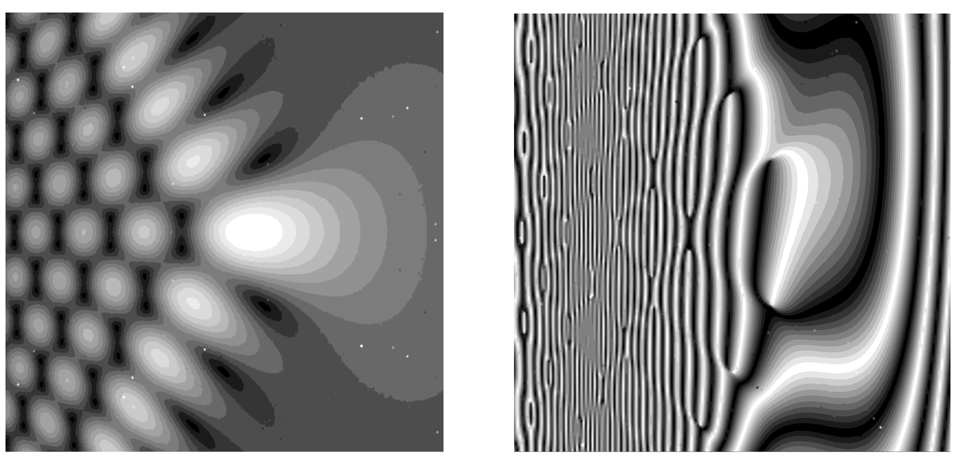


FIGURE 5. Contour maps of the intensity (above left) and phase (above right). In the figure on the left, the black dots represent zeros of the integral (small white dots are computational bugs), whereas in the right figure the black dots represent the centers of whirlpools.

According to Theorem 5.2 (Resonance Curve Theorem), there exist two smooth curves passing through the cusp point, on an infinitesimally small neighborhood of which the 3-web structure is flat. Moreover 3-phases of the wave resonate at some discrete points on those curves. If we suppose the wave in the last figure is propagating from right to left, some “trajectories” seem to be trapped in whirlpools inside the cusp.

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FORMAL AND ANALYTIC NORMAL FORMS OF GERMS OF HOLOMORPHIC NONDICRITIC FOLIATIONS

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To Xavier Gómez-Mont:
source of enthusiasm and creativity!

Abstract. We consider the class $\mathcal{V}_n$ of germs of holomorphic vector fields in $(\mathbb{C}^2, 0)$ with vanishing $(n-1)$-jet at the origin, $n > 1$. For generic germs $\mathbf{v} \in \mathcal{V}_2$ we prove the existence of an analytical orbital normal form whose orbital formal normal form has the form $\mathbf{v}_{c,b}$ given in [ORV4]. Furthermore, fixing one representative $\hat{\mathbf{v}}$ of the analytic class of a germ $\mathbf{v} \in \mathcal{V}_2$ having the y-axis invariant, the corresponding formal normal form $\hat{\mathbf{v}}_{c,\hat{b}}$ is analytic and unique (under strict orbital equivalence). Moreover for generic $\mathbf{v} \in \mathcal{V}_n$, $n \geq 2$ we give a preliminary orbital analytic normal form which is polynomial and of degree at most n in the y-variable.

1. Introduction

The problem of the formal and analytic classification of germs of holomorphic vector fields goes back to Poincaré. He proved that, in the generic situation, such classification relies on the eigenvalues of the linear part of the vector field at the singular point. In such cases, the formal and analytic classification coincides. As it is well-known (see [IY], [ORV3]) the failure of the generic assumptions on the eigenvalues of the linear part of the vector field leads either to simply formal normal forms and complicated analytic ones (and therefore the non coincidence of the formal and analytic classification) or to highly complicated formal and analytic normal forms. In this last situation the formal and analytic classification coincides again: the rigidity phenomena takes place (see [Ce,Mo], [EISV], [M], [Lo1], [Lo2]).

In more complicated situations, when the linear part of the vector field at the singular point is zero (i.e. for degenerated germs of vector fields), the rigidity phenomena takes place again for generic dicritic and nondicritic germs (see [ORV1] and [Vo1] for the classical and orbital rigidity-respectively- of nondicritic germs; [ORV2] for the classical and orbital rigidity of generic dicritic germs of vector fields and [Ca] for orbital rigidity of dicritic germs with higher degeneracies).

In such cases the formal orbital normal form was obtained and Thom's problem on the minimal invariants of the orbital analytic classification of generic dicritic and nondicritic degenerated germs of vector fields was solved (see [ORV2] and [ORV4]). In those works rather simple formal orbital normal forms were obtained and the analytic classification relied in a combination of a finite number of parameters, together with formal invariants related to geometric objects (involutions and separatrices respectively).

The problem on the analyticity of the formal orbital normal forms was solved for generic dicritic germs in [ORV3]. However, the analyticity of the formal orbital normal form for nondicritic generic germs of vector fields given in [ORV4] was still open. In this work we prove the

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analyticity of such normal forms for the generic case: that is, when the formal orbital normal form has quadratic principal part.

For higher degeneracies we give a preliminary orbital analytic normal form (polynomial in the y variable) which does not coincide with the formal orbital normal form given in [ORV4]. We stress that for higher degeneracies one can expect a non coincidence between the formal analytic normal forms. A similar behavior was already observed in the classification of the analytic germs of vector fields with non generic linear part.

As we did in the dicritic case (see [ORV3]), we use surgery of manifolds and Savelev's Theorem for the proof of Theorem 2.1. These ideas were firstly introduced by F.Loray in [Lo2] and [Lo3] for germs at $(\mathbb{C}^2, 0)$ of holomorphic vector fields having a non generic linear term (nilpotent or saddle-node) at the origin.

2. Basic notations.

2.1. Notations.

(1) Let $\mathcal{V}_n$ be the class of holomorphic germs of vector fields in $(\mathbb{C}^2, 0)$ with isolated singularity at the origin, with vanishing $(n-1)$-jet at the origin and non vanishing n-jet, $n \geq 2$.

(2) Given $\mathbf{v} \in \mathcal{V}_n$, we denote by $\mathcal{F}_{\mathbf{v}}$ the germ of foliation generated by $\mathbf{v}$.

(3) Two germs $\mathbf{v}$ and $\mathbf{w}$ in $\mathcal{V}_n$ are *analytically (formally) orbitally equivalent* if there exist an analytic (formal) change of coordinates $H : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ and an analytic function (formal series) $K : (\mathbb{C}^2, 0) \to \mathbb{C}^*$, $(K(0) \neq 0)$ such that $H_*\mathbf{v} = K \cdot \mathbf{w}$, where

$$H_*\mathbf{v}(p) = D_z H\mathbf{v}(z)|_{z=H^{-1}(p)}.$$

(4) The foliations $\mathcal{F}_{\mathbf{v}}, \mathcal{F}_{\mathbf{w}}$ generated by the germs of vector fields $\mathbf{v}, \mathbf{w} \in \mathcal{V}_n$, respectively, are called analytically (formally) equivalent if their corresponding vector fields $\mathbf{v}, \mathbf{w}$ are analytically (formally) orbitally equivalent.

In other words, in the analytic case, if $l_{\mathbf{v},(x,y)} :=$ denotes the leaf through (x, y) of the foliation $\mathcal{F}_{\mathbf{v}}$ then $l_{\mathbf{w},H(x,y)} = H(l_{\mathbf{v},(x,y)})$.

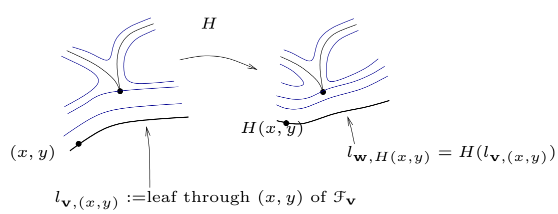

Figure 2.1. analytic equivalence of $\mathcal{F}_{\mathbf{v}}$ and $\mathcal{F}_{\mathbf{w}}$

(5) If the linear part of the germ H is the identity and $K(0) = 1$, we say that the vector fields $\mathbf{v}, \mathbf{w}$ are *strictly* analytically (formally) orbitally equivalent or the foliations $\mathcal{F}_{\mathbf{v}}$, $\mathcal{F}_{\mathbf{w}}$ are *strictly* analytically (formally) equivalent.

(6) In the case when $K \equiv 1$ then the vector fields $\mathbf{v}$, $\mathbf{w}$ are analytically (formally) equivalent.

(7) Let $\mathbf{v} \in \mathcal{V}_n$

$$\mathbf{v} = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y}, \quad P = \sum_{k=n}^{\infty} P_k, \ Q = \sum_{k=n}^{\infty} Q_k \ , \tag{2.1}$$

where P and Q are holomorphic functions and P_k, Q_k are homogeneous polynomials in (x,y) of degree k, $k \geq n$, corresponding to the terms of order k of its Taylor expansion at the origin. Let $R_{n+1} := xQ_n - yP_n$.

(8) We say that the germ of vector field $\mathbf{v}$ is *nondicritic* if

$$R_{n+1} \not\equiv 0 \tag{2.2}$$

and if $R_{m+1} \equiv 0$ we say that the germ of vector field $\mathbf{v}$ is *dicritic*.

Remark 2.1. The condition of nondicriticity is generic in $\mathcal{V}_n$ (it is given by the open condition (2.2)) and has finite codimension in the space $\mathcal{V}_n$. On the contrary the dicritic case is nongeneric in $\mathcal{V}_n$. In this work, unless otherwise stated, one will assume the nondicriticity condition (2.2)

2.2. **Main statements and genericity assumptions.** We state the main results of this work. We begin with the genericity assumptions for the first two theorems:

We say that a holomorphic nondicritic germ of vector field $\mathbf{v} \in \mathcal{V}_n$ of the form (2.1) is *generic nondicritic* if it satisfies the following *genericity assumptions*:

- G1. The homogeneous polynomial $R_{n+1} = xQ_n - yP_n$ is of degree $n+1$ and has only simple factors,
- G2. All the characteristic exponents at the singular points of the blown-up foliations are not zero or positive rational.
- G3. At least at one singular point denoted by p_∞ the blown-up foliation $\tilde{\mathcal{F}}_{\mathbf{v}}$ is generated by a non degenerated vector field holomorphically linearizable and its characteristic exponent λ_∞ is different from -1. This implies that in appropriate coordinates the foliation $\tilde{\mathcal{F}}_{\mathbf{v}}$ at the point p_∞ is locally generated by a linear vector field and the quotient of the corresponding eigenvalues is different from -1.

The main goal of this work is to prove under the genericity assumptions G1,G2,G3 the following theorems:

Theorem 2.1. *(Semi polynomial analytic normal form) Each generic nondicritic germ in $\mathcal{V}_n$, $n \geq 2$, is analytically orbitally equivalent to a germ of vector field of the form*

$$\mathbf{v}_{\mathcal{P},\mathcal{Q}}(x,y) = \quad x\mathcal{P}(x,y)\tfrac{\partial}{\partial x} + y\mathcal{Q}(x,y)\tfrac{\partial}{\partial y}, \tag{2.3}$$

with nondicritic singularity at the origin and $\mathcal{P}, \mathcal{Q}$ polynomials of degree at most $n-1$ in the "y" variable with analytic (on x) coefficients.

Theorem 2.2. *(Semipolynomial analytic normal form for $n = 2$) Any generic nondicritic germ of $\mathcal{V}_2$, is analytically orbitally equivalent to a germ of vector field of the form*

$$\mathbf{v}_{an} = x(\mathcal{P}_1 + x^2\beta(x))\frac{\partial}{\partial x} + y(\mathcal{Q}_1 + x^2\beta(x))\frac{\partial}{\partial y} \tag{2.4}$$

where $\mathcal{P}_1(x,y) = ya_0 + b_1x$, $\mathcal{Q}_1(x,y) = y + d_1x$ are homogeneous polynomials of degree 1, and $\beta(x)$ is a holomorphic function in a neighborhood of the origin. For fixed principal part $x\mathcal{P}_1\frac{\partial}{\partial x} + y\mathcal{Q}_1\frac{\partial}{\partial y}$, the function β is unique (and therefore $\mathbf{v}_{an}$) under strict analytic orbital equivalence.

We stress that any nondicritic generic germ $\mathbf{v} \in \mathcal{V}_n$ can be reduced under, rotation and rectification of one of its separatrix, to a germ

$$P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y}\ , \qquad P(0,y) \equiv 0 \tag{2.5}$$

Denote by $\mathcal{V}_n^0$ the class in $\mathcal{V}_n$ of germs satisfying (2.5). Hence, the problem of classification of generic foliations generated by germs in $\mathcal{V}_n$ is reduced to the equivalent one of the classification of generic foliations generated by germs in $\mathcal{V}_n^0$.

We stress that **strict** formal (and analytic) orbital equivalent germs in $\mathcal{V}_n^0$ have the same n-jet at the origin. Therefore the problem of strict formal (and analytic) orbital classification of germs in $\mathcal{V}_n^0$ is transformed to the analogous one in each class

$$\mathcal{V}(\mathbf{v}_0) = \left\{ \mathbf{v} \in \mathcal{V}_n^0 \ : \ j_0^n(\mathbf{v} - \mathbf{v}_0) = 0 \right\}, \tag{2.6}$$

where $\mathbf{v}_0 := P_n \frac{\partial}{\partial x} + Q_n \frac{\partial}{\partial y}$ is called *the principal part* of $\mathbf{v}$ and P_n, Q_n are homogeneous polynomials of degree n, $P_n(0, y) \equiv 0$. Note that in this case the blow-up $\tilde{\mathbf{v}}$ of $\mathbf{v}$ has a singular point p_∞ at infinity, i.e., at $v = 0, y = 0$, where $v = y/x$.

For generic germs (see Remark 2.2) the solutions to the formal orbital classification problem was given in [ORV4]:

Theorem *(on the formal classification of nondicritic vector fields* [ORV4]*) Each generic holomorphic nondicritic germ $\mathbf{v} \in \mathcal{V}_n$, $n > 1$ is formally orbitally equivalent to a formal series $\mathbf{v}_{c,b}$ of the form*

$$\mathbf{v}_{c,b} = \mathbf{v}_0 + \mathbf{v}_c + \mathbf{v}_b, \tag{2.7}$$

where

(1) $\mathbf{v}_0 := P_n \frac{\partial}{\partial x} + Q_n \frac{\partial}{\partial y}$, *$P_n, Q_n$ are homogeneous polynomials of degree n, and $P_n(0, y) \equiv 0$ is a generic principal part.*
(2) $\mathbf{v}_c = -(\mathcal{H}_c)'_y \frac{\partial}{\partial x} + (\mathcal{H}_c)'_x \frac{\partial}{\partial y}$ *is a Hamiltonian vector field with polynomial Hamiltonian*

$$\mathcal{H}_c(x, y) = \sum c_{i,j} x^i y^j, \ 0 \le i \le n-1, \ 0 \le j \le n-1, \ i + j \ge n+2, \tag{2.8}$$

(3) *and $\mathbf{v}_b = b(x, y)(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y})$ is a radial vector field such that $b(x, y) = \sum_{k=0}^{n-2} b_k(x) y^k x^{n-k}$ is a polynomial on the y variable of degree less or equal to $n-2$ whose coefficients, $b_k(x)$, are formal series on x.*

Moreover any two formal series of the form (2.7) that are formally orbitally equivalent to $\mathbf{v}$ and with the same generic principal part $\mathbf{v}_0$, coincide.

Remark 2.2. The genericity assumptions in this theorem are slightly different:

- $\tilde{G}1$. We ask the principal part $\mathbf{v}_0$ to be such that its blow-up has simple singular points (i.e., the homogeneous polynomial $R_{n+1}(x, y) = x\tilde{R}(x, y)$ of degree $n+1$ has only simple factors, and therefore, in this case, $R_{n+1}(1, u)$ has n simple roots u_j, $j = 1, \ldots, n$, the point at infinity p_∞ is also simple)
- $\tilde{G}2$. All the characteristic exponents corresponding to the singular points are not rational numbers.
- $\tilde{G}3$. Within the proof of Theorem 2.1 we ask that for any $k = 2, \ldots, n+1$, a determinant of $2k+2$ equations to be different from zero (this determinant is a non trivial polynomial on the coefficients of the principal part $\mathbf{v}_0$).

We stress the relevance of $\mathbf{v}_c$ in (2.7): For $\mathbf{v} \in \mathcal{V}(\mathbf{v}_0)$ satisfying the previous genericity assumptions and having nonsolvable projective monodromy group $G_\mathbf{v}$, the tuple $\tau_\mathbf{v} = (\mathbf{v}_c, [G_\mathbf{v}])$ is Thom's invariant on the analytic classification under strict orbital equivalence, where $[G_\mathbf{v}]$ is the class of strict analytic conjugacy of the projective monodromy group $G_\mathbf{v}$ (see [ORV4]).

Remark 2.3. For $n = 2$ the "Hamiltonian" part $\mathbf{v}_c$ in (2.7) is zero. Hence, the strict formal orbital normal form $\mathbf{v}_f := \mathbf{v}_{c,b}$ takes the form:

$$\mathbf{v}_f = (P_2 + x^3 B)\frac{\partial}{\partial x} + (Q_2 + yx^2 B)\frac{\partial}{\partial y} \tag{2.9}$$

where $\mathbf{v}_0 = P_2\frac{\partial}{\partial x} + Q_2\frac{\partial}{\partial y}$, P_2, Q_2 are homogeneous polynomials of degree 2, $P_2(0, y) = 0$, $deg_y Q_2 = 2$, and $B(x) = \sum_{k=0}^{\infty} b_k(x)\, x^k$ is a formal power series.

Therefore $\mathbf{v}_f$ in (2.9) is the strict orbital formal normal form for generic nondicritic vector fields in $\mathcal{V}(\mathbf{v}_0)$. As we state in the next theorem (2.9) is, as well, the orbital strict analytic normal form for generic nondicritic vector fields in $\mathcal{V}(\mathbf{v}_0)$.

As an immediate consequence of Theorem 2.2 and considering the generic assumptions G1, $\tilde{\text{G}}$1, G3, and $\tilde{\text{G}}$3, we have the following:

Theorem 2.3. *(Analyticity of the formal normal form for $n = 2$, $\mathbf{v}_f$) For any generic nondicritic germ in $\mathcal{V}_2$, its strict formal orbital normal form $\mathbf{v}_f$ is analytic. Moreover for fixed $\mathbf{v}_0$ the normal form is unique under strict equivalence.*

2.3. **Structure of the work and acknowledgements.** We begin by giving some properties of the foliation generated by the blow-up of a nondicritic germ satisfying the genericity assumptions needed in the proof of Theorem 2.1. In the section 4 we give a sketch of the proof of Theorem 2.1. In section 5 we give an appropriate extension of $\mathbf{v}$, define an auxiliary foliation, suitable biholomorphisms and domains of definition that allow one to use Savelev's Theorem. Further, we analyze the Savelev's diffeomorphism and apply Weierstrass Preparation Theorem. The end of the proof is given in 5.8. On section 6 we prove Theorem 2.2 and as a consequence of it we get Theorem 2.3.

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3. General properties of nondicritic foliations and prenormalized form.

Following [ORV2], we give in this section a geometric description of the nondicritic foliations as well as their simplest properties.

Let $\mathbf{v}$ be a nondicritic germ in $\mathcal{V}_n$. For any $n > 1$ the singular the linear part of $\mathbf{v}$ at the singular point $0 \in \mathbb{C}^2$ is zero; in 3.1 and 3.2 we introduce its blow-up:

3.1. **Blow-up $\mathcal{B}$ of $(\mathbb{C}^2, 0)$.** We recall that the blow-up of a point $0 \in \mathbb{C}^2$ is the 2-dimensional complex manifold $\mathcal{B}$ obtained from the gluing of two copies of $\mathbb{C}^2$ with coordinates (called standard charts) (x, u) and (y, v) by means of the map $\phi : (x, u) \mapsto (y, v) = (xu, u^{-1})$.

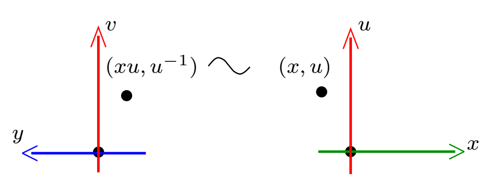

Figure 3.1. Blow-up of $(\mathbb{C}^2, 0)$: $\mathcal{B} = \mathbb{C}^2 \coprod \mathbb{C}^2 / {}_{(y,v)\sim(xu,u^{-1})}$.

The projection $\pi : \mathcal{B} \to (\mathbb{C}^2, 0)$, given in the standard charts by $\pi : (x, u) \mapsto (x, xu)$, $\pi : (y, v) \mapsto (yv, y)$, will be called standard projection as well. The sphere $\mathcal{L} := \pi^{-1}(0) \approx \mathbb{CP}^1$ obtained from the gluing of the regions $\{0\} \times \mathbb{C}$ and $\mathbb{C} \times \{0\}$ by means of $\phi|_{\{0\}\times\mathbb{C}^*}$ will be called the pasted sphere (or the exceptional divisor of the blow-up). The map π is holomorphic and its restriction $\pi|_{\mathcal{B}\setminus\mathcal{L}}$ to the set $\mathcal{B}\setminus\mathcal{L}$ is a biholomorphism whose inverse is denoted by σ and it is: $\sigma := (\pi|_{\mathcal{B}\setminus\mathcal{L}})^{-1}$.

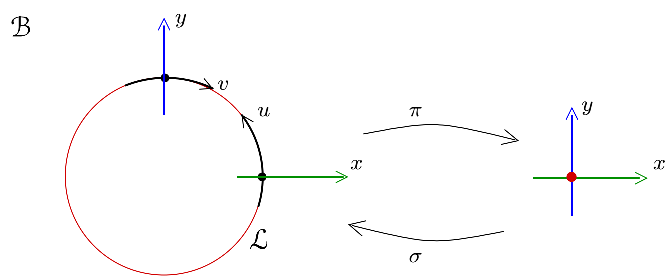

FIGURE 3.2. Blow-up of $(\mathbb{C}^2, 0)$

3.2. **Blow-up of germs of vector fields in $\mathcal{V}_n$.** As it is known, the lifting $\sigma_*\mathbf{v}$ of a germ of vector field $\mathbf{v}$ in $\mathcal{V}_n$ generates, in a neighborhood of the pasted sphere without $\mathcal{L}$, a foliation which can be uniquely extended to $\mathcal{L}$, as a holomorphic foliation $\tilde{\mathcal{F}}_{\mathbf{v}}$ called the blow-up of $\mathcal{F}_{\mathbf{v}}$ at zero (with a finite number of singularities on $\mathcal{L}$, generally speaking). We denote by $\tilde{\mathbf{v}}$ the line field which generates the foliation $\tilde{\mathcal{F}}_{\mathbf{v}}$. We call $\tilde{\mathbf{v}}$ the blow-up of $\mathbf{v}$.

Let $\mathbf{v}$ be a nondicritic germ in $\mathcal{V}_n$. In (x, y)-coordinates, $\mathbf{v}$ has the form (2.1) and the blow-up $\tilde{\mathcal{F}}_{\mathbf{v}}$ of $\mathcal{F}_{\mathbf{v}}$ is given locally, in the standard charts, by the equations

$$\frac{du}{dx} = \frac{xQ(x,ux) - uxP(x,ux)}{x^2P(x,ux)}; \qquad \frac{dv}{dy} = \frac{yP(vy,y) - vyQ(vy,y)}{y^2Q(vy,y)}. \tag{3.1}$$

Let $R_{m+1}(x, y) = xQ_m - yP_m, m = n, n+1, \ldots$. The condition of nondicriticity $R := R_{n+1} \neq 0$ implies that the blow-up $\tilde{\mathcal{F}}_{\mathbf{v}}$, on the region of definition of the standard chart (x, u), is generated by the vector field $\tilde{\mathbf{v}}_+(x, u) = \tilde{P}_+(x, u)\frac{\partial}{\partial x} + \tilde{Q}_+(x, u)\frac{\partial}{\partial u}$, where

$$\tilde{P}_+(x, u) = x[P_n(1, u) + O(x)], \quad \tilde{Q}_+(x, u) = R_{n+1}(1, u) + O(x), \quad \text{for} \quad x \to 0 \ . \tag{3.2}$$

In the same way, on the region of definition of the standard chart (y, v), the foliation $\tilde{\mathcal{F}}_{\mathbf{v}}$ is generated by the vector field $\tilde{\mathbf{v}}_-(y, v) = \tilde{P}_-(y, v)\frac{\partial}{\partial y} + \tilde{Q}_-(y, v)\frac{\partial}{\partial v}$ where

$$\tilde{P}_-(y, v) = y[Q_n(v, 1) + O(y)] \ , \quad \tilde{Q}_-(y, v) = R_{n+1}(v, 1) + O(y) \ , \quad \text{for} \quad y \to 0 \ . \tag{3.3}$$

3.3. **Properties of generic germs (Consequences of the genericity assumptions G1, G2, G3).** For any generic nondicritic germ $\mathbf{v} \in \mathcal{V}_n$, the following statements take place:

(1) The germ $\mathbf{v}$ has exactly $n + 1$ different separatrices, which are smooth at the origin and have pairwise transversal intersection.
(2) A resolution (see [C-S]) of a generic nondicritic germ $\mathbf{v}$ in $\mathcal{V}_n$ consists exactly of one blow-up.
(3) The corresponding blown-up foliation $\tilde{\mathcal{F}}_{\mathbf{v}}$ has exactly $n + 1$ singular points $p_1, \ldots, p_{n+1}$ on the divisor $\mathcal{L}_{\mathbf{v}} \sim \mathbb{CP}^1$ and the characteristic exponents $\lambda_1, \ldots, \lambda_{n+1}$ are neither zero nor rational positive numbers.

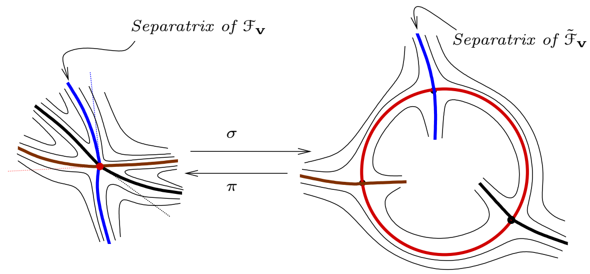

FIGURE 3.3. Phase portrait of a generic nondicritic germ of vector field $\mathbf{v} \in \mathcal{V}_2$ and its blow-up.

(4) **Prenormalized form.** Without loss of generality we will assume that the singular point p_{n+1} is the point at infinity ($p_{n+1} = p_\infty = 0$ in the standard chart (y, v) and thus p_∞ is a nondegenerated singular point of the vector field $\tilde{\mathbf{v}}_-$), and denote by λ_∞ the Camacho-Sad-index with respect to the divisor $\mathcal{L}$. Moreover we assume that the y-axis $(v = 0)$ is the separatrix at p_∞ . We stress that such assumptions can be achieved by performing suitable (analytic) change of coordinates. An additional (analytic) change of coordinates allows one to have the x-axis $(y = 0)$ as separatrix at the origin (as well as in the (x, u) coordinates).

Hence, we assume in what follows that the vector field $\mathbf{v}$ is written in its *prenormalized form*:

$$\mathbf{v}(x,y) = x\hat{P}(x,y)\frac{\partial}{\partial x} + y\hat{Q}(x,y)\frac{\partial}{\partial y}\ , \tag{3.4}$$

where $\hat{P}(x,y), \hat{Q}(x,y)$ are analytic germs at the origin of order $n-1$, $\hat{P} = \sum_{m=n-1}^{\infty} \hat{P}_m$, $\hat{Q} = \sum_{m=n-1}^{\infty} \hat{Q}_m$, where $\hat{P}_m$, $\hat{Q}_m$ are homogeneous polynomials of degree m.

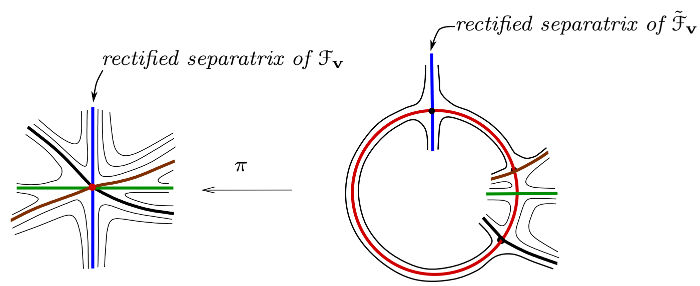

FIGURE 3.4. Phase portrait of a prenormalized nondicritic germ of vector field $\mathbf{v} \in \mathcal{V}_2$ and its blow up.

(5) We stress that the subset $\mathcal{L}_{\mathbf{v}} \setminus \{p_1, ..., p_n, p_\infty\}$ is a leaf of the blown-up foliation $\tilde{\mathcal{F}}_{\mathbf{v}}$. Moreover the polynomial $r(u) = R_{n+1}(1, u)$ has exactly n simple roots; we denote them by $u_1, u_2, ..., u_n$, $r'(u_j) \neq 0$, and their corresponding characteristic exponents which coincide in this case with the Camacho-Sad's index of the foliation $\tilde{\mathcal{F}}_{\mathbf{v}}$ in the singular points with respect to the divisor $\mathcal{L}_{\mathbf{v}}$, λ_j, $\lambda_j = \frac{P_n(1,u_j)}{\frac{\partial R}{\partial y}(1,u_j)} = \frac{p(u_j)}{r'(u_j)}, j = 1, ..., n$, and $\lambda_\infty = -\frac{Q_n(0,1)}{\frac{\partial R}{\partial x}(0,1)}$.

(6) Note that the germ of vector field $\mathbf{v}$ which generates the foliation $\mathcal{F}_{\mathbf{v}}$ has Camacho-Sad's index at the origin with respect to the separatrix $\{x = 0\}$ equal to $1 + \lambda$, where $\lambda = 1/\lambda_\infty$. By the genericity assumption G3 (given in section 2.2) this index is not zero.

4. Sketch of the proof of Theorem 2.1.

Without loss of generality let $\mathbf{v}$ be a generic germ in $\mathcal{V}_n$ written in its prenormalized form (3.3). There exists a cone $\mathcal{C}_{\epsilon_0}$,

$$\mathcal{C}_{\epsilon_0} := \left\{(x, y) \in \mathbb{C}^2 \ : \ \frac{1}{\epsilon_0}|x| \leq |y| \leq \epsilon_0\right\},$$

around the separatrix $\{x = 0\}$ such that, in the blow-up coordinates $(v, y) = (\frac{x}{y}, y)$ the neighborhood $\mathcal{C}_{\epsilon_0}$ takes the form

$$D_{\epsilon_0} \times D_{\epsilon_0} = \{(v, y) \ : \ |v| \leq \epsilon_0, \ |y| \leq \epsilon_0\}.$$

$D_{\epsilon_0} \times D_{\epsilon_0}$ is a neighborhood of the point p_∞ (the origin in the coordinates (v, y)). By the genericity assumptions the blow-up $\tilde{\mathcal{F}}_{\mathbf{v}}$ of $\mathcal{F}_{\mathbf{v}}$ (in the coordinates (v, y)) is locally generated (in a neighborhood of the singular point p_∞) by a linearizable nondegenerated vector field (see generic condition G3.). Hence, for ϵ_0 small enough there exists a biholomorphism G,

$$G : D_{\epsilon_0} \times D_{\epsilon_0} \to (\mathbb{C}^2, 0) \tag{4.1}$$

(preserving the y coordinate) linearizing $\tilde{\mathcal{F}}_{\mathbf{v}}$.

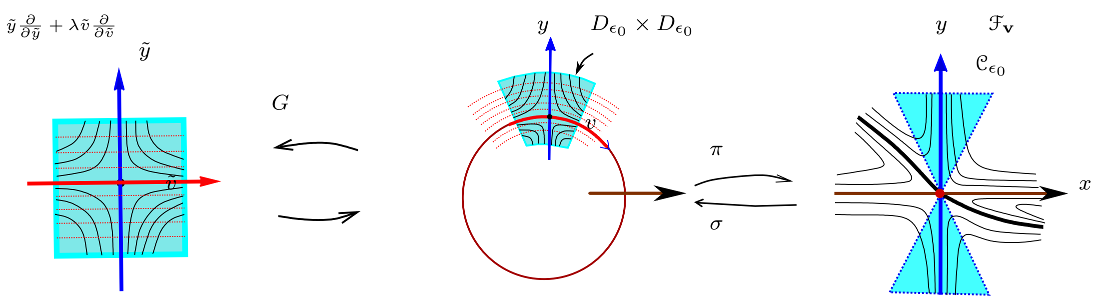

Figure 4.1. Linearizing the foliation $\mathcal{F}_{\mathbf{v}}$ generated by the vector field $\mathbf{v}$ within a conus $\mathcal{C}_{\epsilon_0}$

Let us denote by $\mathbf{v}_\lambda$ the linear vector field such that $\mathbf{v}_\lambda = G_*\tilde{\mathbf{v}}_-$: In the charts $(\tilde{v}, \tilde{y}) := G(v, y)$ the foliation $\tilde{\mathcal{F}}_{\mathbf{v}}$ is thus generated by the vector field:

$$\mathbf{v}_\lambda = \lambda\tilde{v}\frac{\partial}{\partial\tilde{v}} + \tilde{y}\frac{\partial}{\partial\tilde{y}} \tag{4.2}$$

where $\tilde{y} = y$ and $\lambda = \frac{1}{\lambda_\infty}$ is the Camacho-Sad' index of $\mathcal{F}_{\mathbf{v}_\lambda}$ at $(0, 0)$ related to the separatrix $\{v = 0\}$, and λ_∞ is the Camacho-Sad' index of $\mathcal{F}_{\mathbf{v}_\lambda}$ at $(0, 0)$ corresponding to the separatrix $\{y = 0\}$ (the divisor $\mathcal{L}$).

As the vector field $\mathbf{v}_\lambda$ is a linear one, it may be extended to the whole complex manifold $\mathcal{M}$

$$\mathcal{M} := (\mathbb{C} \times D_\epsilon) \sqcup (\mathbb{C} \times D_\epsilon)/_{(\tilde{v},\tilde{y})\sim(\xi=\tilde{v}\tilde{y} \ , \ \eta=\frac{1}{\tilde{y}}), \ \tilde{y}\neq 0},$$

where $G^{-1}(D_\epsilon \times D_\epsilon) \subset D_{\epsilon_0} \times D_{\epsilon_0}$ for ϵ small enough.

Let $\mathcal{M}_+ := \{(\tilde{v}, \tilde{y}) \in D_\epsilon \times \mathbb{C}\}$, $\mathcal{M}_- := \{(\xi, \eta) \in D_\epsilon \times \mathbb{C}\}$.

On $\mathcal{M}$ the vector field $\mathbf{v}_\lambda$ is defined in (4.2) and straight-forward calculations show that $\mathbf{v}_\lambda$ in $\mathcal{M}_-$ is written as

$$\mathbf{v}_{\lambda+1}(\xi,\eta) = (\lambda+1)\xi\frac{\partial}{\partial\xi} - \eta\frac{\partial}{\partial\eta} \tag{4.3}$$

Therefore there is a foliation on $\mathcal{M}$ defined by the extension of $\mathbf{v}_\lambda$, having no more than two singular points: the $(0,0)$ in coordinates $(\tilde{v},\tilde{y})$ and the $(0,0)$ in the coordinates (ξ,η) . We stress that Camacho-Sad's index at the origin with respect to the y axis is λ, and the respective index at the origin in the charts (ξ,η), $\eta = \frac{1}{y}$ is $-(\lambda+1)$. This means that the self-intersection index of the closure $\{y=0\}$ in $\mathcal{M}$ is -1. Hence, $\mathcal{M}$ is the blow-up of a neighborhood of $(\tilde{v},\tilde{y}) = (0,0)$.

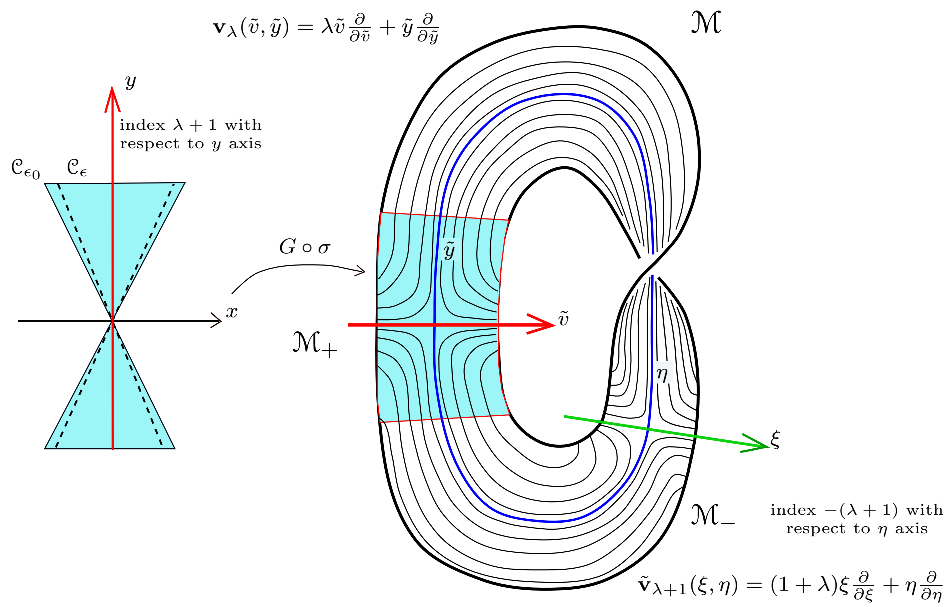

FIGURE 4.2. Extension of the vector field $\mathbf{v}_\lambda$ to $\mathcal{M}$.

We return to the (x,y) coordinates:

Remark that the foliation generated by the vector field $\mathbf{v}$ has Camacho-Sad's index $\lambda+1$ with respect to the y axis. This follows from the correspondence of $\mathcal{F}_{\mathbf{v}}$ with $\mathcal{F}_{\mathbf{v}_\lambda}$ by means of $G\circ\sigma$.

The next goal is to construct an extension of $\mathcal{F}_{\mathbf{v}}$. For this purpose we use the vector field $\mathbf{v}_\lambda$ (see (4.2)) and the following construction:

We define, in a neighborhood of the origin in the $(\tilde{v},\tilde{y})$ coordinates, an annulus $\mathcal{A}_\mu \subset \mathcal{M}$,

$$\mathcal{A}_\mu := D_\epsilon \times D_\epsilon \setminus D_\epsilon \times D_{\epsilon'}, \quad \epsilon' < \epsilon$$

Let $\mathcal{A}$ be the annulus like domain which is the preimage of $\mathcal{A}_\mu$ under $G\circ\sigma$:

$$\mathcal{A} := (G\circ\sigma)^{-1}(\mathcal{A}_\mu), \quad \mathcal{A} \subset \mathcal{C}_\epsilon \ .$$

We stress that $\mathcal{A}_\mu \subset \mathcal{M}_+ \cap \mathcal{M}_-$. Hence by means of $(G\circ\sigma)^{-1}$ we may construct a new manifold by identifying the neighborhood $\mathcal{U}_+$ of the origin in the coordinates (x,y), $\mathcal{A} \subset \mathcal{C}_\epsilon \subset \mathcal{U}_+$, with the open domain $\mathcal{U}_- = D_\epsilon \times D_{\frac{1}{\epsilon}}$ (in the charts (ξ,η)). Namely, we define Φ (see fig 4.3) as the composition

$$\Phi := \beta \circ G \circ \sigma : \mathcal{A} \to \beta(\mathcal{A}_\mu) \subset \mathcal{U}_- \quad , \quad \text{where} \quad \beta(\tilde{v},\tilde{y}) = (\xi = \tilde{v}\tilde{y}, \eta = 1/\tilde{y}). \tag{4.4}$$

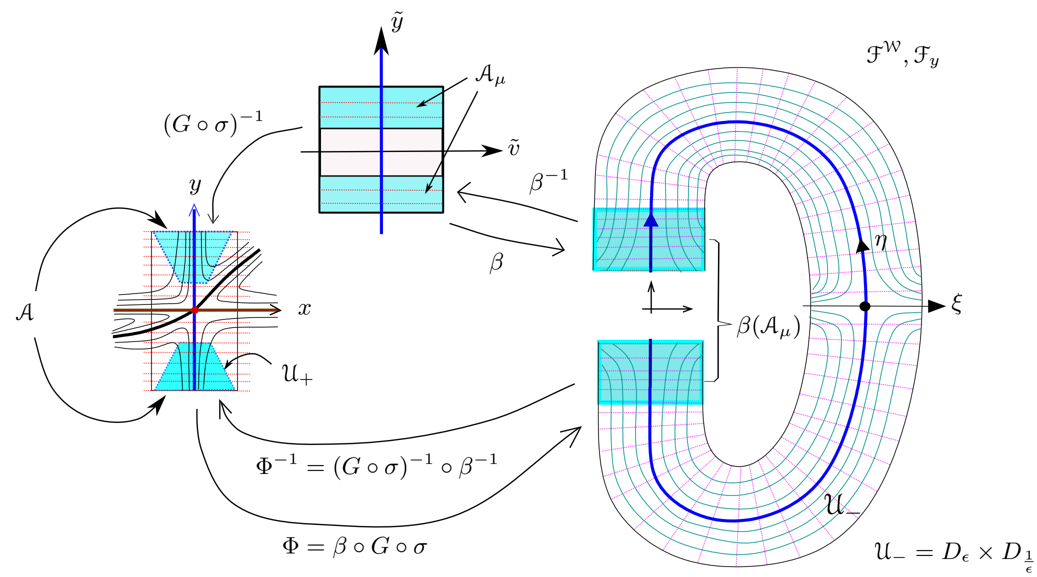

FIGURE 4.3. Extension of the foliation $\mathcal{F}_{\mathbf{v}}$ to the complex surface $\mathcal{W}$ and foliations $\mathcal{F}^{\mathcal{W}}$ and $\mathcal{F}_y$.

We denote by $\mathcal{W}$ the 2-dimensional complex manifold obtained from the domains $\mathcal{U}_+$ and $\mathcal{U}_-$ and the transition maps $\Phi|_{\mathcal{A}}$, $\Phi^{-1}|_{\Phi(\mathcal{A})}$, $\mathcal{A} \subset \mathcal{U}_+$, $\Phi(\mathcal{A}) \subset \mathcal{U}_-$.

The foliation $\mathcal{F}_{\mathbf{v}}$ and $\mathcal{F}_{\mathbf{v}_{\lambda+1}}$ defined by the vector fields $\mathbf{v}$ and $\mathbf{v}_{\lambda+1}$ in $\mathcal{U}_+$ and $\mathcal{U}_-$ respectively, define a global foliation $\mathcal{F}^{\mathcal{W}}$ on $\mathcal{W}$ with exactly two singular points O_+ and O_-: the corresponding singular point of $\mathbf{v}$ and $\mathbf{v}_{\lambda+1}$. Remark that the disk $\{0\} \times D_\epsilon \subset \mathcal{U}_+$ is in correspondence under Φ with the disk $\{0\} \times D_{\frac{1}{\epsilon}} \subset \mathcal{U}_-$ $(\eta = \frac{1}{y})$. This defines a Riemann sphere $\mathcal{L}^{\mathcal{W}}$.

Together with the foliation $\mathcal{F}^{\mathcal{W}}$ we consider the foliation $\mathcal{F}_y$ defined by

$$\{y = cst\} \;\; (\eta = cst \in \mathcal{U}_-).$$

This foliation defines a line bundle (the normal bundle over $\mathcal{L}^{\mathcal{W}}$ on $\mathcal{W}$). To know how $\mathcal{L}^{\mathcal{W}}$ is embedded in $\mathcal{W}$ it is sufficient to calculate Camacho-Sad's index for $\mathbf{v}$ at $(0,0)$ with respect to the separatrix $\mathcal{L}^{\mathcal{W}}$. Namely, Camacho-Sad's index of $\mathbf{v}_{\lambda+1}$ at $(0,0)$ with respect to the separatrix $\{x = 0\}$ is $\lambda + 1$. Hence, by Camacho-Sad's Theorem, the self-intersection index $\mathcal{L}^{\mathcal{W}} \cdot \mathcal{L}^{\mathcal{W}} = (\lambda+1) - (\lambda+1) = 0$.

By Savelev's Theorem [Sa], there exists a biholomorphism Ψ of a neighborhood of $\mathcal{L}^{\mathcal{W}}$ (in $\mathcal{W}$) to the direct product $(\mathbb{C},0) \times \mathbb{CP}^1$ such that $\Psi(\mathcal{L}^{\mathcal{W}}) = \{0\} \times \mathbb{CP}^1$. By reducing, if necessary, the domain of definition in our construction we may assume that Ψ is defined in the whole $\mathcal{W}$, $\Psi : \mathcal{W} \to (\mathbb{C},0) \times \mathbb{CP}^1$.

We denote by $\mathcal{F}$ the foliation induced by $\mathcal{F}^{\mathcal{W}}$ under the transformation Ψ, $\mathcal{F} := \Psi(\mathcal{F}^{\mathcal{W}})$. The foliation $\mathcal{F}$ is defined at the direct product $(\mathbb{C},0) \times \mathbb{CP}^1$ having singular points $\Psi(O_+)$ and $\Psi(O_-)$. In a neighborhood of $\Psi(O_+)$ the foliation $\mathcal{F}$ is generated by the vector field

$$\mathbf{v}_+ := (\Psi \circ t_+^{-1})_* \mathbf{v}$$

and in a neighborhood of $\Psi(O_-)$ the foliation $\mathcal{F}$ is generated by the vector field

$$\mathbf{v}_- := (\Psi \circ t_-^{-1})_* \mathbf{v}$$

where t_+ and t_- are the natural charts in $\mathcal{W}$ corresponding to the domains $\mathcal{U}_+$ and $\mathcal{U}_-$ respectively ($t_+ : t_+^{-1}(\mathcal{U}_+) \to \mathcal{U}_+$, $t_- : t_-^{-1}(\mathcal{U}_-) \to \mathcal{U}_-$).

Let $G : D_{\epsilon_0} \times D_{\epsilon_0} \to (\mathbb{C}^2, 0)$ be the linearizing biholomorphism defined in the beginning of this section. Then, as we will see in 5.8, Theorem 2.1 is a consequence of the following proposition:

Proposition 4.1. *The linearizing biholomorphism $G : D_{\epsilon_0} \times D_{\epsilon_0} \to (\mathbb{C}^2, 0)$, the coordinate system in $(\mathbb{C}, 0) \times \mathbb{CP}^1$ and the domains used in the construction described along this section may be chosen in such way that $\mathbf{v}_+$ ($\mathbf{v}_-$) is orbitally analytically equivalent to a holomorphic vector field, polynomial with respect to the y variable.*

5. Proof of Proposition 4.1

The proof of the Proposition 4.1 is quite long since we require to give explicit biholomorphisms and domains.

The first step is to show that the linearizing biholomorphism G (linearizing $\tilde{\mathbf{v}}_-$ in $\sigma(\mathcal{C}_\epsilon)$) may be chosen, without loss of generality, as the identity in the y variable.

5.1. **Normalization of the biholomorphism G.** Let G be the biholomorphism at the beginning of section 4. G transforms the leaves of the foliation $\mathcal{F}_{\tilde{\mathbf{v}}}$ into the leaves of the foliation $\mathcal{F}_{\mathbf{v}_\lambda}$ ($\mathcal{F}_{\mathbf{v}_\lambda}$ is the foliation generated by the vector field $\mathbf{v}_\lambda$ -see (4.1)-).

As we wish to have a correspondence between the separatrices $\{v = 0\}$ and $\{y = 0\}$ (of the vector field $\tilde{\mathbf{v}}$), and the separatrices $\{\tilde{v} = 0\}$ and $\{\tilde{y} = 0\}$ of the linear vector field $\mathbf{v}_\lambda$, the biholomorphism G must be written as

$$G(v, y) = (vG_1(v, y), yG_2(v, y)),$$

with $G_j(0, 0) \neq 0, j = 1, 2$.

We stress that the phase curves (cy^λ, y) corresponding to the vector field $\mathbf{v}_\lambda$ are invariant under transformations of the form $\Phi_k(v, y) = (v\,k^\lambda, y\,k)$, $k(0, 0) \neq 0$. For this reason (by performing, if needed, the composition $\Phi_k \circ G$ for an appropriate k) we may assume that the map G has the form

$$G(v, y) = (vg(v, y), y), g(0, 0) = 1. \tag{5.1}$$

To give an explicit expression of the function g we observe that, in a neighborhood of the origin in the coordinates (v, y), the foliation $\mathcal{F}_{\tilde{\mathbf{v}}}$ is defined by the integral curves of the equation:

$$\frac{dv}{dy} = \left.\frac{yP(x, y) - xQ(x, y)}{y^2 Q(x, y)}\right|_{x=vy} \quad ;$$

equivalently, $\tilde{\mathcal{F}}_{\mathbf{v}}$ is defined by the vector field

$$\tilde{\mathbf{v}}_- = C(v, y)\frac{\partial}{\partial v} + y\frac{\partial}{\partial y} \quad ,$$

where

$$C(v, y) = \left.\frac{yP(x, y) - xQ(x, y)}{y}\right|_{x=vy} \tag{5.2}$$

$$\frac{\partial C(v, y)}{\partial v}(0, 0) = \lambda.$$

Recalling that the biholomorphism G satisfies

$$G_*\tilde{\mathbf{v}}_- = \mathbf{v}_\lambda \circ G$$

it follows that

$$vyC(v,y)\,\frac{\partial(vg(v,y))}{\partial v} + vy\frac{\partial g}{\partial y}(v,y) = \lambda vg(vy,y) \tag{5.3}$$

As $\{x=0\}$ is a separatrix of the vector field $\mathbf{v}$, then $P(x,y) = x\hat{P}(x,y)$ and so

$$C(v,y) = v\,c(y) + O(v^2), \quad v \to 0,$$

where

$$c(y) = \left.\frac{y\hat{P}(vy,y) - Q(vy,y)}{Q(vy,y)}\right|_{v=0} \quad , \ c(0) = \lambda \ . \tag{5.4}$$

Hence, from (5.3) we get that, for $v = 0$,

$$\frac{g'_y(0,y)}{g(0,y)} = \frac{-c(y)+\lambda}{y} \quad . \tag{5.5}$$

Thus, $g(0,y)$ is a holomorphic function in a neighborhood of $y = 0$,

$$g(0,y) = exp\left(\int \frac{-c(y)+\lambda}{y}dy\right) \quad , \quad g(0,0) = 1$$

5.2. **Gluing biholomorphism $G \circ \sigma$.** After the rectification of the biholomorphism G introduced in section 4, the composition $G \circ \sigma$ that relates the vector fields $\mathbf{v}$ and $\mathbf{v}_\lambda$ is expressed in terms of the holomorphic function g (in the coordinate charts $(\tilde{v},\tilde{y})$ on $\mathcal{M}$) as

$$G \circ \sigma : (x,y) \mapsto (\tilde{v},\tilde{y}) = (\frac{x}{y}g(x/y,y), y).$$

Recall the change of coordinates β introduced in (4.4), $\beta(\tilde{v},\tilde{y}) = (\tilde{v}\tilde{y}, 1/\tilde{y}) = (\xi,\eta)$.

The composition $\Phi = \beta \circ G \circ \sigma$ is expressed in terms of (x,y) as

$$\Phi(x,y) = (\beta \circ G \circ \sigma)(x,y) = (xg(x/y,y), 1/y) \in \mathcal{U}_- \quad . \tag{5.6}$$

Hence,

$$\Phi_*\mathbf{v}(\xi,\eta) = (\beta \circ G \circ \sigma)_*\mathbf{v}(\xi,\eta) = (1+\lambda)\xi\frac{\partial}{\partial\xi} - \eta\frac{\partial}{\partial\eta} \quad .$$

Moreover, if we define the map $\alpha : D_\epsilon \times D_{\frac{1}{\epsilon}} \to D_\epsilon \times (\mathbb{CP}^1,\infty)$, as $\alpha(\xi,\eta) = (\xi, \frac{1}{\eta}x) = (\xi, y)$, then the composition $\alpha \circ \Phi$ is expressed on (x,y) as

$$(\alpha \circ \Phi)_{(x,y)} = (xg(x/y,y), y) \quad . \tag{5.7}$$

Thus

$$((\alpha \circ \Phi)_*\mathbf{v})\,(\xi,\eta) = (1+\lambda)\xi\partial\xi + y\frac{\partial}{\partial y} \quad . \tag{5.8}$$

We now use (5.7) and (5.8) to understand the consequences of Savelev's biholomorphism Ψ.

5.3. **Properties of the Savelev's biholomorphism and its rectification.** As it was mentioned in section 4, Savelev's Theorem guarantees the existence of a biholomorphism

$$\Psi : \mathcal{W} \to (\mathbb{C}, 0) \times \mathbb{CP}^1.$$

At a first glance we do not know much about Ψ; we need to understand its behavior through the charts on $\mathcal{W}$. To this purpose we recall that $\mathcal{W}$ is the result of the identification of the domains $\mathcal{U}_+$ and $\mathcal{U}_-$. We consider the natural projections: $\Pi_\pm : \mathcal{U}_\pm \to \mathcal{W}$, where $\Pi_\pm(p)$ is the class of the point p in the identifying space $\mathcal{W}$. Let $\tilde{\mathcal{U}}_\pm := \Pi_\pm(\mathcal{U}_\pm)$.

Definition 5.1. *We call* $(\Pi_\pm^{-1}, \tilde{\mathcal{U}}_\pm)$ *the "natural charts" of the complex manifold* $\mathcal{W}$.

Note that $\Pi_\pm^{-1} = t_\pm$ (see section 4).

We stress that $\alpha \circ \Phi$ (see (5.7) is just the change of coordinates of the "normal charts" of $\mathcal{W} : \alpha \circ \Phi = \Pi_-^{-1} \circ \Pi_+$ (see fig. 5.1).

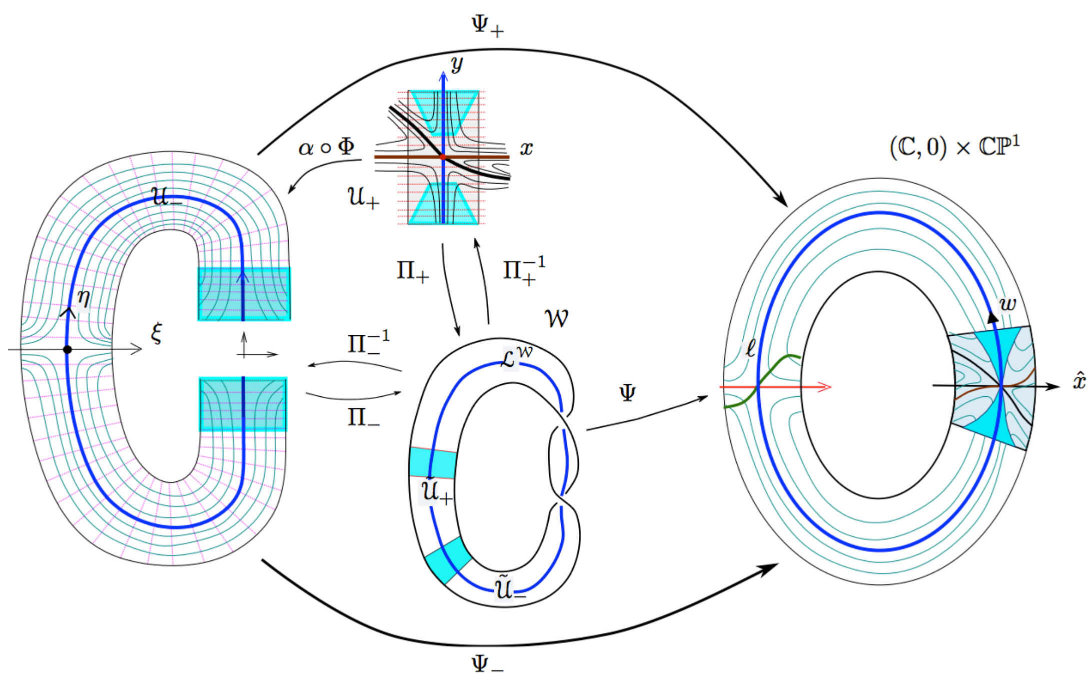

FIGURE 5.1. Savelev's biholomorphism Ψ

Remark that if we define $\Psi_\pm := \Psi \circ \Pi_\pm : \mathcal{U}_\pm \to \Psi(\mathcal{U}_\pm)$, then Ψ_+ and Ψ_- are related by means of $\alpha \circ \Phi$ (where the composition makes sense):

$$\Psi_+ = \Psi_- \circ (\alpha \circ \Phi) \ . \tag{5.9}$$

In order to obtain simple expressions for Ψ_+ and Ψ_- we proceed to give appropriate coordinates in $\hat{\mathcal{W}} := \Psi(\mathcal{W}) = (\mathbb{C}, 0) \times \mathbb{CP}^1$. To this aim we observe that, from Savelev's Theorem we may suppose, without loss of generality, that $\Psi(\mathcal{L}^{\mathcal{W}}) = \{\hat{x} = 0\} \times \mathbb{CP}^1$. Furthermore, we observe that in the charts $\Pi_\pm^{-1}$ the Riemann sphere $\mathcal{L}^{\mathcal{W}}$ is given by $\{\Pi_\pm^{-1} = 0\}$; hence, the restriction of the identifying map $\alpha \circ \Phi = \Pi_-^{-1} \circ \Pi_+$ to the axis $\{x = 0\}$ is the identity. Therefore, it is possible to give coordinates such that

$$\Psi_+ \mid_{x=0} = Id, \quad \Psi_- \mid_{\xi=0} = Id \quad . \tag{5.10}$$

If we consider the image of $\{\eta = 0\}$ under the transformation Ψ_-, $\ell := \Psi_-(\{\eta = 0\})$ we get, from (5.10), that the intersection of ℓ with $\Psi(\mathcal{L}^{\mathcal{W}})$ is transversal in $\Psi_-(\xi, \eta) \mid_{\xi=0,\eta=0}$. Hence if $(\hat{x}, w)$ are the coordinates of $\Psi(\mathcal{W})$, ℓ is expressed as $(\hat{x}, \hat{\gamma}(\hat{x}))$ in a neighborhood of $\hat{x} = 0, w = \infty$. Therefore the curve ℓ may be rectified by means of a Möebius transformation

$$w \mapsto \frac{w}{1 - \hat{\gamma}(\hat{x})w} \ ,$$

so that $\Psi_-(\xi, 0) \in \{w = \infty\}$. Furthermore, under an additional change of coordinates of the form $\hat{x} \mapsto \tilde{\phi}(\hat{x})$ we obtain

$$\Psi_- \mid_{\xi=0} = Id \quad . \tag{5.11}$$

We observe that from (5.9) we get

$$\Psi_+{}^{-1} = (\alpha \circ \Phi)^{-1} \circ \Psi_-{}^{-1}$$

and, as $\alpha \circ \Phi$ is the identity on the second coordinate we get that $\tilde{\Psi}_{+,2} = \tilde{\Psi}_{-,2}$, where

$$\Psi_\pm{}^{-1} = (\tilde{\Psi}_{\pm,1}, \tilde{\Psi}_{\pm,2}).$$

Hence, for small enough fixed $\hat{x}$, the function $\Psi_{\hat{x}}(w) := \tilde{\Psi}_{+,2}(\hat{x}, w)$ may be analytically extended to all $\mathbb{C}$. From (5.11) we get that such extension, which we denote again by $\Psi_{\hat{x}}$, has a pole at $w = \infty$. As Ψ_- is a biholomorphism, then the order of the pole of $\Psi_{\hat{x}}$ is one. Thus, $\Psi_{\hat{x}}$ is a polynomial of degree one on w:

$$\Psi_{\hat{x}}(w) = k(\hat{x})w + \gamma(\hat{x}) \quad ,$$

where k, γ are holomorphic on $\hat{x}$ and $k(0) \neq 0, \gamma(0) = 0$.

In this way the foliation in $\mathcal{U}_+$ given by $\{y = cst\}$ is transformed by the map Ψ_+ to the foliation by curves defined by

$$\begin{aligned} \Psi_+(x, y) &= (\Psi_{+,1}(x, y), \Psi_{+,2}(x, y)) \\ &= (\hat{x}, k(\hat{x})w + \gamma(\hat{x})) \quad . \end{aligned}$$

As $k(0) \neq 0$, $\gamma(0) = 0$, we may define for small enough $\hat{x}$ a rectification biholomorphism

$$\mathbf{r} = \mathbf{r}(\hat{x}, w) = \left(\hat{x}, \frac{w}{k(\hat{x})} - \frac{\gamma(\hat{x})}{k(\hat{x})}\right) \quad ,$$

whose inverse is

$$\mathbf{r}^{-1}(\hat{x}, w) = (\hat{x}, k(\hat{x})\, w + \gamma(\hat{x})) \quad .$$

This biholomorphism sends the curves $k(\hat{x})\, w + \gamma(\hat{x})$, with $w = c$ into the curves $w = c$, $c \in \mathbb{C}$ and fix $w = \infty$.

Finally, as $\mathbf{r} \circ \Psi_+(0, y) = (0, w)$, where $w = w(y)$ is a biholomorphism, we may perform an additional change of coordinates $\mathbf{r}_0(\hat{x}, w) = (\hat{x}, y)$ so that $\mathbf{r}_0 \circ \mathbf{r} \circ \Psi_+(0, y) = (0, y)$. Therefore, in what follows we may assume that

$$\Psi_{+,2}(x, y) \equiv y \equiv \Psi_{-,2}(x, y) \quad . \tag{5.12}$$

Using (5.10) and (5.12) we get

$$\Psi_+(x, y) = (x\alpha_+(x, y), y) \tag{5.13}$$

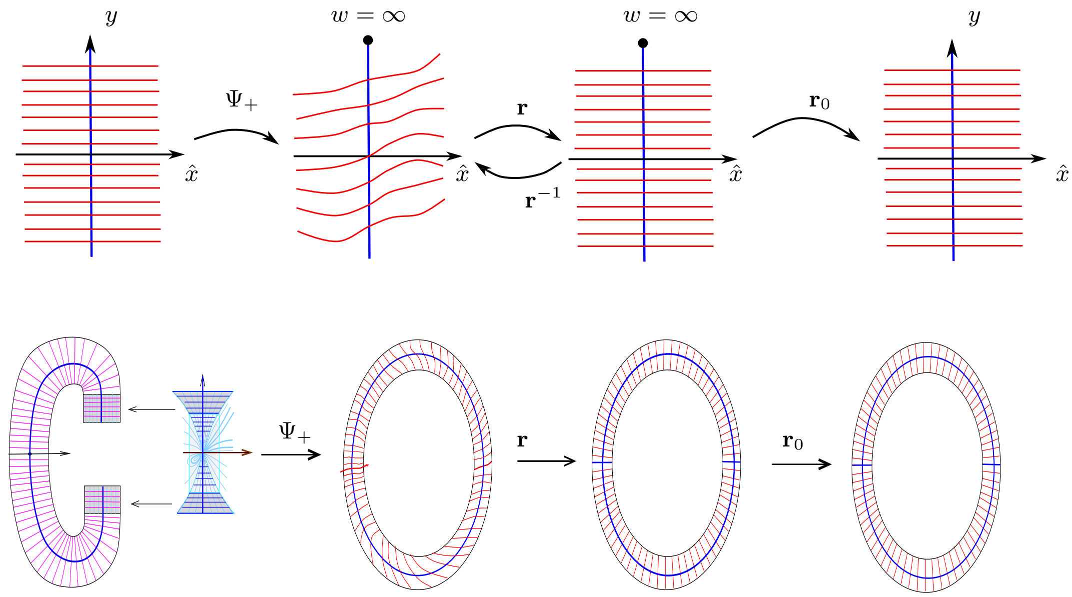

FIGURE 5.2. Rectification process.

$$\Psi_-(x,y) = (\xi\alpha_-(\xi,\eta),\eta) \quad , \tag{5.14}$$

where α_+ is holomorphic in a neighborhood of the disk $\{x=0, |y|<R\} \subset \mathbb{C}\times\mathbb{CP}^1$, $\alpha_+(0,y) \neq 0$ and α_- is holomorphic in a neighborhood of $\{\xi=0, |\eta|<R\} \subset \mathbb{C}\times\mathbb{CP}^1$, $\alpha_-(0,\eta)\neq 0$.

5.4. **Asymptotic of $\alpha_\pm$.** By the substitution of the expression (5.7) for $\alpha\circ\Phi$ and the expressions for Ψ_+ and Ψ_- given in (5.13), (5.14) we get

$$(x\alpha_+(x,y),y) = (\xi\alpha_-(\xi,\eta),\eta)_{(x\,g(x/y,y),y)}$$

i.e.

$$(x\alpha_+(x,y),y) = (xg(x/y,y)\alpha_-(xg(x/y,y),y) \quad ;$$

therefore

$$\alpha_+(x,y) = g(x/y,y)\alpha_-(xg(x/y,y),y) \quad . \tag{5.15}$$

Taking limits when $x\to 0$ we get

$$\alpha_+(0,y) = g(0,y)\alpha_-(0,y) \quad . \tag{5.16}$$

From (5.5) we know that $g(0,y)$ is non vanishing and holomorphic in the disk D_ϵ. Hence,

$$\alpha_-(0,y) = \frac{\alpha_+(0,y)}{g(0,y)}$$

is holomorphic in D_ϵ. The function α_- is holomorphic in $D_{\epsilon'} = \{|y|>\epsilon'\}\cup\{\infty\}$ and coincides with $\frac{\alpha_+(0,y)}{g(0,y)}$ in the annulus given by the intersection $D_\epsilon\cap D_{\epsilon'}$. Therefore α_- can be extended to the closure $\bar{\mathbb{C}}$. By Liouville's Theorem we get $\alpha_-\equiv c\equiv\frac{\alpha_+(0,y)}{g(0,y)}$ for a non zero constant c.

Thus

$$\begin{array}{rcl}\alpha_+(x,y) &=& cg(0,y)+O(x)\\ \alpha_-(x,y) &=& c+O(x) \quad .\end{array} \tag{5.17}$$

5.5. **Action of Ψ_+ on the vector field v.** After all the previous constructions we may look to the action of Ψ_+ on the vector field $\mathbf{v}$, and the action of Ψ_- on $(\alpha\circ\Phi)_*\mathbf{v}$.

We denote

$$\mathbf{v}_+ := {\Psi_+}_*\mathbf{v} \quad \text{and} \quad \mathbf{v}_- := {\Psi_-}_*((\alpha\circ\Phi)_*\mathbf{v}) \quad . \tag{5.18}$$

By construction, $\mathbf{v}_+$ and $\mathbf{v}_-$ generate in their corresponding domain of definition, a complex foliation $\mathcal{F}$ on the complex manifold $\mathcal{W}$. We stress that the definition of $\mathcal{F}$, $\mathbf{v}_0$ and $\mathbf{v}_-$ are in concordance with the definitions introduced at the end of section 4 and in section 5.2.

To get an expression of $\mathbf{v}_\pm(\hat{x},w) = (P_\pm(\hat{x},w), Q_\pm(\hat{x},w))$ we use that ${\Psi_\pm}^{-1}$ may be written as:

$${\Psi_\pm}^{-1} : (\hat{x},w) \mapsto (\hat{x}\ell_\pm(\hat{x},w), w) \quad ,$$

where (using (5.13) and (5.17))

$$\ell_+(\hat{x},w) = ((cg(0,y))^{-1} + O(x) \tag{5.19}$$

$$\ell_-(\hat{x},w) = c^{-1} + O(x) \quad . \tag{5.20}$$

To get an explicit expression for $P_\pm, Q_\pm$ we recall that $\Psi_\pm(x,y) = (x\alpha_\pm(x,y), y)$, thus

$$\begin{aligned}
\mathbf{v}_+(\hat{x},w) &= D\Psi_+\,|_{{\Psi_+}^{-1}(\hat{x},w)}\, \mathbf{v}({\Psi_+}^{-1}(\hat{x},w)) \\
&= \begin{pmatrix} \left(\alpha_+ + x\frac{\partial\alpha_+}{\partial x}\right)\Big|_{{\Psi_+}^{-1}(\hat{x},w)} & x\,\frac{\partial\alpha_+}{\partial y}\Big|_{{\Psi_+}^{-1}(\hat{x},w)} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} P({\Psi_+}^{-1}(\hat{x},w)) \\ Q({\Psi_+}^{-1}(\hat{x},w)) \end{pmatrix} \\
&= \begin{pmatrix} \left[\left(\alpha_+ + x\frac{\partial\alpha_+}{\partial x}\right)P + x\frac{\partial\alpha_+}{\partial y}Q\right]_{(\hat{x}\ell_+(\hat{x},w),w)} \\ Q\left(\hat{x}\ell_+(\hat{x},w),w\right) \end{pmatrix} \quad .
\end{aligned}$$

Therefore, $\mathbf{v}_+(\hat{x},w) = (P_+(\hat{x},w), Q_+(\hat{x},w))$, where

$$P_+(\hat{x},w) = \left[\alpha_+ P + x\frac{\partial\alpha_+}{\partial y}Q + O(x^2)\right]_{(\hat{x}\ell_+(\hat{x},w),w)} \tag{5.21}$$

and

$$Q_+(\hat{x},w) = Q\left(\hat{x}\ell_+(\hat{x},w),w\right) \quad , \quad q(w) := Q(0,w) \quad . \tag{5.22}$$

Analogously, we get explicit expressions for $\mathbf{v}_-(\hat{x},w) = (P_-(\hat{x},w), Q_-(\hat{x},w))$

$$\begin{aligned}
\mathbf{v}_-(\hat{x},w) &= D\Psi_-\,|_{{\Psi_-}^{-1}(\hat{x},w)}\, \mathbf{v}({\Psi_-}^{-1}(\hat{x},w)) \\
&= \begin{pmatrix} \frac{\partial\xi\alpha_-}{\partial\xi} & \frac{\partial\xi\alpha_-}{\partial\eta} \\ 0 & 1 \end{pmatrix}\Bigg|_{{\Psi_-}^{-1}(\hat{x},w)} \begin{pmatrix} (\lambda+1)\hat{x}\ell_-(\hat{x},w) \\ w \end{pmatrix} \\
&= \begin{pmatrix} (\lambda+1)\hat{x}\ell_-(\hat{x},w)\left[\hat{x}\ell_-(\hat{x},w)\frac{\partial\alpha_-}{\partial\xi} + \alpha_-\right] + \hat{x}\,w\,\ell_-(\hat{x},w)\frac{\partial\xi\alpha_-}{\partial\eta} \\ w \end{pmatrix} \quad .
\end{aligned}$$

Therefore,

$$P_-(\hat{x},w) = \hat{x}\ell_-(\hat{x},w)\left[(\lambda+1)\alpha_-(\hat{x}\ell_-(\hat{x},w),w) + w\frac{\partial\xi\alpha_-}{\partial\eta}\right] + O(\hat{x}^2) \tag{5.23}$$

and

$$Q_-(\hat{x},w) = w \quad . \tag{5.24}$$

5.6. **Locus of functions P_- and Q_-.** At this stage it is important to recall that our goal is to prove that $\mathbf{v}_+$ may be written as a polynomial of degree $n-1$ in w with analytic coefficients depending on the x variable. To this sake we will look to the locus of $P_\pm, Q_\pm$ and then use a slightly modified version of Weierstrass Preparation Theorem.

We begin with the study of the locus of P_-, and Q_-:

From (5.24) we know that $Q_-(\hat{x},w) = w$ does not vanish for $|w| > r$. Moreover, from (5.20) we know that $\ell_-(\hat{x},w) = 1/c + \ell_1(\hat{x},w)$, where ℓ_1 is a holomorphic function on

$$\Delta_- = \{|x| < \delta\} \times \{|w| > r\}\,.$$

In particular, ℓ_1 is holomorphic in the point: $x = 0, w = \infty$. From Cauchy's inequalities it follows that in any polydisk $\Delta'_- = \{|x| < \delta'\}\times\{|w| > r'\}$, $r' > r$, $\delta' < \delta$ the inequality $\left|\frac{\partial\alpha_-}{\partial\eta}(\hat{x},w)\right| \leq \frac{\zeta}{|w|}$ is satisfied for an appropriate constant $\zeta = \zeta(\Delta,\Delta')$. By using expressions (5.17) and (5.20) for α_- and ℓ_- in (5.23) we get:

$$\begin{aligned} P_-(\hat{x},w) &= \hat{x}(1/c + \ell_1(\hat{x},w))((\lambda+1)(c+O(\hat{x})) + O(\hat{x}) \\ &= \hat{x}[(\lambda+1) + O(\hat{x})] \quad . \end{aligned} \tag{5.25}$$

Therefore $\hat{P}_-(\hat{x},w) = \frac{P_-(\hat{x},w)}{\hat{x}}$ is holomorphic and does not vanish in Δ'_-.

5.7. **Locus of functions P_+ and Q_+.** We begin by stating a slightly different version of Weierstrass Preparation Theorem:

Lemma 5.1. *Let $F(x,y)$ be a holomorphic function in the polydisk $\Delta_0 = \{|x| < \delta_0\}\times\{|y| < \epsilon_0\}$ such that the function $F(0,y)$ has, at $y = 0$ a zero of order N. If $F(0,y)$ has no more zeros in the disk $\{|y| < \epsilon_0\}$, then, for any ϵ, $0 < \epsilon < \epsilon_0$, there exist $\delta, 0 < \delta < \delta_0$ and holomorphic functions k, W, defined in $\Delta = \{|x| < \delta\}\times\{|y| < \epsilon\}$ such that*

(1) $F = kW$ *in* Δ
(2) $k \neq 0$ *in* Δ
(3) $W_N(x,y) = W(x,y) = y^N + \sum_{j=0}^{N-1} a_j(x)y^j$, $a_j(0) = 0$.

W_N is known as the Weierstrass polynomial (see Shabat pp.123-126).

Let us consider now the series $Q = Q_n + Q_{n+1} + \cdots$, where Q_j denotes the homogeneous polynomial of degree j in the variables (x,y), $j \geq n$, and $Q_n(x,y) = b_0y^n + O(x)$. As we did before, let $q(y) = Q(0,y)$. From the genericity assumptions we know that $b_0 \neq 0$. Hence $q(y)$ has at $y = 0$ a zero of order n.

Let $\Delta_0 = \{|\hat{x}| < \delta\}\times\{|w| < \epsilon_0\}$, $\epsilon < \epsilon_0$ such that Q_+ is holomorphic in Δ_0. From (5.22) we have that $q(w) = Q_+(0,w)$ and for any $\hat{\epsilon} \leq \epsilon$ and $\hat{\delta} < \delta$ it is possible to factorize (by Lemma 5.1) $Q_+(\hat{x},w)$ as

$$Q_+(\hat{x},w) = K_Q(\hat{x},w)W_n(\hat{x},w), \tag{5.26}$$

$(\hat{x},w) \in \Delta_+ = \left\{|\hat{x}| < \hat{\delta}\right\}\times\{|w| < \hat{\epsilon}\}$, where $K_Q \neq 0$ at Δ_+ and W_n is the Weierstrass polynomial (of degree n). In particular Q_+ has, for small enough fixed x, exactly n zeros in $\{|w| < \hat{\epsilon}\}$.

We consider now the zeros of $P_+(\hat{x},w)$ at $\hat{x}=0$. Recall that the set $\{x=0\}$ is invariant for the vector field $\mathbf{v}$, and $\{\hat{x}=0\}$ is invariant for $\Psi_*\mathbf{v}$. Therefore,

$$P(x,y)=x\hat{P}(x,y) \text{ and } P_+(\hat{x},w)=\hat{x}\hat{P}_+(\hat{x},w) \quad ,$$

where (see (5.21))

$$\hat{P}_+(\hat{x},w)=\frac{1}{\hat{x}\ell_+(\hat{x},w)}\left[\alpha_+ P + x\frac{\partial\alpha_+}{\partial y}Q+O(x^2)\right]_{(\hat{x}\ell_+(\hat{x},w),w)} \quad .$$

Hence, for $\hat{x}=0$ we get

$$P_+(0,w)=\alpha_+(0,w)\hat{P}(0,w)+\frac{\partial\alpha_+}{\partial y}(0,w)\,Q(0,w) \quad .$$

Moreover, as $\alpha_+(x,y)=cg(0,y)+O(x)$ (see (5.17)), then $\frac{\partial\alpha_+}{\partial y}(x,y)=cg'_y(0,y)+O(x)$. Therefore, $\hat{P}_+(0,w)=cg(0,w)\left[\hat{P}(0,w)+\frac{g'_y(0,w)}{g(0,w)}Q(0,w)\right]$.

From (5.3) follows that $\hat{P}(0,w)=\frac{c(w)q(w)+q(w)}{w}$ (where $q(w)=Q(0,w)$ as before), and from (5.5) $\frac{g'_y(0,w)}{g(0,w)}=\frac{c(w)q(w)+q(w)}{w}$, where $c(0)=1$. Hence,

$$\begin{aligned}\hat{P}_+(0,w) &= cg(0,w)\left[\tfrac{c(w)q(w)+q(w)}{w}+\tfrac{c(w)q(w)+q(w)}{w}q(w)\right]\\ &= cg(0,w)\left[\tfrac{(\lambda+1)q(w)}{w}\right] \quad .\end{aligned}$$

As $g(0,w)$ satisfies the equation (5.5),

$$g(0,w)=exp\left(\int\frac{-c(y)+\lambda}{y}\right) \quad , \quad and \quad g(0,0)=1.$$

Therefore, $g(0,w)$ does not vanish for small enough w. Hence, as $q(w)$ has a zero of order n at $w=0$, then $\hat{P}_+(0,w)$ has a zero of order $n-1$ for $|w|$ small enough.

From Lemma 5.1, for small enough δ_1 and ϵ_1, $\Delta_0=\{|\hat{x}|<\delta_1\}\times\{|w|<\epsilon_1\}$ there exist $K_p(\hat{x},w)$ and $W_{n-1}(\hat{x},w)$ such that K_P does not vanish in Δ and W_{n-1} is the Weierstrass polynomial of degree $n-1$ such that

$$P_+(\hat{x},w)=K_P(\hat{x},w)W_{n-1}(\hat{x},w) \quad . \tag{5.27}$$

In particular, for fixed $\hat{x}$, $|\hat{x}|<\delta_1$, $P_+(\hat{x},w)$ has exactly $n-1$ zeros in the disk $|w|<\epsilon_1$.

5.8. **End of the proof of Theorem 2.1.** In 5.7 it was proved that the vector fields

$$\mathbf{v}_+=\hat{x}\hat{P}_+\frac{\partial}{\partial\hat{x}}+Q_+\frac{\partial}{\partial w} \quad \text{and} \quad \mathbf{v}_-=\hat{x}\hat{P}_-\frac{\partial}{\partial x}+Q_-\frac{\partial}{\partial w}$$

are generators of the same foliation $\mathcal{F}$ of $\hat{\mathcal{W}}=\Psi(\mathcal{W})$. This implies that in the intersection domain of $\mathbf{v}_+$ and $\mathbf{v}_-$ the following equality must take place:

$$\frac{\hat{x}\hat{P}_-}{Q_-}=\frac{\hat{x}\hat{P}_+}{Q_+}$$

Then, for $\hat{x}\neq 0$,

$$\frac{\hat{P}_-}{Q_-}=\frac{\hat{P}_+}{Q_+}$$

and it can be extended to $\hat{x}=0$. From (5.24), (5.26) and (5.27)

It follows that

$$\frac{\hat{P}_-}{w} = \frac{K_P W_{n-1}}{K_Q W_n} \quad . \tag{5.28}$$

Hence,

$$\frac{\hat{P}_-(\hat{x}, w) W_n(\hat{x}, w)}{w W_{n-1}(\hat{x}, w)} = \frac{K_P(\hat{x}, w)}{K_Q(\hat{x}, w)} \quad . \tag{5.29}$$

We stress that for small enough $\hat{x}$, the right member of (5.27) is holomorphic in the disk $\{|w| > r\}$ for $r > 0$ (see (5.25)). Moreover, as for small enough $\hat{x}$, $W_n(\hat{x}, w)$, $W_{n-1}(\hat{x}, w)$ are polynomials on w, $|w| < \epsilon_1$, they can be extended for any w, $|w| > r$.

Therefore, for small enough fixed $\hat{x}$ the left hand side of (5.29) is holomorphic on $|w| > r$. At the same time, the right hand side of (5.29) is holomorphic on $|w| < \epsilon_1$. Hence, as $r > 0$ is arbitrary we can choose $r < \frac{\epsilon_1}{2}$. Then (5.29) is defined in an annulus and has holomorphic extension for $w \in \mathbb{CP}^1$.

Therefore, by Liouville's Theorem (for $\hat{x}$ fixed) it is constant, $\delta = \delta(\hat{x})$:

$$\frac{\hat{P}_-(\hat{x}, w) W_n(\hat{x}, w)}{w W_{n-1}(\hat{x}, w)} = \delta(\hat{x}) \quad .$$

Thus,

$$\frac{\hat{x} \hat{P}_- W_n}{Q_+ W_{n-1}} = \hat{x} \delta(\hat{x})$$

and

$$\frac{\hat{P}_-(\hat{x}, w)}{Q_+(\hat{x}, w)} = \hat{x} \delta(\hat{x}) \frac{W_{n-1}(\hat{x}, w)}{\hat{x} W_n(\hat{x}, w)} \quad .$$

This last equality implies that the vector field $\mathbf{v}_+$ is proportional (obtained by multiplication by a non vanishing function) to

$$\tilde{\mathbf{v}}_+ = \hat{x} \delta(\hat{x}) W_{n-1}(\hat{x}, w) \frac{\partial}{\partial \hat{x}} + W_n(\hat{x}, w) \frac{\partial}{\partial w}. \tag{5.30}$$

To finish the proof of Theorem 2.1 we stress that by construction $\tilde{\mathbf{v}}_+$ in (5.30) is orbitally analytically equivalent to the original vector field $\mathbf{v}$.

Let $\gamma = \{w = \gamma(\hat{x})\}$ be one separatrix of $\tilde{\mathbf{v}}_+$. The biholomorphism $H(\hat{x}, w) = (\hat{x}, w - \gamma(\hat{x}))$ transforms $\tilde{\mathbf{v}}_+$ to a vector field $\hat{\mathbf{v}}_+ = H_* \tilde{\mathbf{v}}_+$ having $\{\hat{w} = 0\}$, $\hat{w} = w - \gamma(\hat{x})$, as a separatrix. hence, the second component of $\hat{\mathbf{v}}_+$ has the form $\hat{W}_+(\hat{x}, \hat{w}) = \hat{w} \hat{W}_{n-1}(\hat{x}, \hat{w})$, where $\hat{W}_{n-1}(\hat{x}, \hat{w})$ is a Weierstrass polynomial of degree $n-1$. Thus, the vector field $\hat{\mathbf{v}}_+$ has all the required properties. This finishes the proof of Theorem 2.1.

6. Analytic normal form for $n = 2$.

In this section we prove Theorem 2.2. As it was already mentioned in the introduction of this work, Theorem 2.2 shows that (after rotation and rectification of one of its separatrices) nondicritic generic germs of vector fields in $(\mathbb{C}^2, 0)$ have analytic strict orbital normal form given by

$$\mathbf{v}_2(x, y) = (P_2 + xB) \frac{\partial}{\partial x} + (Q_2 + yB) \frac{\partial}{\partial y} \quad ,$$

where P_2, Q_2 are homogeneous polynomials of degree 2, $deg_y Q_2 = 2$, $B(x) = x^2 b(x)$ and

$$b(x) = \sum_{k=0}^{\infty} b_k x^k$$

is analytic.

We begin with the preliminary analytic normal form given in Theorem 2.1 for $n = 2$:

$$\hat{\mathbf{v}}(x,y) = x(a(x)y + b(x))\frac{\partial}{\partial x} \;+\; y(c(x)y + d(x))\frac{\partial}{\partial y} \quad , \tag{6.1}$$

where a, b, c, d are holomorphic functions in $(\mathbb{C}, 0)$, $b(0) = d(0) = 0$.

Let us denote $a_0 = a(0), c_0 = c(0), b_1 = b'(0)$ and $d_1 = d'(0)$. From the genericity assumptions given in section 3.3 it follows that

$$a_0 \neq c_0,\; a_0 \neq 0,\; c_0 \neq 0,\; b_1 \neq 0,\; b_1 \neq d_1 \tag{6.2}$$

Remark 6.1. From $a_0 \neq c_0$ and $b_1 \neq d_1$ we get that the polynomial $R_3(1, u)$ has exactly two (different) roots. If $a_0 = 0$ then $\lambda_\infty = -1$. If $b_1 = 0$ there is a characteristic exponent equal to zero. The same happens for $c_0 = 0$ for the characteristic exponent associated to p_∞.

As $c_0 \neq 0$ we may assume that $c \equiv 1$. Indeed, for x small enough we can divide $\hat{\mathbf{v}}$ by $c(x)$. Moreover, by performing if needed the change of coordinates $x \mapsto g(x) = exp\left(\int \frac{a_0}{xa(x)}dx\right)$ (where g is holomorphic since $\operatorname{res}_{x=0} \frac{a_0}{xa(x)} = 1$) we may assume, without loss of generality that the vector field $\hat{\mathbf{v}}$ defined in (6.1) satisfies

$$c \equiv 1,\; a \equiv a_0\;,\; \text{and from (6.2)} \quad a_0 \neq 1 \quad . \tag{6.3}$$

Proposition 6.1. *Let $\mathbf{v}$ and $\mathbf{w}$ holomorphic vector fields of the form (6.1) satisfying the normalizing conditions (6.3),*

$$\mathbf{v}(x,y) = x(a_0 y + b(x))\tfrac{\partial}{\partial x} \;+\; y(y + d(x))\tfrac{\partial}{\partial y} \quad ,$$

$$\mathbf{w}(x,y) = x(\tilde{a}_0 y + \tilde{b}(x))\tfrac{\partial}{\partial x} \;+\; y(y + \tilde{d}(x))\tfrac{\partial}{\partial y} \quad .$$

The necessary and sufficient conditions for the existence of a holomorphic change of coordinates

$$\begin{gathered} H : (\mathbb{C}, 0) \times \mathbb{CP}^1 \to (\mathbb{C}, 0) \times \mathbb{CP}^1 \\ H : (x, y) \mapsto (\varphi(x), k(x)y) \end{gathered} \tag{6.4}$$

where

$$\varphi(0) = 0; \varphi'(0) = 1; k(0) = 1, \tag{6.5}$$

and such that

$$DH\mathbf{v} = q\,\mathbf{w} \circ H \tag{6.6}$$

where $q = q(x)$ is an holomorphic function

$$q(0) = 1, \tag{6.7}$$

is the solvability of the following equations:

$$\begin{gathered} a_0 = \tilde{a}_0 \\ \tilde{b} \circ \varphi = \left(\tfrac{\varphi(x)}{x}\right)^{\mu} b \end{gathered} \tag{6.8}$$

and

$$(\varphi' \tilde{d} \circ \varphi)(x) = \left(\frac{\varphi(x)}{x}\right)^{\mu+1} \left[\left(\frac{x\varphi'(x)}{\varphi(x)} - 1\right) b(x)\,\mu + d(x)\right] \quad . \tag{6.9}$$

Proof. The substitution of $\mathbf{v}$ and $\mathbf{w}$ and H on (6.6) leads to the equality:

$$\begin{pmatrix} \varphi'(x) & 0 \\ k'(x)y & k(x) \end{pmatrix} \begin{pmatrix} a_0xy + xb(x) \\ y^2 + d(x)y \end{pmatrix} = \begin{pmatrix} q(x)\tilde{a}_0\varphi(x)k(x)\,y + q(x)\varphi(x)\tilde{b}(\varphi(x)) \\ q(x)k^2(x)y^2 + q(x)\tilde{d}(\varphi(x))\,k(x)\,y \end{pmatrix}$$

Therefore,

$$\begin{array}{rcl} \varphi'(x)a_0\,x & = & q(x)\tilde{a}_0k(x)\varphi \\ \varphi'(x)b(x)x & = & q(x)\varphi(x)\tilde{b}(\varphi(x)) \\ k'(x)a_0x + k(x) & = & q(x)k^2(x) \\ k'(x)b(x)x + k(x)d(x) & = & q(x)\tilde{d}(\varphi(x))k(x) \end{array} \tag{6.10}$$

We stress that condition (6.5) and (6.7) imply that $\tilde{a}_0 = a_0$. Hence, the system of equations (6.10) is equivalent to:

$$\frac{x\varphi'}{k\varphi} = q \tag{6.11}$$

$$x\frac{\varphi'}{\varphi} = q\frac{\tilde{b}\circ\varphi}{b} \tag{6.12}$$

$$q = \frac{a_0xk'}{k^2} + \frac{1}{k} \tag{6.13}$$

$$q\,\tilde{d}\circ\varphi = \frac{k'bx}{k} + d \tag{6.14}$$

By substitution of (6.11) in (6.13) we get

$$\frac{\varphi'}{\varphi} = a_0\frac{k'}{k} + \frac{1}{x} \tag{6.15}$$

The integration of (6.15) yields to an explicit expression of φ:

$$\varphi(x) = x\,\left(k(x)\right)^{1/\mu} \quad ,$$

where $\mu = 1/a_0$. Equivalently,

$$k(x) = \left(\frac{\varphi(x)}{x}\right)^{\mu} \tag{6.16}$$

Using (6.16) in (6.11) we get

$$\left(\frac{x}{\varphi(x)}\right)^{\mu+1}\varphi'(x) = q(x) \tag{6.17}$$

The substitution of (6.17) in (6.12), and (6.15), (6.16) in (6.14) yields to the pair of equations:

$$(\tilde{b}\circ\varphi)(x) = \left(\frac{\varphi(x)}{x}\right)^{\mu} b(x)$$

$$(\varphi'\tilde{d}\circ\varphi)\,|_x = \left(\frac{\varphi(x)}{x}\right)^{\mu+1}\left[\left(\frac{x\varphi'(x)}{\varphi(x)} - 1\right)b(x)\,\mu + d(x)\right]$$

This proves the Proposition 6.1 □

In what follows we will prove that generic (in the sense G1,G2,G3) germs of vector fields $\mathbf{v} \in \mathcal{V}_n$ always satisfy the conditions (6.8) and (6.9) of Proposition 6.1. This will imply the existence of an analytic (non-strict) change of coordinates taking the germ $\mathbf{v}$ to its analytic normal form.

Let $\mathbf{v} \in \mathcal{V}_n$ be such that $\mathbf{v}$ satisfies that the generic assumptions G1,G2,G3. By Theorem 2.1 and normalizations (6.2) and (6.3) $\mathbf{v}$ is analytically equivalent to a germ $\mathbf{v}_{norm} \in \mathcal{V}_n$ such that

$$\mathbf{v}_{norm} = x(ya_0 + b(x))\frac{\partial}{\partial x} + y(y + d(x))\frac{\partial}{\partial y} \quad . \tag{6.18}$$

We may write b and d in (6.18) as $b(x) = b_1 x + x^2 b_2(x)$ and $d(x) = d_1 x + x^2 d_2(x)$ where b_2 and d_2 are holomorphic germs in $(\mathbb{C}, 0)$.

Remark 6.2. Any generic germ $\mathbf{v}_{norm}$ as in (6.18) satisfies that: $a_0 \neq 0$ and $b_1\mu - d_1 \neq 0$. Indeed, if $b_1\mu - d_1 = 0$ then the characteristic number at p_∞ is -1. This contradicts the generic assumptions. For a_0 see (6.2).

Lemma 6.1. *There exists a change of coordinates H satisfying the conditions of Proposition 6.1, such that $\mathbf{v}_{norm}$ is analytically equivalent to*

$$\mathbf{v}_{an}(x,y) = x(ya_0 + b_1 x + x^2\beta(x))\frac{\partial}{\partial x} + y(y + d_1 x + x^2\beta(x))\frac{\partial}{\partial y} \quad .$$

Proof. We prove the Lemma by making a direct substitution of

$$\tilde{b}(x) = b_1 x + x^2\beta(x) \qquad \text{and} \qquad \tilde{d}(x) = d_1 x + x^2\beta(x)$$

in the equalities (6.8) and (6.9):

$$b_1\varphi + \varphi^2\beta\circ\varphi = \left(\frac{\varphi}{x}\right)^{\mu} b \tag{6.19}$$

$$\varphi'\,\left(d_1\varphi + \varphi^2\beta\circ\varphi\right) = \left(\frac{\varphi}{x}\right)^{\mu+1}\left[d + b\mu\left(\frac{x\varphi'}{\varphi} - 1\right)\right] \quad . \tag{6.20}$$

Multiplying the equation (6.19) by φ' and substracting it from (6.20) we get

$$\varphi'(d_1 - b_1)\varphi = \left(\frac{\varphi}{x}\right)^{\mu}\left[\varphi'\,b(\mu-1) + (d - b\mu)\frac{\varphi}{x}\right] \quad .$$

Therefore

$$\varphi'\left[xb(\mu-1) + (b_1 - d_1)\frac{x^{\mu+1}}{\varphi^{\mu-1}}\right] = (\mu b - d)\varphi \quad . \tag{6.21}$$

The substitution in (6.21) of the expression for b and d, and $\varphi(x) = x\Psi(x)$ leads to the equality:

$$x\Psi' + \Psi = \frac{[\mu b_1 - d_1 + (\mu b_2(x) - d_2(x))\,x]\Psi}{(\mu-1)(b_1 + xb_2(x)) + (b_1 - d_1)\Psi^{1-\mu}} \tag{6.22}$$

Let us define $F(x,\Psi) = \frac{x\Psi' + \Psi}{\Psi}$. Then Ψ is solution of the differential equation

$$\Psi' = \left[\frac{F(x,\Psi) - 1}{x}\right]\Psi \tag{6.23}$$

with initial condition $\Psi(0) = 1$ (see (6.5)).

Together with equation (6.23) we consider the vector field

$$\xi(x,\Psi) = x\frac{\partial}{\partial x} + (F(x,\Psi) - 1)\,\Psi\frac{\partial}{\partial y} \quad . \tag{6.24}$$

Since $\mu b_1 - d_1 \neq 0$ (see Remark 6.1), then the vector field is holomorphic in a neighborhood of the singular point $x = 0, \Psi(0) = 1$:

$$\xi(0,1) = 0\frac{\partial}{\partial x} + (F(0,1) - 1)\,\Psi\frac{\partial}{\partial y} = (0,0) \quad ,$$

where

$$F(0,1) = \frac{\mu b_1 - d_1}{(\mu-1)b_1 + b_1 - d_1} = 1 \ .$$

The eigenvalues of the linearization at the singular point $(0,1)$ of the vector field ξ are $\lambda_1 = 1$ (for $e_1 = \frac{\partial}{\partial x}$) and $\lambda_2 = F'_\Psi(0,1)$ (for the eigenvector e_2 transversal to $\{x=0\}$).

Since $F'_\Psi = (\mu-1)(b_1-d_1)(b_1\mu - d_1)^{-1} \neq 0$, by the Hadamard-Perron's Theorem, there is a smooth separatrix γ at the singular point $(0,1)$ with tangent direction at $(0,1)$ equal to e_2.

This curve is, locally, the graphic of a holomorphic function $\Psi = \Psi(x)$ satisfying equation (6.23) and such that $\Psi(0) = 1$.

We now substitute this function Ψ in (6.19):

$$\varphi^2 \beta \circ \varphi = \left[\Psi^\mu (b_1 x + b_2(x)x^2) - b_1\varphi\right] \ .$$

Therefore

$$\beta \circ \varphi = \frac{x^2}{\varphi^2}\left[\frac{\Psi^\mu(b_1 + xb_2(x))}{x} - \frac{b_1\Psi}{x}\right] \ .$$

Thus,

$$\beta \circ \varphi = \frac{1}{\Psi^2}\left[\Psi^\mu b_2(x) + \frac{b_1(\Psi^\mu - \Psi)}{x}\right] \ ;$$

and since $\Psi(0) = 1$, $\beta \circ \varphi$ is holomorphic in $(\mathbb{C}, 0)$.

We know that $\varphi = x\Psi$, $\Psi(0) = 1$ is holomorphic, thus

$$\beta = \frac{1}{\Psi^2}\left[\Psi\, b_2(x) + b_1\left(\frac{\Psi^\mu - \Psi}{x}\right)\right] \circ \varphi^{-1} \ .$$

is also holomorphic in $(\mathbb{C}, 0)$, and both, φ and β are solutions of equations (6.19) and (6.20). Therefore, by Proposition 6.1, the vector field $\mathbf{v}_{norm}$ is analytically equivalent at the origin to

$$\mathbf{v}_{an} = x(ya_0 + b_1x + x^2\beta(x))\frac{\partial}{\partial x} + y(y + d_1x + x^2\beta(x))\frac{\partial}{\partial y} \ .$$

Lemma 6.1 is proved. □

Finally, as $\mathbf{v}$ is analytically equivalent to $\mathbf{v}_{norm}$, then it is also analytically equivalent to

$$\mathbf{v}_{an} = x(ya_0 + b_1x + x^2\beta(x))\frac{\partial}{\partial x} + y(y + d_1x + x^2\beta(x))\frac{\partial}{\partial y} \ .$$

Theorem 2.2 is proved.

6.1. **Proof of Theorem 2.3.** To prove Theorem 2.3 we recall that by the Theorem of formal orbital strict classification (see section 2) any generic (in the sense $\tilde{\text{G}}1$, $\tilde{\text{G}}2$, $\tilde{\text{G}}3$) nondicritic germ of vector field $\mathbf{v} \in \mathcal{V}_2$ is formal orbital strict equivalent to a formal vector field $\mathbf{v}_f$

$$\mathbf{v}_f = (P_2 + xB)\frac{\partial}{\partial x} + (Q_2 + yB)\frac{\partial}{\partial y} \tag{6.25}$$

where $\mathbf{v}_0 = P_2\frac{\partial}{\partial x} + Q_2\frac{\partial}{\partial y}$, P_2, Q_2 are homogeneous polynomials of degree 2, $deg_y Q_2 = 2$, $B(x) = x^2 b(x)$, and $b(x) = \sum_{k=0}^{\infty} b_k\, x^k$, $b_k \in \mathbb{C}$, is a formal power series.

We need to prove that if we assume that the singular point at infinity of the blow-up $\tilde{\mathbf{v}}$ of $\mathbf{v}$ is linearizable, then the formal normal form (6.2) is analytic; i.e. we will prove that b is a convergent power series.

By Theorem 2.2 we know that $\mathbf{v}$ is orbitally analytically equivalent (non necessarily strict) to a germ of holomorphic vector field of the form

$$x(\mathcal{P}_1 + x^2\beta(x))\frac{\partial}{\partial x} + y(\mathcal{Q}_1 + x^2\beta(x))\frac{\partial}{\partial y}$$

where $\mathcal{P}_1(x,y) = ya_0 + b_1x$, $\mathcal{Q}_1(x,y) = y + d_1x$, and $\beta(x)$ is a holomorphic function in a neighborhood of the origin.

To finish the proof of Theorem 2.3 we stress that it is always possible to define a linear transformation:

$$L : (x, y) \mapsto (\alpha_0 x, \alpha_1 x + \alpha_2 y) \tag{6.26}$$

such that for appropriate constants a_0, b_1, d_1,

$$\mathbf{v}_{an} = x(ya_0 + b_1x + x^2\beta(x))\frac{\partial}{\partial x} + y(y + d_1x + x^2\beta(x))\frac{\partial}{\partial y}$$

is linearly equivalent to

$$\mathbf{w} = (P_2 + xB)\frac{\partial}{\partial x} + (Q_2 + yB)\frac{\partial}{\partial y} \quad ,$$

where the components of

$$\mathbf{v}_0 = P_2\frac{\partial}{\partial x} + Q_2\frac{\partial}{\partial y}$$

are homogeneous polynomials of degree 2, $P_2(0, y) = 0$, $deg_y Q_2 = 2$, and $B(x) = x^2b(x)$.

The equivalence between $\mathbf{v}$ and $\mathbf{w}$ is strict (orbital and analytic). Then by the uniqueness of the formal normal form under strict orbital equivalence, the formal normal form of $\mathbf{v}$, $\mathbf{v}_f$, and $\mathbf{w}$ must coincide.

Thus, $B(x)$ is analytic and therefore $\mathbf{v}_f$ is analytic too. Theorem 2.3 is proved.

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STRONG TOPOLOGICAL INVARIANCE OF THE MONODROMY GROUP AT INFINITY FOR QUADRATIC VECTOR FIELDS

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ABSTRACT. In this work we consider foliations on $\mathbb{C}\mathrm{P}^2$ which are generated by quadratic vector fields on $\mathbb{C}^2$. Generically these foliations have isolated singularities and an invariant line at infinity. We say that the monodromy groups at infinity of two such foliations having the same singular points at infinity are strongly analytically equivalent provided there exists a germ of a conformal mapping at zero which conjugates the monodromy maps defined *along the same loops* on the infinite leaf.

The object of this paper is to show that topologically equivalent generic foliations from this class must have, after an affine change of coordinates, their monodromy groups at infinity strongly analytically conjugated.

As a corollary it is proved that any two such generic and sufficiently close foliations can only be topologically conjugated if they are affine equivalent. This improves, in the case of quadratic vector fields, the main result of [2] which claims that two generic, topologically equivalent and sufficiently close foliations are affine equivalent provided the conjugating homeomorphism is close enough to the identity map.

1. INTRODUCTION

It is a well known result that every polynomial vector field on $\mathbb{C}^2$ can be analytically extended to a line field on $\mathbb{C}\mathrm{P}^2$. In this paper we will consider holomorphic foliations on $\mathbb{C}\mathrm{P}^2$ which in a *fixed* affine chart are generated by quadratic vector fields.

1.1. Holomorphic foliations from the class $\mathcal{A}_2$.

Definition 1. *Let $\mathbb{I}$ be a line on $\mathbb{C}\mathrm{P}^2$ which will be fixed throughout this paper. The space $\mathcal{A}_n$ is defined to be the class of all foliations on $\mathbb{C}\mathrm{P}^2$ generated by a polinomial vector field of degree n in the affine chart $\mathbb{C}^2 \approx \mathbb{C}\mathrm{P}^2 \setminus \mathbb{I}$ and having only isolated singularities.*

Having fixed this affine chart, the space $\mathcal{A}_n$ can naturally be embedded in the projective vector space of polynomial vector fields of degree at most n; two such vector fields generate the same foliation if and only if they differ only by a scalar multiple.

In this work we will deal exclusively with the class of foliations $\mathcal{A}_2$. Let $\mathcal{A}_2'$ be the subclass of foliations from $\mathcal{A}_2$ which have the line at infinity $\mathbb{I}$ invariant and exactly three singularities on $\mathbb{I}$. The space $\mathcal{A}_2'$ is Zariski open in $\mathcal{A}_2$.

Definition 2. *Two foliations $\mathcal{F}, \widetilde{\mathcal{F}} \in \mathcal{A}_2$ are topologically equivalent provided there exists a homeomorphism $\mathcal{H}\colon \mathbb{C}\mathrm{P}^2 \to \mathbb{C}\mathrm{P}^2$ which preserves the orientation both on the leaves and on $\mathbb{C}\mathrm{P}^2$ and brings the leaves of the first foliation to those of the second one. In such case we will say that the two foliations are topologically conjugated by the homeomorphism $\mathcal{H}$. The foliations are said to be affine equivalent if $\mathcal{H}$ is an affine transformation.*

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Let $\mathcal{F}$, $\widetilde{\mathcal{F}}$ in $\mathcal{A}_2'$ be two foliations having the same singular locus at infinity $\Sigma = \mathrm{Sing}(\mathcal{F}) \cap \mathbb{I}$. Choose a base point b on the infinite leaf $\mathcal{L}_{\mathcal{F}} = \mathbb{I} \setminus \Sigma$ and consider, for each element on the fundamental group $\gamma \in \pi_1(\mathbb{I} \setminus \Sigma, b)$, the monodromy transformations Δ_γ and $\widetilde{\Delta}_\gamma$ corresponding to the foliations $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ respectively.

Definition 3. *We say that the monodromy groups at infinity $G_{\mathcal{F}}$ and $G_{\widetilde{\mathcal{F}}}$ of two foliations having the same singular set at infinity are strongly analytically equivalent provided there exists a germ h of a conformal mapping at zero such that*

$$h \circ \Delta_\gamma = \widetilde{\Delta}_\gamma \circ h$$

for any element γ of the fundamental group of the infinite leaf.

1.2. Main result.

Theorem 1. *If two generic foliations from the class $\mathcal{A}_2$ with the same singular points at infinity are topologically equivalent and the conjugacy fixes these singular points then their monodromy groups at infinity are strongly analytically equivalent.*

This property is called strong topological invariance of the monodromy group at infinity. Note that for any two topologically equivalent foliations in $\mathcal{A}_2'$ we can always assume, after an affine change of coordinates, that both foliations have the same singular points at infinity and that the conjugating homeomorphism preserves these singular points.

Previously the following invariance property was known:

Proposition 1 ([5])**.** *If two foliations from $\mathcal{A}_2'$ with non-solvable monodromy group at infinity are topologically equivalent and have the same singular points at infinity then for any set of generators γ_1, γ_2 of the fundamental group of the infinite leaf there exists another set of generators ρ_1, ρ_2 and an analytic germ $h\colon (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ such that*

$$h \circ \Delta_{\gamma_i} = \widetilde{\Delta}_{\rho_i} \circ h, \quad i = 1, 2.$$

Contrary to this proposition, Theorem 1 claims that the second set of generators can be chosen to coincide with the original set of generators. As an important corollary of Theorem 1 we obtain the following result:

Theorem 2. *A generic foliation from the class $\mathcal{A}_2$ has a neighborhood in this class such that any other foliation in this neighborhood which is topologically equivalent to the first foliation must be affine equivalent to the original foliation.*

Note that in the above theorem no assumptions are being made about the conjugating homeomorphism. This property was introduced in [3] and is called *ideal rigidity*. However, it was stated as an *unknown property* for polynomial foliations.

1.3. Sketch of the proofs.

A topological equivalence between two generic foliations from the class $\mathcal{A}_2$ having the same singular points at infinity restricts to a homeomorphism $H\colon \mathcal{L}_{\mathcal{F}} \to \mathcal{L}_{\widetilde{\mathcal{F}}}$ from the infinite leaf onto itself. Theorem 1 is proved by studying the isomorphisms that such homeomorphism induces on the fundamental group and first homology group of the infinite leaf. It will be shown that if the conjugacy preseves the singular points at infinity then the induced isomorphism on homology is the identity map and therefore an inner automorphism is induced on the fundamental group. From this fact we will easily deduce that the monodromy groups at infinity are strongly analytically equivalent.

Notice that Theorem 1 is stated only for generic foliations from the class $\mathcal{A}_2$. The proof of Theorem 1 cannot be carried out in a similar way for the classes $\mathcal{A}_n$ with $n > 2$ due to an algebraic obstruction; if the fundamental group of the infinite leaf is free on more than two generators a trivial action on homology[1] does not imply that the action on the fundamental group is an inner automorphism. The fact that the action on fundamental group is an inner automorphism is the key ingredient in the proof of Theorem 1.

Ideal rigidity is very closely related to a property called *absolute rigidity* which was introduced in [2], yet in Theorem 2 there are no restrictions on the conjugating homeomorphism.

Definition 4. *A foliation $\mathcal{F} \in \mathcal{A}_n$ is absolutely rigid in the class $\mathcal{A}_n$ provided there exists a neighborhood $U \subseteq \mathcal{A}_n$ of $\mathcal{F}$ and a neighborhood $\mathcal{U}$ of the identity $id\colon \mathbb{C}\mathrm{P}^2 \to \mathbb{C}\mathrm{P}^2$ in the space Homeo$(\mathbb{C}\mathrm{P}^2)$ of homeomorphisms of $\mathbb{C}\mathrm{P}^2$ onto itself such that every foliation $\mathcal{F}' \in U$ topologically conjugated to $\mathcal{F}$ by a homeomorphism $\mathcal{H} \in \mathcal{U}$ is affine equivalent to $\mathcal{F}$.*

Proposition 2 ([2])**.** *A generic foliation from the class $\mathcal{A}_n$ is absolutely rigid.*

In the proof of Proposition 2 the closeness of the topological conjugacy to the identity homeomorphism is required in order to guarantee that the monodoromy groups at infinity are strongly analytically equivalent. In the case of quadratic vector fields, in virtue of Theorem 1, such hypothesis can be dropped and so Theorem 2 is deduced.

1.4. **Genericity assumptions.** Consider the following properties for a foliation $\mathcal{F} \in \mathcal{A}'_2$:

- (i) The monodromy group at infinity $G_{\mathcal{F}}$ is non-solvable;
- (ii) The characteristic numbers of the singular points at infinity are pairwise different;
- (iii) All singularities of $\mathcal{F}$ are hyperbolic;
- (iv) Foliation $\mathcal{F}$ has no algebraic leaves except for the infinite line.

The genericity of conditions (i) and (iv) is discussed in [5]. Foliations having pairwise different characteristic numbers form a complex Zariski open subset of $\mathcal{A}'_2$ and the set of foliations with hyperbolic singularities determines a real Zariski subset of $\mathcal{A}'_2$.

It is proved in [5] that non-solvable groups of germs are topologically rigid, hence condition (i) is sufficient to prove Theorem 1. For Theorem 2 all conditions (i)–(iv) are assumed.

2. Induced automorphisms on the fundamental group and first homology group

In the following constructions we will consider foliations with *close* tuples of singular points at infinity yet not necessarily equal.

Let $\mathcal{F} \in \mathcal{A}'_2$ be a generic foliation and let $\Sigma = \{a_1, a_2, a_3\}$ be its singular locus at infinity. Let D_1, D_2, D_3 be open disks on $\mathbb{I}$ centered at a_1, a_2, a_3 respectively with pairwise disjoint closures and define $D = \cup D_i$. Let b be an arbitrary point in $\mathbb{I} \setminus \overline{D}$.

Denote by $\widetilde{U}$ the set of those foliations in $\mathcal{A}'_2$ with the property of having their singularities at infinity on D and having exactly one singularity on each D_i.

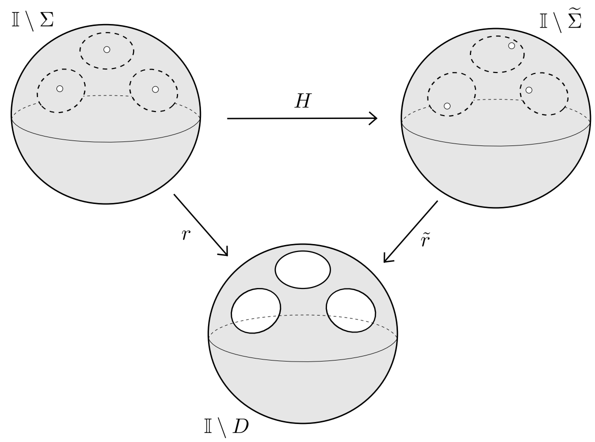
Definition 5. *Denote by $\mathcal{T}_{\mathcal{OP}}(\mathcal{F}, b)$ the set of all pairs $(\widetilde{\mathcal{F}}, \mathcal{H})$ in the product $\widetilde{U} \times$ Homeo$(\mathbb{C}\mathrm{P}^2)$ such that $\mathcal{H}$ is a topological conjugacy between $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ that fixes the point b.*

Choose any $(\widetilde{\mathcal{F}}, \mathcal{H}) \in \mathcal{T}_{\mathcal{OP}}(\mathcal{F}, b)$. The foliation $\mathcal{F}$, and so does $\widetilde{\mathcal{F}}$, has a unique algebraic leaf; the punctured infinite line. This implies that the homeomorphism $\mathcal{H}\colon \mathbb{C}\mathrm{P}^2 \to \mathbb{C}\mathrm{P}^2$ preserves the line $\mathbb{I}$ and maps bijectively the singular set $\Sigma = \mathrm{Sing}(\mathcal{F}) \cap \mathbb{I}$ onto $\widetilde{\Sigma} = \mathrm{Sing}(\widetilde{\mathcal{F}}) \cap \mathbb{I}$.

From now on if $\mathcal{H}$ is a homeomorphism from $\mathbb{C}\mathrm{P}^2$ onto itself which preserves the infinite line $\mathbb{I}$ we shall denote by H its restriction $H = \mathcal{H}|_{\mathbb{I}}$.

If $\Sigma \neq \widetilde{\Sigma}$ the fundamental groups $\pi_1(\mathbb{I} \setminus \Sigma, b)$ and $\pi_1(\mathbb{I} \setminus \widetilde{\Sigma}, b)$ do not coincide. However, both surfaces $\mathbb{I} \setminus \Sigma$ and $\mathbb{I} \setminus \widetilde{\Sigma}$ deformation retract onto $\mathbb{I} \setminus D$ and thus both the fundamental groups $\pi_1(\mathbb{I} \setminus \Sigma, b)$ and $\pi_1(\mathbb{I} \setminus \widetilde{\Sigma}, b)$ are naturally isomorphic to the group $\pi_1(\mathbb{I} \setminus D, b)$.

[1] See Section 2 for the corresponding definitions.



In fact, for every loop γ on $\mathbb{I}\setminus\Sigma$ based on b we can assume, without loss of generality, that it is contained in $\mathbb{I}\setminus D$ and so it can be regarded indistinctly as an element of any of the groups $\pi_1(\mathbb{I}\setminus\Sigma, b)$, $\pi_1(\mathbb{I}\setminus\widetilde{\Sigma}, b)$, $\pi_1(\mathbb{I}\setminus D, b)$. The concept of strong analytic equivalence for the monodromy groups can naturally be extended for pairs of foliations whose singularities at infinity are *close enough*. In particular, this can be done for foliations $\mathcal{F}$, $\widetilde{\mathcal{F}}$ if $\widetilde{\mathcal{F}}$ belongs to the neighborhood $\widetilde{U}$ constructed above.

Definition 3'. *Let $\widetilde{\mathcal{F}} \in \widetilde{U}$. We say that the monodromy groups at infinity $G_{\mathcal{F}}$ and $G_{\widetilde{\mathcal{F}}}$ are strongly analytically equivalent provided there exists a germ h of a conformal mapping at zero such that*

$$h \circ \Delta_\gamma = \widetilde{\Delta}_\gamma \circ h$$

for any element γ of the fundamental group $\pi_1(\mathbb{I}\setminus D, b)$.

We are now going to define *the action that H has on the fundamental group* by assigning to each pair $(\widetilde{\mathcal{F}}, \mathcal{H}) \in \mathcal{T}_{\mathcal{OP}}(\mathcal{F}, b)$ an element of the automorphism group of the group $\pi_1(\mathbb{I}\setminus D, b)$ in the following way:

Let $r\colon \mathbb{I}\setminus\Sigma \to \mathbb{I}\setminus D$ and $\tilde{r}\colon \mathbb{I}\setminus\widetilde{\Sigma} \to \mathbb{I}\setminus D$ be the retractions mentioned above. Since they are homotopy equivalences they induce isomorphisms

$$r_*\colon \pi_1(\mathbb{I}\setminus\Sigma, b) \to \pi_1(\mathbb{I}\setminus D, b) \quad \text{and} \quad \tilde{r}_*\colon \pi_1(\mathbb{I}\setminus\widetilde{\Sigma}, b) \to \pi_1(\mathbb{I}\setminus D, b)$$

on the fundamental groups. The homeomorphism $H|_{\mathbb{I}\setminus\Sigma}$ also induces an isomorphism

$$H_*\colon \pi_1(\mathbb{I}\setminus\Sigma, b) \to \pi_1(\mathbb{I}\setminus\widetilde{\Sigma}, b).$$

There exists a unique group automorphism $\Phi(H)\colon \pi_1(\mathbb{I}\setminus D, b) \to \pi_1(\mathbb{I}\setminus D, b)$ which makes the following diagram commutative:

$$\begin{array}{ccc} \pi_1(\mathbb{I}\setminus\Sigma, b) & \xrightarrow{\ H_*\ } & \pi_1(\mathbb{I}\setminus\widetilde{\Sigma}, b) \\ \downarrow r_* & & \downarrow \tilde{r}_* \\ \pi_1(\mathbb{I}\setminus D, b) & \xrightarrow[\ \Phi(H)\ ]{} & \pi_1(\mathbb{I}\setminus D, b) \end{array}$$

Thus we get a well defined map

$$\Phi\colon \mathcal{T}_{\mathcal{OP}}(\mathcal{F}, b) \to \operatorname{Aut}(\pi_1(\mathbb{I}\setminus D, b)).$$

Here $\mathrm{Aut}(\pi_1(\mathbb{I}\setminus D,b))$ denotes the automorphism group of $\pi_1(\mathbb{I}\setminus D,b)$. For the sake of simplicity we shall write $\Phi(H)$ instead of $\Phi(\widetilde{\mathcal{F}},\mathcal{H})$.

2.1. **Inner automorphisms of the fundamental group.** Let $(\widetilde{\mathcal{F}},\mathcal{H})\in\mathcal{T}_{\mathcal{OP}}(\mathcal{F},b)$ and suppose $\Phi(H)=id$. For any germ of cross-section Γ at b transversal to the leaves of $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ there exists (Proposition 1) an analytic germ

$$h\colon(\Gamma,b)\to(\Gamma,b)$$

induced by $\mathcal{H}$ that conjugates the monodromy groups in the following way

$$h\circ\Delta_{\gamma_i}=\widetilde{\Delta}_{\rho_i}\circ h,\quad i=1,2$$

where ρ_i is defined by the composition $\rho_i=H\circ\gamma_i$ and γ_1, γ_2 are canonical generators of $\pi_1(\mathbb{I}\setminus D,b)$. But the condition $\Phi(H)=id$ implies that the loops ρ_i are homotopic to the corresponding γ_i and so the monodromy groups are strongly analytically equivalent.

The following lemma shows that we can also deduce the strong analytic equivalence of the monodromy groups in the case when the action on the fundamental group is an inner automorphism, not necessarily trivial.

Lemma 1. *If $(\widetilde{\mathcal{F}},\mathcal{H})\in\mathcal{T}_{\mathcal{OP}}(\mathcal{F},b)$ and $\Phi(H)$ is an inner automorphism on $\pi_1(\mathbb{I}\setminus D,b)$ then the monodromy groups $G_{\mathcal{F}}$ and $G_{\widetilde{\mathcal{F}}}$ are strongly analytically equivalent.*

Proof. Let $(\widetilde{\mathcal{F}},\mathcal{H})\in\mathcal{T}_{\mathcal{OP}}(\mathcal{F},b)$ and suppose $\Phi(H)$ is an inner automorphism; namely, there exists an element $\lambda\in\pi_1(\mathbb{I}\setminus D,b)$ such that for any $\gamma\in\pi_1(\mathbb{I}\setminus D,b)$

$$\Phi(H)(\gamma)=\lambda\cdot\gamma\cdot\lambda^{-1}.$$

Since the curve $H\circ\gamma$ is homotopic to $\Phi(H)(\gamma)$ for any $\gamma\in\pi_1(\mathbb{I}\setminus D,b)$ there exists an analytic germ $h\colon(\Gamma,b)\to(\Gamma,b)$ such that

$$h\circ\Delta_\gamma=\widetilde{\Delta}_{\lambda\cdot\gamma\cdot\lambda^{-1}}\circ h.$$

This implies

$$h\circ\Delta_\gamma=\widetilde{\Delta}_{\lambda^{-1}}\circ\widetilde{\Delta}_\gamma\circ\widetilde{\Delta}_\lambda\circ h,$$

and so

$$h_0\circ\Delta_\gamma=\widetilde{\Delta}_\gamma\circ h_0,$$

where h_0 is defined to be $h_0=\widetilde{\Delta}_\lambda\circ h$. □

2.2. **Induced action on homology.** In an analogous way, moving on to the first homology group, we are now going to define a map

$$\begin{aligned}\eta\,:\,\mathcal{T}_{\mathcal{OP}}(\mathcal{F},b)&\longrightarrow\mathrm{Aut}(H_1(\mathbb{I}\setminus D;\mathbb{Z}))\\(\widetilde{\mathcal{F}},\mathcal{H})&\longmapsto\eta(H)\end{aligned}$$

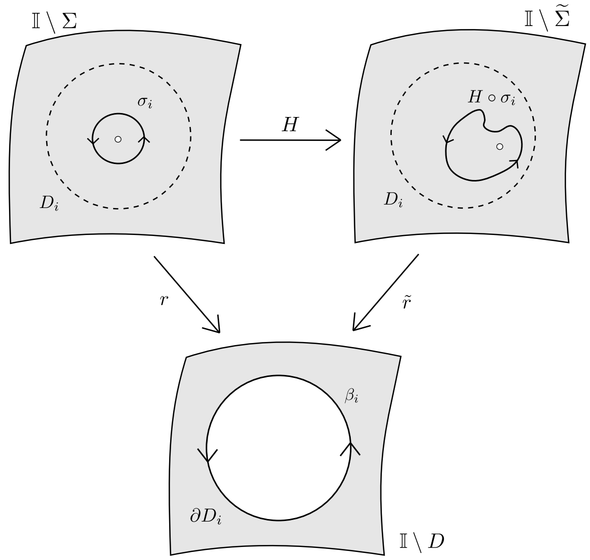
such that $\eta(H)$ is the only automorphism which makes the following diagram commutative:

$$\begin{array}{ccc}H_1(\mathbb{I}\setminus\Sigma;\mathbb{Z})&\xrightarrow{H_*}&H_1(\mathbb{I}\setminus\widetilde{\Sigma};\mathbb{Z})\\ \downarrow r_*&&\downarrow\tilde{r}_*\\ H_1(\mathbb{I}\setminus D;\mathbb{Z})&\xrightarrow[\eta(H)]{}&H_1(\mathbb{I}\setminus D;\mathbb{Z})\end{array}$$

Lemma 2. *Let* $(\widetilde{\mathcal{F}}, \mathcal{H}) \in \mathcal{T}_{\mathcal{OP}}(\mathcal{F}, b)$. *Then* $\eta(H) = id$ *provided that* $H(a_i) \in D_i$ *for each* $i = 1, 2, 3$.

Proof. Let us choose 1-cycles[2] $\sigma_1, \sigma_2 \colon \Delta^1 \to \mathbb{I} \setminus \Sigma$ in such a way that they make up a canonical set of generators of the group $H_1(\mathbb{I} \setminus \Sigma; \mathbb{Z})$ and $\sigma_i(\Delta^1) \subseteq D_i$, $H(\sigma_i(\Delta^1)) \subseteq D_i$.

Define now $\beta_i = r \circ \sigma_i$. In this way β_1, β_2 is a canonical set of generators of the group $H_1(\mathbb{I} \setminus D; \mathbb{Z})$ which satisfies $\beta_i(\Delta^1) \subseteq \partial D_i$.



$H(\sigma_i(\Delta^1)) \subseteq D_i$ and so $\tilde{r} \circ H \circ \sigma_i(\Delta^1) \subseteq \partial D_i$. Therefore $(\tilde{r} \circ H)_* \sigma_i$ must be homologous to an integer multiple of β_i. This implies that the automorphism $\eta(H)$ can be expressed as

$$\eta(H)(\beta_1) = m\beta_1, \quad \eta(H)(\beta_2) = n\beta_2.$$

for some integers m, n.

On the other hand, the composition $\tilde{r} \circ H \colon \mathbb{I} \setminus \Sigma \to \mathbb{I} \setminus D$ is a homotopy equivalence and so it induces an isomorphism on the homology group. Thus $m\beta_1$ and $n\beta_2$ generate $H_1(\mathbb{I} \setminus D; \mathbb{Z})$. This is only possible if $m, n = \pm 1$, i.e. $(\tilde{r} \circ H)_* \sigma_i \simeq \pm \beta_i$, $i = 1, 2$. But both $\tilde{r}$ and H are orientation preserving maps and so we conclude that $(\tilde{r} \circ H)_* \sigma_i \simeq \beta_i$ and thus $\eta(H)$ is the identity automorphism. □

3. Proof of the main results

3.1. **Proof of Theorem 1.**

Proof of Theorem 1. Suppose $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ are generic foliations having the same singular points at infinity, are topologically conjugated by a homeomorphism $\mathcal{H}$ and this topological conjugacy preserves the singular points at infinity. Without loss of generality we can assume it also preserves the base point b. Therefore $(\widetilde{\mathcal{F}}, \mathcal{H}) \in \mathcal{T}_{\mathcal{OP}}(\mathcal{F}, b)$ and clearly the condition $H(a_i) \in D_i$ is satisfied. By Lemma 2 the action on homology $\eta(H)$ is the identity automorphism.

By Hurewicz Theorem $H_1(\mathbb{I} \setminus D; \mathbb{Z})$ is naturally isomorphic to the abelianization of $\pi_1(\mathbb{I} \setminus D, b)$. Let $q \colon \pi_1(\mathbb{I} \setminus D, b) \to H_1(\mathbb{I} \setminus D; \mathbb{Z})$ be the canonical projection. Through q every automorphism f on $\pi_1(\mathbb{I} \setminus D, b)$ descends to a unique automorphism on $H_1(\mathbb{I} \setminus D; \mathbb{Z})$. This assignment gives

[2] Δ^1 is the standard 1-simplex $\Delta^1 = \{(t_0, t_1) \in \mathbb{R}^2 \mid t_0 + t_1 = 1 \text{ and } t_1, t_2 \geq 0\}$.

raise to a natural and surjective homomorphism $T\colon \mathrm{Aut}(\pi_1(\mathbb{I}\setminus D,b)) \to \mathrm{Aut}(H_1(\mathbb{I}\setminus D;\mathbb{Z}))$ such that $\forall f \in \mathrm{Aut}(\pi_1(\mathbb{I}\setminus D,b))$ the diagram commutes:

$$\begin{array}{ccc} \pi_1(\mathbb{I}\setminus D,b) & \xrightarrow{\ f\ } & \pi_1(\mathbb{I}\setminus D,b) \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q} \\ H_1(\mathbb{I}\setminus D;\mathbb{Z}) & \xrightarrow[T(f)]{} & H_1(\mathbb{I}\setminus D;\mathbb{Z}) \end{array}$$

Moreover, the kernel of such homomorphism consists precisely on those automorphisms on $\pi_1(\mathbb{I}\setminus D,b)$ which are inner automorphisms [3] [4]; i.e. $\mathrm{Ker}(T) = \mathrm{Inn}(\pi_1(\mathbb{I}\setminus D,b))$.

The homeomorphism H satisfies $q\circ\Phi(H) = \eta(H)\circ q$,

$$\begin{array}{ccc} \pi_1(\mathbb{I}\setminus D,b) & \xrightarrow{\ \Phi(H)\ } & \pi_1(\mathbb{I}\setminus D,b) \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q} \\ H_1(\mathbb{I}\setminus D;\mathbb{Z}) & \xrightarrow[\eta(H)]{} & H_1(\mathbb{I}\setminus D;\mathbb{Z}) \end{array}$$

and therefore $\eta(H) = T(\Phi(H))$. Since $\eta(H) = id$ then $\Phi(H) \in \mathrm{Ker}(T)$ and so is an inner automorphism on $\pi_1(\mathbb{I}\setminus D,b)$. By Lemma 1 the monodromy groups $G_{\mathcal{F}}$ and $G_{\widetilde{\mathcal{F}}}$ are strongly analytically equivalent. □

3.1.1. *A remark about conjugating homeomorphisms.* Theorem 1 has been proved above by exhibiting explicitly a conformal germ $h_0\colon (\mathbb{C},0)\to(\mathbb{C},0)$ that conjugates the monodromy groups. In fact, this germ can be realized as the transverse component of a global topological conjugacy between $\mathcal{F}$ and $\widetilde{\mathcal{F}}$. Namely, we have the following lemma:

Lemma 3. *Let $(\widetilde{\mathcal{F}},\mathcal{H}) \in \mathcal{T}_{\mathcal{OP}}(\mathcal{F},b)$ and choose a cross-section Γ at b transversal to the leaves of $\mathcal{F}$ and $\widetilde{\mathcal{F}}$. If $H(a_i)\in D_i$ for each $i=1,2,3$ then there exists another topological conjugacy $\mathcal{H}_0\colon \mathbb{C}\mathrm{P}^2\to\mathbb{C}\mathrm{P}^2$ between $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ such that its transverse component*

$$h_0 = {\mathcal{H}_0}^{\pitchfork}_{b}\colon (\Gamma,b) \longrightarrow (\Gamma,b)$$

yields a strong analytic equivalence between the monodromy groups $G_{\mathcal{F}}$ and $G_{\widetilde{\mathcal{F}}}$;

$$h_0\circ\Delta_\gamma = \widetilde{\Delta}_\gamma\circ h_0$$

for any element $\gamma\in\pi_1(\mathbb{I}\setminus D,b)$.

Proof. We have a topological conjugacy $\mathcal{H}$ that satisfies $H(a_i)\in D_i$. By Lemma 2 the action on homology $\eta(H)$ is trivial and so $\Phi(H)$ is an inner automorphism on $\pi_1(\mathbb{I}\setminus D,b)$. By the same arguments used on Section 2.1 there is an analytic germ

$$h\colon (\Gamma,b)\to(\Gamma,b)$$

induced by $\mathcal{H}$ (its transverse component at b) and an element $\lambda\in\pi_1(\mathbb{I}\setminus D,b)$ that the monodromy groups $G_{\mathcal{F}}$ and $G_{\widetilde{\mathcal{F}}}$ are conjugated in the following way

$$h\circ\Delta_\gamma = \widetilde{\Delta}_{\lambda^{-1}}\circ\widetilde{\Delta}_\gamma\circ\widetilde{\Delta}_\lambda\circ h,$$

[3] This statement would not hold if $\pi_1(\mathbb{I}\setminus D,b)$ was a free groups of rank grater than two. This fact is precisely the obstruction for proving the same result in the case of polynomial vector fields of degree $n>2$.

therefore

$$(\widetilde{\Delta}_\lambda \circ h) \circ \Delta_\gamma = \widetilde{\Delta}_\gamma \circ (\widetilde{\Delta}_\lambda \circ h).$$

Suppose we can find a homeomorphism $\widetilde{\mathcal{H}}\colon \mathbb{C}\mathrm{P}^2 \to \mathbb{C}\mathrm{P}^2$ that self-conjugates $\widetilde{\mathcal{F}}$, preserves the cross-section Γ and such that its transverse component at b

$$\widetilde{\mathcal{H}}_b^{\pitchfork}\colon (\Gamma, b) \to (\Gamma, b)$$

coincides with the germ $\widetilde{\Delta}_\lambda$. Then the composition $\mathcal{H}_0 = \widetilde{\mathcal{H}} \circ \mathcal{H}$ would yield a topological conjugacy between $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ whose transverse component at b

$$h_0 = \widetilde{\mathcal{H}}_b^{\pitchfork} \circ h = \widetilde{\Delta}_\lambda \circ h$$

strongly conjugates the monodromy groups $G_{\mathcal{F}}$ and $G_{\widetilde{\mathcal{F}}}$. Such a homeomorphism $\widetilde{\mathcal{H}}$ can easily be constructed in the following way: Consider the monodromy map $\widetilde{\Delta}_\lambda$. Recall that holonomy transformations along a path are defined as a finite composition of correspondence maps

$$\Delta_j\colon (\tau_j, p_j) \to (\tau_{j+1}, p_{j+1})$$

where τ_j, τ_{j+1} are cross-sections at points p_j, p_{j+1} that lay on the same leaf and belong to a same flow box. We can assume this correspondence maps are given by the time-one map of a constant (in the appropriate coordinates) vector field. If the flow box is sufficiently small we can extend such vector field to a smooth (real C^∞) vector field tangent to the leaves of $\widetilde{\mathcal{F}}$ that vanishes outside a compact neighborhood of the flow box. The time-one map of this new vector field is a homeomorphism $\mathcal{H}_j\colon \mathbb{C}\mathrm{P}^2 \to \mathbb{C}\mathrm{P}^2$ which preserves the foliation $\widetilde{\mathcal{F}}$, maps the cross-section (τ_j, p_j) to the cross-section (τ_{j+1}, p_{j+1}) and the restriction

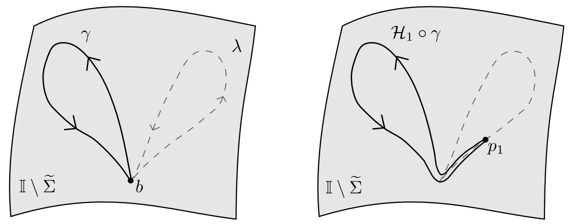
$$\mathcal{H}_j|_{(\tau_j,p_j)}\colon (\tau_j, p_j) \to (\tau_{j+1}, p_{j+1})$$

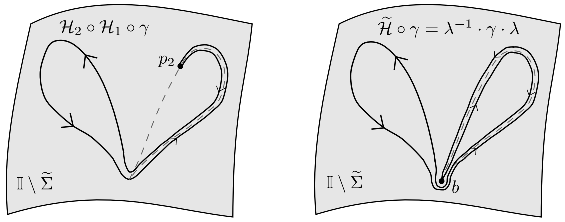
coincides with the correspondence map Δ_j.

The composition of all of the homeomorphisms $\mathcal{H}_j$ will be a homeomorphism $\widetilde{\mathcal{H}}$ which self-conjugates foliation $\widetilde{\mathcal{F}}$ and whose transverse component at b, by construction, coincides with the monodromy map $\widetilde{\Delta}_\lambda$.

Lemma 3 is now proved. □

Remark 1. The homeomorphism $\widetilde{\mathcal{H}}$ constructed above is isotopic to the identity map on $\mathbb{C}\mathrm{P}^2$. Its restriction to the infinite leaf is a map isotopic to the identity and such isotopy is obtained by *sliding* the base point b along the closed loop λ. For any loop $\gamma \in \pi_1(\mathbb{I} \setminus D, b)$ the composition $\widetilde{H} \circ \gamma$ turns out to be homotopic to the loop $\lambda^{-1} \cdot \gamma \cdot \lambda$.





This action is exactly inverse to the one induced by the original conjugacy between $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ and so the composition $\widetilde{\mathcal{H}} \circ \mathcal{H}$ has a trivial action on the fundamental group $\pi_1(\mathbb{I} \setminus D, b)$; this is, $(\widetilde{\mathcal{F}}, \mathcal{H}_0) \in \mathcal{T}_{\mathcal{OP}}(\mathcal{F}, b)$ and $\Phi(H_0) = id$.

3.2. **Topological invariance of the characteristic numbers of the singular points.** On this section we shall define the neighborhood U of $\mathcal{F}$ in $\mathcal{A}_2$ that is claimed to exist in Theorem 2. Its defining property being that if $(\widetilde{\mathcal{F}}, \mathcal{H}) \in \mathcal{T}_{\mathcal{OP}}(\mathcal{F}, b)$ and $\widetilde{\mathcal{F}} \in U$ then the homeomorphism H satisfies $H(a_i) \in D_i$ for $i = 1, 2, 3$. Whenever this situation happens we will say that H preserves the numbering of the singular points at infinity.

Let us denote by $\lambda_1, \lambda_2, \lambda_3$ the characteristic numbers of the singular points a_1, a_2, a_3 respectively. Given any other foliation $\widetilde{\mathcal{F}} \in \widetilde{U}$, denote by $a_i(\widetilde{\mathcal{F}})$ the unique singularity that $\widetilde{\mathcal{F}}$ has on the disk D_i. Let us denote by $\lambda(a_i(\widetilde{\mathcal{F}}))$ the characteristic number of the singularity $a_i(\widetilde{\mathcal{F}})$ corresponding to the foliation $\widetilde{\mathcal{F}}$. We shall keep writing a_i and λ_i instead of $a_i(\mathcal{F})$ and $\lambda(a_i(\mathcal{F}))$.

Let $M \colon \widetilde{U} \to \mathbb{C}^3$ be the map $M(\widetilde{\mathcal{F}}) = (\lambda(a_1(\widetilde{\mathcal{F}})), \lambda(a_2(\widetilde{\mathcal{F}})), \lambda(a_3(\widetilde{\mathcal{F}})))$. Since the characteristic numbers $\lambda_1, \lambda_2, \lambda_3$ are pairwise different there exists $\epsilon > 0$ such that if $j \neq k$ then $|\lambda_j - \lambda_k| \geq 2\epsilon$. Denote by V_i the disk $V_i = \{z \in \mathbb{C} \mid |\lambda_i - z| < \epsilon\}$ and let $U = M^{-1}(V_1 \times V_2 \times V_3)$. The map M is continuous (in fact, it is algebraic [1]) so U is an open neighborhood of $\mathcal{F}$ contained in $\widetilde{U}$.

Lemma 4. *If $\mathcal{F}$ is a generic foliation then for any other foliation $\widetilde{\mathcal{F}} \in U$ topologically conjugated to $\widetilde{\mathcal{F}}$ by a homeomorphism $\mathcal{H} \colon \mathbb{CP}^2 \to \mathbb{CP}^2$ the homeomorphism H preserves the numbering of the singular points at infinity; this is, for every $i = 1, 2, 3$ $H(a_i) \in D_i$.*

Proof. Choose $\widetilde{\mathcal{F}} \in U$ topologically conjugated to $\mathcal{F}$ by $\mathcal{H} \colon \mathbb{CP}^2 \to \mathbb{CP}^2$. The genericity conditions imposed on $\mathcal{F}$ imply that the characteristic numbers of the singularities at infinity are topological invariants in the following sense [1]: if $\mathcal{H}$ is a topological conjugacy between $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ then $\lambda(H(a_i)) = \lambda_i$. Additionally, from the definition of U it follows that

$$|\lambda(a_j(\widetilde{\mathcal{F}})) - \lambda_k| < \epsilon \quad \text{if } j = k$$

$$|\lambda(a_j(\widetilde{\mathcal{F}})) - \lambda_k| \geq \epsilon \quad \text{if } j \neq k,$$

which implies $H(a_i) = a_i(\widetilde{\mathcal{F}})$ for each $i = 1, 2, 3$; this is, H preserves the numbering of the singular points at infinity. □

3.3. **Ideal rigidity of foliations from the class $\mathcal{A}_2$.** In order to conclude that generic foliations from the class $\mathcal{A}_2$ are ideally rigid we will use a modified version of Proposition 2 which appears in [3].

Definition 6. *Let $S = \{a_1, ..., a_{n+1}\} \subseteq \mathbb{I}$ be a finite set of $n + 1$ distinct points; $D_1, ..., D_{n+1}$ a collection of $n + 1$ disjoint open disks covering S; $D = \cup D_i$ and $b \in \mathbb{I} \setminus D$.*

A homeomorphism $H \colon \mathbb{I} \to \mathbb{I}$ is called homotopically trivial over $\mathbb{I} \setminus D$ if $H(b) = b$, for each point a_i its image $H(a_i)$ belongs to the same disk D_i and the images $H(\alpha_i)$ of the segments $\alpha_i = [b, a_i]$ connecting the base point b with each point a_i are homotopic to the corresponding *segments α_i in the class of homotopy with the fixed endpoint b and free endpoint $a_{i,t} \in D$ restricted to the respective disk.*

A homeomorphism is said to be homotopically trivial without specifying the system of disks, if it is homotopically trivial over *some* system of disks.

Definition 7. *A foliation $\mathcal{F} \in \mathcal{A}'_n$ will be called reasonably rigid if there exists a neighborhood U of it in $\mathcal{A}_n$ such that any foliation $\mathcal{F}' \in U$ topologically equivalent to $\mathcal{F}$ is affine equivalent to $\mathcal{F}$ provided that the topological equivalence between $\mathcal{F}$ and $\mathcal{F}'$ induces a homotopically trivial homeomorphism of the infinite line $\mathbb{I}$ onto itself.*

Proposition 3 ([3])**.** *A generic foliation from the class $\mathcal{A}'_n$ is reasonably rigid.*

We now prove Theorem 2.

Proof of Theorem 2. Let $\mathcal{F} \in \mathcal{A}_2$ be a generic foliation. Let U be the neighborhood of $\mathcal{F}$ constructed in Section 3.2. Since foliation $\mathcal{F}$ is reasonably rigid there exists a neighborhood U' of it in $\mathcal{A}_2$ such that any foliation $\mathcal{F}' \in U'$ topologically equivalent to $\mathcal{F}$ is affine equivalent to $\mathcal{F}'$ provided that the topological equivalence between $\mathcal{F}$ and $\mathcal{F}'$ induces a homotopically trivial homeomorphism.

Suppose now that $\widetilde{\mathcal{F}} \in U \cap U'$ is topologically equivalent to $\mathcal{F}$. Without loss of generality we can suppose this conjugacy preserves the base point b. Since $\widetilde{\mathcal{F}} \in U$ the topological conjugacy $\mathcal{H}$ preserves the numbering of the singular points at infinity and, according to Lemma 3 and Remark 1, we can also suppose that the topological conjugacy satisfies $\Phi(H) = id$. This condition is equivalent to H being a homotopically trivial homeomorphism. Since $\widetilde{\mathcal{F}} \in U'$ we conclude that both foliations are affine equivalent. □

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SOLENOIDAL MANIFOLDS

DENNIS SULLIVAN

To Xavier Gómez-Mont who discovers and appreciates beautiful mathematics

Abstract. It is shown that every oriented solenoidal manifold of dimension one is the boundary of a compact oriented solenoidal 2-manifold. For compact solenoidal surfaces one can develop a theory of complex structures parallel to the theory for Riemann surfaces. In particular, there exists a corresponding Teichmüller space. The Teichmüller space of the solenoidal surface $\mathcal{S}$ obtained by taking the inverse limit of all finite pointed covers of a compact surface of genus greater than one is a separable Banach manifold version of the universal Teichmuller space of the upper half plane which is not separable. The commensurability automorphism group of the fundamental group of the surface acts minimally on this solenoidal version of the universal Teichmüller space.

A compact Hausdorff space which is locally homeomorphic to a k-disk cross a compact totally disconnected space is called a *laminar manifold*. If the totally disconnected space is perfect and infinite we call the laminar manifold a *solenoidal manifold*. A laminar manifold is foliated by its path components. We can speak of smooth laminar manifolds, Riemannian laminar manifolds, complex laminar manifolds etc. Such structures are meant to be, respectively, smooth, Riemannian or holomorphic in the leaf direction and continuously varying in the transverse direction.

Examples:

(i) The mapping torus of a homeomorphism of a Cantor set defines a fibration over the circle with fiber the Cantor set. The total space is an oriented solenoidal one-manifold.
(ii) The suspension of a representation of the fundamental group of a closed k-manifold into the group of homeomorphisms of a Cantor set defines a Cantor bundle over the manifold and again the total space is a solenoidal k-manifold.
(iii) The inverse limit of an infinite system of connected finite covers over a closed k-manifold defines a solenoidal k-manifold.

Laminar manifolds may be viewed geometrically in the leaf directions as generalizations of compact manifolds. Manifold concepts and theory can be applied to them. Laminar manifolds may also be viewed in the transverse direction as dynamical systems. The example following Theorem 3 is used seriously in both perspectives.

Theorem 1 (Conversation at IHES with Bob Edwards early 90's)**.** *Any oriented one dimensional solenoidal manifold is the boundary of an oriented solenoidal surface.*

Proof: The argument combines two ideas. First one observes, by choosing enough transversals, that any oriented solenoid is a mapping torus as in example (i). In more detail, choose a finite set of transversals cutting every leaf. By adding more transversals one can be sure that starting at a point on one transversal and going forward (with respect to the orientation) one first meets a different transversal. This picture presents the solenoid as a mapping torus of a homeomorphism on a Cantor set K. Second one writes this homeomorphism of K as a product of g commutators of homeomorphisms of K using the fact that $Homeo(K)$, the group of homeomorphisms of the Cantor set, is equal to its commutator subgroup. This is because R. D. Anderson [1] proved that $Homeo(K)$ is a simple group and in particular it is a perfect group. Now use the fact that the boundary of a surface of genus g with one boundary component is the product of the commutators of the standard generators. Then build using the naturally associated representation of the fundamental group of the surface with boundary into $Homeo(K)$ a compact solenoidal surface with boundary cantor fibred over the surface with boundary. The solenoidal one manifold in question appears over the boundary of the surface.

Problem: Is there a describable cobordism classification of oriented solenoidal manifolds in dimensions $2, 3, \ldots$?

Theorem 2. *In dimension two, the smoothability and the holomorphicity results for compact orientable surfaces extends to orientable laminar surfaces.*

Proof sketch: One may cover a laminar manifold with product charts that have a reasonable size in the leaf direction and a very small size in the transverse direction, moreover they can be chosen to be clopen in the transverse direction by the total discontinuity of the transversals. One sees then that any argument for compact manifolds that is continuous in parameters extends to laminar manifolds. Smoothing a compact surface or realizing it as a complex manifold can be argued to have this continuous form.

Theorem 3 (Candel [3])**.** *For any transversally continuous Riemannian metric on a smooth laminar surface, sometimes both but at least one of the following holds:*

I) the universal cover of every leaf is conformally the disk (compare the example following and Theorem 4);

II) there is a nontrivial tranverse invariant measure (a measure on each transversal so that the germs of transversal holonomy maps along paths are measure preserving).

Sketch of Candel's argument. If I) is not true, some leaf by Ahlfors lemma has a conformal metric with a sequence of neighborhoods of infinity with bounded length boundaries. This leaf is used to define the transverse invariant measure by the 70's argument of Joseph Plante.

Example of a solenoidal surface without transverse invariant measure: Consider the squaring map Q outside the unit disk in the complex plane. On the inverse limit space of the tower of covers defined by iterates of Q, the lift of Q is a bijection defining a properly discontinuous action of Z. The quotient by this action is a solenoidal surface with every leaf hyperbolic and and possessing no transversal invariant measure (cf [5]).

Theorem 4 (Candel and Verjovsky [3], [6])**.** *If every leaf of a laminar Riemannian surface is conformally covered by the disk, then the unique constant curvature minus one metric on each leaf is transversally continuous.*

Remark. To my knowledge H.E. Winkelnkemper was the first person to point out (circa 1976) the very interesting fact that whether a given non compact leaf of a smoothly foliated compact space with a transversally continuous metric has universal cover conformally the disk is independent of the choice of metric and therefore an intrinsic property of the smoothly foliated space. One can show also it is a topological invariant as well.(The hyperbolic plane is not related to the euclidean plane by a homeomorphism which is uniformaly continuous in both directions).

Theorem 5 ([5])**.** *The space of hyperbolic structures on a laminar surface (as in Theorem 4) up to isometries isotopic to the identity has the structure of a separable complex Banach manifold. The metric is the natural Teichmüller metric based on the minimal conformal distortion of a map* *between structures in the fixed isotopy class. The isotopy classes of homeomorphisms preserving a chosen leaf act by isometries of this separable Banach manifold and plays the role of the Teichmuller modular group in the classical case.*

Corollary ([5]): The analog of Riemann's moduli space exists as a Banach orbifold iff this action is appropriately discontinuous (there is a covering by open balls so that under the action only finitely many group elements bring a ball back to intersect itself.)

Consider the solenoidal surface S obtained by taking the inverse limit of all finite pointed covers of a compact surface of genus greater than one and chosen base point. The base points upstairs in the covers determine a point and a distinguished leaf in the inverse limit solenoidal surface.

Theorem 6 (based on Kahn-Marković affirmation of the Ehrenpreis Conjecture)**.** *The space of hyperbolic structures up to isometry preserving the distinguished leaf on this solenoidal surface S is non Hausdorff and any Hausdorff quotient is a point.*

Proof: In affirming the Enrenpreis Conjecture Jeremy Kahn and Vladimir Marković show any two constant negative curvature structures on compact surfaces become almost isometric after taking appropriate finite covers [4]. This shows the group mentioned in Theorem 5 has every orbit dense because it is explained in [5] how every point in the Teichmüller space of Theorem 5 is approximated by a hyperbolic structure on some high finite cover of the inverse system.[These are the transversally locally constant structures of [5]]. The group mentioned in Theorem 5 for this solenoidal surface is the commensurability automorphism group of the fundamental group of any higher genus compact surface (this means all isomorphisms between finite index subgroups). By the affirmation of the Ehrenpreis Conjecture mentioned above this commensurability group acts densely for each transversally locally constant structure in the Teichmüller space of all structures. Since the action is isometric one dense orbit implies all orbits are dense.

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GSV-INDICES AS RESIDUES

TATSUO SUWA

Abstract. We introduce a local invariant for a vector field v on a complete intersection V with an isolated singularity as the residue of the relevant Chern class of the ambient tangent bundle by a frame consisting of v and some natural meromorphic vector fields associated with defining functions of V. We then show that the residue coincides with the GSV-index as well as the virtual index of v so that it provides another interpretation of these indices. As an application, we give an algebraic formula for the GSV-indices of holomorphic vector fields on singular curves.

In this note we introduce a local invariant of a vector field v on a complete intersection V with an isolated singularity. It is the residue arising from the localization of the relevant Chern class of the ambient tangent bundle by a frame consisting of v and some other vector fields. The last ones are naturally associated to defining functions of V and are holomorphic on and normal to the non-singualr part of V (Definition 2.7 below). Although it is a priori of differential geometric nature, defined in the framework of Chern-Weil theory adapted to the Čech-de Rham cohomology, it is directly related to a topological invariant coming from the obstruction theory (cf. (2.10)).

Historically, there is the so-called GSV-index for a vector field v as above ([6], [13]). It is defined topologically, either using the frame consisting of v and the conjugated gradient vector fields of defining functions or referring to the Milnor fiber. On the differential geometric side, there is the virtual index which is the residue arising from the localization by v of the Chern class of the virtual tangent bundle of V (cf. [11]). It coincides with the GSV-index in the case considered here, however it can be defined in more general settings.

The topological aspect of the residue mentioned in the beginning is that it coincides with the GSV-index (Theorem 3.4) and the differential geometric aspect is that it coincides with the virtual index (Theorem 4.4), so that it provides another interpretation of these indices as well as another way of computing them. On the way we show how topological and differential geometric residues of vector fields on complete intersections interacts.

We then apply the above to the case of holomorphic vector fields on singular curves. A direct computation of the residue taking suitable connections shows an integral representation of the GSV-index (Proposition 5.1), which was given by M. Brunella in [4] by a different approach. This in turn gives an algebraic formula for the GSV-index in this case (Corollary 5.2). The formula is somewhat different from the one in this special case of the general algebraic formula obtained as homological index by X. Gómez-Mont in [5] (see [2] for complete intersections). It is only for the case of curves, however the advantage is that each term of it is expressed as the dimension of the quotient of the ring of holomorphic functions by an ideal generated by a regular sequence.

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1. Preliminaries

We recall localization theory of characteristic classes in the framework of Chern-Weil theory adapted to the Čech-de Rham cohomology, as initiated in [10]. Here we adopt the presentation in [15], see also [18].

Connections. Let M be a C^∞ manifold and E a C^∞ complex vector bundle of rank l on M. We denote by $A^p(M,E)$ the $\mathbb{C}$-vector space of complex valued C^∞ p-forms with coefficients in E on M, i.e., C^∞ sections of the bundle $\bigwedge^p(T^c_{\mathbb{R}}M)^* \otimes E$, where $T^c_{\mathbb{R}}M$ denotes the complexification of the tangent bundle of M. In the case $E = \mathbb{C}\times M$, the trivial line bundle, we denote it by $A^p(M)$ so that it is the space of complex valued p-forms on M.

Recall that a *connection* for E is a $\mathbb{C}$-linear map

$$\nabla : A^0(M,E) \longrightarrow A^1(M,E)$$

satisfying the "Leibniz rule"

$$\nabla(fs) = df \otimes s + f\nabla(s), \quad \text{for } \ f \in A^0(M) \ \text{ and } \ s \in A^0(M,E).$$

Note that every vector bundle admits a connection. If ∇ is a connection for E, it induces a $\mathbb{C}$-linear map

$$\nabla : A^1(M,E) \longrightarrow A^2(M,E)$$

satisfying

$$\nabla(\omega \otimes s) = d\omega \otimes s - \omega \wedge \nabla(s), \quad \text{for } \ \omega \in A^1(M) \ \text{ and } \ s \in A^0(M,E).$$

The composition

$$K = \nabla \circ \nabla : A^0(M,E) \longrightarrow A^2(M,E)$$

is called the *curvature* of ∇.

The fact that a connection is a local operator allows us to get local representations of it and its curvature by matrices whose entries are differential forms. Thus suppose that ∇ is a connection for E and that E is trivial on an open set U. If $\boldsymbol{e} = (e_1,\dots,e_l)$ is a frame of E on U, we may write

$$\nabla(e_i) = \sum_{j=1}^{l} \theta_{ji} \otimes e_j$$

with θ_{ij} 1-forms on U. We call $\theta = (\theta_{ij})$ the connection matrix with respect to $\boldsymbol{e}$. Also, from the definition we compute to get

$$K(e_i) = \sum_{j=1}^{l} \kappa_{ji} \otimes e_j, \quad \kappa_{ij} = d\theta_{ij} + \sum_{k=1}^{l} \theta_{ik} \wedge \theta_{kj}.$$

We call $\kappa = (\kappa_{ij})$ the curvature matrix with respect to $\boldsymbol{e}$. If $\boldsymbol{e}' = (e'_1 \dots, e'_l)$ is another frame of E on U', we have $e'_i = \sum_{j=1}^{l} a_{ji} e_j$ for some C^∞ functions a_{ij} on $U \cap U'$. The matrix $A = (a_{ij})$ is non-singular at each point of $U \cap U'$. If we denote by θ' and κ' the connection and curvature matrices of ∇ with respect to $\boldsymbol{e}'$, we have

$$\theta' = A^{-1} \cdot dA + A^{-1}\theta A \quad \text{and} \quad \kappa' = A^{-1}\kappa A \quad \text{in} \quad U \cap U'. \tag{1.1}$$

Chern forms. Since differential forms of even degrees commute one another with respect to exterior product, we may treat κ above as an ordinary matrix. Thus, for $q = 1, \dots, l$, we define a $2q$-form $\sigma_q(\kappa)$ on U by

$$\det(I_l + \kappa) = 1 + \sigma_1(\kappa) + \cdots + \sigma_l(\kappa),$$

where I_l denotes the identity matrix of rank l. In particular, $\sigma_1(\kappa) = \mathrm{tr}(\kappa)$ and $\sigma_l(\kappa) = \det(\kappa)$. Although $\sigma_q(\kappa)$ depends on the connection ∇, it does not depend on the choice of the frame of E by (1.1) and it defines a global $2q$-form on M, which we denote by $\sigma_q(\nabla)$. An important feature of the forms is that they are closed. We set

$$c^q(\nabla) = \left(\frac{\sqrt{-1}}{2\pi}\right)^q \sigma_q(\nabla)$$

and call it the q-th *Chern form*. The total Chern form is defined by $c^*(\nabla) = 1 + \sum_{q=1}^{l} c^q(\nabla)$ so that locally it is given by

$$c^*(\nabla) = \det\Bigl(I_l + \frac{\sqrt{-1}}{2\pi}\kappa\Bigr). \tag{1.2}$$

Note that it is invertible.

If we have two connections ∇_0 and ∇_1 for E, we may construct the *difference form* $c^q(\nabla_0, \nabla_1)$, which is a $(2q-1)$-form with the properties that $c^q(\nabla_1, \nabla_0) = -c^q(\nabla_0, \nabla_1)$ and that

$$d\, c^q(\nabla_0, \nabla_1) = c^q(\nabla_1) - c^q(\nabla_0).$$

In fact the form $c^q(\nabla_0, \nabla_1)$ is constructed as follows. We consider the vector bundle

$$E \times \mathbb{R} \to M \times \mathbb{R}$$

and let $\tilde{\nabla}$ be the connection for it given by $\tilde{\nabla} = (1-t)\nabla_0 + t\nabla_1$, with t a coordinate on $\mathbb{R}$. Then we define

$$c^q(\nabla_0, \nabla_1) = p_* c^q(\tilde{\nabla}),$$

where p_* denotes the integration along the fiber of the projection $p : M \times [0,1] \to M$.

From the above we see that the class $[c^q(\nabla)]$ of the closed $2q$-form $c^q(\nabla)$ in the de Rham cohomology $H_d^{2q}(M)$ depends only on E and not on the choice of the connection ∇. It is the q-th *Chern class* $c^q(E)$ of E.

Remark 1.3. If we use the obstruction theory, the q-th Chern class is defined in the integral cohomology $H^{2q}(M, \mathbb{Z})$. It is shown that the class $c^q(E)$ defined as above is equal to its image by the canonical homomorphism

$$H^{2q}(M, \mathbb{Z}) \longrightarrow H^{2q}(M, \mathbb{C}) \xrightarrow{\sim} H_d^{2q}(M),$$

where the last isomorphism is the de Rham isomorphism (e.g., [18]).

Localization. Let E be a vector bundle of rank l. An r-section of E is an r-tuple $\boldsymbol{s} = (s_1, \dots, s_r)$ of sections of E. A singular point of $\boldsymbol{s}$ is a point where $s_1, \dots, s_r$ fail to be linearly independent. An *r-frame* is an r-section without singularities. An l-frame is simply called a frame, as already used above.

Definition 1.4. Let $\boldsymbol{s} = (s_1, \dots, s_r)$ be a C^∞ r-frame of E on an open set U. We say that a connection ∇ is *trivial with respect to* $\boldsymbol{s}$, or simply $\boldsymbol{s}$*-trivial*, on U, if $\nabla(s_i) = 0,\ i = 1, \dots, r$.

The following is fundamental for the localization we consider:

Proposition 1.5. *If* ∇ *is* $\boldsymbol{s}$*-trivial,*

$$c^q(\nabla) = 0, \quad \textit{for} \ \ q \geq l - r + 1.$$

We explain localization process and the associated residues in the case pertinent to ours. Thus let M be a complex manifold of dimension n and p a point in M. Let $U_0 = M \smallsetminus \{p\}$ and U_1 a neighborhood of p and consider the covering $\mathcal{U} = \{U_0, U_1\}$ of M. We then work in the framework of the Čech-de Rham cohomology of $\mathcal{U}$. Let E be a complex vector bundle of rank l on M. Suppose we have an r-frame $\boldsymbol{s}$ of E on U_0, $r = l - n + 1$. The n-th Chern class $c^n(E)$ of E is represented by the Čech-de Rham cocycle

$$(c^n(\nabla_0), c^n(\nabla_1), c^n(\nabla_0, \nabla_1)),$$

where ∇_i is a connection for E on U_i, $i = 0, 1$. We choose ∇_0 so that it is $\boldsymbol{s}$-trivial. Thus by Proposition 1.5, $c^n(\nabla_0) = 0$ and the cocycle defines a class $c^n(E, \boldsymbol{s})$, called the *localization of* $c^n(E)$ *by* $\boldsymbol{s}$, in the relative cohomology $H^{2n}(M, M \smallsetminus \{p\}; \mathbb{C})$. This in turn gives rise to the residue $\operatorname{Res}_{c^n}(\boldsymbol{s}, E; p)$ as its image by the Alexander isomorphism

$$H^{2n}(M, M \smallsetminus \{p\}; \mathbb{C}) \xrightarrow{\sim} H_0(\{p\}, \mathbb{C}) = \mathbb{C}.$$

The residue is in fact an integer given by

$$\operatorname{Res}_{c^n}(\boldsymbol{s}, E; p) = \int_R c^n(\nabla_1) - \int_{\partial R} c^n(\nabla_0, \nabla_1),$$

where R is a $2n$-disk around p in U_1.

Exact sequence. Let

$$0 \longrightarrow E'' \xrightarrow{\iota} E \xrightarrow{\varphi} E' \longrightarrow 0 \tag{1.6}$$

be an exact sequence of vector bundles, and ∇'', ∇ and ∇' connections for E'', E and E', respectively. We say that $(\nabla'', \nabla, \nabla')$ is *compatible* with (1.6) if

$$\nabla(\iota \circ s'') = (\mathrm{id} \otimes \iota) \circ \nabla''(s'') \quad \text{and} \quad \nabla'(\varphi \circ s) = (\mathrm{id} \otimes \varphi) \circ \nabla(s)$$

for s'' in $A^0(M, E'')$ and s in $A^0(M, E)$.

The following is proved using the expression (1.2) :

Proposition 1.7. *If* $(\nabla'', \nabla, \nabla')$ *is* compatible *with* (1.6),

$$c^*(\nabla) = c^*(\nabla'') \cdot c^*(\nabla').$$

Remark 1.8. Given connections ∇'' and ∇' for E'' and E', it is possible to construct a connection ∇ for E so that $(\nabla'', \nabla, \nabla')$ is compatible with (1.6). Moreover, this can be done under the assumption that the connections be trivial with respect to appropriate frames.

Virtual bundles. Let E and E' be vector bundles and ∇ and ∇' connections for E and E', respectively. We set $\nabla^\bullet = (\nabla, \nabla')$ and define the total Chern form of the virtual bundle $E - E'$ by

$$c^*(\nabla^\bullet) = c^*(\nabla)/c^*(\nabla').$$

For two pairs of connections $\nabla_0^\bullet = (\nabla_0, \nabla_0')$ and $\nabla_1^\bullet = (\nabla_1, \nabla_1')$, we may define the difference form $c^q(\nabla_0^\bullet, \nabla_1^\bullet)$ with similar properties as before. Namely, letting $\tilde{\nabla}^\bullet = (\tilde{\nabla}, \tilde{\nabla}')$ with

$$\tilde{\nabla} = (1 - t)\nabla_0 + t\nabla_1 \qquad \text{and} \qquad \tilde{\nabla}' = (1 - t)\nabla_0' + t\nabla_1',$$

we set

$$c^q(\nabla_0^\bullet, \nabla_1^\bullet) = p_* c^q(\tilde{\nabla}^\bullet).$$

The total Chern class $c^*(E - E')$ of $E - E'$ is the class of $c^*(\nabla^\bullet)$ in the de Rham cohomology $H_d^*(M)$. It is also given by $c^*(E - E') = c^*(E)/c^*(E')$.

2. Residues of vector fields

The localization theory explained in the previous section applies also to the case of singular varieties. We first recall this in the situation relevant to ours. For details we refer to [16], [17] and [18]. Then we define the residue of a vector field on a complete intersection with an isolated singularity using an appropriate frame.

Residues of multi-sections. Let U be a neighborhood of the origin 0 in $\mathbb{C}^m$ and V a subvariety (reduced, but may not be irreducible) of pure dimension n in U. Assume that V contains 0 and that $V \smallsetminus \{0\}$ is non-singular. We take a closed ball B around 0 sufficiently small so that, in particular, $R = B \cap V$ has a cone structure over $\partial R = L$, the link of V at 0 (cf. [12]).

Let E be a C^∞ complex vector bundle of rank l on U and $\boldsymbol{s} = (s_1, \dots, s_r)$ a C^∞ r-frame of E on a neighborhood V' of L in V, $r = l - n + 1$. Then there is a natural localization of the n-th Chern class $c^n(E|_V)$ of $E|_V$ by $\boldsymbol{s}$, which gives rise to a residue, denoted by $\operatorname{Res}_{c^n}(\boldsymbol{s}, E|_V; 0)$. This is given as follows. Let ∇_0 be an $\boldsymbol{s}$-trivial connection for $E|_{V'}$ and ∇_1 a connection for E.

Definition 2.1. The *residue of* $\boldsymbol{s}$ *at* 0 *with respect to* c^n is defined by

$$\operatorname{Res}_{c^n}(\boldsymbol{s}, E|_V; 0) = \int_R c^n(\nabla_1) - \int_{\partial R} c^n(\nabla_0, \nabla_1).$$

Remark 2.2. 1. The definition of the residue above does not depend on the choice of B or the connections involved.

2. In practice, we may assume that E is trivial on U and we may take as ∇_1 the connection trivial with respect to some frame of E. In this case, the first term disappears and we have only an integral on $\partial R = L$.

The fundamental fact is that the residue above coincides with the "topological residue" defined by the obstruction theory. To explain this, we denote by $W_r(\mathbb{C}^l)$ the Stiefel manifold of ordered r-frames in $\mathbb{C}^l$. It is $(2n-2)$-connected and its $(2n-1)$-st homotopy group is naturally isomorphic to $\mathbb{Z}$.

Let us first consider the basic case where $U = V$ and $l = m = n$. Thus $r = 1$ and $\boldsymbol{s}$ consists of a single section s. In this case $L = S^{2n-1}$, a $(2n-1)$-sphere and, if we denote by $h = (h_1, \dots, h_n)$ the components of s with respect to some frame of E, the restriction of h to L defines a map

$$\varphi : L \longrightarrow W_1(\mathbb{C}^n) = \mathbb{C}^n \smallsetminus \{0\}.$$

On the other hand, by appropriate choices of ∇_0 and ∇_1, we may show that $c^n(\nabla_1) = 0$ and $c^n(\nabla_0, \nabla_1) = h^*\beta_n$, where β_n denotes the Bochner-Martinelli form on $\mathbb{C}^n$ (cf. [18, Lemma 3.4.1]). Thus we have

$$\operatorname{Res}_{c^n}(s, E; 0) = \deg \varphi. \tag{2.3}$$

In particular, if $E = TU$, the holomorphic tangent bundle of U, $s = v$ is a vector field and this is the Poincaré-Hopf index $\mathrm{PH}(v, 0)$ of v at 0.

Coming back to the general case, if $V = \bigcup V_i$ is the irreducible decomposition of V, the link L has connected components (L_i) accordingly, each L_i being the link of V_i. The r-frame $\boldsymbol{s}$ defines a map

$$\varphi_i : L_i \longrightarrow W_r(\mathbb{C}^l).$$

Since L_i is a connected real $(2n-1)$-dimensional manifold, we have the degree of φ_i, as an integer. We refer to [18, Theorem 6.3.2] for the following (in [17, Theorem 6.1], we need to assume that V is irreducible):

Lemma 2.4. *We have*

$$\mathrm{Res}_{c^n}(\boldsymbol{s}, E|_V; 0) = \sum \deg \varphi_i.$$

Remark 2.5. 1. In the above E and $\boldsymbol{s}$ may be assumed to be only continuous, as they admit "C^∞ approximations".

2. If E and $\boldsymbol{s}$ are restrictions of holomorphic ones on U, we have an analytic expression of $\mathrm{Res}_{c^n}(\boldsymbol{s}, E|_V; 0)$ as a Grothendieck residue (cf. [16]). Moreover, if V is a complete intersection, or more generally if V admits a smoothing in U, we have an algebraic expression as the dimension of certain analytic algebra (cf. [17]).

Vector fields on complete intersections. Letting U, V and V' be as above, we have an exact sequence

$$0 \longrightarrow TV' \longrightarrow TU|_{V'} \xrightarrow{\pi} N_{V'} \longrightarrow 0, \tag{2.6}$$

where TV' and TU denote the holomorphic tangent bundles of V' and U, and $N_{V'}$ the normal bundle of V' in U.

Let us now assume that V is a complete intersection defined by $f = (f_1, \dots, f_k)$ in U, $k = m - n$. Here we adopt the terminologies in [15, Ch.II, 13] so that V is reduced but may not be irreducible, to make sure.

In a neighborhood of a regular point of f, we may choose $(f_1, \dots, f_k)$ as a part of local coordinates on U so that we have holomorphic vector fields $\frac{\partial}{\partial f_1}, \dots, \frac{\partial}{\partial f_k}$ away from the critical set of f. They are linearly independent and "normal" to the non-singular part of V so that $(\pi(\frac{\partial}{\partial f_1}|_{V'}), \dots, \pi(\frac{\partial}{\partial f_k}|_{V'}))$, which will be simply denoted by $\boldsymbol{\partial}$, form a frame of $N_{V'}$. Here we should note that the restriction means the restriction as a section of the vector bundle TU.

Suppose we have a C^∞ non-singular vector field v on V'. Then the $(k+1)$-tuple of sections

$$\boldsymbol{v} = \Big(v, \frac{\partial}{\partial f_1}\Big|_{V'}, \dots, \frac{\partial}{\partial f_k}\Big|_{V'}\Big)$$

of $TU|_{V'}$ form a $(k+1)$-frame so that we have the residue $\mathrm{Res}_{c^n}(\boldsymbol{v}, TU|_V; 0)$, which we simply call the residue of v:

Definition 2.7. The *residue of* v *at* 0 is defined by

$$\mathrm{Res}(v, 0) = \mathrm{Res}_{c^n}(\boldsymbol{v}, TU|_V; 0).$$

Thus

$$\mathrm{Res}(v, 0) = \int_R c^n(\nabla_1) - \int_{\partial R} c^n(\nabla_0, \nabla_1), \tag{2.8}$$

where ∇_0 is a $\boldsymbol{v}$-trivial connection for $TU|_{V'}$ and ∇_1 a connection for TU.

Remark 2.9. 1. For the frame $\boldsymbol{v}$ above we cannot use the analytic or algebraic expression mentioned in Remark 2.5, 2, even if v admits a holomorphic extension to U, as the vector fields $\frac{\partial}{\partial f_j}$ cannot be extended holomorphically through 0. On the other hand, the topological expression in Lemma 2.4 is still valid:

$$\mathrm{Res}(v, 0) = \sum \deg \varphi_i, \tag{2.10}$$

where φ_i is the map defined by $\boldsymbol{v}$ on each connected component L_i of the link L of V.

2. The above residue is, in some sense, dual to the index for a 1-form introduced in [7].

Proposition 2.11. *If* V *is non-singular at* 0*,*

$$\mathrm{Res}(v, 0) = \mathrm{PH}(v, 0).$$

Proof. In this case, the sequence (2.6) extends to the exact sequence

$$0 \longrightarrow TV \longrightarrow TU|_V \longrightarrow N_V \longrightarrow 0. \tag{2.12}$$

Note that $\mathrm{PH}(v,0)$ is given as the right side of (2.8) with ∇_0 and ∇_1 replaced by a v-trivial connection for TV' and a connection for TV, respectively.

We take a v-trivial connection ∇_0'' for TV', a $\boldsymbol{v}$-trivial connection ∇_0 for $TU|_{V'}$ and the $\boldsymbol{\partial}$-trivial connection ∇_0' for $N_{V'}$ so that $(\nabla_0'', \nabla_0, \nabla_0')$ is compatible with (2.6). Also let ∇_1'' be a connection for TV and ∇_1' the $\boldsymbol{\partial}$-trivial connection for N_V. We take a connection ∇_1 for TU so that $(\nabla_1'', \nabla_1, \nabla_1')$ is compatible with (2.12) (cf. Remark 1.8). Then noting that the total Chern forms satisfy $c^*(\nabla_1) = c^*(\nabla_1'') \cdot c^*(\nabla_1')$ and that $c^*(\nabla_1') = 1$, as ∇_1' is trivial, we have $c^n(\nabla_1) = c^n(\nabla_1'')$. Similarly, from the construction of the difference form and noting that $\nabla_0' = \nabla_1'$ on V', we have $c^n(\nabla_0, \nabla_1) = c^n(\nabla_0'', \nabla_1'')$. □

Remark 2.13. The above may be shown by obstruction theory as well, the essential point being again that $(\frac{\partial}{\partial f_1}, \dots, \frac{\partial}{\partial f_k})$ has no singularities on V, if V is non-singular.

In the global case, this type of residues also appear as relative Chern classes. Let us again start with the basic case. Thus let M be the closure of a relatively compact open set of a complex manifold M_1 of dimension n. Suppose ∂M is (piecewise) C^∞ and we have a non-singular vector field v in a neighborhood M' of ∂M in M_1. Let ∇_0 be a v-trivial connection for TM' and ∇_1 a connection for TM_1 and define

$$\mathrm{PH}(v, M) = \int_M c^n(\nabla_1) - \int_{\partial M} c^n(\nabla_0, \nabla_1). \tag{2.14}$$

We may extend v to all of M with possibly a finite number of singularities p_i and using (2.3), we see that

$$\mathrm{PH}(v, M) = \sum \mathrm{PH}(v, p_i). \tag{2.15}$$

Coming back to the situation before, let $f = (f_1, \dots, f_k) : U \to \mathbb{C}^k$ and B be as above. We denote by $C(f)$ the set of critical points of f and set $D(f) = f(C(f))$, which is a hypersurface in a neighborhood of the origin 0 in $\mathbb{C}^k$. For t sufficiently near 0, we set $V_t = f^{-1}(t)$, which admits at most isolated singularities $C(f) \cap V_t$, all lying in the interior of B. If t is not in $D(f)$, V_t is non-singular, in fact a Milnor fiber F of f (cf. [12], [8]). Let V_t' be a neighborhood of $R_t = B \cap V_t$ in V_t and v_t a non-singular vector field on V_t'. We set $\boldsymbol{v}_t = (v_t, \frac{\partial}{\partial f_1}|_{V_t'}, \dots, \frac{\partial}{\partial f_k}|_{V_t'})$ and define the residue $\mathrm{Res}(v_t, V_t)$ by the formula (2.8) with ∇_0 replaced by a $\boldsymbol{v}_t$-trivial connection for $TU|_{V_t'}$ and R by R_t. The following is proves as Proposition 2.11 :

Proposition 2.16. *If V_t is non-singular,*

$$\mathrm{Res}(v_t, V_t) = \mathrm{PH}(v_t, V_t).$$

3. GSV-index

Let U be a neighborhood of the origin 0 in $\mathbb{C}^{n+k}$ and V a complete intersection in U of dimension n, as in Section 2. Let v be a non-singular continuous vector field on V', a neighborhood in V of the link L of V. For the definition of the GSV-index of v at 0, we adopt the one in [15, Ch.IV, 1]. It is in the spirit of the second definition in [6], involving the Milnor fiber, and is equivalent to the one given in [6] and [13] as the degree of a certain map, *provided that V is irreducible* (see Remark 3.3 below).

Let us consider the situation in the last part of Section 2. Let U' be a neighborhood of L in U. We may assume that U' does not contain critical points of f. Then we have an exact sequence, which extends (2.6), $V' = U' \cap V$:

$$0 \longrightarrow Tf|_{U'} \longrightarrow TU|_{U'} \longrightarrow N|_{U'} \longrightarrow 0, \tag{3.1}$$

where Tf denotes the bundle on $U \smallsetminus C(f)$ of vectors tangent to the fibers of f and N a trivial bundle of rank k on U (cf. Section 4 below). In this situation, $N|_{U'}$ may be thought of as $f^*T\mathbb{C}^k|_{U'}$. Starting from the given non-singular vector field v on V', we may construct a non-singualr vector field $\tilde{v}$ on U' so that it is tangent to V'_t for all t near 0 in $\mathbb{C}^k$. This is done by taking an extension of v to a section of $TU|_{U'}$ and then projecting it to a section of $Tf|_{U'}$ by a splitting of (3.1). Let v_t denote the restriction of $\tilde{v}$ to V'_t. For a regular value t of f we denote V_t by F and v_t by w. Then we have the Poincaré-Hopf index $\mathrm{PH}(w, F)$ (cf. (2.14)).

Definition 3.2. The *GSV-index of v at* 0 is defined by

$$\mathrm{GSV}(v, 0) = \mathrm{PH}(w, F).$$

Remark 3.3. 1. The definition does not depend on the choice of the regular value t (cf. the proof of Theorem 3.4 below).

2. If V is irreducible, then L is connected and the above index $\mathrm{GSV}(v, 0)$ coincides with the degree of the map

$$\psi : L \longrightarrow W_{k+1}(\mathbb{C}^{n+k})$$

given as the restriction to L of $(v, \overline{\mathrm{grad}\, f_1}, \dots, \overline{\mathrm{grad}\, f_k})$, where $\overline{\mathrm{grad}\, f_j}$ denotes the complex conjugate of the gradient vector field of f_j: $\overline{\mathrm{grad}\, f_j} = \sum_{i=1}^{n+k} \overline{\frac{\partial f_j}{\partial z_i}} \frac{\partial}{\partial z_i}$ (cf. [6], [13]). This can be shown by the obstruction theory as in [6]. We could also show this by the Chern-Weil theory as Theorem 3.4 below, considering another residue using the frame $(v, \overline{\mathrm{grad}\, f_1}, \dots, \overline{\mathrm{grad}\, f_k})$ instead of $(v, \frac{\partial}{\partial f_1}, \dots, \frac{\partial}{\partial f_k})$.

3. Suppose V is not irreducible and let $V = \bigcup V_i$ be the irreducible decomposition. Note that this happens only if $k \geq n$, as $V \smallsetminus \{0\}$ is assumed to be non-singular. In this case, L has as many connected components (L_i) and it is not appropriate to consider the degree of ψ as above. However, proceeding as Theorem 3.4 below and using Lemma 2.4, we have $\mathrm{GSV}(v, 0) = \sum \deg \psi_i$, with ψ_i the restriction of ψ to L_i.

To further make comments in this situation, we denote the above index by $\mathrm{GSV}(v, V; 0)$. If each V_i is also a complete intersection, restricting v to V_i, we have $\mathrm{GSV}(v, V_i; 0)$ defined as in Definition 3.2 and it is expressed as the degree of a map as above, however the point is that we have to use the defining functions for V_i (not for V) as $(f_1, \dots, f_k)$. For that reason, $\mathrm{GSV}(v, V; 0) \neq \sum_i \mathrm{GSV}(v, V_i; 0)$, in general. For example, in the case $n = k = 1$, denoting V and V_i by C and C_i, we have

$$\mathrm{GSV}(v, C; 0) = \sum_i \mathrm{GSV}(v, C_i; 0) - \sum_{i \neq j} (C_i \cdot C_j)_0,$$

where $(C_i \cdot C_j)_0$ denotes the intersection number of C_i and C_j at 0 (see [15, Ch.V, 5] and references therein).

Let us note that in the beginning of Section 3.2 of [3], V has to be assumed to be irreducible, even in the higher dimensional case, and that in Remark 3.2.2, loc. cit., there are some misplacements of terms in the second displayed formula : it should be read as above with $\mathrm{GSV}(v, C; 0)$ defined as in Definition 3.2.

Here is the main theorem of this section :

Theorem 3.4. *We have*

$$\mathrm{GSV}(v,0)=\mathrm{Res}(v,0).$$

Proof. We compute $\mathrm{Res}(v_t,V_t)$ using (the restriction to V_t of) connections as follows. Let ∇_0 be a $\tilde{v}$-trivial connection for TU' and ∇_1 a connection for TU. Then we have

$$\mathrm{Res}(v_t,V_t)=\int_{R_t}c^n(\nabla_1)-\int_{\partial R_t}c^n(\nabla_0,\nabla_1).$$

which depends continuously on t. For a regular value t, this is $\mathrm{PH}(v_t,V_t)$ (cf. Proposition 2.16), which is an integer (cf. (2.15)). Thus it does not depend on t, since the regular values are dense. For a regular value this is $\mathrm{GSV}(v,0)$, while for $t=0$, this is equal to $\mathrm{Res}(v,0)$. □

Remark 3.5. 1. The above may be shown by the obstruction theory as well.

2. From the above theorem and (2.10), we see that if V is irreducible, we have another expression of the GSV-index as the degree of a map, which involves the vector field v and the holomorphic vector fields $\frac{\partial}{\partial f_j}$. In the case V is not irreducible, we have again from the above theorem and (2.10), $\mathrm{GSV}(v,0)=\sum\deg\varphi_i$ (cf. Remark 3.3, 3 above).

4. Virtual index

The notion of *virtual index* was introduced in [11]. It can be defined for a vector field on a certain type of local complete intersection V. To be a little more precise, let S be a compact set in V and V_1 a neighborhood of S such that $V_1\smallsetminus S$ is in the non-singular part of V. For a C^∞ vector field v non-singular on $V_1\smallsetminus S$, we may define the virtual index $\mathrm{Vir}(v,S)$ of v at S as the residue arising from the localization of the n-th Chern class of the virtual tangent bundle of V by v, $n=\dim V$.

Here we recall the case of isolated singularities. Thus let U, V and V' be as in Section 2. Assume that V is a complete intersection defined by $f=(f_1,\dots,f_k)$ in U. In this case, the bundle map π in (2.6) has an extension

$$\pi: TU|_V\longrightarrow N|_V$$

with N a trivial vector bundle of rank k on U (e.g., [15, Ch.II, 13]). The extension is natural in the sense that N admits a frame $\boldsymbol{\nu}=(\nu_1,\dots,\nu_k)$ extending the frame $\boldsymbol{\partial}=(\pi(\frac{\partial}{\partial f_1}|_{V'}),\dots,\pi(\frac{\partial}{\partial f_k}|_{V'}))$ of $N_{V'}$. We set $\tau_V=TU|_V-N|_V$ and call it the *virtual tangent bundle* of V. Recall that its total Chern class is given by $c^*(\tau_V)=c^*(TU|_V)/c^*(N|_V)$.

Let v be a C^∞ vector field on V'. Then we will see that the n-th Chern class $c^n(\tau_V)$ of τ_V is localized at 0 to give rise to the virtual index $\mathrm{Vir}(v,0)$ of v at 0. In the sequel, we follow the description of [15, Ch.IV, 3].

We take connections ∇, ∇_0 and ∇'_0 for TV', $TU|_{V'}$ and $N_{V'}$, respectively, so that
(i) ∇ is v-trivial: $\nabla(v)=0$, and that
(ii) the triple $(\nabla,\nabla_0,\nabla'_0)$ is compatible with (2.6).

We set $\nabla^\bullet_0=(\nabla_0,\nabla'_0)$. Recall that the total Chern form of the pair $\nabla^\bullet_0$ of connections is defined by $c^*(\nabla^\bullet_0)=c^*(\nabla_0)/c^*(\nabla'_0)$. By (ii) above, $c^*(\nabla^\bullet_0)=c^*(\nabla)$ so that by (i),

$$c^n(\nabla^\bullet_0)=c^n(\nabla)=0,$$

which is the key fact for the localization. Let ∇_1 and ∇'_1 be connections for TU and N, respectively, and set $\nabla^\bullet_1=(\nabla_1,\nabla'_1)$. The total Chern form $c^*(\nabla^\bullet_1)$ of the pair $\nabla^\bullet_1$ is defined as above and $c^n(\nabla^\bullet_1)$ is a $2n$-form on U. Recall that we have also the difference form $c^n(\nabla^\bullet_0,\nabla^\bullet_1)$. Let B and $R=B\cap V$ be as in Section 2.

Definition 4.1. The *virtual index* of v at 0 is defined by

$$\mathrm{Vir}(v,0)=\int_R c^n(\nabla_1^\bullet)-\int_{\partial R} c^n(\nabla_0^\bullet,\nabla_1^\bullet). \tag{4.2}$$

Remark 4.3. 1. If V is non-singular at 0, we have (cf. [15, Ch.IV, Lemma 3.3]):

$$\mathrm{Vir}(v,0)=\mathrm{PH}(v,0).$$

2. In practice we may take as ∇_1 and ∇_1' connections trivial with respect to some frames of TU and N, respectively. In this case, the first term in (4.2) disappears and we have only an integral on $\partial R = L$.

3. If v is the restriction to V' of some holomorphic vector field on U leaving V invariant, this integral can be expressed as a Grothendieck residue relative to V (cf. [11], [15, Ch.IV, (7.3)]). Moreover in this case, we have the "virtual residues" for Chern polynomials of degree n (cf. [15, Ch.IV, 7]), generalizing the Baum-Bott residues for holomorphic vector fields in [1].

Theorem 4.4. *We have*

$$\mathrm{Vir}(v,0)=\mathrm{Res}(v,0).$$

Proof. We take a v-trivial connection ∇ for TV', a $\boldsymbol{v}$-trivial connection ∇_0 for $TU|_{V'}$ and a $\boldsymbol{\partial}$-trivial connection ∇_0' for $N_{V'}$ so that $(\nabla,\nabla_0,\nabla_0')$ is compatible with (2.6) and set $\nabla_0^\bullet=(\nabla_0,\nabla_0')$. Also, let ∇_1 be an arbitrary connection TU and let ∇_1' be the $\boldsymbol{\nu}$-trivial connection for N and set $\nabla_1^\bullet=(\nabla_1,\nabla_1')$. Here we recall that $\boldsymbol{\nu}$ is a frame extending $\boldsymbol{\partial}$.

From $c^*(\nabla_1^\bullet)=c^*(\nabla_1)/c^*(\nabla_1')$ and $c^*(\nabla_1')=1$, we have

$$c^n(\nabla_1^\bullet)=c^n(\nabla_1). \tag{4.5}$$

To find $c^n(\nabla_0^\bullet,\nabla_1^\bullet)$, recall that it is given by integrating $c^n(\tilde\nabla^\bullet)$ over the 1-simplex $[0,1]$, where $\tilde\nabla^\bullet=(\tilde\nabla,\tilde\nabla')$ with $\tilde\nabla=(1-t)\nabla_0+t\nabla_1$ and $\tilde\nabla'=(1-t)\nabla_0'+t\nabla_1'$. Since $\nabla_0'=\nabla_1'$ on V', we have $\tilde\nabla'=\nabla_0'$ and moreover, $c^*(\tilde\nabla')=1$, as ∇_0' is $\boldsymbol{\partial}$-trivial. Thus we have $c^n(\tilde\nabla^\bullet)=c^n(\tilde\nabla)$ exactly as above. Therefore we have $c^n(\nabla_0^\bullet,\nabla_1^\bullet)=c^n(\nabla_0,\nabla_1)$, which together with (4.5) implies the equality. □

Remark 4.6. The above proof is similar to the one for [9, Theorem 4.3]. We note that the latter can also be simplified as above.

From Theorems 3.4 and 4.4, we recover the following equality, which was initially proved in [11], see also [14]:

Corollary 4.7. *We have*

$$\mathrm{Vir}(v,0)=\mathrm{GSV}(v,0).$$

Remark 4.8. As can be seen from the above, we could use, instead of $(\frac{\partial}{\partial f_1},\dots,\frac{\partial}{\partial f_k})$, an arbitrary k-frame of TU to define the residue $\mathrm{Res}(v,0)$ for similar results, as long as it is normal to the non-singular part of V. An advantage of the use of $(\frac{\partial}{\partial f_1},\dots,\frac{\partial}{\partial f_k})$ is, besides its naturalness, that we have some concrete results as shown in the following section.

5. The case of plane curves

Let C be an analytic curve (reduced but may not be irreducible) defined by $f=0$ in a neighborhood U of 0 in $\mathbb{C}^2=\{(z_1,z_2)\}$, containing 0 as a possibly singular point. Also let

$$\tilde v=a_1\frac{\partial}{\partial z_1}+a_2\frac{\partial}{\partial z_2}$$

be a holomorphic vector field on U, possibly singular at 0 and leaving C invariant. The last condition can be rephrased as $\tilde{v}(f) = hf$ for some holomorphic function h. Let v denote the restriction of $\tilde{v}$ to $C' = C \setminus \{0\}$.

We denote by $\mathcal{O}$ the ring of germs of holomorphic functions at 0 in $\mathbb{C}^2$. We may assume that, changing the coordinates of $\mathbb{C}^2$ if necessary, the germs of f and a_1 are relatively prime in $\mathcal{O}$. In this case f and $\frac{\partial f}{\partial z_2}$ are also relatively prime. We set $\partial_i f = \frac{\partial f}{\partial z_i}$. Let L denote the link of C at 0.

Proposition 5.1. *In the above situation,*

$$\mathrm{GSV}(v, 0) = \frac{1}{2\pi\sqrt{-1}} \int_L \Big(\frac{da_1}{a_1} - \frac{d(\partial_2 f)}{\partial_2 f} \Big).$$

Proof. By Theorem 3.4, we only need to compute $\mathrm{Res}(v, 0) = \mathrm{Res}_{c^n}(\boldsymbol{v}, TU|_C; 0)$ for $\boldsymbol{v} = (v, \frac{\partial}{\partial f}|_{C'})$, as given in (2.8).

Let ∇_0 be the connection for $TU|_{C'}$ trivial with respect to $\boldsymbol{v}$ and ∇_1 the connection for TU trivial with respect to $(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2})$. Then we have $c^1(\nabla_1) = 0$. Now we compute $c^1(\nabla_0, \nabla_1)$. For this, consider the connection $\tilde{\nabla} = (1-t)\nabla_0 + t\nabla_1$ of the bundle $TU|_{C'} \times \mathbb{R}$ on $C' \times \mathbb{R}$. Let θ_0 and θ_1 be the connection matrix of ∇_0 and ∇_1 with respect to the frame $(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2})$. We have $\theta_1 = 0$. We try to find θ_0. We assume that f and a_1 are relatively prime as before. Thus f and $\partial_2 f = \frac{\partial f}{\partial z_2}$ are relatively prime so that (z_1, f) forms a coordinate system on a neighborhood of C' and we may write $\frac{\partial}{\partial f} = (\partial_2 f)^{-1} \frac{\partial}{\partial z_2}$. The matrix A of change of frame from $\boldsymbol{v}$ to $(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2})$ can be computed from $(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}) = \boldsymbol{v}A$ to get

$$A = \frac{\partial_2 f}{a_1} \begin{pmatrix} (\partial_2 f)^{-1} & 0 \\ -a_2 & a_1 \end{pmatrix}.$$

Thus by (1.1), we have

$$\theta_0 = A^{-1} \cdot dA = - \begin{pmatrix} \frac{da_1}{a_1} & 0 \\ * & -\frac{d(\partial_2 f)}{\partial_2 f} \end{pmatrix}.$$

Let $\tilde{\theta}$ and $\tilde{\kappa}$ be the connection and curvature matrices of $\tilde{\nabla}$ with respect to $(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2})$. Then we have $\tilde{\kappa} = d\tilde{\theta} + \tilde{\theta} \wedge \tilde{\theta}$, $\tilde{\theta} = (1-t)\theta_0 + t\theta_1 = (1-t)\theta_0$. The term in $\tilde{\kappa}$ involving dt is $-dt \wedge \theta_0$ so that we have, denoting by p_* the integration along the fiber of the projection $p : C' \times [0,1] \to C'$,

$$c^1(\nabla_0, \nabla_1) = \frac{\sqrt{-1}}{2\pi} p_* \operatorname{tr} \tilde{\kappa} = \frac{1}{2\pi\sqrt{-1}} \operatorname{tr} \theta_0 = -\frac{1}{2\pi\sqrt{-1}} \Big(\frac{da_1}{a_1} - \frac{d(\partial_2 f)}{\partial_2 f} \Big),$$

which proves the proposition. □

Corollary 5.2. *In the above situation,*

$$\mathrm{GSV}(v, 0) = \dim_{\mathbb{C}} \mathcal{O}/(f, a_1) - \dim_{\mathbb{C}} \mathcal{O}/(f, \partial_2 f).$$

Proof. As f and a_1 are relatively prime, $\Gamma_1 = \{\, z \in U \mid f(z) = 0,\ |a_1(z)| = \varepsilon \,\}$ is a 1-cycle on C' homologous to L, for a small positive number ε. Thus

$$\frac{1}{2\pi\sqrt{-1}} \int_L \frac{da_1}{a_1} = \frac{1}{2\pi\sqrt{-1}} \int_{\Gamma_1} \frac{da_1}{a_1}.$$

Then by the projection formula we have (e.g., [18])

$$\frac{1}{2\pi\sqrt{-1}} \int_{\Gamma_1} \frac{da_1}{a_1} = \Big(\frac{1}{2\pi\sqrt{-1}} \Big)^2 \int_{\Gamma} \frac{df}{f} \wedge \frac{da_1}{a_1},$$

where Γ is the 2-cycle on C' given by $\Gamma = \{\, z \in U \mid |f(z)| = |a_1(z)| = \varepsilon \,\}$. The right side above equals $\dim \mathcal{O}/(f, a_1)$. Similarly for the second term. □

Remark 5.3. 1. Proposition 5.1 gives an alternative verification of an integral representation of the GSV-index given in [4, p. 532]. For this, note that we may take $-a_1$ and $\partial_2 f$ as k and g in [4].

2. In fact in [4] the right side of the above formula appears as the difference of the orders of zeros and of poles of a certain vector field on the Milnor fiber. We may also give such an interpretation on the central fiber C as follows. Note that, on $C \smallsetminus \{0\}$, $(a_1\frac{\partial}{\partial z_1}, \frac{\partial}{\partial f})$ as well as $(v, \frac{\partial}{\partial f})$ is a frame of the holomorphic tangent bundle of $\mathbb{C}^2$ and we may write $\frac{\partial}{\partial f} = \frac{1}{\partial_2 f}\frac{\partial}{\partial z_2}$. Thus the first term above may be thought of as the order of zero of the vector field $a_1\frac{\partial}{\partial z_1}$ at 0 on C and the second term as the order of pole of the vector field $\frac{\partial}{\partial f}$ at 0 on C.

3. The general algebraic formula in [5] reads, in this particular case,

$$\mathrm{GSV}(v, 0) = \dim_{\mathbb{C}} \mathcal{O}/(f, a_1, a_2) - \dim_{\mathbb{C}} \mathcal{O}/(f, \partial_1 f, \partial_2 f).$$

Compared with the one in Corollary 5.2, the corresponding terms may be different, however the differences are the same.

4. Also in this case, a general integral formula (cf. [11], [15, Ch.IV, Theorem 7.2]) for the virtual index gives

$$\mathrm{GSV}(v, 0) = \frac{1}{2\pi\sqrt{-1}} \int_{\Gamma_1} \Big(\frac{\partial a_1}{\partial z_1} + \frac{\partial a_2}{\partial z_2} - h\Big)\frac{dz_1}{a_1},$$

where Γ_1 is as in the proof of Corollary 5.2 and may be replaced by L, and h a holomorphic function such that $\tilde{v}(f) = hf$.

5. In this case again, the arguments in [16] are still valid, even if $\frac{\partial}{\partial f}$ does not extend through 0. Thus we may use the formula in the case (2), p.285, loc. cit., to directly obtain the formula in Proposition 5.1, noting that the matrix F there is given by ${}^tA^{-1}$, with A as in the above proof. We should note that F becomes meromorphic in this case.

6. It would be an interesting problem to generalize the above formula to the higher dimensional and codimensional case.

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HYPERSURFACES IN $\mathbb{P}^5$ CONTAINING UNEXPECTED SUBVARIETIES

I. VAINSENCHER

Abstract. Smooth cubic 4-folds in $\mathbb{P}^5$ containing a general pair of 2-planes are known to be rational. They form a family of codimension 2 in $\mathbb{P}^{55}$. We find a polynomial which encodes, for all $d \geq 3$, the degrees of the loci of hypersurfaces in $\mathbb{P}^5$ of degree d containing some plane-pair.

Dedicated to Xavier Gómez-Mont Ávalos
on the occasion of his 60th birthday.

1. INTRODUCTION

The Noether-Lefschetz theorem tells us that a surface of degree at least four is not supposed to contain curves besides its intersection with another surface. Asking surfaces of a given degree to contain say, a (few) line(s), or a conic, a twisted cubic, etc., defines subvarieties, the so called Noether-Lefschetz loci, in the appropriate projective space. There are *polynomial* formulas for their degrees, [4], [14].

Our motivation here stems from a tale told by Joe Harris we were fortunate to attend (cf. [9]). The theme was the lack of knowledge about the rationality of cubic 4-folds in $\mathbb{P}^5$. As an elementary dimension count shows, a general cubic 4-fold $F_3 \subset \mathbb{P}^5$ contains no plane $\mathbb{P}^2 \subset \mathbb{P}^5$. Those F_3 that do contain some $\mathbb{P}^2$ are easily seen to form a hypersurface in $\mathbb{P}^{55}$. Now the smooth cubic 4-folds containing *two* disjoint planes are known to form a family (of dimension 53) of *rational* hypersurfaces. Indeed, through a general point of such hypersurface F, there is exactly one line which meets both planes; conversely, given a choice of a point on each plane, the line joining them meets F in a third point, thus establishing a birational map $F \dashrightarrow \mathbb{P}^2 \times \mathbb{P}^2$. See [11, 1.33, p. 24].

Our aim is to find the degree of the family of cubic 4-folds containing some pair of disjoint planes in $\mathbb{P}^5$. In fact, the answer is given by a polynomial in d which encodes, for each $d \geq 3$, the degree of the locus of hypersurfaces in $\mathbb{P}^5$ of degree d containing some plane-pair, cf. (2).

A naïve, direct application of the formula of double points gives a *wrong* answer, **5 752 908**, instead of **3 371 760**, presently dedicated to Xavier. In fact, it turns out that the cubic 4-folds containing some pair of incident planes contribute a full component to the double point locus. Ditto for those containing a pair of planes meeting along a line.

We employ the tools pioneered by Geir Ellingsrud and Stein A. Strømme, cf. [6] and masterfully used by Maxim Kontsevich in [12].

Let us summarize the main construction. Write $\mathbb{G}$ for the Grassmann variety of planes in $\mathbb{P}^5$. The family of plane-pairs can be parameterized by a double blowup $\widehat{\mathbb{X}} \to \widetilde{\mathbb{X}} \to \mathbb{G}\times\mathbb{G}$, first along the diagonal, then along the strict transform of the locus of plane-pairs containing a line. The resulting variety $\widehat{\mathbb{X}}$ comes equipped with vector bundles $\mathcal{V}_d$, $d \geq 2$. The fiber of $\mathcal{V}_d$ over each

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$q \in \widehat{\mathbb{X}}$ is equal to the vector space of homogeneous polynomials of degree d lying in the ideal of the plane-pair q (or a flat specialization thereof). There are natural $\mathbb{C}^\star$-actions with finitely many explicit fixed points in $\widehat{\mathbb{X}}$. Bott's formula applies to compute the answers for as many values of d as needed to interpolate and find an explicit polynomial. The *a priori* polynomial nature of the answers is deduced from Grothendieck-Riemann-Roch.

2. NOTATION AND PRELIMS

2.1. Let us start with Fermat's $F_3 := x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 = 0$ just to get a feeling. It contains a few pairs of disjoint planes in $\mathbb{P}^5$, *e.g.,* $q = \langle p_1, p_u \rangle$ with

$$\begin{cases} p_1 : x_0 + x_1 = x_2 + x_3 = x_4 + x_5 = 0, \\ p_u : x_0 - ux_1 = x_2 - ux_3 = x_4 - ux_5 = 0, \ (u^3 = -1, u \neq -1). \end{cases}$$

A little calculation shows that in the affine neighborhood of the grassmannian of planes in $\mathbb{P}^5$ defined by

$$p_a : \begin{cases} a_{1,1}x_3 + a_{1,2}x_4 + a_{1,3}x_5 + x_0, \\ a_{2,1}x_3 + a_{2,2}x_4 + a_{2,3}x_5 + x_1, \\ a_{3,1}x_3 + a_{3,2}x_4 + a_{3,3}x_5 + x_2, \end{cases} \tag{1}$$

there are 162 planes contained in F_3, *e.g.,* $x_0 + x_3, x_1 + x_5, x_2 + x_4$, or $x_0 + x_3, x_1 + x_4, ux_5 + x_2$, with $u^2 + u + 1 = 0$.

2.2. Cubic 4-folds in $\mathbb{P}^5$ depend on $\binom{3+5}{3} - 1 = 55$ parameters. Those containing say the plane $p_0 : x_0 = x_1 = x_2 = 0$ can be written uniquely as $a_0x_0 + a_1x_1 + a_2x_2$ where the a_i are homogeneous polynomials of degree 2. We may assume

$$\begin{cases} a_0 = & \text{any quadric,} & \binom{7}{2} = 21 \text{ free coefficients,} \\ a_1 = & \text{has no term with } x_0, & \binom{6}{2} = 15, \\ a_2 = & \text{has no term with } x_0, x_1, & \binom{5}{2} = 10. \end{cases}$$

Hence we get a $\mathbb{P}^{45}$-bundle over the grassmannian $\mathbb{G}$=$\mathbb{G}$r[2,5],

$$X = \{(p, F_3) \in \mathbb{G} \times \mathbb{P}^{55} \mid F_3 \supset p\}.$$

The dimension of the total space is 54. We expect and get a hypersurface in $\mathbb{P}^{55}$ consisting of cubic 4-folds which contain some $\mathbb{P}^2 \subset \mathbb{P}^5$. This comes from an argument of Castelnuovo-Mumford regularity (cf. [5]).

2.2.1. **Proposition.** *Hypersurfaces of degree d containing some $\mathbb{P}^2$ in $\mathbb{P}^5$ form a subvariety of codimension $\binom{d+2}{2} - 9$ and degree*

$7\binom{d+5}{8}\big(50d^{19} + 750d^{18} + 3300d^{17} + 1800d^{16} - 7800d^{15} + 5400d^{14} - 141400d^{13} - 367800d^{12} - 215115d^{11} - 2480805d^{10} + 2686380d^9 - 538110d^8 + 12830747d^7 + 86752281d^6 - 18022266d^5 + 703254420d^4 - 305343432d^3 + 2350054944d^2 - 787739904d + 2821754880\big) / \big(2^9 \cdot 3^{10} \cdot 5^2\big)$.

In particular, cubic $4-$folds containing some $\mathbb{P}^2$ in $\mathbb{P}^5$ form a hypersurface in $\mathbb{P}^{55}$ of degree **3402**.

Proof. Consider the tautological exact sequence of vector bundles over the grassmannian $\mathbb{G}$,

$$\mathcal{S} \rightarrowtail \mathcal{F} \twoheadrightarrow \mathcal{Q}$$

where $\mathcal{F}$ stands for the trivial bundle with fiber $H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(1))$ whereas $\mathcal{S}$ is the subbundle of rank 3 with fiber over each $p \in \mathbb{G}$ equal to the space of linear forms cutting the plane $p := \mathbb{P}^2$ in $\mathbb{P}^5$. The fiber of the quotient bundle is $\mathcal{Q}_p = H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$. Taking symmetric power, we get the exact sequence

$$\mathcal{S}^{(d)} := \ker \rho \rightarrowtail \operatorname{Sym}_d \mathcal{F} \xrightarrow{\rho} \operatorname{Sym}_d \mathcal{Q}\ .$$

Here ρ stands for the map which sends homogeneous forms of degree d in $\mathbb{P}^5$ to their restrictions to a varying $\mathbb{P}^2 \subset \mathbb{P}^5$. The fiber of $\mathcal{S}^{(d)}$ over $p \in \mathbb{G}$ is equal to the space of forms of degree d lying in the homogeneous ideal of the plane p in $\mathbb{P}^5$. The projectivization $\mathbb{P}(\mathcal{S}^{(d)}) \subset \mathbb{G} \times \mathbb{P}^N$ consists of pairs (p, F_d) such that the plane p lies in the hypersurface F_d. The map $\mathbb{P}(\mathcal{S}^{(d)}) \to \mathbb{P}^N$ induced by the projection $p_2 : \mathbb{G} \times \mathbb{P}^N \to \mathbb{P}^N$ is generically injective for all $d \geq 3$ due to an argument of regularity discussed in §2.3.1. Put

$$m := \dim \mathbb{P}(\mathcal{S}^{(d)}) = 9 + \binom{d+5}{5} - \binom{d+2}{2} - 1.$$

The image $\mathbb{Y}_d$ of $\mathbb{P}(\mathcal{S}^{(d)})$ in $\mathbb{P}^N$ has the same dimension m. It consists of all $F_d \subset \mathbb{P}^5$ containing some $\mathbb{P}^2$. The degree of $\mathbb{Y}_d$ is given by $\int h^m \cap [\mathbb{Y}_d]$, where $h := c_1\mathcal{O}_{\mathbb{P}^5}(1)$ is the hyperplane class. By the projection formula, we have $\deg \mathbb{Y}_d = \int p_2^\star h^m \cap [\mathbb{P}(\mathcal{S}^{(d)})]$. Pushing forward via the structure map $\mathbb{P}(\mathcal{S}^{(d)}) \to \mathbb{G}$, it reduces to the calculation of the Segre class: $\deg \mathbb{Y}_d = \int_{\mathbb{G}} s(\mathcal{S}^{(d)})$, cf. [7, p. 30, §3.1]. The latter is equal to $\int_{\mathbb{G}} c_9(\mathrm{Sym}_d\, \mathcal{Q})$ in view of the above exact sequence. With the help of Katz & Strømme's Schubert package (see [10]) we get the formula. Here is a script for Macaulay2 [8]:

```
loadPackage "Schubert2"
pt = base d; X = flagBundle({3,3},pt); (S,Q) = bundles X
g=chern(9,symmetricPower_d Q); use QQ[d][H_(2,3)]
g=substitute(g,QQ[d][H_(2,3)]); g=sub(g,H_(2,3)=>1);
toString factor g
binom=(d,m)->product(i=1..m,i->(d-i+1)/i)
f=binom(d+5,8); f=g/oo ; toString factor f
sub(denominator f,ZZ)
toString factor oo
```

□

2.3. **Double point formula.** In view of the formula in 2.2.1, given two general cubic 4-folds f_1, f_2 there are **3402** elements in the pencil of cubics $\alpha_1 f_1 + \alpha_2 f_2$ which contain some 2-plane $\mathbb{P}^2 \subset \mathbb{P}^5$.

Likewise, given a general net of cubic 4-folds, we ask now for the number of its members which contain *two* 2-planes.

Let $\varphi : X^m \to Y^n$ be a map of varieties of dimensions m, n. The double point locus, $\mathbb{D}(\varphi)$, is defined as the closure in X of

$$\{x \in X \mid \exists x' \neq x, \quad \text{with } \varphi(x) = \varphi(x')\}.$$

Under mild conditions, the above is the support of a cycle in the Chow group, expressed by the double point formula,

$$\mathbb{D}(\varphi) = \varphi^\star \varphi_\star [X] - c_{n-m} T_\varphi \cap [X].$$

Here $T_\varphi = TX - \varphi^\star TY$, the virtual normal bundle, cf. [7, p. 166, 9.3], [13].

With notation as in 2.2.1, try and apply the double point formula to

$$X := \mathbb{P}(\mathcal{S}^{(3)}) = \{(p, F_3) \in \mathbb{G} \times \mathbb{P}^{55} \mid p \subset F_3\} \xrightarrow{\varphi} Y := \mathbb{P}^{55}.$$

The map $\mathbb{P}(\mathcal{S}^{(3)}) \xrightarrow{\varphi} \mathbb{P}^{55}$ is generically injective and its image is the hypersurface $\mathbb{Y}_3 \subset \mathbb{P}^{55}$ of cubic 4-folds containing some $p = \mathbb{P}^2 \subset \mathbb{P}^5$. Look at the double point locus $\mathbb{D}(\varphi)$. We have $\dim \mathbb{D}(\varphi) = 53$. Its image $\overline{\mathbb{D}}(\varphi) \subset \mathbb{P}^{55}$ has the expected dimension 53, and degree **5 752 908**; below is a script.

```
loadPackage "Schubert2"
X = flagBundle({3,3}); TX=tangentBundle X; (S,Q)=bundles X
S3=symmetricPower_3(Q);     R=symmetricPower_3(6*OO_X)-S3
```

```
Y = projectiveBundle'(dual R)     ---(this takes too long)
TY=tangentBundle Y;  TP55=56*OO_Y(1)-OO_Y
N=TP55-TY                          ---virtual normal bundle
H=chern(1,OO_Y(1)); toString oo
d=integral (H^54)  ---degree of image in P55
c1=chern(1,N); toString oo
dblpt=d*H-c1;  integral(H^53*dblpt)/2      ---get 5752908
```

But…It turns out that $\overline{\mathbb{D}}(\varphi)$ is reducible. The point is that, when the two planes are in special position, they impose less conditions on the linear system of cubics. We recall from [5] the following regularity bound.

2.3.1. **Lemma.** *The Castelnuovo-Mumford regularity of the saturated homogeneous ideal of a union of $r \geq 2$ subspaces is at most r.* □

2.3.2. Consider the correspondence

$$Z := \{(q_1, q_2, F_3) \in \mathbb{G} \times \mathbb{G} \times \mathbb{P}^{55} \,|\, q_1 \neq q_2,\, F_3 \supset q_1 \cup q_2\}.$$

Let $q_{1,2} = q_1 \cup q_2$. Now Z decomposes into pieces corresponding to the relative position of the plane-pair as we now describe.

• 2 general planes, *e.g.,*

$$\begin{cases} q_1 : x_0 = x_1 = x_2 = 0, \\ q_2 : x_3 = x_4 = x_5 = 0. \end{cases}$$

The homogeneous ideal is generated by the nine quadrics

$$\langle x_i x_j,\, 0 \leq i \leq 2,\, 3 \leq j \leq 5\rangle.$$

Recall the Hilbert polynomial, $P(t) = 2 + 3t + t^2 = 2\binom{t+2}{2}$ measures, for all t beyond the regularity, the number of independent conditions imposed on hypersurfaces of degree t to lie in the homogeneous ideal of $q_{1,2}$. Thus, the dimension of the fiber of Z over such plane pair is $55 - 2\binom{5}{2} = 35$. This yields a component of Z of dimension $35 + 18 = 53$.

• 2 planes meeting at a point; these form a hypersurface in $\mathbb{G}\times\mathbb{G}$. Now the homogeneous ideal is of the form

$$\langle x_0,\, x_1x_3, x_1x_4, x_2x_3, x_2x_4\rangle.$$

Its Hilbert polynomial is $P(t) = 1 + 3t + t^2$; hence the fiber of Z has dimension 36. We get again a component of dimension 53.

• 2 planes through a line in $\mathbb{P}^5$. The homogeneous ideal is of the form $\langle x_0, x_1, x_2x_3\rangle$. These plane-pairs vary in a subvariety of $\mathbb{G}\times\mathbb{G}$ of codimension 4. The Hilbert polynomial is $P(t) = 1+2t+t^2$. Thus we find a component of Z of dimension $= 14 + 55 - 16 = 53$.

So, that's a situation where diminishing the dimension of the base is compensated by an equal increase in the dimension of the fiber.

It follows that the locus in $\mathbb{P}^{55}$ consisting of cubic 4-folds which contain a plane-pair in $\mathbb{P}^5$ is of dimension 53. Hence we may conclude that the map $(p, F_3) \overset{\varphi}{\mapsto} F_3$ is generically injective and its double point locus receives contribution from the three configurations as described above.

In order to find the degree of the closure of the "good" locus corresponding to two general planes, we shall pursue below a different route. It amounts to building a smooth 2:1 cover of the component of the Hilbert scheme of unions of 2-planes in $\mathbb{P}^5$.

3. Parameter space for unions of 2-planes in $\mathbb{P}^5$

Start with $\mathbb{G} \times \mathbb{G}$, the variety of ordered plane-pairs. Define

$$\mathbb{S} = \{ \text{ plane-pairs with a common line } \}.$$

Let $\widetilde{\mathbb{G}}$ be the blowup of the diagonal $\mathbb{D}$ in $\mathbb{G} \times \mathbb{G}$.

One checks that, though $\mathbb{S}$ is singular along $\mathbb{D}$, its strict transform $\widetilde{\mathbb{S}} \subset \widetilde{\mathbb{G}}$ is smooth. It is isomorphic to a natural $\mathbb{P}^3 \times \mathbb{P}^3$–bundle over $\mathbb{G}(2,6) = \mathbb{G}r[1,5]$, the grassmannian of lines in $\mathbb{P}^5$, blown-up along its diagonal.

We have $\dim \mathbb{S} = \dim \widetilde{\mathbb{S}} = 14$.

Let $\widehat{\mathbb{G}}$ be the blowup of $\widetilde{\mathbb{G}}$ along $\widetilde{\mathbb{S}}$. Then $\widehat{\mathbb{G}}$ parameterizes a flat family with general member the union of two general planes in $\mathbb{P}^5$.

In other words, there is a surjective map $\widehat{\mathbb{G}} \longrightarrow \mathbb{H}$, where $\mathbb{H}$ denotes the Hilbert scheme component of unions of two planes in $\mathbb{P}^5$. See [3] for the case of pairs of subspaces of codimension 2.

The verification of the assertions above can be done using local coordinates. Instead of working with $\mathbb{G} \times \mathbb{G}$, we may fix the 2-plane

$$p_0 := \langle x_0, x_1, x_2 \rangle$$

and consider the variable 2-plane p_a as in (1), where the $a_{i,j}$ stand for affine coordinates in $\mathbb{G}$ around p_0. Equations for the fiber of $\mathbb{S}$ over p_0 are given by the 5×5 minors of the 6×6 matrix of the system $p_0 = p_a = 0$. It's the same as the ideal $\mathcal{J}$ of the 2×2 minors of $p_a = 0$ alone. One sees at once that this is singular precisely at the origin, *i.e.*, $p_0 : a_{i,j} = 0, i, j = 1, 2, 3$. Blowing up the diagonal means now blowing up $\mathbb{G}$ at p_0. We choose $a_{3,3}$ as the local generator of the exceptional ideal and write

$$a_{1,1} = b_{1,1} a_{3,3}, \ldots, a_{3,2} = b_{3,2} a_{3,3},$$

the 8 relations for the blowup $\widetilde{\mathbb{A}^9} \to \mathbb{A}^9 \subset \mathbb{G}$. Presently

$$a_{3,3}, b_{i,j}, i, j = 1, 2, 3, (i, j) \neq (3, 3)$$

are affine coordinates up in the blowup $\widetilde{\mathbb{A}^9}$. Let $\mathcal{J}'$ be the ideal generated by $\mathcal{J}$ upstairs, *i.e.*, upon plugging in the relations $a_{i,j} = a_{3,3} b_{i,j}$. We find that

$$\mathcal{J}' = (a_{3,3})^2 \widetilde{\mathcal{J}},$$

with

$$\widetilde{\mathcal{J}} = \langle b_{2,2} - b_{2,3} b_{3,2}, b_{2,1} - b_{2,3} b_{3,1}, b_{1,2} - b_{1,3} b_{3,2}, b_{1,1} - b_{1,3} b_{3,1} \rangle.$$

This is the ideal of the (fiber over p_0 of the) strict transform $\widetilde{\mathbb{S}}$ up in the blowup of $\mathbb{G} \times \mathbb{G}$ along the diagonal. Given a plane-pair $(q_1, q_2) \in \mathbb{S}$, the intersection $q_1 \cap q_2$ is a line provided $q_1 \neq q_2$. Thus the rational map

$$\begin{array}{ccc} \mathbb{S} & \dashrightarrow & \mathbb{G}r[1,5] \\ (q_1, q_2) & \longmapsto & q_1 \cap q_2 \end{array}$$

is a morphism off the diagonal $\mathbb{D}$. We claim that it induces a morphism

$$\lambda : \widetilde{\mathbb{S}} \longrightarrow \mathbb{G}r[1,5].$$

Indeed, the lifted linear system $p_0 = \widetilde{p}_a = 0$ restricted to $\widetilde{\mathbb{S}}$ yields the system

$$p_0 : x_0 = x_1 = x_2 = b_{3,1} x_3 + b_{3,2} x_4 + x_5 = 0$$

from which we infer there is always a well defined line lying on both planes. The morphism λ lifts to yield a map $\widetilde{\lambda}$ to the natural $\mathbb{P}^3 \times \mathbb{P}^3$-bundle $\mathbb{U}$ over $\mathbb{G}$r[1,5] defined by picking a pair of 2-planes through a line $\ell \in \mathbb{G}r[1,5]$,

$$\mathbb{U} = \{(q_1, q_2, \ell) \in \mathbb{G} \times \mathbb{G} \times \mathbb{G}r[1,5] \,|\, \ell \subseteq q_1 \cap q_2\}.$$

One also checks that $\widetilde{\lambda}$ actually induces an isomorphism $\lambda : \widetilde{\mathbb{S}} \longrightarrow \widetilde{\mathbb{U}}$ where $\widetilde{\mathbb{U}}$ denotes the blowup of the relative diagonal $(q_1 = q_2)$ in $\mathbb{U}$.

Set for short

$$\mathcal{F}_d := \mathrm{Sym}_d \, \mathcal{F} = H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(d)),$$

the space of homogeneous polynomials of degree d. There is a natural rational map

$$\begin{array}{ccc} \mu : \mathbb{G} \times \mathbb{G} & \dashrightarrow & \mathbb{G}r(9, \mathcal{F}_2) \\ (q_1, q_2) & \longmapsto & q_1 \cdot q_2 \end{array}$$

defined by multiplying the 3-dimensional subspaces $q_i \subset \mathcal{F}$ of linear polynomials. The scheme of indeterminacy of μ is equal to $\mathbb{S}$. It lifts to a rational map

$$\widetilde{\mathbb{G}} \dashrightarrow \mathbb{G}r(9, \mathcal{F}_2)$$

with scheme of indeterminacy equal to $\widetilde{\mathbb{S}}$. Let $\widehat{\mathbb{G}}$ be the blowup of $\widetilde{\mathbb{G}}$ along $\widetilde{\mathbb{S}}$. We obtain a morphism

$$\widehat{\mu} : \widehat{\mathbb{G}} \longrightarrow \mathbb{G}r(9, \mathcal{F}_2)$$

Over $\widehat{\mathbb{G}}$, for each degree $d \geq 2$, there is a vector subbundle $\mathcal{V}_d \subset \mathcal{F}_d$ of the trivial bundle of homogeneous polynomials of degree d such that:

- The fiber of $\mathcal{V}_d$ over a plane-pair $p_{1,2} \in \widehat{\mathbb{G}}$ is the space of equations of hypersurfaces of degree d containing $p_{1,2}$;
- $\operatorname{rank}\mathcal{V}_d = \binom{d+5}{5} - 2\binom{d+2}{2}$;
- The image $\mathbb{W}_d$ in $\mathbb{P}^N = \mathbb{P}\mathcal{F}_d$ of the projectivization $\mathbb{P}\mathcal{V}_d \subset \widehat{\mathbb{G}} \times \mathbb{P}^N$ is the variety of hypersurfaces containing a (flat specialization of a) plane-pair.

The variety $\widehat{\mathbb{G}}$ inherits a $\mathbb{C}^\star$-action, with (a lot:-) of isolated fix points. The vector bundles $\mathcal{V}_d \to \widehat{\mathbb{G}}$ are equivariant. Bott localization (cf. [6, §2], [15], [1], [2]) applies, enabling us to find the degree of $\mathbb{W}_d$ in $\mathbb{P}\mathcal{F}_d$, $d \geq 3$, to wit,

$$\deg \mathbb{W}_d = \int_{\widehat{\mathbb{G}}} c_{18}(-\mathcal{V}_d) = \sum_{\text{fixpts}} \frac{c_{18}^T(-\mathcal{V}_d)}{c_{18}^T \tau}\,.$$

Using Maple, in the flavor explained by Meurer [15] we are able to integrate and find, *e.g.*, for $d = 3$, the value $\deg \mathbb{W}_3 = \mathbf{3\,371\,760}$, not in agreement with the double point formula.

An argument employing Grothendieck-Riemann-Roch (cf. [4]) shows that there is a formula for the degree of $\mathbb{W}_d \subset \mathbb{P}\mathcal{F}_d$ as a polynomial in d. Actually we got by interpolation the following polynomial of degree 54 (cf. [16] for a script):

$$\begin{gathered}
\deg \mathbb{W}_d = \tfrac{1}{2^{32}\cdot 3^{23}\cdot 5^6\cdot 7^2\cdot 11\cdot 13\cdot 17}\tbinom{d}{3}\Big(297797500d^{51} + 16974457500d^{50} + \\
438953515000d^{49} + 6750473730000d^{48} + 67557745255000d^{47} + 446469328305000d^{46} \\
+1821546306580000d^{45} + 3261093465630000d^{44} - 5452213497731000d^{43} \\
-26658904130859000d^{42} - 1792499938229000d^{41} - 807392033197659000d^{40} \\
-690452757469587700d^{39} + 3477546191451769500d^{38} + \\
16829310517659656980 0d^{37} - 83205055050390026400d^{36} - \\
418358516692370972 5625d^{35} + 4729797968873046725475d^{34} + \\
93623512083339602708675d^{33} - 210025261579623597041475d^{32} - \\
1497759082084784912756740d^{31} + 6691368991089621694295820d^{30} + \\
10512834434651356253342780d^{29} - 127045484364059052592597740d^{28} + \\
173715078834280290838756586d^{27} + 1128680664343084906757160738d^{26} - \\
4994152749025875809985069838d^{25} + 2356774599575513190792679230d^{24} + \\
37766401805433040109235274520d^{23} - 120118775223192214665021263640d^{22} + \\
641601317593845382598024791 40d^{21} + 507092558093142767480135015700d^{20} - \\
1451146056063090731464859692765d^{19} + 1272606825133942111965200965455d^{18} + \\
3883125863775315719224517949 95d^{17} + 3580712277013841049646053016725d^{16} - \\
20845497262217658319851150940560d^{15} + 12031904188478235409221442162320d^{14} + \\
169428261347272908281967678701280d^{13} - 687428225963718953591666450858400d^{12} + \\
1340013212341098964586554590155520d^{11} - 1087468822379100789481710030842880d^{10} - \\
1847260530393109277960386571454720d^{9} + 8564365120882865993680747841936640d^{8} - \\
18503733474733545031760180202663936d^{7} + 28014319703719681406965987557875712d^{6} - \\
35797411551433954908141875545178112d^{5} + 39893149299289869979218029094174720d^{4} - \\
42655130536947988709557871758540800d^{3} + 40106073268823932733976960565248000d^{2} - \\
24647062713098039616382917672960000d + 9496912828923697566983808614400000\Big).
\end{gathered} \tag{2}$$

There are three other known families of rational cubic 4 folds: impose F_3 to contain either a Del Pezzo of degree 5 or a scroll of degree 4 (2 types) in $\mathbb{P}^5$ (cf. [11]). It would be nice to find their degrees. The difficulty lies in describing appropriate parameter spaces for those families of surfaces.

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ON THE GEOMETRY AND ARITHMETIC OF INFINITE TRANSLATION SURFACES

FERRÁN VALDEZ AND GABRIELA WEITZE-SCHMITHÜSEN

In honorem professoris Xavier Gómez-Mont sexagesimum annum complentis.

Abstract. Precompact translation surfaces, *i.e.* closed surfaces which carry a translation atlas outside of finitely many finite angle cone points, have been intensively studied for about 25 years now. About 5 years ago the attention was also drawn to general translation surfaces. In this case the underlying surface can have infinite genus, the number of finite angle cone points of the translation structure can be infinite, and there can be singularities which are not finite angle cone points. There are only a few invariants one classically associates with precompact translation surfaces, among them certain number fields, *i.e.* fields which are finite extensions of $\mathbb{Q}$. These fields are closely related to each other; they are often even equal. We prove by constructing explicit examples that most of the classical results for the fields associated with precompact translation surfaces fail in the realm of general translation surfaces and investigate the relations among these fields. A very special class of translation surfaces are so called square-tiled surfaces or origamis. We give a characterisation for infinite origamis.

1. Introduction

Let S be a translation surface, in the sense of Thurston [Thu97], and denote by $\overline{S}$ the metric completion with respect to its natural translation invariant flat metric. S is called *precompact* if $\overline{S}$ is homeomorphic to a compact surface. We call translation surfaces *origamis*, if they are obtained from gluing copies of the Euclidean unit square along parallel edges by translations; see Definition 2.6. They are precompact translation surfaces if and only if the number of copies is finite. An important invariant associated with a translation surface S is the *Veech group* $\Gamma(S)$ formed by the differentials of affine diffeomorphisms of S that preserve orientation; as further invariants one considers the *trace field* $K_{\text{tr}}(S)$, the *holonomy field* $K_{\text{hol}}(S)$, the *field of cross ratios of saddle connections* $K_{\text{cr}}(S)$ and the *field of saddle connections* $K_{\text{sc}}(S)$; compare Definition 2.9 and Definition 3.2. For precompact surfaces we have the following characterisation:

Theorem A. [GJ00, Theorem 5.5] *Let S be a precompact translation surface, and let $\Gamma(S)$ be its Veech group. The following statements are equivalent.*

- *(i) The groups $\Gamma(S)$ and $\mathrm{SL}(2,\mathbb{Z})$ are commensurable.*
- *(ii) Every cross ratio of saddle connections is rational. Equivalently the field $K_{cr}(S)$ is equal to $\mathbb{Q}$.*
- *(iii) There exists a translation covering from a puncturing of S to a once-punctured flat torus.*

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(iv) *S is an origami up to an affine homeomorphism,* i.e. *there is a Euclidean parallelogram that tiles S by translations.*

The first result of this article explores what remains of the preceding characterisation if S is a general *tame* translation surface. Tame translation surfaces are the translation surfaces all of whose singularities are cone angle singularities (possibly of infinite angle). This includes surfaces like $\mathbb{R}^2$, but also surfaces whose fundamental group is not finitely generated. We define tameness and the different types of singularities in Section 2. Furthermore, we call S *maximal*, if it has no finite singularities of total angle 2π; compare Definition 2.6.

Theorem 1. *Let S be a maximal tame translation surface. Then,*

(i) *S is affine equivalent to an origami if and only if the set of* developed cone points *is contained in $L + x$, where $L \subset \mathbb{R}^2$ is a lattice and $x \in \mathbb{R}^2$ is fixed.*

(ii) *If S is an origami the following statements (b)-(d) hold. In (a) and (e) we require in addition that there are at least two nonparallel saddle connections on S:*
   (a) *The Veech group of S is commensurable to a subgroup of* $\mathrm{SL}(2, \mathbb{Z})$.
   (b) *The field of cross ratios $K_{\mathrm{cr}}(S)$ is isomorphic to $\mathbb{Q}$.*
   (c) *The holonomy field $K_{\mathrm{hol}}(S)$ is isomorphic to $\mathbb{Q}$.*
   (d) *The saddle connection field $K_{\mathrm{sc}}(S)$ is isomorphic to $\mathbb{Q}$.*
   (e) *The trace field $K_{\mathrm{tr}}(S)$ is isomorphic to $\mathbb{Q}$.*
   However, none of (a)-(e) implies that S is an origami.

In the proof of Theorem 1 we will show that even if we require that in (a) the Veech group of S is equal to $\mathrm{SL}(2, \mathbb{Z})$, this condition does not imply that S is an origami.

If S is precompact, then the four fields $K_{\mathrm{tr}}(S)$, $K_{\mathrm{hol}}(S)$, $K_{\mathrm{cr}}(S)$ and $K_{\mathrm{sc}}(S)$ are number fields and we have the following hierarchy:

$$\mathbb{Q} \subseteq K_{\mathrm{tr}}(S) \subseteq K_{\mathrm{hol}}(S) \subseteq K_{\mathrm{cr}}(S) = K_{\mathrm{sc}}(S) \tag{1.1}$$

Thus by Theorem Theorem A the conditions (a), (b) and (d) in (ii) of Theorem 1 are, for precompact surfaces, equivalent to being an origami. Conditions (c) and (e), however, are even for precompact translation surfaces not equivalent to being an origami. Indeed, recall that the "general" precompact translation surface has trivial Veech group, *i.e.* Veech group $\{I, -I\}$, where I is the identity matrix (see [Möl09, Thm. 2.1]). This implies that (e) is not equivalent to being an origami. Furthermore, in Example 4.5 we construct an explicit example of a precompact surface S that is not an origami and such that $K_{\mathrm{hol}}(S) = \mathbb{Q}$. This shows that (c) is not equivalent to being an origami.

In the case of general tame translation surfaces, the fields $K_{\mathrm{tr}}(S)$, $K_{\mathrm{hol}}(S)$, $K_{\mathrm{cr}}(S)$ and $K_{\mathrm{sc}}(S)$ are not necessarily number fields anymore; compare Proposition 3.6. Furthermore from the hierarchy in (1.1) it just remains true in general that $K_{\mathrm{hol}}(S)$ and $K_{\mathrm{cr}}(S)$ are both subfields of $K_{\mathrm{sc}}(S)$. Some of the other relations in (1.1) hold under extra assumptions on S; compare Corollary 4.7. It follows that, in general, if $K_{\mathrm{sc}}(S)$ is isomorphic to $\mathbb{Q}$, then both $K_{\mathrm{hol}}(S)$ and $K_{\mathrm{cr}}(S)$ are isomorphic to $\mathbb{Q}$. In terms of Theorem 1, part (ii), this is equivalent to say that (d) implies both (b) and (c). Note that furthermore trivially (a) implies (e). We treat the remaining of these implications in the next theorem.

Theorem 2. *There are examples of tame translation surfaces S for which*

(i) *The Veech group $\Gamma(S)$ is equal to $\mathrm{SL}(2, \mathbb{Z})$ and K is not equal to $\mathbb{Q}$, where K can be chosen from $K_{\mathrm{cr}}(S)$, $K_{\mathrm{hol}}(S)$ and $K_{\mathrm{sc}}(S)$.*

(ii) *The fields $K_{\mathrm{sc}}(S)$ (hence also $K_{\mathrm{cr}}(S)$ and $K_{\mathrm{hol}}(S)$) and $K_{\mathrm{tr}}(S)$ are equal to $\mathbb{Q}$, but $\Gamma(S)$ is not commensurable to a subgroup of $\mathrm{SL}(2,\mathbb{Z})$.*

(iii) *$K_{\mathrm{cr}}(S)$ or $K_{\mathrm{hol}}(S)$ is equal to $\mathbb{Q}$, but $K_{\mathrm{sc}}(S)$ is not.*

(iv) *The field $K_{\mathrm{cr}}(S)$ is equal to $\mathbb{Q}$, but $K_{\mathrm{hol}}(S)$ is not or* vice versa*: $K_{\mathrm{hol}}(S)$ is equal to $\mathbb{Q}$, but $K_{\mathrm{cr}}(S)$ is not.*

(v) *The field $K_{\mathrm{tr}}(S)$ is equal to $\mathbb{Q}$, but none of the conditions (a), (b), (c) or (d) in Theorem 1 hold. Moreover, none of the conditions (b), (c) or (d) imply that $K_{\mathrm{tr}}(S)$ is isomorphic to $\mathbb{Q}$.*

The proofs of the preceding two theorems heavily rely on modifications of the construction in [PSV11, Construction 4.9] which was there used to determine all possible Veech groups of tame translation surfaces. We summarise this construction in Section 2.3. One can furthermore modify the construction to prove that any subgroup of $\mathrm{SL}(2,\mathbb{Z})$ is the Veech group of an origami. From this we will deduce the following statement about the oriented outer automorphism group $\mathrm{Out}^+(\mathrm{F}_2)$ of the free group F_2 in two generators:

Corollary 1.1. *Every subgroup of $\mathrm{Out}^+(\mathrm{F}_2)$ is the stabiliser of a conjugacy class of some (possibly infinite index) subgroup of F_2.*

If S is a precompact translation surface, the existence of hyperbolic elements, *i.e.* matrices whose trace is bigger than 2, in $\Gamma(S)$ has consequences for the image of $H_1(S,\mathbb{Z})$ in $\mathbb{R}^2$ under the developing map (also called *holonomy* map; see Section 2) and for the nature of some of the fields associated with S. To be more precise, if S is precompact, the following is known:

(A) If there exists $M \in \Gamma(S)$ hyperbolic, then the holonomy field of S is equal to $\mathbb{Q}[tr(M)]$. In particular, the traces of any two hyperbolic elements in $\Gamma(S)$ generate the same field over $\mathbb{Q}$; see [KS00, Theorem 28].

(B) If there exists $M \in \Gamma(S)$ hyperbolic and $tr(M) \in \mathbb{Q}$, then S is an origami; see [McM03b, Theorem 9.8].

(C) If S is a "bouillabaisse surface" (*i.e.* if $\Gamma(S)$ contains two transverse parabolic elements), then $K_{\mathrm{tr}}(S)$ is totally real; compare [HL06a, Theorem 1.1]. This implies that if there exists an hyperbolic M in $\Gamma(S)$ such that $\mathbb{Q}[tr(M)]$ is not totally real then $\Gamma(S)$ does not contain any parabolic elements; see Theorem 1.2 in *ibid.*

(D) Let Λ and Λ_0 be the subgroups of $\mathbb{R}^2$ generated by the image under the holonomy map of $H_1(\overline{S},\mathbb{Z})$ and $H_1(\overline{S},\Sigma;\mathbb{Z})$, respectively. Here Σ is the set of cone angle singularities of S. If the affine group of S contains a pseudo-Anosov element, then Λ has finite index in Λ_0; see [KS00, Theorem 30].

The third main result of this paper shows that when passing to general tame translation surfaces there are no such consequences. For such surfaces, an element of $\Gamma(S) < \mathrm{GL}_+(2,\mathbb{R})$ will be called hyperbolic, parabolic or elliptic if its image in $\mathrm{PSL}(2,\mathbb{R})$ is hyperbolic, parabolic or elliptic respectively.

Theorem 3. *There are examples of tame translation surfaces S for which (A), (B), (C) or (D) from above do not hold.*

We remark that all tame translation surfaces S that we construct in the proof of the preceding theorem have the same topological type: one end and infinite genus. Such topological surfaces are called Loch Ness monster; see Section 2.

This paper is organised as follows. In Section 2 we review the basics about general translation surfaces, tame translation surfaces and origamis, their singularities and possible Veech groups.

In Section 3 we present the definitions of the fields listed in Theorem 1 for general tame translation surfaces. We prove that the main algebraic properties of these fields which are true for precompact translation surfaces no longer hold for general translation surfaces. For example, we construct examples of tame translation surfaces for which the trace field is not a number field. We furthermore show those inclusions from (1.1) which still are valid for tame translation surfaces. Section 4 deals with the proofs of the three theorems stated in this section. We refer the reader to [HS10], [HLT11] or [HHW13] for recent developments concerning tame translation surfaces.

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2. Preliminaries

2.1. **General translation surfaces and their singularities.** In this section we review some basic notions needed for the rest of the article. For a detailed exposition, we refer to [GJ00] and [Thu97].

A *translation surface* S will be a 2-dimensional real G-manifold with $G = \mathbb{R}^2 = Trans(\mathbb{R}^2)$; that is, a surface on which coordinate changes are translations of the real plane $\mathbb{R}^2$. We can pull back to S the standard translation invariant flat metric of the plane and obtain this way a flat metric on the surface. We denote by $\overline{S}$ the metric completion of S with respect to this natural flat metric. A *translation map* is a G-map between translation surfaces. Every translation map $f : S_1 \longrightarrow S_2$ has a unique continuous extension $\overline{f} : \overline{S_1} \longrightarrow \overline{S_2}$.

Definition 2.1. If $\overline{S}$ is homeomorphic to an orientable compact surface, we say that S is a *precompact translation surface*. Else we say that S is non precompact. Observe that a not precompact translation surface is not necessarily of infinite type. The union of all precompact and not precompact translation surfaces form the set of *general translation surfaces*.

Definition 2.2. Let S be a translation surface. We call the points of $\overline{S}\backslash S$ *singularities* of the translation surface S. A point $x \in \overline{S} \setminus S$ is called a *finite angle singularity* or *finite angle cone point* of total angle $2\pi m$, where $m \geq 1$ is a natural number, if there exists a neighbourhood of x which is isometric to a neighbourhood of the origin in $\mathbb{R}^2$ with a metric that, in polar coordinates (r, θ), has the form $ds^2 = dr^2 + (mrd\theta)$. The set of finite angle singularities of $\overline{S}$ is denoted by Σ_{fin}.

Precompact translation surfaces are obtained by glueing finitely many polygons (deprived of their vertices) along parallel edges by translations. One even obtains all precompact translation surfaces in this way; see [Mas06]. Thus if S is a precompact translation surface, all of its singularities are finite angle singularities. If furthermore $\overline{S}$ has genus at least 2, then, by a simple Euler characteristic calculation, $\overline{S}$ always has singularities. For non precompact translation surfaces, new kinds of singularities will occur. We illustrate this in the following example.

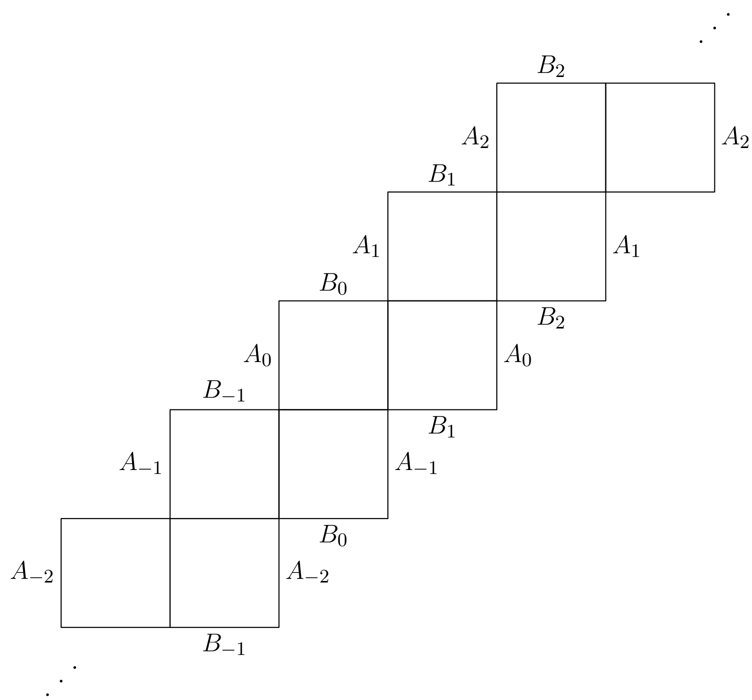


Figure 1. An infinite-type translation surface.

Example 2.3. In Figure 1 we depict a translation surface obtained from infinitely many copies of the Euclidean unit square. More precisely, we remove the vertices from all the squares in the figure. Some pairs of edges are already identified; among the remaining edges we identify opposite ones which are labelled by the same letter by translations. The result is a translation surface S which is not precompact. It is called *infinite staircase* because of its shape. This and similar shaped surfaces have been intensively studied in the literature; see *e.g.* [HS10], [HHW13], [HW12] and [CG12]. S is a prototype for what we will call in this text an infinite origami or infinite square-tiled surface; compare Definition 2.6. The translation surface S comes with a natural cover p to the once punctured torus obtained from glueing parallel edges of the Euclidean unit square again with its vertices removed.
Observe furthermore that the metric completion of the infinite staircase S has four singularities x_1, x_2, x_3 and x_4. Restricted to a punctured neighbourhood of them p is infinite cyclic and the universal cover of a once punctured disk. In this sense the singularities $x_1, \ldots, x_4$ generalise finite angle singularities of angle $2\pi m$. They are prototypes for what we call infinite angle singularities; compare Definition 2.4.

Definition 2.4. Let S be a translation surface. A point $x \in \overline{S}$ is called an *infinite angle singularity* or *infinite angle cone point* if there exists a neighbourhood of x isometric to the neighbourhood of the branching point of the infinite cyclic flat branched covering of $\mathbb{R}^2$. The set of infinite angle singularities of $\overline{S}$ is denoted by Σ_{inf}. Points in the set $\Sigma = \Sigma_{\text{fin}} \cup \Sigma_{\text{inf}}$ will be called *cone angle singularities* of S or just *cone points*.

Definition 2.5. A translation surface S is called *tame* if all points in $\overline{S} \setminus S$ are cone angle singularities (of finite or infinite total angle). A tame translation surface S is said to be of *infinite-type* if the fundamental group of S is not finitely generated.

Every precompact translation surface is tame. There are tame translation surfaces with infinite angle singularities which are not of infinite type. For example consider the infinite cyclic covering of the once punctured plane. Explicit examples of infinite-type translation surfaces arise naturally when studying the billiard on a polygon whose interior angles are not rational multiples of π (see [Val09]). Nevertheless, not all translation surfaces are tame. If one allows infinitely many polygons, *wild* types of singularities may occur. Simple examples of not tame translation surfaces can be found in [Cha04] and [BV13].

In the following we define the very special class of translation surfaces called origamis or square-tiled surfaces. With Example 2.3 we have already seen a specific instance of them.

Definition 2.6. A translation surface is called *origami* or *square-tiled surface*, if it fulfils one of the two following equivalent conditions:

(i) S is a translation surface obtained from glueing (possibly infinitely many) copies of the Euclidean square along edges by translations according to the following rules:
- each left edge is glued to precisely one right edge,
- each upper edge to precisely one lower edge and
- the resulting surface is connected;

and removing all singularities.

(ii) S allows an unramified covering $p : S^* \to T_0$ of the once-punctured unit torus

$$T_0 = (\mathbb{R}^2 \backslash L_0)/L_0,$$

such that p is a translation map. Here S^* is a subset of S such that the complement $S \backslash S^*$ is a discrete set of points on S. L_0 is the lattice in $\mathbb{R}^2$ spanned by the two standard basis $e_1 = (1, 0)$ and $e_2 = (0, 1)$. Furthermore, S is maximal in the sense that $\overline{S} \backslash S$ contains no finite angle singularities of angle 2π.

An origami will be called *finite* if the number of squares needed to construct it is finite or, equivalently, if the unramified covering $p : S \to T_0$ is finite. Else, the origami will be called *infinite*. See [Sch06, Section 1] for a detailed introduction to finite origamis. Infinite origamis were studied *e.g.* in [HS10] and [Gut10].

2.2. **Developed cone points and the Veech group.** In the following we introduce the *set of developed cone points* for tame translation surfaces, which will play an important role in the proof of Theorem 1. Let $\pi_S : \widetilde{S} \longrightarrow S$ be a universal cover of a translation surface S and $\mathrm{Aut}(\pi_S)$ the group of its deck transformations. From now on, $\widetilde{S}$ is endowed with the translation structure obtained as pull-back from the one on S via π_S. Recall from [Thu97, Section 4.3] that for every deck transformation γ, there is a unique translation $\mathrm{hol}(\gamma)$ satisfying

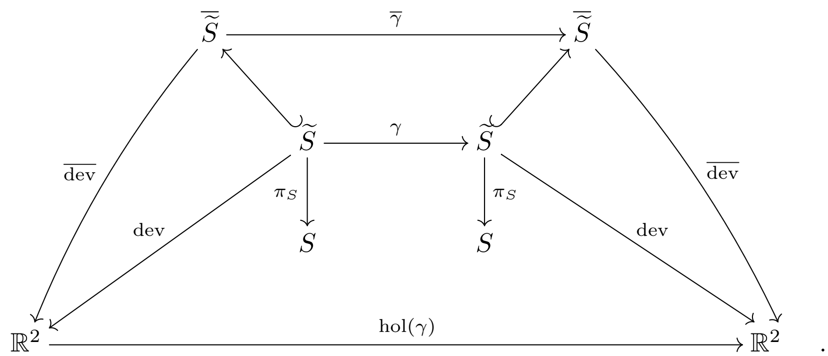
$$\mathrm{dev} \circ \gamma = \mathrm{hol}(\gamma) \circ \mathrm{dev}, \tag{2.2}$$

where $\mathrm{dev} : \widetilde{\mathrm{S}} \longrightarrow \mathbb{R}^2$ denotes the developing map. The map hol: $\mathrm{Aut}(\pi_S) \to \mathrm{Trans}(\mathbb{R}^2) \cong \mathbb{R}^2$ is a group homomorphism. By considering the continuous extension of each map in Equation (2.2) to the metric completion of its domain, we obtain

$$\overline{\mathrm{dev}} \circ \overline{\gamma} = \mathrm{hol}(\gamma) \circ \overline{\mathrm{dev}}. \tag{2.3}$$

Overall we have the following commutative diagram:

(2.4)



Definition 2.7. *The set of developed singularities* of S is the subset of the plane $\mathbb{R}^2$ given by $\overline{\text{dev}}(\overline{\widetilde{S}} \setminus \widetilde{S})$. We denote it by $\widetilde{\Sigma}(S)$. If S is a tame translation surface, we also call $\widetilde{\Sigma}(S)$ the *set of developed cone points*.

Definition 2.8. A *singular geodesic* of a translation surface S is an open geodesic segment in the flat metric of S whose image under the natural embedding $S \hookrightarrow \overline{S}$ issues from a singularity of $\overline{S}$, contains no singularity in its interior and is not properly contained in some other geodesic segment. A *saddle connection* is a finite length singular geodesic.

To each saddle connection we can associate a *holonomy vector*: we 'develop' the saddle connection in the plane by using local coordinates of the flat structure. The difference vector defined by the planar line segment is the holonomy vector. Two saddle connections are *parallel*, if their corresponding holonomy vectors are linearly dependent.

Next, we introduce the *Veech group*, which since Veech's article [Vee89] from 1989 has been studied for precompact translation surfaces as the natural object associated with the surface. Let $\text{Aff}_+(S)$ be the group of affine orientation preserving homeomorphisms of a translation surface S. Consider the map

$$\text{Aff}_+(S) \xrightarrow{D} \mathbf{GL}_+(2, \mathbb{R}) \tag{2.5}$$

that associates to every $\phi \in \text{Aff}_+(S)$ its (constant) Jacobian derivative $D\phi$.

Definition 2.9. Let S be a translation surface. We call $\Gamma(S) = D(\text{Aff}_+(S))$ the *Veech group* of S.

Remark 2.10. The group $\mathbf{GL}_+(2, \mathbb{R})$ naturally acts on the set of translation surfaces: We define $A \cdot S$ to be the translation surface obtained from S by composing each chart in the translation atlas with the linear map $\binom{x}{y} \mapsto A \cdot \binom{x}{y}$. Since the map $\text{id}_A : S \to A \cdot S$ which topologically is the identity map has derivative A, we have that $\Gamma(A \cdot S) = A \cdot \Gamma(S) \cdot A^{-1}$.

2.3. **Constructing tame surfaces with prescribed Veech groups.** The proofs of our main results heavily rely on slight modifications of the construction in the proof of [PSV11, Proposition 4.1]. In this section we review this construction. We will mainly use the notation of [PSV11]. The construction we are about to review proves the following:

Proposition 2.11 ([PSV11, Proposition 4.1])**.** *For any countable subgroup G of* $\text{GL}_+(2, \mathbb{R})$ *disjoint from* $\mathcal{U} = \{g \in \mathbf{GL}_+(2, \mathbb{R}) \ : \ ||\text{g}|| < 1\}$ *there exists a tame translation surface $S = S(G)$, which is homeomorphic to the* Loch Ness monster*, with Veech group G.*

The *Loch Ness monster* is the unique topological surface S (up to homeomorphism) of infinite genus and one end. By one end we mean that for every compact set $K \subset S$ there exists a compact set $K \subset K' \subset S$ such that $S \setminus K'$ is connected. We refer the reader to [Ric63] for a more detailed discussion on surfaces of infinite genus and ends.

First we have to recall a basic geometric operation which will play an important role in the construction: *glueing translation surfaces along marks.*

Definition 2.12. Let S be a tame translation surface. A *mark* on S is an oriented finite length geodesic (with endpoints) on S. The *vector* of a mark is its holonomy vector, which lies in $\mathbb{R}^2$. If m, m' are two disjoint marks on S with equal vectors, we can perform the following operation. We cut S along m and m', which turns S into a surface with boundary consisting of four straight segments. Then we reglue these segments to obtain a tame translation surface S' different from the one we started from. We say that S' is obtained from S by *reglueing along m and m'*. Let $S_0 = S \setminus (m \cup m')$. Then S' admits a natural embedding i of S_0. If $A \subset S_0$, then we say that $i(A)$ is *inherited* by S' from A.

Remark 2.13. If S' is obtained from S by reglueing, then the number of singularities of S' of a fixed angle equals the one of S, except for 4π–angle singularities, whose number in S' is greater by 2 to that in S (we put $\infty + 2 = \infty$). The Euler characteristic of S is greater by 2 than the Euler characteristic of S'.
We can extend the notion of reglueing to ordered families $\mathcal{M} = (m_n)_{n=1}^{\infty}$ and $\mathcal{M}' = (m'_n)_{n=1}^{\infty}$ of disjoint marks, which do not accumulate in $\overline{S}$, and such that the vector of m_n equals the vector of m'_n, for each n.

Outline of the construction. Let $\{a_i\}_{i \in I}$ (with $I \subseteq \mathbb{N}$) be a (possibly infinite) set of generators for G. We make use of the fact that any group G acts on its Cayley graph Γ and turn the graph Γ in a G-equivariant way into a translation surface. In the following we describe the general idea of the construction; below we give the explicit construction for the case that G is generated by two elements. The construction then works just in the same way for general groups; compare [PSV11, Construction 4.9].

- With each vertex g of Γ we associate a translation surface V_g. More precisely we start from some translation surface V_{Id} and define V_g to be its translate $g \cdot V_{\mathrm{Id}}$ by the action of $\mathbf{GL}_+(2, \mathbb{R})$ on the set of translation surfaces described in Remark 2.10. Observe that the linear group G naturally acts via affine homeomorphisms on the disjoint union of the V_g's; an element $h \in G$ maps V_g to $V_{h \cdot g}$. In the next step we will choose disjoint marks on the translation surfaces V_g. Reglueing the disjoint union of the surfaces V_g along these marks will give us a connected surface on which G acts by affine homeomorphisms. At the moment, we can assume V_{Id} just to be the real plane $\mathbb{R}^2$ equipped with an origin and a coordinate system.
- We choose marks on the starting surface V_{Id} in the following way:
  - For each i in I we choose a family $\mathcal{C}^i = \{m^i_j\}_{j \in J}$ (with $J \subseteq \mathbb{N}$) of horizontal marks m^i_j of length 1, *i.e.* the vector of each mark m^i_j is the first standard basis vector e_1.
  - For each i we choose a family $\mathcal{C}^{-i} = \{m^{-i}_j\}_{j \in J}$ of marks with vector $a_i^{-1}(e_1)$, *i.e.* the vector of m^{-i}_j is equal to $a_i^{-1} \cdot e_1$.
  - All marks are disjoint.
- On each V_g we take the corresponding marks $g(m^i_j)$ and $g(m^{-i}_j)$ with $i \in I$ and $j \in \mathbb{N}$. The mark $g(m^i_j)$ has the vector $g \cdot e_1$ and $g(m^{-i}_j)$ has the vector $ga_i^{-1} \cdot e_1$.

- We pair the mark $g(m_j^i)$ on the surface V_g with the mark $ga_i(m_j^{-i})$ on V_{ga_i}. Observe that for both the vector is $g \cdot e_1$.
  We now reglue the disjoint union of the V_g's along these pairs of marks.

This gives us a translation surface S_1 on which the elements of G act via affine homeomorphisms, *i.e.* $\Gamma(S_1)$ contains G. However we are not yet done, but still have the following problems:

(i) The Veech group $\Gamma(S_1)$ can be bigger than G.
(ii) The singularities can accumulate. In this case S_1 is not tame.
(iii) We want the translation surface to have one end.

We resolve the problems in the following way: To enforce that all elements in the Veech group are in G, we will modify the starting surface V_{Id}. We will replace it by a surface obtained from glueing a *decorated surface* $\widetilde{L}'_{\mathrm{Id}}$ (described below) to a plane $A_{\mathrm{Id}} = \mathbb{R}^2$. The surface $\widetilde{L}'_{\mathrm{Id}}$ will be decorated with special singularities. This will guarantee that every orientation preserving affine homeomorphism permutes the set of the singularities on the $\widetilde{L}'_g$'s and with some more care we will establish that it actually acts as one of the elements of G. To avoid accumulation of singularities, we will associate with each edge in the Cayley graph between two vertices g and g' (let us say that $g^{-1}g' = a_i$ is the i-th generator) a *buffer surface* $\hat{E}_g^i$ which connects V_g to $V_{g'}$, but separates them by a definite distance. Finally, we keep track of the end by providing that each V_g and $\hat{E}_g^i$ is one-ended and that after glueing all V_g and $\hat{E}_g^i$, their ends actually merge into one end. This actually is the reason why we have to choose infinite families of marks. If we do not require the surface to be a Loch Ness monster, then it suffices to take one mark from each infinite family.

An illustrative example. In the following paragraphs we carry out the construction for the case where G is generated by two matrices a_1 and a_2. The general case works in the same way; compare [PSV11, Construction 4.9].

Constructing the translation surface V_g: We first construct the surface V_{Id}. We will obtain it by glueing two surfaces A_{Id} and $\widetilde{L}'_{Id}$ along an infinite family of marks. Let A_{Id} be an oriented flat plane, equipped with an origin and the standard basis $e_1 = (1, 0)$ and $e_2 = (0, 1)$. We define the families of marks as follows:

- For $i = 0, 1, 2$ let $\mathcal{C}^i$ be the family of marks on A_{Id} with endpoints $ie_2 + (2n-1)e_1,\ ie_2 + 2ne_1$, for $n \geq 1$. All these marks are pairwise disjoint.
- Given $x_1, y_1 \in \mathbb{R}$, consider the family $\mathcal{C}^{-1}$ of marks on A_{Id} with endpoints
  $$(nx_1, y_1),\ (nx_1, y_1) + a_1^{-1}(e_1),$$
  for $n \geq 1$. We can choose $x_1 > 0$ sufficiently large and $y_1 < 0$ sufficiently small so that all these marks are pairwise disjoint and disjoint from the ones in $\mathcal{C}^i$ for $i = 0, 1, 2$.
- Observe that a translate of the lower half-plane in A_{Id} is avoided by all already constructed marks. In this way we can choose $x_2, -y_2 \in \mathbb{R}$ sufficiently large so that the marks with endpoints $(nx_2, y_2),\ (nx_2, y_2) + a_2^{-1}(e_1)$, for $n \geq 1$, are pairwise disjoint and disjoint with the previously constructed marks. We denote this family by $\mathcal{C}^{-2}$.

Let L_{Id} be an oriented flat plane, equipped with an origin O_{Id}. Let $\widetilde{L}_{\mathrm{Id}}$ be the threefold cyclic branched covering of L_{Id}, which is branched over the origin. Denote the projection map from $\widetilde{L}_{\mathrm{Id}}$ onto L_{Id} by π. Denote by R the closure in $\widetilde{L}_{\mathrm{Id}}$ of one connected component of the preimage under π of the open right half-plane in L_{Id}. On R consider coordinates induced from L_{Id} via π. We define the following family of marks on $\widetilde{L}_{\mathrm{Id}}$:

- Let $\mathcal{C}'$ be the family of marks in R with endpoints $(2n-1)e_1, 2ne_1$, for $n \geq 1$.
- Let t and b be the two marks in $\widetilde{L}_{\mathrm{Id}}$ with endpoints in R with coordinates $e_2, 2e_2$ and $-2e_2, -e_2$, respectively.

Let $\widetilde{L}'_{\mathrm{Id}}$ be the tame flat surface obtained from $\widetilde{L}_{\mathrm{Id}}$ by regluing along t and b. We call $\widetilde{L}'_{\mathrm{Id}}$ the *decorated surface*. Finally, we obtain V_{Id} by glueing A_{Id} with $\widetilde{L}'_{\mathrm{Id}}$ along the families of marks $\mathcal{C}^0$ and $\mathcal{C}'$. For each $g \in G$ we define V_g as the translation surface $g \cdot V_{\mathrm{Id}}$.
Observe that if we denote by $\widetilde{O}_g$ the unique preimage on $\widetilde{L}_g$ of the origin O_g of L_g via the three-fold covering, then $\widetilde{O}_g$ is a singularity of total angle 6π and there are precisely three saddle connections starting in $\widetilde{O}_g$.
Constructing the buffer surface $\hat{E}^i_g$: Let $E_{\mathrm{Id}}, E'_{\mathrm{Id}}$ be two oriented flat planes, equipped with origins that allow us to identify them with $\mathbb{R}^2$. We define the following families of vector e_1 marks on $E_{\mathrm{Id}} \cup E'_{\mathrm{Id}}$.

- Let $\mathcal{S}$ be the family of marks on E_{Id} with endpoints $4ne_1, (4n+1)e_1$, for $n \geq 1$.
- Let $\mathcal{S}_{\mathrm{glue}}$ be the family of marks on E_{Id} with endpoints $(4n+2)e_1, (4n+3)e_1$, for $n \geq 1$.
- Let $\mathcal{S}'$ be the family of marks on E'_{Id} with endpoints $2ne_2, 2ne_2 + e_1$, for $n \geq 1$.
- Finally, let $\mathcal{S}'_{\mathrm{glue}}$ be the family of marks on E'_{Id} with endpoints $(2n+1)e_2, (2n+1)e_2+e_1$, for $n \geq 1$.

Let $\hat{E}_{\mathrm{Id}}$ be the tame flat surface obtained from E_{Id} and E'_{Id} by regluing along $\mathcal{S}_{\mathrm{glue}}$ and $\mathcal{S}'_{\mathrm{glue}}$. We call $\hat{E}_{\mathrm{Id}}$ the *buffer surface*. The surface $\hat{E}_{\mathrm{Id}}$ comes with the distinguished families of marks inherited from $\mathcal{S}$ and $\mathcal{S}'$, for which we retain the same notation. Let $\hat{E}^1_{\mathrm{Id}}$ and $\hat{E}^2_{\mathrm{Id}}$ be two copies of $\hat{E}_{\mathrm{Id}}$ and for each $g \in G$ let $\hat{E}^i_g$ to be the translation surface $g \cdot \hat{E}^i_g$ ($i \in \{1, 2\}$). It is endowed with the two family of marks $\mathcal{S}^i_g$ and $\mathcal{S}'^i_g$.
Construction of the surface S: We finally obtain the desired surface S from the disjoint union of all V_g's and $\hat{E}^i_g$ in the following way:

- Reglue each mark $\mathcal{C}^1_g$ on V_g with $\mathcal{S}^1_g$ on $\hat{E}^1_g$, and each mark $\mathcal{S}'^1_g$ on $\hat{E}^1_g$ with $\mathcal{C}^{-1}_{ga_1}$ on V_{ga_1}.
- Reglue each mark $\mathcal{C}^2_g$ on V_g with $\mathcal{S}^2_g$ on $\hat{E}^2_g$, and each mark $\mathcal{S}'^2_g$ on $\hat{E}^2_g$ with $\mathcal{C}^{-2}_{ga_2}$ on V_{ga_2}.

In [PSV11, Section 4] it is carefully carried out that the construction is well defined and gives the desired result from Proposition 2.11.

3. Fields associated with translation surfaces

There are four subfields of $\mathbb{R}$ in the literature which are naturally associated with a translation surface S. They are called the *holonomy field* $K_{\mathrm{hol}}(S)$, the *segment field* or *field of saddle connections* $K_{\mathrm{sc}}(S)$, the *field of cross ratios of saddle connections* $K_{\mathrm{cr}}(S)$, and the *trace field* $K_{\mathrm{tr}}(S)$; compare [KS00] and [GJ00]. In the following, we extend their definitions to (possibly non precompact) tame translation surfaces.

Remark 3.1. It follows from [PSV11, Lemma 3.2] that there are only three types of tame translation surfaces such that $\overline{S}$ has no singularity: $\mathbb{R}^2$, $\mathbb{R}^2/\mathbb{Z}$ and flat tori. Furthermore, tame translation surfaces with only one singularity are cyclic coverings of $\mathbb{R}^2$ ramified over the origin. Finally, if $\overline{S}$ has at least two singularities, then there exists at least one saddle connection.

Definition 3.2. Let S be a tame translation surface and $\overline{S}$ the metric completion of S.

(i) (Following [KS00, Section 7].) Let Λ be the image of $H_1(\overline{S}, \mathbb{Z})$ in $\mathbb{R}^2$ under the holonomy map h and let n be the dimension of the smallest $\mathbb{R}$-subspace of $\mathbb{R}^2$ containing Λ; in particular n is 0, 1 or 2. The *holonomy field* $K_{\mathrm{hol}}(S)$ is the smallest subfield k of $\mathbb{R}$ such that
$$\Lambda \otimes_{\mathbb{Z}} k \cong k^n.$$
(ii) Let Σ denote the set of all singularities of $\overline{S}$. Using in (i) $H_1(\overline{S}, \Sigma_{\mathrm{fin}}; \mathbb{Z})$, the homology relative to the set of finite angle singularities, instead of the absolute homology $H_1(\overline{S}, \mathbb{Z})$, we obtain the *segment field* or *field of saddle connections* $K_{\mathrm{sc}}(S)$.

(iii) (Following [GJ00, Section 5].) The *field of cross ratios of saddle connections* $K_{\text{cr}}(S)$ is the field generated by the set of all cross ratios $(v_1, v_2; v_3, v_4)$, where the v_i's are four pairwise nonparallel holonomy vectors of saddle connections of S; compare Remark 3.3 iii).

(iv) Finally, the *trace field* $K_{\text{tr}}(S)$ is the field generated by the traces of elements in the Veech group: $K_{\text{tr}}(S) = \mathbb{Q}[\text{tr}(A) | A \in \Gamma(S)]$.

In the rest of this section we mean by a *holonomy vector* always the holonomy vector of a saddle connection.

Remark 3.3. (i) Definition 3.2 (i) is equivalent to the following: If $n = 2$, take any two nonparallel vectors $\{e_1, e_2\} \subset \Lambda$, then $K_{\text{hol}}(S)$ is the smallest subfield k of $\mathbb{R}$ such that every element v of Λ can be written in the form $a \cdot e_1 + b \cdot e_2$, with $a, b \in k$. If $n = 1$, any element v of Λ can be written as $a \cdot e_1$, with $a \in K_{\text{hol}}(S)$ and e_1 any nonzero (fixed) vector in Λ. If $n = 0$, $K_{\text{hol}}(S) = \mathbb{Q}$.
The same is true for $K_{\text{sc}}(S)$, if Λ is the image of $H_1(\overline{S}, \Sigma_{\text{fin}}; \mathbb{Z})$ in $\mathbb{R}^2$.

(ii) Recall that $\overline{S}$ is a topological surface if and only if all of its singularities have finite cone angles. However, if Σ_{inf} (resp. Σ_{fin}) is the set of infinite (resp. finite) angle singularities, then $\widehat{S} = \overline{S} \backslash \Sigma_{\text{inf}} = S \cup \Sigma_{\text{fin}}$ is a surface, possibly of infinite genus. We furthermore have that the fundamental group $\pi_1(\overline{S})$ equals $\pi_1(\widehat{S})$ and thus

$$H_1(\overline{S}, \mathbb{Z}) \cong H_1(\widehat{S}, \mathbb{Z}).$$

Indeed, for every infinite angle singularity $p_0 \in \overline{S}$, there exists by definition a neighbourhood U of p_0 in $\overline{S}$ which is isometric to a neighbourhood of the branching point z_0 of the infinite flat cyclic covering X_0 of $\mathbb{R}^2$ branched over 0. Without loss of generality we may choose the neighbourhood of z_0 as an open ball of radius ε in X_0. We then have that U is homeomorphic to $\{(x, y) \in \mathbb{R}^2 | x > 0\} \cup \{(0, 0)\} \subset \mathbb{R}^2$. In particular, U and $U \backslash \{p_0\}$ are both contractible, and by the Seifert-van Kampen theorem we have $\pi_1(\overline{S} \backslash \{p_0\}) \cong \pi_1(\overline{S})$.

(iii) Recall that the cross ratio r of four vectors $v_1, \ldots, v_4$ with $v_i = (x_i, y_i)$ is equal to the cross ratio of the real numbers $r_1 = y_1/x_1, \ldots, r_4 = y_4/x_4$, *i.e.*

$$(v_1, v_2; v_3, v_4) = \frac{(r_1 - r_3) \cdot (r_2 - r_4)}{(r_2 - r_3) \cdot (r_1 - r_4)}. \tag{3.6}$$

If $r_i = \infty$ for some $i = 1, \ldots, 4$, one eliminates the factors on which it appears in Equation (3.6). For example, if $r_1 = \infty$, then $(v_1, v_2; v_3, v_4) = \frac{r_2 - r_4}{r_2 - r_3}$. If there are no four non parallel holonomy vectors, $K_{\text{cr}}(S)$ is equal to $\mathbb{Q}$.

(iv) The four fields from Definition 3.2 are invariant under the action of $\mathbf{GL}(2, \mathbb{R})$ described in Remark 2.10, *i.e.* for $A \in \mathbf{GL}(2, \mathbb{R})$ we have

$$\begin{array}{rclrcl} K_{\text{hol}}(S) & = & K_{\text{hol}}(A \cdot S), & K_{\text{sc}}(S) & = & K_{\text{sc}}(A \cdot S), \\ K_{\text{cr}}(S) & = & K_{\text{cr}}(A \cdot S), & K_{\text{tr}}(S) & = & K_{\text{tr}}(A \cdot S). \end{array}$$

For $K_{\text{hol}}(S)$ and $K_{\text{sc}}(S)$ this follows from (i). Recall that the cross ratio is invariant under linear transformation. Thus the claim is true for the field $K_{\text{cr}}(S)$. Finally, we have that $\Gamma(A \cdot S)$ is conjugated to $\Gamma(S)$; compare Remark 2.10. Since the trace of a matrix is invariant under conjugation, the claim also holds for $K_{\text{tr}}(S)$.

It follows directly from the definitions that $K_{\text{hol}}(S) \subseteq K_{\text{sc}}(S)$. Furthermore, we see from Remark 3.3 that $K_{\text{cr}}(S) \subseteq K_{\text{sc}}(S)$: Suppose S has two linearly independent holonomy vectors w_1 and w_2. By (iv) in the preceding remark we may assume that $w_1 = e_1$, $w_2 = e_2$ is the

standard basis. Let v_1, v_2, v_3, v_4 be four arbitrary pairwise nonparallel holonomy vectors with $v_i = (x_i, y_i)$. By (i) we have that all the coordinates x_i and y_i are in $K_{\rm sc}(S)$. Thus in particular the cross ratio $(v_1, v_2; v_3, v_4)$ is in $K_{\rm sc}(S)$. If there is no pair (w_1, w_2) of linearly independent holonomy vectors, then $K_{\rm cr}(S) = \mathbb{Q}$ and the inclusion $K_{\rm cr}(S) \subseteq K_{\rm sc}(S)$ trivially holds.
Since the Veech group preserves the set of holonomy vectors, we furthermore have that if there are at least two linearly independent holonomy vectors, then $K_{\rm tr}(S) \subseteq K_{\rm hol}(S)$. However, if all holonomy vectors are parallel, it is not in general true that $K_{\rm tr}(S) \subseteq K_{\rm hol}(S)$. An example of a surface S showing this is given in [PSV11, Lemma 3.7]: The surface S is obtained from glueing two copies of $\mathbb{R}^2$ along horizontal slits l_n of the plane with end points $(4n+1, 0)$ and $(4n+3, 0)$. In particular all saddle connections are horizontal and the fields $K_{\rm hol}(S)$, $K_{\rm cr}(S)$ and $K_{\rm sc}(S)$ are all $\mathbb{Q}$. But the Veech group is very big. It consists of all matrices in $\mathbf{GL}_+(2, \mathbb{R})$ which fix the first standard basis vector e_1; compare [PSV11, Lemma 3.7].

Remark 3.4. The translation surface S from [PSV11, Lemma 3.7] has the following properties:

$$\Gamma(S) = \left\{ \begin{pmatrix} 1 & t \\ 0 & s \end{pmatrix} | t \in \mathbb{R}, s \in \mathbb{R}_+ \right\}$$

and $K_{\rm hol}(S) = K_{\rm cr}(S) = K_{\rm sc}(S) = \mathbb{Q}$. In particular, we have $K_{\rm tr}(S) = \mathbb{R}$.

Finally, in Proposition 3.5 we see that for a large class of translation surfaces we have that $K_{\rm cr}(S) = K_{\rm sc}(S)$. The main argument of the proof was given in [GJ00] for precompact surfaces.

Proposition 3.5. *Let S be a (possibly non precompact) tame translation surface, $\overline{S}$ its metric completion and $\Sigma \subset \overline{S}$ its set of singularities. Suppose that $\overline{S}$ has a geodesic triangulation by countably many triangles Δ_k ($k \in I$ for some index set I) such that the set of vertices equals Σ. We then have $K_{\rm cr}(S) = K_{\rm sc}(S)$.*

Proof. The inclusion "$\subset$" was shown in general in the paragraph below Remark 3.3. The inclusion "$\supset$" follows from [GJ00, Proposition 5.2]. The statement there is for precompact surfaces, but the proof works in the same way if there exists a triangulation as required in this proposition. More precisely, in [GJ00] it is shown that the $K_{\rm cr}(S)$-vector space $V(S)$ spanned by the image of $H_1(\overline{S}, \Sigma; \mathbb{Z})$ under the holonomy map is 2-dimensional over $K_{\rm cr}(S)$. Hence $K_{\rm sc}(S) \subseteq K_{\rm cr}(S)$. □

It follows from Theorem 2 that in general no further inclusions between the four fields from Definition 3.2 hold than those stated above; see Corollary 4.7 for a subsumption of the relations between the fields.

If S is a precompact translation surface of genus g, then $[K_{\rm tr}(S) : \mathbb{Q}] \leq g$. Moreover, the traces of elements in $\Gamma(S)$ are algebraic integers (see [McM03a]). When dealing with tame translation surfaces, such algebraic properties do not hold in general.

Proposition 3.6. *For each $n \in \mathbb{N} \cup \{\infty\}$ there exists a tame translation surface S_n of infinite genus such that the transcendence degree of the field extension $K_{\rm tr}(S_n)/\mathbb{Q}$ is n. S_n can be chosen to be a Loch Ness monster.*

Proof. Let $\{\lambda_1, \ldots, \lambda_n\}$ be $\mathbb{Q}$-algebraically independent real numbers with $|\lambda_i| > 2$. Define

$$G_n := \left\langle \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix} \mid \mu + \mu^{-1} = \lambda_i \text{ with } i \in \{1, \ldots, n\} \right\rangle.$$

G_n is countable and a subgroup of the diagonal group. In particular, G_n is disjoint from the set $\mathcal{U}$ of contraction matrices; recall Proposition 2.11 for the definition of $\mathcal{U}$. Thus, we can apply Proposition 2.11 and obtain a surface S_n with Veech group G_n. We have

$$\mathbb{Q} \subset \mathbb{Q}(\lambda_1, \ldots, \lambda_n) \subseteq K_{\mathrm{tr}}(S_n) \subseteq L = \mathbb{Q}(\mu | \mu + \mu^{-1} = \lambda_i \text{ with } i \in \{1, \ldots, n\}).$$

Since the generators μ of L are algebraic over $\mathbb{Q}(\lambda_1, \ldots, \lambda_n)$, it follows that $L/\mathbb{Q}(\lambda_1, \ldots, \lambda_n)$ and thus also $K_{\mathrm{tr}}(S_n)/\mathbb{Q}(\lambda_1, \ldots, \lambda_n)$ is algebraic and we obtain the claim. $\square$

If μ_1 is one of the two solutions of $\mu + \mu^{-1} = \pi$ and

$$G := \left\langle \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_1^{-1} \end{pmatrix} \right\rangle,$$

then we obtain in the same way the following corollary.

Corollary 3.7. *There are examples of tame translation surfaces S of infinite genus with a cyclic hyperbolic Veech group such that $K_{\mathrm{tr}}(S)$ is not a number field. Again the translation surface can be chosen to be as a Loch Ness monster.*

Transcendental numbers naturally appear also in fields associated with Veech groups arising from a generic triangular billiard. Indeed, let $\mathcal{T} \subset \mathbb{R}^2$ denote the space of triangles parametrised by two angles (θ_1, θ_2). Remark that $\mathcal{T}$ is a simplex. For every $T = T_{(\theta_1,\theta_2)} \in \mathcal{T}$, a classical construction due to Katok and Zemljakov produces a tame flat surface S_T from T [ZK75]. If T has an interior angle which is not commensurable with π, S_T is a Loch Ness monster; compare [Val09].

Proposition 3.8. *The set $\mathcal{T}' \subset \mathcal{T}$ formed by those triangles such that $K_{\mathrm{sc}}(S_T)$, $K_{\mathrm{cr}}(S_T)$ and $K_{\mathrm{tr}}(S_T)$ are not number fields, is of total (Lebesgue) measure in $\mathcal{T}$.*

Proof. Since S_T has a triangulation with countably many triangles satisfying the hypotheses of Proposition 3.5, the fields $K_{\mathrm{sc}}(S_T)$ and $K_{\mathrm{cr}}(S_T)$ coincide. Without loss of generality we can assume that the triangle $T = T_{(\theta_1,\theta_2)}$ has the vertices 0, 1 and $\rho e^{i\theta_1}$ (with $\rho > 0$) in the complex plane $\mathbb{C}$. When doing the Katok-Zemljakov construction we start by reflecting T at its edges. Thus in particular S_T contains the geodesic quadrangle shown in Figure 2.

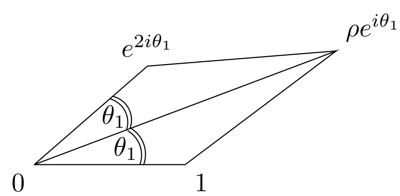


FIGURE 2. Geodesic quadrangle in the surface S_T with T the triangle $T_{(\theta_1,\theta_2)}$

Thus the vectors $v_1 = (1, 0)$, $v_2 = (\rho\cos\theta_1, \rho\sin\theta_1)$ and $v_3 = (\cos 2\theta_1, \sin 2\theta_1)$ are holonomy vectors. Choose $\{v_1, v_2\}$ as basis of $\mathbb{R}^2$. We then have $v_3 = a \cdot v_1 + b \cdot v_2$ with

$$a = -1 \text{ and } b = \frac{2\cos\theta_1}{\rho}.$$

Therefore $\frac{2\cos\theta_1}{\rho}$ is an element of $K_{\mathrm{sc}}(S) = K_{\mathrm{cr}}(S)$. Furthermore, from [Val12] we know that the matrix representing the rotation by θ_1 is in $\Gamma(S_T)$. Hence $2\cos\theta_1$ is in $K_{\mathrm{tr}}(S_T)$. Thus if we choose the values $\frac{\cos\theta_1}{\rho}$, respectively $\cos\theta_1$, to be non algebraic numbers, then $K_{\rm sc}(S) = K_{\rm cr}(S)$, respectively $K_{\rm tr}(S_T)$, are not number fields. $\square$

4. Proof of main results

In this section we prove the results stated in the introduction.

Proof Theorem 1. We begin proving part (i). Let $T_0 = T \setminus \{\infty\}$ be the once punctured torus with $T = \mathbb{R}^2/L$, where L is a lattice in $\mathbb{R}^2$, and with $\infty \in T$ the image of the origin removed. Let $p : S \longrightarrow T_0$ be an unramified translation covering. The existence of such a covering is equivalent to S being affine equivalent to an origami. We use the notation from Section 2. In particular $\pi_S : \widetilde{S} \longrightarrow S$ is a universal cover, $\overline{\widetilde{S}}$ and $\overline{S}$ are the metric completions of $\widetilde{S}$ and S, respectively, and $\widetilde{\Sigma}(S)$ is the set of developed cone points. We then have that the following diagram commutes, since p and π_S are translation maps:

$$(4.7)\qquad \begin{array}{ccccc}
\widetilde{S} & \hookrightarrow & \overline{\widetilde{S}} & & \\
\downarrow \pi_S & & \downarrow \overline{\pi_S} & \searrow \overline{dev} & \\
S & \hookrightarrow & \overline{S} & & \mathbb{R}^2 \\
\downarrow p & & \downarrow \overline{p} & \swarrow \pi_T & \\
T_0 & \hookrightarrow & T & &
\end{array} .$$

Given that $T \setminus T_0 = \infty = \overline{p} \circ \overline{\pi_S}(\overline{\widetilde{S}} \setminus \widetilde{S})$, the projection of $\widetilde{\Sigma}(S)$ to T is just a point. This proves sufficiency.

Equation (2.3) implies that if $\widetilde{\Sigma}(S)$ is contained in $L + x$ then every $\mathrm{hol}(\gamma)$ is a translation of the plane of the form $z \to z + \lambda_\gamma$, where $\lambda_\gamma \in L$. Puncture $\widetilde{S}$ and S at $\mathrm{dev}^{-1}(L + x)$ and $\pi_S(\mathrm{dev}^{-1}(L + x))$ respectively to obtain $\widetilde{S}_0$ and S_0 and denote $\mathbb{R}^2_0 = \mathbb{R}^2 \setminus (L + x)$. Let $\pi_{S|} : \widetilde{S}_0 \longrightarrow S_0$ and $\pi_{T|} : \mathbb{R}^2_0 \longrightarrow T_0$ be the restrictions of the universal covers π_S and π_T. Given that $\widetilde{S}_0$ has the translation structure induced by pull-back of $\pi_{S|}$, the map $\mathrm{dev}_| : \widetilde{S}_0 \longrightarrow \mathbb{R}^2_0$ is a flat surjective map; compare [Thu97, §3.4]. Equation (2.2) implies that

$$(4.8)\qquad \begin{array}{ccc}
\widetilde{S}_0 & \xrightarrow{\mathrm{dev}_|} & \mathbb{R}^2_0 \\
\downarrow \pi_{S|} & & \downarrow \pi_{T|} \\
S_0 & & T_0
\end{array}$$

descends to a flat covering map $p : S_0 \longrightarrow T_0$. Hence $\overline{S} = \overline{S_0}$ defines a covering over a flat torus ramified at most over one point. This proves necessity.

Now we prove part (ii). First we prove that every origami satisfies conditions (a), (b), (c), (d) and (e).

Let $\overline{p} : \overline{S} \to T$ be an origami ramified at most over $\infty \in T$. All saddle connections of S are preimages of closed simple curves on T with a base point at ∞. This implies that all holonomy vectors have integer coordinates. Thus $K_{\rm sc}(S) = K_{\rm hol}(S) = K_{\rm cr}(S) = \mathbb{Q}$. Hence every origami fulfils conditions (b), (c) and (d) in part (ii). If S furthermore has at least two linearly independent holonomy vectors, then the Veech group must preserve the lattice spanned by them. Thus it is commensurable to a (possible infinite index) subgroup of $\mathrm{SL}(2,\mathbb{Z})$ and S fulfils in addition (a) and (e).

We finally prove that none of the conditions in theorem (a) to (e) imply that S is an origami. Example 4.1 shows that neither (a) nor (e) imply that S is an origami. Example 4.2 shows that neither (b), nor (c), nor (d) imply that S is an origami.

Example 4.1. In this example we construct a tame translation surface S whose Veech group $\Gamma(S)$ is $\mathrm{SL}(2,\mathbb{Z})$, hence $K_{\mathrm{tr}}(S)=\mathbb{Q}$, but which is not an origami. We achieve this making a slight modification of the construction presented in Section 2.3. Let $G=\mathrm{SL}(2,\mathbb{Z})$. Apply the construction in Section 2.3 to G but choose the family of marks $\mathcal{C}^{-1}$ in such a way that the there exists $N\in\mathbb{Z}$ and irrational $\alpha>0$ so that (α,N) is a holonomy vector. This is possible since in the cited construction the choice of the point (x_1,y_1) is free. Observe that $v_1=(-1,1)$, $v_2=(0,1)$, $v_3=(1,0)$ and $v_4=(\alpha,N)$ are holonomy vectors of $\overline{S}$. Let l_i be lines in $\mathbb{P}^1(\mathbb{R})$ containing v_i, $i=1,\ldots,4$ respectively. A direct calculation shows that the cross ratio of these four lines is $\frac{\alpha}{\alpha+N}$, which lies in $K_{\mathrm{cr}}(S)$. Hence $K_{\mathrm{cr}}(S)$ is not isomorphic to $\mathbb{Q}$ and therefore S cannot be an origami.

Example 4.2. In this example we construct a surface S whose Veech group is not a discrete subgroup of $\mathrm{SL}(2,\mathbb{R})$ (hence S cannot be an origami, since in addition S has two non parallel saddle connections) but such that

$$K_{\mathrm{cr}}(S)=K_{\mathrm{hol}}(S)=K_{\mathrm{sc}}(S)=K_{\mathrm{tr}}(S)=\mathbb{Q}. \tag{4.9}$$

Consider $G=\mathrm{SL}(2,\mathbb{Q})$ or $G=\mathrm{SO}(2,\mathbb{Q})$. These are non-discrete countable subgroups of $\mathrm{SL}(2,\mathbb{R})$ with no contracting elements. Hence we can apply the construction from Proposition 2.11 to G but choosing the points (x_i,y_i) that define the families of marks $\mathcal{C}^i$ in $\mathbb{Q}\times\mathbb{Q}$ for all $i\geq 1$ indexing a countable set of generators of G. The result is a tame translation surface S whose Veech group is isomorphic to G and whose holonomy vectors S have all coordinates in $\mathbb{Q}\times\mathbb{Q}$. This implies (4.9).

□

Proof Theorem 2. Let us first show that (i) holds. The tame translation surface S in Example 4.1 is such that $\Gamma(S)=\mathrm{SL}(2,\mathbb{Z})$ and $K_{\mathrm{cr}}(S)$ is not isomorphic to $\mathbb{Q}$. Since in general $K_{\mathrm{cr}}(S)$ is a subfield of $K_{\mathrm{sc}}(S)$ this surface also satisfies that $\Gamma(S)=\mathrm{SL}(2,\mathbb{Z})$ and $K_{\mathrm{sc}}(S)$ is not isomorphic to $\mathbb{Q}$. To finish the proof of (i) we consider the following example.

Example 4.3. In this example we construct a tame translation surface such that $\Gamma(S)=\mathrm{SL}(2,\mathbb{Z})$ and $K_{\mathrm{hol}}(S)$ is not isomorphic to $\mathbb{Q}$. Apply the construction described in Section 2.3 to $G=\mathrm{SL}(2,\mathbb{Z})$ but consider the following modification. Let $\{e_1,e_2\}$ be the standard basis of $\mathbb{R}^2$. There exists a natural number $n>0$ such that the mark M in A_{Id} whose end points are $-ne_1$ and $-(n-1)e_1$ does not intersect all other marks used in the construction. On a $[0,\pi]\times[0,e]$ rectangle R, where e is Euler's number, identify opposite sides to obtain a flat torus T. Consider on T a horizontal mark M' of length 1 and glue A_{Id} with T along M and M'. Then proceed with the construction in a $\mathrm{SL}(2,\mathbb{Z})$-equivariant way. This produces a tame translation surface S whose Veech group is $\mathrm{SL}(2,\mathbb{Z})$. The image of $H_1(\overline{S},\mathbb{Z})$ under the holonomy map contains the vectors e_1, $e\cdot e_1$ and $\pi\cdot e_2$. Hence $K_{\mathrm{hol}}(S)$ is not isomorphic to $\mathbb{Q}$.

Part (ii) follows from Example 4.2. We now prove (iii). First we construct S such that $K_{\mathrm{cr}}(S)=\mathbb{Q}$ but $K_{\mathrm{sc}}(S)$ is not. Consider the following example.

Example 4.4. Let P_1, P_2 and P_3 be three copies of $\mathbb{R}^2$; choose on each copy an origin, and let $\{e_1,e_2\}$ be the standard basis. Consider the following:

(i) Let v_n be the mark on the plane P_1 along segments whose end points are $n \cdot e_2$ and $n \cdot e_2 + e_1$ with $n = 0, 1$.
(ii) Marks on P_2 and P_3 along the segments w_0, w_1 whose end points are $(0,0)$ and $(1,0)$, and then along the segments z_0 and z_1 whose end points are $(2,0)$ and $(2+\sqrt{p}, 0)$, for some prime p.

Glue the three planes along slits as follows: v_i to w_i, for $i = 0, 1$ and z_0 to z_1. The result is a surface S for which $\{0, 1, -1, \infty\}$ parametrizes all possible slopes of lines through the origin in $\mathbb{R}^2$ containing holonomy vectors of saddle connections. Hence $K_{\text{cr}}(S) = \mathbb{Q}$. On the other hand, the set of holonomy vectors contains $(1,0)$, $(0,1)$, $(1,1)$ and $(\sqrt{p}, 0)$. Therefore $K_{\text{sc}}(S)$ contains $\mathbb{Q}(\sqrt{p})$ as a subfield.

We finish the proof of (iii) by constructing a precompact tame translation surface such that $K_{\text{hol}}(S) = \mathbb{Q}$, but $K_{\text{sc}}(S)$ is not. Consider the following example.

Example 4.5. Consider two copies L_1 and L_2 of the L-shaped origami tiled by three unit squares; see *e.g.* [HL06b, Example on p. 293]. Consider a point $p_i \in L_i$ at distance $0 < \varepsilon << 1$ from the 6π-angle singularity s_i, $i = 1, 2$. Let m_i be a marking of length ε on L_i defined by a geodesic of length ε joining p_i to s_i, $i = 1, 2$. We can choose p_i so that both markings are parallel and the vector defined by them has irrational coordinates. Glue then L_1 and L_2 along m_1 and m_2 to obtain S. By construction $h(H_1(\overline{S}, \mathbb{Z})) = \mathbb{Z} \times \mathbb{Z}$, hence $K_{\text{hol}}(S) = \mathbb{Q}$, but $h(H_1(\overline{S}, \Sigma; \mathbb{Z}))$ contains an orthonormal basis $\{e_1, e_2\}$ and a vector $h(m_1)$ with irrational coordinates. This implies that $K_{\text{sc}}(S)$ is not isomorphic to $\mathbb{Q}$.

We address (iv) now. Observe that the surface S constructed in Example 4.5 satisfies that $K_{\text{hol}}(S) = \mathbb{Q}$ but $K_{\text{cr}}(S)$ is not equal to $\mathbb{Q}$. Indeed, we have saddle connections of slope 0, 1 and ∞. Since the slope of $h(m_1)$ is irrational, we are done. We now construct S such that $K_{\text{cr}}(S) = \mathbb{Q}$, but $K_{\text{hol}}(S)$ is not. We will furthermore have that S has four pairwise nonparallel holonomy vectors thus $K_{\text{cr}}(S)$ is not trivially $\mathbb{Q}$.

Example 4.6. Take two copies of the real plane P_1 and P_2. Choose an origin and let e_1, e_2 be the standard basis. Let $\mu_i > 1$, $i = 1, 2, 3$ be three distinct irrational numbers and define $\lambda_0 = 0$ and $\lambda_n = \sum_{i=1}^{n} \mu_i$ for $n = 1, 2, 3$. On P_1 consider the markings m_n whose end points are ne_2 and $ne_2 + e_1$ for $n = 0, \ldots, 3$. On P_2 consider the markings m'_n whose end points are $(n + \lambda_n)e_1$ and $(n + \lambda_n + 1)e_1$ for $n = 0, \ldots, 3$. Glue P_1 and P_2 along the markings m_n and m'_n. The result is a tame flat surface S with eight 4π-angle singularities. These singularities lie on P_2 on a horizontal line, and hence we can naturally order them from, say, left to right. Let us denote these ordered singularities by a_j, $j = 1, \ldots, 8$. Let $g_{e_1}(a_i, a_j)$ (respectively $g_{e_2}(a_i, a_j)$) be the directed geodesic in S parallel to e_1 (respectively e_2) joining a_i with a_j. Define in $H_1(\overline{S}, \mathbb{Z})$

- the cycle c_1 as $g_{e_1}(a_3, a_4) g_{e_1}(a_4, a_5) g_{e_2}(a_5, a_3)$,
- the cycle c_2 as $g_{e_1}(a_4, a_3) g_{e_1}(a_3, a_2) g_{e_2}(a_2, a_4)$,
- the cycle c_3 as $g_{e_2}(a_6, a_8) g_{e_1}(a_8, a_7) g_{e_1}(a_7, a_6)$.

Where the product is defined to be the composition of geodesics on S, *i.e.* following one after the other. Note that $h(c_1) = (1 + \mu_2, -1)$, $h(c_2) = (-(1 + \mu_1), 1)$ and $h(c_3) = (-(1 + \mu_3), 1)$. We can choose parameters μ_i, $i = 1, 2, 3$ so that the $\mathbb{Z}$-module generated by these 3 vectors has rank 3. Therefore $K_{\text{hol}}(S)$ cannot be isomorphic to $\mathbb{Q}$.

We address now (v). We construct first a flat surface S for which $K_{\text{tr}}(S) = \mathbb{Q}$ but none of the conditions (a), (b), (c) or (d) in Theorem 1 hold. We achieve this by making a slight modification

on the construction of the surface in Example 4.3 in the following way. First, change $\mathrm{SL}(2,\mathbb{Z})$ for $\mathrm{SL}(2,\mathbb{Q})$. Second, let the added mark M be of unit length and such that the vector defined by developing it along the flat structure neither lies in the lattice $\pi\mathbb{Z}\times e\mathbb{Z}$ nor has rational slope. The result of this modification is a tame translation surface S homeomorphic to the Loch Ness monster for which $\Gamma(S) = \mathrm{SL}(2,\mathbb{Q})$ and such that both $K_{\mathrm{cr}}(S)$ and $K_{\mathrm{hol}}(S)$ (hence $K_{\mathrm{sc}}(S)$ as well) have transcendence degree at least 1 over $\mathbb{Q}$.
Finally, an example of a surface S which satisfies (c), (d) and (b), but with $K_{\mathrm{tr}}(S) \neq \mathbb{Q}$, is given in [PSV11, Lemma 3.7]; see Remark 3.4. We underline that all holonomy vectors in this surface S are parallel and hence $K_{\mathrm{cr}}(S)$ is by definition isomorphic to $\mathbb{Q}$. For the sake of completeness we construct a tame translation surface S where not all holonomy vectors are parallel, such $K_{\mathrm{cr}}(S) = \mathbb{Q}$, but where $K_{\mathrm{tr}}(S)$ is not equal to $\mathbb{Q}$. Let E_0 be a copy of the affine plane $\mathbb{R}^2$ with a chosen origin and (x, y)-coordinates. Slit E_0 along the rays $R_v := (0, y \geq 1)$ and $R_h := (x \geq 1, 0)$ to obtain $\hat{E}_0$. Choose an irrational $0 < \lambda < 1$ and $n \in \mathbb{N}$ so that $1 < n\lambda$. Define

$$M := \begin{pmatrix} \lambda & 0 \\ 0 & n\lambda \end{pmatrix} \qquad R_v^k := M^k R_v \qquad R_h^k := M^k R_h \quad k \in \mathbb{Z}.$$

Here M^k acts linearly on E_0. For $k \neq 0$, slit a copy of E_0 along the rays R_v^k and R_h^k to obtain $\hat{E}_k$. We glue the family of slitted planes $\{\hat{E}_k\}_{k\in\mathbb{Z}}$ to obtain the desired tame flat surface as follows. Each $\hat{E}_k$ has a "vertical boundary" formed by two vertical rays issuing from the point of coordinates $(0,(n\lambda)^k)$. Denote by $R_{v,l}^k$ and $R_{v,r}^k$ the boundary ray to the left and right respectively. Identify by a translation the rays $R_{v,r}^k$ with $R_{v,l}^{k+1}$, for each $k \in \mathbb{Z}$. Denote by $R_{h,b}^k$ and $R_{h,t}^k$ the horizontal boundary rays in $\hat{E}_k$ to the bottom and top respectively. Identify by a translation $R_{h,b}^k$ with $R_{h,t}^{k+1}$ for each $k \in \mathbb{Z}$.
By construction, $\{(-\lambda^k,(n\lambda)^k)\}_{k\in\mathbb{Z}}$ is the set of all holonomy vectors of S. Clearly, all slopes involved are rational; hence $K_{\mathrm{cr}}(S) = \mathbb{Q}$. On the other hand, $M \in \Gamma(S)$ and $tr(M) = (n+1)\lambda$. Note that the surface S constructed in this last paragraph admits no triangulation satisfying the hypotheses of Proposition 3.5. □

Corollary 4.7. *The four fields $K_{\mathrm{tr}}(S)$, $K_{\mathrm{hol}}(S)$, $K_{\mathrm{cr}}(S)$ and $K_{\mathrm{sc}}(S)$ satisfy the following relations:*

- *(i)* $K_{\mathrm{hol}}(S) \subseteq K_{\mathrm{sc}}(S)$ *and* $K_{\mathrm{cr}}(S) \subseteq K_{\mathrm{sc}}(S)$.
- *(ii) For each other pair (i, j), with $i, j \in \{\mathrm{tr}, \mathrm{hol}, \mathrm{cr}, \mathrm{sc}\}$, $(i, j) \neq (\mathrm{hol}, \mathrm{sc})$ and $(i, j) \neq (\mathrm{cr}, \mathrm{sc})$, we can find surfaces S such that $K_i(S) \not\subseteq K_j(S)$. In these examples we can always choose $K_j(S)$ to be $\mathbb{Q}$.*
- *(iii) If S has two non parallel holonomy vectors, then $K_{\mathrm{tr}}(S) \subseteq K_{\mathrm{hol}}(S)$.*
- *(iv) If $\overline{S}$ has a geodesic triangulation by countably many triangles whose vertices form the set Σ of singularities of S, then $K_{\mathrm{cr}}(S) = K_{\mathrm{sc}}(S)$.*

Proof. (i) is shown in Section 3 before Remark 3.4; (ii) is shown in Theorem 2 and in Remark 3.4; (iii) is shown before Remark 3.4 and (iv) is the result of Proposition 3.5. □

Proof Corollary 1.1:
Let Γ be a subgroup of $\mathrm{SL}_2(\mathbb{Z})$. By Proposition 2.11 we know that there is a translation surface S with Veech group Γ. Furthermore in the construction all slits can be chosen such that their end points are integer points in the corresponding plane; thus S is an origami by Theorem 1, part (i). Hence it allows for a subset S^* of S, whose complement is a discrete set of points, an unramified covering $p : S^* \to T_0$ to the once puncture unit torus T_0. Recall that p defines the conjugacy class $[U]$ of a subgroup U of F_2 as follows. Let U be the fundamental group of S^*. It is

embedded into $F_2 = \pi_1(T_0)$ via the homomorphism p_* between fundamental groups induced by p. The embedding depends on the choices of the base points up to conjugation. In [Sch04] this is used to give the description of the Veech group completely in terms of $[U]$; compare Theorem B below. Recall for this that the outer automorphism group $\mathrm{Out}(F_2)$ is isomorphic to $\mathrm{GL}_2(\mathbb{Z})$. Furthermore it naturally acts on the set of the conjugacy classes of subgroups U of F_2.

Theorem B. *The Veech group $\Gamma(S^*)$ equals the stabiliser of the conjugacy class $[U]$ in $SL_2(\mathbb{Z})$ under the action described above.*

This theorem, that can be found in [*Ibid.*], considers only finite origamis, but the proof works in the same way for infinite origamis. Recall furthermore that $\Gamma(S^*) = \Gamma(S) \cap \mathrm{SL}_2(\mathbb{Z})$ and $\Gamma(S) \subseteq \mathrm{SL}_2(\mathbb{Z})$ if and only if the $\mathbb{Z}$-module spanned by the holonomy vectors of the saddle connections equals $\mathbb{Z}^2$. We can easily choose the marks in the construction in [PSV11] such that this condition is fulfilled.

□

Proof Theorem 3: First notice that the translation surfaces constructed in the proof of Theorem 2 parts (i) and (v) are both counterexamples for statements (A) and (B). Furthermore, Proposition 3.6 shows that two hyperbolic elements in $\Gamma(S)$ do not have to generate the same trace field. To disprove (C) we let μ be a solution to the equation $\mu + \mu^{-1} = \sqrt[3]{11}$ and G is the group generated by the matrices

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix}, \tag{4.10}$$

then Proposition 2.11 produces a tame translation surface S with Veech group G for which $K_{\mathrm{tr}}(S) = \mathbb{Q}(\sqrt[3]{11})$ is not totally real and thus is a counterexample for (C).
Finally for disproving (D) we construct a tame translation surface S with a hyperbolic element in its Veech group for which Λ has infinite index in Λ_0. The construction has two steps.
Step 1: Let M be the matrix given by

$$\begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}. \tag{4.11}$$

Let S' be the tame translation surface obtained from Proposition 2.11 for the group G' generated by M. Let Λ' be the image in $\mathbb{R}^2$ under the holonomy map of $H_1(\overline{S'}, \mathbb{Z})$, $\{e_1, e_2\}$ be the standard basis of $\mathbb{R}^2$ and $\beta := G' \cdot \{e_1, e_2\}$. We suppose without loss of generality that e_1 and e_2 lie in Λ'.
Step 2: Let $\alpha = \{v_j\}_{j\in\mathbb{N}} \subset \mathbb{R}^2 \setminus \Lambda'$ be a sequence of $\mathbb{Q}$-linearly independent vectors. We modify the construction in Proposition 2.11 (applied to G') in the following way. We add to the page A_{Id} a family of marks parallel to vectors in α. We can suppose that the new marks lie in the left-half plane $Re(z) < 0$ in A_{Id} and are disjoint by pairs and do not intersect any of the marks in $\mathcal{C}_1$ used in the construction from Step 1. For each $j \in \mathbb{N}$ there exists a natural number k_j such that $2k_j > |v_j|$. Let T_j be the torus obtained from a $2k_j \times 2k_j$ square by identifying opposite sides. Slit each T_j along a vector parallel to v_j and glue it to A_{Id} along the mark parallel to v_j. Denote by A'_{Id} the result of performing this operation for every $j \in \mathbb{N}$, then proceed just the same construction as in Proposition 2.11. Let S be the resulting translation surface. Observe that glueing in the tori T_j produces new elements in $H_1(\overline{S}, \mathbb{Z})$ whose image under the holonomy map lie in $\mathbb{Z} \times \mathbb{Z}$. Thus the subgroups of $\mathbb{R}^2$ generated by the image under the holonomy map of $H_1(\overline{S}', \mathbb{Z})$ and $H_1(\overline{S}, \mathbb{Z})$ are the same. Let Λ be the image in $\mathbb{R}^2$ under the holonomy map of $H_1(S, \mathbb{Z})$. By construction, the index of Λ in Λ_0 is at least the cardinality of α, which is infinite.

□

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COMMENTARIES ON THE PAPER *SOLENOIDAL MANIFOLDS* BY DENNIS SULLIVAN

ALBERTO VERJOVSKY

Abstract. Several remarks and comments on the paper about solenoidal manifolds referred in the title are given. In particular, the fact is emphasized that there is a parallel theory of compact solenoidal manifolds of dimensions one, two and three with the theory of compact manifolds of these dimensions.

A k-dimensional *solenoidal manifold* or *lamination* is a metric space which is locally the product of an euclidean k-disk and an infinite perfect and totally disconnected set (a subset of the Cantor set). These solenoidal manifolds appear naturally in many branches of mathematics. In topology the Vietoris-Van Dantzig solenoid ([13] [15]) is one of the fundamental examples in topology and it motivated the development of homology and cohomology theories which could apply to these spaces, for instance in the paper by Steenrod [10].

Solenoids appear naturally also as Pontryagin duals of discrete locally compact Hausdorff abelian groups. For instance if $\mathbb{Q}$ denotes the rationals with addition as group structure and with the discrete topology then its Pontryagin dual $\mathbb{Q}^*$ is the universal 1-dimensional solenoid which is a compact abelian group which fibers over the circle $\mathbb{S}^1$ via an epimorphism $p : \mathbb{Q}^* \to \mathbb{S}^1$ where the fibre is the Cantor group which is the pro-finite completion of the integers $\mathbb{Z}$. This fact has an important relationship with the *adèles* and *idèles* and its properties are the first steps in Tate's thesis.

Again, solenoids appear naturally also as basic sets of Axiom A diffeomorphisms in the sense of Smale [9]. In particular one-dimensional expanding attractors are solenoidal manifolds and were studied extensively by Bob Williams [17].

Let $\mathcal{H}(K)$ be the group of homeomorphisms of the Cantor K. Let N be a compact manifold and $\rho : \pi_1(N) \to \mathcal{H}(K)$ a homomorphism from the fundamental group of N to $\mathcal{H}(K)$. There is a lamination $\mathcal{L}_\rho$ associated to ρ called the *suspension* of ρ which is obtained by taking the quotient of $\tilde{N} \times K$ under the action of $\pi_1(N)$ given by $\gamma(x, k) = (\gamma(x), \rho(\gamma)(k))$ where $\tilde{N}$ is the universal cover of N and the action of of $\pi_1(N)$ on $\tilde{N}$ is by deck transformations.

One has a natural locally trivial fibration $p : \mathcal{L}_\rho \to N$ with fibre K.

In his paper Dennis Sullivan shows that any compact, *oriented*, 1-dimensional solenoidal manifold $\mathcal{S}$ is a mapping torus of a homeomorphism $h : K \to K$of the Cantor set K. In other words it corresponds to the representation of the fundamental group of the circle into $\mathcal{H}(K)$ induced by h. The proof is done by finding a global transversal in the oriented case. Since the topological dimension of the solenoid is one it follows that $\mathcal{S}$ embeds continuously in $\mathbb{R}^3$. However there is a nicer proof of this last fact using an unpublished idea I learned form Evgeny Shchepin.

Theorem 1. *(Shchepin) Let K be the standard triadic Cantor set in the interval $[0,1] \subset \mathbb{R} \subset \mathbb{R}^2$, and $h : K \to K$ any homeomorphism. Then h extends to a homeomorphism $H : \mathbb{R}^2 \to \mathbb{R}^2$. We think of the Cantor set as contained in the x-axis of the (x, y)-plane.*

Proof 1. *By Tietze extension theorem the map h extends to a continuous map $f : \mathbb{R} \to \mathbb{R}$. Of course f might be neither injective nor onto. The map $F(x, y) = (x, y + f(x))$ is a homeomorphism of $\mathbb{R}^2$ to itself and $F(K)$ is the graph of h. On the other hand the map $h^{-1} : K \to K$ also extends to a map $g : \mathbb{R} \to \mathbb{R}$. The map $G(x, y) = (x - g(y), y)$ is a homeomorphism of $\mathbb{R}^2$. Thus $G \circ F$ sends K to the vertical axis: $G \circ F(x, 0) = (0, h(x))$ if $x \in K$. Then we take $H = T \circ G \circ F$, where $T(x, y) = (y, x)$.* ■

Therefore we see that any oriented one dimensional solenoid $\mathcal{S}$ is contained as a "diffuse braid" in the open solid torus $\mathbb{R}^2 \times \mathbb{S}^1$ which is the mapping torus of H. In this respect one can consult [5].

The fact that any oriented one-dimensional solenoidal manifold is the suspension of a homeomorphism h of the Cantor set implies, as shown in the paper, that any such one-dimensional solenoidal manifold *is cobordant to zero*: there exists a compact two dimensional solenoidal manifold whose boundary is the given solenoidal one-dimensional manifold.

The proof is based on the fact that any homeomorphism of the Cantor set is a product of commutators and therefore there exists a representation $\rho : \pi_1(\Sigma) \to \mathcal{H}(K)$, where Σ is a smooth compact surface Σ with connected boundary a circle, such that the restriction of ρ to the element represented to the boundary is h.

The proof of the fact that de group of homeomorphisms of the Cantor set is perfect is proven in all detail in the paper [2] by R.D. Anderson.

Some of the most interesting and important solenoids are the two dimensional solenoidal manifolds (or solenoidal surfaces). In this respect Dennis himself has constructed one of the most beautiful and natural laminations whose Teichmüller space is remarkable: *The universal commensurability Teichmüller space* [11]. His paper in Acta [3], in collaboration with I. Biswas and S. Nag, is also an essential reference for this subject.

The idea of considering profinite constructions is very natural. If Σ is a compact surface and if we consider the inverse limit corresponding to the tower of all finite index coverings of Σ we obtain a two dimensional solenoidal manifold or surface lamination $\mathcal{L}$: we can consider complex structures on this lamination so that each leaf has a complex structure and the complex structures vary continuously in the transversal direction. There exists a canonical projection $\pi : \mathcal{L} \to \Sigma$. For a dense set of complex structures the restriction to each leaf is a conformal map to a finite cover of the original surface. Moreover the inverse limit of a point $K_z := \pi^{-1}\{z\},\ z \in \Sigma$ is a Cantor set. In fact in this construction one could use, to get the same inverse limit, any co-final set of finite coverings, for instance normal subgroups or even characteristic subgroups. In the latter case K_z is a nonabelian Cantor group.

The lamination $\mathcal{L}$ is the suspension of a homeomorphism $\rho : \pi_1(\Sigma) \to \mathcal{H}(K)$.

If Σ is a surface of genus two we can consider a simple closed curve γ in Σ which separates the surface into two surfaces of genus one with common boundary γ. The restriction of the lamination to γ is an oriented one dimensional solenoid. Thus there exists four homeomorphisms $f_1,\ f_2,\ g_1,\ g_2$ of the Cantor group K_z such that $[f_1, f_2] = [g_1, g_2] := h$ and the one dimensional solenoid is the suspension of h.

To me this is fascinating because these four homeomorphisms of the Cantor set satisfying the commutator relations above determine the universal solenoid. I think that it is a very interesting problem to understand the structure of these homeomorphisms.

Theorem 2 of the paper by Dennis Sullivan gives a sketch of the theorem that every solenoidal surface has a smooth structure, and in fact a laminated complex structure. Of course this is a classical theorem for surfaces (compact or not). This can be attributed to Radó and Kerékjártó since they prove that every surface can be triangulated (i.e. is homeomorphic to a simplicial complex of dimension two). There is a more recent proof of this fact by Thomassen [12]. The triangulation theorem can be adapted to solenoidal surfaces. The definition of a triangulation of a solenoidal surface is the natural one: each leaf is triangulated and the triangulation depends continuously in the transverse direction, in other words, if $L(z)$ denotes the leaf through z one requires:

For every point $z \in \mathcal{L}$ there exists a subcomplex $C \subset L(z)$ which is homeomorphic to a 2-disk and a homeomorphism $\phi : C \times K \to \mathcal{L}$ such that ϕ restricted to $C \times \{k\}$ is a simplicial linear homeomorphsm from $C \times \{k\}$ onto a subcomplex of the triangulated leaf

$$L(\phi(c,k)) \ (c \in C,\ k \in K).$$

Theorem 2. *Let $\mathcal{L}$ be a topological compact solenoidal surface then $\mathcal{L}$ can be triangulated.*

Let me give a sketch of my own proof of this theorem. The Riemann mapping theorem together with Carathéodory's theorem of prime ends imply that any continuous Jordan curve in the plane is locally flat, which implies that every Jordan curve has a topological tubular neighborhood. It is easy to prove - via the Riemann mapping theorem - that given two topological disks which are the images of two topological embeddings $\phi_i : \bar{\Delta} \to S$ $(i = 1, 2)$ of the unit closed disk in the complex plane into a topological surface S one can perturb ϕ_i $(i = 1, 2)$ to two embeddings ϕ_i such that the images of $\mathbb{S}^1 = \partial\bar{\Delta}$ meet topologically transversally (locally like the intersection of the coordinate axis in $\mathbb{R}^2$ at the origin). A Riemann surface can be covered by coordinate charts $\psi_j : \bar{\Delta} \to S$ such that the union of images of the disk of radius $1/2$ still cover the surface and the covering is locally finite. We can perturb slightly the embeddings so that the images of the boundary of the disk of radius $1/2$ meet topologically transversally. The union of the images of the these boundary circles divide the surface into cells with boundary a Jordan curve with a finite number of marked points where two such curves meet transversally. Using these points we can subdivide each cell to triangulate the Riemann surface. For a solenoidal surface $\mathcal{L}$ a similar construction works: we can cover the lamination with laminated charts $f_i : \bar{\Delta} \times K \to \mathcal{L}$ and then we can perturb these charts to have in each leaf a situation like the previous for a Riemann surface.

A triangulated solenoidal surface has a natural flat structure with singularities: we give each triangle of the triangulation the euclidean metric so that it is an equilateral triangle and all of these triangles have edges of equal lengths. This provides each leaf with a flat metric singular at the vertices (a sort of laminated Veech surface). By Riemann extension theorem each leaf is a complex surface and thus each solenoidal surface has a complex structure

Reciprocally every compact smooth solenoidal manifolds $\mathcal{S}$ has a triangulation à la Cairns. Let me give a sketch of the proof which is modeled on Cairns proof. Whitney embedding theorem is valid for smooth solenoidal manifolds: there exists a topological embedding $j : \mathcal{S} \to \mathbb{R}^n$. This follows from the usual fact that smooth real valued functions (in the sense of laminations) separate points. The embedding j when restricted to a leaf is an embedding (not necessarely a proper embedding) and if $\Phi : \mathbb{D}^k \times T \to \mathcal{S}$ (T a closed subset of the Cantor set) is a solenoidal chart the composition $j \circ \phi$ when restricted to a plaque $\mathbb{D}^k \times \{t\},\ \ t \in T$ is an embedding $k_t : \mathbb{D}^k \to \mathbb{R}^n$.

We require that the embeddings of plaques depend continuously on the transverse parameter (i.e. the map $t \mapsto k_t \in \mathrm{C}^\infty(\mathbb{D}^k, \mathbb{R}^n)$ is continuous). Then if we consider the solenoidal manifold $j(\mathcal{S}) \subset \mathbb{R}^n$ we can apply a very large homothetic transformation$T$, $x \mapsto rx\ \ x \in \mathbb{R}^n,\ \ r \in \mathbb{R}$ with $r > 0$ very large so that the curvature of the leaves of $j(\mathcal{S}$ is almost zero). Now we consider the canonical cubulation by unit cubes of $\mathcal{R}^n$ and the intersection of $T(j(\mathcal{S}))$ with each cube of the cubulation. Since we can assume without difficulty that $J(\mathcal{S})$ is transverse to all the skeletons of the cubulation, we see that each leaf is almost an affine subspace of dimension k with respect to a unit cube, so that each leaf meets each cube in a convex polytope of dimension k after subdividing in an obvious way each of these polytopes we get the triangulation of the solenoidal manifold.

Since every solenoidal surface has a smooth structure we can provide each leaf with a Riemannian metric in such a way that the metric is smooth on each leaf and it depends continuously on the transverse parameter. We call such a solenoidal surface with a leaf-wise metric metric g a *solenoidal Riemannian surface* $(\mathcal{S}, g)$.

Given a *compact* solenoidal surface $(\mathcal{S}, g)$ we see that each leaf has a conformal type with respect to g, i.e for any $z \in \mathcal{S}$ the universal covering of the leaf $L(z)$ is conformally equivalent to the Riemann sphere (elliptic leaf) the complex plane (parabolic leaf) or the Poincaré disk (hyperbolic leaf). If g' es any other leaf-wise smooth Riemannian metric the conformal type of the leaf does not change. This is a beautiful observation of Elmar Winkelnkemper (1976). Therefore one can speak of a *hyperbolic solenoidal Riemannian surface* when all the leaves are of hyperbolic type. We have the analog of the uniformization theorem of Koebe-Poincaré for compact hyperbolic solenoidal Riemannian surface.

Theorem 3. *(Candel [4] and Verjovsky [14]). If every leaf of a laminar Riemannian surface is conformally covered by the disk, then the unique constant curvature minus one metric on each leaf is transversally continuous.*

Sullivan states and sketches a proof of the following theorem of Alberto Candel [4]:

Theorem 4. *For any transversally continuous Riemannian metric on a smooth laminar surface, sometimes both but at least one of the following holds:*

(1) *The universal cover of every leaf is conformally the disk.*
(2) *There is a nontrivial tranversal measure (a measure on each transversal so that the germs of transversal holonomy maps along paths are measure preserving).*

Of course there are compact solenoidal surfaces such that every leaf has universal covering conformally equivalent to the euclidean plane. For instance the inverse limit of finite covers of a flat 2-torus. For these laminations some times it is impossible to simultaneously uniformize all the leaves [6].

Sullivan gives an example of a noncompact surface lamination without transverse measure but there is, in my opinion, a better *compact* example which of course Dennis knows since I learned it from him. Let $\mathcal{S}_2$ be the dyadic solenoid given as the inverse limit of

$$\cdots \longrightarrow \mathbb{S}^1 \overset{z\to z^2}{\longrightarrow} \mathbb{S}^1 \overset{z\to z^2}{\longrightarrow} \mathbb{S}^1.$$

Then $\mathcal{S}_2$ is a compact abelian solenoidal group with a canonical metric which induces Haar measure on the group. After choosing an orientation, there is a unit vector field Y tangent to the lamination. The squaring map $F(Z) = Z^2$ is an isomorphism of $\mathcal{S}_2$ onto itself. Its derivative in the sense of laminations expands by two every unit tangent vector $Y(x)$. The suspension of F is defined as the mapping torus of F. It is a two dimensional lamination. There is the canonical suspension flow generated by the vector field X tangent to $\mathcal{S}_2$. In fact the leaves of this lamination are the orbits of a locally free action of the real affine group since we have the Lie bracet relation $[X, Y] = Y$. It is not difficult to prove:

Proposition 1. *If $\mathcal{L}$ is a compact lamination whose leaves are given by a locally free action of the real affine group, then the lamination does not admit a transverse measure.*

The last part of the paper deals with the Teichmüller theory of compact solenoidal (or laminar) surfaces. For a compact laminar surface such that all its leaves are hyperbolic it is possible to develop Teichmüller theory. Almost everything valid for a hyperbolic Riemann surface is also valid for such a lamination. In general the Teichmüller space is infinite dimensional if the transverse structure is a Cantor set.

Thus it is possible to speak of Teichmüller distance, quadratic differentials, etc.

Theorem 5. *The space of hyperbolic structures on a hyperbolic laminar surface (as in Theorem 4) up to isometries isotopic to the identity has the structure of a separable complex Banach manifold. The metric is the natural Teichmüller metric based on the minimal conformal distortion of a map between structures. The isotopy classes of homeomorphisms preserving a chosen leaf act by isometries on this Banach manifold.*

As was remarked before, Sullivan constructs the universal Teichmüller space of the solenoidal surface $\mathcal{S}$ obtained by taking the inverse limit of all finite pointed covers of a compact surface of genus greater than one and chosen base point. The base points upstairs in the covers determine a point and a distinguished leaf L in the inverse limit solenoidal surface. In this space the commensurability automorphism group of the fundamental group of any higher genus compact surface acts by isometries. This group is independent of the genus dy definition.

Theorem 6. *The space of hyperbolic structures up to isometry preserving the distinguished leaf on this solenoidal surface $\mathcal{S}$ is non Hausdorff and any Hausdorff quotient is a point.*

The proof ot this result relies on the recent deep results by Jeremy Kahn and Vladimir Marković on the validity of the Ehrenpreis Conjecture [7].

The remark by Sullivan is that the action of the commensurability automorphism group of the fundamental group is by isometries and minimal. The action is described in the paper in *Acta Mathematica* [3] mentioned before.

Sullivan does not include in his article the role of laminations in holomorphic dynamics, a subject created by him to prove the Feiganbaum universality conjectures, and continued, for instance, in the use of 3-dimensional hyperbolic laminations by Misha Lyubich and Yair Minsky. in [8].

Given any compact manifold M a representation of $\rho : \pi_1(M) \to \mathcal{H}(K)$, where $\mathcal{H}(K)$ is the group of homeomorphisms of the Cantor set, gives rise to a solenoidal manifold. Therefore if M is any compact manifold with residually finite fundamental group (as in the case of a Riemann surface of genus bigger than one or any compact hyperbolic manifold) one has a lamination by considering the inverse limit of the tower of its finite covers. This is, in a sense, the *profinite completion* of a manifold with residually finite fundamental group. The fundamental groups of compact hyperbolic 3-manifolds are residually finite so that we can consider the infinite tower of finite covers.

A direct consequence of the recent results by Ian Agol [1] and Daniel Wise [17] which solve in the affirmative the question by Bill Thurston whether every hyperbolic 3-manifold M virtually fibers over the circle (i.e. there exists a finite covering $\tilde{M}$ and a locally-trivial fibration over the circle $p : \tilde{M} \to \mathbb{S}^1$) we have:

Theorem 7. *Let M be a compact hyperbolic 3-manifold and let $\mathcal{L}(M)$ be the compact 3-dimensional lamination obtained by the inverse limit of the directed set of its finite covers. Then:*

(1) *$\mathcal{L}(M)$ fibers over M with fiber the Cantor set*
(2) *There exists a locally trivial fibration $\pi : \mathcal{L}(M) \to \mathbb{S}^1$ with fiber a laminar surface $\mathcal{S}$.*
(3) *By 2. there exists a homeomorphism $f : \mathcal{S} \to \mathcal{S}$ such that $\mathcal{L}(M)$ is obtained by suspending f.*

I think that the study of the homeomorphism f in *3* above is interesting. It is the lifting, in the tower of coverings of the fibre $p^{-1}(\{1\})$ of the virtual fibration, of the pseudo-Anosov homeomorphism of the fibre which determines the fibration over the circle.

A solenoidal manifold (or lamination) is said to be hyperbolic if there exist a Riemannian metric for which every leaf has constant negative curvature -1.

In view of theorem 7 some natural questions arise:

Question. *Let $\mathcal{L}$ be a compact laminar surface. Let $f : \mathcal{L} \to \mathcal{L}$ be a homeomorphism. Let M be the 3-dimensional compact solenoidal manifold which is obtained by suspending f.*

(1) *When is M a hyperbolic compact solenoidal 3-manifold ?*
(2) *Is there a classification à la Thurston of isotopy classes of homeomorphisms of compact laminar surfaces?*
(3) *Does every compact hyperbolic* 3*-dimensional hyperbolic lamination fibers over the circle?*

Another topic would be to develop the theory of *geodesic laminations* for compact hyperbolic solenoidal surfaces.

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