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Proceedings of the Workshop on Singularities in Geometry and Applications, Będlewo, 5 – 21 May 2011

Editors:
Peter Giblin
Stanisłaus Janeczko
Carmen Romero-Fuster

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Preface

Obviously it is possible to practice pure mathematics without any interest in its applications. There are people convinced that applications might be even dangerous for the purity of mathematical research. However there is also another, very important, tradition in science going back to Archimedes, Newton and Gauss. In this tradition mathematics is considered the language of nature and, by straightforward feedback, applications are the source of fresh ideas for mathematics itself. Singularity theory is considered to be a relatively recent branch of mathematics—it "grew out of the work of Hassler Whitney and René Thom in the 1950s and 1960s, with crucial input from Bernard Malgrange and John Mather who put so many of Thom's beautiful ideas on a sound mathematical footing"[1]. Nevertheless singularity theory has been extensively developed in the applications tradition for more than a half century. Its deep and intriguing results are considered to be extremely interesting and stimulating for interdisciplinary research. There has been fundamental progress in optics, image recognition and processing, control theory, mechanics, relativity theory and numerous other fields of study, including those pertaining to biological, medical and social sciences. New singularity theory methods and techniques for solving theoretical and practical problems are being developed all the time.

With the aim of exploring the current and potential areas of creative interaction between singularity theory and other mathematical disciplines, and of fostering active exchange of ideas among people with different scientific backgrounds, a series of workshops on *Singularities in Generic Geometry and Applications* was proposed by Carmen Romero-Fuster and the first workshop was organized in Spain in Valencia in 2009. The success of this workshop was evident and the need for such a biennial feast of this most fresh and creative branch of mathematics became obvious. As a result, the second workshop on singularities in geometry and applications was organized at the Banach Center in Poland. The workshop brought together in Będlewo, Poland, more than eighty outstanding mathematicians from fourteen countries. The plenary lectures were as follows:

- Jean-Paul Brasselet (Some insights on the Euler local obstruction),
- James Damon (Medial/skeletal linking structures and the geometry of multi-object configurations),
- Peter Donelan (Singularities of robot manipulators: Lie groups and exponential products)
- Andrew du Plessis (Stable unfoldings of map-germs on singular varieties),
- Peter Giblin (In Memoriam Ian R. Porteous 9 October 1930 – 30 January 2011) ,
- Victor Goryunov (Local invariants of maps between 3-manifolds),
- Goo Ishikawa (Singularities of tangent varieties to curves and surfaces),
- Maxim Kazarian (Stabilization of cohomology classes represented by singularity loci)
- Isabel Labouriau (The geometry of fast and slow dynamics in nerve impulse),
- Walter Neumann (Local bilipschitz geometry of complex surfaces),
- Juan Jose Nuño-Ballesteros (Topological $\mathcal{K}$-equivalence of map germs),
- Kentaro Saji (Geometry of wavefronts),
- Federico Sánchez-Bringas (Geometric invariants on Lorentzian surfaces immersed in Minkowski $\mathbb{R}^{3,1}$),

[1] Preface to the Proceedings of the First Workshop on Singularities in Generic Geometry and Applications, Valencia 2009, *Topology and its Applications* **159**, issue 2, 2012

- Zbigniew Szafraniec (Quadric forms and intersection numbers for polynomial immersions),
- Farid Tari (Umbilics of surfaces in the Minkowski 3-space),
- Stephen Yau (New invariants for complex manifolds and its application to complex Plateau problem),
- Michail Zhitomirskii (Normal forms in singularity theory versus differential geometry).

The minicourses were as follows:

- Carmen Romero Fuster (Singularity theory techniques in extrinsic geometry),
- Farid Tari and Alexey Davydov (Singularity theory of implicit differential equations).

Many of the lecturers have now presented their new results in a written form. We are very grateful to the editors of the Journal of Singularities for making possible this special issue containing some of the tangible outcomes of the conference in Będlewo.

Peter Giblin
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Singularities in Geometry and Appl., Będlewo, 2011

A LOCAL BUT NOT GLOBAL ATTRACTOR FOR A $\mathbb{Z}_n$-SYMMETRIC MAP

B. ALARCÓN, S.B.S.D. CASTRO, AND I.S. LABOURIAU

Abstract. There are many tools for studying local dynamics. An important problem is how this information can be used to obtain global information. We present examples for which local stability does not carry on globally. To this purpose we construct, for any natural $n \geq 2$, planar maps whose symmetry group is $\mathbb{Z}_n$ having a local attractor that is not a global attractor. The construction starts from an example with symmetry group $\mathbb{Z}_4$. We show that although this example has codimension 3 as a $\mathbb{Z}_4$-symmetric map-germ, its relevant dynamic properties are shared by two 1-parameter families in its universal unfolding. The same construction can be applied to obtain examples that are also dissipative. The symmetry of these maps forces them to have rational rotation numbers.

1. Introduction

At the end of the 19^{th} century, Lyapunov [12] related the local stability of an equilibrium point to the eigenvalues of the Jacobian matrix of the vector field at that point. This led to the Markus-Yamabe Conjecture [13] in the 1960's, and fifteen years later to a version for maps of the original conjecture, using the relation between stability of fixed points and the eigenvalues of the Jacobian matrix of the map at that point [11]. In the 1990's, this was named, by analogy, the Discrete Markus-Yamabe Conjecture and remains unproven. It may be stated as follows:

Discrete Markus-Yamabe Conjecture: Let f be a C^1 map from $\mathbb{R}^m$ to itself such that $f(0) = 0$. If all the eigenvalues of the Jacobian matrix at every point have modulus less than one, then the origin is a global attractor.

It is known that the original conjecture holds for $m = 2$ and is, in this case, equivalent to the injectivity of the vector field [10], [8]. It is false for $m > 2$ [4], [6]. On the other hand, the Discrete Markus-Yamabe Conjecture holds, for all m, if the Jacobian matrix of the map is triangular and, additionally for $m = 2$, for polynomial maps [7]. It is false in higher dimensions, also for polynomial maps [6]. There exists a counter-example for $m = 2$ that is an injective rational map ([7]). This striking difference between the discrete and continuous versions encouraged the study of the dynamics of continuous and injective maps of the plane that satisfy the hypotheses of the Discrete Markus-Yamabe Conjecture. This is now known as the Discrete Markus-Yamabe Problem. From the results in [1], it follows that the Discrete Markus-Yamabe Problem is true for $m = 2$ for dissipative maps, by introducing as an extra condition the existence of an invariant ray (a continuous curve without self-intersections connecting the origin to infinity). An invariant ray can be, for instance an axis of symmetry.

In the presence of symmetry, that is, when the map is equivariant, the ultimate question can be stated as follows:

Equivariant Discrete Markus-Yamabe Problem: Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a dissipative C^1 equivariant planar map such that $f(0) = 0$. Assume that all eigenvalues of the Jacobian matrix at every point have modulus less than one. Is the origin a global attractor?

Given the results in Alarcón *et al.* [1], the Equivariant Discrete Markus-Yamabe Problem is true if the group of symmetries of f contains a reflection. In this case, the fixed-point space of the reflection plays the role of the invariant ray. This situation is addressed in Alarcón *et al.* [3]. In the present paper, we are concerned with symmetry groups that do not contain a reflection.

The Equivariant Discrete Markus-Yamabe Problem has a negative answer if the reflection is not a group element. In fact, the example constructed by Szlenk and reported in [7] satisfies all the hypotheses of the Discrete Markus-Yamabe Problem, is equivariant (as we show here) under the standard action of $\mathbb{Z}_4$, but the origin is not a global attractor. Indeed, there is an orbit of period 4 and the rotation number defined in [16] is $\frac{1}{4}$. The example has a singularity at the origin with $\mathbb{Z}_4$ codimension 3, and we show that two inequivalent 1-parameter families in its unfolding share these dynamic properties.

We use Szlenk's example to construct differentiable maps on the plane with symmetry group $\mathbb{Z}_n$ for all $n \geq 2$. Each example has an attracting fixed point at the origin and a periodic orbit of minimal period n which prevents local dynamics to extend globally. The construction may be extended to one of the 1-parameter families mentioned above.

We adapt the $\mathbb{Z}_n$ symmetric example to make it dissipative. In that case its symmetry implies that the rotation number is rational. Implications of this fact are discussed in the final section.

1.1. **Equivariant Planar Maps.** The reference for the folllowing definitions and results is Golubitsky *et al.* [9, chapter XII], to which we refer the reader interested in further detail.

Our concern is about groups acting linearly on $\mathbb{R}^2$ and more particularly about the action of $\mathbb{Z}_n$, $n \geq 2$ on $\mathbb{R}^2$. Identifying $\mathbb{R}^2 \simeq \mathbb{C}$, the finite group $\mathbb{Z}_n$ is generated by one element R_n, the rotation by $2\pi/n$ around the origin, with action given by

$$R_n \cdot z = \mathrm{e}^{2\pi i/n} z.$$

A map $f : \mathbb{R}^2 \to \mathbb{R}^2$ is $\mathbb{Z}_n$-equivariant if

$$f(\gamma x) = \gamma f(x) \quad \forall\, \gamma \in \mathbb{Z}_n,\ x \in \mathbb{R}^2.$$

We also say, if the above only holds for elements in $\mathbb{Z}_n$, that $\mathbb{Z}_n$ is the symmetry group of f.

Since most of our results depend on the existence of a unique fixed point for f, the following is a useful result.

Lemma 1.1. *If f is $\mathbb{Z}_n$-equivariant then $f(0) = 0$.*

Proof. We have $f(0) = f(\gamma 0) = \gamma f(0)$, by equivariance. The element $\gamma = \exp 2\pi i/n$ of $\mathbb{Z}_n$ is such that $\gamma x \neq x$ for all $x \neq 0$. It then follows that $f(0) = 0$. □

2. Example with an orbit of period 4

In this section, we explore the properties of an example of a local attractor which is not global since it has an orbit of period 4. This example is due to Szlenk and is reported in [7]. A list of properties for this example is given in Proposition 2.1. We divide this section in two subsections, the first dealing with dynamic properties and the second concerned with the study of the singularity in Szlenk's map.

2.1. **Dynamics.** Before introducing the example it is useful to establish some concepts that will be used in the proofs to come. Let $S_{1,n} \subset \mathbb{R}^2$ be the open sector

$$S_{1,n} = \{(x, y) = (r\cos\theta, r\sin\theta) :\ 0 < \theta < 2\pi/n\}$$

and define $S_{j,n}$, $j = 2, \cdots, n$ recursively by $S_{j,n} = R_n\,(S_{j-1,n})$. Then $\mathbb{R}^2 = \bigcup_{j=1}^{n} \overline{S_{j,n}}$, where $\overline{A}$ is the closure of A. Moreover, $S_{1,n} = R_n\,(S_{n,n})$. Then each $\overline{S_{j,n}}$ is a *fundamental domain* for the action of $\mathbb{Z}_n$, in particular if $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is $\mathbb{Z}_n$-equivariant then f is completely determined by its restriction to $\overline{S_{j,n}}$.

A *line ray* is a half line through the origin, of the form $\{t(\alpha, \beta) : \quad t \geq 0\}$, with $0 \neq (\alpha, \beta) \in \mathbb{R}^2$.

The next Proposition establishes the relevant properties of Szlenk's example that will be used in the construction of other $\mathbb{Z}_n$-equivariant maps in the next section.

Proposition 2.1 (Szlenk's example)**.** *Let $F_4 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be defined by*

$$F_4(x, y) = \left(-\frac{ky^3}{1 + x^2 + y^2}, \frac{kx^3}{1 + x^2 + y^2}\right) \qquad \textit{for} \quad 1 < k < \frac{2}{\sqrt{3}}.$$

The map F_4 has the following properties:

1) *F_4 is of class C^1.*
2) *F_4 is a homeomorphism.*
3) *$Fix(F_4) = \{0\}$.*
4) *$F_4^4(P) = P$ for $P = \left((k-1)^{-1/2}, 0\right)$, with $F_4^j(P) = R_4^j(P) \neq P$ for $j = 2, 3$.*
5) *0 is a local attractor.*
6) *F_4 is $\mathbb{Z}_4$-equivariant.*
7) *The restriction of F_4 to any line ray is a homeomorphism onto another line ray.*
8) *$F_4\left(\overline{S_{j,4}}\right) = \overline{S_{j+1,4}}$ for $j = 1, \cdots, 4 \pmod 4$ with $F_4\left(\partial S_{j,4}\right) = \partial S_{j+1,4}$.*
9) *The curve $F_4(\cos\theta, \sin\theta)$ goes across each line ray and is transverse to line rays at all points $\theta \neq \frac{m\pi}{2}$ for $m = 0, 1, 2, 3$.*

Proof. Some of the statements follow from previously established results. Since we deal with these first, the order of the proof does not follow the numbering in the list above.

Statements 1) and 4) are immediate from the expressions of F_4 and of P, as remarked in [7]. Note that the periodic orbit of P of statement 4) lies in the boundary of the sectors $\bigcup_j \partial S_{j,4}$.

In the appendix of [7] it is shown that the eigenvalues of $DF_4(x, y)$ lie in the open unit disk, establishing 5). Statement 3) follows as a direct consequence of Corollary 2 in [2] and the same estimates on the eigenvalues.

Concerning 6) note that R_4, the generator of $\mathbb{Z}_4$, acts on the plane as $R_4(x, y) = (-y, x)$. In order to prove that $F_4(x, y)$ is $\mathbb{Z}_4$-equivariant we compute

$$F_4(R_4(x, y)) = \left(-\frac{kx^3}{1 + x^2 + y^2}, -\frac{ky^3}{1 + x^2 + y^2}\right)$$

and

$$R_4F_4(x, y) = R_4\left(-\frac{ky^3}{1 + x^2 + y^2}, \frac{kx^3}{1 + x^2 + y^2}\right) = \left(\frac{-kx^3}{1 + x^2 + y^2}, \frac{-ky^3}{1 + x^2 + y^2}\right).$$

Observing that these are equal establishes statement 6).

The behaviour of F_4 on line rays described in 7) is easier to understand if we write (x, y) in polar coordinates $(x, y) = (r\cos\theta, r\sin\theta)$ yielding:

$$F_4(r\cos\theta, r\sin\theta) = \frac{kr^3}{1 + r^2}\left(-\sin^3\theta, \cos^3\theta\right) \quad . \tag{1}$$

From this expression it follows that for each fixed θ, the line ray through $(\cos\theta, \sin\theta)$ is mapped into the line ray through $(-\sin^3\theta, \cos^3\theta)$. The mapping is a bijection, since $r^3/(1+r^2)$ is a monotonically increasing bijection from $[0, +\infty)$ onto itself. In particular, it follows from this that F_4 is injective and that $F_4(\mathbb{R}^2) = \mathbb{R}^2$. Since every continuous and injective map in $\mathbb{R}^2$ is open (see Ortega [15, Chapter 3, Lemma 2]), it follows that F_4 is a homeomorphism, establishing 2).

The behaviour of F_4 on sectors and their boundary is the essence of 8). From the definition of the sectors we have

$$S_{j+1,4} = R_4\left(S_{j,4}\right)$$

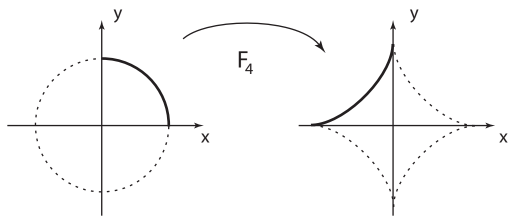

FIGURE 1. Szlenk's example F_4 maps a quarter of the unit circle into a quarter of the astroid $\frac{k}{2}(-\sin^3\theta, \cos^3\theta)$.

and therefore, by $\mathbb{Z}_4$-equivariance,

$$F_4\left(S_{j+1,4}\right) = F_4\left(R_4\left(S_{j,4}\right)\right) = R_4\left(F_4\left(S_{j,4}\right)\right) \ .$$

It then suffices to show that $F_4\left(\overline{S_{1,4}}\right) = \overline{S_{2,4}}$. The sectors $S_{1,4}$ and $S_{2,4}$ have the simple forms

$$S_{1,4} = \{(x,y): \quad x>0, \quad y>0\} \qquad S_{2,4} = \{(x,y): \quad x<0, \quad y>0\}\,.$$

From the expression of F_4 it is immediate that if $x>0$ and $y>0$ then the first coordinate of $F_4(x,y)$ is negative and the second is positive and thus $F_4\left(S_{1,4}\right) \subset S_{2,4}$. It remains to show the equality, which we delay until after the proof of 9).

The expression (1) in polar coordinates shows that the circle $(\cos\theta, \sin\theta)$, $0 \le \theta \le 2\pi$ is mapped by F_4 into the curve $\gamma(\theta) = \frac{k}{2}(-\sin^3\theta, \cos^3\theta)$ known as the *astroid* (Figure 1). The arc $\gamma(\theta)$, $0 \le \theta \le \pi/2$ joins $(0, \frac{k}{2})$ to $(-\frac{k}{2}, 0)$. Since for $\theta \in (0, \pi/2)$ the functions $\cos^3\theta$ and $-\sin^3\theta$ are both monotonically decreasing with strictly negative derivatives, then the $0 \le \theta \le \pi/2$ arc of the astroid has no self intersections and the restriction of F_4 to the quarter of a circle $0 \le \theta \le \pi/2$ is a bijection into this arc (Figure 1).

Moreover, the determinant of the matrix with rows $\gamma(\theta)$ and $\gamma'(\theta)$ is

$$\det\begin{pmatrix} \gamma(\theta) \\ \gamma'(\theta) \end{pmatrix} = \frac{3k^2}{4}\sin^2\theta\cos^2\theta$$

showing that the arc of the astroid is transverse at each point $\gamma(\theta)$, $0 < \theta < \pi/2$ to the line ray through it. Transversality fails at the end points of the arc, but the line rays still go across the astroid at the cusp points — this is assertion 9).

Thus, F_4 induces a bijection between line rays in $S_{1,4}$ and line rays in $S_{2,4}$ and using the radial property 7) it follows that $F_4\left(S_{1,4}\right) = S_{2,4}$. The behaviour on the boundary of $S_{1,4}$ also follows either from the radial property or from a simple direct calculation, concluding the proof of 8). □

2.2. **Universal unfolding of F_4.** In this section we discuss a universal unfolding of the singularity F_4 in the context of $\mathbb{Z}_4$-equivariant maps that fix the origin under contact equivalence. All the preliminaries concerning equivariant unfolding theory, as well as the proof of the result, are deferred to an appendix. The trusting reader may proceed without reading it.

Proposition 2.2. *A $\mathbb{Z}_4$ universal unfolding under contact equivalence of the germ at the origin of the singularity F_4 is given by*

$$G_4(x,y,\alpha,\beta,\delta) = F_4(x,y) + \alpha(x,y) + \left[\beta + \delta(x^2+y^2)\right](-y,x),$$

where parameters α, β and δ are real.

From the point of view of the dynamics, it is important to describe the maps in the unfolding that preserve the dynamic properties of F_4. The first result is immediate from the expression of the derivative of G_4 at the origin:

Lemma 2.3. *The origin is a hyperbolic local attractor for $G_4(x, y, \alpha, \beta, \delta)$ if and only if $\alpha^2 + \beta^2 < 1$.*

Although the unfolding above refers to the germ at the origin, we show below that its expression defines a map that shares some dynamic properties of F_4 for some parameter values. These values lie on two lines in parameter space.

Proposition 2.4. *Let $g(x, y)$ be either $G_4(x, y, \alpha, 0, 0)$ or $G_4(x, y, 0, \beta, 0)$. Then for α or β positive and small enough,*

- *g is a global diffeomorphism;*
- *at every point in $\mathbb{R}^2$ the eigenvalues of the jacobian of g have modulus less than one;*
- *there exists $p \in \mathbb{R}^2$ such that $g^4(p) = p$.*

Proof. The case $\alpha > 0$ is the one adressed in [7, Theorem E]. We treat the case $\beta > 0$ in a similar manner.

The matrix $DF_4(x, y)$ is given in the appendix. In this proof denote it by

$$DF_4 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

If μ is an eigenvalue of Dg then

$$\mu = \frac{1}{2}\left(-\mathrm{tr}(DF_4) \pm \sqrt{\mathrm{tr}^2(DF_4) - 4\det(DF_4) - 4\beta(\beta + c - b)}\right).$$

We know from [7, Theorem D] that all eigenvalues of DF_4 are zero on the coordinate axes and complex otherwise. Furthermore, all eigenvalues of DF_4 have modulus less than $k\sqrt{3}/2 < 1$. The latter statement ensures that, for any k and for small β, the eigenvalues of Dg also have modulus less than one.

We want to show that all eigenvalues of Dg are non-zero. When the eigenvalues of DF_4 are zero it is clear that those of Dg are not. Away from the axes, the eigenvalues of DF_4 are non-zero and $\det(DF_4) > 0$. Since $\det(Dg) = \det(DF_4) + \beta^2 - \beta(b - c)$, the eigenvalues of Dg are zero if and only if

$$\det(DF_4) + \beta^2 = \beta(b - c).$$

Since $b - c < 0$, then for $\beta > 0$, it is always the case that the eigenvalues of Dg are nonzero.

So far, we have shown that g is a local diffeomorphism at every point. In order to show that it is a global diffeomorphism, we show as in [7, Theorem E] that

$$\lim_{|(x,y)|\to\infty} |g(x, y)| = \infty.$$

This implies that g is proper and we may invoke Hadamard's theorem (quoted in [7]) that asserts that a proper local diffeomorphism is a global diffeomorphism.

In order to establish the limit above we use polar coordinates and write

$$g(r, \theta) = \frac{kr^3}{1 + r^2}(-\sin^3\theta, \cos^3\theta) + \beta(-r\sin\theta, r\cos\theta)$$

and hence,

$$|g(r, \theta)|^2 = \frac{k^2 r^6}{(1 + r^2)^2}(\sin^6\theta + \cos^6\theta) + \beta^2 r^2 + 2\beta k \frac{r^4}{1 + r^2}(\sin^4\theta + \cos^4\theta).$$

Noting now that $\sin^6\theta + \cos^6\theta \geq 1/4$ and $\sin^4\theta + \cos^4\theta \geq 1/2$, we use $1 + r^2 < 2r^2$ for $r > 1$ to write

$$|g(r,\theta)|^2 \geq \frac{kr^2}{16} + \beta^2 r^2 + \frac{\beta k r^2}{2} \overset{r\to\infty}{\longrightarrow} \infty.$$

The existence of points of period 4 follows from the hyperbolicity of the period 4 points of F_4. □

3. Construction of $\mathbb{Z}_n$-equivariant examples

The next examples refer to a local attractor, examples with a local repellor may be obtained considering f^{-1}.

Theorem 3.1. *For each $n \geq 2$ there exists $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that:*

a) *f is a differentiable homeomorphism;*
b) *f has symmetry group $\mathbb{Z}_n$;*
c) *$Fix(f) = \{0\}$;*
d) *The origin is a local attractor;*
e) *There exists a periodic orbit of minimal period n.*

Proof. For $n \geq 2$, the map

$$h_n\,(r\cos\theta, r\sin\theta) = \left(r\cos\frac{4\theta}{n}, r\sin\frac{4\theta}{n}\right) \tag{2}$$

is a local diffeomorphism at all points in $\mathbb{R}^2\backslash\{0\}$, is continuous at 0 and $h_n(S_{1,4}) = S_{1,n}$, $h_n(\overline{S_{2,4}}) = S_{2,n}$ with $|h_n(x,y)| = |(x,y)|$. Moreover, the restriction of h_n to $\overline{S_{1,4}}$ is a bijection onto $\overline{S_{1,n}}$ and h_n maps each line ray through the origin into another line ray through the origin.

Similar properties hold for the inverse

$$h_n^{-1}\,(r\cos\theta, r\sin\theta) = \left(r\cos\frac{n\theta}{4}, r\sin\frac{n\theta}{4}\right)$$

with $h_n^{-1}(S_{1,n}) = S_{1,4}$.

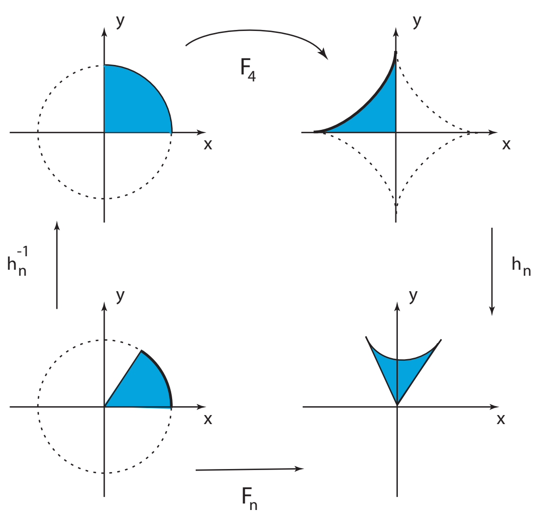

FIGURE 2. Construction of the $\mathbb{Z}_n$-equivariant example F_n in a fundamental domain of the $\mathbb{Z}_n$-action, shown here for $n = 6$.

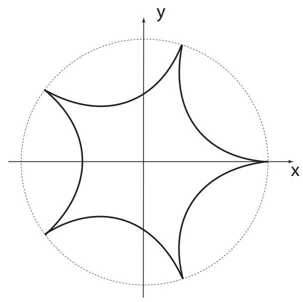

FIGURE 3. Image of the circle $(\sin\theta, \cos\theta)$ by the $\mathbb{Z}_n$-equivariant example F_n, shown here for $n = 5$.

Let $F_n : \overline{S_{1,n}} \longrightarrow \overline{S_{2,n}}$ be defined by (see Figure 2)

$$F_n(x, y) = h_n \circ F_4 \circ h_n^{-1}(x, y) \ . \tag{3}$$

We extend F_n to a $\mathbb{Z}_n$-equivariant map $F_n : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ recursively, as follows.

Suppose for $1 \leq j \leq n-1$ the map F_n is already defined in $S_{j,n}$ with $F_n(S_{j,n}) = S_{j+1,n}$. If $(x, y) \in S_{j+1,n}$ we have $R_n^{-1}(x, y) \in S_{j,n}$ and thus $F_n \circ R_n^{-1}(x, y)$ is well defined, with $F_n \circ R_n^{-1}(x, y) \in S_{j+1,n}$. Define $F_n(x, y)$ for $(x, y) \in S_{j+1,n}$ as $F_n(x, y) = R_n \circ F_n \circ R_n^{-1}(x, y) \in S_{j+2,n}$. Finally, for $(x, y) \in S_{n-1,n}$ we obtain $F_n(x, y) \in S_{1,n}$.

The following properties of F_n now hold by construction, using Proposition 2.1:

- F_n is $\mathbb{Z}_n$-equivariant.
- $Fix(F_n) = \{0\}$.
- The origin is a local attractor.
- $F_n^n(P) = P$ for $P = \left((k-1)^{-1/2}, 0\right)$, with $F_n^j(P) \neq P$ for $j = 2, \ldots, n-1$. Note that all $F_n^j(P)$ lie on the boundaries $\partial S_{j,n}$ of the sectors $S_{j,n}$.
- F_n maps each line ray through the origin onto another line ray through the origin.

Since h_n maps line rays to line rays, to see that F_n is a homeomorphism it is sufficient to observe that $\gamma_n(\theta) = F_n(\cos\theta, \sin\theta)$, $0 \leq \theta \leq 2\pi$ is a simple closed curve that meets each line ray only once and does not go through the origin (Figure 3). This is true because away from the origin both h_n and h_n^{-1} are differentiable with non-singular derivatives. Since h_n and h_n^{-1} map line rays into line rays, it follows from assertion 9) of Proposition 2.1 that γ_n is transverse to line rays except at the cusp points $\gamma_n(\theta)$, $\theta = \frac{2m\pi}{n}$, $m = 0, 1, \ldots, n-1$ where the line ray goes across it.

It remains to show that F_n is everywhere differentiable in $\mathbb{R}^2$. This is done in Lemma 3.2 below. □

Lemma 3.2. *F_n is everywhere differentiable in $\mathbb{R}^2$.*

Proof. First we show that $DF_4(0, 0) = (0)$ (zero matrix) implies that F_n is differentiable at the origin with $DF_n(0, 0) = (0)$. That $DF_4(0, 0) = (0)$ means that for every $\varepsilon > 0$ there is a $\delta > 0$ such that, for every $X \in \mathbb{R}^2$, if $|X| < \delta$ then

$$|F_4(X) - F_4(0, 0) - DF_4(0, 0)X| = |F_4(X)| < \varepsilon\, |X| \ .$$

Since h_n and h_n^{-1} preserve the norm, we have that if $Y = h_n(X)$ then $|Y| = |X|$ and furthermore, for any Y such that $|Y| < \delta$, we obtain

$$|F_n(Y)| = \left|h_n\left(F_4\left(h_n^{-1}(Y)\right)\right)\right| = |h_n\left(F_4(X)\right)| = |F_4(X)| < \varepsilon\, |X| = \varepsilon\, |Y| \, .$$

Therefore, since $F_n(0,0)=(0,0)$ and since this holds for any ε,

$$\lim_{|X|\to 0}\frac{|F_n(X)-F_n(0,0)-(0)X|}{|X|}=0$$

proving our claim.

Recall that in (3) and in the text thereafter the map F_n is made up by gluing different functions on sectors: in $S_{1,n}$ the expression of F_n is given by $h_n\circ F_4\circ h_n^{-1}$ and in $S_{2,n}$ by $R_n\circ h_n\circ F_4\circ h_n^{-1}\circ R_n^{-1}$. Both expressions define differentiable functions away from the origin since both h_n and h_n^{-1} are of class C^1 in $\mathbb{R}^2\backslash\{(0,0)\}$. We have already shown that F_n is differentiable at the origin. It remains to prove that the derivatives of the two functions coincide at the common boundary of $\partial S_{1,n}$ and $\partial S_{2,n}$. At the remaining boundaries the result follows from the $\mathbb{Z}_n$-equivariance of F_n.

Since we are working away from the origin, we may use polar coordinates. The expressions for h_n, R_n and their inverses take the simple forms below, where we use $\widehat{f}$ to indicate the expression of f using polar coordinates in both source and target:

$$\widehat{h}_n(r,\theta)=\left(r,\frac{4\theta}{n}\right)\qquad \widehat{h}_n^{-1}(r,\theta)=\left(r,\frac{n\theta}{4}\right)$$

$$\widehat{R}_n(r,\theta)=\left(r,\theta+\frac{2\pi}{n}\right)\qquad \widehat{R}_n^{-1}(r,\theta)=\left(r,\theta-\frac{2\pi}{n}\right)\ .$$

Let $\widehat{F}_4(r,\theta)=(\Psi_4(r,\theta),\Phi_4(r,\theta))$ be the expression of F_4 in polar coordinates. From (1) we get:

$$\Psi_4(r,\theta)=\frac{kr^3}{1+r^2}\sqrt{\cos^6\theta+\sin^6\theta}=\frac{kr^3}{1+r^2}\sqrt{1-3\cos^2\theta+3\cos^4\theta}\tag{4}$$

$$\Phi_4(r,\theta)=\begin{cases}\arctan\left(-\dfrac{\cos^3\theta}{\sin^3\theta}\right) & \text{if}\quad \theta\neq k\pi\\[2ex] \operatorname{arccot}\left(-\dfrac{\sin^3\theta}{\cos^3\theta}\right) & \text{if}\quad \theta\neq\frac{\pi}{2}+k\pi\ .\end{cases}\tag{5}$$

The derivative $D\widehat{F}_4(r,\theta)$ of $\widehat{F}_4$ is thus,

$$\begin{pmatrix} kr^2\dfrac{3+r^2}{(1+r^2)^2}\sqrt{\cos^6\theta+\sin^6\theta} & \dfrac{kr^3}{1+r^2}\dfrac{3\sin\theta\cos\theta\left(\sin^4\theta-\cos^4\theta\right)}{\sqrt{\cos^6\theta+\sin^6\theta}}\\[3ex] 0 & \dfrac{3\sin^2\theta\cos^2\theta}{\cos^6\theta+\sin^6\theta}\end{pmatrix}\tag{6}$$

where the two alternative forms for $\Phi_4(r,\theta)$ yield the same expression for the derivative.

Note that the Jacobian matrix of $\widehat{h}_n$ is constant and the same is true for its inverse. The derivatives of both $\widehat{R}_n$ and of $\widehat{R}_n^{-1}$ are the identity. Let $(r,2\pi/n)$ be the polar coordinates of a point ξ in $(\partial S_{1,n}\cap\partial S_{2,n})\backslash\{0\}$. In order to show that the derivatives at ξ of $\widehat{h}_n\circ\widehat{F}_4\circ\widehat{h}_n^{-1}$ and of $\widehat{R}_n\circ\widehat{h}_n\circ\widehat{F}_4\circ\widehat{h}_n^{-1}\circ\widehat{R}_n^{-1}$ coincide, we only need to show that $D\widehat{F}_4$ at $\widehat{h}_n^{-1}(r,2\pi/n)=(r,\pi/2)$ equals $D\widehat{F}_4$ at $\widehat{h}_n^{-1}(\widehat{R}_n^{-1}(r,2\pi/n))=(r,0)$. More precisely, for any (r,θ)

$$D\widehat{h}_n(r,\theta)=A_n=\begin{pmatrix}1&0\\0&\frac{4}{n}\end{pmatrix}\qquad D\widehat{h}_n^{-1}(r,\theta)=B_n=\begin{pmatrix}1&0\\0&\frac{n}{4}\end{pmatrix}$$

and thus

$$\begin{aligned}
& D\left(\widehat{R}_n \circ \widehat{h}_n \circ \widehat{F}_4 \circ \widehat{h}_n^{-1} \circ \widehat{R}_n^{-1}\right)(\xi) \\
= \;& D\widehat{R}_n(\widehat{h}_n(\widehat{F}_4((r,0)))D\widehat{h}_n(\widehat{F}_4((r,0))D\widehat{F}_4(r,0)D\widehat{h}_n^{-1}(r,0)D\widehat{R}_n^{-1}(r,2\pi/n) \\
= \;& Id \cdot A_n \cdot D\widehat{F}_4(r,0) \cdot B_n \cdot Id \\
= \;& A_n \cdot D\widehat{F}_4(r,0) \cdot B_n
\end{aligned}$$

and

$$\begin{aligned}
& D\left(\widehat{h}_n \circ \widehat{F}_4 \circ \widehat{h}_n^{-1}\right)(\xi) \\
= \;& D\widehat{h}_n(\widehat{F}_4((r,\pi/2))D\widehat{F}_4(r,\pi/2)D\widehat{h}_n^{-1}(r,2\pi/n) \\
= \;& A_n \cdot D\widehat{F}_4(r,\pi/2) \cdot B_n \ .
\end{aligned}$$

From (6) it follows that

$$D\widehat{F}_4(r,\pi/2) = D\widehat{F}_4(r,0) = \begin{pmatrix} kr^2\dfrac{3+r^2}{(1+r^2)^2} & 0 \\ & \\ 0 & 0 \end{pmatrix}$$

completing our proof. □

The construction in the proof of Theorem 3.1 only works because Szlenk's example F_4 has the special properties 7), 8) and 9) of Proposition 2.1. For instance, identifying $\mathbb{R}^2 \sim \mathbb{C}$ the map $f(z) = \overline{z}^3$ is $\mathbb{Z}_4$-equivariant, but does not have the properties above and $h_5 \circ f \circ h_5^{-1}(z) = f(z)$.

Alarcón *et al.* [1, Theorem 4.4] construct, starting from F_4, an example having the additional property that ∞ is a repelllor. The new example, $H(x,y)$, is of the form

$$H(x,y) = \phi(|F_4(x,y)|)F_4(x,y)$$

where $\phi : [0,\infty) \longrightarrow [0,\infty)$ is described in [1, Lemma 4.6].

Then H has all the properties of Proposition 2.1. Therefore, applying to H the construction of Theorem 3.1 we obtain the following:

Corollary 3.3. *For each $n \geq 2$ there exists a map $f : \mathbb{R}^2 \to \mathbb{R}^2$ satisfying properties a)–e) of Theorem 3.1 and, moreover, for which ∞ is a repellor.*

4. Final comments

It remains an interesting question to find out whether our construction can be applied to G_4 to produce a $\mathbb{Z}_n$ universal unfolding of F_n. A partial answer is given next. The proof is straightforward.

Lemma 4.1. *If $\alpha = 0$ then G_4 has the property that $G_4(S_{1,4}) = S_{2,4}$.*

As a consequence, the previous construction applied to G_4 with $\alpha = 0$ produces other examples with $\mathbb{Z}_n$-symmetry and period n orbits. Furthermore, using Proposition 2.4, if also $\delta = 0$ these new examples are diffeomorphisms.

Note that, even though the unfolding applies only locally, the dynamic properties are robust beyond this constraint as they hold if we use the expression of the unfolding to define a global map.

A very interesting problem in Dynamical Systems is to describe the global dynamics with hypotheses based on local properties of the system. The Markus-Yamabe Conjecture is an example but not the only one. For instance, Alarcón *et al.* [1] prove the existence of a global attractor arising from a unique local attractor, using the theory of free homeomorphisms of the plane. Recently, Ortega and Ruiz del Portal in [16], have studied the global behavior of an orientation preserving homeomorphism introducing techniques based on the theory of prime ends. They define the rotation number for some orientation preserving homeomorphisms of $\mathbb{R}^2$ and show how this number gives information about the global dynamics of the system. In this context, even a list of elementary concepts would be too long to include here. The discussion that follows may be taken as an appetizer for the reader willing to look them up properly in [16], [17] and [5].

The theory of prime ends was introduced by Carathéodory in order to study the complicated shape of the boundary of a simply connected open subset of $\mathbb{R}^2$. When such a subset U is non empty and proper, by the Riemann mapping theorem, there is a conformal homeomorphism from U onto the open unit disk. Usually this homeomorphism cannot be extended to the closed disk. Carathéodory's compactification associates the boundary of U with the space of prime ends $\mathbb{P}$, which is homeomorphic to $\mathbb{S}^1$. In that way, $U \cup \mathbb{P}$ is homeomorphic to the closed unit disk. The correspondence between points in the boundary of U and points in $\mathbb{P}$ may be both multi-valued and not one to one, but if f is an orientation preserving homeomorphism with $f(U) = U$, then f induces an orientation preserving homeomorphism $\tilde{f}$ in $\mathbb{P}$. Since the space of prime ends is homeomorphic to the unit circle, the rotation number of $\tilde{f}$ is well defined and the rotation number of f is defined to be equal to the rotation number of $\tilde{f}$.

The points in $\partial_{\mathbb{S}^2} U$, the boundary of U in the one point compactification of the plane, that play an important role in the dynamics are accessible points. A point $\alpha \in \partial_{\mathbb{S}^2} U$ is *accessible* from U if there exists an arc ξ such that α is an end point of ξ and $\xi \setminus \{\alpha\} \subset U$. Then α determines a prime end $p(\alpha) \in \mathbb{P}$, which may not be unique, such that $\xi \setminus \{p\} \cup \{p(\alpha)\}$ is an arc in $U \cup \mathbb{P}$.

Accessible points are dense in $\partial_{\mathbb{S}^2} U$, but for instance, in the case of fractal boundaries there exist points which are not accessible from U. On the contrary, when the boundary is well behaved, for instance an embedded curve of $\mathbb{R}^2$, accessible points define a unique prime end. That means that accessible periodic points of f are periodic points of $\tilde{f}$ with the same period. Consequently the rotation number of f is 1 divided by the period. See [17] and [5] for more details and definitions.

Proposition 4.2. *The examples F_n in Theorem 3.1 have rotation number $1/n$.*

Proof. Let U be the basin of attraction of the origin for F_4. By construction of the maps in Theorem 3.1, the basin of attraction of the origin for F_n,

$$U_n = \bigcup_{j=0}^{n-1} R_n^j \left(h_n(U) \cap S_{1,n} \right)$$

is invariant by the map F_n and is a non empty and proper simply connected open set. Moreover, as the periodic point P is hyperbolic, the boundary of U is an embedded curve of $\mathbb{R}^2$ in a neighborhood of P. In addition, P is an accessible point from U_n, thus the rotation number of F_n is $\frac{1}{n}$. □

The fact that the symmetry forces the maps in Theorem 3.1 to have a rational rotation number seems to point out at a connection between symmetry and rotation number. It raises the question: for orientation preserving homeomorphisms of the plane with a non global asymptotically stable fixed point, does $\mathbb{Z}_n-$equivariance imply a rational rotation number?

The question is relevant because the rotation number gives strong information about the global dynamics of the system. For instance, consider a dissipative orientation preserving $\mathbb{Z}_n-$equivariant homeomorphism f of the plane with an asymptotically stable fixed point p.

If the question has an affirmative answer, then Proposition 2 of [16] implies that p is a global attractor under f if and only if f has no other periodic point.

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Appendix — Unfolding Theory for $\mathbb{Z}_n$

In order to better understand the singularity for Szlenk's $\mathbb{Z}_4$-equivariant map, we calculate its codimension and provide a universal unfolding. Some of the information below may be retrieved from the D_4 equivariant set-up described for instance in Golubitsky *et al.* [9].

Let $\mathcal{E}(\mathbb{Z}_4)$ be the set of $\mathbb{Z}_4$-invariant function germs from the plane to the reals. This is a ring generated by the following Hilbert basis

$$\mathcal{E}(\mathbb{Z}_4) = \left\langle N = x^2 + y^2, A = x^4 + y^4 - 6x^2y^2, B = (x^2 - y^2)xy \right\rangle \tag{7}$$

in the sense that every germ in $\mathcal{E}(\mathbb{Z}_4)$ can be written in the form $\phi(N, A, B)$ where ϕ is a smooth function of three variables.

The set of $\mathbb{Z}_4$-equivariant map germs is a module over the ring of invariants; it is denoted by $\overrightarrow{\mathcal{E}}(\mathbb{Z}_4)$ and generated by the following

$$\begin{array}{ll} X_1 = (x, y); & X_3 = (x(x^2 - 3y^2), y(y^2 - 3x^2)); \\ X_2 = (-y, x); & X_4 = (-y(y^2 - 3x^2), x(x^2 - 3y^2)). \end{array} \tag{8}$$

Two map-germs, g and h, are $\mathbb{Z}_4$-contact-equivalent if (see Mather [14], even though we follow the notation in [9], chapter XIV) there exists an invertible change of coordinates $x \mapsto X(x)$, fixing the origin and $\mathbb{Z}_4$-equivariant, and a matrix-valued germ $S(x)$ satisfying for all $\gamma \in \mathbb{Z}_4$

$$S(\gamma x)\gamma = \gamma S(x),$$

with $S(0)$ and $dX(0)$ in the same connected component as the identity in the space of linear maps of the plane, and such that

$$g(x) = S(x)h(X(x)).$$

The set of matrices satisfying the $\mathbb{Z}_4$-equivariance described above is denoted and generated as follows

$$\overleftrightarrow{\mathcal{E}}(\mathbb{Z}_4) = \langle S_j; T_j = iS_j, \quad j = 1, \dots 4\rangle ,$$

with

$$T_i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, S_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, S_2 = \begin{pmatrix} x^2 & xy \\ xy & y^2 \end{pmatrix},$$
$$S_3 = \begin{pmatrix} -x^2 & xy \\ xy & -y^2 \end{pmatrix}, S_4 = \begin{pmatrix} 0 & x^3y \\ xy^3 & 0 \end{pmatrix}.$$

Note that, in the Z_4-equivariant context, all map germs preserve the origin. In such cases as these, the tangent space T to the $\mathbb{Z}_4$-contact orbit coincides with the restricted tangent space, RT.

The tangent space to F_4 is

$$T_{\overrightarrow{\mathcal{E}}(\mathbb{Z}^4)}(F_4) = \langle (dF_4)X_i, S_jF_4, T_jF_4\rangle ,$$

where X_i is one of the generators of $\overrightarrow{\mathcal{E}}(\mathbb{Z}_4)$ and S_j and T_j are the generators of $\overleftrightarrow{\mathcal{E}}(\mathbb{Z}_4)$.

Given F_4 and dividing both components by k as it does not affect the singularity, we have

$$dF_4 = \begin{pmatrix} \frac{2xy^3}{(1+x^2+y^2)^2} & -\frac{3y^2(1+x^2+y^2)-2y^4}{(1+x^2+y^2)^2} \\ \frac{3x^2(1+x^2+y^2)-2x^4}{(1+x^2+y^2)^2} & -\frac{2x^3y}{(1+x^2+y^2)^2} \end{pmatrix}.$$

Note that all rows of this matrix have the common factor $1/(1 + x^2 + y^2)^2$, which does not affect the singularity. Also, all the products with F_4 will exhibit the common factor $1/(1 + x^2 + y^2)$, which again does not affect the singularity. We therefore present the generators of $T_{\overrightarrow{\mathcal{E}}(\mathbb{Z}^4)}(F_4)$ after a multiplication by the corresponding common factor. To exemplify,

$$S_1F_4 = (-\frac{y^3}{1 + x^2 + y^2}, \frac{x^3}{1 + x^2 + y^2})$$

is reported as $S_1F_4 = (-y^3, x^3)$. This stated, we have the following list of generators of $T_{\vec{\mathcal{E}}(\mathbb{Z}^4)}(F_4)$, where the symbol $\sim$ indicates that a simplification was made through a product by a non-zero invariant:

$$
\begin{aligned}
(dF_4)X_1 &= 3N(N-1)X_2 + (N-1)X_4 \sim 3NX_2 + X_4;\\
(dF_4)X_2 &= \frac{1}{4}(N(N+1)X_1 - (N+1)X_3) \sim NX_1 - X_3\\
(dF_4)X_3 &= \frac{3}{4}[(N^3 + N^2 + 2A)X_2 + (N^2 + N - \frac{2}{3}A)X_4];\\
(dF_4)X_4 &= \frac{1}{4}[(N^3 + 6A + 3N^2)X_1 + (2A - 3N^2 - 9N)X_3];\\
S_1F_4 &= 3NX_2 + X_4\\
S_2F_4 &= -3BX_1 - AX_2 + NX_4\\
S_3F_4 &= \frac{1}{4}(NX_4 - N^2X_2) \sim N^2X_2 - NX_4\\
S_4F_4 &= (-\frac{1}{16}N^3 - \frac{5}{32}NA)X_2 + \frac{1}{8}BX_3 + \frac{7}{32}N^2X_4\\
T_1F_4 &= \frac{1}{4}(3NX_1 + X_3) \sim 3NX_1 + X_3\\
T_2F_4 &= -BX_2;\\
T_3F_4 &= \frac{1}{4}(A - N^2)X_1 - BX_2;\\
T_4F_4 &= \frac{1}{16}[(NA - N^3)X_1 - 14NBX_2 - 2BX_4].
\end{aligned}
$$

We use a filtration by degree $\mathcal{F} = \{E^j\}_{j\in\mathbb{N}_0}$ of $\vec{\mathcal{E}}(\mathbb{Z}^4)$ where $E^j\backslash E^{j+1}$ is the set of germs in $\vec{\mathcal{E}}(\mathbb{Z}^4)$ with all coordinates homogeneous polynomials of the same degree j and $E^0 = \vec{\mathcal{E}}(\mathbb{Z}^4)$. Note that $E^{2j} = E^{2j+1}$ for all $j \geq 0$ and each E^j is a finitely generated $\mathcal{E}(\mathbb{Z}_4)$-module. Moreover, denoting as $\mathcal{M}(\mathbb{Z}^4)$ the unique maximal ideal in $\vec{\mathcal{E}}(\mathbb{Z}^4)$, we have

$$\mathcal{M}(\mathbb{Z}^4).E^j \subset E^{j+1}.$$

We show that $E^5 \subset T_{\vec{\mathcal{E}}(\mathbb{Z}^4)}(F_4)$ by showing that

$$E^5 \subset T_{\vec{\mathcal{E}}(\mathbb{Z}^4)}(F_4) + \mathcal{M}(\mathbb{Z}^4)E^5$$

and invoking Nakayama's Lemma. We have that E^5 is generated over $\mathcal{E}(\mathbb{Z}_4)$ as

$$E^5 = \langle N^2X_i, AX_i, BX_i, NX_j, AX_j, BX_j \rangle, \quad i = 1, 2;\ j = 3, 4. \tag{9}$$

We point out that there are no equivariants of degree 6 and therefore E^6 contains germs of degree 7 or higher.

Multiply by N the lower order generators of $T_{\vec{\mathcal{E}}(\mathbb{Z}^4)}(F_4)$, that is, $(dF_4)X_1$, $(dF_4)X_2$, S_1F_4 and T_1F_4 and append AS_1F_4 at the end of the list; add or subtract as necessary terms in $\mathcal{M}(\mathbb{Z}^4)E^5$ to the generators of $T_{\vec{\mathcal{E}}(\mathbb{Z}^4)}(F_4)$. After performing these two operations, we obtain the matrix Q below, where the entry (i, j) is the coefficient of generator j in (9) coming from the term i in the list of generators of $T_{\vec{\mathcal{E}}(\mathbb{Z}^4)}(F_4)$:

$$Q = \begin{pmatrix} 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 1 & -2/3 & 0 \\ 3 & 6 & 0 & 0 & 0 & 0 & -9 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -3 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/8 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1/4 & 1/4 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

The matrix Q is of rank 12, establishing our claim that $E^5 \subset T_{\vec{\mathcal{E}}(\mathbb{Z}^4)}(F_4)$.

We can then simplify the generators of $T_{\vec{\mathcal{E}}(\mathbb{Z}^4)}(F_4)$ even further adding the elements in $T_{\vec{\mathcal{E}}(\mathbb{Z}^4)}(F_4) \cap E^3 \backslash E^5$:

$$NX_1, X_3, 3NX_2 + X_4.$$

It is easily seen that there are the following two choices for a complement to $T_{\vec{\mathcal{E}}(\mathbb{Z}^4)}(F_4)$ inside $\vec{\mathcal{E}}\,(\mathbb{Z}^4)$

$$V_1 = \{X_1, X_2, X_4\} \quad \text{and} \quad V_2 = \{X_1, X_2, NX_2\}.$$

Therefore, the $\mathbb{Z}_4$-equivariant codimension of F_4 is 3. A universal unfolding is given by

$$G_4(x, y, \alpha, \beta, \delta) = F_4(x, y) + \alpha X_1 + \beta X_2 + \delta N X_2.$$

Of course a choice using V_1 as a complement is just as good from the point of view of singularity theory. However, our choice yields better results for the construction of an example with symmetry $\mathbb{Z}_n$.

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TOPOLOGICAL $\mathcal{K}$ AND $\mathcal{A}$ EQUIVALENCES OF POLYNOMIAL FUNCTIONS

L. BIRBRAIR AND J.J. NUÑO-BALLESTEROS

Abstract. We give a simple proof that the germs of real polynomial functions from $\mathbb{R}^n$ to $\mathbb{R}$ are C^0-$\mathcal{A}$-equivalent if they are PL-$\mathcal{K}$-equivalent (for example, semialgebraically). No restriction on the polynomial functions is needed.

1. Introduction

Given two smooth map germs $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$, if f, g are $\mathcal{A}$-equivalent, then they are $\mathcal{K}$-equivalent and the same is true if we consider any other reasonable category like real or complex analytic, C^k ($k \geq 1$), C^0, Lipschitz, PL, etc. The converse is known to be true for C^∞-stable map germs (according to the work of J. Mather [8]), but it is also well known that it is false in the general case. Here, we are interested in the function case (i.e., $p = 1$), where it might seem possible to recover the $\mathcal{A}$-class data from the $\mathcal{K}$-class.

In [5] Fukuda proved the finitness theorem for C^0-$\mathcal{A}$- equivalence of polynomial functions from $\mathbb{K}^n$ to $\mathbb{K}$, $\mathbb{K} = \mathbb{R}, \mathbb{C}$. Benedetti and Shiota [2] proved the same result for semialgebraic functions. In [1] the authors gave very simple models for the equivalence classes with respect to C^0-$\mathcal{K}$- equivalence of semialgebraic functions. The motivation of our work is the comparison of C^0-$\mathcal{K}$ and C^0-$\mathcal{A}$ equivalence for function germs.

Notice that for complex analytic functions with isolated singularity, it was shown by Saeki [11] that if $f, g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ are C^0- $\mathcal{V}$-equivalent, then they are C^0-$\mathcal{A}$-equivalent. For real analytic functions with isolated singularity and $n = 2, 3$, it is pointed out by King in [7] that in this case again C^0-$\mathcal{V}$- equivalence implies C^0-$\mathcal{A}$-equivalence. The case $n = 2$ also follows from the works of Prishlyak [10] and Alvarez-Birbrair- Costa-Fernandes [1].

However, it is not true in general that C^0-$\mathcal{K}$-equivalence implies C^0-$\mathcal{A}$-equivalence of functions. For any $n \geq 7$, King [7] gives examples of polynomials $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ with isolated singularity which are C^0-$\mathcal{V}$-equivalent, but not C^0- $\mathcal{A}$-equivalent. This combined with the result of Nishimura [9] that C^0-$\mathcal{V}$-equivalence of smooth functions with isolated singularity implies C^0-$\mathcal{K}$-equivalence provides the desired counterexample. The reason of these counterexamples is that the corresponding zero-sets are homeomorphic but not PL homeomorphic.

In this work we consider the PL classification of polynomial germs $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ with no necessarily isolated singularity. The main result is the following theorem.

Theorem 1.1. *If two polynomials $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ are PL-$\mathcal{K}$-equivalent, then they are PL-$\mathcal{A}$-equivalent.*

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As a consequence, we deduce that for $n = 2, 3$, if two polynomials $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ are C^0-$\mathcal{K}$-equivalent, then they are C^0-$\mathcal{A}$-equivalent. We also obtain another proof of the Finiteness Theorem of Fukuda [5].

2. $\mathcal{A}$-equivalence and triangulation

Let $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be two smooth (or C^∞) map germs. We say that:

- f and g are *$\mathcal{A}$-equivalent* if there exist diffeomorphisms $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ and $k : (\mathbb{R}^p, 0) \to (\mathbb{R}^p, 0)$ such that the following diagram commutes

$$\begin{array}{ccc} (\mathbb{R}^n, 0) & \xrightarrow{f} & (\mathbb{R}^p, 0) \\ h\big\downarrow & & k\big\downarrow \\ (\mathbb{R}^n, 0) & \xrightarrow{g} & (\mathbb{R}^p, 0) \end{array}$$

- f and g are *$\mathcal{K}$-equivalent* if there exist diffeomorphisms $H : (\mathbb{R}^n \times \mathbb{R}^p, 0) \to (\mathbb{R}^n \times \mathbb{R}^p, 0)$ and $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ such that $H(\mathbb{R}^n \times \{0\}) = \mathbb{R}^n \times \{0\}$ and the following diagram is commutative:

$$\begin{array}{ccccc} (\mathbb{R}^n, 0) & \xrightarrow{(\mathrm{id}, f)} & (\mathbb{R}^n \times \mathbb{R}^p, 0) & \xrightarrow{\pi_n} & (\mathbb{R}^n, 0) \\ h\big\downarrow & & H\big\downarrow & & h\big\downarrow \\ (\mathbb{R}^n, 0) & \xrightarrow{(\mathrm{id}, g)} & (\mathbb{R}^n \times \mathbb{R}^p, 0) & \xrightarrow{\pi_n} & (\mathbb{R}^n, 0) \end{array}$$

  where $\mathrm{id} : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ is the identity mapping of $\mathbb{R}^n$ and $\pi_n : (\mathbb{R}^n \times \mathbb{R}^p, 0) \to (\mathbb{R}^n, 0)$ is the canonical projection germ.
- f and g are *$\mathcal{V}$-equivalent* if there exist a diffeomorphism $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ such that $h(f^{-1}(0)) = g^{-1}(0)$.

In these definitions, if we have homeomorphisms (resp. PL homeomorphisms, semialgebraic homeomorphisms) instead of diffeomorphisms, we say that f and g are C^0-$\mathcal{A}$, C^0-$\mathcal{K}$ or C^0-$\mathcal{V}$-equivalent (resp. PL-$\mathcal{A}$, PL-$\mathcal{K}$ or PL- $\mathcal{V}$-equivalent, semialgebraically $\mathcal{A}$, $\mathcal{K}$ or $\mathcal{V}$-equivalent).

We start with a lemma about PL $\mathcal{A}$-equivalence of PL functions. We consider a PL function $f : X \to \mathbb{R}$, where X is any polyhedron.

Lemma 2.1. *Let $f_1 : X_1 \to \mathbb{R}$ and $f_2 : X_2 \to \mathbb{R}$ be two PL functions. Assume there is a PL homeomorphism $h : X_1 \to X_2$ such that $h(f_1^{-1}(0)) = f_2^{-1}(0)$ and moreover the sign of $f_1(x)f_2(h(x))$ is constant on $X_1 \setminus f_1^{-1}(0)$. Then there are neighbourhoods N_i of $f_i^{-1}(0)$ on X_i and V_i of 0 in $\mathbb{R}$ such that the restrictions $f_i : N_i \to V_i$ are PL-$\mathcal{A}$-equivalent.*

Proof. We assume, for simplicity, that the sign of $f_1(x)$ is equal to the sign of $f_2(h(x))$ on $X_1 \setminus f_1^{-1}(0)$, the other case being analogous.

After subdivision, we can take simplicial complexes K_1, K_2, L_1, L_2 with $|K_i| = X_i$ and $|L_i| = \mathbb{R}$, such that $f_1 : K_1 \to L_1$, $f_2 : K_2 \to L_2$ are simplicial maps and $h : K_1 \to K_2$ is a simplicial isomorphism.

We fix neighbourhoods in the target $V_1 = \mathrm{Star}(0, L_1)$ with vertices $a_1 < 0 < b_1$ and $V_2 = \mathrm{Star}(0, L_2)$ with vertices $a_2 < 0 < b_2$. We also take the corresponding neighbourhoods in the source $N_1 = f_1^{-1}(V_1)$ and $N_2 = f_2^{-1}(V_2)$. We denote by $\beta : V_1 \to V_2$ the simplicial isomorphism given by $\beta(a_1) = a_2$ and $\beta(b_1) = b_2$.

We claim that $h(N_1) = N_2$ and that the following diagram is commutative:

$$\begin{array}{ccc} N_1 & \xrightarrow{f_1} & V_1 \\ {\scriptstyle h}\downarrow & & {\scriptstyle \beta}\downarrow \\ N_2 & \xrightarrow{f_2} & V_2. \end{array}$$

In fact, let $\sigma \in N_1$ be a simplex such that $f_1(\sigma) = \{0\}$. Then $\sigma \in f_1^{-1}(0)$ and hence $h(\sigma) \in f_2^{-1}(0) \subset N_2$. Moreover, $\beta(0) = 0$ and the above diagram is obviously commutative on σ.

Otherwise, we take a simplex $\sigma \in N_1$ such that $f_1(\sigma) \neq \{0\}$. Since f_1 is simplicial we must have either $f_1(\sigma) = [0, b_1]$ or $f_1(\sigma) = [a_1, 0]$. If $f_1(\sigma) = [0, b_1]$, then $f_2(h(\sigma)) = [0, b_2]$ by the initial assumption and thus, $h(\sigma) \in N_2$. On the other hand, given any vertex v of σ, if $f_1(v) = 0$ then $\beta(f_1(v)) = 0 = f_2(h(v))$ and if if $f_1(v) = b_1$ then again $\beta(f_1(v)) = b_2 = f_2(h(v))$. This shows that the diagram is also commutative on σ in this case. The other case is analogous. □

Proof of theorem 1.1. We first note that since f, g are polynomials, by Shiota Theorem [12], they are triangulable on a small enough neighbourhood of the origin. Hence we can choose triangulations:

$$\begin{array}{ccc} (\mathbb{R}^n, 0) & \xrightarrow{f} & (\mathbb{R}, 0) \\ {\scriptstyle \alpha_1}\downarrow & & {\scriptstyle \beta_1}\downarrow \\ (X_1, 0) & \xrightarrow{f_1} & (\mathbb{R}, 0), \end{array} \qquad \begin{array}{ccc} (\mathbb{R}^n, 0) & \xrightarrow{g} & (\mathbb{R}, 0) \\ {\scriptstyle \alpha_2}\downarrow & & {\scriptstyle \beta_2}\downarrow \\ (X_2, 0) & \xrightarrow{f_2} & (\mathbb{R}, 0), \end{array}$$

where X_i are polyhedra, $f_i : X_i \to \mathbb{R}$ are PL-maps and α_i, β_i are homeomorphisms.

Now, the hypothesis that f, g are PL $\mathcal{K}$-equivalent implies that there is a commutative diagram:

$$\begin{array}{ccccc} (X_1, 0) & \xrightarrow{(\mathrm{id}, f_1)} & (X_1 \times \mathbb{R}, 0) & \xrightarrow{\pi_1} & (X_1, 0) \\ {\scriptstyle h}\downarrow & & {\scriptstyle H}\downarrow & & {\scriptstyle h}\downarrow \\ (X_2, 0) & \xrightarrow{(\mathrm{id}, f_2)} & (X_2 \times \mathbb{R}, 0) & \xrightarrow{\pi_1} & (X_2, 0) \end{array}$$

where h, H are PL homeomorphisms, id is the identity mapping and π_1 is the projection onto the first factor.

We write $H(x, y) = (h(x), \theta_x(y))$, then we have that $\theta_x : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ is a family of homeomorphisms depending continuously on x in a neighbourhood of the origin. In particular, we have that either: for any x, θ_x is always increasing, or for any x, θ_x is always decreasing (depending on the local degree of H).

With this notation, the $\mathcal{K}$-equivalence is written as

$$\theta_x(f_1(x)) = f_2(h(x)), \quad \forall x \in X_1.$$

Then we have that $h(f_1^{-1}(0)) = f_2^{-1}(0)$ and that the sign of $f_1(x) f_2(h(x))$ is constant on $X_1 \setminus f_1^{-1}(0)$. The result follows now from lemma 2.1. □

We give now some interesting consequences of this theorem. The first one follows from the fact that in dimensions $n = 2, 3$, any homeomorphism between semialgebraic subsets can be triangulated. Therefore, if two polynomial germs are C^0-$\mathcal{K}$-equivalent, then they are PL-$\mathcal{K}$-equivalent.

Corollary 2.2. *Let $n = 2, 3$, if two polynomials $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ are C^0-$\mathcal{K}$-equivalent, then they are C^0-$\mathcal{A}$-equivalent.*

The second corollary is for the case of isolated singularity. In this case, we have that the C^0-$\mathcal{V}$-equivalence implies the C^0-$\mathcal{K}$-equivalence (see [9]) and the same is true in the PL category.

Corollary 2.3. *If two polynomials $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ with isolated singularity are PL-$\mathcal{V}$-equivalent, then they are PL-$\mathcal{A}$-equivalent.*

Another easy consequence is that the theorem is also true if we consider semialgebraic homeomorphisms instead of PL homeomorphisms. This follows from the fact that any semialgebraic homeomorphism of semialgebraic sets can be triangulated.

Corollary 2.4. *If two polynomials $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ are semialgebraically $\mathcal{K}$-equivalent, then they are semialgebraically $\mathcal{A}$-equivalent.*

Finally, we give another proof of the Finitness Theorem of Fukuda [5] about C^0-$\mathcal{A}$-equivalence of polynomial function germs of a given degree (see also Benedetti-Shiota [2]). It is deduced from Hardt work [6] that there is a finite number of topological types of zero-sets up to semialgebraic homeomorphisms. Moreover, there is a finite number of possible choices for the sign of the function on the complement of the zero-set. By theorem 1.1, we have a finite number of C^0-$\mathcal{A}$-classes.

Corollary 2.5. *There is a finite number of C^0-$\mathcal{A}$-classes in the space of all polynomial map germs $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ of degree $\leq k$.*

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ZERO-DIMENSIONAL SYMPLECTIC ISOLATED COMPLETE INTERSECTION SINGULARITIES

WOJCIECH DOMITRZ

Abstract. We study the local symplectic algebra of the 0-dimensional isolated complete intersection singularities. We use the method of algebraic restrictions to classify these symplectic singularities. We show that there are non-trivial symplectic invariants in this classification.

1. Introduction

The problem of symplectic classification of singular varieties was introduced by V. I. Arnold in [A1]. Arnold showed that the A_{2k} singularity of a planar curve (the orbit with respect to the standard $\mathcal{A}$-equivalence of parameterized curves) split into exactly $2k+1$ symplectic singularities (orbits with respect to the symplectic equivalence of parameterized curves). He distinguished different symplectic singularities by different orders of tangency of the parameterized curve with the *nearest* smooth Lagrangian submanifold. Arnold posed a problem of expressing these new symplectic invariants in terms of the local algebra's interaction with the symplectic structure and he proposed to call this interaction the **local symplectic algebra**. This problem was studied by many authors mainly in the case of singular curves.

In [IJ1] G. Ishikawa and S. Janeczko classified symplectic singularities of curves in the 2-dimensional symplectic space. A symplectic form on a 2-dimensional manifold is a special case of a volume form on a smooth manifold. The generalization of results in [IJ1] to volume-preserving classification of singular varieties and maps in arbitrary dimensions was obtained in [DR]. The orbit of the action of all diffeomorphism-germs agrees with the volume-preserving orbit in the $\mathbb{C}$-analytic category for germs which satisfy a special weak form of quasi-homogeneity e.g. the weak quasi-homogeneity of varieties is a quasi-homogeneity with non-negative weights $\lambda_i \geq 0$ and $\sum_i \lambda_i > 0$.

P. A. Kolgushkin classified stably simple symplectic singularities of parameterized curves in the $\mathbb{C}$-analytic category ([K]).

In [DJZ2] the local symplectic algebra of singular quasi-homogeneous subsets of a symplectic space was explained by the algebraic restrictions of the symplectic form to these subsets. The generalization of the Darboux-Givental theorem ([AG]) to germs of arbitrary subsets of the symplectic space obtained in [DJZ2] reduces the problem of symplectic classification of germs of quasi-homogeneous subsets to the problem of classification of algebraic restrictions of symplectic forms to these subsets. For non-quasi-homogeneous subsets there is one more cohomological invariant apart of the algebraic restriction ([DJZ2], [DJZ1]). The method of algebraic restrictions is a very powerful tool to study the local symplectic algebra of 1-dimensional singular analytic varieties since the space of algebraic restrictions of closed 2-forms to a 1-dimensional singular analytic variety is finite-dimensional ([D]). By this method complete symplectic classifications

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of the $A-D-E$ singularities of planar curves and the S_5 singularity were obtained in [DJZ2]. These results were generalized to other 1-dimensional isolated complete intersection singularities: the S_μ symplectic singularities for $\mu > 5$ in [DT1], the $T_7 - T_8$ symplectic singularities in [DT2] and the $W_8 - W_9$ symplectic singularities in [T].

In this paper we show that some non-trivial symplectic invariants appear not only in the case of singular curves but also in the case of multiple points. We consider the symplectic classification of the 0-dimensional isolated complete intersection singularities (ICISs) in the symplectic space $(\mathbb{C}^{2n}, \omega)$. We need to introduce a symplectic V-equivalence to study this problem since we consider the ideals of function-germs that have not got the property of zeros.

We recall that ω is a $\mathbb{C}$-analytic symplectic form on $\mathbb{C}^{2n}$ if ω is a $\mathbb{C}$-analytic nondegenerate closed 2-form, and $\Phi : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ is a symplectomorphism if Φ is a $\mathbb{C}$-analytic diffeomorphism and $\Phi^*\omega = \omega$.

Definition 1.1. Let $f, g : (\mathbb{C}^{2n}, 0) \to (\mathbb{C}^k, 0)$ be $\mathbb{C}$-analytic map-germs on the symplectic space $(\mathbb{C}^{2n}, \omega)$. f, g are **symplectically V-equivalent** if there exist a symplectomorphism-germ $\Phi : (\mathbb{C}^{2n}, 0, \omega) \to (\mathbb{C}^{2n}, 0, \omega)$ and a $\mathbb{C}$-analytic map-germ $M : (\mathbb{C}^{2n}, 0) \to GL(k, \mathbb{C})$ such that such that $f \circ \Phi = M \cdot g$.

If $\Phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^m, 0)$ is a $\mathbb{C}$-analytic map-germ then for an ideal I in the ring of $\mathbb{C}$-analytic function-germs on $\mathbb{C}^m$ we denote by Φ^*I the following ideal $\{f \circ \Phi : f \in I\}$ in the ring of $\mathbb{C}$-analytic function-germs on $\mathbb{C}^n$. The (symplectic) V-equivalence of map-germs $f = (f_1, \cdots, f_k), g = (g_1, \cdots, g_k) : (\mathbb{C}^{2n}, 0) \to (\mathbb{C}^k, 0)$ corresponds to the following (symplectic) equivalence of finitely-generated ideals $< f_1, \cdots, f_k >$ and $< g_1, \cdots, g_k >$ (see [AVG]).

Definition 1.2. Ideals $< f_1, \cdots, f_k >$ and $< g_1, \cdots, g_k >$ of $\mathbb{C}$-analytic function-germs at 0 on the symplectic space $(\mathbb{C}^{2n}, \omega)$ are **symplectically equivalent** if there exists a symplectomorphism-germ $\Phi : (\mathbb{C}^{2n}, 0, \omega) \to (\mathbb{C}^{2n}, 0, \omega)$ such that $\Phi^* < f_1, \cdots, f_k > = < g_1, \cdots, g_k >$.

In this paper we present the complete symplectic classification of the $I_{a,b}$, I_{2a+1}, I_{2a+4}, I_{a+5}, I^*_{10} singularities. For $n = 1$ all V-orbits coincide with symplectic V-orbits. The situation for $n \geq 2$ is different: the $I_{a,b}$ singularities split into two symplectic V-orbits, the I_{2a+1}, I_{2a+4}, I_{a+5} singularities split into three symplectic orbits and finally I^*_{10} singularity splits into four symplectic V-orbits. The symplectic V-orbits of a map $f = (f_1, \cdots, f_{2n})$ are distinguished by the order of vanishing of a pullback of the germ of the symplectic form to a $\mathbb{C}$-analytic non-singular submanifold M of the minimal dimension such that the ideal of $\mathbb{C}$-analytic function-germs vanishing M is contained in the ideal $< f_1, \cdots, f_{2n} >$ (see Definition 3.2).

To obtain these results we need some reformulation and modification of the method of algebraic restrictions. We present it in Section 2. In Section 3 we give the definitions of discrete symplectic invariants which completely distinguish symplectic V-singularities considered in this paper. We recall basic facts on the classification of V-simple maps in Section 4. In Section 5 we prove the symplectic V-classification theorem for 0-dimensional ICISs (Theorem 5.1).

2. The method of algebraic restrictions for the symplectic V-equivalence.

In this section we present basic facts on the method of algebraic restrictions adapted to the case of the symplectic V-equivalence. The proofs of all results are small modifications of the proofs of analogous results in [DJZ2].

Given a germ at 0 of a non-singular $\mathbb{C}$-analytic submanifold M of $\mathbb{C}^m$ denote by $\Lambda^p(M)$ the space of all germs at 0 of $\mathbb{C}$-analytic differential p-forms on M. By $\mathcal{O}(M)$ denote the ring of $\mathbb{C}$-analytic function-germs on M at 0. Given an ideal I in $\mathcal{O}(M)$ introduce the following subspace of $\Lambda^p(M)$:

$$\mathcal{A}^p_0(I, M) = \{\alpha + d\beta : \quad \alpha \in I\Lambda^p(M), \ \beta \in I\Lambda^{p-1}(M).\}$$

The relation $\omega \in I\Lambda^p(M)$ means that $\omega = \sum_{i=1}^k f_i\alpha_i$, where $\alpha_i \in \Lambda^p(M)$ and $f_i \in I$ for $i = 1, ..., k$.

Definition 2.1. Let I be an ideal of $\mathcal{O}(M)$ and let $\omega \in \Lambda^p(M)$. The **algebraic restriction** of ω to I is the equivalence class of ω in $\Lambda^p(M)$, where the equivalence is as follows: ω is equivalent to $\widetilde{\omega}$ if $\omega - \widetilde{\omega} \in \mathcal{A}_0^p(I, M)$.

Notation. The algebraic restriction of the germ of a p-form ω on M to the ideal I in $\mathcal{O}(M)$ will be denoted by $[\omega]_I$. Writing $[\omega]_I = 0$ (or saying that ω has zero algebraic restriction to I) we mean that $[\omega]_I = [0]_I$, i.e. $\omega \in A_0^p(I, M)$.

Definition 2.2. Two algebraic restrictions $[\omega]_I$ and $[\widetilde{\omega}]_{\widetilde{I}}$ are called **diffeomorphic** if there exists the germ of a diffeomorphism $\Phi : M \to \widetilde{M}$ such that $\Phi^*(\widetilde{I}) = I$ and $[\Phi^*\widetilde{\omega}]_I = [\omega]_I$.

Definition 2.3. The germ of a function, a differential k-form, or a vector field α on $(\mathbb{C}^m, 0)$ is **quasi-homogeneous** in a coordinate system $(x_1, \cdots, x_m)$ on $(\mathbb{C}^m, 0)$ with positive integer weights $(\lambda_1, \cdots, \lambda_m)$ if $\mathcal{L}_E\alpha = \delta\alpha$, where $E = \sum_{i=1}^m \lambda_i x_i \frac{\partial}{\partial x_i}$ is the germ of the **Euler vector field** on $(\mathbb{C}^m, 0)$ and the integer δ is called the quasi-degree.

It is easy to show that α is quasi-homogeneous in a coordinate system $(x_1, \cdots, x_m)$ with weights $(\lambda_1, \cdots, \lambda_m)$ if and only if $F_t^*\alpha = t^\delta\alpha$, where

$$F_t(x_1, \cdots, x_m) = (t^{\lambda_1}x_1, \cdots, t^{\lambda_m}x_m). \tag{2.1}$$

Definition 2.4. A finitely generated ideal I of $\mathcal{O}(\mathbb{C}^m)$ is **quasi-homogeneous** if there exist generators of I which are quasi-homogeneous in the same coordinate system $(x_1, \cdots, x_m)$ on $\mathbb{C}^m$ with the same positive integer weights $(\lambda_1, \cdots, \lambda_m)$.

A map-germ $f = (f_1, \cdots, f_k) : (\mathbb{C}^m, 0) \to (\mathbb{C}^k, 0)$ is **quasi-homogeneous** if function-germs $f_1, \cdots, f_k$ are quasi-homogeneous in the same coordinate system $(x_1, \cdots, x_m)$ on $\mathbb{C}^m$ with the same positive integer weights $(\lambda_1, \cdots, \lambda_m)$.

To prove the generalization of Darboux-Givental theorem suitable for the symplectic V-equivalence of maps or the symplectic equivalence of ideals of function-germs we need the following version of the Relative Poincaré Lemma.

Lemma 2.5. *Let I be a finitely generated quasi-homogeneous ideal in $\mathcal{O}(\mathbb{C}^m)$. If $\omega \in I\Lambda^p(\mathbb{C}^m)$ is closed than there exists $\alpha \in I\Lambda^{p-1}(\mathbb{C}^m)$ such that $\omega = d\alpha$.*

Proof. We use the method described in [DJZ1]. We can find a coordinate system $(x_1, \cdots, x_m)$ on $(\mathbb{C}^m, 0)$ and positive integer weights $(\lambda_1, \cdots, \lambda_m)$ and quasi-homogeneous function-germs $f_1, \cdots, f_k \in \mathcal{O}(\mathbb{C}^m)$ (in this coordinate systems with these weights) such that $I =< f_1, \cdots, f_k >$. Let δ_i be a quasi-degree of f_i for $i = 1, \cdots, k$.

Let F_t be a map defined in (2.1) and let V_t be a vector field along F_t for $t \in [0; 1]$ such that $V_t \circ F_t = F_t'$.

Then we have $F_0^*\omega = 0$ and it implies that

$$\omega = F_1^*\omega - F_0^*\omega = \int_0^1 (F_t^*\omega)'dt = \int_0^1 F_t^* d(V_t\rfloor\omega)dt = d\left(\int_0^1 F_t^*(V_t\rfloor\omega)dt\right).$$

Let $\alpha = \int_0^1 F_t^*(V_t\rfloor\omega)dt$, then $\omega = d\alpha$. But ω belongs to $I\Lambda^p(\mathbb{C}^m)$. It implies that there exist germs of p-forms β_i in $\Lambda^p(\mathbb{C}^m)$ for $i = 1, \cdots, k$ such that $\omega = \sum_{i=1}^k f_i\beta_i$. So we have that

$$\alpha = \int_0^1 F_t^*(V_t\rfloor \sum_{i=1}^k f_i\beta_i)dt = \sum_{i=1}^k f_i \int_0^1 t^{\delta_i} F_t^*(V_t\rfloor\beta_i)dt.$$

Thus α belongs to $I\Lambda^{p-1}(\mathbb{C}^m)$. $\square$

The method of algebraic restrictions applied to finitely-generated quasi-homogeneous ideals is based on the following theorem.

Theorem 2.6 (a modification of Theorem A in [DJZ2])**.** *Let I be a finitely generated quasi-homogeneous ideal in $\mathcal{O}(\mathbb{C}^{2n})$.*

(1) *If ω_0, ω_1 are germs at 0 of symplectic forms on $\mathbb{C}^{2n}$ with the same algebraic restriction to I then there exists a $\mathbb{C}$-analytic diffeomorphism-germ Φ of $\mathbb{C}^{2n}$ at 0 of the form $\Phi(x) = (x_1 + \phi_1(x), \cdots, x_{2n} + \phi_{2n}(x))$, where $\phi_i \in I$ for $i = 1, \cdots, 2n$, such that $\Phi^*\omega_1 = \omega_0$.*
(2) *$\mathbb{C}$-analytic quasi-homogeneous map-germs $f = (f_1, \cdots, f_k), g = (g_1, \cdots, g_k) : (\mathbb{C}^{2n}, 0) \to (\mathbb{C}^k, 0)$ on the symplectic space $(\mathbb{C}^{2n}, \omega)$ are symplectically V-equivalent if and only if algebraic restrictions $[\omega]_{<f_1,\cdots,f_k>}$ and $[\omega]_{<g_1,\cdots,g_k>}$ are diffeomorphic.*

Remark 2.7. It is obvious that if $\Phi(x) = (x_1 + \phi_1(x), \cdots, x_{2n} + \phi_{2n}(x))$ where $\phi_i \in I$ for $i = 1, \cdots, 2n$ then $\Phi^* I = I$

A proof of Theorem 2.6 can be obtain by a small modification of the proof of Theorem A in [DJZ2]. One only needs Lemma 2.5 and the following fact.

Lemma 2.8. *Let I be a finitely generated ideal in $\mathcal{O}(\mathbb{C}^m)$. Let $X_t = \sum_{i=1}^m f_{i,t} \frac{\partial}{\partial x_i}$ for $t \in [0;1]$ be a family of germs of $\mathbb{C}$-analytic vector fields on $\mathbb{C}^m$ such that $f_{i,t} \in I$ for $i = 1, \cdots, m$.*

If Φ_t for $t \in [0,1]$ is a family of diffeomorphism-germs of $(\mathbb{C}^m, 0)$ such that

$$\frac{d}{dt}\Phi_t = X_t \circ \Phi_t \tag{2.2}$$

then

$$\Phi_t(x) = (x_1 + \phi_{1,t}(x), \cdots, x_{2n} + \phi_{2n,t}(x)), \tag{2.3}$$

where $\phi_{i,t} \in I$ for $i = 1, \cdots, 2n$.

A sketch of the proof. The map $t \mapsto \Phi_t(x)$ is a solution of ODE $\frac{dy}{dt} = X_t(y)$ with the initial condition $y(0) = x$. So $\Phi_t(x)$ can be obtained as a limit $\lim_{n\to\infty} T^n\Psi$ where $\Psi(t, x) \equiv x$ and $(T\Psi)(t, x) = x + \int_0^t X_s(\Psi(s, x))ds$ is the Picard's operator. It is easy to see that if Ψ has the form (2.3) then $T\Psi$ has the form (2.3) too. The ideal I is finitely generated. Thus Φ_t has also this form. □

Theorem 2.6 reduces the problem of symplectic classification of quasi-homogeneous ideals to the problem of classification of the algebraic restrictions of the germ of the symplectic form to quasi-homogeneous ideals.

The meaning of the zero algebraic restriction is explained by the following theorem.

Theorem 2.9 (a modification of Theorem **B** in [DJZ2])**.** *A finitely generated quasi-homogeneous ideal I of $\mathcal{O}(\mathbb{C}^{2n})$ contains the ideal of $\mathbb{C}$-analytic function-germs vanishing on the germ of a non-singular Lagrangian submanifold of the symplectic space $(\mathbb{C}^{2n}, \omega)$ if and only if the symplectic form ω has zero algebraic restriction to I.*

We now formulate the modifications of basic properties of algebraic restrictions ([DJZ2]). First we can reduce the dimension of the manifold due to the following propositions.

If the ideal I in $\mathcal{O}(\mathbb{C}^m)$ contains an ideal $I(M)$ of function-germs vanishing on a non-singular submanifold $M \subset \mathbb{C}^m$ then the classification of the algebraic restrictions to I of p-forms on $\mathbb{C}^m$ reduces to the classification of the algebraic restrictions to $I|_M = \{f|_M : f \in I\}$ of p-forms on M. At first note that the algebraic restrictions $[\omega]_I$ and $[\omega|_{TM}]_{I|_M}$ can be identified:

Proposition 2.10. *Let I be an ideal in $\mathcal{O}(\mathbb{C}^m)$ which contains an ideal of function-germs vanishing on a non-singular submanifold $M \subset \mathbb{C}^m$ and let ω_1, ω_2 be germs of p-forms on $\mathbb{C}^m$. Then $[\omega_1]_I = [\omega_2]_I$ if and only if $\left[\omega_1|_{TM}\right]_{I|_M} = \left[\omega_2|_{TM}\right]_{I|_M}$.*

The following, less obvious statement, means that the *orbits* of the algebraic restrictions $[\omega]_I$ and $[\omega|_{TM}]_{I|_M}$ also can be identified.

Proposition 2.11. *Let I_1, I_2 be ideals in the ring $\mathcal{O}(\mathbb{C}^m)$, which contain $I(M_1)$ and $I(M_2)$ respectively, where M_1, M_2 are equal-dimensional non-singular submanifolds. Let ω_1, ω_2 be two germs of p-forms. The algebraic restrictions $[\omega_1]_{I_1}$ and $[\omega_2]_{I_2}$ are diffeomorphic if and only if the algebraic restrictions $\left[\omega_1|_{TM_1}\right]_{I_1|_{M_1}}$ and $\left[\omega_2|_{TM_2}\right]_{I_2|_{M_2}}$ are diffeomorphic.*

To calculate the space of algebraic restrictions of germs of 2-forms we will use the following obvious properties.

Proposition 2.12. *If $\omega \in \mathcal{A}_0^k(I, \mathbb{C}^{2n})$ then $d\omega \in \mathcal{A}_0^{k+1}(I, \mathbb{C}^{2n})$ and $\omega \wedge \alpha \in \mathcal{A}_0^{k+p}(I, \mathbb{C}^{2n})$ for any germ of $\mathbb{C}$-analytic p-form α on $\mathbb{C}^{2n}$.*

Then we need to determine which algebraic restrictions of closed 2-forms are realizable by symplectic forms. This is possible due to the following fact.

Proposition 2.13. *Let I be an ideal of $\mathcal{O}(\mathbb{C}^{2n})$. Let r be the minimal dimension of non-singular submanifolds M of $\mathbb{C}^{2n}$ such that I contains the ideal $I(M)$. The algebraic restriction $[\theta]_I$ of the germ of a closed 2-form θ is realizable by the germ of a symplectic form on $\mathbb{C}^{2n}$ if and only if $rank(\theta|_{T_0M}) \geq 2r - 2n$.*

3. Discrete symplectic invariants.

We use discrete symplectic invariants to distinguish symplectic singularity classes. We modify definitions of these invariants introduced in [DJZ2] for the symplectic V-equivalence.

The first invariant is a symplectic multiplicity ([DJZ2]) introduced in [IJ1] as a symplectic defect of a curve.

Let $f : (\mathbb{C}^{2n}, 0) \to (\mathbb{C}^k, 0)$ be the germ of a $\mathbb{C}$-analytic map on the symplectic space $(\mathbb{C}^{2n}, \omega)$.

Definition 3.1. The **symplectic multiplicity** $\mu_{sympl}(f)$ of f is the codimension of the symplectic V-orbit of f in the V-orbit of f.

The second invariant is the index of isotropy [DJZ2].

Definition 3.2. The **index of isotropy** $\iota(f)$ of $f = (f_1, \cdots, f_k)$ is the maximal order of vanishing of the 2-forms $\omega|_{TM}$ over all smooth submanifolds M such that the ideal $< f_1, \cdots, f_k >$ contains $I(M)$.

These invariants can be described in terms of algebraic restrictions.

Proposition 3.3 ([DJZ2])**.** *The symplectic multiplicity of the germ of a quasi-homogeneous map $f = (f_1, \cdots, f_k)$ on the symplectic space $(\mathbb{C}^{2n}, \omega)$ is equal to the codimension of the orbit of the algebraic restriction $[\omega]_{<f_1, \cdots, f_k>}$ with respect to the group of diffeomorphism-germs preserving the ideal $< f_1, \cdots, f_k >$ in the space of the algebraic restrictions of closed 2-forms to $< f_1, \cdots, f_k >$.*

Proposition 3.4 ([DJZ2])**.** *The index of isotropy of the germ of a quasi-homogeneous map $f = (f_1, \cdots, f_k)$ on the symplectic space $(\mathbb{C}^{2n}, \omega)$ is equal to the maximal order of vanishing of closed 2-forms representing the algebraic restriction $[\omega]_{<f_1, \cdots, f_k>}$.*

We will use these invariants to distinguish symplectic singularities.

4. V-simple maps

We recall some results on classification of V-simple germs (for details see [AVG]).

Definition 4.1. The germ $f:(\mathbb{C}^m,0)\to(\mathbb{C}^n,0)$ is said be V**-simple** if its k-jet, for any k, has a neighborhood in the small jet space $J^k_{0,0}(\mathbb{C}^m,\mathbb{C}^n)$ that intersects only a finite number of V-equivalence classes (bounded by a constant independent of k).

Definition 4.2. The p**-parameter suspension of the map-germ** $f:(\mathbb{C}^m,0)\to(\mathbb{C}^n,0)$ is the map germ

$$F:(\mathbb{C}^m\times\mathbb{C}^p,0)\ni(y,z)\mapsto(f(y),z)\in(\mathbb{C}^n\times\mathbb{C}^p,0).$$

Theorem 4.3 (see [AVG])**.** *The V-simple map-germs $(\mathbb{C}^m,0)\to(\mathbb{C}^n,0)$ with $m\geq n$ belong, up to V-equivalence and suspension, to one of the three lists: the $A-D-E$ singularities of map-germs $\mathbb{C}^m\to\mathbb{C}$ (hypersurfaces with an isolated singularity), $S-T-U-W-Z$ singularities of map-germs $\mathbb{C}^3\to\mathbb{C}^2$ (1-dimensional ICISs) and singularities of map-germs $\mathbb{C}^2\to\mathbb{C}^2$ (0-dimensional ICISs) presented in Table 1.*

Notation	Normal form	Restrictions
$I_{a,b}$	(yz,y^a+z^b)	$a\geq b\geq 2$
I_{2a+1}	(y^2+z^3,z^a)	$a\geq 3$
I_{2a+4}	(y^2+z^3,yz^a)	$a\geq 2$
I_{a+5}	(y^2+z^a,yz^2)	$a\geq 4$
I^*_{10}	(y^2,z^4)	-

Table 1. V-simple map-germs $\mathbb{C}^2\to\mathbb{C}^2$.

The normal forms in Table 1 were obtained in [G] by M. Giusti.

5. Symplectic 0-dimensional ICISs

We use the method of algebraic restrictions to obtain a complete classification of singularities presented in Table 1.

Theorem 5.1. *Any map-germ $(\mathbb{C}^{2n},0)\to(\mathbb{C}^{2n},0)$ from the symplectic space $(\mathbb{C}^{2n},\sum_{i=1}^n dp_i\wedge dq_i)$ which is V-equivalent (up to a suitable suspension) to one of the normal forms in Table 1 is symplectically V-equivalent to one and only one of the following normal forms presented in Table 2*

Proof. In the case $n=1$ the proof follows from results in [DR] where it was proved that for quasi-homogeneous singularities in the $\mathbb{C}$-analytic category V-orbits coincide with volume-preserving V-orbits. For general n we present the proof in the case of the I^*_{10} singularity where there are 4 different symplectic singularity classes, and in the case of the I_{a+5} singularity. The proofs in other cases are very similar.

For the I^*_{10} singularity we calculate the space of algebraic restrictions of 2-forms to the ideal $I=<y^2,z^4,x_1,\cdots,x_{2n-2}>$. The ideal generated by $x_1,\cdots,x_{2n-2}$ is contained in I. So by Proposition 2.10 we may consider the following ideal $J=I|_{\{x_1=\cdots=x_{2n-2}=0\}}=<y^2,z^4>$ in the ring $\mathcal{O}(\mathbb{C}^2)$. By Proposition 2.12 germs of 1-forms $d(1/2y^2)=ydy, d(1/4z^4)=z^3dz$ and germs of 2-forms $ydy\wedge dz, z^3dy\wedge dz$ have zero algebraic restriction to J. So any algebraic

Symplectic class	*Normal forms*	*cod*	μ_{sympl}	i
$I^0_{a,b}$, $(n \geq 1)$	$(p_1q_1, p_1^a + q_1^b, p_2, q_2, \cdots, p_n, q_n)$	0	0	0
$I^1_{a,b}$, $(n \geq 2)$	$(p_1p_2, p_1^a + p_2^b, q_1, q_2, p_3, q_3, \cdots, p_n, q_n)$	1	1	∞
I^0_{2a+1}, $(n \geq 1)$	$(p_1^2 + q_1^3, q_1^a, p_2, q_2, \cdots, p_n, q_n)$	0	0	0
I^1_{2a+1}, $(n \geq 2)$	$(p_1^2 + p_2^3, p_2^a, q_1, q_2 + p_1p_2, p_3, q_3, \cdots, p_n, q_n)$	1	1	1
I^2_{2a+1}, $(n \geq 2)$	$(p_1^2 + p_2^3, p_2^a, q_1, q_2, p_3, q_3, \cdots, p_n, q_n)$	2	2	∞
I^0_{2a+4}, $(n \geq 1)$	$(p_1^2 + q_1^3, p_1q_1^a, p_2, q_2, \cdots, p_n, q_n)$	0	0	0
I^1_{2a+4}, $(n \geq 2)$	$(p_1^2 + p_2^3, p_1p_2^a, q_1, q_2 + p_1p_2, p_3, q_3, \cdots, p_n, q_n)$	1	1	1
I^2_{2a+4}, $(n \geq 2)$	$(p_1^2 + p_2^3, p_1p_2^a, q_1, q_2, p_3, q_3, \cdots, p_n, q_n)$	2	2	∞
I^0_{a+5}, $(n \geq 1)$	$(p_1^2 + q_1^a, p_1q_1^2, p_2, q_2, \cdots, p_n, q_n)$	0	0	0
I^1_{a+5}, $(n \geq 2)$	$(p_1^2 + p_2^a, p_1p_2^2, q_1, q_2 + p_1p_2, p_3, q_3, \cdots, p_n, q_n)$	1	1	1
I^1_{a+5}, $(n \geq 2)$	$(p_1^2 + p_2^a, p_1p_2^2, q_1, q_2, p_3, q_3, \cdots, p_n, q_n)$	2	2	∞
I^{*0}_{10}, $(n \geq 1)$	$(p_1^2, q_1^4, p_2, q_2, \cdots, p_n, q_n)$	0	0	0
I^{*1}_{10}, $(n \geq 2)$	$(p_1^2, p_2^4, q_1, q_2 + p_1p_2, p_3, q_3, \cdots, p_n, q_n)$	1	1	1
I^{*2}_{10}, $(n \geq 2)$	$(p_1^2, p_2^4, q_1, q_2 + p_1p_2^2, p_3, q_3, \cdots, p_n, q_n)$	2	2	2
I^{*3}_{10}, $(n \geq 2)$	$(p_1^2, p_2^4, q_1, q_2, p_3, q_3, \cdots, p_n, q_n)$	3	3	∞

TABLE 2. Classification of symplectic 0-dimensional isolated complete intersection singularities, *cod* – codimension of the classes; μ_{sympl}– symplectic multiplicity; i – index of isotropy.

restriction of the germ of a closed 2-forms to J can be presented in the following form $[\omega]_J = A[dy \wedge dz]_J + B[zdy \wedge dz]_J + C[z^2dy \wedge dz]_J$, where $A, B, C \in \mathbb{C}$.

If $A \neq 0$ then we obtain $\Phi^*[\omega]_J = [dy \wedge dz]_J$ by the diffeomorphism-germ of the form $\Phi(y, z) = (y, z(A + 1/2Bz + 1/3Cz^2))$. If $A = 0$ and $B \neq 0$ then we obtain $\Phi^*[\omega]_J = [zdy \wedge dz]_J$ by the diffeomorphism-germ of the form $\Phi(y, z) = (y, z\phi(z))$, where $\phi^2(z) = B + 2/3Cz$. If $A = B = 0$ and $C \neq 0$ then we obtain $\Phi^*[\omega]_J = [z^2dy \wedge dz]_J$ by the diffeomorphism-germ of the form $\Phi(y, z) = (Cy, z)$.

Since the minimal dimension r of the germ of a non-singular submanifold M such that $I(M) \subset I$ is 2 then by Proposition 2.13 for $n = 1$ only the algebraic restriction $[dy \wedge dz]_I$ is realizable by the germ of a symplectic form.

For $n > 1$ all algebraic restrictions are realizable by the following symplectic forms:

$$dy \wedge dz + \sum_{i=1}^{n-1} dx_{2i-1} \wedge dx_{2i}, \tag{5.1}$$

$$zdy \wedge dz + dy \wedge dx_1 + dz \wedge dx_2 + \sum_{i=2}^{n-1} dx_{2i-1} \wedge dx_{2i}, \tag{5.2}$$

$$z^2dy \wedge dz + dy \wedge dx_1 + dz \wedge dx_2 + \sum_{i=2}^{n-1} dx_{2i-1} \wedge dx_{2i}, \tag{5.3}$$

$$dy \wedge dx_1 + dz \wedge dx_2 + \sum_{i=2}^{n-1} dx_{2i-1} \wedge dx_{2i}. \tag{5.4}$$

By a simple change of coordinates we obtain the normal forms in Table 2.

For the I_{a+5} singularity the space algebraic restrictions of germs of closed 2-forms to the ideal $I =< y^2 + z^a, yz^2, x_1, \cdots, x_{2n-2} >$ can calculated in the same way. We obtain that any algebraic restriction of the germs of a closed 2-forms on $\mathbb{C}^2 = \{x_1 = \cdots = x_{2n-2} = 0\}$ to $J = I|_{\{x_1=\cdots=x_{2n-2}=0\}} =< y^2 + z^a, yz^2 >$ can be presented in the following form

$$[\omega]_J = A[dy \wedge dz]_J + B[zdy \wedge dz]_J, \tag{5.5}$$

where $A, B \in \mathbb{C}$.

First assume that $A \neq 0$. Let E denote the germ of the Euler vector field $ay\frac{\partial}{\partial y} + 2z\frac{\partial}{\partial y}$. Then it is easy to check that a flow Φ_t of the germ of a vector field $X = \frac{B}{(a+4)A}zE$ preserves J, $\mathcal{L}_X(Ady \wedge dz) = Bzdy \wedge dz$, $[\mathcal{L}_X(Bzdy \wedge dz)]_J = 0$. Therefore $\Phi_t^*[Ady \wedge dz + tBzdy \wedge dz]_J = [Ady \wedge dz]_J$ for $t \in [0;1]$ (see [D]). Finally by a linear change of coordinates of the form $(y, z) \mapsto (Cy, Dz)$, where for $C, D \in \mathbb{C}$ such that $C^2 = D^a$ and $CD = A$ we show that if $A \neq 0$ then the algebraic restriction (5.5) is diffeomorphic to $[dy \wedge dz]_J$. By a similar change of coordinates preserving J we show that if $A = 0$ and $B \neq 0$ then the algebraic restriction (5.5) is diffeomorphic to $[zdy \wedge dz]_J$. As in the previous case, for $n = 1$ only $[dy \wedge dz]_I$ can be realizable by the germ of a symplectic form . For $n \geq 2$ algebraic restrictions are realizable by (5.1), (5.2) and (5.4). Normal forms in Table 2 are obtained by an obvious change of coordinates. □

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MULTIPLICITIES OF DEGENERATIONS OF MATRICES AND MIXED VOLUMES OF CAYLEY POLYHEDRA

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Abstract. Using a certain Pick-type formula for the mixed volume of Cayley polyhedra, we compute the multiplicity of the isolated common zero of the maximal minors for a matrix of generic homogeneous polynomials of given degrees.

1. Introduction

The local version of D. Bernstein's formula [Ber] expresses the local degree of a germ of a proper analytic map in terms of the Newton polyhedra of its components, provided that the principal parts of its components are in general position (see Theorem 5). We generalize this formula to germs of matrix-valued functions.

Let $A:\mathbb{C}^m\to\mathbb{C}^{n\times k}$ be a germ of an analytic $(n\times k)$-matrix-valued function, where $n\leqslant k$ (we denote the space of all $(n\times k)$-matrices by $\mathbb{C}^{n\times k}$). If $\operatorname{rk}A(0)<n$ and $\operatorname{rk}A(x)=n$ for all $x\neq 0$, then $m\leqslant k-n+1$. Suppose that $m=k-n+1$ (in particular, if $n=1$, then this condition means that $A:\mathbb{C}^m\to\mathbb{C}^m$ is a germ of a proper analytic map). The intersection number $m(A)$ of the germ $A(\mathbb{C}^m)$ and the set of all degenerate matrices in $\mathbb{C}^{n\times k}$ is well defined, because the codimension of degenerate matrices in $\mathbb{C}^{n\times k}$ equals $k-n+1$. In particular, if $n=1$, then $m(A)$ equals the local degree of the map $A:\mathbb{C}^m\to\mathbb{C}^m$.

Definition 1. Let $A:\mathbb{C}^m\to\mathbb{C}^{n\times k}$ be a germ of an analytic $(n\times k)$-matrix-valued function, such that $m=k-n+1$, $\operatorname{rk}A(0)<n$ and $\operatorname{rk}A(x)=n$ for all $x\neq 0$. Then the intersection number $m(A)$ will be called *the multiplicity (of degeneration)* of the germ A.

We recall the relation of this number to algebraic and topological invariants, motivating our interest to it.

1. Relation to Buchsbaum-Rim multiplicities. In the notation of Definition 1, the multiplicity of the matrix A is equal to $\dim_{\mathbb{C}}\mathcal{O}_{\mathbb{C}^m,0}/\langle\text{maximal minors of }A\rangle$, where $\mathcal{O}_{\mathbb{C}^m,0}$ is the ring of germs of analytic functions on $\mathbb{C}^m$ near the origin. In particular, it equals the Buchsbaum-Rim multiplicity of the submodule of $\mathcal{O}^n_{\mathbb{C}^m,0}$, generated by the columns of A (see, for example, Proposition 2.3 in [G]).

2. Relation to characteristic classes. Let v_i be a holomorphic section of a vector bundle $\mathcal{I}$ of rank k on a smooth $(k-n+1)$-dimensional complex manifold M for $i=1,\ldots,n$. Suppose that there is a finite number of points $x\in M$ such that the vectors $v_1(x),\ldots,v_n(x)$ are linearly dependent. Denote the set of all such points by X. Near each point $x\in X$, choosing a local basis $s_1,\ldots,s_k$ in the bundle $\mathcal{I}$, one can represent v_i as a linear combination $v_i=a_{i,1}s_1+\ldots+a_{i,k}s_k$, where $a_{i,j}$ are the entries of an $(n\times k)$-matrix $A:M\to\mathbb{C}^{n\times k}$ defined near x. Denote the multiplicity of A by m_x. Then the Chern number $c_{k-n+1}(\mathcal{I}_q)\cdot[M]$ is equal to the sum of the multiplicities m_x over all points $x\in X$ (see, for example, [GH]).

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The aim of this paper is to present a formula for the multiplicity of a matrix A in terms of the Newton polyhedra of the entries of A, provided that the principal parts of the entries are in general position. In [Biv], a similar formula is given under the assumption that all the entries from the same row of the matrix A have the same Newton polyhedron. [E05] contains a general formula (see Theorem 23), which is somewhat indirect in the sense that one has to increase the dimension of polyhedra under consideration in order to formulate the answer. The aim of this paper is to simplify this answer combinatorially (see Theorem 7), so that no higher-dimensional polyhedra are involved.

In Sections 2 and 3, we present the formula for the multiplicity of a matrix and the condition of general position for the principal parts of the entries of a matrix, respectively. In Sections 5, this formula is deduced from Theorem 23, which expresses the multiplicity of a matrix in terms of the mixed volume of pairs of certain polyhedra (this notion is introduced in Section 4). This requires a formula for the mixed volume of Cayley polyhedra (Theorem 24, the proof given in Section 7), which follows from the Oda equality $(A \cap \mathbb{Z}^n) + (B \cap \mathbb{Z}^n) = (A + B) \cap \mathbb{Z}^n$ for some class of bounded lattice polyhedra $A, B \subset \mathbb{R}^n$ (see Section 6).

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2. Multiplicity in terms of Newton polyhedra

A *polyhedron* in $\mathbb{R}^n$ is the intersection of a finite number of closed half-spaces. A *face* of a polyhedron A is the intersection of A and the boundary of a closed half-space, containing A. Note that the empty set is a face of every polyhedron. *The Minkowski sum* of sets A and B in $\mathbb{R}^n$ is the set $A + B = \{a + b \,|\, a \in A,\, b \in B\}$. Note that $\varnothing + A = \varnothing$ for every A.

Definition 2. Let B_i be a face of a polyhedron $\Delta_i \subset \mathbb{R}^m$ for $i = 1, \ldots, k$. The collection of faces $(B_1, \ldots, B_k)$ is said to be *compatible*, if the sum $B_1 + \ldots + B_k$ is a non-empty bounded face of the sum $\Delta_1 + \ldots + \Delta_k$.

Denote the positive orthants of $\mathbb{R}^m$ and $\mathbb{Z}^m$ by $\mathbb{R}^m_+$ and $\mathbb{Z}^m_+$ respectively. For each point $a = (a_1, \ldots, a_m) \in \mathbb{Z}^m$, denote the monomial $x_1^{a_1} \ldots x_m^{a_m}$ by x^a.

Definition 3. *The Newton polyhedron* Δ_f of a germ of an analytic function $f = \sum_{a \in \mathbb{Z}^m_+} c_a x^a : \mathbb{C}^m \to \mathbb{C}$ is the convex hull of the union $\bigcup\limits_{a \,|\, c_a \neq 0} (a + \mathbb{R}^m_+)$.

Definition 4. *The restriction* $f|_B$ of a germ $f = \sum_{a \in \mathbb{Z}^m_+} c_a x^a$ to a bounded subset B of the Newton polyhedron Δ_f is the polynomial $\sum\limits_{a \in \mathbb{Z}^m \cap B} c_a x^a$. The restriction of f to the union of all bounded faces of Δ_f is called *the principal part* of f. The restriction to the empty set equals zero by definition.

The principal parts of the components of a map $f : \mathbb{C}^m \to \mathbb{C}^m$ form *the principal part* of f, and the principal parts of the entries of an $(n \times k)$-matrix $A : \mathbb{C}^m \to \mathbb{C}^{n \times k}$ form *the principal part* of A.

For a polyhedron $\Delta \subset \mathbb{R}^m_+$, denote the number of integer lattice points in the difference $\mathbb{R}^m_+ \setminus \Delta$ by $I(\Delta)$. Recall the local version of D. Bernstein's formula [Ber] (it can be deduced, for example from M. Oka's formula [O90]):

Theorem 5. *Let* $f = (f_1, \ldots, f_m) : \mathbb{C}^m \to \mathbb{C}^m$ *be a germ of an analytic map near the origin, and the differences* $\mathbb{R}^m_+ \setminus \Delta_{f_i}$ *are bounded.*

1) The local degree of f is greater than or equal to

$$\sum_{0<p\leqslant m}(-1)^{m-p}\sum_{0<i_1<\ldots<i_p\leqslant m} I(\Delta_{f_{i_1}}+\ldots+\Delta_{f_{i_p}}), \qquad (*)$$

provided that f is proper.
2) The germ f is proper, and its local degree equals $(*)$, if and only if, for each compatible collection of faces $B_1,\ldots,B_m$ of the polyhedra $\Delta_1,\ldots,\Delta_m$, the system of polynomial equations $f_1|_{B_1}=\ldots=f_m|_{B_m}=0$ has no roots in $(\mathbb{C}\setminus\{0\})^m$.

Remark. The principal parts, which satisfy the condition from part (2) of this theorem, form a dense algebraic set in the space of principal parts of maps with given Newton polyhedra of components.

The main result of this paper is the following generalization of this fact to multiplicities of matrices.

Definition 6. The *tropical semiring* P of polyhedra is the set of all convex polyhedra in $\mathbb{R}^n$ (including the empty one) with the additive operation

$$A\vee B = \text{ convex hull of } A\cup B$$

and the Minkowski sum as the multiplicative operation

$$A+B=\{a+b\,|\,a\in A,\, b\in B\}.$$

The name is justified by the fact that the support functions of $A\vee B$ and $A+B$ are equal to the maximum and the sum of the support functions of A and B respectively. All the polyhedra A, satisfying the equation $A+\mathbb{R}^m_+=A$, form a subring $P_+\subset P$, and $\mathbb{R}^m_+$ is the unit in this subring. In particular, whenever the sum of polyhedra $A_j\in P_+$ is taken over an empty set of indices $J=\varnothing$, we set $\sum_{j\in J}A_\alpha=\mathbb{R}^m_+$ by definition.

Theorem 7. *Let $A=(a_{i,j}):\mathbb{C}^m\to\mathbb{C}^{n\times k}$ be a germ of an $(n\times k)$-matrix with analytic entries, $m=k-n+1$, and the differences $\mathbb{R}^m_+\setminus\Delta_{a_{i,j}}$ are bounded.*
1) The multiplicity of the matrix A is greater than or equal to

$$\sum_{\substack{J\subset\{1,\ldots,k\}\\ b_1+\ldots+b_n=|J|}}(-1)^{k-|J|}I\Bigl(\bigvee_{\substack{J_1\sqcup\ldots\sqcup J_n=J\\ |J_1|=b_1,\ldots,|J_n|=b_n}}\sum_{\substack{i=1,\ldots,n\\ j\in J_i}}\Delta_{a_{i,j}}\Bigr), \qquad (**)$$

provided that $\operatorname{rk}A(x)=n$ for all $x\neq 0$. Here the first summation is taken over all non-empty $J\subset\{1,\ldots,k\}$ and all collections of non-negative integers b_i that sum up to $|J|$, and $\bigvee$ is taken over all decompositions of J into disjoint sets J_i of size b_i.
2) We have $\operatorname{rk}A(x)=n$ for all $x\neq 0$, and the multiplicity of A equals $(**)$, if and only if the principal part of A is in general position in the sense of Definition 17.

It is a purely combinatorial problem to deduce this fact from Theorem 23, and it will be addressed in Section 5.

Example 8. Theorem 7 appears to be more convenient than Theorem 23 in many important special cases. For instance, in the classical case of homogeneous $a_{i,j}$, Theorem 7 unlike Theorem 23 gives a closed formula for the multiplicity in terms of the degrees $d_{i,j}$ of the components $a_{i,j}$. For $J\subset\{1,\ldots,k\}$ and a decomposition $|J|=b_1+\ldots+b_n$ into non-negative integers, introduce the number

$$d^J_{b_1,\ldots,b_n}=\min_{\substack{J_1\sqcup\ldots\sqcup J_n=J\\ |J_1|=b_1,\ldots,|J_n|=b_n}}\sum_{\substack{i=1,\ldots,n\\ j\in J_i}}d_{i,j}.$$

Corollary 9. *In the setting of Theorem 7, assume that the components $a_{i,j}$ are homogeneous polynomials of degree $d_{i,j}$.*

1) The multiplicity of the matrix A is greater or equal to

$$\sum_{\substack{J\subset\{1,\ldots,k\}\\ b_1+\ldots+b_n=|J|}}(-1)^{k-|J|}\binom{m+d^J_{b_1,\ldots,b_n}-1}{m}.$$

2) The multiplicity is strictly greater than this number or is infinite, if and only if the entries are not in general position in the following sense: there exist integer numbers $\alpha_1,\ldots,\alpha_n$ and $\beta_1,\ldots,\beta_k$ and non-zero $x\in\mathbb{C}^m$ such that $d_{i,j}\geqslant\alpha_i+\beta_j$ for every i and j, and the matrix of the entries $\delta^{\alpha_i+\beta_j}_{d_{i,j}}a_{i,j}(x)$ is effectively degenerate in the sense of Definition 15 (as usual, δ^q_p is 1 if $p=q$ and 0 otherwise).

Example 10. Note that, unlike in the complete intersection case $n=1$, the multiplicity of such a homogeneous matrix can be strictly greater than expected, but still finite. For example, if $(m,n,k)=(2,2,3)$, then the matrix

$$\begin{pmatrix} x+y & (x+y)^2+y^2 & x+y\\ x+y & x+y & (x+y)^2+2y^2\end{pmatrix}$$

has multiplicity 6, which is strictly greater than the answer 3, given by Part 1 for a generic matrix of degree $\begin{pmatrix}1&2&1\\1&1&2\end{pmatrix}$. This is because the matrix above is not in general position (consider $\alpha_1=\alpha_2=1$, $\beta_1=\beta_2=\beta_3=0$ in the notation of Part 2).

Example 11. Let us expand the answer given by Theorem 7 in the simplest case $(m,n,k)=(2,2,3)$. Denote $\Delta_{a_{i,j}}$ by $\Delta_{i,j}$, then $(**)$ equals

$$I(\Delta_{1,1}+\Delta_{1,2}+\Delta_{1,3})+$$
$$+I\Big((\Delta_{2,1}+\Delta_{1,2}+\Delta_{1,3})\vee(\Delta_{1,1}+\Delta_{2,2}+\Delta_{1,3})\vee(\Delta_{1,1}+\Delta_{1,2}+\Delta_{2,3})\Big)+$$
$$+I\Big((\Delta_{1,1}+\Delta_{2,2}+\Delta_{2,3})\vee(\Delta_{2,1}+\Delta_{1,2}+\Delta_{2,3})\vee(\Delta_{2,1}+\Delta_{2,2}+\Delta_{1,3})\Big)+$$
$$+I(\Delta_{2,1}+\Delta_{2,2}+\Delta_{2,3})-I(\Delta_{1,1}+\Delta_{1,2})-I(\Delta_{1,1}+\Delta_{1,3})-I(\Delta_{1,2}+\Delta_{1,3})-$$
$$-I(\Delta_{2,1}+\Delta_{2,2})-I(\Delta_{2,1}+\Delta_{2,3})-I(\Delta_{2,2}+\Delta_{2,3})-I((\Delta_{1,1}+\Delta_{2,2})\vee(\Delta_{1,2}+\Delta_{2,1}))-$$
$$-I((\Delta_{1,1}+\Delta_{2,3})\vee(\Delta_{1,3}+\Delta_{2,1}))-I((\Delta_{1,3}+\Delta_{2,2})\vee(\Delta_{1,2}+\Delta_{2,3}))-$$
$$+I(\Delta_{1,1})+I(\Delta_{1,2})+I(\Delta_{1,3})+I(\Delta_{2,1})+I(\Delta_{2,2})+I(\Delta_{2,3}).$$

Example 12. If $\Delta_{i,j}=\Delta_i$ does not depend on the column j, then the answer, given by Theorems 7 and 23, admits a much simpler form $\sum_{1\leqslant i_1\leqslant\ldots\leqslant i_m\leqslant k}\mathrm{MV}(\Delta_{i_1},\ldots,\Delta_{i_m})$. If $\Delta_{i,j}=\Delta_j$ does not depend on the row i, then the answer, given by Theorems 7 and 23, admits a much simpler form $\sum_{1\leqslant j_1<\ldots<j_m\leqslant k}\mathrm{MV}(\Delta_{j_1},\ldots,\Delta_{j_m})$. Both of these facts can be easily deduced from Theorem 23 (see [E06] and [E09] for details). The latter one was discovered earlier in a much more general setting by Bivià-Ausina ([Biv]).

For example, if the germs $a_{i1}\in\langle x^2,y\rangle$, $a_{i2}\in\langle x,y^3\rangle$, $a_{i3}\in\langle x^2,y^3\rangle$, $i=1,2$, are in general position, then the multiplicity of A equals $4I(\Delta_1+\Delta_2+\Delta_3)-3I(\Delta_1+\Delta_2)-3I(\Delta_1+\Delta_3)-3I(\Delta_2+\Delta_2)+2I(\Delta_1)+2I(\Delta_2)+2I(\Delta_3)=4\cdot16-3\cdot(6+9+11)+2\cdot(2+3+5)=6$ according to Theorem 7 and $\mathrm{MV}(\Delta_1,\Delta_2)+\mathrm{MV}(\Delta_1,\Delta_3)+\mathrm{MV}(\Delta_2,\Delta_3)=1+2+3=6$ according to [Biv].

3. General position of principal parts of matrices

By convention, each polyhedron has the empty face. In particular, some faces $B_{i,j}$ in the following definition may be empty.

Definition 13. Let $B_{i,j}$ be a bounded face of a polyhedron $\Delta_{i,j} \subset \mathbb{R}^m$ for $i = 1, \ldots, n$, $j = 1, \ldots, k$. The collection of faces $B_{i,j}$ is said to be *matrix-compatible*, if there exist vectors $c_1, \ldots, c_n \in \mathbb{Z}^m$ and compatible faces $B_1, \ldots, B_k$ of the convex hulls $\bigvee_i (\Delta_{i,1} + c_i), \ldots, \bigvee_i (\Delta_{i,k} + c_i)$, such that $B_{i,j} = (B_j - c_i) \cap \Delta_{i,j}$ for each $i = 1, \ldots, n$, $j = 1, \ldots, k$.

Example 14. Let $\Delta_{i,j} \subset \mathbb{R}^1$ be the rays

$$\begin{pmatrix} [1,\infty) & [1,\infty) & [1,\infty) \\ [1,\infty) & [2,\infty) & [2,\infty) \\ [1,\infty) & [2,\infty) & [2,\infty) \end{pmatrix},$$

then every face $B_{i,j}$ is either the origin of $\Delta_{i,j}$ (denoted by $*$), or empty (denoted by $\varnothing$). In this case, the matrix-compatible collection of faces are

$$\mathcal{B}_1 = \begin{pmatrix} * & * & * \\ * & \varnothing & \varnothing \\ * & \varnothing & \varnothing \end{pmatrix}, \quad \mathcal{B}_2 = \begin{pmatrix} \varnothing & * & * \\ * & * & * \\ * & * & * \end{pmatrix},$$

and 11 more collections with fewer non-empty faces.

Definition 15. A matrix $M \in \mathbb{C}^{n\times k}$, $n \leqslant k$, is said to be *effectively non-degenerate*, if $(t_1, \ldots, t_n) \cdot M \neq (0, \ldots, 0)$ for all $(t_1, \ldots, t_n) \in (\mathbb{C} \setminus \{0\})^n$.

Example 16. The complex matrix

$$\begin{pmatrix} a & b & c \\ d & 0 & 0 \\ e & 0 & 0 \end{pmatrix}$$

is effectively degenerate if and only if $b = c = 0$ (although it is degenerate for all complex numbers a, b, c, d, e).

For an $(n \times k)$-matrix A with analytic entries $a_{i,j} : \mathbb{C}^m \to \mathbb{C}$ and a collection $\mathcal{B}$ of faces $B_{i,j}$ of the Newton polyhedra $\Delta_{a_{i,j}}$, we denote the matrix with entries $a_{i,j}|_{B_{i,j}}$ by $A|_{\mathcal{B}}$.

Definition 17. The principal part of an $(n \times k)$-matrix A with analytic entries $a_{i,j} : \mathbb{C}^m \to \mathbb{C}$ is said to be *in general position*, if, for each matrix-compatible collection $\mathcal{B}$ of faces of the Newton polyhedra $\Delta_{a_{i,j}}$ and for each $x \in (\mathbb{C} \setminus \{0\})^m$, the matrix $A|_{\mathcal{B}}(x)$ is effectively non-degenerate.

Remark. Principal parts in general position form a dense algebraic set in the space of principal parts of matrices with given Newton polyhedra of entries. However, this is not true, if we replace the effective non-degeneracy of matrices with the conventional one in Definition 17. For instance, if $(m, n, k) = (1, 3, 3)$, and the Newton polyhedra $\Delta_{a_{i,j}}$ are as in the example to Definition 13, then the only non-trivial condition, imposed by Definition 17, corresponds to the second matrix-compatible collection of faces shown in the example:

$$\det(A|_{\mathcal{B}_2}) = \det \begin{pmatrix} 0 & a^0_{0,1} & a^0_{0,2} \\ a^0_{1,0} & a^0_{1,1} & a^0_{1,2} \\ a^0_{2,0} & a^0_{2,1} & a^0_{2,2} \end{pmatrix} \neq 0,$$

where $a^0_{i,j}$ is the leading coefficient of the series $a_{i,j}$. However, if we replace effective nondegeneracy with nondegeneracy in Definition 17, then no matrix A will satisfy it, because the matrix $A|_{\mathcal{B}_1}$ is always degenerate (see the example to Definition 15).

It would be thus interesting to describe a collection of minors of the matrices $A|_{\mathcal{B}}$, such that
1) If the principal part of A is in general position, then these minors vanish.
2) The principal parts for which these minors vanish form a (closed algebraic) set of positive codimension in the space of all principal parts of matrices with given Newton polyhedra of entries.

This reduces to the following problem: given $K \subset \mathbb{N}^2$, assume that $a_{i,j}$ are independent variables for $(i,j) \in K$, and the entries of the matrix A equal $a_{i,j}$ for $(i,j) \in K$ and 0 for $(i,j) \notin K$. Find a collection of minors $\mathcal{A}$ of the matrix A, such that
1) If A is effectively degenerate, then $\mathcal{A} = 0$.
2) We have $\mathcal{A} \neq 0$ for generic $a_{i,j}, (i,j) \in K$.

4. Mixed volumes of pairs of polyhedra

Definition 18. Polyhedra Δ_1 and Δ_2 in $\mathbb{R}^n$ are said to be *parallel* if $a+\Delta_1 \subseteq \Delta_1 \Leftrightarrow a+\Delta_2 \subseteq \Delta_2$ for every point $a \in \mathbb{R}^n$.

Definition 19. ([E05], [E06]) 1) A pair of polyhedra Δ_1, Δ_2 in $\mathbb{R}^n$ is called *bounded* if both $\Delta_1 \setminus \Delta_2$ and $\Delta_2 \setminus \Delta_1$ are bounded. The set of all bounded pairs of polyhedra parallel to a given convex cone $C \subset \mathbb{R}^n$ is denoted by BP_C.

2) *The Minkowski sum* $(\Delta_1, \Delta_2) + (\Gamma_1, \Gamma_2)$ of two pairs from BP_C is the pair $(\Delta_1 + \Gamma_1, \Delta_2 + \Gamma_2) \in \mathrm{BP}_C$.

3) *The volume* $\mathrm{Vol}(\Delta_1, \Delta_2)$ of a bounded pair (Δ_1, Δ_2) is the difference $\mathrm{Vol}(\Delta_1 \setminus \Delta_2) - \mathrm{Vol}(\Delta_2 \setminus \Delta_1)$.

4) *The mixed volume* is the symmetric multilinear (with respect to Minkowski summation) function $\mathrm{MV} : \underbrace{\mathrm{BP}_C \times \ldots \times \mathrm{BP}_C}_{n} \to \mathbb{R}$ such that $\mathrm{MV}(A, \ldots, A) = \mathrm{Vol}(A)$ for every pair $A \in \mathrm{BP}_C$.

There exists a unique such function MV (see [E06], Section 4, Lemma 3 for existance, uniqueness and all other basic facts about the mixed volume of pairs, mentioned below). Recall that a polyhedron is said to be *lattice* if its vertices are integer lattice points. The mixed volume of pairs of n-dimensional lattice polyhedra is a rational number with denominator $n!$.

Example. If C consists of one point, then BP_C consists of pairs of bounded polyhedra, and

$$\mathrm{MV}\big((\Delta_1, \Gamma_1), \ldots, (\Delta_n, \Gamma_n)\big) = \mathrm{MV}(\Delta_1, \ldots, \Delta_n) - \mathrm{MV}(\Gamma_1, \ldots, \Gamma_n),$$

where MV in the right hand side is the classical mixed volume of bounded polyhedra. If C is not bounded, then both terms in the right hand side are infinite, but "their difference makes sense".

One can use the following formula to express the mixed volume of pairs in terms of mixed volumes of polyhedra ([E06], Section 4, Lemma 3).

Lemma 20. *For bounded pairs $(\Delta_i, \Gamma_i) \in \mathrm{BP}_C$, $i = 1, \ldots, n$, let $H \subset \mathbb{R}^n$ be a half-space such that $C \cap H$ is bounded and $\Delta_i \setminus H = \Gamma_i \setminus H$. Then*

$$\mathrm{MV}\big((\Delta_1, \Gamma_1), \ldots, (\Delta_n, \Gamma_n)\big) = \mathrm{MV}(\Delta_1 \cap H, \ldots, \Delta_n \cap H) - \mathrm{MV}(\Gamma_1 \cap H, \ldots, \Gamma_n \cap H),$$

where MV *in the right hand side is the classical mixed volume of bounded polyhedra.*

For a bounded pair of (closed) polyhedra $(\Delta, \Gamma) \in \mathrm{BP}_C$, define $I(\Delta, \Gamma)$ as the number of integer lattice points in the difference $\Delta \setminus \Gamma$ minus the number of integer lattice points in the difference $\Gamma \setminus \Delta$.

Lemma 21. *For bounded pairs of lattice polyhedra $A_i \in \mathrm{BP}_C$, we have*

$$n!\,\mathrm{MV}(A_1, \ldots, A_n) = \sum_{0 < p \leqslant m} (-1)^{n-p} \sum_{0 < i_1 < \ldots < i_p \leqslant n} I(A_{i_1} + \ldots + A_{i_p}).$$

Proof. For the classical mixed volume of bounded polyhedra, this equality is well known (see, for example, [Kh]). The general case can be deduced to the case of bounded polyhedra by the previous lemma. $\square$

5. Proof of Theorem 7

The following theorem is a special case of Theorem 5 from [E06].

Definition 22. For polyhedra $\Delta_1, \ldots, \Delta_n \subset \mathbb{R}^m$, define the *Cayley polyhedron* $\Delta_1 * \ldots * \Delta_n$ as the convex hull of the union

$$\bigcup_i \{b_i\} \times \Delta_i \subset \mathbb{R}^{n-1} \oplus \mathbb{R}^m,$$

where $b_1, \ldots, b_n$ are the points $(1, 0, \ldots, 0), (0, 1, \ldots, 0), \ldots, (0, 0, \ldots, 1)$ and $(0, 0, \ldots, 0)$ in $\mathbb{R}^{n-1}$. Denote $\mathbb{R}^m_+ * \ldots * \mathbb{R}^m_+$ by D.

For germs of analytic functions $a_1, \ldots, a_n$ on $\mathbb{C}^m$ near the origin, denote the sum $t_1 a_1 + \ldots + t_{n-1} a_{n-1} + a_n$ by $a_1 * \ldots * a_n$, where $t_1, \ldots, t_{n-1}$ are the standard coordinates on $\mathbb{C}^{n-1}$.

Theorem 23. *([E05], [E06], [E09]) Let A be an $(n \times k)$-matrix with entries $a_{i,j} : \mathbb{C}^m \to \mathbb{C}$ which are germs of analytic functions near the origin. Suppose that the Newton polyhedra $\Delta_{i,j}$ of the germs $a_{i,j}$ intersect all coordinate axes in $\mathbb{R}^m$.*
1) The multiplicity of A is greater than or equal to

$$(m + n - 1)!\,\mathrm{MV}\big((D, \Delta_{1,1} * \ldots * \Delta_{n,1}), \ldots, (D, \Delta_{1,k} * \ldots * \Delta_{n,k})\big). \qquad (***)$$

2) We have $\operatorname{rk} A(x) = n$ for all $x \neq 0$, and the multiplicity of A equals $(***)$, if and only if, for each compatible collection of faces $B_1, \ldots, B_k$ of the polyhedra $\Delta_{1,1} * \ldots * \Delta_{n,1}, \ldots, \Delta_{1,k} * \ldots * \Delta_{n,k}$, the polynomials $(a_{1,1} * \ldots * a_{n,1})|_{B_1}, \ \ldots, \ (a_{1,k} * \ldots * a_{n,k})|_{B_k}$ have no common zeroes in $(\mathbb{C} \setminus \{0\})^{n-1} \times (\mathbb{C} \setminus \{0\})^m$.

The "only if" part of (2) is actually proved in [E06], but is explicitly formulated and discussed only in [E09], Theorem 1.21.

Recall that $|S \cap \mathbb{Z}^m|$ is denoted by $I(S)$ for a bounded set $S \in \mathbb{R}^m$. If the symmetric difference of (closed) lattice polyhedra Γ and Δ in $\mathbb{R}^m$ is bounded, denote the difference $I(\Gamma \setminus \Delta) - I(\Delta \setminus \Gamma)$ by $I(\Gamma, \Delta)$. For pairs of polyhedra (Γ_i, Δ_i) in $\mathbb{R}^m$, denote the pair $(\bigvee_i \Gamma_i, \bigvee_i \Delta_i)$ by $\bigvee_i (\Gamma_i, \Delta_i)$ and the pair $(\Gamma_1 * \ldots * \Gamma_n, \Delta_1 * \ldots * \Delta_n)$ by $(\Gamma_1, \Delta_1) * \ldots * (\Gamma_n, \Delta_n)$.

Theorem 24. *If $B_{i,j}$, $i = 1, \ldots, n$, $j = 1, \ldots, k$, are bounded lattice polyhedra in $\mathbb{R}^m$ or pairs of lattice polyhedra in BP_C, and $m = k - n + 1$, then the mixed volume of $B_{1,j} * \ldots * B_{n,j}$, $j = 1, \ldots, k$, equals*

$$\frac{1}{k!} \sum_{\substack{J \subset \{1, \ldots, k\} \\ b_1 + \ldots + b_n = |J|}} (-1)^{k - |J|} I\Big(\bigvee_{\substack{J_1 \sqcup \ldots \sqcup J_n = J \\ |J_1| = b_1, \ldots, |J_n| = b_n}} \sum_{\substack{i = 1, \ldots, n \\ j \in J_i}} B_{i,j} \Big).$$

Note that some of $B_{i,j}$ may be empty. The proof is given in Section 7. Theorem 7 follows from Theorems 23 and 24 (one can easily check that the condition of general position in Theorem 23(2) coincides with the one given by Definition 17).

6. Fans and lattice points of polyhedra

Here we prove the equality

$$(A \cap \mathbb{Z}^q) + (B \cap \mathbb{Z}^q) = (A + B) \cap \mathbb{Z}^q$$

for some class of bounded lattice polyhedra $A, B \subset \mathbb{R}^q$ (see [O97] for a conjecture in the general case).

Definition 25. *A (rational) cone* in $\mathbb{R}^q$ generated by (rational) vectors $v_1, \ldots, v_m$ is the set of all linear combinations of $v_1, \ldots, v_m$ with positive coefficients.

Note that, according to this definition, a cone is not a closed set unless it is a vector subspace of $\mathbb{R}^q$, and is not an open set unless it is q-dimensional.

Definition 26. A collection of rational cones $C_1, \ldots, C_p$ in $\mathbb{R}^q$ is said to be $\mathbb{Z}$-*transversal*, if $\sum \dim C_i = q$ and the set $\mathbb{Z}^q \cap \bigcup_i C_i$ generates the lattice $\mathbb{Z}^q$.

Definition 27. *A (rational) fan* Φ in $\mathbb{R}^q$ is a non-empty finite set of nonoverlapping (rational) cones in $\mathbb{R}^q$ such that
1) Each face of each cone from Φ is in Φ,
2) Each cone from Φ is a face of a q-dimensional cone from Φ.

Definition 28. A collection of fans $\Phi_1, \ldots, \Phi_p$ in $\mathbb{R}^q$ is said to be $\mathbb{Z}$-*transversal with respect to shifts* $c_1 \in \mathbb{R}^q, \ldots, c_p \in \mathbb{R}^q$, if each collection of cones $C_1 \in \Phi_1, \ldots, C_p \in \Phi_p$, such that the intersection $(C_1 + c_1) \cap \ldots \cap (C_p + c_p)$ consists of one point, is $\mathbb{Z}$-transversal.

Definition 29. *The dual cone* of a face B of a polyhedron $A \subset \mathbb{R}^q$ is the set of all covectors $\gamma \in (\mathbb{R}^q)^*$ such that $\{a \in A \,|\, \gamma(a) = \min \gamma(A)\} = B$. *The dual fan* of a polyhedron is the set of dual cones of all its faces.

Theorem 30. *If the dual fans of bounded lattice polyhedra $A_1, \ldots, A_p \subset \mathbb{R}^q$ are $\mathbb{Z}$-transversal with respect to some shifts $c_1 \in (\mathbb{R}^q)^*, \ldots, c_p \in (\mathbb{R}^q)^*$ and $\dim(A_1 + \ldots + A_p) = q$, then*

$$(A_1 \cap \mathbb{Z}^q) + \ldots + (A_p \cap \mathbb{Z}^q) = (A_1 + \ldots + A_p) \cap \mathbb{Z}^q.$$

Proof. Consider covectors $c_1 \in (\mathbb{R}^q)^*, \ldots, c_p \in (\mathbb{R}^q)^*$ as linear functions on the polyhedra $A_1 \subset \mathbb{R}^q, \ldots, A_p \subset \mathbb{R}^q$ respectively, and denote their graphs in $\mathbb{R}^q \oplus \mathbb{R}^1$ by $\Gamma_1, \ldots, \Gamma_p$. Denote the projection $\mathbb{R}^q \oplus \mathbb{R}^1 \to \mathbb{R}^q$ by π, and denote the ray $\{(0, \ldots, 0, t) \mid t < 0\} \subset \mathbb{R}^q \oplus \mathbb{R}^1$ by L_-.

Each bounded q-dimensional face B of the sum $\Gamma_1 + \ldots + \Gamma_p + L_-$ is the sum of some faces $B_1, \ldots, B_p$ of polyhedra $\Gamma_1 + L_-, \ldots, \Gamma_p + L_-$. $\mathbb{Z}$-transversality with respect to shifts $c_1 \in (\mathbb{R}^q)^*, \ldots, c_p \in (\mathbb{R}^q)^*$ implies that

$$\big(\pi(B_1) \cap \mathbb{Z}^q\big) + \ldots + \big(\pi(B_p) \cap \mathbb{Z}^q\big) = \pi(B_1 + \ldots + B_p) \cap \mathbb{Z}^q.$$

Since the projections of bounded q-dimensional faces of the sum $\Gamma_1 + \ldots + \Gamma_p + L_-$ cover the sum $A_1 + \ldots + A_p$, it satisfies the same equality:

$$(A_1 \cap \mathbb{Z}^q) + \ldots + (A_p \cap \mathbb{Z}^q) = (A_1 + \ldots + A_p) \cap \mathbb{Z}^q.$$

□

Corollary 31. *Let $S \subset \mathbb{R}^q$ be the standard q-dimensional simplex, let $l_1, \ldots, l_p$ be linear functions on S with graphs $\Gamma_1, \ldots, \Gamma_p$, and let l be the maximal piecewise-linear function on pS, such that its graph Γ is contained in the sum $\Gamma_1 + \ldots + \Gamma_p$. Then, for each integer lattice point $a \in pS$, the value $l(a)$ equals the maximum of sums $l_1(c_1) + \ldots + l_p(c_p)$, where $(c_1, \ldots, c_p)$ runs over all p-tuples of vertices of S such that $c_1 + \ldots + c_p = a$.*

Proof. Denote the projection $\mathbb{R}^q \oplus \mathbb{R}^1 \to \mathbb{R}^q$ by π. A q-dimensional face B of Γ, which contains the point $\big(a, l(a)\big) \in \mathbb{R}^q \oplus \mathbb{R}^1$, can be represented as a sum of faces B_i of simplices Γ_i. Since $\pi(B_1), \ldots, \pi(B_p)$ are faces of the standard simplex, their dual fans are $\mathbb{Z}$-transversal with respect to a generic collection of shifts, and, by Theorem 30,

$$\big(\pi(B_1) \cap \mathbb{Z}^q\big) + \ldots + \big(\pi(B_p) \cap \mathbb{Z}^q\big) = \pi(B) \cap \mathbb{Z}^q.$$

In particular, $a = c_1 + \ldots + c_p$ for some integer lattice points $c_i \in \pi(B_i)$, which implies $l(a) = l_1(c_1) + \ldots + l_p(c_p)$. □

Remark. In particular, if the functions $l_1,\ldots,l_p$ are in general position, then all C^q_{p+q} integer lattice points in the simplex pS are projections of vertices of Γ. Translating this into the tropical language, one can prove again the following well-known fact: p generic tropical hyperplanes in the space $\mathbb{R}^q$ subdivide it into C^q_{p+q} pieces.

Example 32. If S in the formulation of Corollary 31 is not the standard simplex, then the statement is not always true. For example, consider

$$S = \operatorname{conv}\bigl\{(1,1),(1,-1),(-1,1),(-1,-1)\bigr\},$$

$$l_1(x,y) = x+y,\ l_2(x,y) = x-y,\ a=(1,0).$$

If, in addition, we allow functions l_j to be concave piecewise linear with integer domains of linearity, then the statement is not true unless S is the standard simplex. That is why we cannot use computations below to simplify the formula in the statement of Theorem 5 from [E06] in general.

7. Proof of Theorem 24

Rewriting the mixed volume of the pairs $B_{1,i} * \ldots * B_{n,i}$, $i=1,\ldots,k$, as

$$\sum_{0<p\leqslant m} (-1)^{n-p} \sum_{0<i_1<\ldots<i_p\leqslant n} I\bigl((B_{1,i_1} * \ldots * B_{n,i_1}) + \ldots + (B_{1,i_p} * \ldots * B_{n,i_p})\bigr)$$

by Lemma 21, and applying the following Lemma 33 to every term in this sum, we obtain the statement of Theorem 24.

Lemma 33. *For bounded pairs of polyhedra $A_{i,j} = (\Delta_{i,j},\Phi_{i,j}) \in \mathrm{BP}_C$, $i=1,\ldots,n$, $j=1,\ldots,p$,*

$$I\bigl((A_{1,1} * \ldots * A_{n,1}) + \ldots + (A_{1,p} * \ldots * A_{n,p})\bigr) =$$

$$= \sum_{\substack{a_1+\ldots+a_n=p\\ a_1\geqslant 0,\ldots,a_n\geqslant 0}} I\Bigl(\bigvee_{\substack{J_1\sqcup\ldots\sqcup J_n=\{1,\ldots,p\}\\ |J_1|=a_1,\ldots,|J_n|=a_n}} \sum_{\substack{i=1,\ldots,n\\ j\in J_i}} \Delta_{i,j},\ \bigvee_{\substack{J_1\sqcup\ldots\sqcup J_n=\{1,\ldots,p\}\\ |J_1|=a_1,\ldots,|J_n|=a_n}} \sum_{\substack{i=1,\ldots,n\\ j\in J_i}} \Phi_{i,j}\Bigr).$$

Proof. Every integer lattice point, participating in the left hand side, is contained in the plane $\{(a_1,\ldots,a_{n-1})\}\times\mathbb{R}^m \subset \mathbb{R}^{n-1}\oplus\mathbb{R}^m$ for some non-negative integer numbers $a_1,\ldots,a_n$, which sum up to p. Thus, it is enough to describe the intersection of the pair $\bigl((A_{1,1} * \ldots * A_{n,1}) + \ldots + (A_{1,p} * \ldots * A_{n,p})\bigr)$ with each of these planes, using the following fact. □

Lemma 34. *Suppose that polyhedra $\Delta_{i,j}\subset\mathbb{R}^m$ are parallel to each other for $i=1,\ldots,n$, $j=1,\ldots,p$. Then, for each n-tuple of non-negative integer numbers $a_1,\ldots,a_n$ which sum up to p,*

$$\Bigl(\{(a_1,\ldots,a_{n-1})\}\times\mathbb{R}^m\Bigr) \cap \bigl((\Delta_{1,1} * \ldots * \Delta_{n,1}) + \ldots + (\Delta_{1,p} * \ldots * \Delta_{n,p})\bigr) =$$

$$= \{(a_1,\ldots,a_{n-1})\}\times\Bigl(\bigvee_{\substack{J_1\sqcup\ldots\sqcup J_n=\{1,\ldots,p\}\\ |J_1|=a_1,\ldots,|J_n|=a_n}} \sum_{\substack{i=1,\ldots,n\\ j\in J_i}} \Delta_{i,j}\Bigr) \subset \mathbb{R}^{n-1}\oplus\mathbb{R}^m.$$

Proof. For each hyperplane $L\subset\mathbb{R}^m$, denote the projection $\mathbb{R}^{n-1}\oplus\mathbb{R}^m \to \mathbb{R}^{n-1}\oplus\mathbb{R}$ along $\{0\}\oplus L$ by π_L. It is enough to prove that the images of the left hand side and the right hand side under π_L coincide for each L. To prove it, apply Corollary 31, setting q to $n-1$, a to $(a_1,\ldots,a_{n-1})$, and Γ_j to the maximal bounded face of the projection $\pi_L\bigl(\Delta_{1,j} * \ldots * \Delta_{n,j}\bigr)$ for every $j=1,\ldots,p$. □

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CLASSIFICATION OF CURVES ON SURFACES AND FREE LINKS VIA HOMOTOPY THEORY OF WORDS AND PHRASES

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Abstract. In this paper, we introduce Turaev's homotopy theory of words and phrases. As new results, we give the classification of oriented ordered pointed irreducible multi-component curves on surfaces which is called monoliteral type with at most six crossings up to stably equivalence using Turaev's homotopy theory of words and phrases. Moreover we also give the classification of (oriented) ordered pointed irreducible free links of monoliteral type with at most six crossings.

1. Introduction.

A *knot* is the image of a smooth embedding of S^1 into $\mathbb{R}^3$. Further, a k-components *link* is the image of a smooth embedding of the disjoint union of k circles into $\mathbb{R}^3$. When we study knots and links, we often use link diagrams of links. A *knot diagram* is a smooth immersion of S^1 into $\mathbb{R}^2$ with transversal double points such that the two paths at each double point are assigned to be the over path and the under path respectively (we call a double point of such immersion a *crossing*). If a knot diagram D is obtained as the image of a knot by a projection of $\mathbb{R}^3$ to $\mathbb{R}^2$, then we call D a diagram of the knot. A *link diagram* is defined similarly as a smooth immersion of the disjoint union of circles to $\mathbb{R}^2$.

In the paper [14], L. Kauffman introduced the theory of virtual knots and links using combinatorially extended link diagrams which are called virtual link diagrams. A *virtual knot diagram* is a planar graph of valency four endowed with the following structure: Each vertex either has an overcrossing and undercrossing (in other words, *real crossing*) or is marked by a *virtual crossing* (See Figure 3). A *virtual link diagram* is defined similarly. Then, we define *virtual links* by the set of virtual link diagrams quotiented by an equivalence relation generated by the virtual Reidemeister moves (see [14] for more details).

We call a virtual link diagram *pointed* if each component is endowed with a base point distinct from the crossing points. Further, we call a virtual link diagram *ordered* if its components are numerated. We also call a virtual link diagram *flat* when we ignore over/under at real crossings. A *pointed ordered flat virtual link* is defined by a set of pointed ordered flat virtual link diagrams quotiented by an equivalence relation generated by the flat virtual Reidemeister moves which are applied away from the base points.

The theory of flat virtual links is closely related to the theory of curves on surfaces. In fact, for all positive integer k, oriented ordered pointed k-components flat virtual links are in one to one correspondency to stably equivalent classes of oriented ordered pointed k components curves on surfaces (see [13] for example).

In this paper, we introduce the classification of oriented ordered pointed multi-components curves on surfaces up to stable equivalence with some conditions. To do this, we use the theory of nanowords which are introduced by Vladimir Turaev in [18] and [19]. Turaev defined generalized

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words and phrases which are called nanowords and nanophrases. Moreover he introduced an equivalence relation which is called S-homotopy on a set of generalized words and he proved that if we consider some special cases of homotopy of words and phrases, then we obtain the theory of curves on surfaces and link diagrams on surfaces. Therefore we can use the homotopy theory of words and phrases to study curves on surfaces and link diagrams on surfaces. Another applications of the theory of words can be found in [11] and [12]. N. Ito studies curves on a plane and wave fronts on a plane by using Turaev's theory of words. See [11] and [12] for more details.

This paper is organized as follows. In section 2, we review the theory of topology of words. We introduce some important notions to obtain the main result. In section 3, we introduce geometric meanings of the theory of words and phrases. We describe how to construct a bijection from the set of stable equivalence classes of curves on surfaces to the set of homotopy classes of nanophrases. Moreover we introduce flat virtual links. This leads us to a simple presentation of curves on surfaces. In section 4, we introduce the classification of nanowords, nanophrases and monoliteral phrases with some conditions on the length of words and on the number of component of phrases, which is proved by Turaev in [18] and the author in [1], [3] and [4]. In section 5, we introduce some homotopy invariants of nanophrases which was used to classify nanophrases. In section 6, we introduce application of the classification theorems. As a new result, we classify oriented ordered pointed irreducible curves on surfaces of monoliteral type with at most six crossings up to stably equivalent. Moreover we make the list of a complete representative system of oriented ordered pointed irreducible curves on surfaces of monoliteral type with at most six crossings. Moreover in section 8, we give the classification of (oriented) ordered pointed irreducible free links with at most two crossings and the classification of (oriented) ordered pointed irreducible free links of monoliteral type with at most six crossings.

2. Turaev's Homotopy Theory of Words and Phrases

In this section we introduce Turaev's homotopy theory of words and phrases which was introduced by V. Turaev in papers [18] and [19]. We can find a survey of Turaev's theory of words in the paper [20].

2.1. **Étale words and nanowords.** In this paper an *alphabet* means a finite set and a *letter* means an element of an alphabet. For an alphabet $\mathcal{A}$ and $n \in \mathbb{N}$, a *word* on $\mathcal{A}$ of length n is a map $w : \hat{n} \to \mathcal{A}$ where $\hat{n}$ is $\{1, 2, \cdots, n\}$. We denote a word of length n by $w(1)w(2)\cdots w(n)$. Roughly speaking, a word is a finite sequence of elements of an alphabet. We regard the map from empty set to empty set as the word of length 0 and denote it by $\emptyset$. A *phrase* of length k on $\mathcal{A}$ is a sequence of words $w_1, w_2, \cdots, w_k$ on $\mathcal{A}$. We denote this sequence by $(w_1|w_2|\cdots|w_k)$. We call the number $\sum_{i=1}^{k}(\text{length of } w_i)$ *number of letters* of the phrase. Especially if each letter in $\mathcal{A}$ appear exactly twice in a word w on $\mathcal{A}$, then we call this word w a *Gauss word*. Similarly for a phrase P on $\mathcal{A}$ if each letter in $\mathcal{A}$ appear exactly twice in P, then we call P a *Gauss phrase* (C. F. Gauss studied topology of plane curves using Gauss words. See [6] for more details).

In [18] and [19], Turaev introduced generalized words and phrases. Let α be an alphabet endowed with an involution $\tau : \alpha \to \alpha$. Then an *α-alphabet* is a pair of an alphabet $\mathcal{A}$ and a map $|\cdot| : \mathcal{A} \to \alpha$. We call this map $|\cdot|$ *projection* and we denote the image of a letter $A \in \mathcal{A}$ under the projection $|A|$. We also call $|A|$ a *projection of A*. Now we define generalized words (respectively Gauss words) which are called étale words (respectively nanowords). An *étale word* over α is a pair (an α-alphabet $\mathcal{A}$, a word w on $\mathcal{A}$). We call the length of w *length of étale word* $(\mathcal{A}, w)$. Especially if w is a Gauss word on $\mathcal{A}$, then we call a pair $(\mathcal{A}, w)$ a *nanoword* over α. Next we define generalized phrases (respectively Gauss phrases) which are called étale phrases (respectively nanophrases). An *étale phrase* over α is a pair (an α-alphabet $\mathcal{A}$, a phrase P on $\mathcal{A}$). We call length of P the *length of étale phrase* $(\mathcal{A}, P)$. Especially if P is a Gauss phrase on $\mathcal{A}$, then we call a pair $(\mathcal{A}, P)$ a *nanophrase* over α.

Example 2.1. Let α be an alphabet given by $\{a, b\}$ with an involution $\tau : \alpha \to \alpha$ given by $\tau(a)$ is equal to b. Let $\mathcal{A}$ be a α-alphabet given by $\{A, B, C\}$ with a projection given by $|A|$ is equal to a and $|B|$ and $|C|$ are equal to b. Then a pair $(\mathcal{A}, ABCABC)$ is a nanoword over α of length six. Furthermore, a pair $(\mathcal{A}, (AB|AC|BC))$ is a nanophrase over α of length three with six letters. On the other hand, a pair $(\mathcal{A}, ABCBC)$ is an étale word over α of length five, but not nanoword over α since the letter A appear only once in the word $ABCBC$. A pair $(\mathcal{A}, ABAB)$ is not nanoword, since the letter C does not appear. A pair $(\mathcal{A} - \{C\}, ABAB)$ is a nanoword over α of length four.

2.2. ***S*-homotopy of nanophrases and étale phrases.** In the paper [18] Turaev defined an equivalence relation on nanophrases which is called S-homotopy. This is suggested by the Reidemeister moves in the theory on knots. In this subsection, we introduce S-homotopy theory of words and phrases.

To define S-homotopy of nanophrases we prepare some definitions. First we define isomorphism of nanophrases.

Definition 2.1. Let $(\mathcal{A}_1, (w_1|\cdots|w_k))$ and $(\mathcal{A}_2, (v_1|\cdots|v_k))$ be nanophrases of length k over an alphabet α. Then $(\mathcal{A}_1, (w_1|\cdots|w_k))$ and $(\mathcal{A}_2, (v_1|\cdots|v_k))$ are *isomorphic* if there exist a bijection φ between $\mathcal{A}_1$ and $\mathcal{A}_2$ such that $|A| = |\varphi(A)|$ for all $A \in \mathcal{A}_1$ and $v_j = \varphi(w_j)$ for each $j \in \hat{k}$.

Next we define S-homotopy moves of nanophrases.

Definition 2.2. Let S be a subset of $\alpha \times \alpha \times \alpha$. Then we define *S-homotopy moves* (H1) - (H3) of nanophrases as follows:

(H1) $(\mathcal{A}, (xAAy)) \longrightarrow (\mathcal{A} \setminus \{A\}, (xy))$
for all $A \in \mathcal{A}$ and x, y are sequences of letters in $\mathcal{A} \setminus \{A\}$, possibly including the $|$ character.

(H2) $(\mathcal{A}, (xAByBAz)) \longrightarrow (\mathcal{A} \setminus \{A, B\}, (xyz))$
if $A, B \in \mathcal{A}$ satisfy $|B| = \tau(|A|)$. x, y, z are sequences of letters in $\mathcal{A} \setminus \{A, B\}$, possibly including the $|$ character.

(H3) $(\mathcal{A}, (xAByACzBCt)) \longrightarrow (\mathcal{A}, (xBAyCAzCBt))$
if $A, B, C \in \mathcal{A}$ satisfy $(|A|, |B|, |C|) \in S$. x, y, z, t are sequences of letters in $\mathcal{A}$, possibly including the $|$ character.

We call this S *homotopy data*.

Now we define *S-homotopy* of nanophrases.

Definition 2.3. Let $(\mathcal{A}_1, P_1)$ and $(\mathcal{A}_2, P_2)$ be nanophrases over α. Then $(\mathcal{A}_1, P_1)$ and $(\mathcal{A}_2, P_2)$ are *S-homotopic* (denote $\simeq_S$) if they are related by a finite sequence of isomorphism, S-homotopy moves (H1) - (H3) and inverse of (H1) - (H3).

Remark 2.1. S-homotopy moves and isomorphism of nanophrases do not change length of nanophrases. Thus for two different integers k_1 and k_2, a nanophrase of length k_1 and a nanophrase of length k_2 are not homotopic to each other.

Especially if S is the diagonal set of $\alpha \times \alpha \times \alpha$, then we call S-homotopy *homotopy*.

We denote the set $\{$Nanophrases of length k over $\alpha\}/(\text{S}-\text{homotopy})$ by $\mathcal{P}_k(\alpha, S)$ and $\mathcal{P}_1(\alpha, S)$ by $\mathcal{N}(\alpha, S)$.

Example 2.2. Nanophrases $(AB|ADDCBC)$ and $(BA|CACB)$ with $|A| = |B| = |C| \in \alpha$ over α are homotopic. Indeed

$$(AB|A\underline{DD}CBC) \simeq (\underline{AB}|\underline{ACBC}) \simeq (BA|CACB).$$

Next we define S-homotopy of étale phrases. To do so, we define desingularization of étale phrases.

For a nanophrase $(\mathcal{A}, P)$ and a letter A in $\mathcal{A}$, we define *multiplicity* of the letter A by the number of A in the phrase P. We denote multiplicity of A by $m_P(A)$. Let $\mathcal{A}^d$ be an α-alphabet $\{A_{i,j} := (A, i, j) | A \in \mathcal{A}, 1 \leq i < j \leq m_P(A)\}$ with the projection $|A_{i,j}| := |A|$ for all $A_{i,j}$. The phrase P^d is obtained from P by first deleting all $A \in \mathcal{A}$ for which $m_P(A)$ is less than or equal to one. Then for each $A \in \mathcal{A}$ for which $m_P(A)$ is grater than or equal to two and each $i = 1, 2, \ldots m_P(A)$, we replace the i-th entry of A in P by

$$A_{1,i}A_{2,i}\ldots A_{i-1,i}A_{i,i+1}A_{i,i+2}\ldots A_{i,m_P(A)}.$$

The resulting $(\mathcal{A}^d, P^d)$ is a nanophrase with $\sum m_P(A)(m_P(A)-1)$ letters and called a *desingularization of* $(\mathcal{A}, P)$. Note that if $(\mathcal{A}, P)$ is a nanophrase, then desingularization of $(\mathcal{A}, P)$ is isomorphic to itself.

Example 2.3. Let α be an alphabet. Let $\mathcal{A}$ be an α-alphabet given by $\{A, B, C\}$ and P be a phrase given by $(AA|BB|A|C)$. Then desingularization of an étale phrase $(\mathcal{A}, P)$ is given by

$$(\{A_{12}, A_{13}, A_{23}, B_{12}\}, (A_{12}A_{13}A_{12}A_{23}|B_{12}B_{12}|A_{13}A_{23}|\emptyset)),$$

with $|A_{12}| = |A_{13}| = |A_{23}| = |A|$ and $|B_{12}| = |B|$.

Now we define S-homotopy of étale phrases.

Definition 2.4. Two étale phrases $(\mathcal{A}_1, P_1)$ and $(\mathcal{A}_2, P_2)$ over α are *S-homotopic* (denoted $(\mathcal{A}_1, P_1) \simeq (\mathcal{A}_2, P_2)$) if $((\mathcal{A}_2)^d, (P_2)^d)$ can be obtained from $((\mathcal{A}_1)^d, (P_1)^d)$ by a finite sequence of isomorphism, S-homotopy moves (H1) - (H3) and the inverse of moves (H1) - (H3).

Remark 2.2. By the definition of homotopy of étale phrases, every homotopy invariant I of nanophrases extends to a homotopy invariant I of étale phrases by $I(P) := I(P^d)$.

We recall two lemmas from [18] and [19].

Lemma 2.1 (Turaev [18], [19])**.** *Let (α, S) be homotopy data and $\mathcal{A}$ be an α-alphabet. Let A, B, C be distinct letters in $\mathcal{A}$ and let x, y, z, t be words in $\mathcal{A} \setminus \{A, B, C\}$ such that $xyzt$ is a Gauss phrase. Then the following (i)-(iii) hold :*

(i) $(\mathcal{A}, (xAByCAzBCt)) \simeq_S (\mathcal{A}, (xBAyACzCBt))$
if $(|A|, \tau(|B|), |C|) \in S$,
(ii) $(\mathcal{A}, (xAByCAzCBt)) \simeq_S (\mathcal{A}, (xBAyACzBCt))$
if $(\tau(|A|), \tau(|B|), |C|) \in S$,
(iii)$(\mathcal{A}, (xAByACzCBt)) \simeq_S (\mathcal{A}, (xBAyCAzBCt))$
if $(|A|, \tau(|B|), \tau(|C|)) \in S$.

Lemma 2.2 (Turaev [18], [19])**.** *Suppose that $S \cap (\alpha \times b \times b) \neq \emptyset$ for all $b \in \alpha$. Let $(\mathcal{A}, (xAByABz))$ be a nanoword over α with $|B| = \tau(|A|)$ where x, y, z are words in $\mathcal{A} \setminus \{A, B\}$ such that xyz is a Gauss phrase. Then*

$$(\mathcal{A}, (xAByABz)) \simeq_S (\mathcal{A} \setminus \{A, B\}, (xyz)).$$

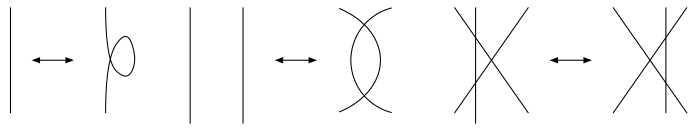

Figure 1. The flat Reidemeister moves.

3. Geometric Interpretation of Homotopy of Nanophrases.

In this section we explain geometric interpretation of S-homotopy of nanophrases which was introduced in the paper [19].

3.1. **Stable equivalence of curves on surfaces.** In this subsection we introduce stable equivalence of curves on surfaces. First we define some terminologies. Through this paper a *curve* means the image of a generic immersion of an oriented circle into an oriented surface. The word "generic" means that the curve has only a finite set of self-intersections which are all double and transversal. A *k-component curve* is defined in the same way as a curve with the difference that they may be formed by k curves. These curves are called *components* of the k-component curve. A k-component curve is *pointed* if each component is endowed with a base point (the origin) distinct from the crossing points of the k-component curve. A k-component curve is *ordered* if its components are numerated. Next we introduce an equivalence relation which is called stably equivalence. Two ordered, pointed curves are *stably homeomorphic* if there is an orientation preserving homeomorphism of their regular neighborhoods in the ambient surfaces mapping the first multi-component curve onto the second one and preserving the order, the origins and the orientations of the components.

Now we define stable equivalence of ordered, oriented, pointed multi-component curves [14]: Two ordered, pointed multi-component curves are *stably equivalent* if they can be related by a finite sequence of the following transformations: (i) a move replacing an ordered, pointed multi-component curve with a stably homeomorphic one; (ii) the flat Reidemeister moves away from the origin as in Figure 1.

We denote the set of stable equivalence classes of ordered, oriented, pointed k-component curves by $\mathcal{C}_k$.

Remark 3.1. The theory of stable equivalence of curves is closely related to the theory of virtual strings. See [17], [21] and Section 3.3 in this paper for more details.

3.2. **Geometric interpretation of S-homotopy of nanophrases.** In the paper [19] Turaev gave geometric meanings of S-homotopy of nanophrases over α with an involution τ for some α, S and τ. More precisely, Turaev proved the following theorem.

Theorem 3.1 (Turaev [19]). *There is a canonical bijection between $\mathcal{C}_k$ and $\mathcal{P}_k(\alpha_0, S_0)$ where α_0 is equal to $\{a, b\}$ with an involution τ_0 where $\tau_0(a)$ is equal to b and S_0 is equal to $\{(a, a, a), (b, b, b)\}$.*

The way of making a nanophrase $P(C)$ from an ordered, oriented, pointed k-component curve C is as follows. Let us label the double points of the curve C by distinct letters $A_1, \cdots, A_n$. Starting at the origin of first component of C and following along C in the positive direction, we write down the labels of double points which we passes until return to the origin. Then we obtain a word w_1. Similarly we obtain words $w_2, \cdots, w_k$ on the alphabet $\mathcal{A} = \{A_1, \cdots, A_n\}$ from second component, $\cdots$, k-th component. Let t_i^1 (respectively, t_i^2) be the tangent vector to C at the double point labeled A_i appearing at the first (respectively, second) passage through this point. Set $|A_i|$ is equal to a, if the pair (t_i^1, t_i^2) is positively oriented, and $|A_i|$ is equal to b otherwise. Then we obtain a required nanophrase $P(C) := (\mathcal{A}, (w_1|\cdots|w_k))$.

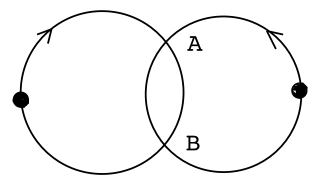

FIGURE 2. An example

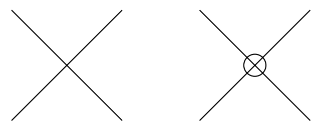

FIGURE 3. A real crossing and a virtual crossing.

Remark 3.2. By the above theorem if we classify the homotopy classes of nanophrases, then we obtain the classification of ordered, pointed multi-component curves under the stable equivalence as a corollary.

Example 3.1. Consider a two-component pointed ordered curve shown in Fig. 2. Assume that a left circle is first component of this curve and a right circle is second component of this curve. Then a nanophrase which corresponds to this curve is $(\{A, B\}, (AB|AB))$ with $|A|$ is equal to b and $|B|$ is equal to a.

Moreover let $\mathcal{L}_k$ be the set of *stable equivalence classes of k-component pointed ordered oriented link diagrams* (definition of the stable equivalence of link diagrams is given in [19] for example). Then Turaev proved following theorem.

Theorem 3.2 (Turaev [19])**.** *There is a canonical bijection between $\mathcal{L}_k$ and $\mathcal{P}_k(\alpha_*, S_*)$ where α_* is equal to $\{a_+, a_-, b_+, b_-\}$ with an involution $\tau_*(a_\pm)$ is equal to $b_\mp$ and S_* is equal to $\{(a_\pm, a_\pm, a_\pm),\ (a_\pm, a_\pm, a_\mp),\ (a_\mp, a_\pm, a_\pm),\ (b_\pm, b_\pm, b_\pm),\ (b_\pm, b_\pm, b_\mp),\ (b_\mp, b_\pm, b_\pm)\}$.*

The method of making nanophrase $P(L)$ from ordered, pointed k-component link L is similar to the case Theorem 3.1. See [19] for more details.

Remark 3.3. We can find another applications of the theory of nanowords and étale words to geometry and topology in papers [11] and [12]. N. Ito used the theory of nanowords to study planar curves and wave fronts on $\mathbb{R}^2$.

3.3. **Presentation of curves on surfaces by virtual strings.** In this subsection, we introduce useful method to illustrate curves on surfaces. To do so, we introduce virtual string diagrams and virtual strings.

A *virtual string diagram* is a planar graph of valency four endowed with the following structure: each vertex either is an unmarked crossing (in other words, *real crossing*) or is marked by a *virtual crossing* (see Figure 3). Then we define a *virtual string* by a virtual string diagram modulo *flat virtual Reidemeister moves* which are illustrated in Figure 4. We also use terminologies *pointed*, *ordered* and *oriented* same as in the case of curves on surfaces.

It is known the stable equivalence theory of pointed ordered curves on surfaces is equivalent to the theory of pointed ordered virtual strings by the correspondence illustrated in Figure 5 (see [13], [19] for example). Therefore in the rest of this paper, we illustrate curves on surfaces as virtual strings diagrams.

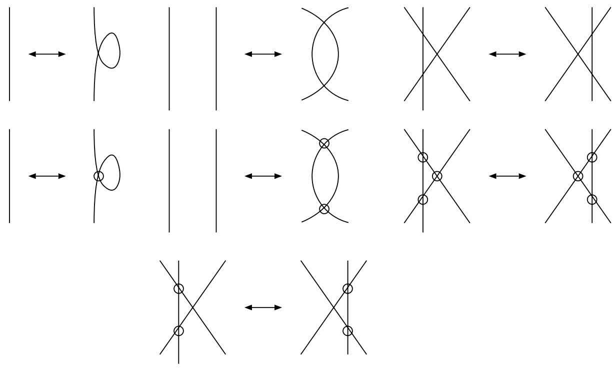

FIGURE 4. Flat virtual Reidemeister moves.

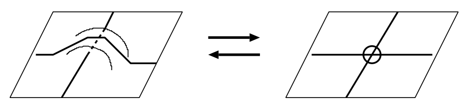

FIGURE 5. The correspondence of virtual strings and curves on surfaces.

4. Classification of Nanophrases and Étale Phrases up to Homotopy.

In this section, we introduce classification theorems of nanowords, nanophrases, étale words and étale phrases up to homotopy which were proved in [1], [3], [4] and [18].

4.1. **Classification of nanowords and étale words.**

First, we introduce classification of nanowords with at most six letters (see [18]). Note that an arbitrary nanoword of length two is homotopic to an empty nanoword $\emptyset$ by a first homotopy move.

Theorem 4.1 (Turaev [18])**.** *Let w be a nanoword of length four over α. Then w is either homotopic to the empty nanoword or isomorphic to the nanoword $w_{a,b} := (\mathcal{A} = \{A, B\}, ABAB)$ where $|A| = a, |B| = b \in \alpha$ with $a \neq \tau(b)$. Moreover for $a \neq \tau(b)$, the nanoword $w_{a,b}$ is non-contractible and two nanowords $w_{a,b}$ and $w_{a',b'}$ are homotopic if and only if $a = a'$ and $b = b'$.*

Next we introduce homotopy classification of nanowords with length less than or equal to six. Pick three letters $a, b, c \in \alpha$ (possibly coinciding). Let $\mathcal{A}$ be an α-alphabet consisting of three letters A, B and C where $|A|$ is a, $|B|$ is b and $|C|$ is c. Consider nanowords over α, $w^1_{a,b,c} = ABCABC$, $w^2_{a,b,c} = ABCACB$, $w^3_{a,b,c} = ABCBAC$, $w^4_{a,b,c} = ABCBCA$, and $w^5_{a,b,c} = ABACBC$. It is easily checked that a nanoword of length six is either homotopic to a nanoword with length less than or equal to four or isomorphic to $w^i_{a,b,c}$ for some $i \in \{1, 2, 3, 4, 5\}$. We now point out obvious sufficient conditions for $w^i_{a,b,c}$ to be isomorphic to the empty word.

If $a = \tau(b)$ or $c = \tau(b)$, then $w^1_{a,b,c} \simeq \emptyset$. We say that an ordered triple $a, b, c \in \alpha$ is 1-*regular* if $a \neq \tau(b) \neq c$.

If $c = \tau(b)$, then $w^2_{a,b,c} \simeq \emptyset$. We say that an ordered triple $a, b, c \in \alpha$ is 2-*regular* if $c \neq \tau(b)$.

If $a = \tau(b)$, then $w^3_{a,b,c} \simeq \emptyset$. We say that an ordered triple $a, b, c \in \alpha$ is 3-*regular* if $a \neq \tau(b)$.

If $c = \tau(b)$, then $w^4_{a,b,c} \simeq \emptyset$. We say that an ordered triple $a, b, c \in \alpha$ is 4-*regular* if $c \neq \tau(b)$. (This coincides with the 2-regularity).

If $a = b = c = \tau(a)$, then $w^5_{a,b,c} \simeq \emptyset$. We say that an ordered triple $a, b, c \in \alpha$ is *singular* if $a = b = c = \tau(a)$ and 5-*regular* otherwise.

The following theorem gives the homotopy classification of nanowords of length six.

Theorem 4.2 (Turaev [18])**.** *For $i \in \{1, 2, 3, 4, 5\}$ and any i-regular triple $a, b, c \in \alpha$, the nanoword $w^i_{a,b,c}$ is neither contractible nor homotopic to a nanoword of length 4. The nanowords w^i corresponding to i-regular triples a, b, c and a', b', c' are homotopic if and only if (a, b, c) is equal to (a', b', c'). For $i \neq j$, the nanowords w^i corresponding to i-regular triples are not homotopic to nanowords w^j corresponding to j-regular triples with one exception: $w^4_{a,b,c}$ is homotopic to $w^5_{a,b,c}$ for $a = b = c \neq \tau(a)$.*

Turaev constructed some homotopy invariants of nanowords, and proved the above classification theorems in [18].

Moreover, Turaev classified words with at most five letters.

Theorem 4.3 (Turaev [18])**.** *A multiplicity-one-free word of length less than or equal to four in the alphabet α has one of the following forms: aa, aaa, aaaa, aabb, abba, abab with distinct $a, b \in \alpha$ The words aa, aabb, abba are contractible. The words aaa and aaaa are contractible if and only if $\tau(a)$ is equal to a. The word abab is contractible if and only if $\tau(a)$ is equal to b. Non-contractible words of type aaa, aaaa and abab are homotopic if and only if they are equal.*

4.2. **Classification of nanophrases and étale phrases.** Next we introduce classification theorems of nanophrases and étale phrases which were proved by the author in [1], [3] and [4].

First, we introduce the homotopy classification of nanophrases with at most four letters without condition on length. Set $P^{1,1;p,q}_a := (\emptyset|\cdots|\emptyset|\ \overset{p}{\breve{A}}\ |\emptyset|\cdots|\emptyset|\ \overset{q}{\breve{A}}\ |\emptyset|\cdots|\emptyset)$ with $|A| = a$ for $1 \leq p < q \leq k$. Classification of nanophrases with at most two letters is described as follows.

Theorem 4.4 ([3])**.** *Let P be a nanophrase of length k with 2 letters. Then P is either homotopic to $(\emptyset|\cdots|\emptyset)$ or isomorphic to $P^{1,1;p,q}_a$ for some $p, q \in \{1, \cdots k\}$, $a \in \alpha$. Moreover $P^{1,1;p,q}_a$ and $P^{1,1;p',q'}_{a'}$ are homotopic if and only if p is equal to p', q is equal to q' and a is equal to a'.*

To describe the classification theorem of nanophrases with four letters without condition on length, we use following notations.

$P^{4;p}_{a,b} := (\emptyset|\cdots|\emptyset|\ A\overset{p}{\breve{B}}AB\ |\emptyset|\cdots|\emptyset)$,

$P^{3,1;p,q}_{a,b} := (\emptyset|\cdots|\emptyset|\ A\overset{p}{\breve{B}}A\ |\emptyset|\cdots|\emptyset|\ \overset{q}{\breve{B}}\ |\emptyset|\cdots|\emptyset)$,

$P^{2,2I;p,q}_{a,b} := (\emptyset|\cdots|\emptyset|\ \overset{p}{\breve{A}}B\ |\emptyset|\cdots|\emptyset|\ \overset{q}{\breve{A}}B\ |\emptyset|\cdots|\emptyset)$,

$P^{2,2II;p,q}_{a,b} := (\emptyset|\cdots|\emptyset|\ \overset{p}{\breve{A}}B\ |\emptyset|\cdots|\emptyset|\ \overset{q}{\breve{B}}A\ |\emptyset|\cdots|\emptyset)$,

$P^{1,3;p,q}_{a,b} := (\emptyset|\cdots|\emptyset|\ \overset{p}{\breve{A}}\ |\emptyset|\cdots|\emptyset|\ B\overset{q}{\breve{A}}B\ |\emptyset|\cdots|\emptyset)$,

$P^{2,1,1I;p,q,r}_{a,b} := (\emptyset|\cdots|\emptyset|\ \overset{p}{\breve{A}}B\ |\emptyset|\cdots|\emptyset|\ \overset{q}{\breve{A}}\ |\emptyset|\cdots|\emptyset|\ \overset{r}{\breve{B}}\ |\emptyset|\cdots|\emptyset)$,

$P^{2,1,1II;p,q,r}_{a,b} := (\emptyset|\cdots|\emptyset|\ \overset{p}{\breve{B}}A\ |\emptyset|\cdots|\emptyset|\ \overset{q}{\breve{A}}\ |\emptyset|\cdots|\emptyset|\ \overset{r}{\breve{B}}\ |\emptyset|\cdots|\emptyset)$,

$P^{1,2,1I;p,q,r}_{a,b} := (\emptyset|\cdots|\emptyset|\ \overset{p}{\breve{A}}\ |\emptyset|\cdots|\emptyset|\ \overset{q}{\breve{A}}B\ |\emptyset|\cdots|\emptyset|\ \overset{r}{\breve{B}}\ |\emptyset|\cdots|\emptyset)$,

$P^{1,2,1II;p,q,r}_{a,b} := (\emptyset|\cdots|\emptyset|\ \overset{p}{\breve{A}}\ |\emptyset|\cdots|\emptyset|\ \overset{q}{\breve{B}}A\ |\emptyset|\cdots|\emptyset|\ \overset{r}{\breve{B}}\ |\emptyset|\cdots|\emptyset)$,

$P^{1,1,2I;p,q,r}_{a,b} := (\emptyset|\cdots|\emptyset|\ \overset{p}{\breve{A}}\ |\emptyset|\cdots|\emptyset|\ \overset{q}{\breve{B}}\ |\emptyset|\cdots|\emptyset|\ \overset{r}{\breve{A}}B\ |\emptyset|\cdots|\emptyset)$,

$P_{a,b}^{1,1,2II;p,q,r} := (\emptyset|\cdots|\emptyset|\overset{p}{\check{A}}|\emptyset|\cdots|\emptyset|\overset{q}{\check{B}}|\emptyset|\cdots|\emptyset|\overset{r}{\check{B}}A|\emptyset|\cdots|\emptyset)$,
$P_{a,b}^{1,1,1,1I;p,q,r,s} := (\emptyset|\cdots|\emptyset|\overset{p}{\check{A}}|\emptyset|\cdots|\emptyset|\overset{q}{\check{A}}|\emptyset|\cdots|\emptyset|\overset{r}{\check{B}}|\emptyset|\cdots|\emptyset|\overset{s}{\check{B}}|\emptyset|\cdots|\emptyset)$,
$P_{a,b}^{1,1,1,1II;p,q,r,s} := (\emptyset|\cdots|\emptyset|\overset{p}{\check{A}}|\emptyset|\cdots|\emptyset|\overset{q}{\check{B}}|\emptyset|\cdots|\emptyset|\overset{r}{\check{A}}|\emptyset|\cdots|\emptyset|\overset{s}{\check{B}}|\emptyset|\cdots|\emptyset)$,
$P_{a,b}^{1,1,1,1III;p,q,r,s} := (\emptyset|\cdots|\emptyset|\overset{p}{\check{A}}|\emptyset|\cdots|\emptyset|\overset{q}{\check{B}}|\emptyset|\cdots|\emptyset|\overset{r}{\check{B}}|\emptyset|\cdots|\emptyset|\overset{s}{\check{A}}|\emptyset|\cdots|\emptyset)$,
with $|A|$ is equal to a and $|B|$ is equal to b. If a is equal to $\tau(b)$, then nanophrases $P_{a,b}^{4;p}$, $P_{a,b}^{2,2I;p,q}$ and $P_{a,b}^{2,2II;p,q}$ are homotopic to $(\emptyset|\cdots|\emptyset)$. So when we write $P_{a,b}^{4;p}$, $P_{a,b}^{2,2I;p,q}$ and $P_{a,b}^{2,2II;p,q}$ we always assume that a is not equal to $\tau(b)$.

Under the above notations the classification of nanophrases with four letter is described as follows.

Theorem 4.5 ([3]). *Let P be a nanophrase of length k with four letters. Then P is either homotopic to nanophrase with less than or equal to two letters or isomorphic to $P_{a,b}^{X;Y}$ for some $X \in \{4,(3,1),\cdots,(1,1,1,1III)\}$, $Y \in \{1,\cdots,k,(1,2),\cdots,(k-3,k-2,k-1,k)\}$. Moreover $P_{a,b}^{X;Y}$ and $P_{a',b'}^{X';Y'}$ are homotopic if and only if $X = X'$, $Y = Y'$, $a = a'$ and $b = b'$.*

Finally we introduce the classification of étale phrases with at most four letters which are called monoliteral type.

An étale phrases P is called *monoliteral* if P has only empty word as its components or consists of a single letter. For example, étale phrases $(AAA|AAAA|\emptyset|AA)$, $(AA|A)$ and $(\emptyset|\emptyset|\emptyset)$ are monoliteral phrases. Now we consider the following étale phrases: $P_a^{1,1;l_1,l_2} := (\emptyset|\cdots|\emptyset|\overset{l_1}{\check{a}}|\emptyset|\cdots|\emptyset|\overset{l_2}{\check{a}}|\emptyset|\cdots|\emptyset)$,
$P_a^{3;l} := (\emptyset|\cdots|\emptyset|\overset{l}{\check{a}^3}|\emptyset|\cdots|\emptyset)$,
$P_a^{2,1;l_1,l_2} := (\emptyset|\cdots|\emptyset|\overset{l_1}{\check{a}^2}|\emptyset|\cdots|\emptyset|\overset{l_2}{\check{a}}|\emptyset|\cdots|\emptyset)$,
$P_a^{1,2;l_1,l_2} := (\emptyset|\cdots|\emptyset|\overset{l_1}{\check{a}}|\emptyset|\cdots|\emptyset|\overset{l_2}{\check{a}^2}|\emptyset|\cdots|\emptyset)$,
$P_a^{1,1,1;l_1,l_2,l_3} := (\emptyset|\cdots|\emptyset|\overset{l_1}{\check{a}}|\emptyset|\cdots|\overset{l_2}{\check{a}}|\emptyset|\cdots|\emptyset|\overset{l_3}{\check{a}}|\emptyset|\cdots|\emptyset)$,
$P_a^{4;l} := (\emptyset|\cdots|\emptyset|\overset{l}{\check{a}^4}|\emptyset|\cdots|\emptyset)$,
$P_a^{3,1;l_1,l_2} := (\emptyset|\cdots|\emptyset|\overset{l_1}{\check{a}^3}|\emptyset|\cdots|\emptyset|\overset{l_2}{\check{a}}|\emptyset|\cdots|\emptyset)$,
$P_a^{2,2;l_1,l_2} := (\emptyset|\cdots|\emptyset|\overset{l_1}{\check{a}^2}|\emptyset|\cdots|\emptyset|\overset{l_2}{\check{a}^2}|\emptyset|\cdots|\emptyset)$,
$P_a^{1,3;l_1,l_2} := (\emptyset|\cdots|\emptyset|\overset{l_1}{\check{a}}|\emptyset|\cdots|\emptyset|\overset{l_2}{\check{a}^3}|\emptyset|\cdots|\emptyset)$,
$P_a^{2,1,1;l_1,l_2,l_3} := (\emptyset|\cdots|\emptyset|\overset{l_1}{\check{a}^2}|\emptyset|\cdots|\overset{l_2}{\check{a}}|\emptyset|\cdots|\emptyset|\overset{l_3}{\check{a}}|\emptyset|\cdots|\emptyset)$,
$P_a^{1,2,1;l_1,l_2,l_3} := (\emptyset|\cdots|\emptyset|\overset{l_1}{\check{a}}|\emptyset|\cdots|\overset{l_2}{\check{a}^2}|\emptyset|\cdots|\emptyset|\overset{l_3}{\check{a}}|\emptyset|\cdots|\emptyset)$,
$P_a^{1,1,2;l_1,l_2,l_3} := (\emptyset|\cdots|\emptyset|\overset{l_1}{\check{a}}|\emptyset|\cdots|\overset{l_2}{\check{a}}|\emptyset|\cdots|\emptyset|\overset{l_3}{\check{a}^2}|\emptyset|\cdots|\emptyset)$,
$P_a^{1,1,1,1;l_1,l_2,l_3,l_4} := (\emptyset|\cdots|\emptyset|\overset{l_1}{\check{a}}|\emptyset|\cdots|\emptyset|\overset{l_2}{\check{a}}|\emptyset|\cdots|\emptyset|\overset{l_3}{\check{a}}|\emptyset|\cdots|\emptyset|\overset{l_4}{\check{a}}|\emptyset|\cdots|\emptyset)$,
where $a \in \alpha$ and l, l_1,l_2,l_3,$l_4 \in \hat{k}$ with $l_1 < l_2 < l_3 < l_4$.

Note that if a is equal to $\tau(a)$, then $P_a^{4;l}$ and $P_a^{3;l}$ are homotopic to the empty phrase. So when we write $P_a^{4;l}$ or $P_a^{3;l}$ we always assume that a is not equal to $\tau(a)$. Now we describe the classification theorem of monoliteral étale phrases with less than or equal to four letters.

Theorem 4.6 ([4]). *Let P be a multiplicity-one-free monoliteral étale phrase over α with less than or equal to four letters. Then P is either homotopic to $(\emptyset)_k$ or isomorphic to one of the following étale phrases: $P_a^{1,1;l_1,l_2}$, $P_a^{4;l}$, $P_a^{3,1;l_1,l_2}$, $P_a^{1,3;l_1,l_2}$, $P_a^{2,1,1;l_1,l_2,l_3}$, $P_a^{1,2,1;l_1,l_2,l_3}$, $P_a^{1,1,3;l_1,l_2,l_3}$, $P_a^{1,1,1,1;l_1,l_2,l_3,l_4}$, $P_a^{3;l}$, $P_a^{2,1;l_1,l_2}$, $P_a^{1,2;l_1,l_2}$ and $P_a^{1,1,1;l_1,l_2,l_3}$ for some l_1, l_2, l_3, $l_4 \in \hat{k}$ and $a \in \alpha$. Moreover they are homotopic if and only if they are equal with one exception : $P_a^{3,1;l_1,l_2}$ and $P_a^{1,3;l_1,l_2}$ are homotopic to $P_a^{1,1;l_1,l_2}$ if a is equal to $\tau(a)$.*

Remark 4.1. A finite sequence of homotopy moves from $P_a^{3,1;l_1,l_2}$ to $P_a^{1,1;l_1,l_2}$ is realized as follows:

$$\begin{aligned}
(P_a^{3,1;l_1,l_2})^d &= (\emptyset|\cdots|\emptyset|A_{12}A_{13}A_{14}A_{12}A_{23}A_{24}A_{13}A_{23}A_{34}|\emptyset|\cdots|\emptyset|A_{14}A_{24}A_{34}|\emptyset|\cdots|\emptyset) \\
&\simeq (\emptyset|\cdots|\emptyset|A_{13}A_{12}A_{14}A_{23}A_{12}A_{24}A_{23}A_{13}A_{34}|\emptyset|\cdots|\emptyset|A_{14}A_{24}A_{34}|\emptyset|\cdots|\emptyset) \\
&\simeq (\emptyset|\cdots|\emptyset|A_{13}A_{12}A_{23}A_{14}A_{12}A_{23}A_{24}A_{13}A_{34}|\emptyset|\cdots|\emptyset|A_{24}A_{14}A_{34}|\emptyset|\cdots|\emptyset) \\
&\simeq (\emptyset|\cdots|\emptyset|A_{13}A_{14}A_{24}A_{13}A_{34}|\emptyset|\cdots|\emptyset|A_{24}A_{14}A_{34}|\emptyset|\cdots|\emptyset) \\
&\simeq (\emptyset|\cdots|\emptyset|A_{13}A_{13}A_{34}|\emptyset|\cdots|\emptyset|A_{34}|\emptyset|\cdots|\emptyset) \\
&\simeq (\emptyset|\cdots|\emptyset|A_{34}|\emptyset|\cdots|\emptyset|A_{34}|\emptyset|\cdots|\emptyset) \\
&= (P_a^{1,1;l_1,l_2})^d.
\end{aligned}$$

Similarly a finite sequence of homotopy moves from $P_a^{1,3;l_1,l_2}$ to $P_a^{1,1;l_1,l_2}$ is realized as follows:

$$\begin{aligned}
(P_a^{1,3;l_1,l_2})^d &= (\emptyset|\cdots|\emptyset|A_{12}A_{13}A_{14}|\emptyset|\cdots|\emptyset|A_{12}A_{23}A_{24}A_{13}A_{23}A_{34}A_{14}A_{24}A_{34}|\emptyset|\cdots|\emptyset) \\
&\simeq (\emptyset|\cdots|\emptyset|A_{12}A_{13}A_{14}|\emptyset|\cdots|\emptyset|A_{12}A_{24}A_{23}A_{13}A_{34}A_{23}A_{14}A_{34}A_{24}|\emptyset|\cdots|\emptyset) \\
&\simeq (\emptyset|\cdots|\emptyset|A_{12}A_{14}A_{13}|\emptyset|\cdots|\emptyset|A_{12}A_{24}A_{23}A_{34}A_{13}A_{23}A_{34}A_{14}A_{24}|\emptyset|\cdots|\emptyset) \\
&\simeq (\emptyset|\cdots|\emptyset|A_{12}A_{14}A_{13}|\emptyset|\cdots|\emptyset|A_{12}A_{24}A_{13}A_{14}A_{24}|\emptyset|\cdots|\emptyset) \\
&\simeq (\emptyset|\cdots|\emptyset|A_{12}|\emptyset|\cdots|\emptyset|A_{12}A_{24}A_{24}|\emptyset|\cdots|\emptyset) \\
&\simeq (\emptyset|\cdots|\emptyset|A_{12}|\emptyset|\cdots|\emptyset|A_{12}|\emptyset|\cdots|\emptyset) \\
&= (P_a^{1,1;l_1,l_2})^d.
\end{aligned}$$

Proof of classification theorems of nanophrases and monoliteral phrases are described in [1], [3] and [4]. In the next section we introduce some invariants and show examples of classifications.

5. Homotopy Invariants of Nanophrases.

In this section we introduce some homotopy invariants for nanophrases which we used to prove the classification theorems.

5.1. **Component length vector.** In this sub-subsection, we define the component length vector of nanophrases (see [1], [3] and [8]).

Let $P = (w_1|w_2|\cdots|w_k)$ be a nanophrase over α. For $l \in \hat{k}$, we define $w(l) \in \mathbb{Z}/2\mathbb{Z}$ by the length of w_l. We call the vector

$$w(P) := (w(1), \cdots, w(k)) \in (\mathbb{Z}/2\mathbb{Z})^k$$

the *component length vector*.

Proposition 5.1 ([3]). *The component length vector is a homotopy invariant of nanophrases.*

Remark 5.1. Note that the component length vector is a S-homotopy invariant of nanophrases for all S.

Example 5.1. Consider nanophrases $(A|A)$ and $(\emptyset|\emptyset)$. Then $w((A|A))$ is equal to $(1, 1)$. On the other hand, $w((\emptyset|\emptyset))$ is equal to $(0, 0)$. Therefore $(A|A)$ and $(\emptyset|\emptyset)$ are not homotopic each other.

5.2. **Linking vector.** In this sub-section we introduce the linking vector of nanophrases (See [3] and [8]). Let π be the group which is defined as follows:

$$\pi := (a \in \alpha | a\tau(a) = 1, ab = ba \ for \ all \ a, b \in \alpha \ ).$$

Let P be a nanophrases $(w_1|w_2|\cdots|w_k)$ of length k over α. We define $l_P(i,j) \in \pi$ for $i < j$ by

$$l_P(i,j) := \prod_{A \in Im(w_i) \cap Im(w_j)} |A|.$$

We call a vector $lk(P) := (l_P(1,2), l_P(1,3), \cdots, l_P(1,k), l_P(2,3), \cdots, l_P(k-1,k)) \in \pi^{\frac{1}{2}k(k-1)}$ the *linking vector* of P.

Proposition 5.2 ([3])**.** *The linking vector of nanophrases is a homotopy invariant of nanophrases.*

Remark 5.2. This invariant is also S-homotopy invariant for all S.

Example 5.2. Consider nanophrases $(A|A)$ and $(B|B)$ over α where $|A|$ is equal to a and $|B|$ is equal to b. Then $lk((A|A))$ is equal to $a \in \pi$ and $lk((B|B))$ is equal to $b \in \pi$. Therefore $(A|A)$ and $(B|B)$ are homotopic if and only if a is equal to b.

6. Gibson's S_o invariant.

In the paper [8], A.Gibson defined a homotopy invariant of nanophrases over the one-element set. First we define some notations. Let $(\mathcal{A}, P = (w_1|\cdots|w_k))$ be a nanophrase over the one-element set. For a letter $A \in \mathcal{A}_i := \{A \in \mathcal{A} | Card(w_i^{-1}(A)) = 2\}$, we define $l_j(A) \in \mathbb{Z}/2\mathbb{Z}$ as follows : When we write P as $xAyAz$ where x, y and z are words in $\mathcal{A}$ possibly including "|" character, $l_j(A)$ is modulo 2 of the number of letters which appear exactly once in y and once in the j-th component of the phrase P. Then we define $l(A) \in (\mathbb{Z}/2\mathbb{Z})^k$ by

$$l(A) := (l_1(A), l_2(A), \cdots, l_k(A)).$$

Let v be a vector in $(\mathbb{Z}/2\mathbb{Z})^k$. Then we define $d_j(v) \in \mathbb{Z}$ by

$$d_j(v) := Card(\{A \in \mathcal{A}_j | l(A) = v\}),$$

and we define $B_j(P) \in 2^{(\mathbb{Z}/2\mathbb{Z})^k}$ by

$$B_j(P) := \{v \in (\mathbb{Z}/2\mathbb{Z})^k \setminus \{0\} | d_j(v) = 1 \ mod \ 2\}.$$

Then we define the $S_o(P) \in (2^{(\mathbb{Z}/2\mathbb{Z})^k})^k$ by

$$S_o(P) := (B_1(P), B_2(P), \cdots, B_k(P)).$$

Theorem 6.1 (Gibson [8])**.** *S_o is a homotopy invariant of nanophrases over the one-element set.*

Example 6.1. Consider nanophrases $(P^{2,1;l_1,l_2})^d$ and $(P^{2,1;l_1,l_2})^d$. Then

$$S_o((P_a^{2,1;l_1,l_2})^d) = (\emptyset, \cdots, \emptyset, \{\overset{l_1}{\check{\mathbf{e}}}_{l_2}\}, \emptyset, \cdots, \emptyset),$$

and

$$S_o((P_a^{2,1;m_1,m_2})^d) = (\emptyset, \cdots, \emptyset, \{\overset{m_1}{\check{\mathbf{e}}}_{m_2}\}, \emptyset, \cdots, \emptyset),$$

where $\mathbf{e}_i$ is equal to $(0, \cdots, 0, \overset{i}{\check{1}}, 0, \cdots, 0)$. Therefore we obtain that $P_a^{2,1;l_1,l_2}$ is not homotopic to $P_a^{2,1;m_1,m_2}$ if (l_1, l_2) is not equal to (m_1, m_2).

Remark 6.1. The author and A.Gibson generalized the S_o invariant for nanophrases over the one element set to a homotopy invariant over an arbitrary α in papers [5] and [10] independently. These two generalizations are equivalent. See [5] and [10] for more details.

6.1. **Invariant R_o for nanophrases over the one-element set.** In this subsection, we introduce an invariant of nanophrases over the one-element set which was defined in [4]. Let $(\mathcal{A}, P)$ be a nanophrase over the one-element set. For two letters $X \in \mathcal{A}_{l_1}$ and $Y \in \mathcal{A}_{l_2}$, we define $dl_P(X,Y) \in \mathbb{Z}/2\mathbb{Z}$ by

$$dl_P(X,Y) = Card\{Z \in \mathcal{A}_{l_1 l_2} | n(X,Z) = 1, n(Y,Z) = -1\} \ mod \ 2,$$

and for integers l_1 and l_2, we define $de_P(l_1,l_2) \in \mathbb{Z}/2\mathbb{Z}$ by

$$de_P(l_1,l_2) = Card\{(X,Y) \in \mathcal{A}_{l_1} \times \mathcal{A}_{l_2} | dl(X,Y) = 1\} \ mod \ 2.$$

Then we define $R_o(P)$ by

$$R_o(P) = (de(l_1,l_2))_{l_1<l_2}.$$

Proposition 6.1 ([4])**.** *The R_o is a homotopy invariant for nanophrases over the one-element set.*

Example 6.2. Consider the étale phrase $P_a^{2,2;l_1,l_2}$. Then

$$(P_a^{2,2;l_1,l_2})^d = (\emptyset|\cdots|\emptyset|A_{12}A_{13}A_{14}A_{12}A_{23}A_{24}|\emptyset|\cdots|\emptyset|A_{13}A_{23}A_{34}A_{14}A_{24}A_{34}|\emptyset|\cdots|\emptyset).$$

We denote $(P_a^{2,2;l_1,l_2})^d$ by P. In this caseãĂĂ

$$dl_P(A_{12}, A_{34}) = Card\{A_{14}\} = 1$$

and

$$de_P(i,j) = \begin{cases} 1 \text{ if } (i,j) = (l_1,l_2), \\ 0 \text{ otherwise.} \end{cases}$$

Therefore $R_o(P)$ is equal to $\mathbf{e}_{(l_1,l_2)}$. On the other hand $R_o((\emptyset|\cdots|\emptyset))$ is equal to $\mathbf{0}$. Therefore, this example shows that $P_a^{2,2;l_1,l_2}$ is not homotopic to the empty phrase.

Using the above invariants and some properties on nanophrases and étale phrases, we can classify nanophrases and monoliteral phrases at most four letters without condition on length.

7. An Application to Curves on Surfaces.

By the theorems in Section 3, if we put α that is equal to α_0 and τ is equal to τ_o, then we obtain the classification of pointed ordered curves on surfaces up to stable equivalence.

7.1. **Applications of the classification of nanophrases.** In the papers [1], [2] and [4], the author proved the following corollaries.

Corollary 7.1 ([1])**.** *There are exactly 19 stable equivalence classes of two-component pointed, ordered, oriented, curves on surfaces with minimum crossing number less than or equal to 2.*

More generally we can prove a following statement.

Corollary 7.2 ([2])**.** *Let k be an positive integer. Then there are exactly*

$$1 + \frac{1}{2}k^2 + k^3 + \frac{1}{2}k^4$$

stable equivalence classes of ordered, pointed, k-component surface curves with minimal crossing number less than or equal to two.

An ordered, pointed multi-component surface-curve is called *irreducible* if it is not stably equivalent to a surface-curve with a simply closed component.

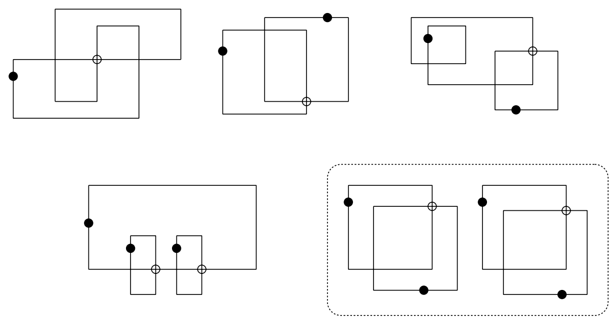

FIGURE 6. The list of irreducible curves. See also Remark 7.1.

Corollary 7.3 ([3]). *Any irreducible ordered, pointed multi-component surface-curve with minimal crossing number less than or equal to two is stably equivalent to one of the ordered, pointed multi-component curves arising from the following list (see also Remark 7.1). There are exactly* 52 *stable equivalence classes of irreducible ordered, pointed, multi-component surface-curves.*

Remark 7.1. We would like to list up the all stable equivalence classes of irreducible ordered, pointed multi-component surface-curves with minimal crossing number less than or equal to two. However there are too many curves to list up. Therefore we make just the list of multi-component curves without order and orientation of the components in Figure 6. If we choose order and orientation of components, then we obtain a ordered, pointed multi-component curve on surface. Two different pictures from Figure 6 never produce equivalent ordered, pointed multi-component curves on surfaces. On the other hand it is possible that two different additional structures (orientation and the order) on the same picture yield equivalent ordered, pointed multi-component curves on surfaces. More precisely, 2 (respectively 2, 8, 4, 24, 12) different ordered, pointed multi-component surface-curves arise from the upper left (respectively upper middle, upper right, lower left, lower middle, lower right) picture. By Theorem 4.5, ordered, pointed multi-component surface-curves arising from pictures in Figure 6 are stably equivalent if and only if nanophrases associated to these curves are homotopic, and we can obtain all of the stable equivalent classes of irreducible ordered, pointed multi-component curves on surfaces with minimal crossing number less than or equal to two by specifying order and orientation for multi-component curves in Figure 6.

7.2. **An application of the classification of monoliteral phrases.** In this sub-section we introduce an application of the classification of monoliteral phrases with at most four letters. To do so, we introduce a notion of monoliteral type curves. A curve on a surface is called of *monoliteral type* if the curve is stably equivalent to a curve which corresponds to a nanophrase obtained by desingularization of a monoliteral phrase. Now we describe the classification of irreducible monoliteral ordered pointed multi-component curves on surfaces with minimal crossing number less than or equal to six.

Corollary 7.4. *Any irreducible monoliteral ordered pointed multi-component curve on a surface with minimal crossing number less than or equal to six is stably equivalent to one of the ordered, pointed multi-component curves in Figures 7 and 8. Therefore there are exactly* 26 *stable equivalence classes of irreducible ordered, pointed, multi-component surface-curves.*

Remark 7.2. Curves in Figure 7 correspond to monoliteral phrases of type $P_a^{X;Y}$ and curves in Figure 8 correspond to monoliteral phrases of type $P_b^{X;Y}$.

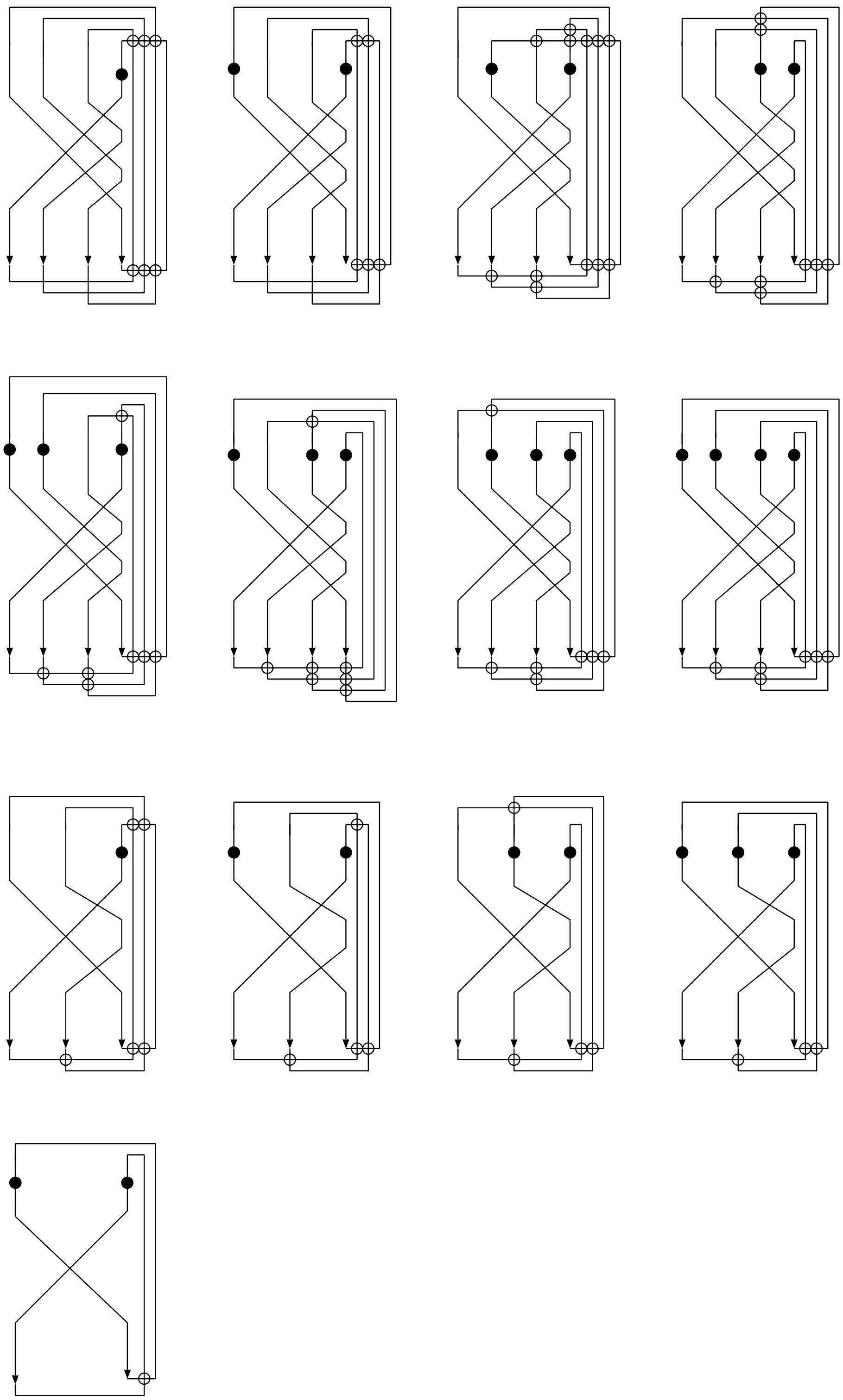

FIGURE 7. The half of list of monoliteral curves. Each component is numerated from right to left.

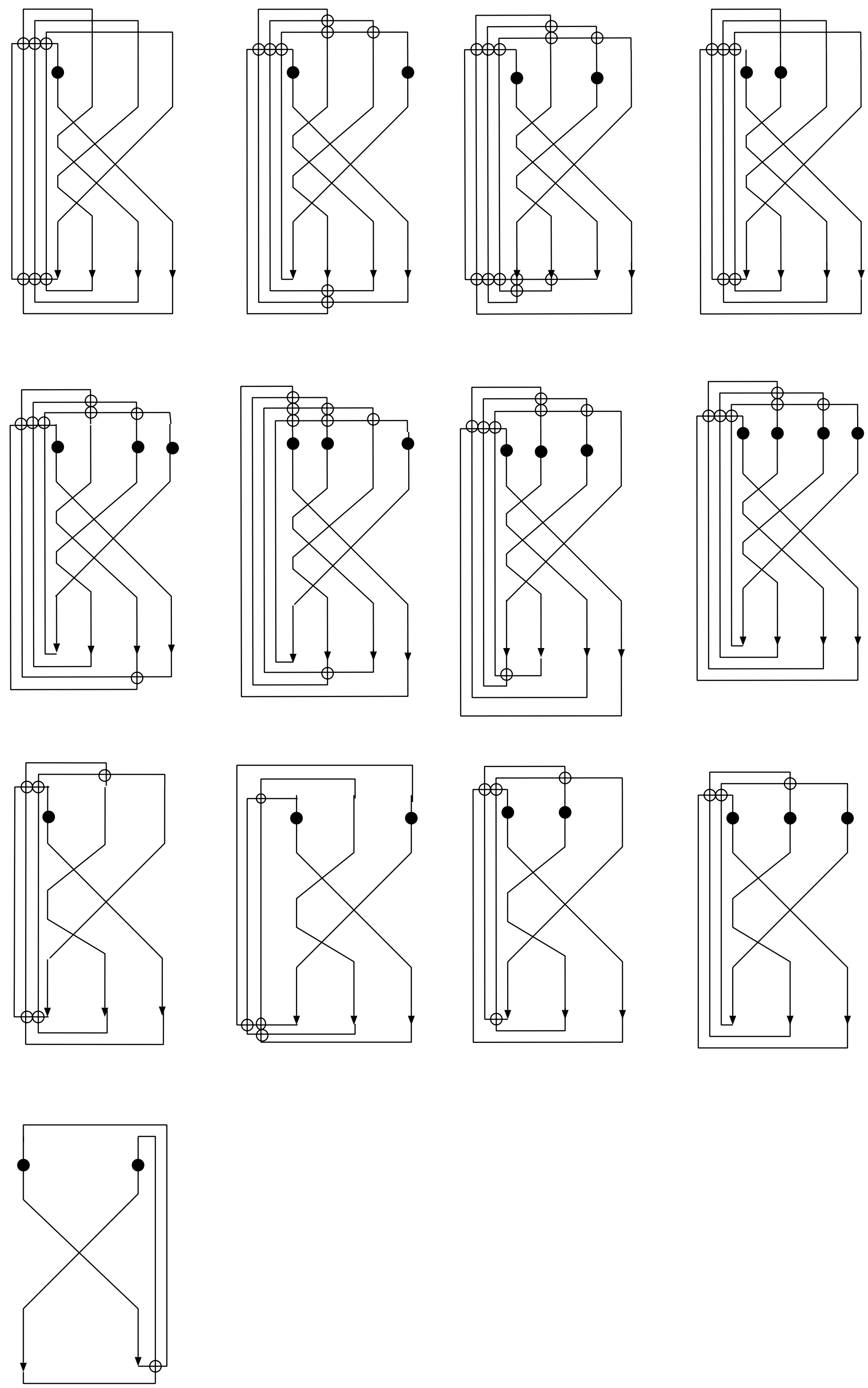

FIGURE 8. The half of list of monoliteral curves. Each component is numerated from left to right.

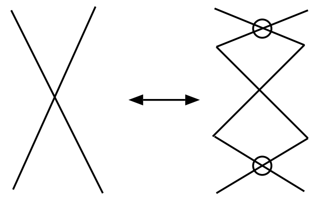

FIGURE 9. A flat virtualization move

Proof. We put that α is equal to α_0, and τ is equal to τ_0, then by Theorem 4.6 we obtain the list of a complete representable system of homotopy class of nanophrases which does not contain empty words as components of phrase as follows: $(aaaa)$, $(aaa|a)$, $(aa|aa)$, $(a|aaa)$, $(aa|a|a)$, $(a|aa|a)$, $(a|a|aa)$, $(a|a|a|a)$, (aaa), $(aa|a)$, $(a|aa)$, $(a|a|a)$, $(a|a)$, $(bbbb)$, $(bbb|b)$, $(bb|bb)$, $(b|bbb)$, $(bb|b|b)$, $(b|bb|b)$, $(b|b|bb)$, $(b|b|b|b)$, (bbb), $(bb|b)$, $(b|bb)$, $(b|b|b)$ and $(b|b)$. Note that in this case $\tau_0(a)$ is not equal to a and $\tau_0(b)$ is not equal to b, therefore $(ccc|c)$, $(c|ccc)$ and $(c|c)$ are not homotopic each other for each $c \in \{a, b\}$. Therefore by Theorem 3.1 there are exactly 26 stable equivalence classes of pointed ordered irreducible curves on surfaces of monoliteral type with at most six crossings.

Moreover by the correspondence of curves and phrases, we obtain the list of curves on surfaces in Figures 7 and 8. □

Remark 7.3. In the paper [8], A. Gibson classified un-pointed oriented flat virtual virtual knots with at most four crossings using the theory of nanowords. See [8] for more details.

8. An application to free links.

In this subsection, we give the classification of ordered pointed free links with some conditions using the classification of nanophrases and monoliteral phrases.

The theory of free knots and links was introduced by V . O. Manturov in [15] and [16]. A *free link* is an equivalence class of flat virtual link diagrams modulo flat virtual Reidemeister moves and *flat virtualization move* which is illustrated in Figure 9. We can define *ordered, pointed, irreducible* and *monoliteral* for free links similarly as in the case for flat virtual links.

It is known that there is a canonical bijection between the set of ordered pointed k-component free links and the set of homotopy classes of nanophrases over the one element set $\{a\}$ with the involution $a \mapsto a$. See [9] and [15] for example.

Now we apply the classification of nanophrases and monoliteral phrases to the classification of ordered pointed irreducible free links.

Corollary 8.1. *There are exactly* 12 *irreducible ordered pointed free links with at most two real crossings.*

Proof. We put α is equal to $\{a\}$, and τ is equal to the identity map on $\{a\}$, then by the Theorem 4.5 we obtain the list of a complete representable system of homotopy classes of nanophrases which does not contain empty words as components of phrase as follows: $(ABA|B)$, $(A|BAB)$, $(AB|A|B)$, $(BA|A|B)$, $(A|AB|B)$, $(A|BA|B)$, $(A|B|AB)$, $(A|B|BA)$, $(A|B|A|B)$, $(A|B|B|A)$, $(A|A|B|B)$ and $(A|A)$ where $|A|$ and $|B|$ are equal to a. Therefore there are 12 irreducible ordered pointed free links with at most two real crossings. □

Corollary 8.2. *There are exactly nine irreducible ordered pointed free links of monoliteral type with at most six real crossings.*

Proof. We put α is equal to $\{a\}$, and τ is equal to the identity map on $\{a\}$, then by Theorem 4.6 we obtain the list of a complete representable system of homotopy classes of nanophrases which does not contain empty words as components of phrase as follows: $(aa|aa)$, $(aa|a|a)$, $(a|aa|a)$, $(a|a|aa)$, $(a|a|a|a)$, $(aa|a)$, $(a|aa)$, $(a|a|a)$ and $(a|a)$. Therefore there are nine irreducible ordered pointed free links of monoliteral type with at most six real crossings. $\square$

Remark 8.1. We can construct the table of irreducible ordered pointed free links of monoliteral type with at most six crossings similarly as in the case of curves on surfaces. It is similar to the Figure 7. If we delete curves which correspond to $(aaaa)$, $(aaa|a)$, $(a|aaa)$ and (aaa), then we obtain the required table. Therefore we avoid drawing the table.

Remark 8.2. From Corollary 8.1, there are no pointed free knots with at most two crossings. Examples of non trivial (pointed) free knots were found by V. O. Manturov and A. Gibson independently. See [15] and [9] for more details. On the other hand, by Corollary 8.2, there are no pointed free knots with at most six crossings. More generally, in the paper [18] Turaev proved there is no pointed free knot of monoliteral type in terms of the theory of nanowords. See [18] for more details.

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SINGULARITIES OF TANGENT VARIETIES TO CURVES AND SURFACES

GOO ISHIKAWA

Abstract. It is given the diffeomorphism classification on generic singularities of tangent varieties to curves with arbitrary codimension in a projective space. The generic classifications are performed in terms of certain geometric structures and differential systems on flag manifolds, via several techniques in differentiable algebra. It is provided also the generic diffeomorphism classification of singularities on tangent varieties to contact-integral curves in the standard contact projective space. Moreover we give basic results on the classification of singularities of tangent varieties to generic surfaces and Legendre surfaces.

1. Introduction

Embedded tangent spaces to a submanifold draw a variety in the ambient space, which is called the *tangent variety* to the submanifold. Tangent varieties appear in various geometric problems and applications naturally. See for instance [1][10][6]. Developable surfaces, varieties with degenerate Gauss mapping and varieties with degenerate projective dual are obtained by tangent varieties. Tangent varieties provide several important examples of non-isolated singularities in applications of geometry. We observe relations of tangent varieties to invariant theory and geometric theory of differential equations (see [29], also see Examples 2.7 and 9.1).

It is known, in the three dimensional Euclidean space, that the tangent variety (tangent developable) to a generic space curve has singularities each of which is locally diffeomorphic to the *cuspidal edge* or to the *folded umbrella* (*cuspidal cross cap*), as is found by Cayley and Cleave [9]. Cuspidal edge singularities appear along ordinary points, while the folded umbrella appears at an isolated point of zero torsion [6][35].

The classification was generalised to more degenerate cases by Mond [32][33] and Scherbak [38][4] and applied to various geometry (see for instance [8][24]). If we consider a curve together with its osculating framings, we are led to the classification of tangent varieties to generic osculating framed curves, possibly with singularities in themselves, in the three dimensional space. Then the list consists of 4 singularities: *cuspidal edge*, *folded umbrella* and moreover *swallowtail* and *Mond surface* ('cuspidal beak to beak') [20]. However the author could not find any literature treating the classification of singularities appearing in tangent varieties to higher codimensional curves.

The diffeomorphism types of tangent varieties to curves are invariant under projective transformations. In this paper, we consider curves in projective spaces and show the classification results on generic singularities of tangent varieties to curves with arbitrary codimension in projective spaces.

The tangent variety can be defined for a 'frontal' variety. A frontal variety has the well-defined embedded tangent space at each point, even where the variety is singular. In Cauchy problem of single unknown function, we have wave-front sets, which are singular hypersurfaces [3]. They are called fronts and form an important class of frontal varieties. Also higher codimensional wave-fronts are examples of frontal varieties, which appear in, for instance, Cauchy problem of several unknown functions, where initial submanifolds of arbitrary codimension evolve to frontal varieties (cf. [13][26]).

First, in §2, we introduce the notion of frontal maps and frontal varieties, generalising that of submanifolds and fronts (Definition 2.1). Moreover we define their tangent maps and tangent varieties (Definition 2.2). Then we give the classification of tangent varieties to generic curves in projective spaces (Theorem 2.6). In fact we find that the tangent variety to a generic curve in $\mathbf{R}P^{N+1}$ has the unique singularity, the higher codimensional cuspidal edge, if $N+1 \geq 4$.

In the geometric theory of curves, however, we usually treat not just curves but we attach an appropriate frame with curves. Thus, to solve the generic classification problem properly, we relate the study of tangent varieties to certain kinds of differential systems on appropriate flag manifolds in §3. Note that the method was initiated by Arnol'd and Scherbak [38]. Also note that it is standard to use flag manifolds in the theory of space curves ([40]). We can utilise various types of flag manifolds. In fact, in this paper, we select three kinds of flag manifolds and three kinds of differential systems, correspondingly to the classes of curves endowed with osculating-frames, with tangent-frames and with tangent-principal-normal-frames. Then we present the classification results on the singularities of which generically appear for these three kinds of classes of curves in projective spaces (Theorems 3.3, 3.4, 3.6).

In §4, the notion of types of curve-germs are recalled. Curves of finite type are frontal and their tangent varieties are frontal. We classify the generic types of curves, and then we show a kind of determinacy of the tangent variety for each generic type of curves.

In §5, we classify the list of types of generic curves satisfying geometric conditions. To do this, we establish the codimension formulae giving the codimension of the set of curves, for given type, which satisfy a given geometric integrality condition in each case. Then the transversality theorem implies the restriction on types of generic curves.

In §6, we introduce the key notion of openings of differentiable map-germs, which has close relations with that of frontal varieties. We collect necessary results on differentiable algebras to solve the generic classification problems treated in this paper. Moreover, in §7, using the method of differentiable algebra, we show the normal forms of tangent varieties appearing in the generic classification problems we have treated in this paper. In particular the main results in this paper, Theorems 2.6, 3.3, 3.4 and 3.6 are proved.

In §8, we treat contact-integral curves and their tangent varieties. If V is a symplectic vector space, then the projective space $P(V)$ has the canonical contact structure. Then we give the generic diffeomorphism classification of singularities on tangent varieties to 'osculating framed contact-integral' curves in $P(V)$ (Theorems 8.5, 8.6). For this, in particular, we show that the diffeomorphism type of $\mathrm{Tan}(\gamma)$ is unique for a curve of type $(1,3,4,6)$ in $\mathbf{R}P^4$ in this paper. Note that it is known that the diffeomorphism type of $\mathrm{TanTan}(\gamma)$ is not unique ([18]).

In §9, we treat the classification problem of singularities of tangent varieties to surfaces, exhibiting several examples and observations. First we observe that the tangent varieties to generic smooth surfaces are not frontal. We characterise the class of surfaces whose tangent varieties are frontal. In particular we show that the tangent varieties to Legendre submanifolds in the five dimensional standard contact projective space $P(\mathbf{R}^6) = \mathbf{R}P^5$ are frontal, if the tangent variety has a dense regular set. Recall that the singularity of tangent variety to a curve along ordinary points is the cuspidal edge. Therefore the singularity of tangent variety at almost any point on a curve is diffeomorphic to cuspidal edge, which is a generic singularity of wave front. We study the analogous problem for tangent varieties to Legendre surfaces. Then we observe that the situation becomes absolutely different. In fact we introduce the notion of hyperbolic and elliptic ordinary points on Legendre surface in $\mathbf{R}P^5$ and show that the transverse section of the tangent variety to the surface, by a 3-plane, has D_4-singularities (Theorem 9.5).

In the last section §10, we collect open problems related to several results obtained in this paper.

In this paper all manifolds and maps are assumed to be of class C^∞ unless otherwise stated.

2. Frontal maps and tangent varieties.

Definition 2.1. Let N and M be manifolds of dimension n and m respectively. Suppose $n \leq m$.

A mapping $f : N^n \to M^m$ is called *frontal* if
(i) the regular locus

$$\mathrm{Reg}(f) = \{x \in N \mid f : (N, x) \to (M, f(x)) \text{ is an immersion}\}$$

of f is dense in N and
(ii) there exists a C^∞ mapping $\widetilde{f} : N \to \mathrm{Gr}(n, TM) = \bigcup_{y \in M} \mathrm{Gr}(n, T_y M)$ satisfying

$$\widetilde{f}(x) = f_*(T_x N), \quad \text{for } x \in \mathrm{Reg}(f).$$

Here $\mathrm{Gr}(n, T_y M)$ is the Grassmannian of n-planes in $T_y M$. Note that the lifting $\widetilde{f}$ is uniquely determined if it exists and is called the *Grassmannian lifting* of f.

We define a subbundle $\mathcal{C} \subset T\mathrm{Gr}(n, TM)$ by setting, for $v \in T_L \mathrm{Gr}(n, TM), L \in T_y M$,

$$v \in \mathcal{C}_L \Longleftrightarrow \pi_*(v) \in L \subset T_y M.$$

The differential system $\mathcal{C}$ is called the *canonical differential system*. The Grassmannian lifting $\widetilde{f}$ is a $\mathcal{C}$-integral map, that is, $\widetilde{f}_*(TN) \subset \mathcal{C}$. We describe the canonical system in the next section (Remark 3.7) in the case M is a projective space.

If f is an immersion, then f is frontal. A wave-front hypersurface is frontal. The key observation for the classification of singularities of tangent varieties is that the tangent variety $\mathrm{Tan}(\gamma)$ to a curve γ of finite type is frontal. The lifting Grassmannian is obtained by taking osculating planes to the curves (See §4). If $n = m$, then f is frontal if the condition (i) is fulfilled, $\widetilde{f}(x)$ being $T_{f(x)}\mathbf{R}^m$.

If $\widetilde{f}$ is an immersion, then the frontal mapping is called a *front*. In [17], we called a frontal hypersurface ($m = \ell + 1$), a "front hypersurface". However we would like to reserve the notion "front" for the case that the Grassmannian lifting is an immersion, as in the Legendre singularity theory. Note that frontal maps are studied also in [28][27][37].

Definition 2.2. Let $f : (\mathbf{R}^n, a) \to (\mathbf{R}^m, b), n \leq m$ be a frontal map-germ and

$$\widetilde{f} : (\mathbf{R}^n, a) \to \mathrm{Gr}(n, T\mathbf{R}^m) \cong \mathbf{R}^m \times \mathrm{Gr}(n, \mathbf{R}^m)$$

be the Grassmannian lifting of f.

A *tangent frame* to f means a system of vector fields $v_1, \ldots, v_n : (\mathbf{R}^n, a) \to T\mathbf{R}^m$ along f such that $v_1(x), \ldots, v_n(x)$ form a basis of $\widetilde{f}(x) \subset T_{f(x)}\mathbf{R}^m$. Then the *tangent map* $\mathrm{Tan}(f, v) : (\mathbf{R}^{2n}, (a, 0)) \to (\mathbf{R}^m, b)$ is defined by

$$\mathrm{Tan}(f, v)(s, x) := f(x) + \sum_{i=1}^{n} s_i v_i(x).$$

If we choose another tangent frame $u_1, \ldots, u_n$ of f and define

$$\mathrm{Tan}(f, u)(s, x) = f(x) + \sum_{i=1}^{n} s_i u_i(x).$$

Then $\mathrm{Tan}(f, u)$ and $\mathrm{Tan}(f, v)$ are right-equivalent. Therefore the *tangent variety* $\mathrm{Tan}(f)$ to a frontal map-germ is uniquely determined as a parametrised variety.

For a frontal map-germ $f : (\mathbf{R}^n, 0) \to \mathbf{R}P^{N+1}$ in a projective space we define the tangent map $\mathrm{Tan}(f) : (\mathbf{R}^n, 0) \to \mathbf{R}P^{N+1}$ by taking a local projective coordinate $(\mathbf{R}P^{N+1}, f(0)) \to (\mathbf{R}^{N+1}, 0)$ (cf. §4).

Remark 2.3. In this paper we treat only tangent varieties, which are closely related to the *secant varieties*. The secant variety of a submanifold $S \subset \mathbf{R}P^n$ is the ruled variety obtained by taking the union of secants connecting two distinct points on S and by taking its closure ([41][12]). See also Example 9.1. The secant variety is parametrised by the 'secant map' and the tangent map is the 'boundary' of secant map in some sense. For the singularities of secant maps, see [14].

Let $\gamma : (\mathbf{R}, 0) \to \mathbf{R}P^{N+1}$ be a germ of immersion and $\gamma(t) = (x_1(t), x_2(t), \dots, x_{N+1}(t))$ be a local representation of γ. Then $\gamma'(t)$ gives the tangent frame of γ. Then the tangent variety to γ is given by $\mathrm{Tan}(\gamma) : (\mathbf{R}^2, 0) \to \mathbf{R}^{N+1}$ defined by

$$\mathrm{Tan}(\gamma)(s,t) = \gamma(t) + s\,\gamma'(t) = (x_i(t) + s\,x_i'(t))_{1 \le i \le N+1}\,.$$

Note that s is the parameter of tangent lines, while t is the parameter of the original curve γ.

If $t = 0$ is a singular point of γ, then the velocity vector $\gamma'(0) = 0$, and hence the above map-germ does not give the parametrisation of the tangent variety. However if there is $k > 0$ such that $v(t) = (1/t^k)\gamma'(t)$ is a tangent frame of γ, then we set

$$\mathrm{Tan}(\gamma)(s,t) = \gamma(t) + s\left(\frac{1}{t^k}\gamma'(t)\right) = \left(x_i(t) + s\left(\frac{1}{t^k}x_i'(t)\right)\right)_{1 \le i \le N+1}.$$

We take $k = 0$ when γ is an immersion at 0.

In the above case, γ is frontal and under a mild condition $\mathrm{Tan}(\gamma)$ is also frontal.

Theorem 2.4. *Let $\gamma : (\mathbf{R}, 0) \to \mathbf{R}P^{N+1}$ be a curve of finite type (§4). Then γ is frontal. Moreover the tangent map $\mathrm{Tan}(\gamma) : (\mathbf{R}^2, 0) \to \mathbf{R}P^{N+1}$ of γ is frontal.*

Theorem 2.4 is proved in §4.

Remark 2.5. Let γ be a curve of finite type. Then it is natural to ask what $\mathrm{Tan}(\mathrm{Tan}(\gamma))$ is, because $\mathrm{Tan}(\gamma)$ is frontal. For a curve γ in $\mathbf{R}P^{N+1}, N \geq 2$, the tangent plane to $\mathrm{Tan}(\gamma)$ along each ruling (tangent line) is constant, that is the osculating 2-plane. Therefore $\mathrm{Tan}(\mathrm{Tan}(\gamma))$ is a 3-fold, not a 4-fold, ruled by osculating 2-planes of the original curve γ ([18]).

We classify the map-germ $\mathrm{Tan}(\gamma)$ by local right-left diffeomorphism equivalence. Two map-germs $f : (N, a) \to (M, b)$ and $f' : (N', a') \to (M', b')$ are called *diffeomorphic* or *right- left equivalent* if there exist diffeomorphism-germs $\sigma : (N, a) \to (N', a')$ and $\tau : (M, b) \to (M', b')$ such that $f' \circ \sigma = \tau \circ f$.

In the followings, I is an open interval.

Theorem 2.6. (1) ([9]) *For a generic curve $\gamma : I \to \mathbf{R}P^3$ in C^∞-topology, the curve γ is of finite type at each point in I and the tangent variety $\mathrm{Tan}(\gamma)$ to γ at each point in I is locally diffeomorphic to the cuspidal edge or to the folded umbrella (cuspidal cross cap).*

(2) *Let $N + 1 \geq 4$. For a generic curve $\gamma : I \to \mathbf{R}P^{N+1}$ in C^∞-topology, the curve γ is of finite type at each point in I and the tangent variety $\mathrm{Tan}(\gamma)$ to γ at each point of I is locally diffeomorphic to the cuspidal edge.*

The genericity means the existence of an open dense subset $\mathcal{O} \subset C^\infty(I, \mathbf{R}P^{N+1})$ such that any $\gamma \in \mathcal{O}$ satisfies the consequence.

The *cuspidal edge* is parametrised by the map-germ $(\mathbf{R}^2, 0) \to (\mathbf{R}^{N+1}, 0), (N+1 \geq 3)$ defined by

$$(u, x) \mapsto (u,\ x^2,\ x^3,\ 0, \dots,\ 0).$$

Note that it is diffeomorphic (right-left equivalent) to the germ

$$(t, s) \mapsto (t+s,\ t^2+2st,\ t^3+3st^2,\ \dots,\ t^{N+1}+(N+1)st^N),$$

and also to

$$(t, s) \mapsto (t+s,\ t^2+2st,\ t^3+3st^2,\ 0, \dots,\ 0),$$

A *folded umbrella* is parametrised by the germ $(\mathbf{R}^2, 0) \to (\mathbf{R}^3, 0)$ defined by

$$(t, s) \mapsto (t+s, t^2+2st, t^4+4st^3),$$

which is diffeomorphic to

$$(u, x) \mapsto (u,\ x^2+ux,\ \frac{1}{2}x^4+\frac{1}{3}ux^3).$$

A folded umbrella is often called a cuspidal cross cap.

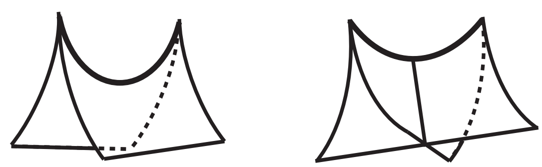

FIGURE 1. cuspidal edge and folded umbrella.

Theorem 2.6 is proved in §7.

Example 2.7. (umbilical bracelet) Let

$$V^{N+2} = \{a_0x^{N+1} + a_1x^Ny + \cdots + a_Nxy^N + a_{N+1}y^{N+1}\} \cong \mathbf{R}^{N+2}$$

be the space of homogeneous polynomials of degree $N+1$ in two variables x, y. The polynomials with zeros of multiplicity $N+1$ form a curve C in $P(V) \cong \mathbf{R}P^{N+1}$. The tangent variety $\mathrm{Tan}(C)$ to C coincides with the set of polynomials with zeros of multiplicity $\geq N$. The surface $\mathrm{Tan}(C)$ has cuspidal edge singularities along C. In particular in the case $N+1=3$, the tangent variety $\mathrm{Tan}(C)$ to C is called the *umbilical bracelet*([35][11]). If $N+1 \geq 4$, $\mathrm{Tan}(\mathrm{Tan}(C)) \subset P(V^{N+2})$ coincides with of polynomials with with zeros of multiplicity $\geq N-1$.

Remark 2.8. The tangent surface to a curve is obtained as a union of strata of envelope generated by the dual curve to the original curve. The generating family associated to the dual curve is determined, up to parametrised $\mathcal{K}$-equivalence in several cases. We recall the notion of types of curves in a projective space in §4. If the type $\mathbf{A} = (a_1, \dots, a_{N+1})$ of a curve in $\mathbf{R}P^{N+1}$ is one of followings

$$\begin{array}{lcl}
(\mathrm{I})_{N,r} & : & (1, 2, \dots, N, N+r),\ (r = 0, 1, 2, \dots), \\
(\mathrm{II})_{N,i} & : & (1, 2, \dots, i, i+2, \dots, N+1, N+2),\ (0 \leq i \leq N-1), \\
(\mathrm{III})_{N} & : & (3, 4, \dots, N+2, N+3),
\end{array}$$

then the generating family is determined by the type of the curve [16]. In each case, a normal form of the tangent variety can be obtained from the generating family

$$F(t, x) = t^{a_{N+1}} + x_1t^{a_{N+1}-a_1} + x_2t^{a_{N+1}-a_2} + \cdots + x_Nt^{a_{N+1}-a_N} + x_{N+1} = 0,$$

by solving

$$F = 0, \quad \frac{\partial F}{\partial t} = 0, \quad \dots, \quad \frac{\partial^{N-1} F}{\partial t^{N-1}} = 0,$$

deleting the divisor $\{t = 0\}$ if necessary. For example, for the type $(\mathrm{II})_{3,2} : (1, 2, 4, 5)$, we have generating family

$$F(t, x) = t^5 + x_1 t^4 + x_2 t^3 + x_3 t + x_4.$$

Then the tangent variety is obtained by solving

$$\begin{cases} t^5 + x_1 t^4 + x_2 t^3 + x_3 t + x_4 = 0, \\ 5t^4 + 4x_1 t^3 + 3x_2 t^2 + x_3 = 0, \\ 20t^3 + 12x_1 t^2 + 6x_2 t = 0. \end{cases}$$

In fact, from these equations, we get a map-germ $(\mathbf{R}^2, 0) \to (\mathbf{R}^4, 0)$ by

$$x_2 = -\frac{10}{3} t^2 - 2x_1 t, \quad x_3 = 5t^4 + 2x_1 t^3, \quad x_4 = -\frac{8}{3} t^5 - x_1 t^4,$$

which is diffeomorphic to the open folded umbrella (see Theorems 3.6, 7.2).

3. Differential systems on flag manifolds.

First we recall the flag manifolds and the canonical differential systems on flag manifolds. For the generality on differential systems, see [23].

Let V be a vector space of dimension n and $0 < n_1 < n_2 < \cdots < n_\ell < n$. Then we define the flag manifold

$$\mathcal{F} = \mathcal{F}_{n_1, n_2, \dots, n_\ell}(V) := \left\{ V_{n_1} \subset V_{n_2} \subset \cdots \subset V_{n_\ell} \subset V \mid \dim(V_{n_j}) = n_j, (1 \leq j \leq \ell) \right\}.$$

Note that

$$\dim(\mathcal{F}) = n_1(n - n_1) + (n_2 - n_1)(n - n_2) + \cdots + (n_\ell - n_{\ell-1})(n - n_\ell).$$

Denote by $\pi_i : \mathcal{F}_{n_1, n_2, \dots, n_\ell}(V) \to \mathrm{Gr}(n_i, V)$ the canonical projection to the i-th member of the flag. The canonical differential system $\mathcal{C} = \mathcal{C}_{n_1, n_2, \dots, n_\ell} \subset T\mathcal{F}$ is defined by, for $v \in T_{\mathbf{V}}\mathcal{F}, \mathbf{V} \in \mathcal{F}$,

$$v \in \mathcal{C}_{\mathbf{V}} \Longleftrightarrow \pi_{i*}(v) \in T\mathrm{Gr}(n_i, V_{n_{i+1}}) (\subset T\mathrm{Gr}(n_i, V)), (1 \leq i \leq \ell - 1).$$

Then $\mathcal{C}$ is a bracket-generating (completely non-integrable) subbundle of $T\mathcal{F}$ with

$$\mathrm{rank}(\mathcal{C}) = n_1(n_2 - n_1) + (n_2 - n_1)(n_3 - n_2) + \cdots + (n_\ell - n_{\ell-1})(n - n_\ell).$$

A C^∞ curve $\Gamma : I \to \mathcal{F}$ from an open interval I is called a $\mathcal{C}$-*integral curve* if $\Gamma'(t) \in \mathcal{C}_{\Gamma(t)}$ for any $t \in I$. A $\mathcal{C}$-integral curve can be phrased as a C^∞-family

$$c(t) = (V_{n_1}(t), V_{n_2}(t), \dots, V_{n_\ell}(t))$$

of flags in $\mathcal{F}$ such that each $V_{n_i}(t)$ moves along $V_{n_{i+1}}(t)$ at every moment infinitesimally.

Let V be an $(N + 2)$-dimensional vector space. For the study of tangent varieties to curves, it is natural to regard the following flag manifolds

$$\mathcal{F}_{1,2} = \mathcal{F}_{1,2}(V) := \{V_1 \subset V_2 \subset V \mid \dim(V_i) = i\},$$

and

$$\mathcal{F}_{1,2,3} = \mathcal{F}_{1,2,3}(V) := \{V_1 \subset V_2 \subset V_3 \subset V \mid \dim(V_i) = i\}.$$

The canonical systems $\mathcal{T} = \mathcal{C}_{1,2}$ and $\mathcal{N} = \mathcal{C}_{1,2,3}$ are defined as follows: For $(V_1, V_2) \in \mathcal{F}_{1,2}$,

$$v \in \mathcal{T}_{(V_1, V_2)} \Longleftrightarrow \pi_{1*}(v) \in TP(V_2) (\subset TP(V)).$$

For $(V_1, V_2, V_3) \in \mathcal{F}_{1,2,3}$,

$$w \in \mathcal{N}_{(V_1,V_2,V_3)} \Longleftrightarrow \pi_{1*}(w) \in TP(V_2)(\subset TP(V)),\ \pi_{2*}(w) \in T\mathrm{Gr}(2, V_3)(\subset T\mathrm{Gr}(2, V)).$$

Then we have

Proposition 3.1. *Let $\gamma : (\mathbf{R}, 0) \to P(V^{N+2}) \cong \mathbf{R}P^{N+1}$ be a C^∞ curve. Suppose* $\mathrm{Reg}(\gamma)$ *is dense in $(\mathbf{R}, 0)$. Then γ is frontal if and only if $\gamma = \pi_1 \circ c$ for some $\mathcal{C}_{1,2}$-integral curve $c : (\mathbf{R}, 0) \to \mathcal{F}_{1,2}(V)$.*

In fact c gives a tangent frame of γ. In this case, γ is called *tangent-framed*.

Proposition 3.2. *Let $\gamma : (\mathbf{R}, 0) \to P(V^{N+2}) \cong \mathbf{R}P^{N+1}$ be a frontal curve. Suppose* $\mathrm{Reg}(\mathrm{Tan}(\gamma))$ *is dense in $(\mathbf{R}^2, 0)$. Then* $\mathrm{Tan}(\gamma)$ *is frontal if and only if $\gamma = \pi_1 \circ \kappa$ for some $\mathcal{C}_{1,2,3}$-integral curve $\kappa : (\mathbf{R}, 0) \to \mathcal{F}_{1,2,3}(V)$.*

In fact, if $\mathrm{Tan}(\gamma)$ is frontal, then $V_1(t) = \gamma(t)$, the tangent (projective) line $V_2(t)$ to γ at t and the tangent (projective) plane to $\mathrm{Tan}(\gamma)$ at $(t, 0)$ form a $\mathcal{C}_{1,2,3}$-integral lifting of γ. Conversely if $c(t) = (V_1(t), V_2(t), V_3(t))$ is a $\mathcal{C}_{1,2,3}$-integral curve, then $\mathrm{Tan}(\gamma)$ has the constant tangent plane $V_3(t)$ along each ruling, and $(t, s) \mapsto T_{V_1(t)}P(V_3)(t)$ gives the Grassmannian lifting of $\mathrm{Tan}(\gamma)$.

The projection of a $\mathcal{C}_{1,2,3}$-integral curve is called a *tangent-principal-nomal-framed* curve.

Theorem 3.3. (1) *Let $N + 1 = 3$. For a generic $\mathcal{C}_{1,2}$-integral curve $c : I \to \mathcal{F}_{1,2}(V^4)$ in C^∞-topology, the tangent variety* $\mathrm{Tan}(\gamma)$ *to the tangent-framed curve $\gamma = \pi_1 \circ c : I \to P(V^4) = \mathbf{R}P^3$ at each point is locally diffeomorphic to the cuspidal edge, the folded umbrella or the swallowtail.*

(2) *Let $N + 1 \geq 4$. For a generic $\mathcal{C}_{1,2}$-integral curve $c : I \to \mathcal{F}_{1,2}(V^{N+2})$ in C^∞-topology, the tangent variety* $\mathrm{Tan}(\gamma)$ *to the tangent-framed curve $\gamma = \pi_1 \circ c : I \to P(V) = \mathbf{R}P^{N+1}$ at each point is locally diffeomorphic to the cuspidal edge or the open swallowtail.*

The *swallowtail* $(\mathbf{R}^2, 0) \to (\mathbf{R}^3, 0)$ is given by

$$(t, s) \mapsto (t^2 + 2s,\ t^3 + 3st,\ t^4 + 4st^2),$$

which is diffeomorphic to

$$(u, x) \mapsto (u,\ x^3 + ux,\ \frac{3}{4}x^4 + \frac{1}{2}ux^2).$$

The *open swallowtail* $(\mathbf{R}^2, 0) \to (\mathbf{R}^{N+1}, 0), N + 1 \geq 4$ is given by

$$(t, s) \mapsto (t^2 + 2s,\ t^3 + 3st,\ t^4 + 4st^2,\ t^5 + 5st^3,\ 0,\ \ldots,\ 0),$$

which is diffeomorphic to

$$(u, x) \mapsto (u,\ x^3 + ux,\ \frac{3}{4}x^4 + \frac{1}{2}ux^2,\ \frac{3}{5}x^5 + \frac{1}{3}ux^3,\ 0,\ \ldots,\ 0).$$

Theorem 3.4. (1) *Let $N + 1 = 3$. For a generic $\mathcal{C}_{1,2,3}$-integral curve $\kappa : I \to \mathcal{F}_{1,2,3}(V^4)$ in C^∞-topology, the tangent variety* $\mathrm{Tan}(\gamma)$ *to the tangent-principal-normal-framed curve $\gamma = \pi_1 \circ \kappa : I \to P(V^4) = \mathbf{R}P^3$ at each point is locally diffeomorphic to the cuspidal edge, the folded umbrella, the Mond surface or the swallowtail.*

(2) *Let $N + 1 \geq 4$. For a generic $\mathcal{C}_{1,2,3}$-integral curve $\kappa : I \to \mathcal{F}_{1,2,3}(V^{N+2})$ in C^∞-topology, the tangent variety* $\mathrm{Tan}(\gamma)$ *to the tangent-principal-normal-framed curve $\gamma = \pi_1 \circ \kappa : I \to P(V) = \mathbf{R}P^{N+1}$ at each point is locally diffeomorphic to the cuspidal edge, the open Mond surface or the open swallowtail.*

The *Mond surface* $(\mathbf{R}^2, 0) \to (\mathbf{R}^3, 0)$ is given by

$$(t + s,\ t^3 + 3st^2,\ t^4 + 4st^3),$$

which is diffeomorphic to

$$(u,x) \mapsto (u,\ x^3+ux^2,\ \frac{3}{4}x^4+\frac{2}{3}ux^3)).$$

The Mond surface is called also *cuspidal beaks* ([25]) or *cuspidal beak to beak* ('bec à bec').

The *open Mond surface* $(\mathbf{R}^2,0)\to(\mathbf{R}^{N+1},0), N+1\geq 4$ is given by

$$(t,s)\mapsto(t+s,\ t^3+3st^2,\ t^4+4st^3,\ t^5+5st^4,\ 0,\ \ldots,\ 0),$$

$$(u,x)\mapsto(u,\ x^3+ux^2,\ \frac{3}{4}x^4+\frac{2}{3}ux^3,\ \frac{3}{5}x^5+\frac{1}{2}ux^4,\ 0,\ \ldots,\ 0).$$

Now we recall on osculating-framed curves (cf. [20]). Let V be an $(N+2)$-dimensional real vector space. Consider the complete flag manifold:

$$\mathcal{F}=\mathcal{F}_{1,2,\ldots,N+1}(V):=\{V_1\subset V_2\subset\cdots V_{N+1}\subset V \mid \dim(V_i)=i, 1\leq i\leq N+1\}.$$

Then $\dim\mathcal{F}=\frac{(N+1)(N+2)}{2}$. We denote by $\pi_i:\mathcal{F}\to\mathrm{Gr}(i,V)$ the canonical projection

$$\pi_i(V_1,V_2,\ldots,V_{N+1})=V_i.$$

The canonical system $\mathcal{C}=\mathcal{C}_{1,2,\ldots,N+1}\subset T\mathcal{F}$ is defined by

$$v\in\mathcal{C}_{(V_1,\ldots,V_{N+1})}\Longleftrightarrow \pi_{i*}(v)\in T\mathrm{Gr}(i,V_{i+1})(\subset T\mathrm{Gr}(i,V)),(1\leq i\leq N).$$

For a C^∞ curve $\gamma:I\to P(V)=\mathbf{R}P^{N+1}$, if we consider Frenet-Serret frame, or the osculating projective moving frame, $\Gamma=(e_0(t),e_1(t),\ldots,e_{N+1}(t)):I\to\mathrm{GL}(\mathbf{R}^{N+2})=\mathrm{GL}(N+2,\mathbf{R})$, $\gamma(t)=[e_0(t)]$, then, setting $V_i(t):=\langle e_0(t),e_1(t),\ldots,e_{i-1}(t)\rangle_{\mathbf{R}},(1\leq i\leq N+1)$, we have a $\mathcal{C}$-integral lifting $\widetilde{\gamma}:I\to\mathcal{F}$ of γ for the projection $\pi_1:\mathcal{F}\to P(V)$, by $\widetilde{\gamma}(t)=(V_1(t),V_2(t),\ldots,V_{N+1}(t))$. In this case, γ is called *osculating-framed*. Note that the framing of an osculating-framed curve is uniquely determined if an orientation of the curve and a metric on $P(V)$ are given.

Theorem 3.5. ([20]) *Let $N+1=3$. For a generic $\mathcal{C}$-integral curve $c:I\to\mathcal{F}(V^4)$ in C^∞-topology, the tangent variety* $\mathrm{Tan}(\gamma)$ *to the osculating-framed curve $\gamma=\pi_1\circ c:I\to P(V^4)=\mathbf{R}P^3$ at each point of I is locally diffeomorphic to the cuspidal edge, the folded umbrella, the swallowtail or to the Mond surface* (Figure 2).

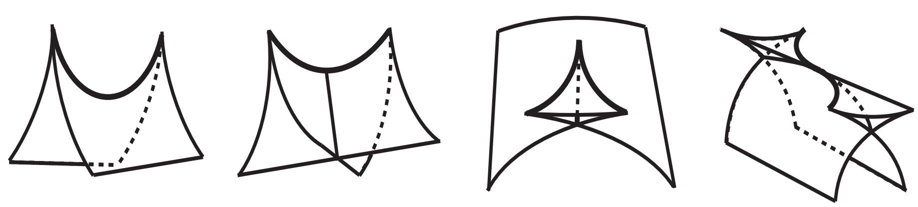

FIGURE 2. cuspidal edge, folded umbrella, swallowtail and Mond surface in $\mathbf{R}^3$.

In this paper we treat higher codimensional cases, and we show the following

Theorem 3.6. *Let $N+1\geq 4$. For a generic $\mathcal{C}$-integral curve $c:I\to\mathcal{F}$ in C^∞-topology, the tangent variety to the osculating-framed curve $\gamma=\pi_1\circ c:I\to P(V^{N+2})=\mathbf{R}P^{N+1}$ at each point is locally diffeomorphic to the cuspidal edge, the open folded umbrella* (*cuspidal non-cross cap*), *the open swallowtail or to the open Mond surface* (Figure 3).

The *open folded umbrella* $(\mathbf{R}^2, 0) \to (\mathbf{R}^{N+1}, 0), N \geq 3$ is given by

$$(t, s) \mapsto (t + s,\ t^2 + 2st,\ t^4 + 4st^3,\ t^5 + 5st^4,\ 0,\ \dots,\ 0),$$

which is diffeomorphic to

$$(u, x) \mapsto (u,\ x^2 + ux,\ \frac{1}{2}x^4 + \frac{1}{3}ux^3,\ \frac{2}{5}x^5 + \frac{1}{4}ux^4,\ 0,\ \dots,\ 0).$$

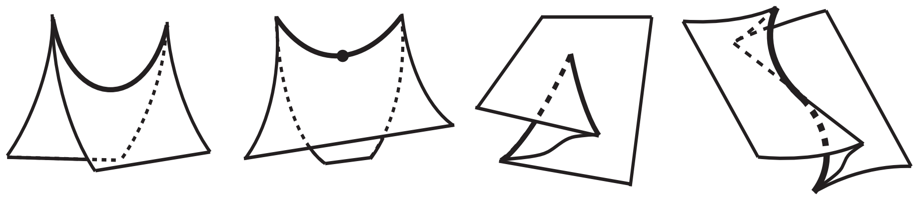

FIGURE 3. cuspidal edge, open folded umbrella, open swallowtail and open Mond surface in $\mathbf{R}^4$.

Our main results, Theorems 3.3, 3.4, 3.6 are proved in §7.

Lastly in this section, we describe the canonical system $\mathcal{C} = \mathcal{C}_{1,2,\dots,k+1}$ on $\mathcal{F}_{1,2,\dots,k+1}(V^{N+2})$. Let $\mathbf{V}_1 = (V_{11}, V_{21}, \dots, V_{k+1,1}) \in \mathcal{F}_{1,2,\dots,k+1}(V^{N+2})$. Fix a flag $V^{N+2} \supset W_{N+1} \supset W_N \supset W_{N-k+1}$ such that $W_{N-i+1} \cap V_{i+1\ 1} = \{0\}$, $i = 0, 1, \dots, k$. Take the open neighbourhood U of $\mathbf{V}_1$ defined by

$$U := \left\{(V_1, V_2, \dots, V_{k+1}) \in \mathcal{F}_{1,2,\dots,k+1}(V^{N+2}) \ \mid \ W_{N-i+1} \cap V_{i+1} = \{0\},\ i = 0, 1, \dots, k\right\}.$$

Take non-zero vectors $e_0 \in V_{11}, e_1 \in V_{21} \cap W_{N+1}, e_2 \in V_{31} \cap W_N, \dots, e_k \in V_{k+1\,1} \cap W_{N-k+2}$. Adding a basis $(e_{k+1}, \dots, e_{N+1})$ of W_{N-k+1}, we get a basis $(e_0, e_1, e_2, \dots, e_k, e_{k+1}, \dots, e_{N+1})$ of V. Then, for each $\mathbf{V} = (V_1, V_2, \dots, V_{k+1})$, V_i has a basis $v_0, v_1, \dots, v_{i-1}$ (a 'moving frame') of the form

$$v_i = e_i + \sum_{j=i+1}^{N+1} x_j^{\ i} e_j, \quad 0 \leq i \leq k.$$

Then the condition that a curve in $\mathcal{F}_{1,2,\dots,k+1}(V^{N+2})$ $\mathcal{C}$-integral is equivalent to that the components of the curve satisfies the conditions

$$(v_{i-1})' = \sum_{j=i}^{N+1} (x_j^{\ i-1})' e_j \in \langle v_0, v_1, \dots, v_i \rangle_{\mathcal{E}_1}, \quad 1 \leq i \leq k.$$

Thus we see that the differential system $\mathcal{C} = \mathcal{C}_{1,2,\dots,k+1}$ is defined by

$$dx_j^{\ i-1} - x_j^{\ i} dx_i^{\ i-1} = 0, \quad (1 \leq i \leq k, i + 1 \leq j \leq N + 1),$$

for the system of local coordinates $\left(x_j^{\ i}\right)_{0 \leq i \leq k, i+1 \leq j \leq N+1}$ of $\mathcal{F}_{1,2,\dots,k+1}(V^{N+2})$.

Remark 3.7. For a $(N+2)$-dimensional vector space V, the Grassmannian bundle $\mathrm{Gr}(n, TP(V^{N+2}))$ over $P(V^{N+2})$ is identified with the flag manifold $\mathcal{F}_{1,n+1}(V^{N+2})$,

$$\mathcal{F}_{1,n+1}(V^{N+2}) = \{V_1 \subset V_{n+1} \subset V^{N+2} \mid \dim(V_1) = 1, \dim(V_{n+1}) = n + 1\}.$$

Remark that the Grassmannian liftings of frontal maps $N^n \to P(V^{N+2})$ are $\mathcal{C}$-integral of the canonical system $\mathcal{C} = \mathcal{C}_{1,n+1}$.

The canonical system $\mathcal{C}_{1,n+1}$ on $\mathcal{F}_{1,n+1}(V^{N+2})$ is locally given by

$$dx_{i+1}^{0} - \sum_{j=1}^{n} x_{i+1}^{j} dx_{j}^{0} = 0, \quad (n \le i \le N),$$

for a system of local coordinates $x_{i+1}^{0}, (0 \le i \le N), x_{i+1}^{j}, (1 \le j \le n, n \le i \le N)$. The projection $\pi_1 : \mathrm{Gr}(n, TP(V^{N+2})) \to P(V^{N+2})$ is represented by $({x_1}^0, \dots, x_{N+1}^{0})$. If we write $x_i = {x_i}^0$ $(1 \le i \le n), y_k = x_{n+k}^{0}$ $(1 \le k \le N-n+1)$ and ${p_k}^i = x_{n+k}^{i}$ $(1 \le k \le N-n+1, 1 \le i \le n)$, then we have

$$dy_k - \sum_{i=1}^{n} {p_k}^i dx_i = 0, \quad 1 \le k \le N-n+1.$$

Therefore the condition that a map $F : L^n \to \mathrm{Gr}(n, TP(V^{N+2}))$ is $\mathcal{C}$-integral is expressed by

$$d(y_k \circ F) - \sum_{i=1}^{n} ({p_k}^i \circ F)\, d(x_i \circ F) = 0, \quad 1 \le k \le N-n+1.$$

4. Type of a curve in a space with flat projective structure.

Let M be an m-dimensional C^∞ manifold. A *flat projective structure* on M is given by an atlas $\{(U_\alpha, \varphi_\alpha)\}$ where $M = \bigcup_\alpha U_\alpha$, $\varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha) \subset \mathbf{R}^m$, and transition functions $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ are fractional linear with a common denominator. Then an admissible chart is called a *system of projective local coordinates*. The projective space $P(V^{m+1})$ for a vector space V^{m+1} has the canonical flat projective structure. Also Grassmannians and Lagrange Grassmannians have flat projective structures (cf. [22]).

Let $\gamma : I \to M$ be a C^∞-curve in a manifold M with a flat projective structure. Take a system of projective local coordinates $(x_1, x_2, \dots, x_m)$ centred at $\gamma(t_0)$ and the local affine representation $(\mathbf{R}, t_0) \to (\mathbf{R}^m, 0)$,

$$\gamma(t) = {}^T(x_1(t), x_2(t), \dots, x_m(t))$$

of γ. Consider the $(m \times k)$-matrix

$$W_k(t) := \Big(\gamma'(t_0), \gamma''(t_0), \cdots, \gamma^{(k)}(t_0)\Big)$$

for any integer $k \ge 1$ and $k = \infty$. Note that the rank of $W_k(t_0)$ is independent of the choice on representations for γ.

Definition 4.1. We call γ *of finite type* at $t = t_0 \in I$ if the $(m \times \infty)$-matrix

$$W_\infty(t_0) = \Big(\gamma'(t_0),\ \gamma''(t_0),\ \cdots,\ \gamma^{(k)}(t_0),\ \cdots\cdots\Big)$$

is of rank m. Define, for $1 \le i \le m$, $a_i := \min\{k \mid \mathrm{rank}\, W_k(t_0) = i\}$. Then we have a sequence of natural numbers $1 \le a_1 < a_2 < \cdots < a_m$, and we call γ *of type* $(a_1, a_2, \dots, a_m)$ at $t = t_0 \in I$.

If $(a_1, a_2, \dots, a_m) = (1, 2, \dots, m)$, then $t = t_0$ is called an *ordinary point* of γ.

It is easy to see

Lemma 4.2. *A curve-germ $\gamma : (\mathbf{R}, 0) \to M$ in a manifold M with a flat projective structure, is of type $(a_1, a_2, \dots, a_m)$ at 0 if and only if there exists a system of projective local coordinates $(x_1, x_2, \dots, x_m)$ centred at $\gamma(0)$ such that*

$$x_1(t) = t^{a_1} + o(t^{a_1}),\ x_2(t) = t^{a_2} + o(t^{a_2}),\ \dots,\ x_m(t) = t^{a_m} + o(t^{a_m}).$$

Lemma 4.3. *Let $\gamma : (\mathbf{R}, 0) \to P(\mathbf{R}^{N+2}) = \mathbf{R}P^{N+1}$ be a curve and $\widetilde{\gamma} : (\mathbf{R}, 0) \to \mathbf{R}^{N+2} \setminus \{0\}$ be a lifting of γ. Set $\widetilde{W_r}(t) = (\widetilde{\gamma}(t), \widetilde{\gamma}'(t), \cdots, \widetilde{\gamma}^{(r)}(t))$. Then γ is of type $\mathbf{A} = (a_1, a_2, \dots, a_{N+1})$ if and only if $a_i = \min\{r \mid \operatorname{rank} \widetilde{W_r}(t_0) = i + 1\}, 1 \le i \le N + 1$.*

Moreover we see

Lemma 4.4. *Let $\gamma : (\mathbf{R}, 0) \to P(\mathbf{R}^{N+2}) = \mathbf{R}P^{N+1}$ a curve of finite type. There is unique $\mathcal{C}_{1,2,3,\dots,N+1}$-integral C^∞ lifting $\Gamma : (\mathbf{R}, 0) \to \mathcal{F}_{1,2,3,\dots,N+1}(\mathbf{R}^{N+2})$ of γ. Moreover by the projection of Γ, we have $\mathcal{C}_{1,2,3}$-integral lifting $\kappa : (\mathbf{R}, 0) \to \mathcal{F}_{1,2,3}(\mathbf{R}^{N+1})$ and $\mathcal{C}_{1,2}$-integral lifting $c : (\mathbf{R}, 0) \to \mathcal{F}_{1,2}(\mathbf{R}^{N+1})$ of γ.*

Proof: The first half is proved in [20] (Lemma 6.1). We take the lifting $\widetilde{\gamma} : (\mathbf{R}, 0) \to \mathbf{R}^{N+2} \setminus \{0\}$ defined by

$$\widetilde{\gamma}(t) = {}^T \left(1,\ t^{a_1} + o(t^{a_1}),\ t^{a_2} + o(t^{a_2}),\ \dots,\ t^{a_{N+1}} + o(t^{a_{N+1}})\right).$$

of γ. Consider the $(N + 2) \times (N + 2)$-matrix

$$A(t) = \left(\widetilde{\gamma}(t), \frac{1}{a_1!}\widetilde{\gamma}^{(a_1)}(t), \cdots, \frac{1}{a_{N+1}!}\widetilde{\gamma}^{(a_{N+1})}(t)\right).$$

Let $V_i(t)$ be the linear subspace of $\mathbf{R}^{N+2}$ generated by the first i- columns of $A(t)$. Then $\Gamma : (\mathbf{R}, 0) \to \mathcal{F}_{1,2,3,\dots,N+1}(\mathbf{R}^{N+2})$ is uniquely determined by $\Gamma(t) = (V_1(t), V_2(t), \dots, V_{N+1}(t))$. The lower triangle components of $A(t)$ give the local representation of Γ, therefore Γ is C^∞. Moreover $\kappa(t) = (V_1(t), V_2(t), V_3(t))$ and $c(t) = (V_1(t), V_2(t))$. $\square$

Proof of Theorem 2.4 : Theorem 2.4 follows from Lemma 4.4 and Proposition 3.2. Here we give concretely the Grassmannian lifting of $\operatorname{Tan}(\gamma)$ in term of Wronskian.

Lemma 4.5. *Let $\gamma : (\mathbf{R}, 0) \to \mathbf{R}P^{N+1}$ be a curve-germ of type $(a_1, a_2, \dots, a_{N+1})$ and*

$$\gamma(t) = (x_1(t), x_2(t), \dots, x_{N+1}(t))$$

be a local affine representation of γ. Then the tangent variety to γ is parametrised by

$$f(s, t) = \operatorname{Tan}(\gamma)(s, t) := \gamma(t) + s\,\frac{1}{\alpha(t)}\gamma'(t) = \left(x_i(t) + s\,\frac{1}{\alpha(t)}x_i'(t)\right)_{1\le i\le N+1},$$

where $\alpha(t) = t^{a_1-1}$. We set $f_i(s, t) = x_i(t) + \dfrac{s}{\alpha(t)}x_i'(t)$. Then we have

$$\frac{W_{i2}}{W_{12}}\,df_1 + \frac{W_{1i}}{W_{12}}\,df_2.$$

Here

$$W_{ij}(t) = \begin{vmatrix} x_i'(t) & x_j'(t) \\ x_i''(t) & x_j''(t) \end{vmatrix}.$$

Proof: We have

$$df_i(s, t) = \frac{x_i'(t)}{\alpha(t)}\,ds + \left(x_i'(t) + s\left(\frac{x_i'(t)}{\alpha(t)}\right)'\right)dt.$$

Then we have

$$\begin{vmatrix} x_1' & x_2' \\ x_1' + s\left(\frac{x_1'}{\alpha}\right)' & x_2' + s\left(\frac{x_2'}{\alpha}\right)' \end{vmatrix} = \frac{s}{\alpha}\,W_{12}.$$

Therefore we have, for $3 \leq i \leq n+1$,

$$
\begin{aligned}
df_i &= \frac{\alpha}{sW_{12}}\,(df_1,\ df_2)\begin{pmatrix} x_2' + s\left(\frac{x_2'}{\alpha}\right)' & -x_2' \\ -x_1' - s\left(\frac{x_1'}{\alpha}\right)' & x_1' \end{pmatrix}\begin{pmatrix} x_i' \\ x_i' + s\left(\frac{x_i'}{\alpha}\right)' \end{pmatrix} \\
&= \frac{W_{i2}}{W_{12}}\,df_1 + \frac{W_{1i}}{W_{12}}\,df_2.
\end{aligned}
$$

□

Remark 4.6. Note that $\dfrac{W_{i2}}{W_{12}}$ and $\dfrac{W_{1i}}{W_{12}}$ are C^∞ functions on t of order $a_i - a_1, a_i - a_2$ respectively. The above formula gives the Grassmannian lifting $\widetilde{f} : (\mathbf{R}^2, 0) \to \mathrm{Gr}(2, T\mathbf{R}^{N+1})$ of $f = \mathrm{Tan}(\gamma)$.

Remark 4.7. If we set $g : (\mathbf{R}^2, 0) \to (\mathbf{R}^2, 0),\ g(s,t) = (f_1(s,t), f_2(s,t))$. Then we have that $f_3, \ldots, f_{n+1} \in \mathcal{R}_g$ and that f is an opening of g in the sense of §6.

5. Codimension formulae and the genericity.

We consider the jet space $J^r(I, \mathbf{R}P^{N+1})$. Let $\mathbf{A} = (a_1, a_2, \ldots, a_N, a_{N+1})$ be a strictly increasing sequence of positive integers. For $r > a_{N+1}$, we define

$$\Sigma(\mathbf{A}) = \{ j^r\gamma(t_0) \mid t_0 \in I, \gamma : (I, t_0) \to \mathbf{R}P^{N+1} \text{ is of type } \mathbf{A} \text{ at } t_0 \}.$$

Theorem 5.1. ([38]) $\Sigma(\mathbf{A})$ *is a semi-algebraic submanifold of codimension* $\sum_{i=1}^{N+1}(a_i - i)$ *in the jet space* $J^r(I, \mathbf{R}P^{N+1})$.

Proof: Let $J^r(1, N+1)$ be the fibre of the projection $\pi : J^r(I, \mathbf{R}P^{N+1}) \to I \times \mathbf{R}P^{N+1}$. Then $J^r(1, N+1)$ is identified with the space $\mathbf{R}^{(N+1)r}$ of $(N+1) \times r$-matrices. Then there exists an affine subspace $\Lambda \subset \mathbf{R}^{(N+1)r}$ such that $\Sigma(\mathbf{A})$ is an image of the polynomial embedding $\mathrm{GL}(N+1, \mathbf{R}) \times \Lambda \to \mathbf{R}^{(N+1)r}$ defined by $(A, W) \mapsto AW$ for $A \in \mathrm{GL}(N+1, \mathbf{R}), W \in \Lambda$. Therefore $\Sigma(\mathbf{A})$ is a semi- algebraic manifold.

The codimension of the set consisting of jets with $\mathrm{rank}(W_{a_1-1}) = 0$ is equal to $(N+1)(a_1 - 1)$. The codimension of the set consisting of jets with $\mathrm{rank}(W_{a_1-1}) = 0, \mathrm{rank}(W_{a_1}) = 1$ and $\mathrm{rank}(W_{a_2-1}) = 1$ is equal to $(N+1)(a_1-1)+N(a_2-a_1-1)$. Thus we have that the codimension of $\Sigma(\mathbf{A})$ is calculated as

$$(N+1)(a_1 - 1) + N(a_2 - a_1 - 1) + (N-1)(a_3 - a_2 - 1) + \cdots + (a_{N+1} - a_N - 1),$$

which is equal to $\sum_{i=1}^{N+1}(a_i - i)$. □

Corollary 5.2. *For a generic curve* $\gamma : I \to \mathbf{R}P^{N+1}$*, and for any* $t_0 \in I$*, the type of* γ *at* t_0 *is equal to*

$$(1, 2, 3, \ldots, N, N+1) \ \text{ or } \ (1, 2, 3, \ldots, N, N+2).$$

Proof: By the transversality theorem, there exists an open dense subset $\mathcal{O} \subset C^\infty(I, \mathbf{R}P^{N+1})$ in C^∞-topology such that for any $\gamma \in \mathcal{O}$ and for any $t_0 \in I$, the type $\mathbf{A}$ of γ at t_0 satisfies $\sum_{i=1}^{N+1}(a_i - i) \leq 1$. Then we have $a_i = i, 1 \leq i \leq N$ and $a_{N+1} = N+1$ or $a_{N+1} = N+2$, and thus we have the required result. □

To treat osculating-framed curves, we consider the jet space of $\mathcal{C}$-integral curves, $\mathcal{C} = \mathcal{C}_{1,2,\ldots,N+1}$, $J^r_{\mathcal{C}}(I, \mathcal{F}) \subset J^r(I, \mathcal{F})$. Define

$$\Sigma_{\mathcal{C}}(\mathbf{A}) := \{ j^r\Gamma(t_0) \mid \Gamma : (\mathbf{R}, t_0) \to \mathcal{F} \text{ is } \mathcal{C}\text{-integral}, \ \ \pi_1 \circ \Gamma \text{ is of type } \mathbf{A} \}$$

in $J^r_{\mathcal{C}}(I, \mathcal{F})$ for sufficiently large r.

Theorem 5.3. ([20]) *$J^r_{\mathcal{C}}(I,\mathcal{F})$ is a submanifold of $J^r(I,\mathcal{F})$ and the codimension of $\Sigma_{\mathcal{C}}(\mathbf{A})$ in $J^r_{\mathcal{C}}(I,\mathcal{F})$ is equal to $a_{N+1}-(N+1)$.*

Remark 5.4. Since any curve of finite type lifts to an $\mathcal{C}$-integral curve, $\Sigma_{\mathcal{C}}(\mathbf{A})$ is not empty for any $\mathbf{A}$.

By the transversality theorem for $\mathcal{C}$-integral curves, we have the following result:

Theorem 5.5. *For a generic $\mathcal{C}$-integral curve $\Gamma : I \to \mathcal{F}_{1,2,\dots,N+1}(V^{N+2})$, the type $\mathbf{A}$ of $\pi_1\circ\Gamma$ at any point of I is given by one of the following:*

$$\mathbf{A}=(1,2,3,\dots,N,N+1),\quad (1,2,\dots,i,i+2,\dots,N+1,N+2),(i=0,\dots,N).$$

Proof: By Theorem 5.3, for a genetic Γ, the type of $\pi_1\circ\Gamma$ at a point in I satisfies that $a_{N+1}-(N+1)\leq 1$, namely that $a_{N+1}\leq N+2$. Then we have the list of types. □

In general, we consider the canonical system $\mathcal{C}=\mathcal{C}_{1,2,\dots,k+1}$ on $\mathcal{F}=\mathcal{F}_{1,2,\dots,k+1}(V^{N+2})$, we consider the jet space of $\mathcal{C}$-integral curves, $J^r_{\mathcal{C}}(I,\mathcal{F})\subset J^r(I,\mathcal{F})$. Define

$$\Sigma_{\mathcal{C}}(\mathbf{A}) := \{j^rc(t_0) \mid c : (\mathbf{R},t_0)\to\mathcal{F} \text{ is } \mathcal{C}\text{-integral},\ \ \pi_1\circ c \text{ is of type } \mathbf{A}\}$$

in $J^r_{\mathcal{C}}(I,\mathcal{F})$ for sufficiently large r.

Theorem 5.6. *$J^r_{\mathcal{C}}(I,\mathcal{F})$ is a submanifold of $J^r(I,\mathcal{F})$ and the codimension of $\Sigma_{\mathcal{C}}(\mathbf{A})$ in $J^r_{\mathcal{C}}(I,\mathcal{F})$ is equal to*

$$\sum_{i=k}^{N+1}(a_i-i)-(N-k+1)(a_k-k).$$

Note that, if $k=N$, the formula is reduced to $a_{N+1}-(N+1)$ (Theorem 5.3).

Proof of Theorem 5.6: Recall that $\mathcal{C}=\mathcal{C}_{1,2,\dots,k+1}$ is defined by

$$dx_j^{\,i-1}-x_j^{\,i}dx_i^{\,i-1}=0,\quad (1\leq i\leq k, i+1\leq j\leq N+1)$$

for the system of local coordinates $\left(x_j^{\,i}\right)_{0\leq i\leq k, i+1\leq j\leq N+1}$ of $\mathcal{F}_{1,2,\dots,k+1}(V^{N+2})$ (§3). Then a $\mathcal{C}$-integral curve $\Gamma: I\to\mathcal{F}$ is obtained just form $x_i^{\,i-1}$-components, $1\leq i\leq k$, and $x_j^{\,k}$-components, by integration. Then we see, at each point $t_0\in I$, $\mathrm{ord}(x_j^{\,0}=\sum_{\ell=1}^{j}\mathrm{ord}(x_j^{\,j-1})$. We have that the type of Γ at t_0 is equal to $\mathbf{A}=(a_1,\dots,a_{N+1})$ if and only if

$$\mathrm{ord}(x_1^{\,0})=a_1,\ \mathrm{ord}(x_2^{\,1})=a_2-a_1,\ \dots,\ \mathrm{ord}(x_k^{\,k-1})=a_k-a_{k-1},$$

and the type of the curve $(x^k_{k+1},\dots,x^k_{N+1}) : (I,t_0)\to\mathbf{R}^{N-k}$ is of type $(a_{k+1}-a_k,\dots,a_{N+1}-a_k)$. Thus the codimension of $\Sigma_{\mathcal{C}}(\mathbf{A})$ is calculated as

$$(a_1-1)+(a_2-a_1-1)+\cdots+(a_k-a_{k-1}-1)+\sum_{k+1}^{N+1}(a_j-a_k-(j-k))=\sum_{i=k}^{N+1}(a_i-i)-(N-k+1)(a_k-k).$$

□

Remark 5.7. *Let $\pi : \mathcal{F}_{1,2,\dots,k,k+1}\to\mathcal{F}_{1,2,\dots,k}$ be the canonical projection defined by*

$$\pi(V_1,V_2,\dots,V_k,V_{k+1})=(V_1,V_2,\dots,V_k).$$

Then the π-fibres are projective subspaces of the flag manifold $\mathcal{F}_{1,2,\dots,k+1}$. In the above proof, the functions $x^k_{k+1},\dots,x^k_{N+1}$ form a system of local projective coordinates of the π-fibre.

By the transversality theorem for $\mathcal{C}$-integral curves, we have the following results:

Theorem 5.8. *For a generic $\mathcal{C}_{1,2}$-integral curve c, the type $\mathbf{A}$ of the tangent-framed curve $\pi_1 \circ c$ at any point of I is given by one of the following:*

$$(1, 2, 3, \dots, N, N+1),\ (1, 2, 3, \dots, N, N+2),\ (2, 3, 4, \dots, N+1, N+2).$$

Proof: By Theorem 5.6, for a genetic c, the type of $\pi_1 \circ c$ at a point in I satisfies that $\sum_{i=1}^{N+1}(a_i - i) - N(a_1 - 1) \leq 1$, namely that $\sum_{i=1}^{N+1}(a_i - i) \leq N(a_1 - 1) + 1$. Then $(N+1)(a_1 - 1) \leq \sum_{i=1}^{N+1}(a_i - i) \leq N(a_1 - 1) + 1$. Therefore $a_1 \leq 2$ and, if $a_1 = 2$, then $\mathbf{A} = (2, 3, 4, \dots, N+1, N+2)$. If $a_1 = 1$, then $\sum_{i=1}^{N+1}(a_i - i) \leq 1$. Therefore we have the result. □

Theorem 5.9. *For a generic $\mathcal{C}_{1,2,3}$-integral curve κ, the type $\mathbf{A}$ of the tangent-principal-normal-framed curve $\pi_1 \circ \kappa$ at any point of I is given by one of the following:*

$$(1, 2, 3, \dots, N, N+1), (1, 2, 3, \dots, N, N+2), (1, 3, 4, \dots, N+1, N+2), (2, 3, 4, \dots, N+1, N+2).$$

Proof: By Theorem 5.6, for a genetic c, the type of $\pi_1 \circ c$ at a point in I satisfies that $\sum_{i=2}^{N+1}(a_i - i) - (N-1)(a_2 - 2) \leq 1$, namely that $\sum_{i=2}^{N+1}(a_i - i) \leq (N-1)(a_2 - 2) + 1$. Then $N(a_2 - 2) \leq \sum_{i=2}^{N+1}(a_i - i) \leq (N-1)(a_2 - 2) + 1$, and we have $a_2 \leq 3$. If $a_2 = 3$, then $\mathbf{A} = (1, 3, 4, \dots, N+1, N+2)$ or $(2, 3, 4, \dots, N+1, N+2)$. If $a_2 = 2$, then $\mathbf{A} = (1, 2, 3, \dots, N, N+1)$ or $(1, 2, 3, \dots, N, N+2)$. □

Remark 5.10. We observe that, in all lists of the generic classifications of types, there are just three possibilities of the leading two digits: $(1, 2), (1, 3)$ and $(2, 3)$. These cases correspond to the cases where the projection of the tangent variety to the osculating plane is diffeomorphic to the map-germ $(\mathbf{R}^2, 0) \to (\mathbf{R}^2, 0)$, the fold singularities $(x, u) \mapsto (x^2, u)$, 'beak to beak' $(x, u) \mapsto (x^3 + ux^2, u)$ and Whitney's cusp map $(x, u) \mapsto (x^3 + ux, u)$ respectively.

6. Opening procedure of differentiable map-germs.

To describe singularities of frontal mappings, we introduce the notion of "openings" of mappings.

The tangent variety to a curve in $\mathbf{R}P^{N+1}$ projects locally to the tangent variety to a space curve in the osculating 3-space, and to a plane curve in the osculating 2- plane. Then the tangent variety in $\mathbf{R}P^{N+1}$ can be regarded as an "opening" of a tangent variety to a space curve and to a plane curve. For example, the open swallowtail, which is an opening of the swallowtail, appears in many context. It appears as a singular Lagrangian variety [2], and as a singular solution to certain partial differential equation [13]. The open folded umbrella appears as a 'frontal-symplectic singularity' ([21]).

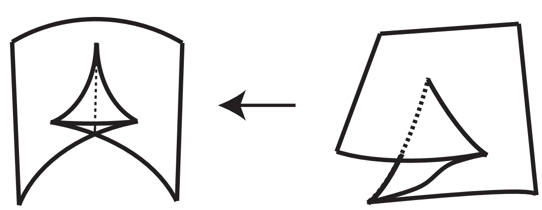

Figure 4. Opening of swallowtail.

We denote by $\mathcal{E}_a$ the $\mathbf{R}$-algebra of C^∞ function-germs on $(\mathbf{R}^n, a)$ with the maximal ideal $\mathfrak{m}_a$. If a is the origin, then we use $\mathcal{E}_n, \mathfrak{m}_n$ instead of $\mathcal{E}_a, \mathfrak{m}_a$ respectively.

Definition 6.1. ([15][19]) Let $f : (\mathbf{R}^n, a) \to (\mathbf{R}^m, b)$ be a C^∞ map-germ with $n \leq m$. We define the *Jacobi module* of f:

$$\mathcal{J}_f = \{ \sum_{j=1}^{m} p_j \ df_j \ \mid \ a_j \in \mathcal{E}_a, (1 \leq j \leq m) \ \} \subset \Omega^1_a,$$

in the space Ω^1_a of 1-form germs on $(\mathbf{R}^n, a)$. Further we define the *ramification module* $\mathcal{R}_f$ by

$$\mathcal{R}_f := \{h \in \mathcal{E}_a \mid dh \in \mathcal{J}_f\}.$$

Note that $\mathcal{J}_f$ is just the first order component of the graded differential ideal $\mathcal{J}^\bullet_f$ in $\Omega^\bullet_a$ generated by $df_1, \dots, df_m$. Then the singular locus is given by $\Sigma_f = \{x \in (\mathbf{R}^n, a) \mid \mathrm{rank}\mathcal{J}_f(x) < n\}$. Also we consider the *Kernel field* $\mathrm{Ker}(f_* : T\mathbf{R}^n \to T\mathbf{R}^m)$, of f near a. Then we see that, for another map-germ $f' : (\mathbf{R}^n, a) \to (\mathbf{R}^{m'}, b')$ with $\mathcal{J}_{f'} = \mathcal{J}_f$, $n \leq m'$, we have $\Sigma_{f'} = \Sigma_f$ and $\mathrm{Ker}(f'_*) = \mathrm{Ker}(f_*)$.

Related notion was introduced in [34].

Lemma 6.2. *Let $f : (\mathbf{R}^n, a) \to (\mathbf{R}^m, b)$ be a C^∞ map-germ.*

(1) *$f^*\mathcal{E}_b \subset \mathcal{R}_f \subset \mathcal{E}_a$ and $\mathcal{R}_f$ is an $\mathcal{E}_b$-module via f^*.*

(2) *For another map-germ $f' : (\mathbf{R}^n, a) \to (\mathbf{R}^{m'}, b')$, $\mathcal{J}_{f'} = \mathcal{J}_f$ if and only if $\mathcal{R}_{f'} = \mathcal{R}_f$.*

(3) *If $\tau : (\mathbf{R}^m, b) \to (\mathbf{R}^m, b')$ is a diffeomorphism-germ, then $\mathcal{R}_{\tau\circ f} = \mathcal{R}_f$. If $\sigma : (\mathbf{R}^n, a') \to (\mathbf{R}^n, a)$ is a diffeomorphism-germ, then $\mathcal{R}_{f\circ\sigma} = \sigma^*(\mathcal{R}_f)$.*

Proof: (1) follows from that, if $h \in \mathcal{R}_f$ and $dh = \sum_{j=1}^m p_j df_j$, then we have

$$d\{(k \circ f)h\} = \sum_{j=1}^{m} \{(k \circ f)p_j + h\,(\partial k/\partial y_j)\}\, df_j.$$

(2) It is clear that $\mathcal{J}_{f'} = \mathcal{J}_f$ implies $\mathcal{R}_{f'} = \mathcal{R}_f$. Conversely suppose $\mathcal{R}_{f'} = \mathcal{R}_f$. Then any component f'_j of f' belongs to $\mathcal{R}_{f'} = \mathcal{R}_f$, hence $df_j \in \mathcal{J}_f$. Therefore $\mathcal{J}_{f'} \subset \mathcal{J}_f$. By the symmetry we have $\mathcal{J}_{f'} = \mathcal{J}_f$.

(3) follows from that $\mathcal{J}_{\tau\circ f} = \mathcal{J}_f$ and $\mathcal{J}_{f\circ\sigma} = \sigma^*(\mathcal{J}_f)$. ☐

Definition 6.3. Let $f : (\mathbf{R}^n, a) \to (\mathbf{R}^m, b)$, $n \leq m$ be a C^∞ map-germ. Given $h_1, \dots, h_r \in \mathcal{R}_f$, the map-germ $F : (\mathbf{R}^n, a) \to \mathbf{R}^m \times \mathbf{R}^r = \mathbf{R}^{m+r}$ defined by

$$f = (f_1, \dots, f_m, h_1, \dots, h_r)$$

is called an *opening* of f, while f is called a *closing* of F.

An opening $F = (f, h_1, \dots, h_r)$ of f is called a *versal opening* (resp. *mini-versal opening*) of f, if $1, h_1, \dots, h_r$ form a (minimal) system of generators of $\mathcal{R}_f$ as an $\mathcal{E}_b$-module via f^*.

Note that an opening of an opening of f is an opening of f.

Here we summarise known results on the ramification module. A map-germ $f : (\mathbf{R}^n, a) \to (\mathbf{R}^m, b)$ is called *finite* if $\dim_{\mathbf{R}} \mathcal{E}_a/(f^*\mathfrak{m}_b)\mathcal{E}_a < \infty$.

Proposition 6.4. (Theorem 1.3 of [17], Corollary 2.4 of [19]) *If $f : (\mathbf{R}^n, a) \to (\mathbf{R}^m, b)$ is finite and of corank at most one. Then we have*

(1) *$\mathcal{R}_f$ is a finite $\mathcal{E}_b$-module. Therefore there exists a versal opening of f.*

(2) *$1, h_1, \dots, h_r \in \mathcal{R}_f$ generate $\mathcal{R}_f$ as $\mathcal{E}_b$-module if and only if they generate the vector space $\mathcal{R}_f/f^*(\mathfrak{m}_b)\mathcal{R}_f$ over $\mathbf{R}$.*

Remark 6.5. By Proposition 6.4, we see that $1, h_1, \ldots, h_r \in \mathcal{R}_f$ form a minimal system of generators of $\mathcal{R}_f$ as $\mathcal{E}_b$-module if and only if they form a basis of $\mathbf{R}$-vector space $\mathcal{R}_f/f^*(\mathfrak{m}_b)\mathcal{R}_f$.

Let $k \geq 0, m \geq 0$. To present the normal forms of Morin map, consider variables $t, \lambda = (\lambda_1, \ldots, \lambda_{k-1}), \mu = (\mu_{ij})_{1 \leq i \leq m, 1 \leq j \leq k}$ and polynomials

$$F(t,\lambda) = t^{k+1} + \sum_{i=1}^{k-1} \lambda_j t^j, \quad G_i(t,\mu) = \sum_{j=1}^{k} \mu_{ij} t^j, (1 \leq i \leq m).$$

Let $f : (\mathbf{R}^{k+km}, 0) \to (\mathbf{R}^{m+k+km}, 0)$ be a Morin map defined by

$$f(t,\lambda,\mu) := (F(t,\lambda), G(t,\mu), \lambda, \mu),$$

for the above polynomials F and G.

For $\ell \geq 0$, we denote by $F_{(\ell)}, G_{i(\ell)}$ the polynomials

$$F_{(\ell)}(t,\lambda) = \int_0^t s^\ell F(s,\lambda) ds, \quad G_{i\,(\ell)}(t,\mu) = \int_0^t s^\ell G_i(s,\mu) ds.$$

Then we have:

Proposition 6.6. (Theorem 3 of [15]) *The ramification module $\mathcal{R}_f$ of the Morin map f is minimally generated over $f^*\mathcal{E}_{m+k+km}$ by the $1 + k + (k-1)m$ elements*

$$1,\ F_{(1)},\ \ldots,\ F_{(k)},\ G_{1\,(1)},\ \ldots,\ G_{1\,(k-1)},\ \ldots,\ G_{m\,(1)},\ \ldots,\ G_{m\,(k-1)}.$$

The map-germ $\mathbf{F} : (\mathbf{R}^{k+mk}, 0) \to (\mathbf{R}^{m+k+km} \times \mathbf{R}^{k+(k-1)m}, 0) = (\mathbf{R}^{2(k+km)}, 0)$ defined by

$$\mathbf{F} = \left(f,\ F_{(1)}, \ldots, F_{(k)}, G_{1\,(1)}, \ldots, G_{1\,(k-1)}, \ldots, G_{m\,(1)}, \ldots, G_{m\,(k-1)}\right)$$

is a versal opening of f.

Remark 6.7. It is shown in [15] moreover that $\mathbf{F}$ is an isotropic map for a symplectic structure on $\mathbf{R}^{2(k+km)}$.

Proposition 6.8. (cf. Proposition 1.6 of [17], Lemma 2.4 of [18]) *Let $f : (\mathbf{R}^n, a) \to (\mathbf{R}^m, b)$ be a C^∞ map-germ and $F : (\mathbf{R}^{n+\ell}, (a,0)) \to (\mathbf{R}^{m+\ell}, (b,0))$ be an unfolding of f: $F(x,u) = (F_1(x,u), u)$ and $F_1(x,0) = f(x)$. Let $i : (\mathbf{R}^n, a) \to (\mathbf{R}^{n+\ell}, (a,0))$ be the inclusion, $i(x) = (x,0)$. Then we have:*

(1) *$i^*\mathcal{R}_F \subset \mathcal{R}_f$.*

(2) *If f is of corank ≤ 1 with $n \leq m$, then $i^*\mathcal{R}_F = \mathcal{R}_f$. If $1, H_1, \ldots, H_r$ generate $\mathcal{R}_F$ via F^*, then $1, i^*H_1, \ldots, i^*H_r$ generate $\mathcal{R}_f$ via f^*.*

(3) *Let ℓ be a positive integer and $F = (F_1(t,u), u) : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ an unfolding of $f : (\mathbf{R}, 0) \to (\mathbf{R}, 0), f(t) = F_1(t,0) = t^\ell$. Suppose $H_1, \ldots, H_r \in \mathcal{R}_F \cap \mathfrak{m}_n$. Then $1, H_1, \ldots, H_r$ generate $\mathcal{R}_F$ via F^* if and only $i^*H_1, \ldots, i^*H_r$ generate $\mathfrak{m}_1^{\ell+1}/\mathfrak{m}_1^{2\ell}$. In particular $F_{1(1)}, \ldots, F_{1(\ell-1)}$ form a system of generators of $\mathcal{R}_F$ via F^* over $\mathcal{E}_n$.*

Proof: (1) is clear. (2) Let $H \in \mathcal{R}_F$. Then $dH \in \mathcal{J}_F$. Hence $d(i^*H) = i^*(dH) \in i^*\mathcal{J}_F \subset \mathcal{J}_f$. Therefore $i^*H \in \mathcal{R}_f$. Let f be of corank at most one. Suppose $h \in \mathcal{R}_f$. Then $dh = \sum_{j=1}^m a_j df_j$ for some $a_j \in \mathcal{E}_a$. There exist $A_j, B_k \in \mathcal{E}_{(a,0)}$ such that $i^*A_j = a_j$ and the 1-form $\sum_{j=1}^m A_j d(F_1)_j + \sum_{k=1}^\ell B_k d\lambda_k$ is closed (cf. Lemma 2.5 of [19]). Then there exists an $H \in \mathcal{E}_{(a,0)}$ such that $dH = \sum_{j=1}^m A_j d(F_1)_j + \sum_{k=1}^\ell B_k d\lambda_k \in \mathcal{J}_F$ and $d(i^*H) = i^*(dH) = dh$. Then there exists $c \in \mathbf{R}$ such that $h = i^*H + c = i^*(H+c)$, and $H + c \in \mathcal{R}_F$. Therefore $h \in i^*\mathcal{R}_F$. Since i^* is a homomorphism over $j^* : \mathcal{E}_{(b,0)} \to \mathcal{E}_b$, where $j : (\mathbf{R}^m, 0) \to (\mathbf{R}^{m+\ell}, 0)$ is the inclusion $j(y) = (y,0)$, we have the consequence. (3) It is easy to show that $\mathcal{R}_f = \mathbf{R} + \mathfrak{m}_1^\ell$. By

Proposition 6.4 (2), $1, H_1, \dots, H_r$ generate $\mathcal{R}_F$ as $\mathcal{E}_n$-module via F^* if and only if they generate $\mathcal{R}_F/F^*(\mathfrak{m}_n)\mathcal{R}_F$ over $\mathbf{R}$. Since

$$\mathcal{R}_F/F^*(\mathfrak{m}_n)\mathcal{R}_F \cong (\mathbf{R}+\mathfrak{m}_1^\ell)/(f^*\mathfrak{m}_1)(\mathbf{R}+\mathfrak{m}_1^\ell) \cong \mathfrak{m}_1^{\ell+1}/\mathfrak{m}_1^{2\ell}$$

we have the consequence. □

Proposition 6.9. *Let $f : (\mathbf{R}^n, a) \to (\mathbf{R}^m, b)$, $n \leq m$ be a C^∞ map-germ.*
(1) *For any versal opening $F : (\mathbf{R}^n, a) \to (\mathbf{R}^{m+r}, F(a))$ of f and for any opening $G : (\mathbf{R}^n, a) \to (\mathbf{R}^{m+s}, G(a))$, there exists an affine bundle map $\Psi : (\mathbf{R}^{m+r}, F(a)) \to (\mathbf{R}^{m+s}, G(a))$ over $(\mathbf{R}^m, f(a))$ such that $G = \Psi \circ F$.*
(2) *For any mini-versal openings $F : (\mathbf{R}^n, a) \to (\mathbf{R}^{m+r}, F(a))$ and $F' : (\mathbf{R}^n, a) \to (\mathbf{R}^{m+r}, F'(a))$ of f, there exists an affine bundle isomorphism $\Phi : (\mathbf{R}^{m+r}, F(a)) \to (\mathbf{R}^{m+r}, F'(a))$ over $(\mathbf{R}^m, f(a))$ such that $F' = \Psi \circ F$. In particular, the diffeomorphism class of mini-versal opening of f is unique.*
(3) *Any versal openings $F'' : (\mathbf{R}^n, a) \to (\mathbf{R}^{m+s}, F''(a))$ of f is diffeomorphic to $(F, 0)$ for a mini-versal opening of f.*

Proof: (1) Let $F = (f, h_1, \dots, h_r)$ and $G = (f, k_1, \dots, k_s)$. Since $k_j \in \mathcal{R}_f$, there exist

$$c_j^{\,0}, c_j^{\,1}, \dots, c_j^{\,r} \in \mathcal{E}_b$$

such that $k_j = c_j^{\,0} \circ f + (c_j^{\,1} \circ f)h_1 + \cdots + (c_j^{\,r} \circ f)h_r$. Then it suffices to set

$$\Psi(y, z) = (y, (c_j^{\,0}(y) + c_j^{\,1}(y)z_1 + \cdots + c_j^{\,r}(y)z_r)_{1\leq j \leq s}).$$

(2) By (1) there exists an affine bundle map Ψ with $F' = \Psi \circ F$. From the minimality, we have that the matrix $(c_j^{\,i}(b))$ is regular. (See Remark 6.5). Therefore Ψ is a diffeomorphism-germ.
(3) Let $F = \Psi \circ F''$ for some affine bundle map Ψ. Then the matrix $(c_j^{\,i}(b))$ is of rank r. Therefore F'' is diffeomorphic to $(F, k_1, \dots, k_{s-r})$ for some $k_j \in \mathcal{R}_f$. Write each $k_j = K_j \circ F$ for some $K_j \in \mathcal{E}_{F(a)}$. Then we set $\Xi(y, z, w) = (y, z, w - K \circ F)$. Then Ξ is a local diffeomorphism on $\mathbf{R}^{m+r+(s-r)}$ and $\Xi \circ (F, k_1, \dots, k_{s-r}) = (F, 0)$. □

7. Normal forms of tangent surfaces.

According to a geometric restriction expressed in differential system, we have imposed on curves in projective spaces a system of differential equations (§3). The genericity, in such a restricted class of curves, naturally implies a restriction on types of curves (§5). Then we use the following results to solve the classification problem. For the concrete expression of normal forms, see §3.

Theorem 7.1. (1) *In $\mathbf{R}P^3$, the tangent variety of a curve of type $(1,2,3)$ (resp. $(1,2,4)$, $(2,3,4)$, $(1,3,4)$) is locally diffeomorphic to the cuspidal edge (the folded umbrella, the swallowtail, the Mond surface) in $\mathbf{R}^3$.*

(2) (Higher codimensional case.) *In $\mathbf{R}P^{N+1}$, $N+1 \geq 4$,*

(i) *the tangent variety of a curve of type $(1, 2, 3, a_4, \dots, a_{N+1})$ is locally diffeomorphic to the cuspidal edge $(\mathbf{R}^2, 0) \to (\mathbf{R}^3, 0)$ composed with the inclusion to $(\mathbf{R}^{N+1}, 0)$.*

(ii) *the tangent variety of a curve of type $(1, 3, 4, 5, a_5, \dots, a_{N+1})$ is locally diffeomorphic to the open Mond surface $(\mathbf{R}^2, 0) \to (\mathbf{R}^4, 0)$ composed with the inclusion to $(\mathbf{R}^{N+1}, 0)$.*

(iii) *the tangent variety of a curve of type $(2, 3, 4, 5, a_5, \dots, a_{N+1})$ is locally diffeomorphic to the open swallowtail $(\mathbf{R}^2, 0) \to (\mathbf{R}^4, 0)$ composed with the inclusion to $(\mathbf{R}^{N+1}, 0)$.*

Proof: (1) is proved in Theorem 1 ($n = 2$) in [18]. (2) In each case, the idea is to show that the tangent map-germ $\mathrm{Tan}(\gamma)$ is diffeomorphic to a mini-versal opening of an appropriate map-germ:

(i) the fold map-germ $(\mathbf{R}^2, 0) \to (\mathbf{R}^2, 0)$.

(ii) the Mond surface $(\mathbf{R}^2,0)\to(\mathbf{R}^3,0)$.
(iii) the swallowtail $(\mathbf{R}^2,0)\to(\mathbf{R}^3,0)$.
Then, by Proposition 6.9, the diffeomorphism class of the tangent map-germ is unique and we get the required results.

Let $\gamma:(\mathbf{R},0)\to\mathbf{R}P^{N+1}$ be a curve-germ of type $(a_1,a_2,\dots,a_{N+1})$,

$$\gamma(t)=(x_1(t),x_2(t),\dots,x_{N+1}(t))$$

a local affine representation of γ as in Lemma 4.2, and

$$f(s,t)=(f_1(s,t),f_2(s,t),\dots,f_{N+1}(s,t))=\left(x_i(t)+s\,\frac{1}{\alpha(t)}x_i'(t)\right)_{1\le i\le N+1},$$

the parametrisation of the tangent variety to γ, where $\alpha(t)=t^{a_1-1}$. We may suppose $x_1(t)=t^{a_1}$.

We define $g':(\mathbf{R}^2,0)\to(\mathbf{R}^2,0)$ by $g'=(f_1,f_2)$. Then, by Lemma 4.5 and Remark 4.7, we see that $f_3,\dots,f_{N+1}\in\mathcal{R}_{g'}$. Note that $f_1(s,t)=x_1(t)+a_1s$ is a regular function. We regard $f_1(s,t)$ as an unfolding parameter u. Then there exist diffeomorphism-germ $\sigma:(\mathbf{R}^2,0)\to(\mathbf{R}^2,0)$ and $\tau:(\mathbf{R}^2,0)\to(\mathbf{R}^2,0)$ such that σ is of form $\sigma(u,t)=(\sigma_1(u),t\sigma_2(u,t))$ and $g=\tau\circ g'\circ\sigma$ is equal to (i) $(u,t)\mapsto(u,t^2+ut)$, (ii) $(u,t)\mapsto(u,t^3+ut^2)$, (iii) $(u,t)\mapsto(u,t^3+ut)$. Then $f_3\circ\sigma,\dots,f_{N+1}\circ\sigma$ belongs $\mathcal{R}_g=\mathcal{R}_{g'\circ\sigma}$. Then, by Lemma 6.8, (i) $F=(f_1\circ\sigma,f_2\circ\sigma,f_3\circ\sigma)$, (ii)(iii) $F=(f_1\circ\sigma,f_2\circ\sigma,f_3\circ\sigma,f_4\circ\sigma)$, are versal opening of g respectively. Note that in cases (ii) and (iii), F is a versal opening of also Mond surface and swallowtail respectively. Then, by Proposition 6.8 (3), we have that $f\circ\sigma$ is diffeomorphic to (i) $(u,t^2+ut,\frac{2}{3}t^3+\frac{1}{2}ut^2,0,\dots,0)$, (ii) $(u,t^3+ut^2,\frac{3}{4}t^4+\frac{2}{3}ut^3,\frac{3}{5}t^5+\frac{1}{2}ut^4,0,\dots,0)$, (iii) $(u,t^3+ut^2,\frac{3}{4}t^4+\frac{1}{2}ut^2,\frac{3}{5}t^5+\frac{1}{3}ut^3,0,\dots,0)$, as required. □

Theorem 7.2. *The tangent variety of a curve of type $(1,2,4,5,a_5,\dots,a_{N+1})$ is locally diffeomorphic to the open folded umbrella $(\mathbf{R}^2,0)\to(\mathbf{R}^4,0)$ composed with the inclusion to $(\mathbf{R}^{N+1},0)$.*

Proof: We argue as in Theorem 7.1. However in this case $(f_1\circ\sigma,f_2\circ\sigma,f_3\circ\sigma,f_4\circ\sigma)$ is not a versal opening of $g=(u,t^2+ut)$. (In fact the open folded umbrella is not a versal opening of the folded umbrella $(\mathbf{R}^2,0)\to(\mathbf{R}^3,0)$.)

To show Theorem 7.2, we define

$$\mathcal{R}_g^{(2)}:=\left\{h\in t^2\mathcal{E}_2 \mid dh\in t^2\mathcal{J}_g\right\}.$$

Then $f_i\circ\sigma\in\mathcal{R}_g^{(2)},(i\ge 3)$. We see that $f_3\circ\sigma,f_4\circ\sigma$ generate $\mathcal{R}_g^{(2)}$ over $g^*\mathcal{E}_2$. In fact $h_1,\dots,h_r$ generate $\mathcal{R}_g^{(2)}$ as $\mathcal{E}_2$-module if and only if $i^*h_1,\dots,i^*h_r$ generate $\mathfrak{m}_1^4/\mathfrak{m}_1^6$ over $\mathbf{R}$. (See Lemma 2.4 of [18]). Also $h_1=\frac{1}{2}t^4+\frac{1}{3}ut^3, h_2=\frac{2}{5}t^5+\frac{1}{4}ut^4$ generate $\mathcal{R}_g^{(2)}$. We write $f_i\circ\sigma=(a_i\circ g)h_1+(b_i\circ g)h_2,(i\ge 3)$, for some $a_i,b_i\in\mathcal{E}_2$. We define $\Psi:(\mathbf{R}^{N+1},0)\to(\mathbf{R}^{N+1},0)$ by

$$\begin{aligned}\Psi(x) \;=\; &(x_1,x_2,a_3(x_1,x_2)x_3+b_3(x_1,x_2)x_4,a_4(x_1,x_2)x_3+b_4(x_1,x_2)x_4,\\ &x_i-a_i(x_1,x_2)x_3+b_i(x_1,x_2)x_4(5\ge i)).\end{aligned}$$

Then Ψ is a diffeomorphism-germ and $\Psi\circ f\circ\sigma=(g,h_1,h_2,0)$. Thus we have that $f\circ\sigma$ is diffeomorphic to $(g,h_1,h_2,0)=(u,t^2+ut,\frac{1}{2}t^4+\frac{1}{3}ut^3,\frac{2}{5}t^5+\frac{1}{4}ut^4,0,\dots,0)$ as required. □

Proofs of the classification theorems. Theorems 2.6, 3.3, 3.4, 3.6 follow from Theorems 5.8, 5.9, 5.5 and Theorems 7.1, 7.2.

We are led, in our generic classifications in a geometric setting, to find the following result, which we use in §8.

Theorem 7.3. *The tangent variety of a curve of type* $(1, 3, 4, 6, a_5, \dots, a_{N+1})$ *in* $\mathbf{R}P^{N+1}, N+1 \geq 4$*, has unique diffeomorphism class.*

We may call it the 'unfurled Mond surface', distinguished with the open Mond surface. The normal form $(\mathbf{R}^2, 0) \to (\mathbf{R}^{N+1}, 0)$ of the *unfurled Mond surface* is given by

$$(s,t) \mapsto \left(t+s,\ t^3+3st^2,\ t^4+4st^3,\ t^6+6st^5,\ 0,\dots,0\right),$$

which is diffeomorphic to

$$(x,u) \mapsto \left(u,\ t^3+ut^2,\ \frac{3}{4}t^4+\frac{2}{3}ut^3,\ \frac{1}{2}t^6+\frac{2}{5}ut^5,\ 0,\dots,0\right).$$

To show Theorem 7.3, we prepare the following:

Lemma 7.4. (cf. Lemma 2.4 of [18]) *Let* $g : (\mathbf{R}^2, 0) \to (\mathbf{R}^2, 0)$ *be the map-germ defined by* $g(t, u) = (u, t^3 + ut^2)$*. We set*

$$\mathcal{R}_g^{(3)} := \left\{h \in t^3\mathcal{E}_2 \mid dh \in t^3\mathcal{J}_g\right\}.$$

and set $T = t^3 + ut^2, T_i = \frac{3}{i+3}t^{i+3} + \frac{2}{i+2}ut^{i+2}, (i = 1, 2, 3, \dots)$*. Then we have (1)* $\mathcal{R}_g^{(3)} = \mathcal{R}_g \cap t^5\mathcal{E}_2$*. (2)* $\mathcal{R}_g^{(3)}$ *is a finite* $\mathcal{E}_2$*-module via* $g^* : \mathcal{E}_2 \to \mathcal{E}_2$ *generated by* T_3, TT_1, T_1^2*. (3) Let* $\iota : (\mathbf{R}, 0) \to (\mathbf{R}^2, 0), \iota(t) = (t, 0)$*. Then* $h_1, \dots, h_\ell \in \mathcal{R}_g^{(3)}$ *generate* $\mathcal{R}_g^{(3)}$ *as* $\mathcal{E}_2$*-module via* g^* *if and only if* $\iota^*h_1, \dots, \iota^*h_\ell$ *generate* $t^6\mathcal{E}_1/t^9\mathcal{E}_1$ *over* $\mathbf{R}$*. (Note that* $T_1 \notin \mathcal{R}_g^{(3)}$*.)*

Proof: (1) First note that $\mathcal{R}_g^{(3)} = \left\{h \in t^3\mathcal{E}_2 \;\middle|\; \dfrac{\partial h}{\partial t} \in t^3\dfrac{\partial T}{\partial t}\mathcal{E}_2\right\}$. Let $h \in \mathcal{R}_g^{(3)}$. Then $\dfrac{\partial h}{\partial t} \in t^4\mathcal{E}_2$ and $h \in t^3\mathcal{E}_2$. Therefore $h \in \mathcal{R}_g \cap t^5\mathcal{E}_2$. Conversely let $h \in \mathcal{R}_g \cap t^5\mathcal{E}_2$. Then $\dfrac{\partial h}{\partial t} = t^3\dfrac{\partial T}{\partial t}K$ for some $K \in \mathcal{E}_2$. Since $h(0,0) = 0$, we have $\dfrac{\partial h}{\partial u} \in t^5\mathcal{E}_2$. Therefore $dh \in t^3\mathcal{J}_g$ and $h \in \mathcal{R}_g^{(3)}$. Thus we have the equality.

(2) Let $h \in \mathcal{R}_g^{(3)}$. Then $h = a\circ g + b\circ gT_1 + c\circ gT_2$, for some $a, b, c \in \mathcal{E}_2$. Since $h \in t^5\mathcal{E}_2$, $h = \widetilde{a}\circ gT^3 + \widetilde{b}\circ gTT_1 + \widetilde{c}\circ gTT_2$, for some $\widetilde{a}, \widetilde{b}, \widetilde{c} \in \mathcal{E}_2$. Note that $T^3, TT_1, TT_2 \in \mathcal{R}_g^{(3)}$. Moreover we have directly

$$T^3 = \frac{32}{15}uT_1^2 + 2TT_3 + \frac{14}{3}T_4,\ TT_2 = \frac{16}{15}T_1^2 + \frac{7}{3}uT_4,\ T_4 = \frac{4}{7}TT_1 - \frac{20}{21}uT_3.$$

Therefore we have

$$TT_2 = -\frac{20}{9}u^2T_3 + \frac{4}{3}uTT_1 + \frac{16}{15}T_1^2,\ T^3 = \left(2T - \frac{40}{9}u^3\right)T_3 + \frac{8}{3}u^2TT_1 + \frac{32}{15}uT_1^2.$$

(3) $\iota^* : \mathcal{E}_2 \to \mathcal{E}_1$ induces $\iota^* : \mathcal{R}_g^{(3)} \to t^6\mathcal{E}_1$, which is clearly surjective. Moreover we have $(\iota^*)^{-1}(t^9\mathcal{E}_1) = g^*\mathfrak{m}_2\mathcal{R}_g^{(3)}$. Therefore ι^* induces an isomorphism $\mathcal{R}_g^{(3)}/g^*\mathfrak{m}_2\mathcal{R}_g^{(3)} \cong t^6\mathcal{E}_1/t^9\mathcal{E}_1$ as $\mathbf{R}$-vector spaces. By (2) and by Malgrange-Mather's preparation theorem [5], we have the required result. □

Proof of Theorem 7.3: We give the proof for the case $N + 1 = 4$. In general case we can argue similarly.

Let $\gamma : (\mathbf{R}, 0) \to \mathbf{R}P^4$ be a curve of type $(1, 3, 4, 6)$. The tangent map-germ $\mathrm{Tan}(\gamma)$ is an opening of a Mond surface. However it is not versal. So we need a specialised idea to show the determinacy result in this situation. Let

$$\gamma(t) = (t, t^3 + \varphi(t), t^4 + \psi(t), t^6 + \rho(t)),$$

with $\varphi \in \mathfrak{m}_1^4, \psi \in \mathfrak{m}_1^5, \rho \in \mathfrak{m}_1^7$. Then $f = \mathrm{Tan}(\gamma)$ is given by

$$f(s,t) = \left(t+s,\ t^3+3st^2+\Phi(t),\ t^4+4st^3+\Psi(t),\ t^6+6st^5+R(t)\right),$$

where $\Phi(s,t) = \varphi(t)+s\varphi'(t), \Psi(s,t) = \psi(t)+s\psi'(t), R(s,t) = \rho(t)+s\rho'(t)$. We set $u = t+s$. Then

$$f(u,t) = \left(u,\ -2t^3+3ut^2+\widetilde{\Phi}(t),\ -3t^4+4ut^3+\widetilde{\Psi}(t),\ -5t^6+6ut^5+\widetilde{R}(t)\right),$$

where $\Phi(s,t) = \varphi(t)+(u-t)\varphi'(t), \Psi(s,t) = \psi(t)+(u-t)\psi'(t), R(s,t) = \rho(t)+(u-t)\rho'(t)$. From the determinacy of tangent varieties to curves of type $(1,3,4)$ in $\mathbf{R}^3$ ([33], [16]), we have that there exist diffeomorphism-germ $\sigma : (\mathbf{R}^2,0) \to (\mathbf{R}^2,0)$ of form $\sigma(u,t) = (\sigma_1(u), t\sigma_2(u,t))$ and a diffeomorphism-germ $\tau : (\mathbf{R}^4,0) \to (\mathbf{R}^4,0)$ such that

$$f\circ\sigma(u,t) = (u,\ T(u,t),\ T_1(u,t),\ T_3(u,t)+S_3(u,t)),$$

with

$$T = t^3+ut^2, \quad T_1 = \frac{3}{4}t^4+\frac{2}{3}ut^3, \quad T_3 = \frac{1}{2}t^6+\frac{2}{5}ut^5,$$

$S_3 \in \mathcal{R}_g^{(3)}$, $g = (u,\ t^3+ut^2)$, $\iota^*S_3 \in \mathfrak{m}_1^7$. Then we have, by Lemma 7.4,

$$S_3 = A_3\circ g\ T_3 + B_3\circ g\ TT_1 + C_3\circ g\ T_1^2,$$

for some $A_3, B_3, C_3 \in \mathcal{E}_2$ with $A_3(0,0) = 0$. Define $\Xi : (\mathbf{R}^4,0) \to (\mathbf{R}^4,0)$ by

$$\begin{aligned}\Xi(x_1,x_2,x_3,x_4) \quad = \quad & \big(x_1,\ x_2,\ x_3+A_1(x_1,x_2)x_4+B_1(x_1,x_2)x_2x_3+C_1(x_1,x_2)x_3^2,\\ & \quad x_4+A_3(x_1,x_2)x_4+B_3(x_1,x_2)x_2x_3+C_3(x_1,x_2)x_3^2\big).\end{aligned}$$

Then the Jacobi matrix of Ξ is the unit matrix, so Ξ is a diffeomorphism-germ and

$$\Xi^{-1}\circ f\circ\sigma = \left(u,\ t^3+ut^2,\ \frac{3}{4}t^4+\frac{2}{3}ut^3,\ \frac{1}{2}t^6+\frac{2}{5}ut^5\right).$$

□

8. Singularities on tangent varieties to osculating framed contact- integral curves.

We give results on the classification of singularities of tangent varieties to contact-integral curves (resp. osculating framed contact-integral curves) in a contact projective space.

Let V be a symplectic vector space of dimension $2n+2$. Consider the isotropic flag manifold:

$$\mathcal{F}_{\mathrm{Lag}} = \mathcal{F}_{\mathrm{Lag}}(V) := \{V_1 \subset V_2 \subset \cdots \subset V_n \subset V_{n+1} \subset V \mid V_{n+1} \text{ is Lagrangian}\}.$$

Note that $\mathcal{F}_{\mathrm{Lag}}$ is a finite quotient of $\mathrm{U}(n+1), \dim(\mathcal{F}_{\mathrm{Lag}}) = (n+1)^2$ and that $\mathcal{F}_{\mathrm{Lag}}(V)$ is embedded into $\mathcal{F}(V) = \mathcal{F}_{1,2,\dots,n+1,\dots,2n+1}(V)$ by taking symplectic orthogonals:

$$(V_1,V_2,\dots,V_n,V_{n+1}) \mapsto (V_1,V_2,\dots,V_n,V_{n+1},V_n^s,\dots,V_2^s,V_1^s),$$

Define a differential system $\mathcal{E} \subset T\mathcal{F}_{\mathrm{Lag}}$ by

$$v \in \mathcal{E}_{(V_1,\dots,V_{n+1})} \Longleftrightarrow \pi_{i*}(v) \in T\,\mathrm{Gr}(i,V_{i+1}) (\subset T\,\mathrm{IGr}(i,V)), (1 \le i \le n).$$

where IGr means the isotropic Grassmannian, $\pi_i : \mathcal{F}_{\mathrm{Lag}} \to \mathrm{IGr}(i,V)$ is the canonical projection. Then $\mathrm{rank}(\mathcal{E}) = n+1$ and $\mathcal{E}$ is bracket generating.

If $n = 1$, then we have $\dim\mathcal{F}_{\mathrm{Lag}} = 4$ and $\mathcal{E}$ is an Engel structure on $\mathcal{F}_{\mathrm{Lag}}$ ([22]).

An $\mathcal{E}$-integral curve $c : I \to \mathcal{F}_{\mathrm{Lag}}$ is a C^∞ family

$$(V_1(t),V_2(t),\dots,V_n(t),V_{n+1}(t))$$

of isotropic flags in the symplectic vector space V such that $V_i(t)$ moves momentarily in $V_{i+1}(t)$.

Remark 8.1. The projective space $P(V^{2n+2}) \cong \mathbf{R}P^{2n+1}$ has the canonical contact structure $\mathcal{D} \subset T(P(V))$: For $V_1 \in P(V)$ and for $v \in T_{V_1}P(V)$, we define

$$v \in \mathcal{D}_{V_1} \iff \pi_{1*}(v) \in T(P(V_1^s))(\subset T(P(V))).$$

If $c : I \to \mathcal{F}_{\rm Lag}(V)$ is an $\mathcal{E}$-integral curve, then $\gamma = \pi_1 \circ c : I \to P(V)$ is a $\mathcal{D}$-integral curve.

We consider the space $J^r_{\mathcal{E}}(I, \mathcal{F}_{\rm Lag}(\mathbf{R}^{2n+2})$ of $\mathcal{E}$-integral jets in $J^r(I, \mathcal{F}_{\rm Lag}(\mathbf{R}^{2n+2}))$ and set

$$\Sigma_{\mathcal{E}}(\mathbf{A}) := \{ j^r\Gamma(t_0) \mid t_0 \in I, \Gamma : (\mathbf{R}, t_0) \to \mathcal{F}_{\rm Lag}(\mathbf{R}^{2n+2}) \text{ is } \mathcal{E}\text{-integral}, \ \ \pi_1 \circ \Gamma \text{ is of type } \mathbf{A}\}.$$

Then we have the codimension formula for osculating framed contact-integral curves.

Theorem 8.2. *The set of $\mathcal{E}$-integral curves $c : I \to \mathcal{F}_{\rm Lag}(\mathbf{R}^{2n+2})$ such that the osculating-framed contact-integral curve $\pi_1 \circ c : I \to P(V^{2n+2})$ is of type $\mathbf{A} = (a_1, a_2, \dots, a_{2n+1})$ is not empty if and only if*

$$a_{n+j} = a_{n+1} + a_n - a_{n+1-j}, \quad (2 \le j \le n+1),$$

and then its codimension in the jet space of $\mathcal{E}$-integral curves is given by $a_{n+1} - (n+1)$.

Proof: To show Theorem 8.2, first we give systems of projective coordinates on $\mathcal{F}_{\rm Lag}(V)$. For the case $n = 1$, refer the paper [22].

We fix a flag $\mathbf{V}_0 = (V_{10}, V_{20}, \dots, V_{n+1\,0}) \in \mathcal{F}_{\rm Lag}(V)$. Then we take the open set $U \subset \mathcal{F}_{\rm Lag}(V)$ defined by

$$U := \left\{ (V_1, V_2, \dots, V_{n+1}) \in \mathcal{F}_{\rm Lag}(V) \mid V_1 \cap V_{10}^s = \{0\}, V_2 \cap V_{20}^s = \{0\}, \dots, V_{n+1} \cap V_{n+1\,0}^s = \{0\} \right\}.$$

Take $\mathbf{V}_1 = (V_{11}, V_{21}, \dots, V_{n+1\,1}) \in U$. Then we have the decomposition $V = V_{n+1\,1} \oplus V_{n+1\,0}$ into Lagrangian subspaces, and the decomposition

$$\begin{aligned} V_{n+1\,1} &= V_{11} \oplus (V_{21} \cap V_{10}^s) \oplus (V_{31} \cap V_{20}^s) \oplus \cdots \oplus (V_{n+1\,1} \cap V_{n\,0}^s), \\ V_{n+1\,0} &= V_{10} \oplus (V_{20} \cap V_{11}^s) \oplus (V_{30} \cap V_{21}^s) \oplus \cdots \oplus (V_{n+1\,0} \cap V_{n\,1}^s), \end{aligned}$$

of each Lagrangian subspace into one-dimensional subspaces. Take non-zero vectors $e_0 \in V_{11}$, $e_i \in V_{i+1\,1} \cap V_{i\,0}^s, (1 \le i \le n)$, $f_0 \in V_{10}$ and $f_i \in V_{i+1\,0} \cap V_{i\,1}^s, (1 \le i \le n)$, to get a symplectic basis $(e_0, e_1, \dots, e_n; f_0, f_1, \dots, f_n)$ of V.

Then, for each $\mathbf{V} = (V_1, V_2, \dots, V_{n+1}) \in U_{\mathbf{V}_0}$, V_{n+1} has a basis $(v_0, v_1, \dots, v_n)$ uniquely expressed as

$$v_i = e_i + \sum_{j=0}^{n} x_j{}^i f_j, \quad (0 \le i \le n),$$

for some $(x_j{}^i)_{0 \le i,j \le n}$. Since V_{n+1} is a Lagrangian subspace of V, we have that $x_j{}^i = x_i{}^j, 0 \le i, j \le n$. Then there exist uniquely $\lambda_i{}^k, (1 \le k \le i \le n)$, such that

$$w_k = v_{k-1} + \sum_{i=k}^{n} \lambda_i{}^k v_i, \quad (1 \le k \le n+1),$$

form a basis of V_{n+1} such that $V_k = \langle w_1, \dots, w_k \rangle_{\mathbf{R}}, (1 \le k \le n+1)$. Then actually we have

$$w_k = e_{k-1} + \sum_{i=k}^{n} \lambda_i{}^k e_i + \sum_{j=0}^{n} \left(x_j{}^{k-1} + \sum_{i=k}^{n} \lambda_i{}^k x_j{}^i \right) f_j, \quad (1 \le k \le n+1).$$

Thus, given $\mathbf{V}_0, \mathbf{V}_1 \in \mathcal{F}_{\rm Lag}(V)$, we have a chart $U \to \mathbf{R}^{(n+1)^2}$ of $\mathcal{F}_{\rm Lag}(V)$, given by the symmetric matrix $(x_j{}^i)_{0 \le i,j \le n}$ and $\lambda_i{}^k, (1 \le k \le i \le n)$. From another choice of $\mathbf{V}_0, \mathbf{V}_1 \in \mathcal{F}_{\rm Lag}(V)$, we have another chart with fractional linear transition functions.

The projection $\pi_1 : \mathcal{F}_{\mathrm{Lag}}(V) \to P(V)$ is expressed by

$$\left(x_j{}^i, \lambda_i{}^k\right) \mapsto \left[1 : \lambda_1{}^1 : \cdots : \lambda_1{}^n : x_0{}^0 + \sum_{i=1}^{n} \lambda_i{}^1 x_0{}^i : \cdots : x_n{}^0 + \sum_{i=1}^{n} \lambda_i{}^1 x_n{}^i\right].$$

We set $X_j{}^k := x_j{}^k + \sum_{i=k+1}^{n} \lambda_i{}^{k+1} x_j{}^i, (0 \leq j \leq n, 0 \leq k \leq n)$. Then the differential system $\mathcal{E}$ is locally given by

$$\begin{cases} d\lambda_i{}^k - \lambda_i{}^{k+1} d\lambda_k{}^k = 0, & 1 \leq k \leq n, k+1 \leq i \leq n, \\ dX_j{}^{k-1} - X_j{}^k d\lambda_k{}^k = 0, & 1 \leq k \leq n, 0 \leq j \leq n. \end{cases}$$

We see that each $\mathcal{E}$-integral curve Γ is obtained from the components $\lambda_k{}^k, 1 \leq k \leq n$, and the $x_n{}^n$-component, by iterative integrations.

The type $(a_1, a_2, \ldots, a_n, a_{n+1}, a_{n+2}, \ldots, a_{2n+1})$ of $\gamma = \pi_1 \circ \Gamma$ is expressed in terms of

$$u_k := \mathrm{ord}(\lambda_k{}^k),\ 1 \leq k \leq n, \quad v := \mathrm{ord}(x_n{}^n)$$

by

$$\begin{array}{rcl} a_i & = & u_1 + u_2 + \cdots + u_i,\ (1 \leq i \leq n) \\ a_{n+1} & = & u_1 + u_2 + \cdots + u_n + v, \\ a_{n+1+j} & = & u_1 + u_2 + \cdots + 2u_{n-j+1} + \cdots + 2u_n + v,\ (1 \leq j \leq n), \end{array}$$

Let $\mathbf{A} = (a_1, \ldots, a_n, a_{n+1}, a_{n+2}, \ldots, a_{2n}, a_{2n+1})$ be a strictly increasing sequence of positive integers. Then The above system of equations has an integer solution $(u_1, \ldots, u_n, v)$ if and only if $a_{n+1+i} - a_{n+i} = a_n - a_{n-i}$. If the non-empty condition is fulfilled, then the codimension of the set

$$\Sigma(\mathbf{A}) = \{j^r\Gamma(t_0) \mid \Gamma : (I, t_0) \to \mathcal{F}_{\mathrm{Lag}}(V) \text{ is } \mathcal{E}\text{-integral},\ \mathrm{type}(\pi_1 \circ \Gamma) = \mathbf{A}\}$$

in $J^r_{\mathcal{E}}(I, \mathcal{F}_{\mathrm{Lag}}(V))$ is calculated by

$$a_1 - 1 + (a_2 - a_1 - 1) + \cdots + (a_{n+1} - a_n - 1) = a_{n+1} - (n+1).$$

$\square$

By Theorem 8.2 and by the transversality theorem for $\mathcal{E}$-integral curves, we have the following result: We separate cases into three groups from the classification viewpoint of singularities.

Theorem 8.3. ([22]) *Let $2n + 1 = 3$. For a generic $\mathcal{E}$-integral curve $c : I \to \mathcal{F}_{\mathrm{Lag}}(\mathbf{R}^4)$ in C^∞-topology, the type $\mathbf{A}$ of $\pi_1 \circ c$ at any point $t \in I$ is given by*

$$\mathbf{A} = (1, 2, 3), (1, 3, 4), (2, 3, 5).$$

The tangent varieties to the osculating-framed Legendre curve $\gamma = \pi_1 \circ c : I \to P(V) \cong \mathbf{R}P^3$ is locally diffeomorphic to the cuspidal edge, to the Mond surface or to the generic folded pleat (Figure 5).

Remark 8.4. In the above Theorem 8.3, the type of the curve γ is restricted to $(1, 2, 3), (1, 3, 4)$ or $(2, 3, 5)$. The local diffeomorphism class of the tangent variety $\mathrm{Tan}(\gamma)$ is determined if $\mathrm{type}(\gamma) = (1, 2, 3)$ or $(1, 3, 4)$, but it is not determined if $\mathrm{type}(\gamma) = (2, 3, 5)$ and there are exactly two diffeomorphism classes, *generic* one and *non-generic* one.

Note that we have obtained in [22] also the generic classification of singularities of tangent varieties to $\pi_2 \circ c : I \to \mathrm{LG}(V)$ in Lagrangian Grassmannian.

In the higher codimensional case, we have:

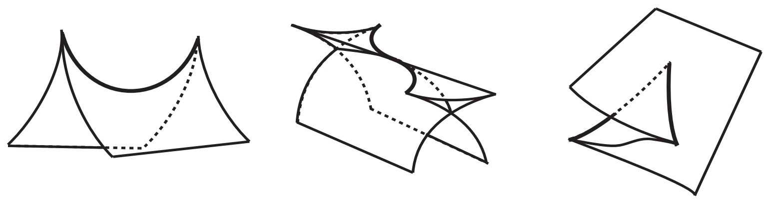

FIGURE 5. cuspidal edge, Mond surface and generic folded pleat in $\mathbf{R}^3$.

Theorem 8.5. *Let $2n+1 \geq 7$. For a generic $\mathcal{E}$-integral curve $c : I \to \mathcal{F}_{\rm Lag}(\mathbf{R}^{2n+2})$ in C^∞-topology, the type of osculating-framed contact-integral curve $\gamma = \pi_1 \circ c : I \to P(V) \cong \mathbf{R}P^{2n+1}$ at each point of I is given by one of*

$$\begin{array}{rcc}\mathbf{A} & = & (1, 2, 3, 4, \dots, n, n+1, n+2, \dots, 2n+1), \\ & & (1, 2, 3, 4, \dots, n, \quad n+2, n+3, \dots, 2n+2), \\ & & \cdots\cdots\cdots \\ & & (1, 2, 4, 5, \dots, n+1, n+2, n+3, \dots, 2n+2), \\ & & (1, 3, 4, 5, \dots, n+1, n+2, n+3, \dots, 2n+2), \\ & & (2, 3, 4, 5, \dots, n+1, n+2, n+3, \dots, 2n+2).\end{array}$$

Moreover the tangent variety $\mathrm{Tan}(\gamma)$ *to the osculating-framed contact- integral curve γ is locally diffeomorphic to the cuspidal edge, the open folded umbrella, the open Mond surface, or to the open swallowtail.*

We should be careful in the low codimensional case:

Theorem 8.6. *Let $2n+1 = 5$. For a generic $\mathcal{E}$-integral curve $c : I \to \mathcal{F}_{\rm Lag}(\mathbf{R}^6)$ in C^∞-topology, the type of osculating-framed contact-integral curve $\gamma = \pi_1 \circ c : I \to P(V) \cong \mathbf{R}P^5$ at each point of I is given by one of*

$$(1, 2, 3, 4, 5),\ (1, 2, 4, 5, 6),\ (1, 3, 4, 6, 7),\ (2, 3, 4, 5, 7).$$

Moreover the tangent variety $\mathrm{Tan}(\gamma)$ *to the osculating-framed contact- integral curve γ is locally diffeomorphic to the cuspidal edge, the open folded umbrella, the unfurled Mond surface, or to the open swallowtail.*

Proofs of Theorems 8.5, 8.6: By the transversality theorem, we reduce the list in each case from Theorem 8.2. In each case, we have the uniqueness of the diffeomorphism class of tangent varieties by Theorem 7.1, except for the case $\mathbf{A} = (1, 3, 4, 6, 7)$. For the case $\mathbf{A} = (1, 3, 4, 6, 7)$, we use Theorem 7.3. □

It is natural to consider the generic classification of tangent varieties to contact-integral curves $I \to P(V) = \mathbf{R}P^{2n+1}$. Here, we give just the result on non-framed three dimensional case $(n = 1)$:

Proposition 8.7. *For a generic contact-integral curve $\gamma : I \to P(V^4) \cong \mathbf{R}P^3$, and for any $t_0 \in I$, the type of γ at t_0 is equal to $(1, 2, 3)$ or to $(1, 3, 4)$ and the tangent variety* $\mathrm{Tan}(\gamma)$ *of γ is locally diffeomorphic to the cuspidal edge or to the Mond surface.*

Proof: Take the local coordinates λ, μ, ν of $P(V)$ such that the contact structure is given by $d\mu = \nu d\lambda - \lambda d\nu$. We express $\gamma(t) = (\lambda(t), \mu(t), \nu(t))$. Since γ is contact-integral, we have that $\mu'(t) = \nu(t)\lambda'(t) - \lambda(t)\nu'(t)$. Therefore $\mu''(t) = \nu(t)\lambda''(t) - \lambda(t)\nu''(t)$ and

$$\mu'''(t) = \nu'(t)\lambda''(t) + \nu(t)\lambda'''(t) - \lambda'(t)\nu''(t) - \lambda(t)\nu'''(t).$$

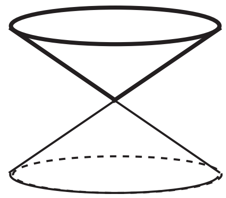

FIGURE 6. Tangent variety of Veronese surface.

Then

$$\det\begin{pmatrix} \lambda' & \mu' & \nu' \\ \lambda'' & \mu'' & \nu'' \\ \lambda''' & \mu''' & \nu''' \end{pmatrix} = (\lambda'\nu'' - \lambda''\nu')^2.$$

Therefore, if $\text{type}(\lambda(t), \nu(t)) = (1, 2)$, then $\text{type}(\gamma(t)) = (1, 2, 3)$. Moreover we have

$$\mu'''' = 2\nu'\lambda''' + \nu\lambda'''' - 2\lambda'\nu''' - \lambda\nu''''.$$

Then

$$\text{rank}\begin{pmatrix} \lambda' & \mu' & \nu' \\ \lambda'' & \mu'' & \nu'' \\ \lambda''' & \mu''' & \nu''' \\ \lambda'''' & \mu'''' & \nu'''' \end{pmatrix} = \text{rank}\begin{pmatrix} \lambda' & \nu' & 0 \\ \lambda'' & \nu'' & 0 \\ \lambda''' & \nu''' & \lambda'\nu'' - \lambda''\nu' \\ \lambda'''' & \nu'''' & \lambda'\nu''' - \lambda'''\nu' \end{pmatrix}.$$

Therefore the rank of the above matrix is 3 at t if and only if $\lambda'\nu'' - \lambda''\nu' \neq 0$ or $\lambda'\nu''' - \lambda'''\nu' \neq 0$ at t. By the transversality theorem, we have that, for a generic γ and for any $t_0 \in I$, (a) $\lambda'(t_0)\nu''(t_0) - \lambda''(t_0)\nu'(t_0) \neq 0$ or (b) $\lambda'(t_0)\nu''(t_0) - \lambda''(t_0)\nu'(t_0) = 0$ and $\lambda'(t_0)\nu'''(t_0) - \lambda'''(t_0)\nu'(t_0) \neq 0$. In case (a), $\text{type}(\gamma) = (1, 2, 3)$ at t_0. In case (b), $\text{type}(\gamma) = (1, 3, 4)$ at t_0. Then, by Theorem 7.1(1), we have the required result. □

9. SINGULARITIES OF TANGENT VARIETIES TO SURFACES.

First we observe that the tangent varieties to a generic smooth surface are not frontal.

Example 9.1. Let $V = \left\{ A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} \,\middle|\, 3 \times 3,\ \text{symmetric} \right\}$,

the vector space of quadratic forms of variables x, y, z. Then $\dim(V) = 6$. Let $S = P(\{\text{rank}(A) = 1\}) \subset P(V) \cong \mathbf{R}P^5$ be the Veronese surface. Then we see that the tangent variety consists of the projection of the locus of semi- indefinite matrices of rank 2 and S. Note that the secant variety $\text{Sec}(S)$, the closure of the union of secants connecting any pair of points on S, consists of the projection of the locus of matrices of rank ≤ 2 :

$$\begin{aligned} \text{Tan}(S) &= S \cup P(\{\text{rank}(A) = 2,\ \text{semi-indefinite}\}) \\ &\subsetneq \text{Sec}(S) = P(\{\text{rank}(A) \leq 2\}) \subsetneq P(V). \end{aligned}$$

See Figure 6. The tangent variety $\text{Tan}(S)$ is not frontal. Note that, even if S is algebraic, $\text{Tan}(S)$ is semi-algebraic in general over the real numbers. For a generic surface $S \in \mathbf{R}P^5$, tangent varieties $\text{Tan}(S)$ are perturbed into a non-frontal hypersurface.

Therefore the tangent variety $\text{Tan}(S)$ to a generic surface $S \subset \mathbf{R}P^5$ is never frontal.

Let V be a $(N+3)$-dimensional vector space. Let us consider a flag manifold

$$\mathcal{F} = \mathcal{F}_{1,3}(V) := \{V_1 \subset V_3 \subset V\} \cong \text{Gr}(2, T(P(V))),$$

$\mathcal{F}_{1,3}(V) = 3N+2$, with local coordinates $x_1, x_2, y_1, \dots, y_N, p_1, \dots, p_N, q_1, \dots, q_N$. The canonical differential system $\mathcal{T} = \mathcal{C} = \mathcal{C}_{1,3}$ is given by $dy_i = p_i dx_1 + q_i dx_2, \quad (1 \le i \le N)$. A frontal map-germ $f : (\mathbf{R}^2, 0) \to P(V) = \mathbf{R}P^{N+2}$ lifts to a $\mathcal{C}_{1,3}$-integral map-germs, therefore f is an opening of $g = (x_1 \circ f, x_2 \circ f) : (\mathbf{R}^2, 0) \to \mathbf{R}^2$ with the dense set of regular points.

Thus it is possible to study the singularities of tangent varieties to frontal surfaces as the singularity theory on $\mathcal{C}_{1,3}$-integral mappings. The general studies from this viewpoint are left to a forthcoming paper.

Now, let us consider another type of flag manifold: $\mathcal{F}_{1,3,5}(V) = \{V_1 \subset V_3 \subset V_5 \subset V\}$. and the canonical system $\mathcal{N} = \mathcal{C}_{1,3,5} \subset T(\mathcal{F}_{1,3,5}(V))$ defined by

$$v \in \mathcal{C}_{1,3,5}{}_{(V_1,V_3,V_5)} \Longleftrightarrow \pi_{i*}(v) \in T(\mathrm{Gr}(i, V_{i+2}))(\subset T(\mathrm{Gr}(i, \mathbf{R}^6)),\ i = 1, 3.$$

If $N = 3$, then $\dim(\mathcal{F}_{1,3,5}(\mathbf{R}^6)) = 13$ and $\mathrm{rank}(\mathcal{C}_{1,3,5}) = 8$. In fact, $\mathcal{N}$ is given by

$$\begin{cases} d{x_3}^0 &=& {x_3}^1 d{x_1}^0 + {x_3}^2 d{x_2}^0 \\ d{x_4}^0 &=& {x_4}^1 d{x_1}^0 + {x_4}^2 d{x_2}^0 \\ d{x_5}^0 &=& {x_5}^1 d{x_1}^0 + {x_5}^2 d{x_2}^0 \\ d{x_5}^1 &=& {x_5}^3 d{x_3}^1 + {x_5}^4 d{x_4}^1 \\ d{x_5}^2 &=& {x_5}^3 d{x_3}^2 + {x_5}^4 d{x_4}^2 \end{cases}$$

for a system of projective local coordinates

$${x_1}^0, {x_2}^0, {x_3}^0, {x_4}^0, {x_5}^0, {x_3}^1, {x_4}^1, {x_5}^1, {x_3}^2, {x_4}^2, {x_5}^2, {x_5}^3, {x_5}^4$$

of $\mathcal{F}_{1,3,5}(V^6)$.

Proposition 9.2. *Let $f : (\mathbf{R}^2, 0) \to P(V^{N+3})$ be a frontal map-germ. Suppose that the regular locus of the tangent map* $\mathrm{Tan}(f) : (\mathbf{R}^4, 0) \to P(V)$ *is dense. Then* $\mathrm{Tan}(f)$ *is frontal if and only if f is the projection of a $\mathcal{C}_{1,3,5}$-integral map by $\pi_1 : \mathcal{F}_{1,3,5}(V) \to P(V)$.*

Proof: Suppose $\mathrm{Tan}(f)$ is frontal and $g : (\mathbf{R}^4, 0) \to \mathrm{Gr}(4, T(P(V))) = \mathcal{F}_{1,5}(V)$ is the Grassmannian lifting of $\mathrm{Tan}(f)$. Then $g|_{\mathbf{R}^2 \times 0}$ lifts a $\mathcal{C}_{1,3,5}$-integral map $F : (\mathbf{R}^2, 0) \to \mathcal{F}_{1,3,5}(V)$ and $\pi_1 \circ F = f$. Conversely if $\pi_1 \circ F = f$ for a $\mathcal{C}_{1,3,5}$-integral map F, then $\mathrm{Tan}(f)$ lifts to $G : (\mathbf{R}^4, 0) \to \mathcal{F}_{1,3,5}(V)$ by $G(s_1, s_2, t_1, t_2) = F(0, 0, t_1, t_2)$. □

Let V^6 be a symplectic vector space. Let us consider the canonical contact structure on $P(V) = \mathbf{R}P^5$. Let $S \subset \mathbf{R}P^5$ be a Legendre surface. Then S lifts to a $\mathcal{C}_{1,3,5}$-integral surface. Therefore, by Theorem 9.2, we have:

Corollary 9.3. *Let $i : (\mathbf{R}^2, 0) \to \mathbf{R}P^5$ be a Legendre immersion-germ. Suppose the regular locus* $\mathrm{Reg}(\mathrm{Tan}(i))$ *of the tangent variety is dense in* $(\mathbf{R}^2, 0)$*. Then the tangent variety* $\mathrm{Tan}(i) :$ $(\mathbf{R}^2, 0) \to \mathbf{R}P^5$ *is a frontal.*

Definition 9.4. A point p of a Legendre surface S in $\mathbf{R}P^5$ is called an *ordinary point* if there exists a local projective-contact coordinates x_1, x_2, x_3, x_4, x_5 and a C^∞ local coordinates (u, v) of S centred p such that locally S is given by

$$\begin{cases} x_1 &=& u, \\ x_2 &=& v, \\ x_3 &=& \frac{1}{2}au^2 + buv + \frac{1}{2}cv^2 \ + \text{ higher order terms}, \\ x_4 &=& \frac{1}{2}bu^2 + cuv + \frac{1}{2}ev^2 \ + \text{ higher order terms}, \\ x_5 &=& -(\frac{1}{6}au^3 + \frac{1}{2}bu^2v + \frac{1}{2}cuv^2 + \frac{1}{6}ev^3) \ + \text{ higher order terms}, \end{cases}$$

with

$$\mathcal{D} = \{dx_5 - x_1 dx_3 - x_2 dx_4 + x_3 dx_1 + x_4 dx_2 = 0\},$$

and

$$\operatorname{rank}\begin{pmatrix} a & b & c \\ b & c & e \end{pmatrix} = 2.$$

An ordinary point p is called *hyperbolic* (resp. *elliptic*, *parabolic*), if moreover

$$H := 4(ac - b^2)(be - c^2) - (ae - bc)^2$$

is negative (resp. positive, zero).

Note that the set of hyperbolic (resp. elliptic) ordinary points is an open subset in S. Then we have the following fundamental result:

Theorem 9.5. *The tangent variety* $\mathrm{Tan}(S)$ *to a Legendre surface* S *in* $\mathbf{R}P^5$ *at a hyperbolic ordinary point* (*resp. an elliptic ordinary point*) *is locally diffeomorphic to* (D_4^+*-singularity in* $\mathbf{R}^3$) $\times\, \mathbf{R}^2$ (*resp.* (D_4^-*-singularity in* $\mathbf{R}^3$) $\times\, \mathbf{R}^2$) *in* $\mathbf{R}^5$.

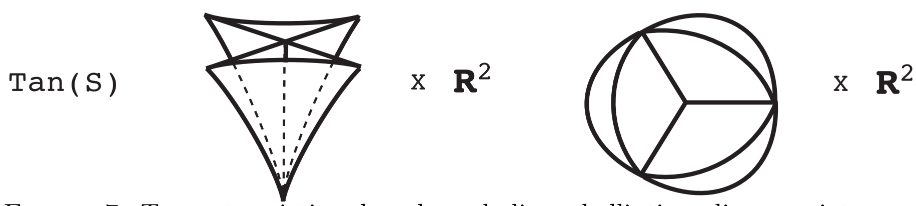

FIGURE 7. Tangent varieties along hyperbolic and elliptic ordinary points on a surface in $\mathbf{R}P^5$.

In [36], a simple criterion on D_4 has been found by Saji. The $D_4^{\pm}$-singularity in $\mathbf{R}^3$ is given by the map-germ $(\mathbf{R}^2, 0) \to (\mathbf{R}^3, 0)$

$$(u, v) \mapsto (uv, u^2 \pm 3v^2, u^2v \pm v^3).$$

Theorem 9.6. ([36]) *Let* $f : (\mathbf{R}^2, 0) \to (\mathbf{R}^3, 0)$ *be a front and* $(f, \nu) : (\mathbf{R}^2, 0) \to \mathbf{R}^3 \times S^2$ *a Legendre lift of* f*. Then* f *is diffeomorphic to* D_4^+ (*resp.* D_4^-) *if and only if* f *is of rank zero at* 0 *and the Hessian determinant of*

$$\lambda(u, v) := \det\left(\frac{\partial f}{\partial u}(u, v), \frac{\partial f}{\partial v}(u, v), \nu(u, v)\right)$$

at $(0, 0)$ *is negative* (*resp. positive*).

Note that D_4-singularity is not a generic singularity of wave-fronts in $\mathbf{R}^3$, but is a generic singularity of wave-fronts in $\mathbf{R}^4$. The criterion for D_4-singularities in $\mathbf{R}^4$ is also given in [36]. Moreover we remark that Saji's criterion is valid also for the case with parameters and it characterises the trivial deformation of D_4-singularity. In fact the same line of proof in [36] works as well for the case with parameters:

Theorem 9.7. *Let* $F = (f_t)_{t \in (\mathbf{R}^r, 0)} : (\mathbf{R}^2 \times \mathbf{R}^r, 0) \to (\mathbf{R}^3, 0)$ *be a family of fronts and* $(F, N) = (f_t, \nu_t) : (\mathbf{R}^2 \times \mathbf{R}^r, 0) \to \mathbf{R}^3 \times S^2$ *a family of Legendre lifts of* F*. Then* F *is diffeomorphic to the trivial deformation of* D_4^+ (*resp.* D_4^-) *if and only if* f_t *is of rank zero at* 0 *and the Hessian determinant of*

$$\lambda(u, v, t) := \det\left(\frac{\partial f_t}{\partial u}(u, v), \frac{\partial f_t}{\partial v}(u, v), \nu_t(u, v)\right)$$

with respect to (u, v) *at* $(0, 0, t)$ *is negative* (*resp. positive*)*, for any* $t \in (\mathbf{R}^r, 0)$.

Proof of Theorem 9.5: Let $x_1 = u, x_2 = v$,

$$\begin{array}{rcl} x_3 & = & \frac{1}{2}au^2 + buv + \frac{1}{2}cv^2 + \varphi(u,v), \\ x_4 & = & \frac{1}{2}bu^2 + cuv + \frac{1}{2}ev^2 + \psi(u,v), \\ x_5 & = & -(\frac{1}{6}au^3 + \frac{1}{2}bu^2v + \frac{1}{2}cuv^2 + \frac{1}{6}ev^3) + \rho(u,v), \end{array}$$

$\mathrm{ord}(\varphi) \geq 3, \mathrm{ord}(\psi) \geq 3$, and

$$\rho_u = u\varphi_u + v\psi_u - \varphi, \quad \rho_u = u\varphi_u + v\psi_u - \psi.$$

As an integrability condition, we have that $\varphi_v = \psi_u$. The tangent map of S is given by $x_1 = u + s, x_2 = v + t$,

$$\begin{array}{rcl} x_3 & = & \frac{1}{2}au^2 + buv + \frac{1}{2}cv^2 + \varphi + s\left(au + bv + \varphi_u\right) + t\left(bu + cv + \varphi_v\right), \\ x_4 & = & \frac{1}{2}bu^2 + cuv + \frac{1}{2}ev^2 + \psi + s\left(bu + cv + \psi_u\right) + t\left(cu + ev + \psi_v\right), \\ x_5 & = & -(\frac{1}{6}au^3 + \frac{1}{2}bu^2v + \frac{1}{2}cuv^2 + \frac{1}{6}ev^3) + \rho \\ & & +s\left(-\frac{1}{2}au^2 - buv - \frac{1}{2}cv^2 + \rho_u\right) + t\left(-\frac{1}{2}bu^2 - cuv - \frac{1}{2}ev^2 + \rho_v\right). \end{array}$$

Take the transversal slice $s = -u, t = -v$. Then we have map-germ $g : (\mathbf{R}^2, 0) \to (\mathbf{R}^3, 0)$,

$$\begin{array}{rcl} g_1(u,v) & = & -\frac{1}{2}au^2 - buv - \frac{1}{2}cv^2 + \varphi - u\varphi_u - v\varphi_v, \\ g_2(u,v) & = & -\frac{1}{2}bu^2 - cuv - \frac{1}{2}ev^2 + \psi - u\psi_u - v\psi_v, \\ g_3(u,v) & = & \frac{1}{3}au^3 + bu^2v + cuv^2 + \frac{1}{3}ev^3 + \rho - u\rho_u - v\rho_v. \end{array}$$

We show that g is diffeomorphic to D_4-singularity, by using Saji's criterion (Theorem 9.6).

First, we have $dg_3 = -udg_1 - vdg_2$. Therefore g is a front and we can take $\nu = \frac{1}{\sqrt{u^2+v^2+1}}(u, v, 1)$. Second, we see f is of rank zero. Third,

$$\begin{array}{rcl} \lambda(u,v) & = & \det(g_u, g_v, \nu) = \det\begin{pmatrix} g_{1u} & g_{1v} & u \\ g_{2u} & g_{2v} & v \\ 0 & 0 & \sqrt{u^2+v^2+1} \end{pmatrix} \\ & = & \sqrt{u^2+v^2+1}\,(g_{1u}g_{2v} - g_{1v}g_{2u}) \end{array}$$

The 2-jet of $h := g_{1u}g_{2v} - g_{1v}g_{2u}$ at 0 is given by

$$j^2h(0) = (ac - b^2)u^2 + (ae - bc)uv + (be - c^2)v^2 \quad (\mathrm{mod.}\ \mathfrak{m}_2^3).$$

Therefore we have that the Hessian determinant of λ at 0 is given by

$$H = \det\begin{pmatrix} 2(ac - b^2) & ae - bc \\ ae - bc & 2(be - c^2) \end{pmatrix}$$

By Theorem 9.6, we see that g is diffeomorphic to $D_4^{\pm}$ if and only if $\mp H > 0$. Moreover, we can show similarly that, regarding S as the parameter space, the tangent map-germ is diffeomorphic to the trivial unfolding of D_4- singularity with two parameters, by using Theorem 9.7. Hence we have Theorem 9.5. $\square$

10. Tangent maps to frontal maps and open problems.

Let V be a $(N+2n)$-dimensional vector space with positive natural numbers N, n. Consider the flag manifolds:

$$\mathcal{F}_{1,n+1,2n+1} = \mathcal{F}_{1,n+1,2n+1}(V) := \{V_1 \subset V_{n+1} \subset V_{2n+1} \subset V\},$$

with the canonical differential system $\mathcal{C}_{1,n+1,2n+1}$, and

$$\mathcal{F}_{1,n+1} = \mathcal{F}_{1,n+1}(V) := \{V_1 \subset V_{n+1} \subset V\},$$

with the canonical differential system $\mathcal{C}_{1,n+1}$. Note that $\mathcal{F}_{1,n+1}$ is identified with the Grassmannian bundle $\mathrm{Gr}(n, T(P(V)))$. Consider the canonical projections

$$\mathcal{F}_{1,n+1,2n+1} \xrightarrow{\Pi} \mathcal{F}_{1,n+1} \xrightarrow{\pi} \mathcal{F}_1 = P(V) = \mathbf{R}P^{N+2n-1}.$$

Similarly to the proof of Proposition 9.2, we have

Proposition 10.1. *Let $f : (\mathbf{R}^n, 0) \to \mathbf{R}P^{N+2n-1}$ be a frontal map-germ. Suppose the regular locus $\mathrm{Reg}(\mathrm{Tan}(f))$ of the tangent map $\mathrm{Tan}(f) : (\mathbf{R}^{2n}, 0) \to \mathbf{R}P^{N+2n-1}$ is dense in $(\mathbf{R}^{2n}, 0)$. Then $\mathrm{Tan}(f)$ is frontal if and only if the Grassmannian lift $\widetilde{f} : (\mathbf{R}^n, 0) \to \mathcal{F}_{1,n+1}$ of f for π, lifts to a $\mathcal{C}_{1,n+1,2n+1}$-integral lift $\mathbf{f} : (\mathbf{R}^n, 0) \to \mathcal{F}_{1,n+1,2n+1}$ for Π.*

It is natural to proceed to consider the tangent varieties to Legendre submanifolds.

Let V be a $(2n+2)$-dimensional symplectic vector space. Consider the Lagrange (isotropic) flag manifold:

$$\mathcal{F}_{\mathrm{Lag}} = \mathcal{F}_{\mathrm{Lag}}(V) := \{V_1 \subset V_{n+1} \subset V \mid V_{n+1} \text{ is Lagrange.}\},$$

with the canonical differential system $\mathcal{E} \subset T\mathcal{F}_{\mathrm{Lag}}$. In general we have

Corollary 10.2. *Let $g : (\mathbf{R}^n, 0) \to \mathcal{F}_{\mathrm{Lag}}$ be $\mathcal{E}$-integral and $\mathrm{Tan}(\pi_1 \circ g) : (\mathbf{R}^{2n}, 0) \to P(V)$ the tangent map-germ of $\pi_1 \circ g : (\mathbf{R}^n, 0) \to P(V)$. Suppose that $\mathrm{Reg}(\mathrm{Tan}(\pi_1 \circ g))$ is dense in $(\mathbf{R}^n, 0)$. Then $\mathrm{Tan}(\pi_1 \circ g)$ is frontal.*

Proof: Note that $\mathcal{F}_{\mathrm{Lag}}$ is embedded in $\mathcal{F}_{1,n+1,2n+1}$ by $(V_1, V_{n+1}) \mapsto (V_1, V_{n+1}, V_1^s)$, where V_1^s is the symplectic skew-orthogonal to V_1, and $\mathcal{E}$ is the restriction of $\mathcal{C}_{1,n+1,2n+1}$. Therefore Proposition 10.2 follows from Proposition 10.1.

Here we give alternative direct proof. Since f is Legendre, $f = (\lambda, \mu, \nu)$ satisfies $d\mu = \sum_{i=1}^n (\nu_i d\lambda_i - \lambda_i d\nu_i)$. The tangent map-germ $\mathrm{Tan}(f) = (\Lambda, M, N)$ is given by

$$\begin{pmatrix} \Lambda \\ M \\ N \end{pmatrix} = \begin{pmatrix} \lambda \\ \mu \\ \nu \end{pmatrix} + \sum_{j=1}^n s_j \begin{pmatrix} \partial\lambda/\partial u_j \\ \partial\mu/\partial u_j \\ \partial\nu/\partial u_j \end{pmatrix}.$$

Then we have

$$\begin{aligned} dM &= d\mu + \sum_{j=1}^n s_j d\left(\partial\mu/\partial u_j\right) + \textstyle\sum_{j=1}^n \left(\partial\mu/\partial u_j\right) ds_j \\ &= d\mu + \sum_{i=1}^n \sum_{j=1}^n s_j \left(\nu_i d\left(\partial\lambda_i/\partial u_j\right) - \lambda_i d\left(\partial\nu_i/\partial u_j\right)\right) \\ &\qquad + \sum_{i=1}^n \sum_{j=1}^n \left(\nu_i \left(\partial\lambda_i/\partial u_j\right) - \lambda_i \left(\partial\nu_i/\partial u_j\right)\right) ds_j \\ &= \sum_{i=1}^n \left(\nu_i d\Lambda_i - \lambda_i dN_i\right). \end{aligned}$$

Thus $M \in \mathcal{R}_{(\Lambda,N)}$ and $\mathrm{Tan}(f)$ is frontal. □

Then Corollary 10.2 implies

Corollary 10.3. *Let $f : (\mathbf{R}^n, 0) \to P(V) = \mathbf{R}P^{2n+1}$ be a germ of Legendre immersion and $\mathrm{Tan}(f) : (\mathbf{R}^{2n}, 0) \to P(V)$ the tangent map-germ of f. Suppose that $\mathrm{Reg}(\mathrm{Tan}(f))$ is dense in $(\mathbf{R}^n, 0)$. Then $\mathrm{Tan}(f)$ is frontal.*

We conclude the paper by posing open generic classification problems, which remain to be solved first:

Problem 1: Classify the singularities of tangent varieties to generic contact-integral curves in $P(V^{2n+2}) \cong \mathbf{R}P^{2n+1}$ for a symplectic vector space V of dimension $2n+2$, under diffeomorphisms and contactomorphisms.
Problem 2: Classify the singularities of tangent varieties to generic surfaces in $\mathbf{R}P^5$. It would be natural to relate singularities of tangent variety to the method of height function or hight family (cf. [39][31]).
Problem 3: Classify the singularities of tangent varieties to generic frontal surfaces (projections of generic $\mathcal{C}_{1,3}$-integral surfaces in $\mathcal{F}_{1,3}(\mathbf{R}^6)$) in $\mathbf{R}P^5$.
Problem 4: Classify the singularities of tangent varieties to projections in $\mathbf{R}P^5$ of generic $\mathcal{C}_{1,3,5}$-integral surfaces in $\mathcal{F}_{1,3,5}(\mathbf{R}^6)$.
Problem 5: Classify the singularities of tangent varieties to Legendre surfaces in $\mathbf{R}P^5$ along parabolic ordinary points. Moreover classify the singularities of tangent varieties of generic Legendre surfaces in $\mathbf{R}P^5$. (See §9.)

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PEDAL FOLIATIONS AND GAUSS MAPS OF HYPERSURFACES IN EUCLIDEAN SPACE

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Abstract. The singular point of the Gauss map of a hypersurface in Euclidean space is the parabolic point where the Gauss-Kronecker curvature vanishes. It is well-known that the contact of a hypersurface with the tangent hyperplane at a parabolic point is degenerate. The parabolic point has been investigated in the previous research by applying the theory of Lagrangian or Legendrian singularities. In this paper we give a new interpretation of the singularity of the Gauss map from the view point of the theory of wave front propagations.

1. Introduction

The singular point of the Gauss map of a hypersurface in Euclidean space is the parabolic point of the hypersurface where the Gauss-Kronecker curvature vanishes [1, 12]. There have been many researches on singularities of Gauss maps [2, 3, 17, 19]. The pedal of the hypersurface (cf. [6, 12]) is the wavefront set whose singular points are the same as the parabolic points of the hypersurface. Actually, we can show that the pedal is defined in $S^{n-1}\times\mathbb{R}$. We call it a *cylindrical pedal* (or, *dual hypersurface*) of the hypersurface [5, 12, 20]. By definition, the Gauss map is the S^{n-1}-component of the cylindrical pedal. In this paper we consider the $\mathbb{R}$-component of the cylindrical pedal which defines a function on the hypersurface. We call it a *pedal height function* on the hypersurface. The pedal height function is traditionally called the *support function* of the hypersurface with respect to the origin. We investigate, in this paper, geometric meanings of the singularities of the pedal height function. A *pedal foliation* is the foliation defined by the level set of the pedal height function.

On the other hand, we investigated relationships between caustics and wave front propagations as an application of the theory of graphlike Legendrian unfoldings in [11, 14]. The image of the pedal foliation by the Gauss map is considered to be a wave front propagation of a certain graphlike Legendrian unfolding (cf. §5). By applying the results in [11, 14], we obtain a new interpretation of the singularity of the Gauss map from the view point of the theory of wave front propagations (cf. §6). In §4, we briefly review the essential part of the theories of Lagrangian singularities and graphlike Legendrian unfoldings which we use in this paper. Especially, we give a correct proof of Proposition 4.1 in [14], which is one of the key propositions in the theory of graphlike Legendrian unfoldings (Proposition 4.3). In §6 we focus on the case for surfaces in $\mathbb{R}^3$. We give a classification of the surface with the constant pedal height function (i.e., the most degenerate case). Moreover, we give extra new conditions which characterize cusps of Gauss maps (cf. [2]).

We shall assume throughout the whole paper that all maps and manifolds are C^∞ unless the contrary is explicitly stated.

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2. Hypersurfaces in Euclidean space

In this section we review the classical theory of differential geometry on hypersurfaces in Euclidean space and introduce some singular mappings associated to geometric properties of hypersurfaces.

Let $\boldsymbol{X} : U \to \mathbb{R}^n$ be an embedding, where $U \subset \mathbb{R}^{n-1}$ is an open subset. We denote that $M = \boldsymbol{X}(U)$ and identify M and U through the embedding $\boldsymbol{X}$. The tangent space of M at $p = \boldsymbol{X}(u)$ is

$$T_pM = \langle \boldsymbol{X}_{u_1}(u), \boldsymbol{X}_{u_2}(u), \dots, \boldsymbol{X}_{u_{n-1}}(u)\rangle_{\mathbb{R}}.$$

For any $\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_{n-1} \in \mathbb{R}^n$, we define

$$\boldsymbol{a}_1 \times \boldsymbol{a}_2 \times \cdots \times \boldsymbol{a}_{n-1} = \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \cdots & \boldsymbol{e}_n \\ a_1^1 & a_2^1 & \cdots & a_n^1 \\ a_1^2 & a_2^2 & \cdots & a_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ a_1^{n-1} & a_2^{n-1} & \cdots & a_n^{n-1} \end{vmatrix},$$

where $\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_n\}$ is the canonical basis of $\mathbb{R}^n$ and $\boldsymbol{a}_i = (a_1^i, a_2^i, \dots, a_n^i)$. It follows that we can define the unit normal vector field

$$\boldsymbol{n}(u) = \frac{\boldsymbol{X}_{u_1}(u) \times \cdots \times \boldsymbol{X}_{u_{n-1}}(u)}{\|\boldsymbol{X}_{u_1}(u) \times \cdots \times \boldsymbol{X}_{u_{n-1}}(u)\|}$$

along $\boldsymbol{X} : U \to \mathbb{R}^n$. A map $\mathbb{G} : U \to S^{n-1}$ defined by $\mathbb{G}(u) = \boldsymbol{n}(u)$ is called the *Gauss map* of $M = \boldsymbol{X}(U)$. Since $\boldsymbol{n}(u)$ is the unit normal vector of S^{n-1}, we can identify T_pM and $T_{\boldsymbol{n}(u)}S^{n-1}$. Under this identification, the derivative of the Gauss map $d\mathbb{G}(u)$ can be interpreted as a linear transformation on the tangent space T_pM at $p = \boldsymbol{X}(u)$. We call the linear transformation $S_p = -d\mathbb{G}(u) : T_pM \to T_pM$ the *shape operator* (or *Weingarten map*) of $M = \boldsymbol{X}(U)$ at $p = \boldsymbol{X}(u)$. We denote the eigenvalues of S_p by $\kappa_i(p)$ $(i = 1, \dots, n-1)$ which we call *principal curvatures*. We call the eigenvector of S_p the *principal direction*. By definition, $\kappa_i(p)$ is a principal curvature if and only if $\det(S_p - \kappa_i(p)I) = 0$. *The Gauss-Kronecker curvature* of $M = \boldsymbol{X}(U)$ at $p = \boldsymbol{X}(u)$ is defined to be $K(p) = \det S_p = \Pi_{i=1}^{n-1}\kappa_i(p)$.

We say that a point $p = \boldsymbol{X}(u) \in M$ is an *umbilical* point if $S_p = \kappa(p)1_{T_pM}$. We also say that M is *totally umbilical* if all points of M are umbilical. Then we have the following proposition (cf. [9, page 147, Proposition 4] for $n = 3$). For general dimensions, the proof is given by the same method as that of [9].

Proposition 2.1. *Suppose that $M = \boldsymbol{X}(U)$ is totally umbilical, then $\kappa(p)$ is a constant κ. Under this condition, we have the following classification:*

(1) *If $\kappa \neq 0$, then M is a part of a hypersphere.*
(2) *If $\kappa = 0$, then M is a part of a hyperplane.*

In the extrinsic differential geometry, totally umbilical hypersurfaces are considered to be the model hypersurfaces in Euclidean space. Since the set $\{\boldsymbol{X}_{u_1}, \dots, \boldsymbol{X}_{u_{n-1}}\}$ is linearly independent, we induce the Riemannian metric (*first fundamental form*) $ds^2 = \sum_{i,j=1}^{n-1} g_{ij}du_idu_j$ on $M = \boldsymbol{X}(U)$, where $g_{ij}(u) = \langle \boldsymbol{X}_{u_i}(u), \boldsymbol{X}_{u_j}(u)\rangle$ for any $u \in U$. We define the *second fundamental invariant* by $h_{ij}(u) = \langle -\boldsymbol{n}_{u_i}(u), \boldsymbol{X}_{u_j}(u)\rangle$ for any $u \in U$. We have the following *Weingarten formula*:

$$\mathbb{G}_{u_i}(u) = -\sum_{j=1}^{n-1} h_i^j(u)\boldsymbol{X}_{u_j}(u),$$

where $(h_i^j(u)) = (h_{ik}(u))(g^{kj}(u))$ and $(g^{kj}(u)) = (g_{kj}(u))^{-1}$. By the Weingarten formula, the Gauss-Kronecker curvature is given by

$$K(p) = \frac{\det(h_{ij}(u))}{\det(g_{\alpha\beta}(u))}.$$

For a hypersurface $\boldsymbol{X} : U \to \mathbb{R}^n$, we say that a point $u \in U$ or $p = \boldsymbol{X}(u)$ is a *flat point* (or, a *geodesic point*) if $h_{ij}(u) = 0$ for all i, j. Therefore, $p = \boldsymbol{X}(u)$ is a flat point if and only if p is an umbilical point with the vanishing principal curvature. We say that a point $p = \boldsymbol{X}(u) \in M$ is a *parabolic point* if $K(p) = 0$.

The *cylindrical pedal* of $M = \boldsymbol{X}(U)$ is defined by

$$\mathbb{CP}_M : U \to S^{n-1} \times \mathbb{R}\ ;\ \mathbb{CP}_M(u) = (\boldsymbol{n}(u), \langle \boldsymbol{X}(u), \boldsymbol{n}(u)\rangle).$$

We remark that $\mathbb{CP}_M$ is called the *dual* of $M = \boldsymbol{X}(U)$ (cf. [5, 7]). For a plane curve $\boldsymbol{\gamma}(s)$, $Pe_\gamma(s) = \langle \boldsymbol{\gamma}(s), \boldsymbol{n}(s)\rangle \boldsymbol{n}(s)$ is called the *pedal curve* of $\boldsymbol{\gamma}$ (cf. [6]), so that we call $\mathbb{CP}_M$ the cylindrical pedal. We have the following result (cf. [12]):

Proposition 2.2. *Let $M = \boldsymbol{X}(U)$ be a hypersurface in $\mathbb{R}^n$. Then the following are equivalent:*
(1) *M is totally umbilical with $\kappa = 0$.*
(2) *The Gauss map of $M = \boldsymbol{X}(U)$ is a constant map.*
(3) *The cylindrical pedal of $M = \boldsymbol{X}(U)$ is a point.*
(4) *M is a part of a hyperplane.*

We can easily show that a point $p = \boldsymbol{X}(u)$ is a parabolic point of $M = \boldsymbol{X}(U)$ (i.e., a singular point of the Gauss map) if and only if it is a singular point of the cylindrical pedal. Therefore we have the following proposition:

Proposition 2.3. *Let $M = \boldsymbol{X}(U)$ be a hypersurface in $\mathbb{R}^n$. Then the following are equivalent:*
(1) *$p = \boldsymbol{X}(u)$ is a parabolic point of M (i.e., $K(u) = 0$).*
(2) *$p = \boldsymbol{X}(u)$ is a singular point of the Gauss map of $M = \boldsymbol{X}(U)$.*
(3) *$p = \boldsymbol{X}(u)$ is a singular point of the cylindrical pedal of $M = \boldsymbol{X}(U)$.*

The Gauss map $\mathbb{G}(u)$ is the first component of the cylindrical pedal $\mathbb{CP}_M(u)$. We have a natural question as follows:
Question. What kind of information are provided by the second component of the cylindrical pedal?

We define a function $h^\pi : U \to \mathbb{R}$ by $h^\pi(u) = \langle \boldsymbol{X}(u), \boldsymbol{n}(u)\rangle$. It has been called h^π the *support function* of $M = \boldsymbol{X}(U)$ with respect to the origin. Since h^π is the second component of the cylindrical pedal, we call it the *pedal height function* of $M = \boldsymbol{X}(U)$ here. We remark that h^π is invariant under the $SO(n)$-action and not invariant under the Euclidean motions.

3. Pedal foliations

A *pedal foliation* is the foliation in U (or $M = \boldsymbol{X}(U)$) defined by the level set of the pedal height function h^π. We write

$$\mathscr{F}^\pi(M) = \{(h^\pi)^{-1}(t_0) \mid h^\pi(u_0) = t_0 \in \mathbb{R}\}.$$

as the pedal foliation and denote by $\mathcal{L}^\pi_{u_0}(M)$ the leaf through u_0 with $h^\pi(u_0) = t_0$. We call $\mathcal{L}^\pi_{u_0}(M)$ a *pedal leaf* of $M = \boldsymbol{X}(U)$ through $u_0 \in U$. The pedal foliation might be singular in general. The singular point of the pedal foliation is a critical point of the pedal height function h^π.

In order to explain the critical point of the pedal height function h^π, we decompose $\boldsymbol{X}(u)$ into the tangent component $\boldsymbol{X}^T(u)$ and the normal component $\boldsymbol{X}^\perp(u)$. For any $p = \boldsymbol{X}(u)$, we have $\boldsymbol{X}(u) = \boldsymbol{X}^T(u) + \boldsymbol{X}^\perp(u)$ where $\boldsymbol{X}^T(u) \in T_pM$ and $\boldsymbol{X}^\perp(u) \in T_pM^\perp$. Then we have the following proposition.

Proposition 3.1. *Let $\boldsymbol{X} : U \to \mathbb{R}^n$ be a hypersurface. Then $u \in U$ is a singular point of the pedal function h^π if and only if $\boldsymbol{X}^T(u) \in \operatorname{Ker} S_p$.*

Proof. By definition, there exist μ_i $(i = 1, \ldots, n-1)$ such that $\boldsymbol{X}^T(u) = \sum_{i=1}^{n-1} \mu_i \boldsymbol{X}_{u_i}(u)$. Since we have $\langle \boldsymbol{X}(u), \boldsymbol{X}_{u_j}(u)\rangle = \sum_{i=1}^{n-1} \mu_i g_{ij}(u)$, we have $\sum_{j=1}^{n-1} g^{kj}(u)\langle \boldsymbol{X}(u), \boldsymbol{X}_{u_j}(u)\rangle = \mu_k$. It follows that

$$\boldsymbol{X}^T(u) = \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} g^{ij}(u)\langle \boldsymbol{X}(u), \boldsymbol{X}_{u_j}(u)\rangle \boldsymbol{X}_{u_i}(u).$$

By the Weingarten formula $\mathbb{G}_{u_i}(u) = -\sum_{j=1}^{n-1} h_i^j(u)\boldsymbol{X}_{u_j}(u)$, we have

$$\begin{aligned}\frac{\partial h^\pi}{\partial u_i}(u) &= \langle \boldsymbol{X}(u), \mathbb{G}_{u_i}(u)\rangle = -\sum_{j=1}^{n-1} h_i^j(u)\langle \boldsymbol{X}(u), \boldsymbol{X}_{u_j}(u)\rangle \\ &= -\sum_{j=1}^{n-1}\sum_{k=1}^{n-1} h_{ik}(u)g^{kj}(u)\langle \boldsymbol{X}(u), \boldsymbol{X}_{u_j}(u)\rangle = -\sum_{k=1}^{n-1} h_{ik}(u)\mu_k.\end{aligned}$$

Therefore, we have

$$\sum_{i=1}^{n-1} g^{ji}(u)\frac{\partial h^\pi}{\partial u_i}(u) = -\sum_{k=1}^{n-1}\left(\sum_{i=1}^{n-1} g^{ji}(u)h_{ik}(u)\right)\mu_k = -\sum_{k=1}^{n-1} h_k^j(u)\mu_k.$$

Thus, $\partial h^\pi/\partial u_i(u) = 0$ for $i = 1, \ldots, n-1$ if and only if $\sum_{k=1}^{n-1} h_k^j(u)\mu_k = 0$ for $j = 1, \ldots, n-1$. This completes the proof. □

Then we have the following corollary.

Corollary 3.2. *Let $p = \boldsymbol{X}(u)$ be a singular point of h^π. Then $\boldsymbol{X}(u)$ is a normal vector of M at p or $K(p) = 0$ and $\boldsymbol{X}^T(u) \in \operatorname{Ker} S_p$.*

Proof. If $\boldsymbol{X}^T(u) \neq \boldsymbol{0}$, then $\operatorname{Ker} S_p \neq \emptyset$. This means that $K(p) = 0$. If $\boldsymbol{X}^T(u) = \boldsymbol{0}$, then $\boldsymbol{X}(u) = \boldsymbol{X}^\perp(u)$. □

We can show that the pedal foliation is non-singular in generic.

Corollary 3.3. *Let $\boldsymbol{X} : U \to \mathbb{R}^n$ be an embedding from an open region $U \subset \mathbb{R}^{n-1}$. Suppose that $p = \boldsymbol{X}(u)$ is a singular point of h^π and non-geodesic point (i.e., non-flat umbilical point). Then, under a small Euclidean motion of $M = \boldsymbol{X}(U)$, h^π is non-singular at $p = \boldsymbol{X}(u)$.*

Proof. By the assumption, $\operatorname{Ker} S_p \neq T_pM$. If $K(p) \neq 0$, the position vector $\boldsymbol{X}(u)$ is not a normal vector at $p = \boldsymbol{X}(u)$ under a small Euclidean motion of M.

Suppose that $K(p) = 0$. If we rotate $M = \boldsymbol{X}(U)$ around the normal direction (i.e, fixining the direciton of $\boldsymbol{n}(u)$) at the point $p = \boldsymbol{X}(u)$, then $p = \boldsymbol{X}^\perp(u) + \boldsymbol{X}^T(u)$ (i.e., of course $\boldsymbol{X}^T(u)$) does not move but $\operatorname{Ker} S_p$ moves. Therefore, we have $\boldsymbol{X}^T(u) \notin \operatorname{Ker} S_p$ by a small Euclidean motion of $M = \boldsymbol{X}(U)$. □

By the above corollary, the pedal foliation is non-singular in generic at least locally, so that we are interested in differential geometric properties of leaves.

We now consider the restriction $\mathbb{G}|\mathcal{L}^\pi_{u_0}(M)$ of the Gauss map $\mathbb{G}$ on the pedal leaf through $u_0 \in U$, which is called the *pedal Gauss map* of $M = \boldsymbol{X}(U)$ at $u_0 \in U$.

4. Graphlike Legendrian unfoldings

In order to apply the theories of Lagrangian singularities and graphlike Legendrian unfoldings, we explain the essential parts of the theories which we need in this paper. The detailed descriptions and the results are referred to be the articles [1, 11, 14, 22, 23].

Firstly, we consider the cotangent bundle $\pi : T^*\mathbb{R}^n \to \mathbb{R}^n$. Let $(x,p) = (x_1,\dots,x_n,p_1,\dots,p_n)$ be the canonical coordinate on $T^*\mathbb{R}^n$. Then the canonical symplectic structure on $T^*\mathbb{R}^n$ is given by the *canonical two form* $\omega = \sum_{i=1}^n dp_i \wedge dx_i$. Let $i : L \subset T^*\mathbb{R}^n$ be a submanifold. We say that i is a *Lagrangian submanifold* if $\dim L$ and $i^*\omega = 0$. Let $F : (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be an n-parameter unfolding of function germs. We say that F is a *Morse family of functions* if the map germ

$$\Delta F = \left(\frac{\partial F}{\partial q_1}, \dots, \frac{\partial F}{\partial q_k}\right) : (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}^k, 0)$$

is a non-singular, where $(q,x) = (q_1,\dots,q_k,x_1,\dots,x_n) \in (\mathbb{R}^k \times \mathbb{R}^n, 0)$. In this case, we have a smooth n-dimensional submanifold germ $C(F) = (\Delta F)^{-1}(0) \subset (\mathbb{R}^k \times \mathbb{R}^n, 0)$ and a map germ $L(F) : (C(F), 0) \to T^*\mathbb{R}^n$ defined by

$$L(F)(q,x) = \left(x, \frac{\partial F}{\partial x_1}(q,x), \dots, \frac{\partial F}{\partial x_n}(q,x)\right).$$

We can show that $L(F)(C(F))$ is a Lagrangian submanifold germ. We say that F is a *generating family* of $L(F)(C(F))$.

We now define an equivalence relation among Lagrangian submanifold germs. Let $F, G : (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be Morse families of functions. Then the Lagrangian submanifold germs $(L(F)(C(F)), \mathbf{0})$ and $(L(G)(C(G)), \mathbf{0})$ are said to be *Lagrangian equivalent* if there exist a symplectic diffeomorphism germ $\hat{\tau} : (T^*\mathbb{R}^n, p) \to (T^*\mathbb{R}^n, p')$ and a diffeomorphism germ $\tau : (\mathbb{R}^n, \pi(p)) \to (\mathbb{R}^n, \pi(p'))$ such that $\hat{\tau}(L(F)(C(F)) = L(G)(C(G))$ and $\pi \circ \hat{\tau} = \tau \circ \pi$, where $\hat{\tau}$ is a symplectic diffeomorphism germ if $\hat{\tau}^*\omega = \omega$. By using the Lagrangian equivalence, we can define the notion of Lagrangian stability for Lagrangian submanifold germs by the ordinary way (see, [1, Part III]).

We can interpret the Lagrangian equivalence by using the notion of generating families. Let $\mathcal{E}_x$ be the ring of function germs of $x = (x_1,\dots,x_n)$ variables at the origin. Let $F, G : (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be function germs. We say that F and G are *P-$\mathcal{R}^+$-equivalent* if there exist a diffeomorphism germ $\Phi : (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}^k \times \mathbb{R}^n, 0)$ of the form $\Phi(q,x) = (\phi_1(q,x), \phi_2(x))$ and a function germ $h : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ such that $G(q,x) = F(\Phi(q,x)) + h(x)$. For any $F_1 : (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ and $F_2 : (\mathbb{R}^{k'} \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$, Let $F : (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be a function germ. We say that F is an *$\mathcal{R}^+$-versal deformation* of $f = F|_{\mathbb{R}^k \times \{0\}}$ if

$$\mathcal{E}_q = J_f + \left\langle \frac{\partial F}{\partial x_1}|\mathbb{R}^k \times \{0\}, \dots, \frac{\partial F}{\partial x_n}|\mathbb{R}^k \times \{0\} \right\rangle_{\mathbb{R}} + \langle 1 \rangle_{\mathbb{R}},$$

where

$$J_f = \left\langle \frac{\partial f}{\partial q_1}(q), \dots, \frac{\partial f}{\partial q_k}(q) \right\rangle_{\mathcal{E}_q}.$$

Then we have the following theorem[1, page 304 and 325]:

Theorem 4.1. *Let $F : (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ and $G : (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be Morse families of functions. Then we have the following:*
(1) $L(F)(C(F))$ and $L(G)(C(G))$ are Lagrangian equivalent if and only if F and G are P-$\mathcal{R}^+$-equivalent.
(2) $L(F)(C(F))$ is a Lagrange stable if and only if F is an $\mathcal{R}^+$-versal deformation of f.

In [1], the assertion (1) of the above theorem is a slightly different. It is used the notion of stable P-$\mathcal{R}^+$-equivalences among Morse families. However, the above assertion is enough for our situation.

Secondly, we now give a brief review on the theory of graphlike Legendrian unfoldings. The notion of graphlike Legendrian unfoldings is defined in the projective cotangent bundle $\overline{\pi}$: $PT^*(\mathbb{R}^n\times\mathbb{R}) \to \mathbb{R}^n\times\mathbb{R}$ (cf. [11]). We remark that the affine open subset $U_\tau = \{((x,t),[\xi:\tau])|\tau \neq 0\}$ of $PT^*(\mathbb{R}^n\times\mathbb{R})$ is canonically identified with the 1-jet space $J^1(\mathbb{R}^n,\mathbb{R})$, see in [11, 14]. For a Morse family of functions $F : (\mathbb{R}^k\times\mathbb{R}^n,0)\to(\mathbb{R},0)$, we define a map $\mathfrak{L}_F : (C(F),0)\to J^1(\mathbb{R}^n,\mathbb{R})$ by

$$\mathfrak{L}_F(q,x) = \left(x, F(q,x), \frac{\partial F}{\partial x_1}(q,x),\ldots,\frac{\partial F}{\partial x_n}(q,x)\right).$$

Then $(\mathfrak{L}_F(C(F)),\mathbf{0})$ is a Legendrian submanifold germ which is called a *graphlike Legendrian unfolding*. We call the set germ $W(\mathfrak{L}_F) = \overline{\pi}(\mathfrak{L}_F(C(F))$ the *graphlike wave front* of $\mathfrak{L}_F(C(F))$. A graphlike Legendrian unfolding $(\mathfrak{L}_F(C(F)),\mathbf{0})$ is said to be *non-degenerate* if $F|C(F)$ is non-singular. We say that F is a *generating family* of the graphlike Legendrian unfolding $\mathfrak{L}_F(C(F))$. We can use all equivalence relations introduced in the previous paper [13, 14, 15]. Especially, the $S.P^+$-Legendrian equivalence among graphlike Legendrian unfoldings was given in the above context. Since we do not need the definition here, we omit to give the definition (see [13]). We also consider the stability of graphlike Legendrian unfolding with respect to $S.P^+$ -Legendrian equivalence which is analogous to the stability of Lagrangian submanifold germs with respect to Lagrangian equivalence (cf. [1, Part III]). We denote that $\overline{F}(q,x,t) = F(q,x)-t$ and $\overline{f}(q,t) = f(q)-t$ for $f(q) = F(q,0)$. We can represent the extended tangent space of $\overline{f} : (\mathbb{R}^k\times\mathbb{R},0)\to(\mathbb{R},0)$ relative to $S.P^+$-$\mathcal{K}$ by

$$T_e(S.P^+\text{-}\mathcal{K})(\overline{f}) = \left\langle \frac{\partial f}{\partial q_1}(q),\ldots,\frac{\partial f}{\partial q_k}(q), f(q)-t\right\rangle_{\mathcal{E}_{(q,t)}} + \langle 1\rangle_{\mathbb{R}}.$$

For an unfolding $\overline{F} : (\mathbb{R}^k\times\mathbb{R}^n,0)\to(\mathbb{R},0)$ of f, $\overline{F}$ is $S.P^+$-$\mathcal{K}$-versal deformation of $\overline{f}$ if

$$\mathcal{E}_{(q,t)} = T_e(S.P^+\text{-}\mathcal{K})(\overline{f}) + \left\langle \frac{\partial F}{\partial x_1}|\mathbb{R}^k\times\{0\},\ldots,\frac{\partial F}{\partial x_n}|\mathbb{R}^k\times\{0\}\right\rangle_{\mathbb{R}}.$$

Then we have the following theorem [11, 14, 23].

Theorem 4.2. *Let $F : (\mathbb{R}^k\times\mathbb{R}^n,0)\to(\mathbb{R},0)$ be a Morse family of functions. Then $\mathcal{L}_F(C(F))$ is $S.P^+$-Legendre stable if and only if $\overline{F}$ is a $S.P^+$-$\mathcal{K}$-versal deformation of $\overline{f}$.*

We gave a proof of the following proposition in [14]. However, there are some gaps on the arguments of the proof. Here we give a correct proof of Proposition 4.1 in [14].

Proposition 4.3. *Let $F : (\mathbb{R}^k\times\mathbb{R}^n,0)\to(\mathbb{R},0)$ be a Morse family of functions. If $\mathfrak{L}_F(C(F))$ is a $S.P^+$-Legendre stable, then $L(F)(C(F))$ is a Lagrange stable.*

Proof. Since $\mathfrak{L}_F(C(F))$ is a $S.P^+$-Legendre stable,

$$\dim_{\mathbb{R}} \frac{\mathcal{E}_{(q,t)}}{\left\langle \frac{\partial f}{\partial q_1}(q),\ldots,\frac{\partial f}{\partial q_k}(q), f(q)-t\right\rangle_{\mathcal{E}_{(q,t)}} + \langle 1\rangle_{\mathbb{R}}} < \infty.$$

It follows that $\dim_{\mathbb{R}}\mathcal{E}_q/\langle \frac{\partial f}{\partial q_1}(q),\ldots,\frac{\partial f}{\partial q_k}(q), f(q)\rangle_{\mathcal{E}_q} < \infty$, namely, f is a $\mathcal{K}$-finitely determined (see the definition [8, 18]). It is a well-known result that f is a $\mathcal{K}$-finitely determined if and only if f is an $\mathcal{R}^+$-finitely determined, see [8]. Under the condition that f is an $\mathcal{R}^+$-finitely determined,

F is an $\mathcal{R}^+$-versal deformation of f if and only if F is an $\mathcal{R}^+$-transversal deformation of f, namely, there exists a number $\ell \in \mathbb{N}$ such that

$$\mathcal{E}_q = J_f + \left\langle \frac{\partial F}{\partial x_1}|\mathbb{R}^k \times \{0\}, \dots, \frac{\partial F}{\partial x_n}|\mathbb{R}^k \times \{0\} \right\rangle_{\mathbb{R}} + \langle 1 \rangle_{\mathbb{R}} + \mathcal{M}_q^{\ell+1}. \tag{1}$$

Hence it is enough to show the equality (1) by Theorem 4.1. Let $g(q) \in \mathcal{E}_q$. Since $g(q) \in \mathcal{E}_{(q,t)}$, there exist $\lambda_i(q,t), \mu(q,t) \in \mathcal{E}_{(q,t)}$ $(i = 1, \dots, k)$ and $c, c_j \in \mathbb{R}$ $(j = 1, \dots, n)$ such that

$$g(q) = \sum_{i=1}^{k} \lambda_i(q,t) \frac{\partial f}{\partial q_i}(q) + \mu(q,t)(f(q) - t) + c + \sum_{j=1}^{n} c_j \frac{\partial F}{\partial x_j}(q,0). \tag{2}$$

Differentiating the equality (2) with respect to t, we have

$$0 = \sum_{i=1}^{k} \frac{\partial \lambda_i}{\partial t}(q,t) \frac{\partial f}{\partial q_i}(q) + \frac{\partial \mu}{\partial t}(q,t)(f(q) - t) - \mu(q,t). \tag{3}$$

We put $t = 0$ in (3), $0 = \sum_{i=1}^{k} (\partial \lambda_i/\partial t)(q,0)(\partial f/\partial q_i)(q) + (\partial \mu/\partial t)(q,0) f(q) - \mu(q,0)$. Also we put $t = 0$ in (2), then

$$\begin{aligned} g(q) &= \sum_{i=1}^{k} \lambda_i(q,0) \frac{\partial f}{\partial q_i}(q) + \mu(q,0) f(q) + c + \sum_{j=1}^{n} c_j \frac{\partial F}{\partial x_j}(q,0) \\ &= \sum_{i=1}^{k} \alpha_i(q) \frac{\partial f}{\partial q_i}(q) + \frac{\partial \mu}{\partial t}(q,0) f^2(q) + c + \sum_{j=1}^{n} c_j \frac{\partial F}{\partial x_j}(q,0), \end{aligned} \tag{4}$$

for some $\alpha_i \in \mathcal{E}_q, i = 1 \dots, k$. Again differentiating (3) with respect to t and put $t = 0$, then

$$0 = \sum_{i=1}^{k} \frac{\partial^2 \lambda_i}{\partial t^2}(q,0) \frac{\partial f}{\partial q_i}(q) + \frac{\partial^2 \mu}{\partial t^2}(q,0) f(q) - 2\frac{\partial \mu}{\partial t}(q,0).$$

Hence (4) is equal to

$$\sum_{i=1}^{k} \beta_i(q) \frac{\partial f}{\partial q_i}(q) + \frac{1}{2}\frac{\partial^2 \mu}{\partial t^2}(q,0) f^3(q) + c + \sum_{j=1}^{n} c_j \frac{\partial F}{\partial x_j}(q,0),$$

for some $\beta_i \in \mathcal{E}_q, i = 1, \dots, k$. Inductively, we take ℓ-times differentiate (3) with respect to t and put $t = 0$, then we have

$$g(q) = \sum_{i=1}^{k} \gamma_i(q) \frac{\partial f}{\partial q_i}(q) + \frac{1}{\ell!}\frac{\partial^\ell \mu}{\partial t^\ell}(q,0) f^{\ell+1}(q) + c + \sum_{j=1}^{n} c_j \frac{\partial F}{\partial x_j}(q,0),$$

for some $\gamma_i \in \mathcal{E}_q, i = 1, \dots, k$. It follows that $g(q)$ is contained in the right hand of (1). This completes the proof. □

We consider a relationship of the equivalence relations between Lagrangian immersion germs and corresponding graphlike Legendrian unfoldings. Let $\mathfrak{L}_F : (C(F), 0) \to (J^1(\mathbb{R}^n, \mathbb{R}), p_0)$ and $\mathfrak{L}_G : (C(G), 0) \to (J^1(\mathbb{R}^n, \mathbb{R}), q_0)$ be graphlike Legendrian unfolding germs. We say that graphlike wave fronts $W(\mathfrak{L}_F)$ and $W(\mathfrak{L}_G)$ are *$S.P^+$-diffeomorphic* if there exists a diffeomorphism germ $\Phi : (\mathbb{R}^n \times \mathbb{R}, \overline{\pi}(p_0)) \to (\mathbb{R}^n \times \mathbb{R}, \overline{\pi}(q_0))$ of the form $\Phi(x,t) = (\phi_1(x), t + \alpha(x))$ such that $\Phi(W(\mathfrak{L}_F)) = W(\mathfrak{L}_G)$. Then we have the following result:

Theorem 4.4. ([14]) *Suppose that $L(F)(C(F))$ and $L(G)(C(G))$ are Lagrange stable. Then Lagrangian submanifold germs $L(F)(C(F))$ and $L(G)(C(G))$ are Lagrangian equivalent if and only if graphlike wave fronts $W(\mathfrak{L}_F)$ and $W(\mathfrak{L}_G)$ are $S.P^+$-diffeomorphic.*

5. Height functions

We respectively define two functions

$$H : U \times S^{n-1} \to \mathbb{R}$$

by $H(u, \boldsymbol{v}) = \langle \boldsymbol{X}(u), \boldsymbol{v}\rangle$ and

$$\widetilde{H} : U \times (S^{n-1} \times \mathbb{R}) \to \mathbb{R}$$

by $\widetilde{H}(u,(\boldsymbol{v},t)) = H(u,\boldsymbol{v}) - t = \langle \boldsymbol{X}(u), \boldsymbol{v}\rangle - t$. We call H a *family of height functions* and $\widetilde{H}$ a *family of extended height functions* of $M = \boldsymbol{X}(U)$. We denote that $h_{\boldsymbol{v}}(u) = H(u,\boldsymbol{v})$ and $\widetilde{h}_{(\boldsymbol{v},t)}(u) = \widetilde{H}(u,(\boldsymbol{v},t))$. By the straightforward calculations, we can show the following proposition:

Proposition 5.1. *Let $M = \boldsymbol{X}(U)$ be a hypersurface in $\mathbb{R}^n$. Then*
(1) $(\partial h_{\boldsymbol{v}}/\partial u_i)(u) = 0$ $(i = 1, \ldots, n-1)$ *if and only if* $\boldsymbol{v} = \pm \boldsymbol{n}(u)$.
(2) $\widetilde{h}_{(\boldsymbol{v},t)}(u) = (\partial \widetilde{h}_{(\boldsymbol{v},t)}/\partial u_i)(u) = 0$ $(i = 1, \ldots, n-1)$ *if and only if* $(\boldsymbol{v},t) = \pm(\boldsymbol{n}(u), \langle \boldsymbol{n}(u), \boldsymbol{X}(u)\rangle)$.

For $\boldsymbol{v} = \mathbb{G}(u)$, we have

$$\frac{\partial^2 H}{\partial u_i \partial u_j}(u,\boldsymbol{v}) = \langle \boldsymbol{X}_{u_i u_j}(u), \boldsymbol{v}\rangle = -\langle \boldsymbol{X}_{u_i}(u), \boldsymbol{n}_{u_j}(u)\rangle = h_{ij}(u).$$

Therefore, for any $\boldsymbol{v} = \mathbb{G}(u)$, $\det(\mathcal{H}(h_v)(u)) = \det((\partial^2 H/\partial u_i \partial u_j)(u,\boldsymbol{v})) = 0$ if and only if $K(p) = 0$ (i.e., $p = \boldsymbol{X}(u)$ is a parabolic point), where $\mathcal{H}(h_v)(u)$ is the Hessian matrix of h_v at a point u. By the above calculation, we have the following results [12]:

Proposition 5.2. *For any $p = \boldsymbol{X}(u)$, we have the following assertions:*
Suppose that $\boldsymbol{v} = \mathbb{G}(u)$, then
(1) *p is a parabolic point if and only if* $\det(\mathcal{H}(h_v)(u)) = 0$.
(2) *p is a flat point if and only if* $\operatorname{rank} \mathcal{H}(h_v)(u) = 0$.

We now consider the relationship with the theories of Lagrangian singularities and graphlike Legendrian unfoldings. By [12, Proposition 4.1], we have the following proposition.

Proposition 5.3. *Let $\boldsymbol{X} : U \to M$ be an embedding.*
(1) *The family of height functions $H : U \times S^{n-1} \to \mathbb{R}$ of $M = \boldsymbol{X}(U)$ is a Morse family of functions.*
(2) *The family of extended height functions $\widetilde{H} : U \times (S^{n-1} \times \mathbb{R}) \to \mathbb{R}$ of $M = \boldsymbol{X}(U)$ is a graphlike Morse family of hypersurfaces.*

By the arguments in §4, we have a graphlike Legendrian unfolding whose generating family is the height function of $M = \boldsymbol{X}(U)$. By Proposition 5.1, we have

$$C(H) = \{(u, \pm \boldsymbol{n}(u)) \in U \times S^{n-1} \mid u \in U\}.$$

It follows that we have a graphlike Legendrian unfolding $\mathfrak{L}_H : C(H) \to T^*S^{n-1} \times \mathbb{R} \cong J^1(S^{n-1}, \mathbb{R})$ defined by

$$\mathfrak{L}_H(u, \pm \boldsymbol{n}(u)) = (L(H)(u, \pm \boldsymbol{n}(u)), \langle \pm \boldsymbol{n}(u), \boldsymbol{X}(u)\rangle),$$

where $L(H) : C(H) \to T^*S^{n-1}$ is the corresponding Lagrangian immersion. By definition, we have the following corollary of the above proposition:

Corollary 5.4. *Under the above notations, $\mathfrak{L}_H(C(H))$ is a graphlike Legendrian unfolding such that the height function $H : U \times S^{n-1} \to \mathbb{R}$ of $M = \boldsymbol{X}(U)$ is a generating family of $\mathfrak{L}_H(C(H))$.*

By Corollary 5.4 and Proposition 5.1, we have the graphlike Legendrian unfolding $\mathfrak{L}_H(C(H))$ whose graphlike wave front is the cylindrical pedal $\pm\mathbb{CP}_M$ of $M = \boldsymbol{X}(U)$. We call $\mathfrak{L}_H(C(H))$ the *Legendrian lift* of the cylindrical pedal $\mathbb{CP}_M$ of $M = \boldsymbol{X}(U)$. By definition, we have $H(u, \pm\boldsymbol{n}(u)) = \pm\langle \boldsymbol{X}(u), \boldsymbol{n}(u)\rangle = \pm h^\pi(u)$. Therefore, we have the following proposition.

Proposition 5.5. *The restriction of the height function $H|C(H)$ is non-singular at $u \in U$ if and only if the pedal height function h^π is non-singular at $u \in U$.*

It follows that the graphlike Legendrian unfolding $\mathfrak{L}_H(C(H))$ is non-degenerate if and only if the pedal height function h^π is non-singular.

6. Families of wave fronts induced by Gauss maps

In this section, we consider general geometric properties of singularities of the pedal foliation of a hypersurface in Euclidean space. Let $\mathscr{F}^\pi(M)$ be the pedal foliation on a hypersurface $M = \boldsymbol{X}(U)$. Suppose that $p = \boldsymbol{X}(u_0) \in M$ is a non-singular point of the pedal height function h^π, so that the germ of the pedal foliation $(\mathscr{F}^\pi(M), p)$ is non-singular. We call the germ of the pedal leaf $\mathcal{L}^\pi_{u_0}(M)$ through p the *central pedal leaf* of the pedal foliation germ $(\mathscr{F}^\pi(M), p)$. We consider the family of pedal Gauss map germs $\{\mathbb{G}|\mathcal{L}^\pi_u(M)\}_{h^\pi(u)\in(\mathbb{R},h^\pi(u_0))}$. Let $\pi_1 : S^{n-1} \times \mathbb{R} \to S^{n-1}$ and $\pi_2 : S^{n-1} \times \mathbb{R} \to \mathbb{R}$ be the canonical projections. Then $\mathbb{G}(\mathcal{L}^\pi_u(M)) = \pi_1(\pi_2^{-1}(t) \cap \mathbb{CP}_M)$ for each $t \in (\mathbb{R}, h^\pi(u_0))$ is the small front of the non-degenerate graphlike Legendrian unfolding $\mathfrak{L}_H(C(H))$. Thus, the family of the image of pedal Gauss map germs $\{\mathbb{G}|\mathcal{L}^\pi_u(M)\}_{h^\pi(u)\in(\mathbb{R},h^\pi(u_0))}$ is a family of wave fronts corresponding to the graphlike Legendrian unfolding $\mathfrak{L}_H(C(H))$. We can apply the theory of graphlike Legendrian unfoldings.

On the other hand, in order to understand the geometric meaning of singularities of Gauss maps (or equivalently, cylindrical pedal), we review the theory of contact of submanifolds with foliations [10, 13, 14]. Let X_i $(i = 1, 2)$ be submanifolds of $\mathbb{R}^n$ with $\dim X_1 = \dim X_2$, $g_i : (X_i, \bar{x}_i) \to (\mathbb{R}^n, \bar{y}_i)$ be immersion germs and $f_i : (\mathbb{R}^n, \bar{y}_i) \to (\mathbb{R}, 0)$ be submersion germs. For a submersion germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$, we denote that $\mathcal{F}(f)$ is the regular foliation defined by f; i.e., $\mathcal{F}(f) = \{f^{-1}(c) \mid c \in (\mathbb{R}, 0)\}$. We say that *the contact of X_1 with the regular foliation* $\mathcal{F}(f_1)$ at $\bar{y}_1$ is the *same type* as *the contact of X_2 with the regular foliation* $\mathcal{F}(f_2)$ at $\bar{y}_2$ if there is a diffeomorphism germ $\Phi : (\mathbb{R}^n, \bar{y}_1) \to (\mathbb{R}^n, \bar{y}_2)$ such that $\Phi(X_1) = X_2$ and $\Phi(Y_1(c)) = Y_2(c)$ for each $c \in (\mathbb{R}, 0)$, where $Y_i(c) = f_i^{-1}(c)$. In this case we write

$$K(X_1, \mathcal{F}(f_1); \bar{y}_1) = K(X_2, \mathcal{F}(f_2); \bar{y}_2).$$

We apply the method of Goryunov [10, Appendix] to the case for $\mathcal{R}^+$-equivalence among function germs, so that we have the following:

Proposition 6.1. ([13]) *Let X_i $(i = 1, 2)$ be submanifolds of $\mathbb{R}^n$ with $\dim X_1 = \dim X_2 - 1$, $g_i : (X_i, \bar{x}_i) \to (\mathbb{R}^n, \bar{y}_i)$ be immersion germs and $f_i : (\mathbb{R}^n, \bar{y}_i) \to (\mathbb{R}, 0)$ be submersion germs. We assume that $\bar{x}_i$ are singularities of function germs $f_i \circ g_i : (X_i, \bar{x}_i) \to (\mathbb{R}, 0)$. Then $K(X_1, \mathcal{F}(f_1); \bar{y}_1) = K(X_2, \mathcal{F}(f_2); \bar{y}_2)$ if and only if $f_1 \circ g_1$ and $f_2 \circ g_2$ are $\mathcal{R}^+$-equivalent.*

We consider a function $\mathfrak{H} : \mathbb{R}^n \times S^{n-1} \to \mathbb{R}$ defined by $\mathfrak{H}(\boldsymbol{x}, \boldsymbol{v}) = \langle \boldsymbol{x}, \boldsymbol{v}\rangle$. For a hypersurface $\boldsymbol{X} : U \to \mathbb{R}^n$, we have $H = \mathfrak{H} \circ (\boldsymbol{X} \times 1_{S^{n-1}})$. We denote $\mathfrak{h}_{\boldsymbol{v}}(\boldsymbol{x}) = \mathfrak{H}(\boldsymbol{x}, \boldsymbol{v})$ for $\boldsymbol{v} \in S^{n-1}$. Suppose that $\boldsymbol{v}_0 = \boldsymbol{n}(u_0)$ and $t_0 = h^\pi(u_0) = \langle \boldsymbol{X}(u_0), \boldsymbol{v}_0\rangle$. By Proposition 5.1, $\mathfrak{h}^{-1}_{\boldsymbol{v}_0}(t_0)$ is tangent to M at $p_0 = \boldsymbol{X}(u_0)$. We denote the tangent hyperplane of M at p_0 by TM_{p_0}. Then $\mathfrak{h}^{-1}_{\boldsymbol{v}_0}(t)$ for

$t \in (t_0 - \varepsilon, t_0 + \varepsilon)$ is a hyperplane parallel to TM_{p_0}. Therefore, we have a foliation $\mathcal{PF}(TM_{p_0})$ consists of the family of hyperplane parallel to TM_{p_0}:

$$\mathcal{PF}(TM_{p_0}) = \{\mathfrak{h}_{\boldsymbol{v}_0}^{-1}(t) \mid t \in (t_0 - \varepsilon, t_0 + \varepsilon)\}.$$

We call $\mathcal{PF}(TM_{p_0})$ the *foliation of parallel tangent hyperplanes* of M at p_0

We now give a characterization of singularities of Gauss maps and cylindrical pedals. Let $\boldsymbol{X}^i : (U^i, u^i) \to (\mathbb{R}^n, p^i)$, $(i = 1, 2)$ be hypersurface germs and $H^i : (U^i \times S^{n-1}, (u^i, \boldsymbol{v}^i)) \to \mathbb{R}$ be families of height functions on $M^i = \boldsymbol{X}^i(U^i)$, where $\boldsymbol{v}^i = \mathbb{G}^i(u^i)$, $(i = 1, 2)$. Then we have Lagrangian submanifold germs $L(H^i)(C(H^i))$ in T^*S^{n-1} which cover the Gauss maps $\mathbb{G}^i$ as Lagrangian maps. We also have the graphlike Legendrian unfoldings $\mathfrak{L}_{H^i}(C(H^i))$ whose graphlike wavefronts are the cylindrical pedals $\mathbb{CP}_{M^i}(U^i)$.

Theorem 6.2. *Suppose that the Lagrange submanifold germs $L(H^i)(C(H^i))$ are Lagrangian stable. Then the following conditions are equivalent:*

(1) *$L(H^1)(C(H_1))$ and $L(H^2)(C(H_2))$ are Lagrangian equivalent,*
(2) *$(\mathbb{CP}_{M^1}(U^1), (\boldsymbol{v}^1, t^1))$ and $(\mathbb{CP}_{M^2}(U^2), (\boldsymbol{v}^2, t^2))$ are $S.P^+$-$\mathcal{K}$-diffeomorphic,*
(3) *$\mathfrak{L}_{H^1}(C(H^1))$ and $\mathfrak{L}_{H^2}(C(H^2))$ are $S.P^+$-Legendrian equivalent,*
(4) *$K(M^1, \mathcal{PF}(TM^1_{p^1}); p^1) = K(M^2, \mathcal{PF}(TM^2_{p^2}); p^2)$.*

Proof. By Proposition 6.1, the conditions (4) is equivalent to the condition that the height function germs $h^1_{\boldsymbol{v}^1}$ and $h^1_{\boldsymbol{v}^2}$ are $\mathcal{R}^+$-equivalent. By the assumption and Theorem 4.1, H^i is an $\mathcal{R}^+$-versal unfolding of $h^i_{\boldsymbol{v}^i}$ as germs. Then the uniqueness theorem of the $\mathcal{R}^+$-versal unfoldings (cf. [4, 18]) asserts that $h^1_{\boldsymbol{v}^1}$ and $h^1_{\boldsymbol{v}^2}$ are $\mathcal{R}^+$-equivalent if and only if H^1 and H^2 are P-$\mathcal{R}^+$-equivalent. The last condition is equivalent to the condition (1) by Theorem 4.1. Since the cylindrical pedal is the graphlike wave front of the graphlike Legendrian unfolding generated by the family of height functions, the conditions (1) and (2) are equivalent by Theorem 4.4. Moreover, by [14, Proposition 3.5], the conditions (2) and (3) are equivalent. This completes the proof. □

We remark that the condition (1) of the above theorem implies that the corresponding Gauss maps $\mathbb{G}^1$ and $\mathbb{G}^2$ are $\mathcal{A}$-equivalent. Here two map germs $f, g : (\mathbb{R}^n, \mathbf{0}) \to (\mathbb{R}^p, \mathbf{0})$ are said to be $\mathcal{A}$-*equivalent* if there exist diffeomorphism germs $\phi : (\mathbb{R}^n, \mathbf{0}) \to (\mathbb{R}^n, \mathbf{0})$ and $\psi : (\mathbb{R}^p, \mathbf{0}) \to (\mathbb{R}^p, \mathbf{0})$ such that $f \circ \phi = \psi \circ g$. Therefore, the singular sets of the Gauss maps (i.e., the parabolic sets of M^i) correspond to each other by the condition (1). In general, the $\mathcal{A}$-equivalence of the Gauss maps does not imply the Lagrangian equivalence of the corresponding Lagrangian submanifolds. Moreover, the above theorem asserts that the pictures of the family of the images of the pedal Gauss maps (the wave front propagations) are also corresponding. In the next section we consider the detailed properties of the pedal foliations in the case for surfaces in $\mathbb{R}^3$.

7. Surfaces in Euclidean 3-space

In this section we consider surfaces in the Euclidean 3-space. Let $\boldsymbol{X} : U \to \mathbb{R}^3$ be an embedding, where $U \subset \mathbb{R}^2$ is an open set. In §3 we introduced the notion of pedal foliations. When h^π is constant? This the case that codimension of the pedal foliation is zero. We give a classification of surfaces such that h^π is constant.

Proposition 7.1. *Let $\boldsymbol{X} : U \to \mathbb{R}^3$ be an regular surface. Suppose that h^π is constant. Then we have the following cases:*

(1) *$M = \boldsymbol{X}(U)$ is a subset of a plane,*
(2) *$M = \boldsymbol{X}(U)$ is a subset of a sphere around the origin,*

(3) $M = \boldsymbol{X}(U)$ *is a subset of a circular cylinder around the origin* $\mathrm{CCY}(r)$, *where* $\mathrm{CCY}(r)$ *is given by a rotation of the standard circular cylinder*

$$\mathrm{SCCY}(r) = \{\boldsymbol{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 = r^2 \}$$

around the origin, where $r > 0$,

(4) $M = \boldsymbol{X}(U)$ *is a subset of a circular cone* $\mathrm{CCO}(r, a)$, *where* $\mathrm{CCO}(r, a)$ *is given by a rotation of the standard circular cone*

$$\mathrm{SCCO}(r, a) = \{\boldsymbol{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 - r^2(x_3 - a)^2 = 0,\ x_3 > a \}$$

around the origin, where $r > 0$,

Proof. Suppose that $\boldsymbol{X}^T \equiv \boldsymbol{0}$. Then $\langle \boldsymbol{X}^\perp(u), \mathbb{G}(u)\rangle = \langle \boldsymbol{X}(u), \mathbb{G}(u)\rangle = h^\pi(u) = r$, so that $\boldsymbol{X}^\perp(u) = r\mathbb{G}(u)$. Therefore, $\langle \boldsymbol{X}(u), \boldsymbol{X}(u)\rangle = r^2$. This means that $M = \boldsymbol{X}(U)$ is a subset of a sphere around the origin with the radius $|r|$.

We consider the case $\boldsymbol{X}^T \not\equiv \boldsymbol{0}$. Then we have two sets $U_1 = \{u \in U \mid \boldsymbol{X}^T(u) \neq 0\}$ and $U_2 = \{u \in U \mid \boldsymbol{X}^T(u) = 0\}$. It is clear that U_1 is an open set and $U_1 \cup U_2 = U$. By the assumption, $U_1 \neq \emptyset$. Moreover, by Corollary 3.3, $U_1 \subset K^{-1}(0)$. If $U_2 = \emptyset$, then M is a developable surface. Suppose that $U_2 \neq \emptyset$. If $U_2 \cap K^{-1}(0)$ has an interior point $v_0 \in U$, then there exists an open neighbourhood V of v_0 such that $\boldsymbol{X}^T|V \equiv \boldsymbol{0}$. By the previous argument, $\boldsymbol{X}(V)$ is a part of a sphere, so that $K(v) \neq 0$ on V. This is a contradiction, so that $U_2 \cap K^{-1}(0)$ has no interior points. Thus, we have $\partial U_1 = U_2 \cap K^{-1}(0)$. Moreover, if $U \setminus K^{-1}(0)$ is non-empty, then $U \setminus K^{-1}(0) \subset U_2$. By the above arguments, we have $\mathrm{Int}\, U_2 = U \setminus K^{-1}(0)$. In this case $\boldsymbol{X}(\mathrm{Int}\, U_2)$ is a part of a sphere and $\boldsymbol{X}(U_1)$ is a developable surface. Therefore, we may suppose that $U_2 = \emptyset$, so that M is a developable surface.

It is classically known that developable surfaces is determined completely as follows [21]: A developable surface is classified into one of the following cases:

(1) a part of a cylindrical surface,
(2) a part of a cone,
(3) a part of a tangent surface,
(4) the glue of the above three surfaces.

Suppose that M is a part of a cylindrical surface. It is parametrized at least locally by $\boldsymbol{X}(s, u) = \boldsymbol{\gamma}(s) + u\boldsymbol{e}$, where $\gamma(s)$ is a unit speed space curve and $\boldsymbol{e}$ is a unit constant vector. Moreover, we can choose that $\gamma(s)$ is a planar curve such that $\boldsymbol{t}(s) \perp \boldsymbol{e}$, where $\boldsymbol{t}(s) = \boldsymbol{\gamma}'(s)$ is a unit tangent vector. In this case the unit normal of $\boldsymbol{X}(s, u)$ is given by $\boldsymbol{n}(s, u) = \boldsymbol{t}(s) \times \boldsymbol{e}$. Therefore, we have $h^\pi(s, u) = \langle \boldsymbol{\gamma}(s) + u\boldsymbol{e}, \boldsymbol{t}(s) \times \boldsymbol{e}\rangle = \langle \boldsymbol{\gamma}(s), \boldsymbol{t}(s) \times \boldsymbol{e}\rangle = \langle \boldsymbol{\gamma}(s) \times \boldsymbol{t}(s), \boldsymbol{e}\rangle = r$. By the Frenet-Serret formulae, we have $0 = (\partial h^\pi/\partial s)(s, u) = \langle \gamma(s) \times \kappa(s)\boldsymbol{n}(s), \boldsymbol{e}\rangle$, where $\boldsymbol{n}(s)$ is the principal normal vector. Since $\boldsymbol{\gamma}$ is a planar curve and $\boldsymbol{t}(s) \perp \boldsymbol{e}$, we have $\boldsymbol{\gamma}(s) \times \kappa(s)\boldsymbol{n}(s) \equiv \boldsymbol{0}$. If $\kappa \equiv 0$ on some interval I, then $\boldsymbol{\gamma}$ is a line on I, so that $\boldsymbol{X}|I \times \mathbb{R}$ is a part of a plane. If $\kappa \neq 0$ on some interval I, $\gamma(s)$ and $\boldsymbol{n}(s)$ are parallel on I. There exists $\lambda(s)$ such that $\boldsymbol{\gamma}(s) = \lambda(s)\boldsymbol{n}(s)$, so that $\pm\lambda(s) = \langle \lambda(s)\boldsymbol{e}, \boldsymbol{e}\rangle = \langle \lambda(s)\boldsymbol{n} \times \boldsymbol{t}(s), \boldsymbol{e}\rangle = \langle \boldsymbol{\gamma}(s) \times \boldsymbol{t}(s), \boldsymbol{e}\rangle = r$. It follows that $\boldsymbol{X}(s, u) = r\boldsymbol{n}(s) + u\boldsymbol{e}$. This means that $\boldsymbol{X}(s, u)$ is on a circular cylinder around the origin for $s \in I$.

Suppose that M is a part of a cone. It is parametrized at least locally by $\boldsymbol{X}(s, u) = \boldsymbol{a} + u\boldsymbol{\delta}(s)$, where $\boldsymbol{\delta}(s)$ is a unit speed spherical curve and $\boldsymbol{a}$ is a constant vector. Then $\boldsymbol{t}(s) = \boldsymbol{\delta}'(s)$ is a unit vector such that $\boldsymbol{\delta}(s)$ and $\boldsymbol{t}(s)$ are orthogonal. In this case the unit normal of $\boldsymbol{X}(s, u)$ is given by $\boldsymbol{n}(s, u) = \boldsymbol{\delta}(s) \times \boldsymbol{t}(s) = \boldsymbol{d}(s)$. The moving frame $\{\boldsymbol{\delta}, \boldsymbol{t}, \boldsymbol{d}\}$ is called a *Sabban frame* along the spherical curve $\boldsymbol{\delta}$ [16]. We have the Frenet-Serret type formulae:

$$\begin{cases} \boldsymbol{\delta}'(s) = \boldsymbol{t}(s), \\ \boldsymbol{t}'(s) = \kappa_g(s)\boldsymbol{d}(s) - \boldsymbol{\delta}(s), \\ \boldsymbol{d}'(s) = -\kappa_g(s)\boldsymbol{t}(s), \end{cases}$$

where $\kappa_g(s) = \langle \boldsymbol{t}'(s), \boldsymbol{d}(s)\rangle$ is the *geodesic curvature* of $\boldsymbol{\delta}(s)$. By the assumption, we have $h^\pi(s,u) = \langle \boldsymbol{a}, \boldsymbol{d}(s)\rangle = r$ and $0 = (\partial h^\pi/\partial s)(s,u) = \langle \boldsymbol{a}, \boldsymbol{d}'(s)\rangle = \langle \boldsymbol{a}, -\kappa_g(s)\boldsymbol{t}(s)\rangle$. If $\kappa_g \equiv 0$, then $\boldsymbol{\delta}$ is a geodesic in the unit sphere, so that it is a great circle. Moreover, $\boldsymbol{d}(s)$ is a constant vector. This means that $\boldsymbol{X}(s,u)$ is in the plane $\langle \boldsymbol{x}, \boldsymbol{d}\rangle = r$. If $\kappa_g(s) \neq 0$ on some interval I, then $\langle \boldsymbol{a}, \boldsymbol{t}(s)\rangle = 0$ on I. Therefore $\boldsymbol{t}(s)$ is a planer curve, so that $\boldsymbol{\delta}$ is a curve in the plane $\langle \boldsymbol{x}, \boldsymbol{a}\rangle = c$. If we consider a vector $\widetilde{\boldsymbol{a}} = (c/\|\boldsymbol{a}\|^2)\boldsymbol{a}$, then $\boldsymbol{\delta}(s) - \widetilde{\boldsymbol{a}}$ is orthogonal to $\boldsymbol{a}$. Moreover, we have

$$\langle \boldsymbol{\delta}(s) - \widetilde{\boldsymbol{a}}, \boldsymbol{\delta}(s) - \widetilde{\boldsymbol{a}}\rangle = \langle \boldsymbol{\delta}(s), \boldsymbol{\delta}(s)\rangle - 2\langle \boldsymbol{\delta}(s), \widetilde{\boldsymbol{a}}\rangle + \langle \widetilde{\boldsymbol{a}}, \widetilde{\boldsymbol{a}}\rangle = 1 - \frac{c^2}{\|\boldsymbol{a}\|^2}.$$

This means that $\boldsymbol{\delta}(s)$ is in the circle in the plane $\langle \boldsymbol{x}, \boldsymbol{a}\rangle = c$. We have the case (4).

Finally, we suppose that M is a part of a tangent surface. It is parametrized at least locally by $\boldsymbol{X}(s,u) = \boldsymbol{\gamma}(s) + u\boldsymbol{t}(s)$, where $\boldsymbol{\gamma}(s)$ is a unit speed curve with $\kappa(s) \neq 0$ and $\boldsymbol{t}(s)$ is the unit tangent vector of $\boldsymbol{\gamma}$. We denote the principal normal vector by $\boldsymbol{n}(s)$ and the binormal vector by $\boldsymbol{b}(s)$ of $\boldsymbol{\gamma}$. It is known that the unit normal vector of $\boldsymbol{X}(s,u)$ is the binormal vector $\boldsymbol{b}(s)$ of $\boldsymbol{\gamma}$. Therefore, we have $h^\pi(s,u) = \langle \boldsymbol{\gamma}(s) + u\boldsymbol{t}(s), \boldsymbol{b}(s)\rangle = \langle \boldsymbol{\gamma}(s), \boldsymbol{b}(s)\rangle = r$. Thus, we have $\partial h^\pi/\partial s(s,u) = \langle \boldsymbol{t}(s), \boldsymbol{b}(s)\rangle + \langle \boldsymbol{\gamma}(s), \boldsymbol{t}'(s)\rangle = -\tau(s)\langle \boldsymbol{\gamma}(s), \boldsymbol{n}(s)\rangle = 0$, where $\tau(s)$ is the torsion of $\boldsymbol{\gamma}$. If $\tau \equiv 0$, then $\boldsymbol{\gamma}$ is a planer curve, so that $\boldsymbol{t}(s)$ is also planer. Therefore $\boldsymbol{X}(s,u)$ is part of a plane. If $\tau(s) \neq 0$ on an interval I, then $\langle \boldsymbol{\gamma}(s), \boldsymbol{n}(s)\rangle = 0$, so that there exist $\lambda(s), \mu(s)$ such that $\boldsymbol{\gamma}(s) = \lambda(s)\boldsymbol{t}(s) + \mu(s)\boldsymbol{b}(s)$. Since $\mu(s) = \langle \boldsymbol{\gamma}(s), \boldsymbol{b}(s)\rangle = r$, we have $\boldsymbol{\gamma}(s) = \lambda(s)\boldsymbol{t}(s) + r\boldsymbol{b}(s)$ for $s \in I$. It follows that

$$\boldsymbol{t}(s) = \boldsymbol{\gamma}'(s) = \lambda'(s)\boldsymbol{t}(s) + \lambda(s)\boldsymbol{t}'(s) + r\boldsymbol{b}'(s) = \lambda'(s)\boldsymbol{t}(s) + \kappa(s)(\lambda(s) - r)\boldsymbol{n}(s).$$

Therefore, we have $\lambda'(s) = 1$ and $\lambda(s) = r$. This is a contradiction. If $\kappa(s) = 0$ on an interval J, then $\boldsymbol{\gamma}(s)$ is a line such that the direction is given by $\boldsymbol{t}(s)$. Then $\boldsymbol{X}(s,u)$ is a line on $J \times \mathbb{R}$, which is singular. This completes the proof. □

Since the leaf of the pedal foliation on a surface is a regular curve in generic, we consider generic properties of regular curves on a surface. Let $\boldsymbol{\gamma} : I \to U \subset \mathbb{R}^2$ be a regular curve such that $\overline{\boldsymbol{\gamma}} = \boldsymbol{X} \circ \boldsymbol{\gamma}$ is a unit speed curve. Then $\boldsymbol{t}(s) = \overline{\boldsymbol{\gamma}}'(s)$ is the unit tangent vector field. Let $\boldsymbol{n}_\gamma(s)$ is the unit normal vector field of $M = \boldsymbol{X}(U)$ along $\overline{\boldsymbol{\gamma}}$. We define the *relative normal vector field* of $\overline{\boldsymbol{\gamma}}$ by $\boldsymbol{e}(s) = \boldsymbol{n}_\gamma(s) \times \boldsymbol{t}(s)$. Then we have the following Frenet-Serret type formulae:

$$\begin{cases} \boldsymbol{t}'(s) = \kappa_g(s)\boldsymbol{e}(s) + \kappa_n(s)\boldsymbol{n}_\gamma(s), \\ \boldsymbol{e}'(s) = -\kappa_g(s)\boldsymbol{t}(s) + \tau_g(s)\boldsymbol{n}_\gamma(s), \\ \boldsymbol{n}_\gamma'(s) = -\kappa_n(s)\boldsymbol{t}(s) - \tau_g(s)\boldsymbol{e}(s), \end{cases}$$

where $\kappa_n(s)$ is the *normal curvature*, $\kappa_g(s)$ is the *geodesic curvature* and $\tau_g(s)$ is the *geodesic torsion*. The frame $\{\boldsymbol{t}(s), \boldsymbol{e}(s), \boldsymbol{n}_\gamma(s)\}$ is called the *Darboux frame*. It is known that

1) $\boldsymbol{\gamma}$ is an asymptotic curve of M if and only if $\kappa_n = 0$,
2) $\boldsymbol{\gamma}$ is a geodesic of M if and only if $\kappa_g = 0$,
3) $\boldsymbol{\gamma}$ is a principal curve of M if and only if $\tau_g = 0$.

By the Frenet-Serret type formulae, $\mathbb{G} \circ \boldsymbol{\gamma} = \boldsymbol{n}_\gamma$ is singular at s if and only if $\kappa_n(s) = \tau_g(s) = 0$.

Proposition 7.2. *Let $\mathcal{L}^\pi_{u_0}(M)$ be a non-singular pedal leaf through $u_0 \in U$. Let $\boldsymbol{\gamma} : I \to U$ be a regular curve such that $\overline{\boldsymbol{\gamma}} = \boldsymbol{X} \circ \boldsymbol{\gamma}$ is a parametrization of the pedal leaf $\mathcal{L}^\pi_{u_0}(M)$ with $\boldsymbol{\gamma}(s_0) = u_0$. Then the following conditions are equivalent:*

(1) *The pedal Gauss map* $\mathbb{G}|\mathcal{L}^\pi_{u_0}(M)$ *is singular at* $p = \boldsymbol{X}(u_0)$,
(2) $\kappa_n(s_0) = \tau_g(s_0) = 0$,
(3) *The tangent line* $T_p\mathcal{L}^\pi_{u_0}(M)$ *gives the principal direction with the zero principal curvature.*

Proof. Since $\mathbb{G}|\mathcal{L}^\pi_{u_0}(M)$ is parametrized by $\mathbb{G} \circ \boldsymbol{\gamma} = \boldsymbol{n}_\gamma$, it has been already shown that the conditions (1) and (2) are equivalent. The condition (2) means that $\boldsymbol{t}(s_0)$ directs both the asymptotic and the principal directions. Here, we have $T_p\mathcal{L}^\pi_{u_0}(M) = \langle \boldsymbol{t}(s_0)\rangle_{\mathbb{R}}$ Therefore, the conditions (2) and (3) are equivalent. □

We now revisit the characterizations of the cusp point of the Gauss map in [2]. There are some geometric characterization of the cusp point of the Gauss map. We add extra new conditions to the characterizations of the cusp point of the Gauss map here.

Theorem 7.3. *Suppose that the Gauss map* $\mathbb{G}$ *of* $M = \boldsymbol{X}(U)$ *is stable. Then the parabolic locus* $K^{-1}(0)$ *is a regular curve and the following conditions are equivalent:*
(1) $p = \boldsymbol{X}(u_0)$ *is a cusp of the Gauss map* $\mathbb{G}$,
(2) $p = \boldsymbol{X}(u_0)$ *is a swallowtail of the cylindrical pedal* $\mathbb{CP}_M$,
(3) *The central pedal leaf* $\mathcal{L}^\pi_{u_0}(M)$ *is tangent to the parabolic locus* $K^{-1}(0)$. *Moreover, for any* $\varepsilon > 0$, *there exist* $u_1 \neq u_2 \in U$ *such that* $|u_1 - u_2| < \varepsilon$, *the tangent planes at* u_1, u_2 *are different but* $h^\pi(u_1) = h^\pi(u_2)$ *and the pedal leaf* $\mathcal{L}^\pi_{u_1}(M) = \mathcal{L}^\pi_{u_2}(M)$ *is transverse to the parabolic locus* $K^{-1}(0)$,
(4) *The family of the images of the pedal Gauss maps* $\{\mathbb{G}|\mathcal{L}^\pi_u(M)\}_{h^\pi(u)=t\in(\mathbb{R},t_0)}$ *is the swallowtail bifurcation on* S^2.

Proof. In the previous contexts (cf. [12]), it has been already known that the conditions (1) and (2) are equivalent. Since the cylindrical pedal $\mathbb{CP}_M$ is a graphlike wave front and the family of images of Pedal Gauss maps is the family of corresponding small fronts, so the conditions (2) and (4) are equivalent by using the classification of non-degenerate graphlike Legendrian unfoldings in [11, Theorem 2.3]. By Proposition 7.2, the tangent line $T_p\mathcal{L}^\pi_{u_0}(M)$ gives the principal direction with the vanishing principal curvature. There is a characterization of the cusp point of the Gauss map in [3] that the principal direction with the vanishing principal curvature is tangent to the parabolic point at the point and transverse at the other points. This means that the conditions (1) and (3) are equivalent. □

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LOCAL CLASSIFICATION OF CONSTRAINED HAMILTONIAN SYSTEMS ON 2-MANIFOLDS

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Abstract. We give local classification results for Constrained Hamiltonian Systems, i.e., for differential systems of the form $X \lrcorner \omega = df$, where ω is a singular 2-form and f is a function, both defined and smooth (analytic) on a 2-dimensional manifold M.

1. Introduction-Main Results

All the objects considered in this paper belong in any fixed category (that is smooth, real or complex analytic). For convenience we fix real objects in the smooth (C^∞)-category, unless otherwise stated. There exist many definitions of Constrained Hamiltonian Systems (CHS) most of them extrinsic, as restrictions of Hamiltonian systems on submanifolds of their symplectic phase spaces representing the constraints (c.f. [8], [9], [11], [14], [17], [18]). In our case the following intrinsic definition of CHS (without reference to an ambient symplectic manifold) is more convenient (see [14], [15] for the Hamiltonian case and [13], [16], [20] for the general, not necessarily Hamiltonian, case):

Definition 1.1. *A CHS on a manifold M is a differential system of the form:*

$$X \lrcorner \omega = df \tag{1.1}$$

defined by a pair $\gamma = (f, \omega) \in C^\infty(M) \times \wedge^2(M)$, consisting of a function f (or generally a closed 1-form α) and a closed 2-form ω.

If we view the 1-form df as a section of the cotangent bundle $T^*M \to M$, any such pair $\gamma = (f, \omega)$ determines the diagram:

$$\begin{tikzcd} TM \arrow[r, "\Omega := \lrcorner\omega"] \arrow[d, "\pi"] & T^*M \\ M \arrow[r, "Id"] & M \arrow[u, "df"'] \end{tikzcd}$$

where Ω is the skew-symmetric vector bundle morphism[1] between the tangent and cotangent bundles over M (equivalently, a morphism between the modules of vector fields and 1-forms), defined by interior multiplication $\lrcorner$ with ω. A solution of the CHS is then a vector field $X : M \to TM$ that makes the above diagram commutative. In local coordinates the 1-form df is the gradient covector $\nabla f : x \mapsto \partial f / \partial x$ and if we denote by $t \mapsto x(t)$ the phase curve of X, then equation (1.1) above is written in coordinate form as:

$$\Omega(x)\dot{x} = \nabla f(x) \tag{1.2}$$

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[1] i.e., a vector bundle map, skew-symmetric linear in the fibers and inducing the identity on the base

where $\Omega(x) = (\omega_{ij}(x))_{1\le i,j\le n}$ is the smooth skew-symmetric matrix-valued map (a $n\times n$ skew-symmetric matrix with coefficients ω_{ij} smooth functions of x) associated to the 2-form ω in the fixed basis of $\mathbb{R}^n$.

Notice that smooth solutions X of the CHS (1.1) might not exist, even locally. The obstruction is the singular locus $\Sigma(\omega)$ of the 2-form ω, i.e., the locus of points where the rank of ω is smaller than the dimension of M. For this reason we call the set $\Sigma(\omega)$ impasse set[2] and any of its points, impasse point.

In this paper we give initial results for the classification problem of generic (typical) singularities of CHS $\gamma = (f,\omega)$ at impasse points $\Sigma(\omega)$. We restrict our study on 2-dimensional manifolds M and the first occuring singularities of 2-forms (Martinet singularities). The impasse set in this case is a smooth curve $\Sigma(\omega)\subset M$, also called the Martinet curve.

By equivalence of (germs of) CHS we mean equivalence of (germs of) pairs $\gamma = (f,\omega)$ and $\gamma' = (f',\omega')$ by (germs of) diffeomorphisms Φ (changes of coordinates preserving the point of application) acting on the space of (germs of) pairs $C^\infty(M)\times \wedge^2(M)$:

$$\Phi^*\gamma' := (\Phi^* f', \Phi^*\omega') = (f,\omega) =: \gamma.$$

The topology of the space of CHS is the usual Whitney C^∞-topology.

In the nondegenerate case where ω defines a symplectic structure, the CHS reduces to a Hamiltonian system with Hamiltonian f and the local classification problem reduces to the well known Hamiltonian normal forms (c.f. [1], [4]): there exist coordinates (Darboux coordinates) such that the germ of the Hamiltonian system $\gamma = (f,\omega)$ is equivalent to the normal form

$$\gamma = (x_1(\text{or }\ x_2), \quad dx_1\wedge dx_2) \tag{1.3}$$

at the regular points of f and to

$$\gamma = (\lambda(f_2), \quad dx_1\wedge dx_2) \tag{1.4}$$

at its nondegenerate critical points, where $f_2 = f_2(x_1,x_2)$ is a nondegenerate quadratic form (the first term in the Taylor expansion of f at the origin) and λ is a function of 1-variable such that $\lambda(0) = 0$, $\lambda'(0)\neq 0$. The proof of the normal form (1.3) follows easily from the Darboux theorem (c.f. [1]). Normal form (1.4) may be obtained also from the Morse-Darboux lemma (c.f. [6]). The result holds in the smooth and analytic (real or complex) categories. The germ λ in the normal form (1.4) is a functional modulus, characteristic for the pair $\gamma = (f,\omega)$ (c.f. [6] and references therein).

In the degenerate case where ω vanishes along the points of the smooth line $\Sigma(\omega)$, the singularities of functions are defined (for the 2-dimensional case) by the relative positions of the curve $f^{-1}(0)$ with the characteristic vector field X_ω of ω:

$$span\{X_\omega\}(x) = T_x\Sigma(\omega)\cap Ker_x(\omega) = T_x\Sigma(\omega),$$

(i.e., by the relative positions of $f^{-1}(0)$ with the Martinet curve $\Sigma(\omega)$). We fix germs at the origin $0\in\Sigma(\omega)$. The following cases (singularity classes) occur typically:

(i) $f^{-1}(0)$ is smooth and transversal to the Martinet curve at the origin:

$$X_\omega(f)(0)\neq 0$$

(ii) the germ $f^{-1}(0)$ is smooth and tangent to the Martinet curve at the origin, with 1st-order (nondegenerate) tangency:

$$X_\omega(f)(0) = 0, \quad X^2_\omega(f)(0)\neq 0,$$

where $X^2_\omega(f) = X_\omega(X_\omega(f))$.

[2]see [16] and [20] for a survey of impasse singularities for the case of general constrained systems, defined by tangent bundle endomorphisms.

Martinet ([12]) has shown that a generic germ of a singular 2-form ω at a point on the curve $\Sigma(\omega)$ is equivalent to the normal form

$$\omega = x_1 dx_1 \wedge dx_2. \tag{1.5}$$

In these coordinates the germ of the Martinet curve is given by $\Sigma(\omega) = \{x_1 = 0\}$ and the characteristic vector field by $X_\omega = \partial/\partial x_2$.

Let now f be a function germ at a generic point $0 \in \Sigma(\omega)$, i.e., satisfying the transversality condition (i). The following theorem implies that it is possible to reduce f to a simple normal form by a diffeomorphism leaving the Martinet 2-form $\omega =$ (1.5) fixed.

Theorem 1.2. *All germs of CHS $\gamma = (f, \omega)$ at impasse points of type (i) are equivalent to the normal form*

$$\gamma = (\pm x_2, \quad x_1 dx_1 \wedge dx_2). \tag{1.6}$$

Moreover, the diffeomorphism bringing γ to its normal form is unique.

Remark 1.3. The theorem holds in both smooth and real analytic categories. The existence of the "$\pm$" sign is related to the canonical orientation of the Martinet curve $\Sigma(\omega) = \{x_1 = 0\}$ induced by the opposite orientations of the two symplectic half spaces $\Sigma_0^+ = \{x_1 > 0\}$, $\Sigma_0^- = \{x_1 < 0\}$, defined by the restriction of ω on each one of them. In particular there does not exist a germ of a diffeomorphism preserving the Martinet germ $\omega =$(1.5) and sending x_2 to $-x_2$. In the complex analytic category such an orientation is not defined and the theorem still holds true after we drop the "$\pm$" sign from the normal form (1.6); indeed, the diffeomorphism $(x_1, x_2) \mapsto (ix_1, -x_2)$ conjugates x_2 to $-x_2$ and leaves ω invariant.

Consider now a germ f at a point $0 \in \Sigma(\omega)$ of type (ii). The condition of 1st-order tangency of the pair of curves $(\Sigma(\omega), f^{-1}(0))$ implies that the restriction of f on $\Sigma(\omega)$ has a nodegenerate (Morse) critical point at the origin. Notice that any such singularity is reducible by a diffeomorphism preserving $\Sigma(\omega) = \{x_1 = 0\}$ to the classical normal form $f = x_1 \pm x_2^2$. The next theorem implies that it is impossible to achieve this normal form (or any normal form with a finite number of parameters) under the action of diffeomorphisms preserving also the Martinet 2-form $\omega =$(1.5).

Theorem 1.4. *Germs of CHS $\gamma = (f, \omega)$ at impasse points of type (ii) are not finitely determined. In particular, for any germ γ there exists a function germ λ of 1-variable and with vanishing 1-jet:*

$$\lambda(t) = \sum_{i \geq 2} \lambda_i t^i, \quad \lambda_2 \neq 0,$$

such that γ is equivalent to the invariant normal form

$$\gamma = (x_1 + \lambda(x_2), \quad x_1 dx_1 \wedge dx_2). \tag{1.7}$$

Moreover, the diffeomorphism bringing γ to its normal form is unique.

Remark 1.5. The theorem holds in both smooth and analytic (real or complex) categories. Invariance of the normal form (1.7) means that it cannot contain (as a singularity class) equivalent germs γ and γ' with different λ and λ'. In the analytic category this means that: *two germs f and f' will be equivalent by an analytic diffeomorphism preserving ω if and only if the corresponding series λ and λ' are exactly the same.* It is convenient to express λ invariantly, in terms of the pair $\gamma = (f, \omega)$ as:

$$\lambda(t) = \sum_{i \geq 2} X_\omega^i(f)(0) t^i, \quad X_\omega^2(f)(0) \neq 0.$$

It follows that for any $l \geq 2$ the system of coefficients $\{X_\omega^2(f)(0), ..., X_\omega^l(f)(0)\}$ is a complete invariant for the classification of l-jets of germs f under diffeomorphisms preserving ω. The existence of the modulus λ and in particular of its first order term $\lambda_2 = X_\omega^2(f)(0)$ admits a nice geometric description in terms of action integrals for $\omega = d\alpha$ (for some primitive 1-form α):

$$A(c) = \int_c \alpha,$$

along "half-cycles" c, i.e., along smooth curves with at least two points of intersection with $\Sigma(\omega)$ (in a sufficiently small neighborhood of the origin).

Theorems 1.2 and 1.4 along with the Hamiltonian normal forms (1.3) and (1.4) give a complete classification of generic singularities (of codimension ≤ 2) of pairs $\gamma = (f, \omega)$ on 2-manifolds. Germs (1.3) (of codimension 0) and germs (1.6) (of codimension 1) are both stable, $(1,0)$ and $(1,1)$-determined respectively[3]. The isolated singularities (1.4) and (1.7) (of codimension 2) are unstable and not finitely determined[4].

For the proofs of the theorems we use the homotopy method. We review some basic facts in Section 2 and we prove Theorems 1.2 and 1.4. In Section 3 we give the geometric description of the first term of the modulus λ. In the last Section 4 we discuss the weaker classification problem of germs of phase portraits (orbital equivalence) of CHS and we show how to get a list of simple normal forms, even for non-generic (degenerate) singularities.

2. The Homotopy Method-Proofs of Theorems

Fix $\mathbb{K} = \mathbb{R}$ (or $\mathbb{C}$) and consider germs of pairs $\gamma = (f, \omega)$ at the origin $0 \in \Sigma(\omega)$ of the plane $(\mathbb{K}^2, 0)$, where $f : (\mathbb{K}^2, 0) \to (\mathbb{K}, 0)$ vanishes at the origin. We will say that $\gamma_0 = (f_0, \omega_0)$ and $\gamma_1 = (f_1, \omega_1)$ are equivalent if there exists a germ of a diffeomorphism $\Phi : (\mathbb{K}^2, 0) \to (\mathbb{K}^2, 0)$ fixing the origin ($\Phi(0) = 0$) and such that $\Phi^*\gamma_1 = \gamma_0$. To find such equivalences we connect γ_0 and γ_1 by a curve

$$\gamma_t = (f_t, \omega_t), \quad t \in [0, 1],$$

where we may write:

$$f_t = f_0 + t\phi, \quad \omega_t = \omega_0 + td\alpha,$$

for the pair $(\phi, d\alpha) = (f_1 - f_0, \omega_1 - \omega_0)$. We seek for 1-parameter families of diffeomorphisms Φ_t depending smoothly (analytically) on $t \in [0, 1]$, fixing the origin ($\Phi_t(0) = 0$ for all $t \in [0, 1]$), inducing the identity on $(\mathbb{K}^2, 0)$ for $t = 0$ ($\Phi_0 = Id$) and such that $\Phi_t^*\gamma_t = \gamma_0$. Let v_t be the 1-parameter family of vector fields generated by any such diffeomorphism Φ_t:

$$\frac{d\Phi_t(x)}{dt} = v_t(\Phi_t(x)), \quad \Phi_0(x) = x.$$

This family depends smoothly (analytically, e.t.c.) on $t \in [0, 1]$ and it has a singular point at the origin $v_t(0) = 0$, for all $t \in [0, 1]$ (since the origin is a fixed point for Φ_t). Then the following proposition is well known:

Proposition 2.1. *If there exists a solution v_t, $v_t(0) = 0$ of the equations*

$$v_t \lrcorner df_t = -\phi, \tag{2.1}$$

$$v_t \lrcorner \omega_t = -\alpha + dh, \tag{2.2}$$

for some function germ $h = h(x_1, x_2)$ vanishing at the origin, then the pairs γ_0 and γ_1 are equivalent.

[3] (i, j)-determinacy means that the pair (f, ω) is equivalent to its (i, j)-jet $(j^i f, j^j \omega)$.

[4] some authors call these singularity classes "wild" (c.f. [19]).

Proof. The time 1-map of the flow Φ_t generated by v_t is the desired diffeomorphism. □

In problems of classification of pairs it is convenient to fix a singularity of one object and classify the other object relative to the symmetries of the first. Here[5] we fix the singularity f and we normalise ω relative to the symmetries of f. This simplifies calculations due to the following simple:

Lemma 2.2. *Let f be a generic function germ at the origin $0 \in \Sigma(\omega)$ of the plane (of type (i) or (ii)). Then the pair $\gamma = (f, \omega)$ is equivalent to the preliminary normal form*

$$\gamma = (\pm x_2, \quad \phi(x_1, x_2)dx_1 \wedge dx_2) \tag{2.3}$$

in the (i) case and to

$$\gamma = (x_1 \pm x_2^2, \quad \phi(x_1, x_2)dx_1 \wedge dx_2) \tag{2.4}$$

in the (ii) case, where ϕ is a nonsingular function germ at the origin, vanishing along the Martinet curve: $\phi|_{x_1=0} = 0$.

Proof. The diffeomorphism $\Phi = (\Phi_1, \Phi_2)$ bringing f to normal form preserves $\Sigma(\omega) = \{x_1 = 0\}$ and sends the Martinet normal form (1.5) to $\phi(x_1, x_2)dx_1 \wedge dx_2$, where $\phi = \Phi_1 det \Phi_*$ vanishes on $x_1 = 0$ as desired. □

2.1. **Case (i).**

Proof of Theorem 1.2. We consider the "+"-sign case. The "-"-sign case follows the same lines. We fix the singularity $(\Sigma(\omega) = \{x_1 = 0\}, f = x_2)$ and we consider 1-parameter families of 2-forms $\omega_t = \omega + tda$, where $\omega_0 = $ (1.5) is the Martinet germ and $\omega_1 = da$ is a 2-form which can be chosen to vanish on $x_1 = 0$ by the previous lemma. Write $d\alpha = \phi_1(x_1, x_2)dx_1 \wedge dx_2$. Since $\omega_1 = d(\alpha + d\xi)$ for any function ξ, we may always choose the primitive α in the form

$$\alpha = \alpha_1(x_1, x_2)dx_2,$$

where α_1 is a function germ vanishing to second order on $x_1 = 0$, i.e., such that

$$\alpha_1|_{x_1=0} = \frac{\partial \alpha_1}{\partial x_1}|_{x_1=0} = \phi_1|_{x_1=0} = 0. \tag{2.5}$$

It follows that the 1-parameter family of 2-forms $\omega_t = \phi_t(x_1, x_2)dx_1 \wedge dx_2$ may be always chosen so that

$$\phi_t(x_1, x_2) = x_1(1 + t\phi_{11}(x_1, x_2)),$$

where ϕ_{11} is the smooth germ defined by division $\phi_1 = x_1\phi_{11}$. Consider now a 1-parameter family $v_t = (v_{t,1}, v_{t,2})$ of germs of vector fields at the origin preserving the pair $(\{x_1 = 0\}, x_2)$. Since v_t preserves x_2 we have that $v_{t,2} = 0$. Since v_t is also tangent to $x_1 = 0$ we have that the first coordinate $v_{t,1}$ vanishes on $x_1 = 0$. After the substitution $v_t = (v_{t,1}, 0)$ in the homological equation (2.2) we arrive to the system:

$$\begin{array}{rcl} \phi_t v_{t,1} & = & \alpha_1 - \frac{\partial h}{\partial x_2} \\ 0 & = & \frac{\partial h}{\partial x_1}, \end{array}$$

where h is some arbitrary germ. We differentiate along the x_1-axes and we arrive to the simplest Cauchy problem for the unknown function $\psi = \phi_t v_{t,1}$:

$$\left\{ \begin{array}{l} \frac{\partial \psi}{\partial x_1} = \phi_1, \\ \psi|_{x_1=0} = 0. \end{array} \right. \tag{2.6}$$

[5]after the kind suggestion of the reviewer.

This admits a unique smooth (analytic) solution:

$$\psi(x_1, x_2) = \int_0^{x_1} \phi_1(s, x_2) ds.$$

By the fact that ϕ_t vanishes on $x_1 = 0$ for all $t \in [0, 1]$ and that ψ vanishes on $x_1 = 0$ to second order (since $\phi_1|_{x_1=0} = 0$), it follows that there exists a unique smooth (analytic) solution $v_{t,1}$ of the homological equation (2.2), vanishing on $x_1 = 0$ and represented by:

$$v_{t,1}(x_1, x_2) = \frac{\int_0^{x_1} \phi_1(s, x_2) ds}{\phi_t(x_1, x_2)}.$$

This finishes the proof. □

2.2. **Case (ii).** The proof of the previous theorem relies on the existence and uniqueness of solutions of the simplest Cauchy problem (2.6), i.e., on the fact that the origin is a non-characteristic point of the initial manifold $x_1 = 0$ for the characteristic vector field[6] $E_f = \pm\partial/\partial x_1$. In the case (ii) where $f^{-1}(0)$ has 1st-order tangency with $x_1 = 0$ at the origin (and so does the characteristic vector field $E_f = (\pm 2x_2, -1)$) existence and uniqueness of solutions of the analogous Cauchy problem is not guaranteed[7]. The following lemma gives the necessary and sufficient conditions for existence and uniqueness of smooth (resp. single-valued analytic) solutions.

Lemma 2.3. *Let $\mu = \mu(x_1, x_2)$ be an arbitrary smooth (resp. analytic) function germ vanishing at the origin. Then the Cauchy problem*

$$\left\{ \begin{array}{c} 2x_2 \frac{\partial \xi}{\partial x_1} - \frac{\partial \xi}{\partial x_2} = \mu, \\ \xi|_{x_1=0} = 0, \end{array} \right. \tag{2.7}$$

admits a smooth (resp. analytic) solution ξ if and only if μ vanishes on $x_1 = 0$. Moreover, the solution is unique.

Proof. Notice first that the origin is an isolated characteristic point and thus if a solution ξ of the Cauchy problem exists, then it will be unique. We consider the associated Cauchy problem for ξ with initial conditions along the transversal $x_2 = 0$:

$$2x_2 \frac{\partial \xi}{\partial x_1} - \frac{\partial \xi}{\partial x_2} = \mu,$$
$$\xi|_{x_2=0} = \xi_1(x_1),$$

where ξ_1 is a an arbitrary function germ vanishing at the origin. For this Cauchy problem the origin is a non-characteristic point and thus a unique smooth (resp. analytic) solution ξ exists for any function germ μ in the right-hand side. We seek necessary and sufficient conditions on μ such that ξ vanishes on $x_1 = 0$. If we parametrise the x_1-axis by τ, then the characteristic curves $t \mapsto (-t^2 + x_1(0), -t + x_2(0))$ of the characteristic vector field E_f emanating from this axis define a map $F(t, \tau) = (x_1(t, \tau), x_2(t, \tau))$ given by

$$F(t, \tau) = (-t^2 + \tau, -t).$$

This map is obviously a diffeomorphism germ at the origin with inverse

$$F^{-1}(x_1, x_2) = (-x_2, x_1 + x_2^2).$$

In the (t, τ)-plane the solution is given by

$$\tilde{\xi}(t, \tau) = \int_0^t \tilde{\mu}(s, \tau) ds + \tilde{\xi}_1(0),$$

[6] i.e., the characteristic direction is transversal to the initial manifold at that point c.f. [7]

[7] see [10] for a solution of the problem in the complex analytic case, in terms of multivalued functions.

and it projects on the (x_1, x_2)-plane to the single valued (smooth, analytic) solution

$$\xi(x_1, x_2) = \tilde{\xi}(F^{-1}(x_1, x_2)).$$

The preimage of the curve $x_1 = 0$ on the (t, τ)-plane consists of the two branches $t = \pm\sqrt{\tau}$ of the parabola (in the real case $\tau \geq 0$). It follows that the solution ξ vanishes on $x_1 = 0$ if and only if

$$\tilde{\xi}(\pm\sqrt{\tau}, \tau) = \int_0^{\pm\sqrt{\tau}} \tilde{\mu}(s, \tau) ds = 0.$$

We view this expression symbolically as a function of $\epsilon = \pm\sqrt{\tau}$:

$$\zeta(\epsilon) = \int_0^{\epsilon} \tilde{\mu}(s, \epsilon^2) ds.$$

Since $\zeta(0) = 0$ we have that $\zeta(\epsilon) = 0$ if and only if $\zeta'(\epsilon) = \tilde{\mu}(\epsilon, \epsilon^2) = 0$, which in turn is equivalent to $\mu(0, x_2) = 0$. Thus we have determined a unique smooth solution ξ for the specific choice of the transversal. Now we show that the solution does not depend on the choice of the transversal, which means that the solution ξ is also unique. If we choose another transversal for initial manifold of the associated Cauchy problem, say $x_2 = g(x_1)$, with new initial condition $\xi|_{x_2=g(x_1)} = \xi_2(x_1)$, then, we arrive as above to a unique solution ξ' which will vanish on $x_1 = 0$ if and only if μ does. Thus we have specified two solutions ξ and ξ' of the same Cauchy problem (2.7). Write $\Xi = \xi' - \xi$ for their difference. Then Ξ satisfies the homogeneous Cauchy problem

$$2x_2 \frac{\partial \Xi}{\partial x_1} - \frac{\partial \Xi}{\partial x_2} = 0,$$

$$\Xi|_{x_1=0} = 0,$$

which obviously, does not admit any nonzero solutions. Thus $\Xi = \xi' - \xi = 0$ and the lemma is proved. $\square$

Proof of Theorem 1.4. Fix the pair $(\Sigma(\omega) = \{x_1 = 0\}, f = x_1 + x_2^2)$ (the "-"-sign case follows again the same lines) and consider 1-parameter families of vector fields $v_t = (v_{t,1}, v_{t,2})$ preserving this pair and having a singular point at the origin. Such a general family can be represented as $v_t = (-2x_2 v_{t,2}, v_{t,2})$, where $v_{t,2}$ is a function germ, vanishing again on $x_1 = 0$. The homological equation (2.2) reduces in that case to the system

$$\begin{array}{rcl} -2x_2\phi_t v_{t,2} & = & \alpha_1 - \frac{\partial h}{\partial x_2} \\ \phi_t v_{t,2} & = - & \frac{\partial h}{\partial x_1}, \end{array}$$

for some arbitrary germ h. Write $\psi = \phi_t v_{t,2}$. It has second order vanishing on $x_1 = 0$ (since both ϕ_t and the unknown $v_{t,2}$ must vanish on $x_1 = 0$.) Then by the integrability condition for h we have that the unknown function ψ must be a solution of the following Cauchy problem:

$$\left\{ \begin{array}{c} 2x_2 \frac{\partial \psi}{\partial x_1} - \frac{\partial \psi}{\partial x_2} = \phi_1, \\ \psi|_{x_1=0} = \frac{\partial \psi}{\partial x_1}|_{x_1=0} = 0, \end{array} \right. \tag{2.8}$$

Since ϕ_1 vanishes on $x_1 = 0$ we have from the previous lemma that there exists a unique solution ψ vanishing on $x_1 = 0$. Since we want our solution to vanish on $x_1 = 0$ to second order, we differentiate the equation along the x_1-axis and we put $\xi = \partial\psi/\partial x_1$. Then ξ must be a solution of the Cauchy problem (2.7) of the above lemma, where we put $\mu = \partial\phi_1/\partial x_1$ in the right hand-side. It follows that the Cauchy problem (2.8) admits a unique smooth solution ψ if and only if ϕ_1 vanishes to second order on $x_1 = 0$. If this is the case, then from the initial substitution $\psi = \phi_t v_{t,2}$, we have have determined a unique smooth (resp. analytic) solution $v_{t,2}$ of the homological equation (2.2), given by the formula

$$v_{t,2}(x_1, x_2) = \frac{\psi(x_1, x_2)}{\phi_t(x_1, x_2)}.$$

Since ψ vanishes to second order on $x_1 = 0$ we have that $v_{t,2}$ vanishes also on $x_1 = 0$ as required. In particular, any 1-parameter family of 2-forms of the form $\omega_t = \omega_0 + tda$ can be reduced by a unique diffeomorphism preserving the pair $(x_1 = 0, x_1 + x_2^2)$ to the normal form:

$$\Phi_t^* \omega_t|_{t=1} = x_1 \tilde{\lambda}(x_2) dx_1 \wedge dx_2, \tag{2.9}$$

where $\tilde{\lambda}$ is an arbitrary function germ of 1-variable, $\tilde{\lambda}(0) \neq 0$. The invariance of this normal form is implied also by the previous lemma; indeed if $\omega_t = x_1 \tilde{\lambda}_t(x_2) dx_1 \wedge dx_2$ is any 1-parameter family of 2-forms, then any 1-parameter family of diffeomorphisms preserving f and realising equivalences between them, whould generate a 1-parameter family of vector fields v_t which should satisfy the homological equation (2.2) and in particular the Cauchy problem (2.8) with a righ hand-side of the form $\phi_1 = x_1 \tilde{\phi}(x_2)$, where $\tilde{\phi}$ is an arbitrary germ. Non-existence of smooth (resp. single-valued analytic) solutions $\psi = \phi_t v_{t,2}$ of the latter Cauchy problem is guaranteed by the previous lemma. Thus the germs ω_0 and ω_1 will be equivalent if and only if $\lambda_0 = \lambda_1$.

To obtain the initial normal form (1.7) of the theorem we consider the diffeomorphism $(x_1, x_2) \mapsto (x_1, \int_0^{x_2} \tilde{\lambda}(s) ds)$ which sends $\omega =$(2.9) to the Martinet 2-form $\omega =$(1.5) and the germ $f = x_1 + x_2^2$ to:

$$f = x_1 + \lambda(x_2),$$

where

$$\lambda(x_2) = (\int_0^{x_2} \tilde{\lambda}(s) ds)^2. \tag{2.10}$$

Obviously $\lambda(0) = \lambda'(0) = 0$ and $\lambda''(0) = 2(\tilde{\lambda}(0))^2 \neq 0$. The theorem is proved. $\square$

2.3. **Geometric Description of the First Modulus λ_2.** Fix the Martinet germ ω_0 and write $\Sigma_0(\omega) = \Sigma_0^+ \sqcup \Sigma_0^-$ for the germs of the symplectic half-spaces so that $\Sigma_0^+ = \{x_1 > 0\}$ and $\Sigma_0^- = \{x_1 < 0\}$. Let $c = c(t)$ be any "half-cycle", i.e., a curve lying in the neighborhood of the origin with two points of intersection with the Martinet curve, such as for example $c = f^{-1}(\epsilon) = \{x_1 + x_2^2 = \epsilon\}$ for some $\epsilon > 0$ fixed (say $\epsilon = 1$). Let $\bar{c}$ be the closed curve obtained by the union of c and the segment of the Martinet curve $x_1 = 0$ lying between the two intersection points. Write $D \subset \Sigma_0^+$ (or $D \subset \Sigma_0^-$) for the closed region whose boundary is $\bar{c}$ and $A_0(D)$ for the signed integral:

$$A_0(D) = \int_D \omega_0.$$

Since ω_0 vanishes on $x_1 = 0$, this integral will be equal to the action integral along the half-cycle c:

$$A_0(D) = \int_c \alpha_0,$$

where α_0 is a primitive of ω_0. We may choose $\alpha_0 = x_1^2 dx_2/2$ and for the specific choice of $c = f^{-1}(1)$ we compute the action to be $A_0(D) = 8/15$.

Consider now a 2-form germ ω_1 with the same Martinet curve $x_1 = 0$ and let Φ be a germ of diffeomorphism sending ω_0 to ω_1. Since Φ preserves $x_1 = 0$ then the following formula holds:

$$A_0(D) = \int_D \omega_0 = \int_D \Phi^* \omega_1 = \int_{\Phi(D)} \omega_1 = A_1(\Phi(D)) \quad \Leftrightarrow$$

$$\Leftrightarrow \quad \int_c \alpha_0 = \int_{\Phi(c)} \alpha_1.$$

It follows that if the diffeomorphism Φ may be chosen to preserve also the germ $f = x_1 + x_2^2$, then $\Phi(c) = c$ and thus the signed action integrals have to be equal:

$$A_0(D) = A_1(D).$$

For the 1-jet $\omega_1 = \tilde{\lambda}_0 \omega_0$ in particular, we have $A_1(D) = \tilde{\lambda}_0 8/15$ and hence, ω_1 cannot be equivalent to ω_0 for any $\tilde{\lambda}_0 \neq 1$ (the orbits of ω_0 and ω_1 under the action of symmetries of $f = x_1 + x_2^2$, belong to different cohomology classes for any $\tilde{\lambda}_0 \neq 1$).

Remark 2.4. The same result can be obtained if we fix ω_0 and vary the half-cycles $c(\lambda_2) = \{x_1 + \lambda_2 x_2^2 = 1\}$, $\lambda_2 > 0$; it suffices to substitute $\tilde{\lambda_0} = \sqrt{\lambda_2/2}$ in the calculation of $A_0(c(\lambda_2))$.

3. Orbital Equivalence of CHS

We fix real objects in C^∞-category. The results of the previous section show that the classification problem of CHS $\gamma = (f, \omega)$ at impasse points $\Sigma(\omega)$ becomes wild for all singularities of codimension ≥ 1. Despite this fact, if we are interested in the configuration of phase curves in a neighborhood of an impasse point (orbital equivalence), then the classification problem admits simple normal forms (without moduli) even for arbitrary deep singularities. Notice that for any germ f at the origin, there is a well defined Hamiltonian vector field $X^\pm$ in any of the symplectic half-spaces $\Sigma_0^\pm$ with Hamiltonian $f^\pm = f|_{\Sigma_0^\pm}$ and symplectic form the restriction $\omega^\pm = \omega|_{\Sigma_0^\pm}$ of the Martinet 2-form ω on each one of them. If f is a generic function germ (or any germ whose differential does not vanish on $\Sigma(\omega) = \{x_1 = 0\}$) then there does not exist a smooth extension of $X^\pm$ along $\Sigma(\omega)$ to a smooth vector field $X = X_f$ such that

$$X_f \lrcorner \omega = df. \tag{3.1}$$

The singularities of this type are called impasse singularities in the literature of constrained systems (c.f. [16], [20] and references therein). The notion of orbital (phase) equivalence for constrained (not necessarilly Hamiltonian) systems has been also introduced in these references. For the Hamiltonian case we need the following modifications.

Definition 3.1. *Let $\gamma = (f, \omega)$ and $\gamma' = (f', \omega')$ be two germs of CHS at the origin of the plane. Then γ will be called orbitally (or phase) equivalent with γ' if there exists a germ of a diffeomorphism Φ fixing the origin, sending the impasse curve $\Sigma(\omega)$ of γ to the impasse curve $\Sigma(\omega')$ of γ' and the oriented phase curves of γ in Σ_0 to the oriented phase curves of γ' in Σ_0'*

Remark 3.2. The definition implies that orbital equivalence of CHS is exactly orbital equivalence of the Hamiltonian vector fields X and X' defined on the symplectic components Σ_0 and Σ_0' respectively. Since the diffeomorphism Φ sends oriented phase curves of X to oriented phase curves of X' it sends the germ of the foliation by level curves $\{f = c\}$ to the foliation $\{f' = c'\}$. The diffeomorphism Φ is not required to preserve the symplectic structures $\omega^\pm$ and $\omega'^\pm$ of the components. In particular the following cases are possible:

(a)
$$\Phi(\Sigma_0^\pm(\omega)) = \Sigma_0^\pm(\omega')$$
and Φ sends oriented phase curves of $X^\pm$ to oriented phase curves of $X'^\pm$, or

(b)
$$\Phi(\Sigma_0^\pm(\omega)) = \Sigma_0^\mp(\omega')$$
and Φ sends oriented phase curves of $X^\pm$ to oriented phase curves of $X'^\mp$.

3.1. **The Extended Vector Field.** Despite the fact that the Hamiltonian vector fields $X^{\pm}$ do not extend to a smooth vector field X_f satisfying equation (3.1), there exist many smooth extensions E with the following property: oriented phase curves of E coincide with the oriented phase curves of X^+ in the symplectic half-space Σ_0^+ and with the oriented phase curves of $-X^-$ in Σ_0^-. Following [20]:

Definition 3.3. *Let $\omega = \sigma(x)v$ be any singular 2-form with smooth impasse curve $\sigma^{-1}(0)$, where v is an area form of the plane. The Extended Vector Field E_f of the CHS $\gamma = (f, \omega)$ is the smooth vector field defined by the equation:*

$$E_f \lrcorner \omega = \sigma df, \tag{3.2}$$

or equivalently, by the Hamiltonian system

$$E_f \lrcorner v = df, \tag{3.3}$$

The fact that E_f is indeed an extension of the CHS γ as defined above, follows from the relation (in Martinet coordinates)

$$X = \frac{1}{x_1} E_f, \quad x_1 \neq 0,$$

that is, multiplication by the positive (resp. negative) function x_1 at points of the half-space Σ_0^+ (resp. Σ_0^-).

Let now $\Gamma = (E_f, \Sigma)$ and $\Gamma' = (E_{f'}, \Sigma')$ be two pairs consisting of the germs at the origin of the extended vector fields and the impasse curves of γ and γ' respectively.

Definition 3.4. *The pairs Γ and Γ' will be called orbitally equivalent if there exists a germ of a diffeomorphism Φ fixing the origin, sending Σ to Σ' and sending oriented phase curves of E_f to oriented phase curves of $E_{f'}$, i.e., there exists a nonvanishing function germ Q at the origin such that:*

$$\Phi_* E_f = Q E_{f'}. \tag{3.4}$$

The following proposition allows us to reduce the problem of orbital equivalence of CHS γ to the orbital classification of the corresponding pairs Γ (as in [20]):

Proposition 3.5. *The germs of the CHS γ and γ' are phase equivalent iff the germs of the pairs Γ and Γ' are phase equivalent.*

Proof. Let γ and γ' be phase equivalent and suppose that the diffeomorphism Φ satisfies (a). Let $x(t)$ be an oriented phase curve of the extended vector field E_f in $\Sigma_0^+(\omega)$. Then it is also an oriented phase curve of X^+ and thus $\Phi(x(t)) \in \Sigma_0^+(\omega')$ is an oriented phase curve of X'^+ and thus of $E_{f'}$. Let now $x(t)$ be an oriented phase curve of E_f in $\Sigma_0^-(\omega)$. It is also a phase curve of $-X^-$ and thus $\Phi(x(t)) \subset \Sigma_0^-(\omega')$ is a phase curve of $-X'^-$. It follows that $\Phi(x(t))$ is an oriented phase curve of $E_{f'}$ which proves the required phase equivalence of the pairs Γ and Γ'. In the case where the diffeomorphism Φ satisfies (b), one obtains in a similar way a diffeomorphism of the oriented phase curves of E_f with those of $-E_{f'}$. The converse of the proposition is proved in a similar way. □

Write $\mathcal{G}(\Sigma)$ for the pseudogroup of symmetries of the impasse curve Σ, i.e., diffeomorphism germs preserving $\{x_1 = 0\}$ and fixing the origin. The orbital classification of pairs Γ is then equivalent to the problem of classification of germs of extended vector fields E_f relative to $\mathcal{G}(\Sigma)$-action. This problem in turn contains (for $Q = 1$ in equation (3.4)) the classification of the defining functions germs f relative to $\mathcal{G}(\Sigma)$-action. The answer to the latter problem is very well known and has been given by V. I. Arnold in [3]. For the 2-dimensional case the results may be summarised in the following list of simple singularities (see also [2], [5]):

$$\begin{array}{ll} \pm x_2 & \\ x_1 \pm x_2^{k+1}, & k \geq 1 \\ \pm x_1^k \pm x_2^2, & k \geq 2 \\ x_1 x_2 \pm x_2^k, & k \geq 2 \\ x_2^3 \pm x_1^2. & \end{array} \tag{3.5}$$

It follows:

Corollary 3.6. *Let $\gamma = (f, \omega)$ be a germ of a CHS at an impasse point where the germ of f is a simple boundary singularity (relative to $\mathcal{G}(\Sigma)$). Then γ is orbitally equivalent to the normal form*

$$\gamma = (f, \quad x_1 dx_1 \wedge dx_2),$$

where f is a germ from the list (3.5) above.

In the figures below the phase portraits for singularities for $k \leq 3$ are presented. To draw them, we draw the phase portrait of the extended vector field E_f and we change the orientation of the phase curves to one of the two half-spaces. The impasse curve is represented by the bolded vertical line. The dotted origin corresponds to the singular point of the f.

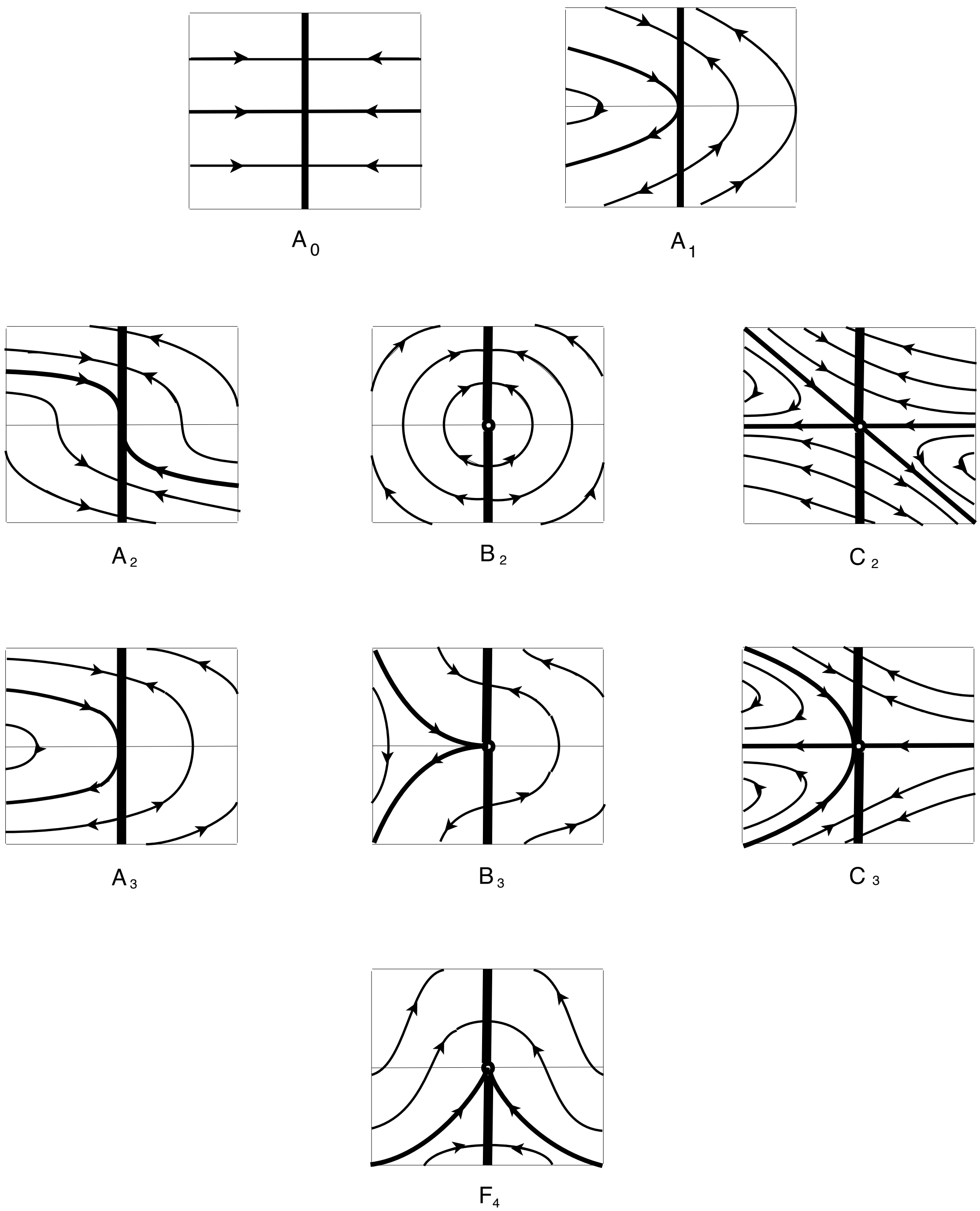

FIGURE 1. Phase portraits of simple singularities for $k \leq 3$ with the "-" sign.

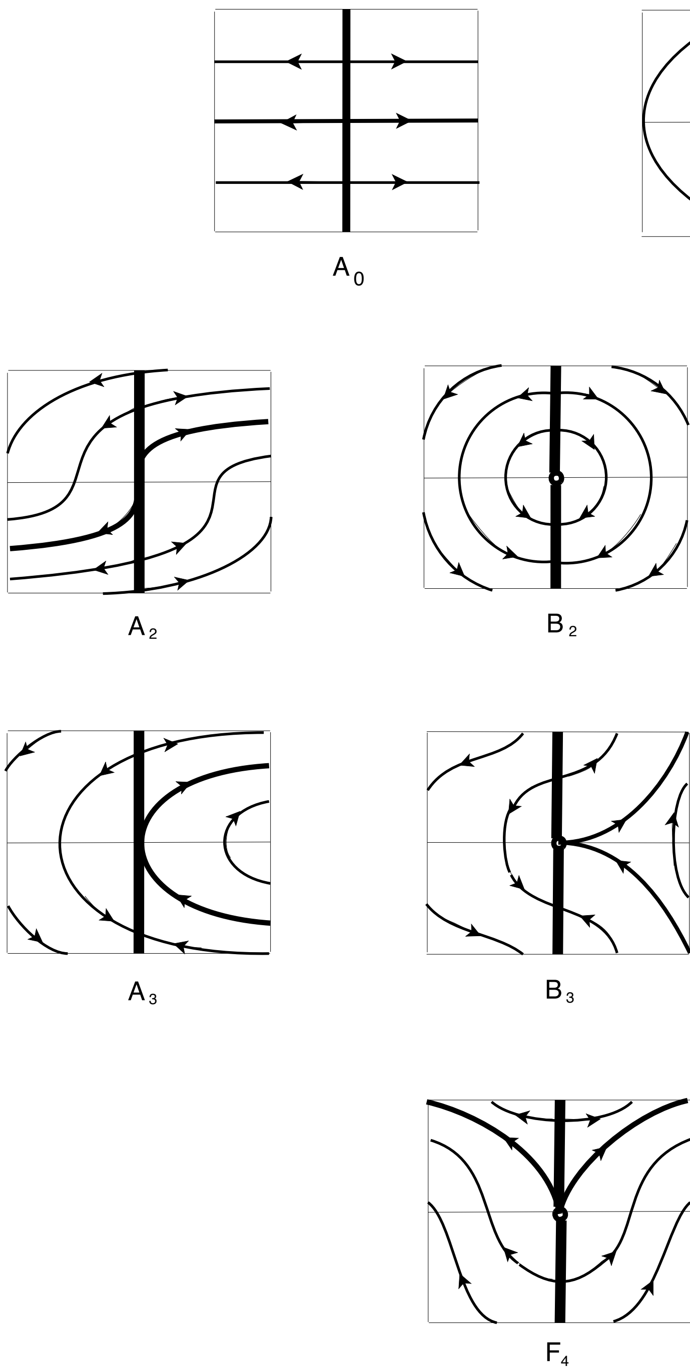

FIGURE 2. Phase portraits of simple singularities for $k \leq 3$ with the "+" sign.

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TOPOLOGICAL TRIVIALITY OF FAMILIES OF MAP GERMS FROM $\mathbb{R}^2$ TO $\mathbb{R}^2$

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Abstract. We show that a 1-parameter unfolding $F : (\mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R}^2 \times \mathbb{R}, 0)$ of a finitely determined map germ f is topologically trivial if it is excellent in the sense of Gaffney and the family of the discriminant curves $\Delta(f_t)$ is topologically trivial. We also give a formula to compute the number of cusps of 1-parameter unfoldings.

1. Introduction

In a previous paper [10], we consider the topological classification of finitely determined map germs $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$, by means of the analysis of the associated link. The link is obtained by taking a small enough representative $f : U \subset \mathbb{R}^2 \to \mathbb{R}^2$ and the restriction of f to $\tilde{S}^1_\epsilon = f^{-1}(S^1_\epsilon)$, where S^1_ϵ is a small enough sphere centered at the origin. It follows that the link is a stable map $\gamma : S^1 \to S^1$ which is well defined up to $\mathcal{A}$-equivalence and that f is topologically equivalent to the cone of its link. We also describe the topology of such links by using an adapted version of the Gauss word.

In this paper we consider a 1-parameter unfolding of f, that is, a map germ $F : (\mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R}^2 \times \mathbb{R}, 0)$ of the form $F(x, t) = (f_t(x), t)$ and such that $f_0 = f$. We are interested in the topological triviality of F, which means that it is topologically equivalent as an unfolding to the constant unfolding. Our main result is that F is topologically trivial if it is excellent in the sense of Gaffney [4] and moreover, the family of the discriminant curves $\Delta(F)$ is a topologically trivial deformation of $\Delta(f)$. This can be seen as a real version of the same result obtained by Gaffney for complex analytic map germs [4, Theorem 9.9]. In fact, since $\Delta(f)$ is a plane curve, the topological triviality of F in the complex case is equivalent to the constancy of the Milnor number $\mu(\Delta(f_t))$. In the real case, we show that this is also a sufficient condition, although it is not necessary in general. In order to have a necessary and sufficient condition we should need an invariant which controls the topological triviality of a family of real plane curves. In the last section we consider unfoldings which are not topologically trivial and give a result about the number of cusps that appear in f_t.

The techniques used to prove this result have been already used by the second named author in [11], where he gets a sufficient condition for the topological triviality in the case $\mathbb{R}^2$ to $\mathbb{R}^3$. The topological triviality of plane-to-plane has been also studied by Fukuda in [3]. We also refer to the work of Ikegami and Saeki [6] for related results.

For simplicity, all map germs considered are real analytic except otherwise stated, although most of the results here are also valid for C^∞-map germs, if they are finitely determined. We adopt the notation and basic definitions that are usual in singularity theory (e.g., $\mathcal{A}$-equivalence, stability, finite determinacy, etc.), as the reader can find in Wall's survey paper [12].

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2. The link of a finitely determined map germ

Two smooth map germs $f, g : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ are $\mathcal{A}$-equivalent if there exist diffeomorphism germs $\phi, \psi : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ such that $g = \psi \circ f \circ \phi^{-1}$. If ϕ, ψ are homeomorphisms instead of diffeomorphisms, then we say that f, g are topologically equivalent.

We say that $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ is k-determined if for any map germ g with the same k-jet, we have that g is $\mathcal{A}$-equivalent to f. We say that f is finitely determined if it is k-determined for some k.

Let $f : U \to V$ be a smooth proper map, where $U, V \subset \mathbb{R}^2$ are open subsets. We denote by $S(f) = \{p \in U : Jf_p = 0\}$ the singular set of f, where Jf is the Jacobian determinant. It is a consequence of the Whitney's work [13] that f is *stable* if and only if the following two conditions hold:

(1) 0 is a regular value of Jf, so that $S(f)$ is a smooth curve in U.
(2) The restriction $f|_{S(f)} : S(f) \to V$ is an immersion with only transverse double points, except at isolated points, where it has simple cusps.

We denote $\Delta(f) = f(S(f))$ and we define $X(f)$ as the closure of $f^{-1}(\Delta(f)) \setminus S(f)$. If f is stable, then $S(f)$ is a smooth plane curve and $\Delta(f), X(f)$ are plane curves whose only singularities are simple cusps or transverse double points. In figure 1 we present the stable map $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x, y) = (x, xy + y^4 - y^2/2)$, which has two cusps and one transverse double fold.

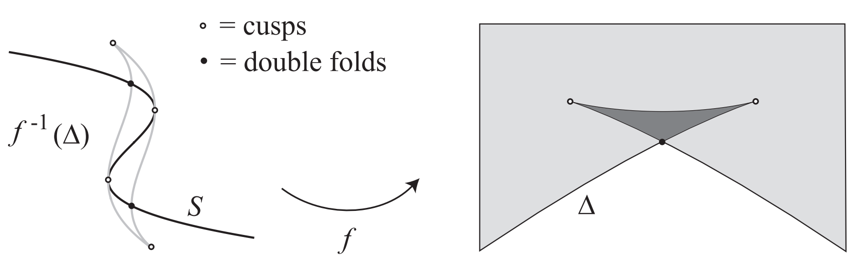

Figure 1.

Given a finitely determined map germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$, if it is real analytic, we can consider its complexification $\hat{f} : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$. It is well known that $\hat{f}$ is also finitely determined as a complex analytic map germ. Then, by the Mather-Gaffney geometric criterion [12], it has an isolated instability. In other words, we can find a small enough representative $\hat{f} : U \to V$, where U, V are open sets, such that

(1) $\hat{f}^{-1}(0) = \{0\}$,
(2) the restriction $\hat{f}|_{U \setminus \{0\}}$ is stable.

From the condition (2), both the cusps and the double folds are isolated points in $U \setminus \{0\}$. By the curve selection lemma [9], we deduce that they are also isolated in U. Thus, we can shrink the neighbourhood U if necessary and get a representative such that $\hat{f}|_{U \setminus \{0\}}$ is stable with only simple folds. Coming back to the real map f, we have the following immediate consequence.

Corollary 2.1. *Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a finitely determined map germ. Then there is a representative $f : U \to V$, where $U, V \subset \mathbb{R}^2$ are open sets, such that*

(1) $f^{-1}(0) = \{0\}$,
(2) $f : U \to V$ *is proper,*
(3) *the restriction $f|_{U \setminus \{0\}}$ is stable with only simple folds.*

Definition 2.2. We say that $f : U \to V$ is a *good representative* for a finitely determined map germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$, if the conditions (1), (2) and (3) of corollary 2.1 hold.

If f is finitely determined, then the three set germs $S(f)$, $\Delta(f)$ and $X(f)$ are plane curves with an isolated singularity at the origin and they will play an important role in the topological classification of f. In the complex case, given a plane curve $(X, 0)$ with reduced equation $h(u, v) = 0$ in $(\mathbb{C}^2, 0)$, its Milnor number is the colength of the ideal generated by the partial derivatives h_u, h_v, that is,

$$\mu(X, 0) = \dim_{\mathbb{C}} \frac{\mathcal{O}_2}{\langle h_u, h_v \rangle}.$$

Definition 2.3. If $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ is a finitely determined map germ, we denote by $\mu(\Delta(f))$ the Milnor number of the discriminant $\Delta(\hat{f})$ of the complexification $\hat{f} : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$.

Example 2.4. Let us consider $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ given by $f(x, y) = (x, x^2y + y^3/3)$. We have that $S(f)$ has defining equation $x^2 + y^2 = 0$ and hence, $\Delta(f)$ is given by $4u^6 + 9v^2 = 0$. Although $\Delta(f) = \{0\}$ as set germs, we have that $\mu(\Delta(f)) = 5$, which is the Milnor number of the complex curve given by this equation.

We finish this section with an important result due to Fukuda [1], which tell us that any finitely determined map germ, $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$, with $n \leq p$, has a conic structure over its link. In order to simplify the notation, we only state the result in our case $n = p = 2$.

Given $\epsilon > 0$, we denote:

$$S^1_\epsilon = \{x \in \mathbb{R}^2 : \|x\|^2 = \epsilon\}, \quad D^2_\epsilon = \{x \in \mathbb{R}^2 : \|x\|^2 \leq \epsilon\}.$$

and given a map germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ we consider a representative $f : U \to V$ and put:

$$\widetilde{S}^1_\epsilon = f^{-1}(S^1_\epsilon), \quad \widetilde{D}^2_\epsilon = f^{-1}(D^2_\epsilon).$$

Theorem 2.5. *Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a finitely determined map germ. Then, up to $\mathcal{A}$-equivalence, there is a representative $f : U \to V$ and $\epsilon_0 > 0$, such that, for any ϵ with $0 < \epsilon \leq \epsilon_0$ we have:*

1. *$\widetilde{S}^1_\epsilon$ is diffeomorphic to S^1.*
2. *The map $f|_{\widetilde{S}^1_\epsilon} : \widetilde{S}^1_\epsilon \to S^1_\epsilon$ is stable, in other words, it is a Morse function all of whose critical values are distinct.*
3. *$f|_{\widetilde{D}^2_\epsilon}$ is topologically equivalent to the cone of $f|_{\widetilde{S}^1_\epsilon}$.*

Definition 2.6. Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a finitely determined map germ. We say that the stable map $f|_{\widetilde{S}^1_\epsilon} : \widetilde{S}^1_\epsilon \to S^1_\epsilon$ is the *link* of f, where f is a representative such that (1), (2) and (3) of theorem 2.5 hold for any ϵ with $0 < \epsilon \leq \epsilon_0$. This link is well defined, up to $\mathcal{A}$-equivalence. We also say that ϵ_0 is a *Milnor-Fukuda radius* for f.

Since any finitely determined map germ is topologically equivalent to the cone of its link, we have the following immediate consequence.

Corollary 2.7. *Two finitely determined map germs $f, g : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ are topologically equivalent if and only if their associated links are topologically equivalent.*

Remark 2.8. If we consider a multigerm $f : (\mathbb{R}^2, S) \to (\mathbb{R}^2, 0)$, with $S = \{x_1, \dots, x_r\}$, the construction of the link can be done in an analogous way. By reviewing carefully Fukuda's arguments, we see that the only difference is the condition (1) of theorem 2.5: now $\widetilde{S}^1_\epsilon$ is not diffeomorphic to S^1 anymore, but it is diffeomorphic to a disjoint union of r copies $S^1 \sqcup \dots \sqcup S^1$. However, the other conditions (2) and (3) are still valid in this case.

3. Gauss words

In this section we recall briefly (for more information and examples see [10]) how we define an adapted version of the Gauss word in our particular case of study and some consequences of such definition.

Definition 3.1. Let $\gamma : S^1 \to S^1$ be a stable map, that is, such that all its singularities are of Morse type and its critical values are distinct. We fix orientations in each S^1 and we also choose base points $z_0 \in S^1$ in the source and $a_0 \in S^1$ in the target.

Suppose that γ has r critical values labeled by r letters $a_1, \dots, a_r \in S^1$ and let us denote their inverse images by $z_1, \dots, z_k \in S^1$. We assume they are ordered such that $a_0 \leq a_1 < \cdots < a_r$ and $z_0 \leq z_1 < \cdots < z_k$ and following the orientation of each S^1.

We define a map $\sigma : \{1, \dots, k\} \to \{a_1, \dots, a_r, \overline{a}_1, \dots, \overline{a}_r\}$ in the following way: given $i \in \{1, \dots, k\}$, then $\gamma(z_i) = a_j$ for some $j \in \{1, \dots, r\}$; we define $\sigma(i) = a_j$, if z_i is a regular point and $\sigma(i) = \overline{a}_j$, if z_i is a singular point (i.e., the bar $\overline{a}_j$ is used to distinguish whether the inverse image of the critical value is regular or singular). We call *Gauss word* to the sequence $\sigma(1) \dots \sigma(k)$.

For instance, the link of the cusp $f(x, y) = (x, xy+y^3)$ has two critical values with four inverse images and the associated Gauss word is $a\overline{b}\overline{a}b$ (see figure 2).

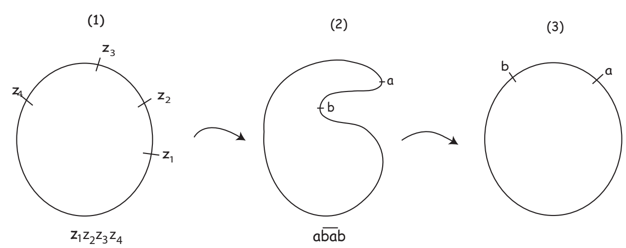

Figure 2.

It is obvious that the Gauss word is not uniquely determined, since it depends on the chosen orientations and base points in each S^1. Different choices will produce the following changes in the Gauss word:

(1) a cyclic permutation in the letters $a_1, \dots, a_r$;
(2) a cyclic permutation in the sequence $\sigma(1) \dots \sigma(k)$;
(3) a reversion in the set of the letters $a_1, \dots, a_r$;
(4) a reversion in the sequence $\sigma(1) \dots \sigma(k)$.

We say that two Gauss words are equivalent if they are related through these four operations. Under this equivalence, the Gauss word is now well defined.

In order to simplify the notation, given a stable map $\gamma : S^1 \to S^1$, we denote by $w(\gamma)$ the associated Gauss word and by $\simeq$ the equivalence relation between Gauss words. We also denote by $\deg(\gamma)$ the topological degree. Then, we can state the main result of this section (see [10]).

Theorem 3.2. *Let $\gamma, \delta : S^1 \to S^1$ be two stable maps. Then γ, δ are topologically equivalent if and only if*

$$\begin{cases} w(\gamma) \simeq w(\delta), & \text{if } \gamma, \delta \text{ are singular,} \\ \\ |\deg(\gamma)| = |\deg(\delta)|, & \text{if } \gamma, \delta \text{ are regular.} \end{cases}$$

Remark 3.3. By following step by step the proof of this theorem in [10] we can observe the following fact: if $\gamma, \delta : S^1 \to S^1$ are stable maps with $w(\gamma) \simeq w(\delta)$ and if we fix any homeomorphism in the target $\psi : S^1 \to S^1$ such that $\psi(\Delta(\gamma)) = \Delta(\delta)$, then there is a unique homeomorphism in the source $\phi : S^1 \to S^1$ such that $\psi \circ \gamma \circ \phi^{-1} = \delta$.

By combining this observation with corollary 2.7 we have an analogous result for map germs: let $f, g : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be two finitely determined map germs that are topologically equivalent. If we fix any homeomorphism in the target $\psi : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ such that $\psi(\Delta(f)) = \Delta(g)$, then there is a unique homeomorphism in the source $\phi : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ such that $\psi \circ f \circ \phi^{-1} = g$.

4. Cobordism of links

We recall that a cobordism between two smooth manifolds M_0, M_1 is a smooth manifold with boundary W such that $\partial W = M_0 \sqcup M_1$. Analogously, a cobordism between smooth maps $f_0 : M_0 \to N_0$ and $f_1 : M_1 \to N_1$ is another smooth map $F : W \to Q$ such that W, Q are cobordisms between M_0, M_1 and N_0, N_1 respectively, and for each $i = 0, 1$, $F^{-1}(N_i) = M_i$ and the restriction $F|_{M_i} : M_i \to N_i$ is equal to f_i. In the case that f_0, f_1 belong to some special class of maps (for instance, immersions, embeddings, stable maps, etc.), then we also require that the cobordism F belongs to the same class.

Definition 4.1. Given two stable maps $\gamma_0, \gamma_1 : S^1 \to S^1$, a cobordism between γ_0 and γ_1 is a stable map $\Gamma : S^1 \times I \to S^1 \times I$, where $I = [0, 1]$ and such that for $i = 0, 1$,

$$\Gamma^{-1}(S^1 \times \{i\}) = S^1 \times \{i\}, \quad \Gamma|_{S^1 \times \{i\}} = \gamma_i \times \{i\}.$$

The first condition implies that $\Gamma(S^1 \times \{0\}) \subset S^1 \times \{0\}$, $\Gamma(S^1 \times \{1\}) \subset S^1 \times \{1\}$ and $\Gamma(S^1 \times (0, 1)) \subset S^1 \times (0, 1)$, but in general, Γ is not level preserving (see figure 3).

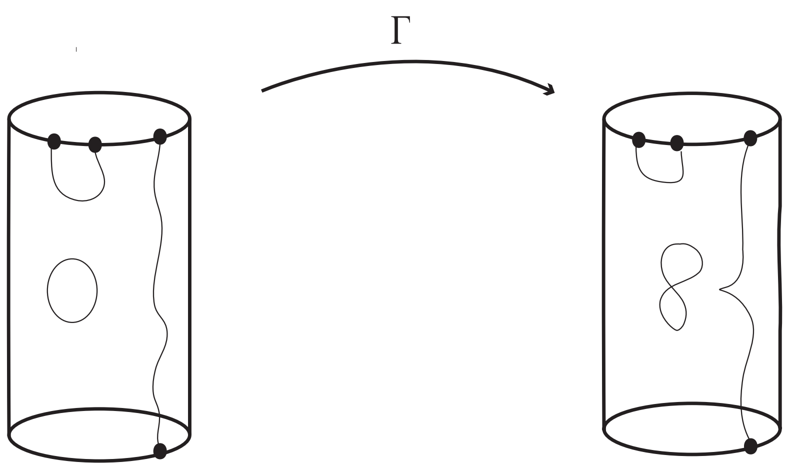

Figure 3.

Lemma 4.2. *Let Γ be a cobordism between γ_0, γ_1. If $\Delta(\Gamma)$ is diffeomorphic to $\Delta(\gamma_0) \times I$, then γ_0, γ_1 are topologically equivalent.*

Proof. Since $\Delta(\Gamma)$ is diffeomorphic to $\Delta(\gamma_0) \times I$, Γ cannot have cusps or double folds. Thus, Γ restricted to $\Gamma^{-1}(\Delta(\Gamma))$ is a local diffeomorphism and it follows that $\Gamma^{-1}(\Delta(\Gamma))$ is also diffeomorphic to $\gamma_0^{-1}(\Delta(\gamma_0)) \times I$.

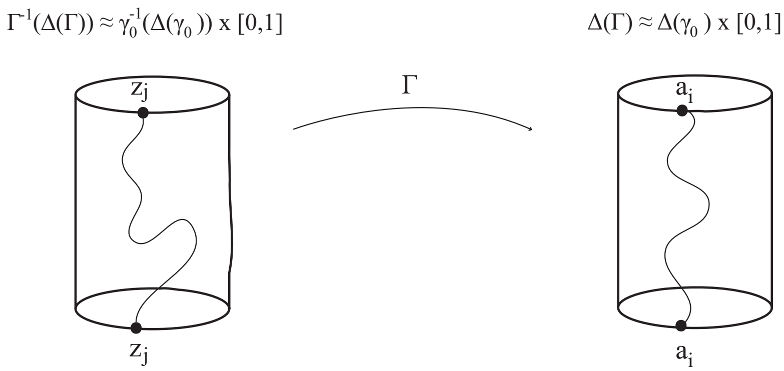

FIGURE 4.

In particular, for each critical value or each inverse image of γ_0 there is a unique arc joining the point in $S^1\times\{0\}$ with a point in $S^1\times\{1\}$ corresponding to a critical value or an inverse image of γ_1 respectively. We choose the orientations and the base points of γ_0,γ_1 in such a way that if two critical values are joined by an arc, then they share the same label a_i and if two inverse images are joined by an arc, then they share the same label z_j (see figure 4).

With these choices, it follows that $w(\gamma_0)=w(\gamma_1)$ and hence γ_0 and γ_1 are topologically equivalent by theorem 3.2. $\square$

Remark 4.3. If Γ is a cobordism between γ_0,γ_1 such that $\Delta(\Gamma)$ is diffeomorphic to $\Delta(\gamma_0)\times I$, then it can be shown that Γ is trivial, that is, Γ is $\mathcal{A}$-equivalent to the product cobordism $\gamma_0\times\mathrm{id}: S^1\times I\to S^1\times I$ by diffeomorphisms $\Phi,\Psi: S^1\times I\to S^1\times I$ such that $\Phi|_{S^1\times\{0\}},\Psi|_{S^1\times\{0\}}=\mathrm{id}$.

To show this, we first choose a diffeomorphism $\psi:\Delta(\gamma_0)\times I\to\Delta(\Gamma)$ such that $\psi(p,0)=(p,0)$, for all $p\in\Delta(\gamma_0)$. We denote by $\phi:\gamma_0^{-1}(\Delta(\gamma_0))\times I\to\Gamma^{-1}(\Delta(\Gamma))$ the induced diffeomorphism by Γ in such a way that $\phi(s,0)=(s,0)$, for all $s\in\gamma_0^{-1}(\Delta(\gamma_0))$ and the following diagram is commutative:

$$\begin{array}{ccc}\Gamma^{-1}(\Delta(\Gamma)) & \xrightarrow{\Gamma} & \Delta(\Gamma)\\ \uparrow{\scriptstyle\phi} & & \uparrow{\scriptstyle\psi}\\ \gamma_0^{-1}(\Delta(\gamma_0))\times I & \xrightarrow{\gamma_0\times id} & \Delta(\gamma_0)\times I\end{array}$$

We extend the diffeomorphisms ϕ,ψ to $S^1\times I$. This can be done by using standard arguments of extensions of vector fields. Details are left to the reader.

5. EXTENDING THE CONE STRUCTURE

Let $f:U\to V$ be a good representative of a finitely determined map germ $f:(\mathbb{R}^2,0)\to(\mathbb{R}^2,0)$. Since $\Delta(f)$ is a 1-dimensional analytic subset, we can also shrink the neighborhoods U,V so that this set is contractible. In this case $\Delta(f)\setminus\{0\}$ has a finite number of connected components, each one of them is an edge joining the origin with the boundary of V. We orient each one of this edges from 0 to ∂V. We denote by $X:\Delta(f)\setminus\{0\}\to\mathbb{R}^2$ the unit normal vector field of $\Delta(f)\setminus\{0\}$ with respect to this orientation (see figure 5).

Definition 5.1. Let $f:U\to V$ be a good representative of a finitely determined map germ $f:(\mathbb{R}^2,0)\to(\mathbb{R}^2,0)$ such that $\Delta(f)$ is contractible. We say that $\epsilon>0$ is a *convenient* radius for f if the following conditions hold:

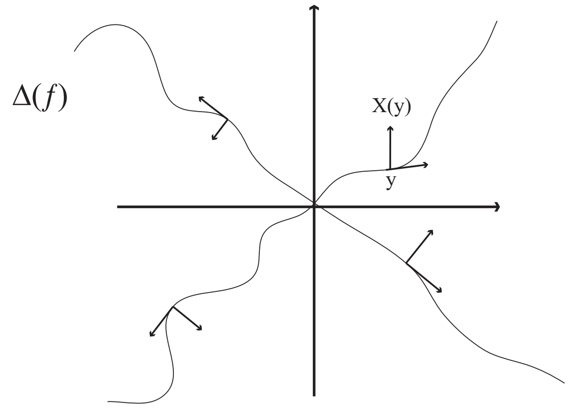

FIGURE 5.

(1) S^1_ϵ is transverse to $\Delta(f)$,
(2) $\widetilde{S}^1_\epsilon$ is diffeomorphic to S^1,
(3) S^1_ϵ intersects $\Delta(f)$ *properly*, that is, S^1_ϵ intersects each connected component of $\Delta(f)\setminus\{0\}$ at exactly one point.

It is easy to see that S^1_ϵ intersects $\Delta(f)$ properly if and only if S^1_ϵ cuts each point of $\Delta(f)$ following the orientation of the outward-pointing normal of S^1_ϵ. In other words, S^1_ϵ cuts $\Delta(f)$ properly if and only if

$$\det(X(y), y) > 0, \quad \forall y \in S^1_\epsilon \cap \Delta(f).$$

If ϵ_0 is a Milnor-Fukuda radius, then S^1_ϵ intersects $\Delta(f)$ properly for any $0 < \epsilon \leq \epsilon_0$, but in general, this may not be true.

Theorem 5.2. *Let $f : U \to V$ be a good representative of a finitely determined map germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ such that $\Delta(f) \subset V$ is contractible and let $\epsilon > 0$ be a convenient radius for f. Then,*

(1) *$f|_{\widetilde{S}^1_\epsilon} : \widetilde{S}^1_\epsilon \to S^1_\epsilon$ is topologically equivalent to the link of f.*
(2) *$f|_{\widetilde{D}^2_\epsilon} : \widetilde{D}^2_\epsilon \to D^2_\epsilon$ is topologically equivalent to the cone of $f|_{\widetilde{S}^1_\epsilon}$.*

Proof. Let $\epsilon_0 > 0$ be a Milnor-Fukuda radius for f. If $\epsilon \leq \epsilon_0$, then the result follows from theorem 2.5. We assume $\epsilon > \epsilon_0$ and take $0 < \delta < \epsilon_0$. We consider the two associated links $\gamma_0 = f|_{\widetilde{S}^1_\delta}$ and $\gamma_1 = f|_{\widetilde{S}^1_\epsilon}$ and we denote by

$$C^2_{\delta,\epsilon} = \{y \in \mathbb{R}^2 : \delta \leq \|y\|^2 \leq \epsilon\}, \quad \widetilde{C}^2_{\delta,\epsilon} = f^{-1}(C^2_{\delta,\epsilon}),$$

and $\Gamma = f|_{\widetilde{C}^2_{\delta,\epsilon}} : \widetilde{C}^2_{\delta,\epsilon} \to C^2_{\delta,\epsilon}$, which defines a cobordism between γ_0 and γ_1. We only need to show that γ_0 and γ_1 are topologically equivalent, since in this case we have that the cone structure of $f|_{\widetilde{D}^2_\delta}$ can be extended to $f|_{\widetilde{D}^2_\epsilon}$.

Let $\Delta_1, \ldots, \Delta_r$ be the connected components of $\Delta(f) \setminus \{0\}$. Since $\Delta(f) \subset V$ is closed, contractible and regular outside the origin, we have that each Δ_i is diffeomorphic to an open interval, whose end points are the origin and another point of ∂V. Now, both S^1_δ and S^1_ϵ intersect $\Delta(f)$ properly, so that $S^1_\delta \cap \Delta_i = \{x_i\}$ and $S^1_\epsilon \cap \Delta_i = \{x'_i\}$ for each $i = 1, \ldots, r$. It follows that

$$\Delta(\Gamma) = \overline{x_1 x'_1} \cup \cdots \cup \overline{x_r x'_r},$$

where $\overline{x_i x_i'}$ is the closed interval in Δ_i joining the points x_i and x_i' (see figure 6). Therefore, $\Delta(\Gamma)$ is diffeomorphic to $\{x_1, \ldots, x_r\} \times [\delta, \epsilon]$ and γ_0 and γ_1 are topologically equivalent by lemma 4.2. □

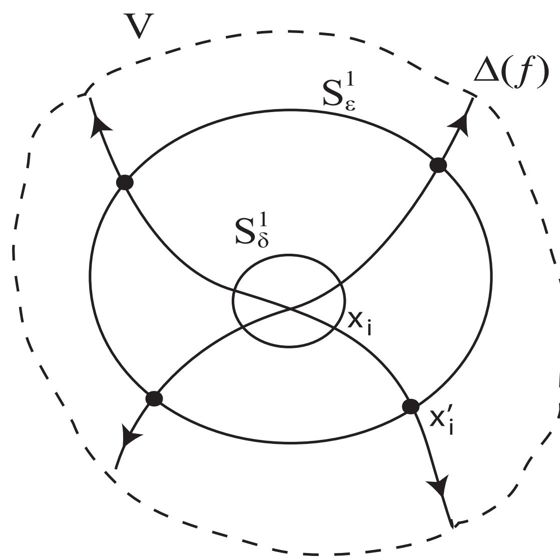

FIGURE 6.

6. TOPOLOGICAL TRIVIALITY OF FAMILIES

Given a map germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$, a 1- parameter unfolding is a map germ $F : (\mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R}^2 \times \mathbb{R}, 0)$ of the form $F(x,t) = (f_t(x), t)$ and such that $f_0 = f$. Here, we consider that the unfolding is origin preserving, that is, $f_t(0) = 0$ for any t. Hence, we have a 1-parameter family of map germs $f_t : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$.

Definition 6.1. Let F be a 1-parameter unfolding of a finitely determined map germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$.

(1) We say that F is *excellent* if there is a representative $F : U \to V \times I$, where U, V, I are open neighborhoods of the origin in $\mathbb{R}^2 \times \mathbb{R}, \mathbb{R}^2$ and $\mathbb{R}$ respectively, such that for any $t \in I$, $f_t : U_t \to V$ is a good representative in the sense of definition 2.2.
(2) We say that F has *constant topological type* if for any $t \neq t'$, the map germs f_t and f_t' are topologically equivalent.
(3) We say that F is *topologically trivial* if there are homeomorphism germs $\Psi, \Phi : (\mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R}^2 \times \mathbb{R}, 0)$ such that they are unfoldings of the identity and $F = \Psi \circ (f \times \mathrm{id}) \circ \Phi$.
(4) We say that F is *μ-constant* if the Milnor number $\mu(\Delta(f_t))$ is independent of t.

Example 6.2. Any topologically trivial unfolding F has constant topological type, but the converse is not true in general. Let us consider $h_t : \mathbb{R}^2 \to \mathbb{R}$ the equation of $S(f_t)$ for each t, given by

$$h_t(x,y) = (x + 3y)(5x - 2y)s_t(x,y),$$

with $s_t(x,y) = ((x-2)^2 + (y-3)^2)t - \epsilon^2 t$ (see figure 7). Then, we set:

$$f_t(x,y) = (x, \int h_t(x,y)dy).$$

It is not difficult to check that the Gauss word is constant $w(f_t) = a\overline{ba}bc\overline{dc}d$. As a consequence, the map germs f_t and $f_{t'}$ are topologically equivalent for any $t \neq t'$. However, it is clear that our family is not topologically trivial.

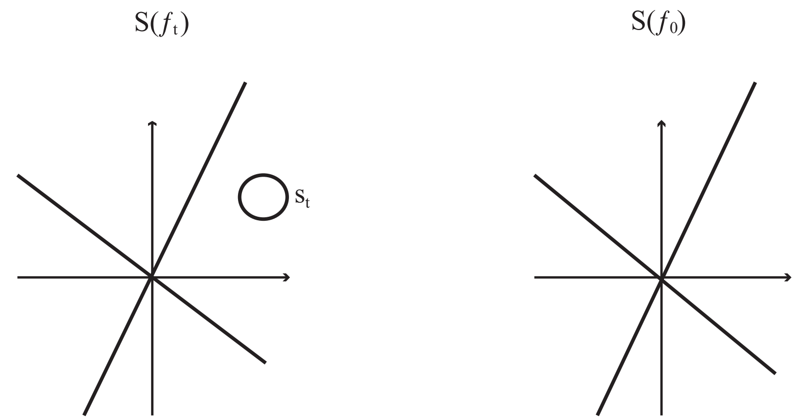

FIGURE 7.

Theorem 6.3. *Let F be an excellent unfolding of a finitely determined map germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$. If $\Delta(F)$ is topologically trivial, then F is topologically trivial.*

Proof. Let $F : U \to V \times I$ be a representative of the unfolding F, where U, V, I are open neighborhoods of the origin in $\mathbb{R}^2 \times \mathbb{R}, \mathbb{R}^2$ and $\mathbb{R}$ respectively, and such that $f_t : U_t \to V$ is a good representative of the map germ f_t, for any $t \in I$. We can shrink the neighborhoods if necessary and assume that $\Delta(f_0) \subset V$ is contractible.

On the other hand, since $\Delta(F)$ is topologically trivial, by shrinking again the neighbourhoods if necessary, there is a homeomorphism $\Psi : V \times I \to V \times I$ of the form $\Psi = (\psi_t, t)$ such that $\psi_0 = \mathrm{id}$ and $\psi_t(\Delta(f_t)) = \Delta(f)$, for any $t \in I$. In particular, $\Delta(f_t)$ is homeomorphic to $\Delta(f_0)$ and it is also contractible.

We take $X : (V \setminus \{0\}) \times I \to \mathbb{R}^2$ such that $X_t(y) = X(y,t)$ is the unit normal vector at each point $y \in \Delta(f_t) \setminus \{0\}$ as in definition 5.1. We also denote by $g_t : U_t \to \mathbb{R}$ the function $g_t(x) = \|f(x)\|^2$ and $G : U \to \mathbb{R}$, given by $G(x,t) = g_t(x)$.

Let $\epsilon_0 > 0$ be a Milnor-Fukuda radius for f and let $0 < \epsilon \leq \epsilon_0$. We have that ϵ is a regular value of g_0, $\widetilde{S}^1_\epsilon = g_0^{-1}(\epsilon)$ is diffeomorphic to S^1 and that S^1_ϵ intersects properly to $\Delta(f)$, that is,

$$\det(X_0(y), y) > 0, \ \forall y \in S^1_\epsilon \cap \Delta(f).$$

Once ϵ is fixed, we can choose $\delta > 0$ such that for any $t \in (-\delta, \delta)$, ϵ is also a regular value of g_t and

$$\det(X_t(y), y) > 0, \ \forall y \in S^1_\epsilon \cap \Delta(f_t).$$

By the fibration theorem, we have that $\widetilde{S}^1_{\epsilon,t} = g_t^{-1}(\epsilon)$ is diffeomorphic to $\widetilde{S}^1_\epsilon$, and hence to S^1. Moreover, the above condition gives that S^1_ϵ is transverse to $\Delta(f_t)$ and that S^1_ϵ intersects $\Delta(f_t)$ properly. In conclusion, we have shown that ϵ is a convenient radius for f_t, for any $t \in (-\delta, \delta)$.

By theorem 5.2, $\gamma_{\epsilon,t} = f_t|_{\widetilde{S}^1_{\epsilon,t}}$ is the link of f_t and $f_t|_{\widetilde{D}^2_{\epsilon,t}}$ is topologically equivalent to the cone of $\gamma_{\epsilon,t}$.

Since $\gamma_{\epsilon,t} : \widetilde{S}^1_{\epsilon,t} \to S^1_\epsilon$, with $t \in (-\delta, \delta)$, is stable, we have that this family of links is trivial. Hence, each $f_t|_{\widetilde{D}^2_{\epsilon,t}}$ is topologically equivalent to $f|_{\widetilde{D}^2_\epsilon}$. By remark 3.3, there is a unique homeomorphism in the source ϕ_t such that $\psi_t \circ f_t \circ \phi_t^{-1} = f$. Note that the unicity of ϕ_t implies that it depends continuously on t. We consider now $\Phi = (\phi_t, t) : F^{-1}(D^2_\epsilon \times (-\delta, \delta)) \to \widetilde{D}^2_\epsilon \times (-\delta, \delta)$. Then Φ is a homeomorphism, it is an unfolding of the identity and $\Psi \circ F \circ \Phi^{-1} = f \times \mathrm{id}$. □

Corollary 6.4. *Any μ-constant unfolding F of a finitely determined map germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ is topologically trivial.*

Proof. Any μ-constant unfolding F is excellent. This is known to be true in the complex case by the results of Gaffney [4]. Since F is analytic we are able to consider its complexification $\widehat{F}$ and we have that $\mu(\Delta(\widehat{f_t})) = \mu(\Delta(f_t))$ is constant. Then, $\widehat{F}$ is excellent, and as a consequence, F is also excellent. On the other hand, the μ-constant condition in the family of plane curves $\Delta(F)$ implies its topological triviality by the results of [7]. By theorem 6.3, F is topologically trivial. □

It is well known that in the complex case, any family of plane curves is topologically trivial if and only if the Milnor number is constant in the family. Hence, the converse of corollary 6.4 is also true in the complex case. In the real case, this is not true in general, as shown in the following example.

Example 6.5. Consider the family $f_t(x, y) = (x, x^4y + y^5 + t^2y^3)$. We have $f_t^{-1}(0) = \{0\}$, $Jf = x^4 + 5y^4 + 3t^2y^2 = 0$ and $S(f_t) = \{0\}$, for any $t \in \mathbb{R}$. Thus, the unfolding $F = (f_t, t)$ is excellent. Moreover, $\Delta(f_t) = \{0\}$ for any $t \in \mathbb{R}$, and hence F is topologically trivial by theorem 6.3.

On the other hand, the discriminant $\Delta(\hat{f}_t)$ of the complexification $\hat{f}_t$ is given by equation:

$$108t^{10}v^2 + 16t^8u^{12} - 900t^6u^4v^2 - 128t^4u^{16} + 2000t^2u^8v^2 + 256u^{20} + 3125v^4 = 0.$$

We have that $\mu(\Delta(f_t)) = 11$ for $t \neq 0$, but $\mu(\Delta(f_0)) = 57$.

7. The number of cusps of an unfolding

In this last section, we follow the arguments of the proof of theorem 6.3 to give a formula for the parity of the number of cusps of an unfolding $F : (\mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R}^2 \times \mathbb{R}, 0)$ of a finitely determined map germ f. Here, we do not assume that F is excellent, but we only assume the following condition $(*)$: there is a representative $F : U \to V \times I$, where U, V, I are open neighbourhoods of the origin in $\mathbb{R}^2 \times \mathbb{R}, \mathbb{R}^2, \mathbb{R}$ respectively, such that $f_t : U_t \to V$ is proper and its restriction to $f_t^{-1}(V \setminus \{0\})$ is stable.

Given an unfolding satisfying this condition $(*)$, we introduce the following notation:

1. $c(f_t^+)$ (respectively $c(f_t^-)$) is the number of cusps of f_t on $f_t^{-1}(V \setminus \{0\})$ for $t > 0$ (respectively $t < 0$).
2. $r(f_t^+)$ (respectively $r(f_t^-)$) is the number of points of $f_t^{-1}(0)$ for $t > 0$ (respectively $t < 0$).
3. $\#S(f_t^+)$ (respectively $S(f_t^-)$) is the number of branches of $S(f_t)$ at $f_t^{-1}(0)$ for $t > 0$ (respectively $t < 0$).
4. $\#S(f_0)$ is the number of branches of $S(f_0)$ at 0.

If the neighbourhoods U, V, I are small enough, then these numbers are well defined. We also denote the multiplicity of a map germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ by

$$m(f) = \dim_{\mathbb{R}} \frac{\mathcal{E}_2}{\langle f_1, f_2 \rangle}.$$

We have the following congruences, which can be also deduced from the arguments of [5, Proof of Theorem 1.12]

Proposition 7.1. *Let $F : (\mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R}^2 \times \mathbb{R}, 0)$ be a 1-parameter unfolding of a finitely determined map germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ satisfying condition $(*)$. Then,*

$$c(f_t^{\pm}) \equiv 1 - r(f_t^{\pm}) + \#S(f_0) + \#S(f_t^{\pm}) \mod 2.$$

Moreover, if $m(f_t)$ is constant for each $t \in \mathbb{R}$ we have that

$$c(f_t^{\pm}) \equiv \#S(f_0) + \#S(f_t^{\pm}) \mod 2.$$

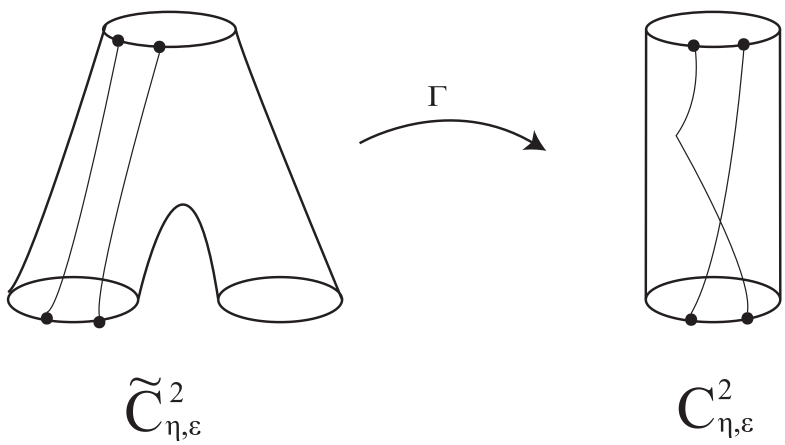

FIGURE 8.

Proof. Let $\epsilon_0 > 0$ be a Milnor-Fukuda radius for f and take $0 < \epsilon \le \epsilon_0$. There is $\delta > 0$ such that if $t \in (-\delta, \delta)$, then ϵ is a convenient radius for the multigerm $f_t : (\mathbb{R}^2, Z_t) \to (\mathbb{R}^2, 0)$, where $f_t^{-1}(0) = Z_t$.

We fix $0 < t < \delta$, the case $-\delta < t < 0$ being analogous. Take $0 < \eta < \epsilon$, where $\eta \le \eta_0$, a Milnor Fukuda radius for f_t. We denote:

$$\gamma_0 = f_t|_{\widetilde{S}^1_\epsilon} : \widetilde{S}^1_\epsilon \to S^1_\epsilon,$$
$$\gamma_1 = f_t|_{\widetilde{S}^1_\eta} : \widetilde{S}^1_\eta \to S^1_\eta,$$
$$\Gamma = f_t|_{\widetilde{C}^2_{\eta,\epsilon}} : \widetilde{C}^2_{\eta,\epsilon} \to C^2_{\eta,\epsilon}.$$

We have that γ_0 is topologically equivalent to the link of the map germ f, γ_1 is the link of the multigerm f_t and Γ is a cobordism between γ_0, γ_1 (see figure 8). Since Γ is a stable map between compact oriented connected surfaces with boundary, we can apply a result due to Fukuda - Ishikawa [2]:

$$c(\Gamma) \equiv \chi(\widetilde{C}^2_{\eta,\epsilon}) + \deg(\Gamma|_{\partial \widetilde{C}^2_{\eta,\epsilon}})\chi(C^2_{\eta,\epsilon}) + \frac{1}{2}\#(S(\Gamma|_{\partial \widetilde{C}^2_{\eta,\epsilon}})) \mod 2,$$

where $c(\Gamma)$ is the number of cusps of Γ. We have $c(\Gamma) = c(f_t^+)$, $\chi(\widetilde{C}^2_{\eta,\epsilon}) = 1 - r(f_t^+)$, $\chi(C^2_{\eta,\epsilon}) = 0$ and

$$\frac{1}{2}\#(S(\Gamma|_{\partial\widetilde{C}^2_{\eta,\epsilon}})) = \#S(f_0) + \#S(f_t^+).$$

Thus, we arrive to

$$c(f_t^+) \equiv 1 - r(f_t^+) + \#S(f_0) + \#S(f_t^{\pm}) \mod 2.$$

If $m(f_t)$ is constant, we have that $\{f_t^{-1}(0)\} = \{0\}$, $r(f_t^+) = 1$ and hence,

$$c(f_t^+) \equiv \#S(f_0) + \#S(f_t^+) \mod 2.$$

□

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A CONJECTURE ON THE ŁOJASIEWICZ EXPONENT

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Abstract. In this paper, we present a conjecture connecting the Łojasiewicz exponent of an isolated nondegenerate singularity with some geometrical characteristics of the Newton diagram associated with this singularity. We prove the conjecture for a class of surface singularities.

1. Introduction

Let $f = f(z_1, \dots, z_n) \in \mathbb{C}\{z_1, \dots, z_n\}$ be a convergent power series defining an isolated singularity at the origin $0 \in \mathbb{C}^n$. The *Łojasiewicz exponent* $\pounds_0(f)$ *of* f is by definition the smallest $\theta > 0$ such that there exist a neighbourhood U of $0 \in \mathbb{C}^n$ and a constant $c > 0$ such that

$$|\nabla f(z)| \geq c\,|z|^\theta \quad \text{for all } z \in U,$$

where $\nabla f = (f'_{z_1}, \dots, f'_{z_n})$. It is an important discrete invariant of isolated singularities: it is a rational number [L-JT], it is a biholomorphic invariant, $\pounds_0(f) + 1$ is equal to the maximal polar invariant of f [T], it is attained on analytic paths centered at 0 [L-JT], $[\pounds_0(f)] + 1$ is C^0-degree of sufficiency of f [ChL, T]. In spite of its importance $\pounds_0(f)$ is not well known (in contrast to the Milnor number) even among experts in singularity theory. An interesting mathematical problem is to give formulas for $\pounds_0(f)$ (in terms of another invariants of f) or an algorithm to compute it. Almost all is known on $\pounds_0(f)$ for the plane curve singularities ($n = 2$) (see [CK1, CK2, K, GKP]). For $n \geq 3$ there are only estimations of $\pounds_0(f)$ [P1, P2]. A standard technique in singularity theory is the method of Newton diagrams, developed by the Moscow School (Kouchnirenko, Varchenko, Khovansky and others). In the paper we propose a conjecture that the Łojasiewicz exponent of a nondegenerate singularity could be read off from its Newton diagram. It is true in the case $n = 2$ (Lenarcik [L]). For general n only estimations of $\pounds_0(f)$ in terms of Newton diagrams (see [A, B, BE, F, O1, O2]) are known. On the other hand a counter-example to it would disprove the Teissier conjecture that $\pounds_0(f)$ is a topological invariant of f.

For $n = 2$ Lenarcik computes $\pounds_0(f)$ from the Newton diagram of f by removing from it some exceptional segments. The main difficulty with the extension of his method to n dimensions is to define exceptional faces appropriately. The third-named author proposed a definition in [O2] which we claim to be the right one. Using this definition we prove our conjecture for surface ($n = 3$) nondegenerate singularities that have only one unexceptional face. We also give a formula for the Łojasiewicz exponent of semi-weighted homogeneous surface singularities.

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2. Preliminaries

Let us recall that if $(w_1,\dots,w_n)$ is a sequence of n rational positive numbers (called *weights*) then a polynomial $f\in\mathbb{C}[z_1,\dots,z_n]$ is called *weighted homogeneous of type* $(w_1,\dots,w_n)$ if it is a linear combination of monomials $z_1^{\alpha_1}\dots z_n^{\alpha_n}$ with $\alpha_1/w_1+\dots+\alpha_n/w_n=1$.

A nonzero holomorphic function $f:(\mathbb{C}^n,0)\longrightarrow(\mathbb{C},0)$ defined in some open neighbourhood of $0\in\mathbb{C}^n$ is a *singularity* if $f(0)=0$ and $\nabla f(0)=0$. A singularity f is an *isolated singularity* if it has an isolated critical point at the origin i.e. $\nabla f(z)\neq 0$ for $z\neq 0$ near 0. Let $\sum_{\nu\in\mathbb{N}^n}a_\nu z^\nu$ be the Taylor expansion of f at 0. We define $\Gamma_+(f):=\operatorname{conv}\{\nu+\mathbb{R}^n_+: a_\nu\neq 0\}\subset\mathbb{R}^n$ and call it the *Newton diagram* of f. Let $u\in\mathbb{R}^n_+\setminus\{0\}$. Put $l(u,\Gamma_+(f)):=\inf\{<u,v>: v\in\Gamma_+(f)\}$ and $\Delta(u,\Gamma_+(f)):=\{v\in\Gamma_+(f):<u,v>=l(u,\Gamma_+(f))\}$. We say that $S\subset\mathbb{R}^n$ is a *face* of $\Gamma_+(f)$, if $S=\Delta(u,\Gamma_+(f))$ for some $u\in\mathbb{R}^n_+\setminus\{0\}$. The vector u is called a *primitive vector* of S. It is easy to see that S is a closed and convex set and $S\subset\operatorname{Fr}(\Gamma_+(f))$, where $\operatorname{Fr}(A)$ denotes the boundary of A. One can prove that a face $S\subset\Gamma_+(f)$ is compact if and only if all coordinates of its primitive vector u are positive. We call the family of all compact faces of $\Gamma_+(f)$ the *Newton boundary* of f and denote it by $\Gamma(f)$. We denote by $\Gamma^k(f)$ the set of all compact k-dimensional faces of $\Gamma(f)$, $k=0,\dots,n-1$. For every compact face $S\in\Gamma(f)$ we define weighted homogeneous polynomial $f_S:=\sum_{\nu\in S}a_\nu z^\nu$. A singularity f is *nondegenerate on the face* $S\in\Gamma(f)$ if the system of equations $(f_S)'_{z_1}=\dots=(f_S)'_{z_n}=0$ has no solution in $(\mathbb{C}^*)^n$, where $\mathbb{C}^*=\mathbb{C}\setminus\{0\}$. A singularity f is *nondegenerate in the Kouchnirenko sense* (shortly *nondegenerate*) if it is nondegenerate on each face of $\Gamma(f)$. A singularity f is *semi-weighted homogeneous* if there exists a face S of $\Gamma(f)$ such that f_S is an isolated singularity.

Let $i\in\{1,\dots,n\}$, $n\geq 2$. We say that $S\in\Gamma^{n-1}(f)\subset\mathbb{R}^n$ is an *exceptional face for f with respect to the axis OX_i* if one of its vertices is at distance 1 to the axis OX_i and the remaining vertices define $(n-2)$-dimensional face which lies in one of the coordinate hyperplanes including the axis OX_i.

Example 2.1. *Let $f(z_1,z_2,z_3)=z_1z_3^4+z_2^2z_3^6+z_2^4z_3+z_1^6$. It is easy to check that $\Gamma^2(f)=\{S_1,S_2\}$, where $S_1=\operatorname{conv}\{(0,4,1),(0,2,6),(1,0,4)\}$ is an exceptional face for f with respect to OX_3 and $S_2=\operatorname{conv}\{(0,4,1),(1,0,4),(6,0,0)\}$ is not an exceptional face. Let us notice that f_{S_2} is an isolated singularity, so f is a semi-weighted homogeneous singularity.*

A face $S\in\Gamma^{n-1}(f)$ is an *exceptional face for f* if there exists $i\in\{1,\dots,n\}$ such that S is an exceptional face for f with respect to the axis OX_i. Denote by E_f the set of all exceptional faces for f. We call a face $S\in\Gamma^{n-1}(f)$ *unexceptional for f* if $S\notin E_f$.

A singularity f is *convenient* (resp. *nearly convenient*) if its Newton diagram has nonempty intersection with every coordinate axis (resp. its distance to every coordinate axis doesn't exceed 1).

For every $(n-1)$-dimensional compact face $S\in\Gamma(f)$ we shall denote by $x_1(S),\dots,x_n(S)$ the coordinates of intersection of the hyperplane determined by the face S with the coordinate axes $OX_1,\dots,OX_n$. We put $m(S):=\max\{x_1(S),x_2(S),\dots,x_n(S)\}$. It is easy to see that

$$x_i(S)=\frac{l(u,\Gamma_+(f))}{u_i},\ i=1,\dots,n, \tag{1}$$

where u is a primitive vector of S.

3. Main results

An interesting problem concerning the Łojasiewicz exponent is to compute $\mathcal{L}_0(f)$ for nondegenerate isolated singularities f in terms of the Newton diagram $\Gamma_+(f)$. In this paper we propose the following conjecture.

Conjecture. *Let $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}, 0)$ be an isolated nondegenerate singularity. If $\Gamma^{n-1}(f) \setminus E_f \neq \emptyset$, then*

$$\text{(2)} \qquad Ł_0(f) = \max_{S \in \Gamma^{n-1}(f) \setminus E_f} m(S) - 1.$$

There are some results that confirm our conjecture:

- Lenarcik [L] improved a bound for $Ł_0(f)$ obtained by Lichtin [Lt] and proved formula (2) for $n = 2$.
- The third-named author proved in [O2] the inequality

$$\text{(3)} \qquad Ł_0(f) \leq \max_{S \in \Gamma^{n-1}(f) \setminus E_f} m(S) - 1$$

  for $n = 3$.
- For weighted homogeneous singularities the Conjecture is true [KOP].
- Fukui [F] proved a weaker bound for $Ł_0(f)$ for any $n \geq 2$. His result was improved in [O1, O2]. Abderrahmane [A] gave another result of this type.

The main result of this note is the proof of the Conjecture in the case of nondegenerate surface singularities with one unexceptional face.

Theorem 3.1. *Let $f : \left(\mathbb{C}^3, 0\right) \longrightarrow (\mathbb{C}, 0)$ be an isolated nondegenerate singularity such that $\#(\Gamma^2(f) \setminus E_f) = 1$. Then*

$$Ł_0(f) = m(S) - 1,$$

where S is the unique unexceptional face for f.

Example 3.2. *The isolated singularity in Example 2.1 satisfies the assumptions of the above theorem. We easily check that $Ł_0(f) = m(S_2) - 1 = 5$.*

The proof of Theorem 3.1 is based on the following formula for the Łojasiewicz exponent of a semi-weighted homogeneous singularity.

Theorem 3.3. *Let $f : \left(\mathbb{C}^3, 0\right) \longrightarrow (\mathbb{C}, 0)$ be a semi-weighted homogeneous singularity. Then*

$$Ł_0(f) = Ł_0(f_S),$$

where S is a face of $\Gamma(f)$ such that f_S is an isolated singularity.

To calculate $Ł_0(f_S)$ one can use the main result of [KOP].

Remark 3.4. *Theorem 3.3 is also true for $n = 2$ (one can prove it using Cor. 4 in [KOP]). It is an open question if $Ł_0(f_S) = Ł_0(f)$ for $n > 3$.*

4. Proofs of the main results

First we prove an auxiliary inequality (see Cor. 4.8 in [BE] for another proof) for any dimension.

Proposition 4.1. *Let $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}, 0)$ be a semi-weighted homogeneous singularity and let $S \in \Gamma(f)$ be a face such that f_S is an isolated singularity. Then*

$$\text{(4)} \qquad Ł_0(f_S) \leq Ł_0(f).$$

Proof. Let $v = (v_1, \ldots, v_n)$ be a primitive vector of S such that $v_i \in \mathbb{N}_+$. We expand f in the form

$$f = f^{[d]} + f^{[d+1]} + \ldots, \quad f^{[d]} \neq 0,$$

where $f^{[i]}$ are weighted homogeneous polynomials of type $(v_1, \dots, v_n)$, $\deg_v f^{[i]} = i$, $i = d, d+1, \dots$. Of course $f^{[d]} = f_S$. Take the following family of singularities

$$f_t := f(z_1 t^{v_1}, \dots, z_n t^{v_n})/t^d, \ t \in \mathbb{C} \setminus \{0\}$$

and $f_0 := f^{[d]}$. Notice that

$$f_t = f^{[d]} + t f^{[d+1]} + t^2 f^{[d+2]} + \dots, \quad t \in \mathbb{C}.$$

The family (f_t) has the following properties:

- (f_t) is a holomorphic family with respect to t,
- f_t are semi-weighted homogeneous singularities,
- $\mu_0(f_t) = \mu_0(f^{[d]})$ for $t \in \mathbb{C}$ ([AGV], Thm. in Section 12.2), where $\mu_0(f)$ is the Milnor number of a singularity f,
- $f_0 = f_S$.

By the semicontinuity of the Łojasiewicz exponent in holomorphic μ-constant families of isolated singularities [T, P3] we obtain

$$\mathcal{L}_0(f_0) \leq \mathcal{L}_0(f_t)$$

for t sufficiently close to 0. On the other hand $\mathcal{L}_0(f_t) = \mathcal{L}_0(f)$ for $t \neq 0$, because

$$f_t = \alpha \cdot (f \circ L),$$

where $\alpha \in \mathbb{C} \setminus \{0\}$ and L is a linear change of coordinates in $\mathbb{C}^n$. Hence for any sufficiently small $t \neq 0$ we have

$$\mathcal{L}_0(f_S) = \mathcal{L}_0(f_0) \leq \mathcal{L}_0(f_t) = \mathcal{L}_0(f).$$

□

Now, we are ready to prove Theorem 3.3.

Proof of Theorem 3.3. Let $L \subset \mathbb{R}^3 : \alpha_1/w_1 + \alpha_2/w_2 + \alpha_3/w_3 = 1$ be a supporting plane to $\Gamma_+(f)$ along the face S (if S is 2-dimensional then L and $w = (w_1, w_2, w_3)$ are uniquely determined). Since $\operatorname{supp}(f_S) \subset L$, f_S is a weighted homogeneous polynomial of type (w_1, w_2, w_3). Write $f = f_S + f'$, where all monomials appearing in the Taylor expansion of f' lie above the plane L. Now, by ([KOP], Thm. 3) we get

$$\mathcal{L}_0(f_S) = \min\big(\max_{i=1}^{3} w_i - 1, \prod_{i=1}^{3}(w_i - 1)\big). \tag{5}$$

Using ([P2], Prop. 2.2) we obtain $\mathcal{L}_0(f) \leq \max_{i=1}^3 w_i - 1$. By ([P1], Thm. 1), ([AGV], Thm. in Section 12.2) and the Milnor-Orlik formula [MO] we get $\mathcal{L}_0(f) \leq \mu_0(f) = \mu_0(f_S) = \prod_{i=1}^3 (w_i - 1)$. Consequently

$$\mathcal{L}_0(f) \leq \min\big(\max_{i=1}^{3} w_i - 1, \prod_{i=1}^{3}(w_i - 1)\big) \tag{6}$$

On the other hand by Proposition 4.1 we get

$$\mathcal{L}_0(f_S) \leq \mathcal{L}_0(f) \tag{7}$$

By (5), (6), (7) we obtain the assertion of the theorem. □

To prove Theorem 3.1 we give some lemmas and properties.

Property 4.2. *Every isolated singularity $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}, 0)$ is nearly convenient.*

Proof. It suffices to show that for every $i = 1, 2, \ldots, n$ there exists $j \in \{1, 2, \ldots, n\}$ and $k \geq 1$ such that monomial $z_j z_i^k$ appears in the Taylor expansion of f with a non-zero coefficient. Indeed, suppose to the contrary that for some $i \in \{1, 2, \ldots, n\}$ no monomial $z_j z_i^k$ appears in the expansion of f for every $j \in \{1, 2, \ldots, n\}$ and $k \geq 1$. Then one can easily check that $f'_{z_j}(0, \ldots, 0, z_i, 0, \ldots, 0) \equiv 0$, $j = 1, \ldots, n$, which is impossible since ∇f has an isolated zero at 0. □

For a series $\phi \in \mathbb{C}\{t\}$, $\phi \neq 0$, by $\operatorname{info} \phi$ (resp. $\operatorname{inco} \phi$) we mean the initial form of ϕ (resp. the non-zero coefficient of $\operatorname{info} \phi$).

Lemma 4.3. *Let $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}, 0)$, $n \geq 3$, be a singularity and $\nabla f \circ \phi = 0$ for some $\phi = (\phi_1, \ldots, \phi_n) \in \mathbb{C}\{t\}^n$, $\phi(0) = 0$, $\phi_1, \ldots, \phi_k \neq 0$, $\phi_{k+1} = \ldots = \phi_n = 0, k \geq 2$, and $f(z_1, \ldots, z_k, 0, \ldots, 0) \not\equiv 0$. Then there exists $S \in \Gamma(f)$ on which f is degenerate.*

Proof. We can represent f in the form

$$f(z_1, \ldots, z_n) = g(z_1, \ldots, z_k) + z_{k+1} h_{k+1}(z_1, \ldots, z_n) + \ldots + z_n h_n(z_1, \ldots, z_n)$$

By the assumption we get $g \neq 0$, $g(0) = 0$, $\nabla g(\phi_1, \ldots, \phi_k) = 0$. By [O2, Cor. 2.4] there exists $S \in \Gamma(g)$, such that $(\operatorname{ord} \phi_i)_{i=1}^k$ is a primitive vector of S and

$$\nabla g_S(\operatorname{info} \phi_1, \ldots, \operatorname{info} \phi_k) = 0. \tag{8}$$

By [O2, Property 2.10] we get $S \in \Gamma(f)$. Of course $f_S = g_S$. Therefore we have

$$(f_S)'_{z_i}(\operatorname{info} \phi_1(t), \ldots, \operatorname{info} \phi_k(t), t, \ldots, t) \equiv 0,\ i = k+1, \ldots, n$$

and by (8) we get

$$(f_S)'_{z_i}(\operatorname{info} \phi_1(t), \ldots, \operatorname{info} \phi_k(t), t, \ldots, t) \equiv 0,\ i = 1, \ldots, k.$$

Hence

$$(f_S)'_{z_i}(\operatorname{inco} \phi_1, \ldots, \operatorname{inco} \phi_k, 1, \ldots, 1) = 0, \quad i = 1, \ldots, n,$$

thus f is degenerate on S. □

Proposition 4.4. *Let $f : \left(\mathbb{C}^3, 0\right) \longrightarrow (\mathbb{C}, 0)$ be a nondegenerate nearly convenient singularity such that $\Gamma_+(f) \cap OX_iX_j \neq \emptyset$ for $i \neq j$. Then f is an isolated singularity.*

Proof. Suppose to the contrary that f is not an isolated singularity. Then there exists a non-zero parametization ϕ such that $\nabla f \circ \phi = 0$. It is not possible for ϕ to have two coordinates equal to zero, because if for example $\phi = (0, 0, \phi_3)$, $\phi_3 \neq 0$, then by Property 4.2 we get that monomial $z_i z_3^k$ appears in the Taylor expansion of f with a non-zero coefficient for some $i \in \{1, 2, 3\}$ and $k \geq 1$. Then one can check that $\operatorname{info} f'_{z_i}(0, 0, \phi_3(t)) = (\operatorname{info} \phi_3(t))^k \neq 0$. Hence $f'_{z_i}(0, 0, \phi_3) \neq 0$, which contradicts the hypothesis $\nabla f \circ \phi = 0$. Therefore we may assume that $\phi = (\phi_1, \phi_2, \phi_3)$ and $\phi_i \neq 0, \phi_j \neq 0$ for some $i \neq j$. Without loss of generality we may assume that $\phi_1 \neq 0, \phi_2 \neq 0$. Then by Lemma 4.3 we have that f is degenerate on some face $S \in \Gamma(f)$, which contradicts the assumption on f. □

Lemma 4.5. *Let $f : \left(\mathbb{C}^3, 0\right) \longrightarrow (\mathbb{C}, 0)$ be a singularity. Suppose there exists an unexceptional face S for f such that f_S is an isolated singularity. Put $w_i := x_i(S)$ for $i = 1, 2, 3$. Then*

$$m(S) - 1 = \min\left(\max_{i=1}^{3} w_i - 1, \prod_{i=1}^{3}(w_i - 1)\right). \tag{9}$$

Proof. Since f_S is an isolated singularity, therefore $\operatorname{ord} f_S \geq 2$ and hence $x_i(S) > 1$, $i = 1, 2, 3$. We consider two cases.

If $w_i \geq 2$, $i = 1, 2, 3$, then

$$\prod_{i=1}^{3}(w_i - 1) \geq \max_{i=1}^{3} w_i - 1 = \max_{i=1}^{3} x_i(S) - 1 = m(S) - 1,$$

which gives (9).

If $w_i < 2$ for some $i \in \{1, 2, 3\}$, say $i = 1$, then $1 < x_1(S) < 2$ and by Property 4.2 there exists a monomial z_1z_2 or z_1z_3, say z_1z_2, appearing in the Taylor expansion of f with a non-zero coefficient. Then $(1, 1, 0)$ lies on the plane $\alpha_1/w_1 + \alpha_2/w_2 + \alpha_3/w_3 = 1$. Hence $(w_1 - 1)(w_2 - 1) = 1$ and thus $\prod_{i=1}^{3}(w_i - 1) = w_3 - 1$. Since S is an unexceptional face, there exists a point $(1, 0, k) \in \operatorname{supp}(f_S)$, $k \geq 1$. Therefore $x_3(S) \geq x_2(S)$ and obviously $x_2(S) > 2$. Hence $m(S) = x_3(S) = w_3$. □

Proof of Theorem 3.1. Using the Lemma about the choice of an unexceptional face (Lemma 3.1 in [O2]) one can check that f_S is nearly convenient and $\Gamma_+(f_S) \cap OX_iX_j \neq \emptyset$ for $i \neq j$. Then by Proposition 4.4 we get that f_S has an isolated singularity. Therefore by Theorem 3.3 and by Theorem 3 in [KOP] we get

$$Ł_0(f) = Ł_0(f_S) = \min\big(\max_{i=1}^{3} w_i - 1, \prod_{i=1}^{3}(w_i - 1)\big),$$

where $w_i = x_i(S)$, $i = 1, 2, 3$. Since S is an unexceptional face, by Lemma 4.5 we have

$$m(S) - 1 = \min\big(\max_{i=1}^{3} w_i - 1, \prod_{i=1}^{3}(w_i - 1)\big).$$

Summing up we get

$$Ł_0(f) = m(S) - 1.$$

□

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PROPAGATIONS FROM A SPACE CURVE IN THREE SPACE WITH INDICATRIX A SURFACE

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Abstract. Generic singularities of rays emanating from a space curve in $\mathbb{R}^3$ in all directions with the rate determined by an indicatrix (independent of the point in $\mathbb{R}^3$) defined by a surface are classified. Similarly rays emanating from surface defined by an indicatrix given by a curve are also considered. Some applications to control theory are indicated.

1. Introduction

In this paper we solve two problems on the classification of local geometrical singularities that are related to control theory. We use some techniques from the singularity theory of caustics and wave fronts to study singularities of exponential mappings in a class of control problems which correspond to special integrable Hamiltonian systems with straight lines as extremals.

The first problem concerns a control system on a three-dimensional affine space with points $q \in \mathbb{R}^3$. We identify the tangent space $T_q\mathbb{R}^3$ with $\mathbb{R}^3$ itself. At each point q we choose an indicatrix I_q of admissible velocities $\dot{q} = \frac{\partial q}{\partial \mu}$ of motion which we assume is independent of the point q itself. Assume that this set is parametrised locally by a regular mapping $(x, y) \mapsto r_2(x, y)$ whose image is a surface M. We shall now write M in place of I_q.

An *admissible motion* is a smooth curve $\gamma(\mu) \in \mathbb{R}^3$, parametrised by a segment of the (affine) time axis μ, such that the velocity at each point $\dot{\gamma}$ belongs to the set of admissible velocities M.

Let $q_b(\mu)$ be the trajectory of an admissible motion of an initial point $b \in N$, issuing at $\mu = 0$ from an initial set N, where N is a space curve which is a submanifold in $\mathbb{R}^3$.

For a fixed value $\mu = \mu_0$ let $\mathcal{C}_b$ be the Banach manifold of all admissible trajectories defined on the segment $[0, \mu_0]$ from an initial point $b \in N$.

Consider the endpoint mapping $\mathcal{E}_b : \mathcal{C}_b \to \mathbb{R}^3$ which associates the endpoint $q_b(\mu_0)$ to a trajectory $q_b(\mu)$.

A corollary of the Pontryagin maximum principle, see [1, 9], states that critical values of $\mathcal{E}$ for all μ_0 trace extremal trajectories. In our case these are projections to $\mathbb{R}^3$ of solutions of the associated Hamiltonian canonical equations on the cotangent bundle

$$\dot{q} = \frac{\partial H_*(p, q)}{\partial p}, \quad \dot{p} = -\frac{\partial H_*(p, q)}{\partial q}.$$

Here the Hamiltonian function $H_*(p, q)$ on the cotangent bundle $T^*\mathbb{R}^3$ is the restriction (multivalued in general) to the subset $\{(p, q) \,|\, \exists (x, y) : \frac{\partial H(p,q,x,y)}{\partial x} = \frac{\partial H(p,q,x,y)}{\partial y} = 0\}$ of the function $H = \langle p, r_2(x, y)\rangle$, provided that the initial conditions (p_0, q_0) satisfy the relation $\langle p_0, v\rangle = 0$ for each vector v tangent to N at b. The angle brackets $\langle -, -\rangle$ denote the standard pairing of vectors $\mathbb{R}^3$ and covectors p of the dual space $(\mathbb{R}^3)^\wedge$.

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In our case extremals are straight lines (parametrised by $\mu \in \mathbb{R}$) $b + \mu v, b \in N, v \in M$, such that there is a covector p_0, which annihilates both tangent spaces $T_b N$ and $T_v M$. Points on these lines with fixed μ form a wave front E_μ of the Legendre variety L_μ, which is the image of the Legendre submanifold L_0 of the initial conditions under the Hamiltonian flow.

The envelope $B(N, M)$ of these extremals is the union of singular points of sets of critical values of $\mathcal{E}_b$ for all b and μ_0.

Rephrasing the above in physical terms, consider an initial space curve in three space which emits rays from every point, and such that the speed of a ray at any point is completely determined by its direction. The *boundary of the set of attainability* of the rays after a given time μ will be the wave front E_μ. The caustics or focal sets correspond to the singularities of this set of attainability.

In this first problem as described above we consider an initial space curve with a velocity indicatrix defined by a surface. The classification where the indicatrix was also a space curve (independent of the point in $\mathbb{R}^3$) was given in [8]. The case of an initial surface and a velocity indicatrix described by a surface was studied in [3]. For completeness we also consider in the present paper a second problem interchanging the surface and the curve, i.e. the indicatrix is defined by a space curve and the initial manifold is a surface.

At present this second problem seems to have fewer applications than the first despite the fact that away from the initial surface the classification coincides with that of the first problem.

In the first problem the dimension of the indicatrix M is one less than the dimension of the ambient space $\mathbb{R}^3$, so the wave fronts E_μ form a family of equidistants in Finsler geometry. However as the dimension of the indicatrix in the second problem is not one less than the ambient space the wave fronts do not form a family of equidistants in Finsler geometry.

In this paper we classify the possible generic singularities of the envelopes $B(M, N)$ and of the family of wave fronts in both problems. We also classify the generic singularities near the initial surface itself in the second problem.

The method of classification of the singularities is similar to that of a related problem in [7]. In that paper the wave fronts were taken to be the closure of an affine ratio of pairs of points, one from a curve and the other from a surface that share parallel tangent planes. Here we consider in the first problem the surface, and then in the second problem the curve, to be at infinity. The computations were largely omitted from [7] and since in the present context they are slightly easier to write down we take this opportunity to include more details.

1.1. **Main definitions and results.** Let M be a smooth surface and let N be a smooth space curve both embedded in affine three space.

Consider a pair a, b of points $a \in M$ and $b \in N$ such that the plane tangent to the surface M at a is parallel to some plane tangent to N at b. The pair a, b is called a *parallel pair* and the straight line through a, b is called a *chord*. The envelope of the family of all chords is called the Minkowski set of M and N. In this paper we shall classify its generic singularities.

The chord $l(a, b)$ joining the parallel pair is defined by

$$l(a, b) = \{q \in \mathbb{R}^3 \mid q = \mu a + (1 - \mu) b, \mu \in \mathbb{R}\}. \tag{1}$$

The following definitions are valid for the propagating from the space curve case but similar definitions, by replacing μ with $1 - \mu = \lambda$ hold in the propagating from the surface case. The points which correspond to parallel tangent plane is the furthest point of the wave front from the curve. The wave front E_μ is the boundary of where the rays have reached at time μ. In the previous papers [3, 4, 7] barycentric coordinates were introduced on to the chords. Here however, we omit λ and just use the coordinate μ where $\mu = 0$ corresponds to the point on the curve N and $\mu = \infty$ corresponds to the point on the surface M.

A germ of the *affine μ-equidistant* E_μ of the pair (M, N) is the set of points $q \in \mathbb{R}^3$ such that $q = \mu a + b$ for given $\mu \in \mathbb{R}$ and for all parallel pairs (a, b) close to (a_0, b_0). Note that E_0 is the germ of N at b_0.

The space $\mathbb{R}^4_e = \mathbb{R} \times \mathbb{R}^3$ with coordinate $\mu \in \mathbb{R}$ (affine time), on the first factor is called the *extended affine space*. Denote by
$\rho : (\mu, q) \mapsto \mu$ the projection of $\mathbb{R}^4_e$ to the first factor and by $\pi : (\mu, q) \mapsto q$ the projection to the second factor.

The *affine extended wave front* $W(M, N)$ of the pair M, N is the union of all affine equidistants each embedded into its own slice of the extended affine space:

$$W(M, N) = \{(\mu, E_\mu)\} \subset \mathbb{R}^4_e.$$

The *bifurcation set* $B(M, N)$ of a family of affine equidistants (or of the family of chords) of the pair (M, N) is the image under π of the locus of the critical points of the restriction $\pi_r = \pi|_{W(M,N)}$. A point is *critical* if π_r at this point fails to be a regular projection of a smooth submanifold. In general $B(M, N)$ consists of two components: the *caustic* Σ is the projection of the singular locus of the extended wave front $W(M, N)$ and the *criminant* $\triangle$ is the (closure of the) image under π_r of the set of regular points of $W(M, N)$ which are the critical points of the projection π restricted to the regular part of $W(M, N)$. The caustic consists of the singular points of the momentary equidistants E_μ while the criminant is the envelope of the family of regular parts of the momentary equidistants. Besides being swept out by the momentary equidistants, the affine extended wave front is swept out by the liftings to $\mathbb{R}^{3+1}_e$ of chords. Each of them has a regular projection to the configuration space $\mathbb{R}^3$. Hence the bifurcation set $B(M, N)$ is essentially the envelope of the family of chords.

In the generic setting the distinguished chords split into three distinct sub-cases: In the first (*transversal*) case the base points $a_0 \in M$ and $b_0 \in N$ are distinct and the chord through them is transversal to both M and N. In the second (*tangential*) case the base points $a_0 \in M$ and $b_0 \in N$ are distinct but the tangent line to N lies in the tangent plane to M. A subcase of the tangential case called the *supertangential* case occurs when the line tangent to the curve N at b contains the point a, i.e. the chord and the tangent line are the same.

Definition 1.1. Two germs of families F_1 and F_2 in parameters μ, q are called *space-time contact equivalent* if there exists a nonzero function $\phi(z, \mu, q)$ and diffeomorphism $\widehat{\theta} : (z, \mu, q) \mapsto (Z(z, \mu, q), P(\mu, q), Q(q))$ such that $F_1 = \phi F_2 \circ \widehat{\theta}$.

Notice that the diffeomorphism $\widehat{\theta} : (\mu, q) \mapsto (P(\mu, q), Q(q))$ of the total parameter space $\mathbb{R}^{3+1}$ maps the extended wave front of the first family to the extended wave front of the second family and the diffeomorphism $\widehat{\theta} : q \mapsto Q(q)$ of the q-parameter space $\mathbb{R}^3$ maps the bifurcation set of the first family to the bifurcation set of the second family.

Definition 1.2. Two germs of families F_1 and F_2 are called *time-space contact equivalent* if there exists a nonzero function $\phi(z, \mu, q)$ and diffeomorphism $\widetilde{\theta} : (z, \mu, q) \mapsto (Z(z, \mu, q), P(\mu), Q(\mu, q))$ such that $F_1 = \phi F_2 \circ \widetilde{\theta}$.

The diffeomorphism $\widetilde{\theta} : (\mu, q) \mapsto (P(\mu), Q(\mu, q))$ preserves the fibration of the μ, q space into fibres parallel to the q space. If two families are time-space contact equivalent then their respective families of momentary wave fronts are diffeomorphic.

The main results are as follows: The first theorem which concerns the wave fronts follows immediately from the results of [7].

Theorem 1.3. *The families of wave fronts and their bifurcations in the propagating from the curve and in the propagating from the surface cases are diffeomorphic to those in the affine ratio case* [7]*. In fact the generating functions are time-space contact equivalent.*

The following theorems all concern the projection π and are related to the caustics. The theorems are the complete classification of generic singularities in the various settings. The list is the same as in [7]. Unlike theorem 1.3 they do not follow immediately from the previous papers and require separate calculation.

Theorem 1.4. *In the propagating from the curve transversal case outside M and N the germ at any point of the envelope of the family of chords for generic M and N is diffeomorphic to one of the standard caustics of A_k type with $k = 2, 3$ or 4 (regular surface, cuspidal edge or swallowtail).*

Theorem 1.5. *In the propagating from the curve tangential case the germ at any point outside M and N of the envelope of the family of chords for generic N and M is diffeomorphic to one of the standard caustics of the boundary singularities of the types B_2, B_3, B_4, C_3, C_4 or F_4. If moreover the tangent line to the curve coincides with the chord (supertangential case) then generically only B_2 and C_3 occur.*

Theorem 1.6. *In the propagating from the surface transversal case outside M and N the germ at any point of the envelope of the family of chords for generic M and N is diffeomorphic to one of the standard caustics of A_k type with $k = 2, 3$ or 4 (regular surface, cuspidal edge or swallowtail).*

Theorem 1.7. *In the propagating from the surface transversal case the envelope of the family of chords transversally intersects the surface M when $\lambda = 0$ generically at either its smooth points or at points of a cuspidal ridge.*

Theorem 1.8. *In the propagating from the surface tangential setting the germ at any point outside M and N of the envelope of the family of chords for generic N and M is diffeomorphic to one of the standard caustics of the boundary singularities of the types B_2, B_3, B_4, C_3, C_4 or F_4. If moreover the tangent line to the curve coincides with the chord (supertangential case) then generically only B_2 and C_3 occur.*

1.2. **Generating families.** Now consider the following generating family $\mathcal{F}_1$ in the propagating from the curve case. The family has variables $n \in (\mathbb{R}^3)^\wedge \setminus \{0\}$, t and (x, y), and parameters $(\mu, q) \in \mathbb{R} \times \mathbb{R}^3$;

$$\mathcal{F}_1(n, t, x, y, \mu, q) = \langle r_1(t) + \mu r_2(x, y) - q, n\rangle \tag{2}$$

where $r_1(t)$ is the embedding with the image N, and $r_2(x, y)$ is the embedding with image M.

In the propagating from the surface case we use the generating family

$$\mathcal{F}_2(n, t, x, y, \lambda, q) = \langle \lambda r_1(t) + r_2(x, y) - q, n\rangle \tag{3}$$

with the same variables as $\mathcal{F}_1$ but parameters $(\lambda, q) \in \mathbb{R} \times \mathbb{R}^3$.

In the paper [7] the affine ratio case was studied and the generic bifurcations of the wave fronts were classified. There the generating family used was

$$\widetilde{\mathcal{F}}(n, t, x, y, \mu, q) = \langle (1 - \mu) r_1(t) + \mu r_2(x, y) - q, n\rangle. \tag{4}$$

We now show that the two family germs $\widetilde{\mathcal{F}}$ and $\mathcal{F}_1$ are time-space equivalent (see theorem 1.3) and hence the classification of their generic wave fronts and their bifurcations are in fact the same.

Proof of theorem 1.3.
Assuming that $\mu \neq 1$ we can divide the family (4) by $(1-\mu)$ to give

$$\widehat{\mathcal{F}}(n,t,x,y,\mu,q) = \langle r_1(t) + \frac{\mu}{1-\mu} r_2(x,y) - \frac{q}{1-\mu}, n\rangle.$$

This is time-space contact equivalent to

$$\widehat{\mathcal{F}_1} = \langle r_1(t) + \tilde{\mu} r_2(x,y) - \tilde{q}, n\rangle$$

with $\tilde{\mu} = \frac{\mu}{1-\mu}$ and $\tilde{q} = -\frac{q}{1-\mu}$. □

If $\mu = 1$ then this is equivalent to the present case at infinity and so does not appear in this "non-projective" setting. Similar considerations show that the family $\widetilde{\mathcal{F}}$ is time-space contact equivalent to the family $\mathcal{F}_2$.

2. Propagating from the curve in the transversal setting

In the transversal setting up to an appropriate affine transformation of $\mathbb{R}^3$ we can always assume that in some coordinate system (x,y,z) the base parallel pair a_0, b_0 coincides with the pair of points $(0,0,-1),(0,0,0)$, the tangent plane to the surface M at a_0 is parallel to the (x,y)-coordinate plane and the tangent line to the curve N at b_0 coincides with the x-axis.

In these coordinates the surface M in the neighbourhood of a_0 is the graph $M = \{(x,y,z)|z = f(x,y)-1)$ of the function f with vanishing 1-jet. Let $f(x,y) = \sum_{i+j\geq 2} f_{ij}x^iy^j$ be the Taylor decomposition of the germ of f at the origin. Define the curve N to be the set $\{(t,\alpha(t),\beta(t))\}$ with the functions $\alpha(t) = \alpha_2 t^2 + \alpha_3 t^3 + \ldots$ and $\beta(t) = \beta_2 t^2 + \beta_3 t^3 + \ldots$ starting with at least quadratic terms in t.

Proposition 2.1. *The germ of the family $\mathcal{F}_1$ at a point corresponding to a point on the base chord is stably-equivalent to the product of the family germ $\Phi(t,\mu,q) = \beta(t) + \mu[f(\widehat{x},\widehat{y}) - 1] - q_3$ at the subset $\widehat{S_0} = \{t = 0, q_1 = q_2 = 0\}$ with a nonzero factor. Here we use the substitution $\widehat{x} = \frac{q_1 - t}{\mu}, \widehat{y} = \frac{q_2 - \alpha(t)}{\mu}$.*

Proof. Writing the family $\mathcal{F}_1$ in the coordinate form we get

$$\mathcal{F}_1 = An_1 + Bn_2 + Cn_3$$

where

$$A = t + \mu x - q_1$$
$$B = \alpha(t) + \mu y - q_2$$

and

$$C = \beta(t) + \mu[f(x,y) - 1] - q_3$$

For $\mu \neq 0$ the functions A and B are regular and we can choose A, B as the coordinate functions instead of x and y. In particular we can write $x = \frac{A+q_1-t}{\mu}$ and $y = \frac{B+q_2-\alpha(t)}{\mu}$.

So in the new coordinates we have $\mathcal{F}_1 = An_1 + Bn_2 + C(A,B,t,\mu,q)n_3$. The function C does not depend on n_1 and n_2 and the Hadamard lemma implies $C(A,B,t,\mu,q) = C(0,0,\mu,t,q) + A\varphi_1 + B\varphi_2$, where φ_1 and φ_2 are smooth functions in A,B,t,μ and q which vanish at the origin.

Now the function $\mathcal{F}_1$ takes the form $\mathcal{F}_1 = A(n_1 + \varphi_1 n_3) + B(n_2 + \varphi_2 n_3) + C(0,0,t,\mu,q)$ where the first two terms represent a non degenerate quadratic form in the independent variables $A, (n_1 + \varphi_1 n_3), B$ and $(n_2 + \varphi_2 n_3)$ in the vicinity of the point on the base chord.

Therefore, the function $\mathcal{F}_1$ is stably-equivalent to the function $\Phi = C(0,0,t,\mu,q)$ being the restriction of the function C to the subspace $A = B = 0$. So to study the envelope of chords and the families of wave fronts we can study the family germ

$$\Phi(t,\mu,q) = \beta(t) + \mu\left[f\left(\frac{q_1 - t}{\mu}, \frac{q_2 - \alpha(t)}{\mu}\right) - 1\right] - q_3 \qquad \Box.$$

For the family germ Φ at the point $m_0 = (0,\mu_0,0,0,q_3 = -\mu_0)$, on the base chord $l(a_0,b_0)$ denote by $g(t) = \Phi(t,\mu_0,0,0,q_3)$ at t the respective organising centre function.

To determine the singularity type of the generating family germ Φ at the point m_0 and the respective versality conditions denote by $\Phi_k(\mu,q)$ the coefficients at t^k in the Taylor decomposition of Φ with respect to t at the origin.

$$\Phi = \Phi_0 + \Phi_1 t + \Phi_2 t^2 + \Phi_3 t^3 + \Phi_4 t^4 + \Phi_5 t^5 + ...$$

The first few formulas where terms of second order or greater in q_1 and q_2 are denoted by dots are as follows:

$$\begin{aligned}
\Phi_0 &= -\mu - q_3\\
\Phi_1 &= \frac{1}{\mu}(-2f_{20}q_1 + f_{11}q_2) + ...\\
\Phi_2 &= \beta_2 + \frac{1}{\mu}(f_{20} + \alpha_2 f_{11}q_1 - 2\alpha_2 f_{02}q_2) + \frac{1}{\mu^2}(3f_{30}q_1 + f_{21}q_2) + ...\\
\Phi_3 &= \beta_3 + \frac{1}{\mu}(\alpha_2 f_{11} - \alpha_3 f_{11}q_1 - 2\alpha_3 f_{02}q_2) + \frac{1}{\mu^2}(-f_{30} + 2\alpha_2 f_{21}q_1 + 2\alpha_2 f_{12}q_2)\\
&\quad + \frac{1}{\mu^3}(-4f_{40}q_1 - f_{31}q_2) + ...
\end{aligned}$$

Setting in these formulas $q_1 = q_2 = 0$ we get the expressions of the Taylor coefficients of the organising centre $g_k = \Phi_k|_{q_1=q_2=0}$ at a chord point m_0.

2.1. **Normal forms of the Minkowski set.** The following proposition together with explicit calculations from the normal forms prove theorem 1.4.

Proposition 2.2. *For a generic pair of M and N at any point q of a base chord (a_0,b_0) except the point b_0 itself ($\mu = 0$) the germ of the respective generating family Φ is space-time contact equivalent to one of the standard versal deformations in parameters $(\mu,q) \in \mathbb{R}\times\mathbb{R}^3$ of the function germs at the origin in the variable t of the type A_k for $k = 1,\dots,4$ as follows:*

$$\begin{aligned}
A_1 : \Phi &= t^2 + \mu; & A_2 : \Phi &= t^3 + q_1 t + \mu;\\
A_3 : \Phi &= t^4 + q_2 t^2 + q_1 t + \mu; & A_4 : \Phi &= t^5 + q_3 t^3 + q_2 t^2 + q_1 t + \mu.
\end{aligned}$$

2.2. **Recognition of transversal singularities.** If β_2 is nonzero, that is the base tangent plane is not the osculating plane to the curve N then we always get a unique A_2 singularity at the point $\mu_c = -\frac{f_{20}}{\beta_2}$. If however $\beta_2 = 0$ then no caustic point occurs on the chord unless additionally $f_{20} = 0$ in which case the whole chord is of type A_2 and therefore belongs to the caustic. These are isolated chords and the situation when these occur at $f_{20} = \beta_2 = 0$ is called the *flattening case*.

If the condition $\beta_3 = \frac{f_{30}\beta_2^2}{f_{20}^2} + \frac{\alpha_2 f_{11}\beta_2}{f_{20}}$ holds then the caustic point at μ_c will be of the type A_3. If in addition to the condition for an A_3 singularity the condition $\beta_4 = \frac{f_{40}{\beta_2^3}^3}{f_{20}} + \frac{\alpha_2 f_{21}{\beta_2}^2}{{f_{20}}^2} + \frac{\alpha_3 f_{11}\beta_2}{f_{20}} + \frac{{\alpha_2}^2 f_{02}\beta_2}{f_{20}}$ also holds, together with g_5 being nonzero, then the caustic point at μ_c will be of the type A_4. The singularities of this type occur at isolated points due to genericity.

In the flattening case $f_{20} = \beta_2 = 0$, in addition to the whole chord being of type A_2, there also exist two points where A_3 singularities occur at $\mu = \frac{-f_{11}\alpha_2 \pm \sqrt{f_{11}^2\alpha_2^2 + 4\beta_3 f_{30}}}{2\beta_3}$.

Proof of proposition 2.2

The proof of the proposition uses the property that $\frac{\partial \Phi}{\partial \lambda} \neq 0$ which holds in the transversal case. The stability with respect to space-time contact equivalence of the germ Φ with this property coincides with its stability with respect to standard contact equivalence. Therefore to show stability with respect to space-time contact equivalence we proceed by proving that each singularity in turn is generically versal with respect to standard contact equivalence, (see [2]).

Let $\Phi : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^3 \mapsto \mathbb{R}$ be an unfolding of a function $g(t), t \in \mathbb{R}$ with parameters $\mu, q \in \mathbb{R} \times \mathbb{R}^3$ and let $g(t)$ have an A_k singularity at the origin.

Denote by δ_{ij} the coefficients of the k-jet of Φ at the origin where

$$\delta_{i1} = \frac{\partial^{i+1}\Phi}{\partial t^i \partial q_3}, \delta_{i2} = \frac{\partial^{i+1}\Phi}{\partial t^i \partial \mu}, \delta_{i3} = \frac{\partial^{i+1}\Phi}{\partial t^i \partial q_1} \text{ and } \delta_{i4} = \frac{\partial^{i+1}\Phi}{\partial t^i \partial q_2}.$$

The jet matrix for the family of functions Φ shall be denoted M_4 and let M_k with $k \leq 4$ be the matrix consisting of the first k rows of M_4. We only consider $k \leq 4$ due to genericity.

The matrix $M_4 = (\delta_{ij})$ up to a factor of the rows for μ nonzero is given by

$$M_4 = \begin{pmatrix} -1 & -1 & 0 & 0 \\ 0 & 0 & -2f_{20} & -f_{11} \\ 0 & \delta_{32} & \delta_{33} & \delta_{34} \\ 0 & \delta_{42} & \delta_{43} & \delta_{44} \end{pmatrix}$$

where

$$\begin{aligned} \delta_{32} &= -f_{20}, \quad \delta_{33} = -\alpha_2 f_{11}\mu + 3f_{30}, \quad \delta_{34} = -2\alpha_2 f_{02}\mu + f_{21}, \\ \delta_{42} &= -\alpha_2 f_{11}\mu + 2f_{30}, \quad \delta_{43} = -\alpha_3 f_{11}\mu^2 + 2\alpha_2 f_{21}\mu - 4f_{40}, \quad \delta_{44} = -2\alpha_3 f_{02}\mu^2 + 2\alpha_2 f_{12}\mu - f_{31}. \end{aligned}$$

Then function Φ is right-versal if and only if the matrix M_k has rank k. Notice that the conditions $g_1 = 0, ..., g_k = 0$ define a Whitney stratification in the jet space of the embeddings. In fact each of these conditions outside λ being zero defines a regular hyper-surface in the space of germs and moreover those hyper-surfaces are mutually transversal since each equation $g_i = 0$ involves only one variable β_i and can be solved for it in terms of coefficients f_{jl} and α_s.

Versality of an A_1 singularity

The proof is immediate because the matrix

$$M_1 = \begin{pmatrix} -1 & 1 & 0 & 0 \end{pmatrix}$$

always has the maximal rank of 1. $\square$

Versality of an A_2 singularity

The versality of the A_2 singularities is determined by whether the matrix M_2 has maximal rank 2 where

$$M_2 = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & -2f_{20} & -f_{11} \end{pmatrix}.$$

The matrix M_2 has non-maximal rank only if both f_{11} and f_{20} vanish. If f_{20} vanishes and $\beta_2 \neq 0$ then recall the caustic occurs at $\mu = 0$ on the curve itself. If $f_{20} = \beta_2 = 0$ then the whole chord belongs to the caustic. In this case the vanishing of β_2, f_{20} and f_{11} is non-generic. Therefore, away from the curve and surface, A_2 singular points at μ_c are versal. $\square$

Versality of an A_3 singularity
If $f_{20} \neq 0$ then clearly the minor consisting of the first three columns of M_3 as nonzero determinant. In the flattening case $f_{20} = \beta_2 = 0$ the derivative matrix M_3 takes the form

$$M_3 = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & f_{11} \\ 0 & 0 & -\mu\alpha_2 f_{11} + 3f_{30} & -2\mu\alpha_2 f_{02} + f_{21} \end{pmatrix}$$

and recall that the A_3 singularities occur at $\mu = \frac{-\alpha_2 f_{11} \pm \sqrt{\alpha_2^2 f_{11}^2 - 4\beta_3 f_{30}}}{2\beta_3}$. For this value of μ the determinant of both of the minors of M_3 vanishes if either $f_{11} = 0$ or $\beta_3 = -\frac{4{f_{11}}^2{\alpha_2}^2}{9f_{30}}$. Neither of these conditions holds generically,so generically A_3 singularities are versal.

Versality of an A_4 singularity.
An A_4 singularity occurs on the base chord at the point m_0 when $g_2 = g_3 = 0$ and

$$g_4 = \beta_4 + \frac{1}{\mu}(\alpha_3 f_{11} + \alpha_2^2 f_{02}) - \frac{1}{\mu^2}\alpha_2 f_{21} + \frac{1}{\mu^3} f_{40} = 0,$$

but $g_5 \neq 0$. Notice also that A_4 cannot happen in the flattening case due to genericity.

The versality of A_4 singularities holds if the determinant

$$\det(M_4) = 2f_{20}[\delta_{32}\delta_{44} - \delta_{34}\delta_{42}] - f_{11}[\delta_{32}\delta_{43} - \delta_{33}\delta_{42}]$$

is nonzero. Since generically A_4 singularities cannot occur in the flattening case we assume that β_2 and f_{20} are nonzero. The condition that the determinant is zero can be solved for f_{31} as a function of the other terms.

The codimension of the stratum which corresponds to an A_4 singularity together with the vanishing of $\det(M_4)$ is greater than 3 so A_4 singularities are generically versal. This completes the proof of proposition 2.2. Explicit calculations from the normal forms completes the proof of theorem 1.4. $\square$

2.3. **Propagating from the space curve in the tangential setting.** In the tangential case we use the same family as was used in the transversal case

$$\mathcal{F}_1(n, t, x, y, \mu, q) = \langle r_1(t) + \mu r_2(x, y) - q, n\rangle.$$

Here we assume that the base chord lies in the plane tangent to M at a_0 (tangential setting). If the base chord and the tangent line to the curve N at b_0 are not collinear then in some coordinate system (x, y, z) the base points a_0, b_0 coincide with the points $(0, 1, 0), (0, 0, 0)$, the curve N at the origin is tangent to the x-axis and the tangent plane to the surface M coincides with the (x, y)-coordinate plane. Now the surface M is defined by the embedding

$$r_2 : U \to \mathbb{R}^3,\ r_2(x, y) = (x, y + 1, f(x, y)),\ (x, y) \in U \subset \mathbb{R}^2$$

where the function $f(x, y)$ has zero 1-jet, and the curve N is defined by the embedding

$$r_1 : V \to \mathbb{R}^3,\ r_1 : t \mapsto (t, \alpha(t), \beta(t)),\ t \in V \subset \mathbb{R}$$

of some neighbourhood V of the origin in $\mathbb{R}$ where $\alpha(t)$ and $\beta(t)$ start with second order terms.

After an appropriate stabilisation the initial generating family germ $\mathcal{F}_1$ at the point $\mu = \mu_0, t = 0, q = 0$ reduces to the form

$$\Phi(t, \varepsilon, q) = \beta(t) + (\mu_0 - \varepsilon) f\left(\frac{q_1 - t}{\mu_0 - \varepsilon}, \frac{\widetilde{q_2} - \alpha(t) - \varepsilon}{\mu_0 - \varepsilon}\right) - q_3 \tag{5}$$

where $\varepsilon = \mu_0 - \mu$ varies in the vicinity of the origin and $\widetilde{q_2} = q_2 - \mu_0$.

Consider the organising centre $g(t,\varepsilon) = \Phi|_{q_1=\widetilde{q_2}=q_3=0}$ of the family and decompose it as $g(t,\varepsilon) = \sum_{i+j\geq 2} a_{ij}t^i\varepsilon^j$ where the first few terms are:

$$\begin{aligned} a_{20} &= \beta_2 + \frac{1}{\mu_0} f_{20}, \quad a_{11} = -\frac{1}{\mu_0} f_{11}, \quad a_{02} = \frac{1}{\mu_0} f_{02}, \\ a_{30} &= \beta_3 + \frac{1}{\mu_0}\alpha_2 f_{11} - \frac{1}{\mu_0^2} f_{30}, \\ a_{21} &= -\frac{2}{\mu_0}\alpha_2 f_{02} + \frac{1}{\mu_0^2}(f_{20} + f_{21}), \end{aligned}$$

The space-time contact equivalence of the families of type Φ corresponds to *fibred contact equivalence* of the respective organising centres $g(t,\varepsilon)$: diffeomorphisms of the form $(t,\varepsilon) \to (\hat{t}(t,\varepsilon), \hat{\varepsilon}(\varepsilon))$ and multiplications by nonzero functions act on g.

The well-known Arnold-Goryunov low dimensional fibred contact classification (which coincides with simple boundary classes) provides all generic space-time contact stable families depending on three parameters (here $k = 2, 3$ or 4):

$$\begin{aligned} B_k: &\quad \pm t^2 + \varepsilon^k + q_{k-2}\varepsilon^{k-2} + \dots + q_3, \\ C_2 \approx &\quad B_2, \\ C_3: &\quad t^3 + t\varepsilon + q_1\varepsilon + q_3, \\ C_3: &\quad t^4 + t\varepsilon + q_2 t^2 + q_1\varepsilon + q_3, \\ F_4: &\quad t^3 + \varepsilon^2 + q_2 t\varepsilon + q_1 t + q_3. \end{aligned} \tag{6}$$

The proof of theorem 1.5 consists of checking the versality and genericity conditions for germs of the family Φ.

Singularities of the type B_k occur $a_{20} \neq 0$. When the quadratic form of $g(t,\varepsilon)$ is non-degenerate then the singularity of type B_2 occurs. If the quadratic form is degenerate, that is $4a_{20}a_{02} - a_{11}^2 = 0$, the singularity is of type B_3. This occurs when $\mu_0 = \frac{f_{11}^2 - 4f_{20}f_{02}}{4\beta_2 f_{02}}$ so every chord in the tangential setting has a singularity of type B_3 (which may occur on the curve or at infinity). The B_3 singularity can become more degenerate at isolated points to form the B_4 type. This condition can be solved for β_3 as a function of the other terms. Any further degenerations are excluded due to genericity.

The C_3 singularity occurs when $a_{20} = 0$ and both $a_{20} \neq 0$ and $a_{11} \neq 0$. This happens at a single point on the chord when $\mu_0 = -\frac{f_{20}}{\beta_2}$. This can become more degenerate if $a_{30} = 0$ and $a_{40} \neq 0$ to form the singularity of type C_4. The singularity of type F_4 belongs to the intersection of the closures of the B_3 and C_3 singularities and occurs when $\mu_0 = -\frac{f_{20}}{\beta_2}$ and $f_{11} = 0$. Similar considerations using slightly different embeddings show that in the supertangential case, away from the curve and surface, only singularities of type B_2 and C_3 occur generically.

3. Propagating from the surface in the transversal setting

We now turn our attention to the case where our initial starting manifold is a surface and the indicatrix of admissible velocities at each point defined by a space curve. Recall that in this case we use the generating function

$$\mathcal{F}_2(n,t,x,y,\lambda,q) = \langle \lambda r_1(t) + r_2(x,y) - q, n\rangle$$

where as before $r_1(t)$ is the embedding with the image of the space curve N, and $r_2(x,y)$ is the embedding with image the surface M. In this case we choose affine coordinates so that near a distinguished chord the surface is at the origin and the tangent plane is the (x,y)-coordinate plane, and the space curve contains the point $(0,0,1)$ and has tangent vector in the direction of the x-axis.

Proposition 3.1. *The family germ $\mathcal{F}_2$ at a point corresponding to a point on the base chord is stably-equivalent to the product of the family germ $\Phi(t,\mu,q) = \lambda(\beta(t)+1) + [f(\widehat{x},\widehat{y})] - q_3$ at the subset $\widehat{S_0} = \{t = x = y = 0, q_1 = q_2 = 0\}$ with a nonzero factor. Here we use the substitution $\widehat{x} = q_1 - \lambda t, \widehat{y} = q_2 - \lambda\alpha(t)$.*

Proof. Writing the family $\mathcal{F}_2$ in the coordinate form we get

$$\mathcal{F} = An_1 + Bn_2 + Cn_3$$

where

$$\begin{array}{rcl} A & = & \lambda t + x - q_1 \\ B & = & \lambda\alpha(t) + y - q_2 \\ C & = & \lambda(\beta(t)+1) + f(x,y) - q_3 \end{array}$$

As in the previous case we make an appropriate substitution, this time $x = q_1 - \lambda t$ and $y = q_2 - \lambda\alpha(t)$, and use the Hadamard lemma to show that this is stably equivalent to the family

$$\Phi(t,\lambda,q) = \lambda\beta(t) + \lambda + [f\,(q_1 - \lambda t, q_2 - \lambda\alpha(t))] - q_3.$$

Expanding the function Φ as a Taylor decomposition with respect to t at the origin where $\Phi = \Sigma_{n=0}^{\infty}\Phi_k t^k$, up to linear terms in q_1 and q_2, has the first few coefficients:

$$\begin{array}{rcl} \Phi_0 & = & \lambda - q_3 \\ \Phi_1 & = & -\lambda(2f_{20}q_1 + f_{11}q_2) \\ \Phi_2 & = & \lambda(\beta_2 - f_{11}\alpha_2 q_1 - 2f_{02}\alpha_2 q_2 + \lambda f_{20} + 3\lambda f_{30}q_1 + f_{21}\lambda q_2) \end{array}$$

Setting in these formulas $q_1 = q_2 = 0$ we get the expressions of the Taylor coefficients of the organising centre $g_k = \Phi_k|q_1 = q_2 = 0$ at a chord point m_0.

As with the propagating from the space curve case away from the initial starting manifold an A_2 singularity occurs on the chord at $\lambda_c = \frac{-\beta_2}{f_{20}}$. This becomes more degenerate as an A_3 singularity if additionally $\beta_3 = \frac{{\beta_2}^2 f_{30}}{{f_{20}}^2} + \frac{\beta_2 f_{11}\alpha_2}{f_{20}}$ and type A_4 if also $\beta_4 = -\lambda^3 f_{40} + \lambda^2\alpha_2 f_{21} - \lambda\alpha_3 f_{11} - \lambda\alpha_2^2 f_{02}$. Notice that these conditions for the singularity to be more degenerate are the same as those in the propagating from the space curve case. In the flattening case when $f_{20} = \beta_2 = 0$ the whole chord belongs to the caustic and is type A_2 everywhere except two points $\lambda = \frac{\alpha_2 f_{11} \pm \sqrt{{\alpha_2}^2 {f_{11}}^2 + 4\beta_3 f_{30}}}{2f_{30}}$ where singularities of type A_3 occur.

Consider the derivative matrix given by

$$M_4 = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & -2f_{20} & -f_{11} \\ 0 & \delta_{32} & \delta_{33} & \delta_{34} \\ 0 & \delta_{42} & \delta_{43} & \delta_{44} \end{pmatrix}$$

where

$$\begin{array}{rcl} \delta_{32} & = & 2\lambda f_{20} + \beta_2, \quad \delta_{33} = 3\lambda^2 f_{30} - \lambda\alpha_2 f_{11}, \quad \delta_{34}\lambda^2 f_{21} - 2\lambda\alpha_2 f_{02}, \\ \delta_{42} & = & -3\lambda^2 f_{30} + 2\lambda\alpha_2 f_{11} + \beta_3, \quad \delta_{43} = -4\lambda^3 f_{40} + 2\lambda^2\alpha_2 f_{21} - \lambda\alpha_3 f_{11}, \\ \delta_{44} & = & -\lambda^3 f_{31} + 2\lambda^2\alpha_2 f_{12} - 2\lambda\alpha_3 f_{02}. \end{array}$$

We can use the same arguments as we used in the propagating from the space curve case to show that all the generic A_k singularities are versal.

3.1. **Propagating from the surface in the tangential setting.** Assume that the base chord is in the tangential setting, that is it lies in the plane tangent to M at a_0 but it is not collinear with the tangent line to the curve N at b_0. In some coordinate system the surface contains the origin at a_0 and the tangent plane at this point is the (x, y)-coordinate plane, the curve passes through the point $(0, 1, 0)$ at b_0 and the tangent line has the same direction as the x-axis. Using appropriate embeddings $r_1(t)$ and $r_2(x, y)$ the generating $\mathcal{F}_2$ can be expanded as a vector to give

$$\mathcal{F}_2 = (x + \lambda t - q_1)n_1 + (y + \lambda(\alpha(t) + 1) - q_2)n_2 + ((f(x, y) + \lambda\beta(t) - q_3)n_3.$$

Proposition 3.2. *Using the substitution $x = q_1 - \lambda t$, $y = q_2 - \lambda(\alpha(t) + 1)$ the family $\mathcal{F}_2$ at the point $\lambda = \lambda_0, t = 0, q = 0$ is stably equivalent to the family*

$$\Phi(t, \varepsilon, q) = (\lambda_0 + \varepsilon)\beta(t) + f(q_1 - (\lambda_0 + \varepsilon)t, \widetilde{q_2} - \lambda_0\alpha(t) - \varepsilon\alpha(t) - \varepsilon). \tag{7}$$

where $\lambda = (\lambda_0 + \varepsilon)$ varies in the vicinity of the origin and $q_2 = \widetilde{q_2} + \lambda_0$.

Consider the organising centre $g(t, \varepsilon) = \Phi|_{q_1=\widetilde{q_2}=q_3=0}$ of the family and decompose it as $g(t, \varepsilon) = \sum_{i+j\geq 2} a_{ij}t^i\varepsilon^j$ where the first few terms are

$$\begin{aligned}
a_{20} &= f_{20}\lambda_0^2 + \lambda_0\beta_2, \quad a_{11} = f_{11}\lambda_0, \quad a_{02} = f_{02},\\
a_{30} &= \lambda_0\beta_3 + f_{11}\lambda_0^2\alpha_2 - f_{30}\lambda_0^3,\\
a_{21} &= \beta_2 + 2f_{02}\lambda_0\alpha_2 - f_{21}\lambda_0^2 + 2f_{20}\lambda_0,\\
a_{12} &= f_{11} - f_{12}\lambda_0, \quad a_{03} = -f_{03}.
\end{aligned}$$

The list of generic singularities coincides with the list (6).

The whole of each chord in the tangential setting is of type at B_2 except for at most two points which can be more degenerate. Generically each chord will consist of a B_3 singularity and a C_3 singularity. At isolated chords one of these can be more degenerate to form either B_4 or C_4. Also at isolated chords the B_3 and C_3 singularities can occur at the same point to give an F_4 singularity.

When the quadratic form is degenerate, that is $4a_{20}a_{02} - a_{11}^2 = 0$, a singularity of type B_3 occurs. This happens at $\lambda = \frac{4f_{02}\beta_2}{f_{11}^2 - 4f_{02}f_{20}}$. For isolated chords one of these can be more degenerate giving the singularity of type B_4.

This condition can be solved for f_{03} as a function of the other terms.

Singularities of type C_3 occur at $\lambda = -\frac{\beta_2}{f_{20}}$. This can be more degenerate to form a C_4 singularity if $\beta_3 = \frac{\beta_2^2 f_{30}}{f_{20}^2} + \frac{\beta_2 f_{11}\alpha_2}{f_{20}}$. An F_4 singularity will result if both the conditions for a C_3 and a B_3 singularity occur, namely if $\lambda = -\frac{\beta_2}{f_{20}}$ and $f_{11} = 0$. Further degenerations are excluded due to genericity.

Checking the versality and genericity conditions for germs of the family Φ completes the proof of Theorem 1.8.

Similar considerations using different embeddings show that in the supertangential case, away from the curve and surface, generically only singularities of type B_2 and C_3 occur. □

3.2. **Propagating from the surface in the transversal case in the vicinity of the surface.** Up until now we have assumed that λ is nonzero and have classified the singularities away from the surface and space curve. In this section we study the generic caustic near the surface itself, that is when λ is close to zero. We use the standard generating family in the propagating from the surface case $\mathcal{F}_2$ and proposition 3.1 implies the generating family is stably equivalent to

$$\Phi(t,\lambda,q) = \lambda(\beta(t)+1) + f(q_1-\lambda t, q_2-\lambda\alpha(t)) - q_3.$$

This can be written as

$$\Phi = \lambda\left(\frac{f-f_0}{\lambda} + \beta + 1\right) + f_0 - q_3$$

where $f_0 = f|_{q_1=q_2=0}$ and $\frac{f-f_0}{\lambda}$ is smooth. Introduce the new parameter $\widetilde{q_3} = -q_3 + f_0$ which vanishes on the surface, yielding

$$\Phi = \lambda\left(\frac{f-f_0}{\lambda} + \beta + 1\right) + \widetilde{q_3}.$$

Denote by $\Phi_0, \Phi_1, \ldots$ the terms of the power series decomposition in λ of the contents of the brackets. With terms of order greater than 4 in t or greater than 1 in q_1 and q_2 denoted by dots the generating function Φ is written

$$\mathcal{F} = \widetilde{q_3} + \lambda\left(\Phi_0 + \ldots + \lambda(\Phi_1 + \ldots) + \lambda^2(\Phi_2 + \ldots) + \ldots\right) \tag{8}$$

where

$$\Phi_0 = 1 + \beta(t) + (-2f_{02}\alpha(t) - f_{11}t)\,q_2 + (-2f_{20}t - f_{11}\alpha(t))\,q_1$$

and

$$\Phi_1 = f_{11}t\alpha(t) + f_{20}t^2 + f_{02}\alpha(t)^2 + \\ \left(2f_{12}t\alpha(t) + f_{21}t^2 + 3f_{03}\alpha(t)^2\right) q_2 + \left(2f_{21}t\alpha(t) + f_{12}\alpha(t)^2 + 3f_{30}t^2\right) q_1.$$

Proposition 3.3. *The family germ $\mathcal{F}$ can be written in the form $\mathcal{F} = \lambda + \widetilde{q_3}H(t,q_1,q_2,\widetilde{q_3})$ where the lower degree terms with respect to $\widetilde{q_3}$ of function H are: $H = \frac{1}{\Phi_0} + \frac{\Phi_1\widetilde{q_3}}{\Phi_0^3} + \ldots$*

Lemma 3.4. *Assume $H(t,q_1,q_2,\widetilde{q_3})$ is $\mathcal{R}^+$-versal with respect to q_1 and q_2 only; then the family germ $\mathcal{F} = \lambda + \widetilde{q_3}H$ is space-time stable with respect to deformations inside the space $W = \lambda + \widetilde{q_3}\widetilde{H}(t, q1, q2, \widetilde{q_3})$ such that $\frac{\partial W}{\partial \widetilde{q_3}} \neq 0$.*

Proposition 3.5. *For generic curve and surface germs in the transversal setting the function $H(t,\lambda,q)$ is versal for standard $\mathcal{R}^+$-equivalence with respect to q_1 and q_2 only.*

The first few terms of the Taylor decomposition of H with respect to t at the origin, namely $H = \Sigma_{k=0}H_k(t,q)t^k$, up to first order terms in q_i, are as follows.

$$\begin{aligned} H_0 &= 1, \\ H_1 &= 2f_{20}q_1 + f_{11}q_2, \\ H_2 &= -\beta_2 + f_{11}\alpha_2 q_1 + 2f_{02}\alpha_2 q_2 + f_{20}\widetilde{q_3}, \\ H_3 &= -\beta_3 + (-4\beta_2 f_{20} + f_{11}\alpha_3)\,q_1 + (-2\beta_2 f_{11} + 2f_{02}\alpha_3)\,q_2 + f_{11}\alpha_2\widetilde{q_3}, \end{aligned}$$

Setting in these formulas $q_1 = q_2 = \widetilde{q_3} = 0$ we get the following expressions of the Taylor coefficients of the organising centre $h_k = H_k|_{q_1=q_2=\widetilde{q_3}=0}$:

$$h_0 = 1, \quad h_1 = 0, \quad h_2 = -\beta_2, \quad h_3 = -\beta_3, \quad h_4 = \beta_2^2 - \beta_4$$

The function H has a singularity of type A_2 if $\beta_2 = 0$ and $\beta_3 \neq 0$. If $\beta_2 = \beta_3 = 0$ and $\beta_4 \neq 0$ then the function H has a singularity of type A_3. More degenerate singularities are excluded due to genericity.

In order for H to be $\mathcal{R}^+$-versal with respect to q_1 and q_2 only at an A_k singularity for $k = 2, 3$ we need the first $k-1$ rows of the jet matrix

$$M_{k-1} = \begin{pmatrix} \frac{\partial^2 H}{\partial q_1 \partial t} & \frac{\partial^2 H}{\partial q_2 \partial t} \\ \\ \frac{\partial^3 H}{\partial q_1 \partial t^2} & \frac{\partial^3 H}{\partial q_2 \partial t^2} \end{pmatrix}$$

to have maximal rank $k-1$.

Versality of A_2 singularities on the surface M
An A_2 singularity occurs if $\beta_2 = 0$ and $\beta_3 \neq 0$. Recall that this is the necessary and sufficient condition that the tangent plane to the curve is the osculating plane with 3 point contact. The A_2 singularities are versal if the matrix

$$M_1 = \begin{pmatrix} 2f_{20} & f_{11} \end{pmatrix}$$

has rank 1.

Clearly the vanishing of β_2, f_{20} and f_{11} provide a set of non-generic conditions so A_2 singularities in the vicinity of the surface are versal.

Versality of A_3 singularities on the surface M
An A_3 singularity occurs if $\beta_2 = 0, \beta_3 = 0$ and $\beta_4 \neq 0$. This is the condition that the tangent plane to curve is the osculating plane and has 4 point contact (at a torsion zero). The A_3 singularities are versal if the matrix

$$M_2 = \begin{pmatrix} 2f_{20} & f_{11} \\ \alpha_2 f_{11} & 2\alpha_2 f_{02} \end{pmatrix}$$

has rank 2. The condition $\det(M_2) = 0$ together with the necessary conditions $\beta_2 = \beta_3 = 0$ singularity provide a non-generic condition so A_3 singularities are versal.

Since the generic singularities of the function H are $\mathcal{R}^+$-versal, lemma 3.4 implies that the generic singularities of the function $\mathcal{F}$ are space-time stable inside the space W.

At A_2 type points on the surface the caustic is smooth and transversally intersects the surface M. The respective generating family germ is space-time equivalent to the normal form:

$$\mathcal{F} = \lambda + \widetilde{q_3}(t^3 + q_1 t + 1)$$

At A_3 type points on the surface the caustic has a cuspidal edge that transversally intersects the surface M. The respective generating family germ is space-time equivalent to the normal form:

$$\mathcal{F} = \lambda + \widetilde{q_3}(t^4 + q_1 t^2 + q_2 t + 1). \quad \square$$

3.3. **Propagating from the surface in the Tangential Case in the vicinity of the surface.** In this case the caustic is space-time contact equivalent to one of the following normal forms (see [6]):

$$\widehat{B_2}: \quad \widetilde{q_3} + \lambda(t^2 + q_1); \qquad \widehat{B_3}: \quad \widetilde{q_3} + \lambda(t^2 \pm \lambda^2 + \lambda q_1 + q_2);$$

$$\widehat{C_3}: \quad \widetilde{q_3} + \lambda(t^3 + \lambda t + \lambda + q_1 t + q_2).$$

The caustic at a $\widehat{B_2}$ singularity consists only of the surface M and the criminant is a smooth surface with first order tangency with the surface M. At a $\widehat{B_3}$ singularity the criminant is diffeomorphic to a semi-cubic cylinder and has second order tangency with the surface M at a_0 (see figure 1). At a $\widehat{C_3}$ singularity the criminant is diffeomorphic to a folded Whitney umbrella and the caustic is a smooth surface. The cuspidal edge of the folded Whitney umbrella has first

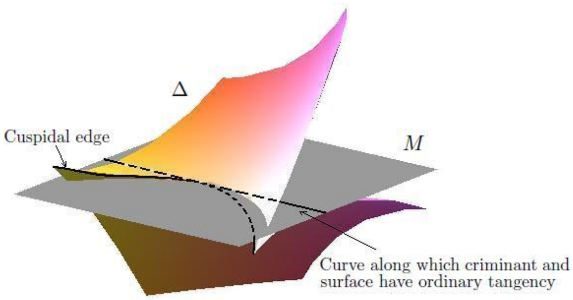

FIGURE 1. The envelope $B(M, N)$ at a $\widehat{B_3}$ singularity near the surface M (plane in figure). Here the caustic is empty and the criminant Δ is a cuspidal edge with second order tangency with the surface at a_0.

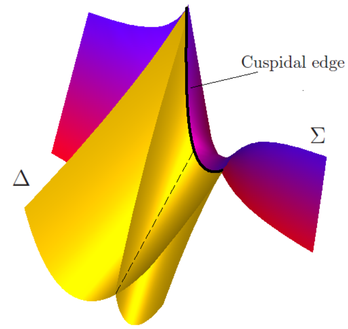

FIGURE 2. The caustic and the criminant shown together at a $\widehat{C_3}$ singularity. Here the criminant Δ is a folded Whitney umbrella and the caustic Σ is a smooth surface. The cuspidal edge of Δ has third order tangency with Σ.

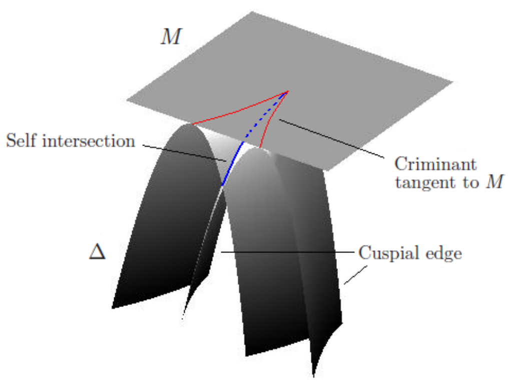

FIGURE 3. The criminant and the surface M (plane in figure) are shown together at a $\widehat{C_3}$ singularity. Here the criminant and M have ordinary tangency along a cusp.

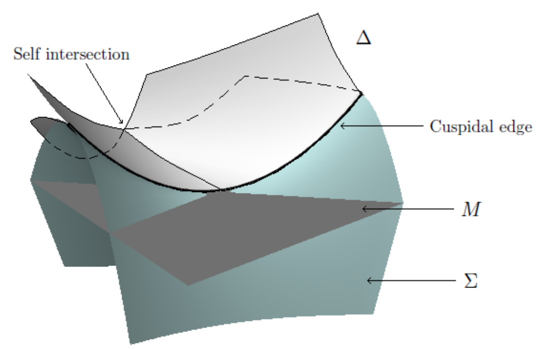

FIGURE 4. The envelope $B(M, N)$ at a $\widehat{C_3}$ singularity near the surface M (plane in figure). Here the criminant Δ is a folded Whitney umbrella and the caustic Σ is a smooth surface.

order tangency with the surface M at a_0 and third order tangency with the caustic (see figure 2). Two additional views are shown in figures 3 and 4.

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SINGULARITIES OF ABEL–JACOBI MAPS AND GEOMETRY OF DISSOLVING VORTICES

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Abstract. Gauged vortices are configurations of fields for certain gauge theories in fibre bundles over a surface Σ. Their moduli spaces support natural L^2-metrics, which are Kähler, and whose geodesic flow approximates vortex scattering at low speed. This paper focuses on vortices in line bundles, for which the moduli spaces are modeled on the spaces $\Sigma^{(k)}$ of effective divisors on Σ with a fixed degree k; we describe the behaviour of the underlying L^2-metrics in a "dissolving limit" where the L^2-geometry simplifies. In such limit, the metrics degenerate precisely at the singular locus of the Abel–Jacobi map AJ of Σ at degree k, and their geometry can be understood in terms of the variety $W_k = \mathrm{AJ}(\Sigma^{(k)})$ inside the Jacobian of Σ. Some intuition about the behaviour of the geodesic flow close to a singularity is provided through the study of the simplest example (resolution of a double point on a surface), corresponding to two dissolving vortices moving on a hyperelliptic curve of genus three.

1. Introduction

The vortex equations originate in the Ginzburg–Landau theory of superconductivity [4] and describe static, stable solutions of certain (2+1)-dimensional gauge theories [13, 9, 33, 23]. In the simplest example, the equations relate a connection d_a on a principal U(1)-bundle over a smooth surface with Kähler structure $(\Sigma, j_\Sigma, \omega_\Sigma)$, which we will assume to be compact, and a section ϕ of an associated line bundle $\mathcal{L} \to \Sigma$. As part of the geometric setup, one fixes a Hermitian structure on this line bundle, which equips each fibre $\mathcal{L}_P \cong \mathbb{C}$ with a symplectic structure preserved by the U(1)-action. This action is Hamiltonian, and a moment map $\mu : \mathcal{L} \to \mathfrak{u}(1)^* \cong \mathbb{R}$ is specified globally as

$$\mu(w) = \frac{1}{2}(\langle w, w\rangle - \tau), \qquad \text{for} \quad w \in \mathcal{L}_P, \quad P \in \Sigma \tag{1.1}$$

where $\tau \in \mathbb{R}$ is a constant (which remains arbitrary a priori). In this setup, the vortex equations read

$$\bar{\partial}_a \phi = 0\,, \tag{1.2}$$

$$F_a + (\mu \circ \phi)\omega_\Sigma = 0\,. \tag{1.3}$$

The first equation expresses that the section $\phi : \Sigma \to \mathcal{L}$ is holomorphic, i.e. annihilated by the operator $\bar{\partial}_a : \Omega^0(\Sigma, \mathcal{L}) \to \Omega^1(\Sigma, \mathcal{L})$ defined from the unitary connection d_a and the complex structure on Σ [8], while the second equation relates the curvature $F_a = \mathrm{d}a$ of the connection to the moment map evaluated on the values of the section and the area form ω_Σ.

By integrating (1.3) over Σ, one finds that the squared L^2-norm $\|\phi\|^2_{L^2} := \int_\Sigma \langle \phi, \phi\rangle \omega_\Sigma$ satisfies

$$\|\phi\|^2_{L^2} = \tau V - 4\pi k \tag{1.4}$$

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where $V := \int_\Sigma \omega_\Sigma$ is the total area of the surface and $k = \frac{1}{2\pi}\int_\Sigma F_a$ is the first Chern class (or degree) of the line bundle. Sometimes, k is referred to as the vortex number. Since the squared L^2-norm is nonnegative, (1.4) implies that a necesssary condition for solutions of the vortex equations to exist is $\tau \geq \frac{4\pi k}{V}$. A theorem of Bradlow [6] (see also [10]) asserts that, if we take $\tau > \frac{4\pi k}{V}$, one can find for any effective divisor D of degree k on Σ a unique solution of the equations up to gauge equivalence which satisfies $(\phi) = D$; this is what is called a k-vortex. The moduli space $\mathcal{M}_k$ of k-vortices is therefore the symmetric product $\Sigma^{(k)} = \Sigma^k/\mathfrak{S}_k$, a smooth complex manifold with complex dimension k.

The divisor of zeroes (ϕ) is the most basic object one can assign to a k-vortex and it gives the precise location of k individual vortex cores, but these objects should be thought of as extending over Σ and interacting with each other. Interesting information about vortex interactions is encoded in a natural metric on the moduli space, which is induced from the trivial L^2-metric on the space of all fields (d_a, ϕ) by an infinite-dimensional analogue of symplectic reduction. The induced metric is nontrivial and also Kähler with respect to the complex structure on $\Sigma^{(k)}$ induced from j_Σ. We use the term 'L^2-geometry' to refer to this family of Kähler structures on each $\mathcal{M}_k$, which is parametrised by $\tau \in \,]\frac{4\pi k}{V}, \infty[$.

To be more precise, the L^2-metrics are defined at each k-vortex solution (d_a, ϕ) by

$$\|(\dot{a}, \dot{\phi})\|^2_{(\mathrm{d}_a,\phi),L^2} = \int_\Sigma \left(\frac{1}{2}\,\dot{a} \wedge \star\dot{a} + \langle \dot{\phi}, \dot{\phi}\rangle \omega_\Sigma\right) \tag{1.5}$$

where $\dot{a} \in \Omega^1(\Sigma)$ and $\dot{\phi} \in \Gamma(\Sigma, \mathcal{L})$ are fields representing tangent vectors in $\mathrm{T}_{(\mathrm{d}_a,\phi)}\mathcal{M}_k$ (they satisfy the linearisation of the vortex equations about (d_a, ϕ) and are L^2-orthogonal to the orbit of the gauge group through this point), and $\star$ is the Hodge star of the Kähler metric on Σ. Integrals over Σ such as (1.5) would seem hopeless to compute directly, but it turns out that they localise onto the support of the divisor (ϕ) associated to the vortex [30, 29]. This feature has been invaluable to understand the L^2-metrics and their physical content. Even though an explicit calculation of the metrics seems to be beyond reach as yet, some results have been obtained in certain regimes, adding to our intuition about the geometry underlying the vortex equations. For example, formulas for the symplectic volume of the moduli spaces $\mathcal{M}_k$ have been established by Manton and Nasir [21] exploring localisation:

$$\mathrm{Vol}(\mathcal{M}_k) = (2\pi)^{2k} \sum_{n=0}^{\min\{k,g\}} \frac{g!}{n!(k-n)!(g-n)!} \left(\frac{\tau V}{4\pi} - k\right)^{k-n}. \tag{1.6}$$

The L^2-metrics encode precious information about infinite-dimensional dynamical models that incorporate solitons. For instance, their geodesic flow is of direct physical interest, since it gives a good approximation to the slow dynamics in the Abelian Higgs model in 2+1 dimensions [23, 31].

One regime in which the L^2-geometry becomes somewhat tractable is what we call the dissolving limit, which corresponds to taking

$$\tau \to \frac{4\pi k}{V} \tag{1.7}$$

for Σ compact. This was considered in [32] and [3] when Σ has genus $g = 0$; and for genus $g \geq 1$ first by Nasir [26], and then by Manton and Romão [22]. In the following, we shall give an account of the results in [22] — we will add little to the presentation in the original paper, and shall focus on the $k > 1$ (multivortex) case, which brings in some interesting issues that relate to the theory of singularities.

2. Vortices in the dissolving limit

We would like to understand the geometry of the moduli space of vortices in the dissolving limit (1.7), which will turn out to be a simplification of the full L^2-geometry. It is instructive to look first at the space of solutions of the equations when one sets $\tau = \frac{4\pi k}{V}$. Then equation (1.4) implies that $\phi = 0$, so the first vortex equation (1.2) is trivially satisfied: the zero section is always holomorphic. Note that the action of $\star$ on $\Omega^1(\Sigma)$ depends on j_Σ alone, so when $\phi \to 0$ we expect that the L^2-metric, defined by the expression (1.5), will only depend on the conformal class of the metric given on Σ. In the following, we shall make this observation more precise.

The second vortex equation (1.3) simplifies to

$$F_a = \mathrm{d}a = \frac{\tau}{2}\,\omega_\Sigma\,, \tag{2.1}$$

which says that the curvature of the connection d_a is a constant multiple of the area form ω_Σ. Notice that the constant of proportionality $\frac{\tau}{2} = \frac{2\pi k}{V}$ is determined by the topology and the normalisation V. This is still a crude approximation to the degeneration of the moduli space of k-vortices that we are interested in; we introduce the following terminology:

Definition 2.1. A *dissolved k-vortex* is a solution d_a to equation (2.1) in a line bundle of degree k.

Dissolved vortices correspond to "constant curvature" or "projectively flat" connections with respect to the 2-form ω_Σ, and they are parametrised by the dual to the Jacobian variety of Σ, a complex g-torus if Σ has genus g. Recall that the Jacobian is defined by [11]

$$\mathrm{Jac}(\Sigma) = H^0(\Sigma, K_\Sigma)^*/H_1(\Sigma,\mathbb{Z}). \tag{2.2}$$

Here, K_Σ denotes the (canonical) sheaf of holomorphic 1-forms, and the inclusion $H_1(\Sigma,\mathbb{Z}) \hookrightarrow H^0(\Sigma, K_\Sigma)^*$ is provided by integration over 1-cycles: $\lambda \mapsto \oint_\lambda$. If we are given a solution d_a (in a unitary trivialisation, $\mathrm{d}_a = \mathrm{d} - ia$ for a real 1-form a) of equation (2.1), for example constructed out of local symplectic potentials of ω_Σ obtained from Kähler potentials, we can write any other solution modulo gauge transformations as $\mathrm{d}_{a+\alpha}$ where α is a global harmonic 1-form (in other words, through twisting by a flat line bundle with connection d_α); α satisfies

$$\mathrm{d}\alpha = 0 \qquad \text{and} \qquad \mathrm{d} \star \alpha = 0. \tag{2.3}$$

The first equation in (2.3) follows from (2.1), while the second equation provides a section from the space of gauge orbits.

Different dissolved vortices have the same curvature 2-form but different holonomies around 1-cycles in Σ. In fact, one should identify dissolved vortices if they have the same holonomies, and this corresponds to quotienting the real $2g$-dimensional vector space of harmonic 1-forms α by the lattice of rank $2g$ defined by the relations

$$\oint_\lambda \alpha \in 2\pi\mathbb{Z}, \qquad \forall \lambda \in H_1(\Sigma,\mathbb{Z}), \tag{2.4}$$

thus we end up with the dual torus to $\mathrm{Jac}(\Sigma)$, as claimed. For some purposes, it is useful to think of a harmonic 1-form as the real part of a holomorphic 1-form on Σ, and so there is also a complex structure involved (more explicitly, $\star$ plays the role of complex structure at each point of the torus). Thus we are really dealing with the geometry of Abelian varieties [11].

Now the dual space $H^0(\Sigma, K_\Sigma)^*$ has a canonical inner product, namely the *polarisation* of the Jacobian [11]. One can think of it as the flat Kähler metric associated to the natural complex structure induced by j_Σ, together with the symplectic form obtained by extending the intersection pairing on $H_1(\Sigma,\mathbb{Z})$ to real coefficients (note that $H_1(\Sigma,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R} = H^0(\Sigma, K_\Sigma)^*$, cf. (2.2)); of course, this structure is invariant under translations. We shall denote by Ω_J the $(1,1)$-form of this Kähler metric on $\mathrm{Jac}(\Sigma)$. The map

$$\omega \mapsto \frac{1}{2\pi}\int \alpha \wedge \omega \tag{2.5}$$

provides an isomorphism relating infinitesimal flat connections α and elements of the dual space $H^0(\Sigma, K_\Sigma)^*$, and therefore a pull-back of the polarisation to a Kähler structure on the dual torus to $\mathrm{Jac}(\Sigma)$, yielding a metric at each point

$$(\alpha, \beta) \mapsto \int_\Sigma \alpha \wedge \star\beta. \tag{2.6}$$

It is not hard to see that this induced metric coincides with the natural L^2-geometry on the space of dissolved vortices (see Section 3 of [22] for the explicit argument). This geometry on the dual Jacobian is independent of the first Chern class k, the vortex number of the dissolved vortex.

To understand the L^2-geometry of k-vortices in the dissolving limit (1.7), a more insightful notion is the following.

Definition 2.2. A *dissolving k-vortex* is a unitary connection d_a on a line bundle of degree k whose induced holomorphic structure $\bar\partial_a$ has nontrivial kernel.

In other words, for a dissolving vortex one requires the existence of a nonzero holomorphic section for the induced holomorphic structure. So to a dissolv*ing* vortex one can always associate a dissolv*ed* vortex, but not conversely, and we should be able to think of it as a limit $(\mathrm{d_a}, 0)$ of a sequence of k-vortices as $\tau \to \frac{4\pi k}{V}$.

Recall that the Jacobian variety $\mathrm{Jac}(\Sigma)$ plays another important role, namely that of classifying holomorphic line bundles over Σ of a given degree [11]. Holomorphic line bundles are determined by divisor classes (i.e. divisors on Σ modulo *linear equivalence*, where two divisors of the same degree are identified if their difference is the divisor of zeroes and poles of a global meromorphic function on Σ). The relation between divisors on Σ and $\mathrm{Jac}(\Sigma)$ is achieved via the Abel–Jacobi map, which depends on the choice of a basepoint $P_0 \in \Sigma$: to a divisor $D = D_+ - D_-$, where $D_+ = \sum_i P_i$ and $D_- = \sum_j Q_j$ are effective divisors, $\mathrm{AJ}(D)$ is defined by a linear functional on holomorphic 1-forms via the Abelian integrals

$$\mathrm{AJ}(D) : \omega \mapsto \sum_i \int_{P_0}^{P_i} \omega - \sum_j \int_{P_0}^{Q_j} \omega.$$

The value determined by this quantity in the Jacobian variety does not depend on the choice of paths connecting each P_i or Q_j to P_0 since the ambiguity lies on the image of $H_1(\Sigma, \mathbb{Z})$ in $H^0(\Sigma, K_\Sigma)^*$. Moreover, a different choice of basepoint P_0 simply leads to a translation in the Jacobian. We will be interested in the restriction of the Abel–Jacobi maps AJ_k to the spaces of effective divisors of degree $k > 0$, which are the symmetric products $\Sigma^{(k)}$ and can be identified with moduli spaces of k-vortices.

Note that the maps AJ_k are holomorphic. Their images

$$W_k := \mathrm{AJ}_k(\Sigma^{(k)}) \;\subset\; \mathrm{Jac}(\Sigma) \tag{2.7}$$

are complex subvarieties of dimension $\min\{k, g\}$, and they can be regarded as the spaces of dissolving k-vortices. It is a classical theorem of Abel [11] that the map

$$\mathrm{AJ}_1 : \Sigma^{(1)} = \Sigma \;\longrightarrow\; \mathrm{Jac}(\Sigma)$$

is an embedding. So the flat Kähler structure associated to the polarisation of the Jacobian variety, discussed above, induces a Kähler structure $\mathrm{AJ}_1^*\Omega_J$ on Σ. In [22], the following result is proven:

Theorem 2.3. *In the dissolving limit (1.7), the L^2-metric on $\mathcal{M}_1$ converges to a natural Bergman metric on Σ, regarded as the moduli space of one dissolving vortex. It coincides with the Kähler metric obtained by pulling back the polarisation of the Jacobian via the Abel–Jacobi embedding $\mathrm{AJ}_1 : \Sigma \hookrightarrow \mathrm{Jac}(\Sigma)$.*

The idea of the proof is to relate Hecke modifications performing shifts of the line bundle associated to a dissolving 1-vortex to complex gauge transformations; such a shift can also be described by addition of harmonic 1-forms at the level of the connections associated to the holomorphic structures, and their length for an infinitesimal shift describes the L^2-metric, which can be computed in holomorphic coordinates. For our purposes, a Bergman metric [18, 14] on a compact Riemann surface of genus $g \geq 1$ is a Riemannian structure of the form

$$\mathrm{d}s^2 = \sum_{j=1}^{g} \omega_j \bar{\omega}_j, \tag{2.8}$$

which is associated to any basis $\{\omega_1, \ldots, \omega_g\}$ of $H^0(\Sigma, K_\Sigma) \cong \mathbb{C}^g$; see Appendix A in [22]. Note that (2.8) is constant on $\mathrm{U}(g)$-orbits of the space of bases. The particular Bergman metric in our result is the one coming from an orthogonal basis with respect to the metric on 1-forms given by (2.6), up to the global factor of $(2\pi)^2$.

3. Geometry of dissolving multivortices

Important details about the complex geometry of the Riemann surface Σ are captured by the dissolving limit of the L^2-geometry of the moduli spaces of vortices. We have argued that the limit Kähler structure should depend only on the intrinsic complex structure of Σ, and it would be interesting to understand how this dependence is reflected qualitatively in its curvature properties and the geodesic flow, for example. In the $k = 1$ case, the Bergman metric on $\mathcal{M}_1 = \Sigma$ in Theorem 2.3 is known to have nonpositive Gauß curvature [19]. This follows from general facts: the image of a holomorphic embedding $\Sigma \hookrightarrow \mathrm{Jac}(\Sigma)$ of a complex curve in a Kähler manifold must be a minimal surface, so its principal curvatures at each point must be symmetric. If $g > 1$, it turns out that the curvature vanishes at most at a finite number of points, which are precisely the Weierstraß points [11] of Σ if X is hyperelliptic (otherwise the curvature never vanishes) [19]. For Kähler structures of dissolving multivortices, one should be able to obtain results in this spirit, but the geometry in higher dimensions will be richer.

In what follows, we shall explore the dissolving limit (1.7) for multivortices, assuming that the two inequalities

$$1 < k < g \tag{3.1}$$

hold. We have already stated that the image (2.7) of the moduli space of vortices $\mathcal{M}_k = \Sigma^{(k)}$ under the Abel–Jacobi map (well defined once a base point $P_0 \in \Sigma$ is chosen, and holomorphic) is a complex subvariety of the Jacobian. However, as we will make more precise in a moment, for large enough k this map is no longer an embedding, in contrast to the $k = 1$ case, and then the images W_k in (2.7) are singular subvarieties. Among these objects, perhaps the most familiar one is W_{g-1}, which is a translation of the Θ-divisor [11] in $\mathrm{Jac}(\Sigma)$ and has singularities if $g > 3$ (see Example 3.2 below for a discussion of the $g = 3$ case).

Whenever the Abel–Jacobi map has singular points, the $(1,1)$-form $\mathrm{AJ}_k^*\Omega_J$ obtained as pullback of the polarisation of the Jacobian is degenerate, i.e. its rank drops down. Then one is left with a degenerating Kähler metric on $\Sigma^{(k)}$, for which the existence and uniqueness of geodesics associated to any point and direction may not hold. (The corresponding $(1,1)$-form is still closed, as it is the pull-back of the closed 2-form Ω_{J}). We argue in Section 7 of [22], following essentially the same steps of the proof of Theorem 2.3, that $\mathrm{AJ}_k^*\Omega_J$ describes once again the dissolving limit of the L^2-geometry on the moduli space of k-vortices.

In the multivortex case, over the sets of regular points of each AJ_k one thus obtains Kähler metrics that can be regarded as higher-dimensional generalisations of the Bergman metric described above. Effective divisors on the subset where the metrics are regular represent line bundles that do not admit independent holomorphic sections (with different divisors of zeroes). In contrast, in the language of algebraic geometry [1], the induced metric of dissolving vortices is degenerate over *special* effective divisors, which run or move in nontrivial linear systems. The directions of degeneracy on $\Sigma^{(k)}$ are precisely those along the complete linear system associated with a special divisor D. The sets of special divisors D, sitting on exceptional fibres of the Abel–Jacobi map, are complex projective spaces whose dimension ℓ can be related to sheaf cohomology via the Riemann–Roch theorem [11]:

$$\begin{aligned}
\ell &= \dim_{\mathbb{C}} \mathbb{P}(H^0(\Sigma, \mathcal{O}(D))) && (3.2)\\
&= \dim_{\mathbb{C}} H^1(\Sigma, \mathcal{O}(D)) + \deg D - g + 1 - 1 && (3.3)\\
&= \dim_{\mathbb{C}} H^1(\Sigma, \mathcal{O}(D)) + k - g\,. && (3.4)
\end{aligned}$$

The divisor D is special precisely when the following strict inequality holds:

$$\dim_{\mathbb{C}} H^1(\Sigma, \mathcal{O}(D)) = \dim_{\mathbb{C}} H^0(\Sigma, \mathcal{O}(K_\Sigma - D))^* > g - k\,. \tag{3.5}$$

The relations among the geometry of linear systems on Σ, exceptional fibres of the Abel–Jacobi map, and singularities of the subvarieties $W_k \subset \mathrm{Jac}(\Sigma)$ are summarised in the beautiful Riemann–Kempf theorem, which essentially says that a point $w \in W_k$ is a singularity of multiplicity $\binom{g-k+\ell}{\ell}$, its tangent cone being the union of images of the tangent spaces $\mathrm{T}_D\Sigma^{(k)}$ by the differential of the Abel–Jacobi map, where the effective divisor D runs over the complete linear system associated with (i.e. is the fibre over) w. The subvarieties $W_k \subset \mathrm{Jac}(\Sigma)$ are locally given by determinantal equations, and their structure is an important topic in the modern algebraic geometry of curves [1].

In particular, the answer to the natural question of whether $W_k(\Sigma) \subset \mathrm{Jac}(\Sigma)$ will happen to be singular or not (i.e. whether special divisors exist) depends on k, the genus g of Σ and the complex structure on Σ, and it is part of a rich subject that goes under the name of Brill–Noether theory [1]. A sufficient condition for existence of singularities is given by in following result:

Theorem 3.1. *If the inequality $k \geq \frac{g}{2} + 1$ is satisfied, then W_k is a singular algebraic variety, irrespective of the complex structure of Σ.*

The first proof of this statement was presented by Meis [24] and used complex analysis on Teichmüller spaces, resorting to certain specific models of Riemann surfaces in separate cases of odd and even genus. Subsequently, a number of more conceptual and algebraic proofs were given, some of them generalising Meis's result to linear systems of higher dimension. Kleiman and Laksov constructed a very clean proof [16] that should appeal most to singularity theorists. It makes crucial use of Porteous' formula [27] for the Thom polynomial giving the class of the scheme parametrising special divisors inside the Chow ring of $\Sigma^{(k)}$, under weaker assumptions [15] than the transversality conditions assumed in the original paper [27].

To illustrate more concretely the behaviour of the Abel–Jacobi map for $k > 1$ and the structure of its image W_k as a complex k-fold inside the Jacobian, we briefly describe examples of the possible behaviours at low vortex number k. Typically, the qualitative behaviour at a given genus depends crucially on the complex structure of Σ, e.g. on whether Σ is hyperelliptic, and on what kind of linear systems the geometry of Σ allows. Needless to say, the situations at higher k and g will be considerably more complicated than these examples. For more information, the reader is referred to the textbooks [1, 25].

Example 3.2. For $k = 2$, the lowest-genus case where (3.1) is satisfied is $g = 3$. Note that these values of k and g do not obey the inequality in Theorem 3.1. In this situation there are two subcases. If Σ is a nonhyperelliptic curve (the generic situation), the image $W_2 \subset \mathrm{Jac}(\Sigma)$ of the Abel–Jacobi map is smooth, and just a copy of the moduli space $\mathcal{M}_2 = \Sigma^{(2)}$ inside the Jacobian. In fact, this is the only case with $k > 1$ where the 2-form $\mathrm{AJ}_2^*\Omega_J$ is globally nondegenerate on a Θ-divisor, and the dissolving limit metric is regular everywhere. If $g = 3$ but Σ is hyperelliptic, then W_2 already has a singularity. W_2 is the singular complex surface got from the smooth surface $\Sigma^{(2)}$ by blowing down a copy of $\mathbb{CP}^1$ to a point, which is a double point in W_2 [28]. The exceptional $\mathbb{CP}^1$ fibre that is blown down is the pencil of degree two divisors that are orbits of the hyperelliptic involution (a g_2^1); the space of orbits is the quotient of Σ by the hyperelliptic involution, which is a $\mathbb{CP}^1$ that embeds in $\Sigma^{(2)}$ holomorphically with noncontractible image. This exceptional fibre has an analogue for any moduli space of 2-vortices on a hyperelliptic curve Σ [5].

Example 3.3. If $k = 3$, the simplest situation requires $g = 4$. Since $3 \geq \frac{4}{2} + 1$, Theorem 3.1 guarantees that W_3 will always contain singularities. In fact, there are three subcases to consider. If Σ is not hyperelliptic, one can show that it can be obtained as an intersection of a quadric Q and a cubic C in $\mathbb{CP}^3$. The first subcase is when Q is smooth, hence biholomorphic to $\mathbb{CP}^1 \times \mathbb{CP}^1$. Then C meets each projective line of the form $\{P_1\} \times \mathbb{CP}^1$ or $\mathbb{CP}^1 \times \{P_2\}$ in Q at three points, so $\Sigma = Q \cap C$ projects to either of the two $\mathbb{CP}^1$ factors of Q as a 3-cover. The pre-images of points in $\mathbb{CP}^1$ by the two projections form effective divisors of degree 3 moving in two pencils (i.e. parametrised by two projective lines), and describe two copies F_1, F_2 of $\mathbb{CP}^1$ inside $\Sigma^{(3)}$, which are g_3^1's on Σ . These are the exceptional fibres of the Abel–Jacobi map. The image W_3 can be obtained by blowing down these rational curves F_1, F_2 to two points, which are ordinary double points of the 3-fold. The second subcase is when Σ is not hyperelliptic, hence $\Sigma = Q \cap C$ as before, but now Q is singular (a quadric cone); then Q can be described as a family of projective lines parametrised by a $\mathbb{CP}^1$ and all meeting at the singular point. Each line in the family again meets C at three points, and so Σ inherits one pencil of degree 3 effective divisors (a g_3^1), which is the only exceptional fibre of the Abel–Jacobi map. The image W_3 in this case is again got by blowing down this $\mathbb{CP}^1$ fibre, and this results in a double point in the 3-fold which has higher multiplicity. The third and last subcase occurs when Σ is hyperelliptic. The exceptional fibres here form a complex surface inside $\Sigma^{(3)}$, namely, the locus of effective divisors on Σ consisting of adding any point of Σ to the $\mathbb{CP}^1$ of hyperelliptic orbits described in the previous example; this can be described as a family of pencils (i.e. g_3^1's) parametrised by Σ. Then W_3 is obtained from $\Sigma^{(3)}$ by blowing down this surface to a curve isomorphic to Σ.

4. Dissolving multivortices near a singularity

One peculiar aspect of the geometry of multivortices is the degeneration of the underlying Kähler structures at the singularities of the Abel–Jacobi map, as described above, and this will be our focus in the present section. To understand the behaviour of the geodesic flow in a neighbourhood of a singularity, we shall analyse in detail the simplest situation, which occurs in the scattering of two dissolving vortices on a hyperelliptic Riemann surface of genus three.

We start by recalling that the image W_2 of the Abel–Jacobi map for degree two effective divisors

$$\mathrm{AJ}_2 : \Sigma^{(2)} \longrightarrow \mathrm{Jac}(\Sigma) \tag{4.1}$$

on a hyperelliptic curve Σ with $g = 3$ has a double point, whose blow-up is the exceptional fibre in $\Sigma^{(2)}$, which is a projective line (see Example 3.2). This fact essentially goes back to Klein [17]; see e.g. [7] for a modern perspective. Since we are only interested in the leading local behaviour near this critical locus, we will not need to use theta-functions, and will instead take the standard algebraic model

$$t_3^2 = t_1 t_2 \tag{4.2}$$

for the double point, using local coordinates $t_i : U \to \mathbb{C}$ centred at the singularity; so (4.2) gives a local equation for the image of $W_2 \cap U \subset \mathrm{Jac}(\Sigma)$ under the coordinate system, which we may regard as a hypersurface W_2' in an open neighbourhood U' of the origin of $\mathbb{C}^3$. Now we blow up $(0,0,0) \in U'$, to obtain a 3-fold $\widetilde{U}'$ together with a holomorphic map $\pi : \widetilde{U'} \to U'$ which has $\pi^{-1}(0,0,0) = \mathbb{P}(\mathrm{T}_{(0,0,0)}U') \cong \mathbb{CP}^2$ but is one-to-one everywhere else. For the benefit of the reader, we recall how this is constructed [2].

The manifold $\widetilde{U}'$ can be regarded as the subset of $U' \times \mathbb{CP}^2$ defined by the incidence relation

$$t_i v_j = t_j v_i \qquad \text{for all} \quad i, j \in \{1,2,3\} \tag{4.3}$$

where v_j are homogeneous coordinates on the projectivisation $\mathbb{CP}^2$ of the tangent space at the origin, and the map π is simply the projection $\mathrm{pr}_{U'}$ onto the first factor. In the open set of $U' \times \mathbb{CP}^2$ where $v_3 \neq 0$, for example, $\widetilde{U}'$ is described by the system of equations

$$t_1 = \frac{v_1}{v_3} t_3, \quad t_2 = \frac{v_2}{v_3} t_3 \tag{4.4}$$

which has constant rank 2, and this determines a 3-dimensional submanifold. Since the incidence relation (4.3) is trivially satisfied for $(t_1, t_2, t_3) = (0,0,0)$, we get indeed the whole of the $\mathbb{CP}^2$ factor as exceptional fibre.

Imposing the equation (4.2), we obtain a surface $\widetilde{W_2'} \cap \widetilde{U}'$ which is smooth; the singularity is replaced by the conic $v_3^2 = v_1 v_2$ in the exceptional fibre $\mathbb{CP}^2$, which is itself a projective line $\mathbb{CP}^1$, and the restriction

$$\pi|_{\widetilde{W_2'} \cap \widetilde{U}'} : \widetilde{W_2'} \cap \widetilde{U}' \to W_2' \cap U' \tag{4.5}$$

provides a local resolution of the double point on the surface. To find the resolution map explicitly, we should use a system of two local coordinates where a dense subset of the exceptional fibre is visible; for example, an affine coordinate on the $\mathbb{CP}^1$ factor, say $q = \frac{v_3}{v_1}$, together with one of the coordinates on the first factor, say $p = t_1$. In these coordinates, the projection is given by

$$(p, q) \mapsto (t_1, t_2, t_3) = (p, pq^2, pq) \in U' \,. \tag{4.6}$$

Working on such local patches, it is not hard to see that the projection of $\widetilde{W_2'} \cap \widetilde{U}'$ onto the second factor of $\widetilde{U}' \times \mathbb{CP}^2$ can be understood as a restriction of the standard projection

$$\mathrm{T}^*\mathbb{CP}^1 \longrightarrow \mathbb{CP}^1 \tag{4.7}$$

to a neighbourhood of the (image of the) zero section, which gives a very concrete picture of the resolution. The exceptional fibre of AJ_2 is identified with the zero section, parametrised by q, and our complex coordinate p parametrises the cotangent fibres.

We want to understand the effect of pulling back a Kähler metric on U' to the blow-up $\widetilde{U}'$, and in particular the behaviour of the geodesic flow near the exceptional fibre where the metric becomes degenerate. The Kähler metric we consider is the standard euclidean metric on U', $g_0 = |\mathrm{d}t_1|^2 + |\mathrm{d}t_2|^2 + |\mathrm{d}t_3|^2$, as the qualitative behaviour of the flow will not depend on anisotropy factors. Pulling back to $\widetilde{U}'$ we obtain

$$\begin{aligned} \widetilde{g} = \pi^* g_0 &= (1+|q|^2+|q|^4)\mathrm{d}p\,\mathrm{d}\bar{p} + |p|^2(1+4|q|^2)\mathrm{d}q\,\mathrm{d}\bar{q} \\ &\quad + \bar{p}q(1+2|q|^2)\mathrm{d}p\,\mathrm{d}\bar{q} + p\bar{q}(1+2|q|^2)\mathrm{d}q\,\mathrm{d}\bar{p}\,. \end{aligned} \tag{4.8}$$

As expected, this tensor defines a Kähler metric in the complement of the complex line with equation $p = 0$, but its rank (over $\mathbb{R}$) drops from 4 to 2 on this line, which corresponds to an affine piece of the exceptional $\mathbb{CP}^1$ fibre of the Abel–Jacobi map. To understand the geodesic flow, we should first compute the Christoffel symbols. For a Kähler metric this calculation simplifies, and moreover Christoffel symbols mixing holomorphic and anti-holomorphic directions automatically vanish [2]. We find:

$$\widetilde{\Gamma}^q_{pq} = \widetilde{\Gamma}^q_{qp} = \frac{1}{p}, \qquad \widetilde{\Gamma}^p_{pq} = \widetilde{\Gamma}^p_{qp} = \widetilde{\Gamma}^q_{pp} = \widetilde{\Gamma}^p_{pp} = 0\,, \tag{4.9}$$

$$\widetilde{\Gamma}^p_{qq} = -\frac{2p\bar{q}^2}{1+4|q|^2+|q|^4}\,, \qquad \widetilde{\Gamma}^q_{qq} = \frac{2\bar{q}(2+|q|^2)}{1+4|q|^2+|q|^4}\,. \tag{4.10}$$

These lead to the following geodesic equations:

$$\ddot{p} - \frac{2p\bar{q}^2\dot{q}^2}{1+4|q|^2+|q|^4} = 0\,, \tag{4.11}$$

$$\ddot{q} + \frac{2\dot{p}\dot{q}}{p} + \frac{2\bar{q}(2+|q|^2)\dot{q}^2}{1+4|q|^2+|q|^4} = 0\,, \tag{4.12}$$

where the derivatives are with respect to a parameter s, say.

An obvious integral of motion is the kinetic energy of the geodesic flow (up to a constant factor),

$$(1+|q|^2+|q|^4)|\dot{p}|^2 + |p|^2(1+4|q|^2)|\dot{q}|^2 + (1+2|q|^2)(\bar{p}q\dot{p}\dot{\bar{q}} + p\bar{q}\dot{\bar{p}}\dot{q})\,, \tag{4.13}$$

and there are further integrals of motion arising from the invariance of $\widetilde{g}$ under phase rotations of p and of q. The conservation of the kinetic energy already implies that the motion on the exceptional fibre $\mathbb{CP}^1$ (parametrised by the coordinate q) is suppressed in its tangent directions: as $p \to 0$, all the kinetic energy must be transferred to motion along the transverse directions parametrised by the complex coordinate p. In particular, any geodesic intersecting the exceptional fibre must do so at isolated points of the fibre.

To demonstrate that there are indeed geodesics crossing the exceptional fibre, we note that the geodesic equations above are satisfied by the rays of the tangent cone to W_2', i.e. paths of the form $s \mapsto (p,q) = (c_1 s, c_2)$ for constants $c_1 \in \mathbb{C}^*$ and $c_2 \in \mathbb{C}$. These correspond to lifts of real straight lines on U' towards the singularity, which hit a point on the exceptional fibre corresponding to the complex tangent direction their velocity represents, and then continue along the same real direction. Since the exceptional fibre is reached in finite time, the metric on the complement of the exceptional fibre in $\widetilde{W_2}$ is not complete.

In fact, such straight ray geodesics are the only geodesics reaching the exceptional fibre $\mathbb{CP}^1$. To see this, note first that, as long as $\dot{p}$ is not constant, (4.11) implies that $\dot{q}$ cannot be zero. Dividing equation (4.12) by $\dot{q}$ (assumed to be nonzero) and extracting the real part of the resulting equation, we obtain a new differential equation,

$$\frac{\ddot{q}}{\dot{q}} + \frac{\ddot{\bar{q}}}{\dot{\bar{q}}} + \frac{2\dot{p}}{p} + \frac{2\dot{\bar{p}}}{\bar{p}} + \frac{2(2+|q|^2)(\bar{q}\dot{q} + q\dot{\bar{q}})}{1+4|q|^2+|q|^4} = 0, \tag{4.14}$$

which can be integrated to conclude that

$$(1+4|q|^2+|q|^4)|p|^4|\dot{q}|^2 \tag{4.15}$$

is another integral of motion. Thus for p to reach zero, $\dot{q}$ would have to blow up, which cannot happen. Initial conditions that try to reach the exceptional fibre with initial velocities having nontrivial tangent component along the $\mathbb{CP}^1$ will be forced to flow rapidly around this 2-sphere as they approach it transversely.

In terms of vortex motion, the effect of the singularity is that motion along the special linear system is suppressed. So whenever two vortices reach points on the surface that are related by the hyperelliptic involution, they will be unable to move to neighbouring pairs of points that are also related by the involution. In particular, in the dissolving limit it will be impossible to make vortices collide head-on onto a Weierstraß point of the surface: these are precisely the branch points of the two-fold holomorphic branched cover $\sigma : \Sigma \to \mathbb{CP}^1$, and geodesics through them are tangentially preserved by the hyperelliptic involution near the branch point. More precisely, we know from the discussion above that the only geodesics through the $\mathbb{CP}^1$ with equation $p = 0$ must cross with $\dot{q} = 0$, whereas we have:

Proposition 4.1. *A frontal collision of two vortices at a fixed point $W \in \Sigma$ of the hyperelliptic involution occurs at right angles and with $\dot{q} \neq 0$.*

Proof. Let $z \in \mathcal{O}_{\Sigma,W}$ denote a local parameter in Σ at the point W, a generator of the maximal ideal $\mathfrak{n}_{\Sigma,W}$ in the local ring [20]. We have been using q to denote any coordinate on the exceptional fibre $\mathbb{CP}^1$ of the Abel–Jacobi map, and now we shall also assume without loss of generality that its image in the local ring $\mathcal{O}_{\mathbb{CP}^1,\sigma(W)}$ is a local parameter. Since the map σ has ramification index two at W, one has $\sigma^* q = uz^2$ for some unit $u \in \mathcal{O}^\times_{\Sigma,W}$.

We denote by Δ the natural embedding via the diagonal inclusion

$$\Delta : \Sigma \hookrightarrow \Sigma \times \Sigma \xrightarrow{\tilde{\pi}} \Sigma^{(2)} = \Sigma^2/\mathfrak{S}_2.$$

Note that z induces local parameters z_1, z_2 in Σ^2 at (W,W) in the obvious way, and from them one obtains a system of local parameters s_1, s_2 on $\Sigma^{(2)}$ at $\Delta(W)$ via the fundamental theorem on symmetric functions, i.e. the map of local rings induced by $\tilde{\pi}$ relates $\tilde{\pi}^*(s_1) = z_1 + z_2, \tilde{\pi}^*(s_2) = z_1 z_2$. The image $\Delta(\Sigma)$ is described by the equation $s_1^2 - 4s_2 = 0$ locally at $\Delta(W)$ in $\Sigma^{(2)}$, and we can compute

$$\begin{aligned} \mathrm{T}^*_{\Delta(W)}\Delta(\Sigma) &= \mathfrak{n}_{\Sigma^{(2)},\Delta(W)}/(\mathfrak{n}^2_{\Sigma^{(2)},\Delta(W)} + (s_1^2 - 4s_2)) \\ &= (s_1, s_2)/(s_1^2, s_2^2, s_1 s_2, s_1^2 - 4s_2) \\ &= (s_1)/(s_1^2). \end{aligned}$$

Let $\iota : \mathbb{CP}^1 \hookrightarrow \Sigma^{(2)}$ denote the inclusion of the g^1_2. It induces a surjective map of local rings $\iota^* : \mathcal{O}_{\Sigma^{(2)},\Delta(W)} \to \mathcal{O}_{\mathbb{CP}^1,\sigma(W)}$. Since the intersection of the images $\Delta(\Sigma)$ and $\iota(\mathbb{CP}^1)$ is transverse at $\Delta(W) \in \Sigma^{(2)}$, the calculation above implies that $\iota^* s_2$ must be a local parameter; so there is also a unit $v \in \mathcal{O}^\times_{\mathbb{CP}^1,\sigma(W)}$ with $\iota^* s_2 = vq$. Hence we obtain in $\mathcal{O}_{\Sigma,W}$

$$z^2 = \tilde{u}\,(\iota \circ \sigma)^* s_2 \tag{4.16}$$

with $\tilde{u} = u\,\sigma^* v \in \mathcal{O}^\times_{\Sigma,W}$.

A collision of two vortices at $W \in \Sigma$ can be described by a parametrisation $t \mapsto (z_1(t), z_2(t))$ with $t \in (-\epsilon, \epsilon)$, $\epsilon > 0$ and $z_1(0) = z_2(0) = 0$; the collision is frontal if moreover $\dot{z}_1(0) = -\dot{z}_2(0)$, which implies $\dot{s}_1(0) = 0$. Then necessarily $\dot{s}_2(0) \neq 0$. From equation (4.16) we obtain infinitesimally close positions of the vortices by taking square roots, which justifies the assertion on the scattering at right angles. (We note in passing that scattering at right angles is a well-known feature of the frontal scattering of vortices for regular L^2-metrics on their moduli spaces [12].) Finally, we obtain in the local ring at $\sigma(W)$ (or the pull-back to $\Sigma^{(2)}$)

$$\dot{q}(0) = \dot{v}(0)s_2(0) + v(0)\dot{s}_2(0) = v(0)\dot{s}_2(0) \neq 0.$$

□

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SYMPLECTIC W_8 AND W_9 SINGULARITIES

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Abstract. We use the method of algebraic restrictions to classify symplectic W_8 and W_9 singularities. We use discrete symplectic invariants to distinguish symplectic singularities of the curves. We also give the geometric description of symplectic classes.

1. Introduction

In this paper we examine the singularities which are in the list of the simple 1-dimensional isolated complete intersection singularities in the space of dimension greater than 2, obtained by Giusti ([G], [AVG]). Isolated complete intersection singularities (ICIS) were intensively studied by many authors (e. g. see [L]), because of their interesting geometric, topological and algebraic properties. Here using the method of algebraic restrictions we obtain the complete symplectic classification of the singularities of type W_8 and W_9. We calculate discrete symplectic invariants for symplectic orbits of the curves and we give their geometric description. It allows us to explore the specific singular nature of these classical singularities that only appears in the presence of the symplectic structure.

We study the symplectic classification of singular curves under the following equivalence:

Definition 1.1. Let N_1, N_2 be two germs of subsets of the symplectic space $(\mathbb{R}^{2n}, \omega)$. N_1, N_2 are **symplectically equivalent** if there exists a symplectomorphism-germ $\Phi : (\mathbb{R}^{2n}, \omega) \to (\mathbb{R}^{2n}, \omega)$ such that $\Phi(N_1) = N_2$.

We recall that ω is a symplectic form if ω is a smooth nondegenerate closed 2-form, and $\Phi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is a symplectomorphism if Φ is a diffeomorphism and $\Phi^*\omega = \omega$.

Symplectic classification of curves was first studied by V. I. Arnold. In [A1] and [A2] the author studied singular curves in symplectic and contact spaces and introduced the local symplectic and contact algebra. In [A2] V. I. Arnold discovered new symplectic invariants of singular curves. He proved that the A_{2k} singularity of a planar curve (the orbit with respect to standard $\mathcal{A}$-equivalence of parameterized curves) split into exactly $2k+1$ symplectic singularities (orbits with respect to symplectic equivalence of parameterized curves). He distinguished different symplectic singularities by different orders of tangency of the parameterized curve to the *nearest* smooth Lagrangian submanifold. Arnold posed a problem of expressing these invariants in terms of the local algebra's interaction with the symplectic structure and he proposed calling this interaction 'the local symplectic algebra'.

In [IJ1] G. Ishikawa and S. Janeczko classified symplectic singularities of curves in the 2-dimensional symplectic space. All simple curves in this classification are quasi-homogeneous.

We recall that a subset N of $\mathbb{R}^m$ is **quasi-homogeneous** if there exist a coordinate system $(x_1, \cdots, x_m)$ on $\mathbb{R}^m$ and positive numbers $w_1, \cdots, w_m$ (called weights) such that for any point

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$(y_1, \cdots, y_m) \in \mathbb{R}^m$ and any $t > 0$ if $(y_1, \cdots, y_m)$ belongs to N then the point $(t^{w_1} y_1, \cdots, t^{w_m} y_m)$ belongs to N.

The generalization of results in [IJ1] to volume-preserving classification of singular varieties and maps in arbitrary dimensions was obtained in [DR]. A symplectic form on a 2-dimensional manifold is a special case of a volume form on a smooth manifold.

In [K] P. A. Kolgushkin classified the stably simple symplectic singularities of parameterized curves (in the $\mathbb{C}$-analytic category). Symplectic singularity is stably simple if it is simple and if it remains simple if the ambient symplectic space is symplectically embedded (i.e. as a symplectic submanifold) into a larger symplectic space.

In [Z] was developed the local contact algebra. The main results were based on the notion of the algebraic restriction of a contact structure to a subset N of a contact manifold.

In [DJZ2] new symplectic invariants of singular quasi-homogeneous subsets of a symplectic space were explained by the algebraic restrictions of the symplectic form to these subsets.

The algebraic restriction is an equivalence class of the following relation on the space of differential k-forms:

Differential k-forms ω_1 and ω_2 have the same **algebraic restriction** to a subset N if $\omega_1 - \omega_2 = \alpha + d\beta$, where α is a k-form vanishing on N and β is a $(k-1)$-form vanishing on N.

The generalization of the Darboux-Givental theorem ([AG]) to germs of arbitrary subsets of the symplectic space was obtained in [DJZ2]. This result reduces the problem of symplectic classification of germs of quasi-homogeneous subsets to the problem of classification of algebraic restrictions of symplectic forms to these subsets. For non-quasi-homogeneous subsets there is one more cohomological invariant except the algebraic restriction ([DJZ2], [DJZ1]). The dimension of the space of algebraic restrictions of closed 2-forms to a 1-dimensional quasi-homogeneous isolated complete intersection singularity C is equal to the multiplicity of C ([DJZ2]). In [D] it was proved that the space of algebraic restrictions of closed 2-forms to a 1-dimensional (singular) analytic variety is finite-dimensional. In [DJZ2] the method of algebraic restrictions was applied to various classification problems in a symplectic space. In particular the complete symplectic classification of classical A-D-E singularities of planar curves and S_5 singularity were obtained. Most of different symplectic singularity classes were distinguished by new discrete symplectic invariants: the index of isotropy and the symplectic multiplicity.

In [DT1] following ideas from [A1] and [D] new discrete symplectic invariants - the Lagrangian tangency orders were introduced and used to distinguish symplectic singularities of simple planar curves of type A-D-E, symplectic T_7 and T_8 singularities.

In [DT2] was obtained the complete symplectic classification of the isolated complete intersection singularities S_μ for $\mu > 5$.

In this paper we obtain the detailed symplectic classification of W_8 and W_9 singularities. The paper is organized as follows. In Section 2 we recall discrete symplectic invariants (the symplectic multiplicity, the index of isotropy and the Lagrangian tangency orders). Symplectic classification of W_8 and W_9 singularity is presented in Sections 3 and 4 respectively. The symplectic sub-orbits of this singularities are listed in Theorems 3.1 and 4.1. Discrete symplectic invariants for the symplectic classes are calculated in Theorems 3.2 and 4.2. The geometric descriptions of the symplectic orbits is presented in Theorems 3.5 and 4.4. In Section 5 we recall the method of algebraic restrictions and use it to classify symplectic singularities.

2. Discrete symplectic invariants.

We can use discrete symplectic invariants to characterize symplectic singularity classes.

The first invariant is the symplectic multiplicity ([DJZ2]) introduced in [IJ1] as a symplectic defect of a curve.

Let N be a germ of a subset of $(\mathbb{R}^{2n}, \omega)$.

Definition 2.1. The **symplectic multiplicity** $\mu^{sym}(N)$ of N is the codimension of a symplectic orbit of N in an orbit of N with respect to the action of the group of local diffeomorphisms.

The second invariant is the index of isotropy [DJZ2].

Definition 2.2. The **index of isotropy** $ind(N)$ of N is the maximal order of vanishing of the 2-forms $\omega|_{TM}$ over all smooth submanifolds M containing N.

This invariant has geometrical interpretation. An equivalent definition is as follows: the index of isotropy of N is the maximal order of tangency between non-singular submanifolds containing N and non-singular isotropic submanifolds of the same dimension. The index of isotropy is equal to 0 if N is not contained in any non-singular submanifold which is tangent to some isotropic submanifold of the same dimension. If N is contained in a non-singular Lagrangian submanifold then the index of isotropy is ∞.

The symplectic multiplicity and the index of isotropy can be expressed in terms of algebraic restrictions (Propositions 5.6 and 5.7 in Section 5).

There is one more discrete symplectic invariant, introduced in [D] (following ideas from [A2]) which is defined specifically for a parameterized curve. This is the maximal tangency order of a curve $f : \mathbb{R} \to M$ to a smooth Lagrangian submanifold. If $H_1 = ... = H_n = 0$ define a smooth submanifold L in the symplectic space then the tangency order of a curve $f : \mathbb{R} \to M$ to L is the minimum of the orders of vanishing at 0 of functions $H_1 \circ f, \cdots, H_n \circ f$. We denote the tangency order of f to L by $t(f, L)$.

Definition 2.3. The **Lagrangian tangency order** $Lt(f)$ **of a curve** f is the maximum of $t(f, L)$ over all smooth Lagrangian submanifolds L of the symplectic space.

The Lagrangian tangency order of a quasi-homogeneous curve in a symplectic space can also be expressed in terms of the algebraic restrictions (Proposition 5.8 in Section 5).

In [DT1] the above invariant was generalized for germs of curves and multi-germs of curves which may be parameterized analytically since the Lagrangian tangency order is the same for every 'good' analytic parametrization of a curve.

Consider a multi-germ $(f_i)_{i\in\{1,\cdots,r\}}$ of analytically parameterized curves f_i. We have r-tuples $(t(f_1, L), \cdots, t(f_r, L))$ for any smooth submanifold L in the symplectic space.

Definition 2.4. For any $I \subseteq \{1, \cdots, r\}$ we define **the tangency order of the multi-germ** $(f_i)_{i\in I}$ to L:

$$t[(f_i)_{i\in\ I}, L] = \min_{i\in\ I} t(f_i, L).$$

Definition 2.5. The **Lagrangian tangency order** $Lt((f_i)_{i\in\ I})$ **of a multi-germ** $(f_i)_{i\in I}$ is the maximum of $t[(f_i)_{i\in\ I}, L]$ over all smooth Lagrangian submanifolds L of the symplectic space.

3. Symplectic W_8-singularities

Denote by (W_8) the class of varieties in a fixed symplectic space $(\mathbb{R}^{2n}, \omega)$ which are diffeomorphic to

$$W_8 = \{x \in \mathbb{R}^{2n\geq 4}\ : x_1^2 + x_3^3 = x_2^2 + x_1x_3 = x_{\geq 4} = 0\}. \tag{3.1}$$

This is the simple 1-dimensional isolated complete intersection singularity W_8 ([G], [AVG][1]). Here N is quasi-homogeneous with weights $w(x_1) = 6,\ w(x_2) = 5,\ w(x_3) = 4$.

We used the method of algebraic restrictions to obtain the complete classification of symplectic singularities of (W_8) presented in the following theorem.

Theorem 3.1. *Any submanifold of the symplectic space $(\mathbb{R}^{2n}, \sum_{i=1}^{n} dp_i \wedge dq_i)$ where $n \geq 3$ (respectively, $n = 2$) which is diffeomorphic to W_8 is symplectically equivalent to one and only one of the normal forms $W_8^i, i = 0, 1, \cdots, 8$ (respectively, $i = 0, 1, 2a, 2b$) listed below. The parameters c, c_1, c_2 of the normal forms are moduli:*

$W_8^0:\ p_1^2 + p_2q_1 = 0,\ \ p_2^2 + q_1^3 = 0,\ \ q_2 = c_1q_1 + c_2p_1,\ \ p_{\geq 3} = q_{\geq 3} = 0;$

$W_8^1:\ q_1^2 + p_1q_2 = 0,\ \ p_1^2 + q_2^3 = 0,\ \ p_2 = c_1p_1 + c_2q_1q_2,\ \ p_{\geq 3} = q_{\geq 3} = 0,\ c_1 \neq 0;$

$W_8^{2a}:\ p_2^2 \pm p_1q_1 = 0,\ \ p_1^2 + q_1^3 = 0,\ \ q_2 = \frac{c_1}{2}q_1^2 + \frac{c_2}{3}q_1^3,\ \ p_{\geq 3} = q_{\geq 3} = 0;$

$W_8^{2b}:\ q_1^2 + p_1q_2 = 0,\ \ p_1^2 + q_2^3 = 0,\ \ p_2 = c_1q_1q_2 + \frac{c_2}{2}q_1^2,\ \ p_{\geq 3} = q_{\geq 3} = 0;$

$W_8^3:\ p_2^2 + p_1p_3 = 0,\ p_1^2 + p_3^3 = 0,\ q_1 = q_2 = 0, q_3 = -p_2p_3 - \frac{c_1}{2}p_2^2 - c_2p_1p_2,\ p_{>3} = q_{>3} = 0;$

$W_8^4:\ p_2^2 + p_1p_3 = 0,\ p_1^2 + p_3^3 = 0,\ q_1 = q_2 = 0, q_3 = \mp\frac{1}{2}p_2^2 - c_1p_1p_2 - c_2p_2p_3^2,\ p_{>3} = q_{>3} = 0;$

$W_8^5:\ p_2^2 + p_1p_3 = 0,\ p_1^2 + p_3^3 = 0,\ q_1 = q_2 = 0, q_3 = -p_1p_2 - cp_2p_3^2,\ p_{>3} = q_{>3} = 0;$

$W_8^6:\ p_2^2 + p_1p_3 = 0,\ p_1^2 + p_3^3 = 0,\ q_1 = q_2 = 0, q_3 = -p_2p_3^2 - \frac{c}{3}p_2^3,\ p_{>3} = q_{>3} = 0;$

$W_8^7:\ p_2^2 + p_1p_3 = 0,\ p_1^2 + p_3^3 = 0,\ q_1 = q_2 = 0, q_3 = -\frac{1}{3}p_2^3,\ p_{>3} = q_{>3} = 0;$

$W_8^8:\ p_2^2 + p_1p_3 = 0,\ p_1^2 + p_3^3 = 0,\ p_{>3} = q_{>0} = 0.$

In Section 3.1 we use the symplectic invariants (in particular the Lagrangian tangency order) to distinguish the symplectic singularity classes. In Section 3.2 we propose a geometric description of these singularities that confirms the classification. Some of the proofs are presented in Section 5.

3.1. **Distinguishing symplectic classes of W_8 by the Lagrangian tangency order and the index of isotropy.** A curve $N \in (W_8)$ can be described as a parametrical curve $C(t)$. Its parametrization is given in the second column of Table 1. To characterize the symplectic classes we use the following invariants:

- $L_N = Lt(N) = \max_L(t(C(t), L))$;
- ind - the index of isotropy of N.

Here L is a smooth Lagrangian submanifold of the symplectic space.

Theorem 3.2. *A stratified submanifold $N \in (W_8)$ of a symplectic space $(\mathbb{R}^{2n}, \omega_0)$ with the canonical coordinates $(p_1, q_1, \cdots, p_n, q_n)$ is symplectically equivalent to one and only one of the curves presented in the second column of Table 1. The parameters c, c_1, c_2 are moduli. The index of isotropy and the Lagrangian tangency order of the curve N are presented in the third and fourth column of Table 1.*

Remark 3.3. The invariants can be calculated by knowing the algebraic restrictions for the symplectic classes. We use Proposition 5.7 to calculate the index of isotropy. The Lagrangian tangency order we can calculate using Proposition 5.8 or by applying directly the definition of the Lagrangian tangency order and finding a Lagrangian submanifold the nearest to the curve $C(t)$.

[1] There is a mistake in the description of W_8 singularity in [AVG]. We find there $W_8 = \{x \in \mathbb{R}^{2n \geq 4} : x_1^2 + x_2^3 = x_2^2 + x_1x_3 = x_{\geq 4} = 0\}$ which is not an isolated complete intersection singularity.

class		parametrization of N	ind	L_N
$(W_8)^0$	$2n \geq 4$	$(t^5, -t^4, t^6, -c_1t^4 + c_2t^5, 0, \cdots)$	0	5
$(W_8)^1$	$2n \geq 4$	$(t^6, t^5, c_1t^6 - c_2t^9, -t^4, 0, \cdots)$	0	6
$(W_8)^{2a}$	$2n \geq 4$	$(\pm t^6, -t^4, t^5, \frac{c_1}{2}t^8 - \frac{c_2}{3}t^{12}, 0, \cdots)$	0	6
$(W_8)^{2b}$	$2n \geq 4$	$(t^6, t^5, -c_1t^9 + \frac{c_2}{2}t^{10}, -t^4, , 0, \cdots)$	0	6
$(W_8)^3$	$2n \geq 6$	$(t^6, 0, t^5, 0, -t^4, t^9 - \frac{c_1}{2}t^{10} - c_2t^{11}, 0, \cdots)$	1	9
$(W_8)^4$	$2n \geq 6$	$(t^6, 0, t^5, 0, -t^4, \mp t^{10} - c_1t^{11} - c_2t^{13}, 0, \cdots)$	1	10
$(W_8)^5$	$2n \geq 6$	$(t^6, 0, t^5, 0, -t^4, -t^{11} - ct^{13}, 0, \cdots)$	1	11
$(W_8)^6$	$2n \geq 6$	$(t^6, 0, t^5, 0, -t^4, -t^{13} - \frac{c}{3}t^{15}, 0, \cdots)$	2	13
$(W_8)^7$	$2n \geq 6$	$(t^6, 0, t^5, 0, -t^4, -t^{15}, 0, \cdots)$	2	15
$(W_8)^8$	$2n \geq 6$	$(t^6, 0, t^5, 0, -t^4, 0, 0, \cdots)$	∞	∞

TABLE 1. The symplectic invariants for symplectic classes of W_8 singularity.

Remark 3.4. The comparison of invariants presented in Table 1 shows that the Lagrangian tangency order distinguishes more symplectic classes than the index of isotropy. Symplectic classes $(W_8)^{2a}$ and $(W_8)^{2b}$ can not be distinguished by any of the invariants but we can distinguish them by geometric conditions.

3.2. **Geometric conditions for the classes $(W_8)^i$.** We can characterize the symplectic classes $(W_8)^i$ by geometric conditions independent of any local coordinate system. Let $N \in (W_8)$. Denote by W the tangent space at 0 to some (and then any) non-singular 3-manifold containing N. We can define the following subspaces of this space: ℓ − the tangent line at 0 to the curve N, V − the 2-space tangent at 0 to the curve N. For $N = W_8 =$(3.1) it is easy to calculate

$$W = \text{span}(\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3), \ \ell = \text{span}(\partial/\partial x_3), \ V = \text{span}(\partial/\partial x_2, \partial/\partial x_3). \tag{3.2}$$

The classes $(W_8)^i$ satisfy special conditions in terms of the restriction $\omega|_W$, where ω is the symplectic form.

Theorem 3.5. *If a stratified submanifold $N \in (W_8)$ of a symplectic space $(\mathbb{R}^{2n}, \omega)$ belongs to the class $(W_8)^i$ then the couple (N, ω) satisfies the corresponding conditions in the last column of Table 2.*

Sketch of the proof of Theorem 3.5. We have to show that the conditions in the row of $(W_8)^i$ are satisfied for any $N \in (W_8)^i$.

Each of the conditions in the last column of Table 2 is invariant with respect to the action of the group of diffeomorphisms in the space of pairs (N, ω). Because each of these conditions depends only on the algebraic restriction $[\omega]_N$ we can take the simplest 2-forms ω^i representing the normal forms $[W_8]^i$ for algebraic restrictions: ω^0, ω^1, $\omega^{2,a}$, $\omega^{2,b}$, ω^3, ω^4, ω^5, ω^6, ω^7, ω^8 and we can check that the pair $(W_8, \omega = \omega^i)$ satisfies the condition in the last column of Table 2.

We note that in the case $N = W_8 =$ (3.1) one has the description (3.2) of the subspaces W, ℓ and V. By simple calculation and observation of the Lagrangian tangency order we obtain that the conditions corresponding to the classes $(W_8)^i$ are satisfied. □

class	normal form	geometric conditions
$(W_8)^0$	$[W_8]^0 : [\theta_1 + c_1\theta_2 + c_2\theta_3]_{W_8}$	$\omega\vert_V \neq 0$
$(W_8)^1$	$[W_8]^1 : [c_1\theta_2 + \theta_3 + c_2\theta_4]_{W_8}$, $c_1 \neq 0$	$\omega\vert_V = 0$ and $\ker\omega \neq \ell$
$(W_8)^{2a}$	$[W_8]^{2a} : [\pm\theta_2 + c_1\theta_4 + c_2\theta_7]_{W_8}$	$\omega\vert_V = 0$ and $\ker\omega \neq \ell$
$(W_8)^{2b}$	$[W_8]^{2b} : [\theta_3 + c_1\theta_4 + c_2\theta_5]_{W_8}$	$\omega\vert_V = 0$ and $\ker\omega = \ell$
		$\omega\vert_W = 0$
$(W_8)^3$	$[W_8]^3 : [\theta_4 + c_1\theta_5 + c_2\theta_6]_{W_8}$	$L_N = 9$
$(W_8)^4$	$[W_8]^4 : [\pm\theta_5 + c_1\theta_6 + c_2\theta_7]_{W_8}$	$L_N = 10$
$(W_8)^5$	$[W_8]^5 : [\theta_6 + c\theta_7]_{W_8}$	$L_N = 11$
$(W_8)^6$	$[W_8]^6 : [\theta_7 + c\theta_8]_{W_8}$	$L_N = 13$
$(W_8)^7$	$[W_8]^7 : [\theta_8]_{W_8}$	$L_N = 15$
$(W_8)^8$	$[W_8]^8 : [0]_{W_8}$	N is contained in a smooth Lagrangian submanifold

TABLE 2. Geometric interpretation of singularity classes of W_8. (W is the tangent space to a non-singular 3-dimensional manifold in $(\mathbb{R}^{2n\geq 4}, \omega)$ containing $N \in (W_8)$. The forms $\theta_1, \ldots, \theta_8$ are described in Theorem 5.10 on the page 168.)

4. Symplectic W_9-singularities

Denote by (W_9) the class of varieties in a fixed symplectic space $(\mathbb{R}^{2n}, \omega)$ which are diffeomorphic to

$$W_9 = \{x \in \mathbb{R}^{2n\geq 4} : x_1^2 + x_2x_3^2 = x_2^2 + x_1x_3 = x_{\geq 4} = 0\}. \tag{4.1}$$

This is the simple 1-dimensional isolated complete intersection singularity W_9 ([G], [AVG]). Here N is quasi-homogeneous with weights $w(x_1)=5$, $w(x_2)=4$, $w(x_3)=3$.

We present the complete classification of the symplectic singularities of (W_9) which was obtained using the method of algebraic restrictions.

Theorem 4.1. *Any submanifold of the symplectic space $(\mathbb{R}^{2n}, \sum_{i=1}^n dp_i \wedge dq_i)$ where $n \geq 3$ (respectively, $n = 2$) which is diffeomorphic to W_9 is symplectically equivalent to one and only one of the normal forms $W_9^i, i = 0, 1, \cdots, 9$ (respectively, $i = 0, 1, 2$) listed below. The parameters c, c_1, c_2 of the normal forms are moduli:*

W_9^0 : $p_1^2 + p_2q_2^2 = 0$, $p_2^2 + p_1q_2 = 0$, $q_1 = c_1q_2 + c_2p_2$, $p_{\geq 3} = q_{\geq 3} = 0$;

W_9^1 : $p_1^2 + p_2q_1^2 = 0$, $p_2^2 \pm p_1q_1 = 0$, $q_2 = -c_1p_1 + \frac{c_2}{2}q_1^2$, $p_{\geq 3} = q_{\geq 3} = 0$;

W_9^2 : $p_1^2 + q_1p_2^2 = 0$, $q_1^2 + p_1p_2 = 0$, $q_2 = c_1q_1p_2 - c_2p_1p_2$, $p_{\geq 3} = q_{\geq 3} = 0$;

W_9^3 : $p_1^2 + p_2p_3^2 = 0$, $p_2^2 + p_1p_3 = 0, q_3 = \mp p_2p_3 - c_1p_1p_3 - c_2p_1p_2$, $q_1 = q_2 = p_{>3} = q_{>3} = 0$;

W_9^4 : $p_1^2 + p_2p_3^2 = 0$, $p_2^2 + p_1p_3 = 0$, $q_3 = -p_1p_3 - c_1p_1p_2 - c_2p_2p_3^2, q_1 = q_2 = p_{>3} = q_{>3} = 0$;

W_9^5 : $p_1^2 + p_2p_3^2 = 0$, $p_2^2 + p_1p_3 = 0$, $q_3 = \mp p_1p_2 - c_1p_2p_3^2 - c_2p_1p_3^2, q_1 = q_2 = p_{>3} = q_{>3} = 0$;

W_9^6 : $p_1^2 + p_2p_3^2 = 0$, $p_2^2 + p_1p_3 = 0$, $q_1 = q_2 = 0, q_3 = -p_2p_3^2 - cp_1p_3^2$, $p_{>3} = q_{>3} = 0$;

W_9^7 : $p_1^2 + p_2p_3^2 = 0$, $p_2^2 + p_1p_3 = 0$, $q_1 = q_2 = 0, q_3 = \mp p_1p_3^2 - cp_2p_3^3$, $p_{>3} = q_{>3} = 0$;

W_9^8 : $p_1^2 + p_2p_3^2 = 0$, $p_2^2 + p_1p_3 = 0$, $q_1 = q_2 = 0, q_3 = \mp p_2p_3^3$, $p_{>3} = q_{>3} = 0$;

W_9^9 : $p_1^2 + p_2p_3^2 = 0$, $p_2^2 + p_1p_3 = 0$, $p_{>3} = q_{>0} = 0$.

In Section 4.1 we use the Lagrangian tangency orders to distinguish the symplectic classes. In Section 4.2 we propose a geometric description of the symplectic singularities. Some of the proofs are presented in Section 5.

4.1. **Distinguishing symplectic classes of W_9 by Lagrangian tangency orders.** The Lagrangian tangency orders were used to distinguish the symplectic classes of (W_9). A curve $N \in (W_9)$ may be described as a union of two parametrical branches: C_1 and C_2. The curve C_1 is nonsingular and the curve C_2 is singular. Their parametrization in the coordinate system $(p_1, q_1, p_2, q_2, \cdots, p_n, q_n)$ is presented in the second column of Table 3. To characterize the symplectic classes of this singularity we use the following two invariants:

- $L_N = Lt(C_1, C_2) = \max\limits_L(\min\{t(C_1, L), t(C_2, L)\})$,
- $L_2 = Lt(C_2) = \max\limits_L t(C_2, L)$.

Here L is a smooth Lagrangian submanifold of the symplectic space.

Theorem 4.2. *A stratified submanifold $N \in (W_9)$ of the symplectic space $(\mathbb{R}^{2n}, \omega_0)$ with the canonical coordinates $(p_1, q_1, \cdots, p_n, q_n)$ is symplectically equivalent to one and only one of the curves presented in the second column of Table 3. The parameters c, c_1, c_2 are moduli. The Lagrangian tangency orders are presented in the third and fourth columns of the table.*

class	parametrization of branches	L_N	L_2
$(W_9)^0$	$C_1: (0, c_1t, 0, t, 0, 0, \cdots)$, $C_2: (t^5, -c_1t^3 - c_2t^4, -t^4, -t^3, 0, \cdots)$	4	4
$(W_9)^1$	$C_1: (0, \pm t, 0, \frac{c_2}{2}t^2, 0, \cdots)$, $C_2: (t^5, \mp t^3, -t^4, -c_1t^5 + \frac{c_2}{2}t^6, 0, \cdots)$	5	5
$(W_9)^2$	$C_1: (0, 0, t, 0, 0, \cdots)$, $C_2: (t^5, -t^4, -t^3, -c_1t^7 + c_2t^8, 0, \cdots)$	5	5
$(W_9)^3$	$C_1: (0, 0, 0, 0, t, 0, \cdots)$, $C_2: (t^5, 0, -t^4, 0, t^3, \mp t^7 + c_1t^8 + c_2t^9, 0, \cdots)$	7	7
$(W_9)^4$	$C_1: (0, 0, 0, 0, t, 0, \cdots)$, $C_2: (t^5, 0, -t^4, 0, t^3, t^8 + c_1t^9 + c_2t^{10}, 0, \cdots)$	8	8
$(W_9)^5$	$C_1: (0, 0, 0, 0, t, 0, \cdots)$, $C_2: (t^5, 0, -t^4, 0, t^3, \pm t^9 + c_1t^{10} - c_2t^{11}, 0, \cdots)$	9	∞
$(W_9)^6$	$C_1: (0, 0, 0, 0, t, 0, \cdots)$, $C_2: (t^5, 0, -t^4, 0, t^3, t^{10} - ct^{11}, 0, \cdots)$	10	∞
$(W_9)^7$	$C_1: (0, 0, 0, 0, t, 0, \cdots)$, $C_2: (t^5, 0, -t^4, 0, t^3, \mp t^{11} - ct^{13}, 0, \cdots)$	11	∞
$(W_9)^8$	$C_1: (0, 0, 0, 0, t, 0, \cdots)$, $C_2: (t^5, 0, -t^4, 0, t^3, \mp t^{13}, 0, \cdots)$	13	∞
$(W_9)^9$	$C_1: (0, 0, 0, 0, t, 0, \cdots)$, $C_2: (t^5, 0, -t^4, 0, t^3, 0, 0, \cdots)$	∞	∞

TABLE 3. The Lagrangian tangency orders for symplectic classes of W_9 singularity.

Remark 4.3. The invariants can be calculated by knowing the parametrization of branches C_1 and C_2. We apply directly the definition of the Lagrangian tangency order finding a Lagrangian submanifold the nearest to the branches.

4.2. **Geometric conditions for the classes $(W_9)^i$.**

Let $N \in (W_9)$. Denote by W the tangent space at 0 to some (and then any) non-singular 3-manifold containing N. We can define the following subspaces of this space:

ℓ – the tangent line at 0 to both branches of N,

V – 2-space tangent at 0 to the singular branch of N.

The classes $(W_9)^i$ satisfy special conditions in terms of the restriction $\omega|_W$, where ω is the symplectic form.

Theorem 4.4. *A stratified submanifold $N \in (W_9)$ of the symplectic space $(\mathbb{R}^{2n}, \omega)$ belongs to the class $(W_9)^i$ if and only if the couple (N, ω) satisfies the corresponding conditions in the last column of Table 4.*

class	normal form	geometric conditions
$(W_9)^0$	$[W_9]^0 : [\theta_1 + c_1\theta_2 + c_2\theta_3]_{W_9}$	$\omega\|_V \neq 0$ (2-space tangent to N is not isotropic)
$(W_9)^1$	$[W_9]^1 : [\pm\theta_2 + c_1\theta_3 + c_2\theta_4]_{W_9}$	$\omega\|_V = 0$ and $\ker\omega \neq \ell$
$(W_9)^2$	$[W_9]^2 : [\theta_3 + c_1\theta_4 + c_2\theta_5]_{W_9}$	$\omega\|_V = 0$ and $\ker\omega = \ell$
		$\omega\|_W = 0$
$(W_9)^3$	$[W_9]^3 : [\pm\theta_4 + c_1\theta_5 + c_2\theta_6]_{W_9}$	$L_N = 7$
$(W_9)^4$	$[W_9]^4 : [\theta_5 + c_1\theta_6 + c_2\theta_7]_{W_9}$	$L_N = 8$
$(W_9)^5$	$[W_9]^5 : [\pm\theta_6 + c_1\theta_7 + c_2\theta_8]_{W_9}$	$L_N = 9$
$(W_9)^6$	$[W_9]^6 : [\theta_7 + c\theta_8]_{W_9}$	$L_N = 10$
$(W_9)^7$	$[W_9]^7 : [\pm\theta_8 + c\theta_9]_{W_9}$	$L_N = 11$
$(W_9)^8$	$[W_9]^8 : [\pm\theta_9]_{W_9}$	$L_N = 13$
$(W_9)^9$	$[W_9]^9 : [0]_{W_9}$	N is contained in a smooth Lagrangian submanifold

TABLE 4. Geometric characterization of symplectic classes of W_9 singularity. (The forms $\theta_1, \ldots, \theta_9$ are described in Theorem 5.23 on the page 173.)

Sketch of the proof of Theorem 4.4. The conditions on the pair (ω, N) in the last column of Table 4 are disjoint. It suffices to prove that these conditions in the row of $(W_9)^i$, are satisfied for any $N \in (W_9)^i$.

We can take the simplest 2-forms ω^i representing the normal forms $[W_9]^i$ for algebraic restrictions and we can check that the pair $(W_9, \omega = \omega^i)$ satisfies the condition in the last column of Table 4.

We note that in the case $N = W_9 = $ (4.1) one has

$\ell = \mathrm{span}(\partial/\partial x_3)$, $V = \mathrm{span}(\partial/\partial x_2, \partial/\partial x_3$, $W = \mathrm{span}(\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3)$.

By simple calculation and observation of the Lagrangian tangency orders we obtain that the conditions corresponding to the classes $(W_9)^i$ are satisfied. □

5. PROOFS

5.1. **The method of algebraic restrictions.** In this section we present basic facts on the method of algebraic restrictions, which is a very powerful tool for the symplectic classification. The details of the method and proofs of all results of this section can be found in [DJZ2].

Given a germ of a non-singular manifold M denote by $\Lambda^p(M)$ the space of all germs at 0 of differential p-forms on M. Given a subset $N \subset M$ introduce the following subspaces of $\Lambda^p(M)$:

$$\Lambda^p_N(M) = \{\omega \in \Lambda^p(M) : \ \ \omega(x) = 0 \text{ for any } x \in N\};$$

$$\mathcal{A}^p_0(N, M) = \{\alpha + d\beta : \ \ \alpha \in \Lambda^p_N(M),\ \beta \in \Lambda^{p-1}_N(M).\}$$

Definition 5.1. Let N be the germ of a subset of M and let $\omega \in \Lambda^p(M)$. The **algebraic restriction** of ω to N is the equivalence class of ω in $\Lambda^p(M)$, where the equivalence is as follows: ω is equivalent to $\widetilde{\omega}$ if $\omega - \widetilde{\omega} \in \mathcal{A}^p_0(N, M)$.

Notation. The algebraic restriction of the germ of a p-form ω on M to the germ of a subset $N \subset M$ will be denoted by $[\omega]_N$. By writing $[\omega]_N = 0$ (or saying that ω has zero algebraic restriction to N) we mean that $[\omega]_N = [0]_N$, i.e. $\omega \in A_0^p(N, M)$.

Definition 5.2. Two algebraic restrictions $[\omega]_N$ and $[\widetilde{\omega}]_{\widetilde{N}}$ are called **diffeomorphic** if there exists the germ of a diffeomorphism $\Phi : \widetilde{M} \to M$ such that $\Phi(\widetilde{N}) = N$ and $\Phi^*([\omega]_N) = [\widetilde{\omega}]_{\widetilde{N}}$.

The method of algebraic restrictions applied to singular quasi-homogeneous subsets is based on the following theorem.

Theorem 5.3 (Theorem A in [DJZ2])**.** *Let N be the germ of a quasi-homogeneous subset of $\mathbb{R}^{2n}$. Let ω_0, ω_1 be germs of symplectic forms on $\mathbb{R}^{2n}$ with the same algebraic restriction to N. There exists a local diffeomorphism Φ such that $\Phi(x) = x$ for any $x \in N$ and $\Phi^*\omega_1 = \omega_0$.*

Two germs of quasi-homogeneous subsets N_1, N_2 of a fixed symplectic space $(\mathbb{R}^{2n}, \omega)$ are symplectically equivalent if and only if the algebraic restrictions of the symplectic form ω to N_1 and N_2 are diffeomorphic.

Theorem 5.3 reduces the problem of symplectic classification of germs of singular quasi-homogeneous subsets to the problem of diffeomorphic classification of algebraic restrictions of the germ of the symplectic form to the germs of singular quasi-homogeneous subsets.

The geometric meaning of the zero algebraic restriction is explained by the following theorem.

Theorem 5.4 (Theorem **B** in [DJZ2])**.** *The germ of a quasi-homogeneous set N of a symplectic space $(\mathbb{R}^{2n}, \omega)$ is contained in a non-singular Lagrangian submanifold if and only if the symplectic form ω has zero algebraic restriction to N.*

In the remainder of this paper we use the following notations:

- $\left[\Lambda^2(\mathbb{R}^{2n})\right]_N$: the vector space consisting of the algebraic restrictions of germs of all 2-forms on $\mathbb{R}^{2n}$ to the germ of a subset $N \subset \mathbb{R}^{2n}$;
- $\left[Z^2(\mathbb{R}^{2n})\right]_N$: the subspace of $\left[\Lambda^2(\mathbb{R}^{2n})\right]_N$ consisting of the algebraic restrictions of germs of all closed 2-forms on $\mathbb{R}^{2n}$ to N;
- $\left[\mathrm{Symp}(\mathbb{R}^{2n})\right]_N$: the open set in $\left[Z^2(\mathbb{R}^{2n})\right]_N$ consisting of the algebraic restrictions of germs of all symplectic 2-forms on $\mathbb{R}^{2n}$ to N.

To obtain a classification of the algebraic restrictions we use the following proposition.

Proposition 5.5. *Let $a_1, \cdots, a_p$ be a quasi-homogeneous basis of quasi-degrees $\delta_1 \leq \cdots \leq \delta_s < \delta_{s+1} \leq \cdots \leq \delta_p$ of the space of algebraic restrictions of closed 2-forms to quasi-homogeneous subset N. Let $a = \sum_{j=s}^{p} c_j a_j$, where $c_j \in \mathbb{R}$ for $j = s, \cdots, p$ and $c_s \neq 0$.*

If there exists a tangent quasi-homogeneous vector field X over N such that $\mathcal{L}_X a_s = r a_k$ for $k > s$ and $r \neq 0$ then a is diffeomorphic to $\sum_{j=s}^{k-1} c_j a_j + \sum_{j=k+1}^{p} b_j a_j$, for some $b_j \in \mathbb{R}$, $j = k+1, \cdots, p$.

Proposition 5.5 is a modification of Theorem 6.13 formulated and proved in [D]. It was formulated for algebraic restrictions to a parameterized curve but we can generalize this theorem for any quasi-homogeneous subset N. The proofs of the cited theorem and Proposition 5.5 are based on the Moser homotopy method.

For calculating discrete invariants we use the following propositions.

Proposition 5.6 ([DJZ2]). *The symplectic multiplicity of the germ of a quasi-homogeneous subset N in a symplectic space is equal to the codimension of the orbit of the algebraic restriction $[\omega]_N$ with respect to the group of local diffeomorphisms preserving N in the space of algebraic restrictions of closed 2-forms to N.*

Proposition 5.7 ([DJZ2]). *The index of isotropy of the germ of a quasi-homogeneous subset N in a symplectic space $(\mathbb{R}^{2n}, \omega)$ is equal to the maximal order of vanishing of closed 2-forms representing the algebraic restriction $[\omega]_N$.*

Proposition 5.8 ([D]). *Let f be the germ of a quasi-homogeneous curve such that the algebraic restriction of a symplectic form to it can be represented by a closed 2-form vanishing at 0. Then the Lagrangian tangency order of the germ of a quasi-homogeneous curve f is the maximum of the order of vanishing on f over all 1-forms α such that $[\omega]_f = [d\alpha]_f$*

5.2. Proofs for W_8 singularity.

5.2.1. *Algebraic restrictions to W_8 and their classification.* One has the following relations for (W_8)-singularities:

$$[d(x_2^2 + x_1x_3)]_{W_8} = [2x_2dx_2 + x_1dx_3 + x_3dx_1]_{W_8} = 0, \tag{5.1}$$

$$[d(x_1^2 + x_3^3)]_{W_8} = [2x_1dx_1 + 3x_3^2dx_3]_{W_8} = 0. \tag{5.2}$$

Multiplying these relations by suitable 1-forms we obtain the relations in Table 5.

δ	relations	proof
14	$[x_2dx_2 \wedge dx_3]_N = -\frac{1}{2}[x_3dx_1 \wedge dx_3]_N$	(5.1)$\wedge dx_3$
15	$[x_1dx_2 \wedge dx_3]_N = [x_3dx_1 \wedge dx_2]_N$	(5.1)$\wedge dx_2$
16	$[x_2dx_1 \wedge dx_2]_N = -\frac{1}{2}[x_1dx_1 \wedge dx_3]_N = 0$	(5.2)$\wedge dx_3$ and (5.1)$\wedge dx_1$
17	$[x_3^2dx_2 \wedge dx_3]_N = \frac{2}{3}[x_1dx_1 \wedge dx_2]_N$	(5.2)$\wedge dx_2$
18	$[x_3^2dx_1 \wedge dx_3]_N = 2[x_2x_3dx_2 \wedge dx_3]_N = 0$	(5.2)$\wedge dx_1$ and (5.1)$\wedge x_3dx_3$
19	$[x_2^2dx_2 \wedge dx_3]_N = -\frac{1}{2}[x_2x_3dx_1 \wedge dx_3]_N$ $[x_2^2dx_2 \wedge dx_3]_N = -[x_1x_3dx_2 \wedge dx_3]_N =$ $= -[x_3^2dx_1 \wedge dx_2]_N$	(5.1)$\wedge x_2dx_3$ (5.1)$\wedge x_3dx_2$ and $[x_2^2 + x_1x_3]_N = 0$
20	$[\alpha]_N = 0$ for all 2-forms α of quasi-degree 20	relations for $\delta \in \{14, 15, 16\}$ and $[x_2^2 + x_1x_3]_N = 0$
21	$[\alpha]_N = 0$ for all 2-forms α of quasi-degree 21	relations for $\delta \in \{15, 16, 17\}$ and $[x_1^2+x_3^3]_N = [x_2^2+x_1x_3]_N = 0$
22	$[\alpha]_N = 0$ for all 2-forms α of quasi-degree 22	relations for $\delta \in \{16, 17, 18\}$ and $[x_1^2+x_3^3]_N = [x_2^2+x_1x_3]_N = 0$
23	$[\alpha]_N = 0$ for all 2-forms α of quasi-degree 23	relations for $\delta \in \{17, 18, 19\}$ and $[x_1^2 + x_3^3]_N = 0$
24	$[\alpha]_N = 0$ for all 2-forms α of quasi-degree 24	relations for $\delta \in \{18, 19, 20\}$ and $[x_1^2 + x_3^3]_N = 0$
25	$[\alpha]_N = 0$ for all 2-forms α of quasi-degree 25	relations for $\delta \in \{19, 20, 21\}$
$\delta > 25$	$[\alpha]_N = 0$ for all 2-forms α of quasi-degree $\delta > 25$	relations for $\delta > 19$

TABLE 5. Relations towards calculating $[\Lambda^2(\mathbb{R}^{2n})]_N$ for $N = W_8$.

Using the method of algebraic restrictions and Table 5 we obtain the following proposition.

Proposition 5.9. *The space $[\Lambda^2(\mathbb{R}^{2n})]_{W_8}$ is a 9-dimensional vector space spanned by the algebraic restrictions to W_8 of the 2-forms:*

$$\theta_1 = dx_2 \wedge dx_3,\ \ \theta_2 = dx_1 \wedge dx_3,\ \ \theta_3 = dx_1 \wedge dx_2,\ \ \ \theta_4 = x_3 dx_2 \wedge dx_3,\ \ \theta_5 = x_2 dx_2 \wedge dx_3,$$
$$\sigma_1 = x_1 dx_2 \wedge dx_3,\quad \sigma_2 = x_2 dx_1 \wedge dx_3,\quad \theta_7 = x_3^2 dx_2 \wedge dx_3,\quad \theta_8 = x_2^2 dx_2 \wedge dx_3.$$

Proposition 5.9 and results of Section 5.1 imply the following description of the space $[Z^2(\mathbb{R}^{2n})]_{W_8}$ and the manifold $[\mathrm{Symp}(\mathbb{R}^{2n})]_{W_8}$.

Theorem 5.10. *The space $[Z^2(\mathbb{R}^{2n})]_{W_8}$ is an 8-dimensional vector space spanned by the algebraic restrictions to W_8 of the quasi-homogeneous 2-forms θ_i of degree δ_i:*

$\theta_1 = dx_2 \wedge dx_3, \quad \delta_1 = 9,$

$\theta_2 = dx_1 \wedge dx_3, \quad \delta_2 = 10,$

$\theta_3 = dx_1 \wedge dx_2, \quad \delta_3 = 11,$

$\theta_4 = x_3 dx_2 \wedge dx_3, \quad \delta_4 = 13,$

$\theta_5 = x_2 dx_2 \wedge dx_3, \quad \delta_5 = 14,$

$\theta_6 = \sigma_1 + \sigma_2 = x_1 dx_2 \wedge dx_3 + x_2 dx_1 \wedge dx_3, \quad \delta_6 = 15,$

$\theta_7 = x_3^2 dx_2 \wedge dx_3, \quad \delta_7 = 17,$

$\theta_8 = x_2^2 dx_2 \wedge dx_3, \quad \delta_8 = 19.$

If $n \geq 3$ then $[\mathrm{Symp}(\mathbb{R}^{2n})]_{W_8} = [Z^2(\mathbb{R}^{2n})]_{W_8}$. The manifold $[\mathrm{Symp}(\mathbb{R}^4)]_{W_8}$ is an open part of the 8-space $[Z^2(\mathbb{R}^4)]_{W_8}$ consisting of algebraic restrictions of the form $[c_1\theta_1 + \cdots + c_8\theta_8]_{W_8}$ such that $(c_1, c_2, c_3) \neq (0, 0, 0)$.

Theorem 5.11.

(i) Any algebraic restriction in $[Z^2(\mathbb{R}^{2n})]_{W_8}$ can be brought by a symmetry of W_8 to one of the normal forms $[W_8]^i$ given in the second column of Table 6.

(ii) The codimension in $[Z^2(\mathbb{R}^{2n})]_{W_8}$ of the singularity class corresponding to the normal form $[W_8]^i$ is equal to i, the symplectic multiplicity and the index of isotropy are given in the fourth and fifth columns of Table 6.

(iii) The singularity classes corresponding to the normal forms are disjoint.

(iv) The parameters c, c_1, c_2 of the normal forms $[W_8]^i$ are moduli.

In the first column of Table 6 we denote by $(W_8)^i$ a subclass of (W_8) consisting of $N \in (W_8)$ such that the algebraic restriction $[\omega]_N$ is diffeomorphic to some algebraic restriction of the normal form $[W_8]^i$, where i is the codimension of the class. Classes $(W_8)^{2a}$ and $(W_8)^{2b}$ have the same codimension equal to 2 but they can be distinguished geometrically (see Table 2).

The proof of Theorem 5.11 is presented in Section 5.2.3.

5.2.2. *Symplectic normal forms.* Let us transfer the normal forms $[W_8]^i$ to symplectic normal forms. Fix a family ω^i of symplectic forms on $\mathbb{R}^{2n}$ realizing the family $[W_8]^i$ of algebraic restrictions. We can fix, for example,

$\omega^0 = \theta_1 + c_1\theta_2 + c_2\theta_3 + dx_1 \wedge dx_4 + dx_5 \wedge dx_6 + \cdots + dx_{2n-1} \wedge dx_{2n};$

$\omega^1 = c_1\theta_2 + \theta_3 + c_2\theta_4 + dx_3 \wedge dx_4 + dx_5 \wedge dx_6 + \cdots + dx_{2n-1} \wedge dx_{2n}, \quad c_1 \neq 0;$

$\omega^{2,a} = \pm\theta_2 + c_1\theta_4 + c_2\theta_7 + dx_2 \wedge dx_4 + dx_5 \wedge dx_6 + \cdots + dx_{2n-1} \wedge dx_{2n};$

$\omega^{2,b} = \theta_3 + c_1\theta_4 + c_2\theta_5 + dx_3 \wedge dx_4 + dx_5 \wedge dx_6 + \cdots + dx_{2n-1} \wedge dx_{2n};$

symplectic class	normal forms for algebraic restrictions	cod	$\mu^{\rm sym}$	ind
$(W_8)^0 \quad (2n \geq 4)$	$[W_8]^0 : [\theta_1 + c_1\theta_2 + c_2\theta_3]_{W_8}$,	0	2	0
$(W_8)^1 \quad (2n \geq 4)$	$[W_8]^1 : [c_1\theta_2 + \theta_3 + c_2\theta_4]_{W_8},\ c_1 \neq 0$	1	3	0
$(W_8)^{2,a} \quad (2n \geq 4)$	$[W_8]^{2,a} : [\pm\theta_2 + c_1\theta_4 + c_2\theta_7]_{W_8}$,	2	4	0
$(W_8)^{2,b} \quad (2n \geq 4)$	$[W_8]^{2,b} : [\theta_3 + c_1\theta_4 + c_2\theta_5]_{W_8}$,	2	4	0
$(W_8)^3 \quad (2n \geq 6)$	$[W_8]^3 : [\theta_4 + c_1\theta_5 + c_2\theta_6]_{W_8}$	3	5	1
$(W_8)^4 \quad (2n \geq 6)$	$[W_8]^4 : [\pm\theta_5 + c_1\theta_6 + c_2\theta_7]_{W_8}$	4	6	1
$(W_8)^5 \quad (2n \geq 6)$	$[W_8]^5 : [\theta_6 + c\theta_7]_{W_8}$	5	6	1
$(W_8)^6 \quad (2n \geq 6)$	$[W_8]^6 : [\theta_7 + c\theta_8]_{W_8}$	6	7	2
$(W_8)^7 \quad (2n \geq 6)$	$[W_8]^7 : [\theta_8]_{W_8}$	7	7	2
$(W_8)^8 \quad (2n \geq 6)$	$[W_8]^8 : [0]_{W_8}$	8	8	∞

TABLE 6. Classification of symplectic W_8 singularities.
cod – codimension of the classes; μ^{sym}– symplectic multiplicity;
ind – the index of isotropy.

$$\omega^3 = \theta_4 + c_1\theta_5 + c_2\theta_6 + dx_1 \wedge dx_4 + dx_2 \wedge dx_5 + dx_3 \wedge dx_6 + dx_7 \wedge dx_8 + \cdots + dx_{2n-1} \wedge dx_{2n};$$
$$\omega^4 = \pm\theta_5 + c_1\theta_6 + c_2\theta_7 + dx_1 \wedge dx_4 + dx_2 \wedge dx_5 + dx_3 \wedge dx_6 + dx_7 \wedge dx_8 + \cdots + dx_{2n-1} \wedge dx_{2n};$$
$$\omega^5 = \theta_6 + c\theta_7 + dx_1 \wedge dx_4 + dx_2 \wedge dx_5 + dx_3 \wedge dx_6 + dx_7 \wedge dx_8 + \cdots + dx_{2n-1} \wedge dx_{2n};$$
$$\omega^6 = \theta_7 + c\theta_8 + dx_1 \wedge dx_4 + dx_2 \wedge dx_5 + dx_3 \wedge dx_6 + dx_7 \wedge dx_8 + \cdots + dx_{2n-1} \wedge dx_{2n};$$
$$\omega^7 = \theta_8 + dx_1 \wedge dx_4 + dx_2 \wedge dx_5 + dx_3 \wedge dx_6 + dx_7 \wedge dx_8 + \cdots + dx_{2n-1} \wedge dx_{2n};$$
$$\omega^8 = dx_1 \wedge dx_4 + dx_2 \wedge dx_5 + dx_3 \wedge dx_6 + dx_7 \wedge dx_8 + \cdots + dx_{2n-1} \wedge dx_{2n}.$$

Let $\omega_0 = \sum_{i=1}^m dp_i \wedge dq_i$, where $(p_1, q_1, \cdots, p_n, q_n)$ is the coordinate system on $\mathbb{R}^{2n}, n \geq 3$ (resp. $n = 2$). Fix, for $i = 0, 1, \cdots, 8$ (resp. for $i = 0, 1, 2$) a family Φ^i of local diffeomorphisms which bring the family of symplectic forms ω^i to the symplectic form ω_0: $(\Phi^i)^*\omega^i = \omega_0$. Consider the families $W_8^i = (\Phi^i)^{-1}(W_8)$. Any stratified submanifold of the symplectic space $(\mathbb{R}^{2n}, \omega_0)$ which is diffeomorphic to W_8 is symplectically equivalent to one and only one of the normal forms $W_8^i, i = 0, 1, \cdots, 8$ (resp. $i = 0, 1, 2$) presented in Theorem 3.1. By Theorem 5.11 we obtain that parameters c, c_1, c_2 of the normal forms are moduli.

5.2.3. *Proof of Theorem 5.11.* In our proof we use vector fields tangent to $N \in W_8$. Any vector fields tangent to $N \in W_8$ can be described as $V = g_1 E + g_2 \mathcal{H}$ where E is the Euler vector field and $\mathcal{H}$ is a Hamiltonian vector field and g_1, g_2 are functions. It was shown in [DT1] (Prop. 6.13) that the action of a Hamiltonian vector field on the algebraic restriction of a closed 2-form to any 1-dimensional complete intersection is trivial.

The germ of a vector field tangent to W_8 of non trivial action on algebraic restrictions of closed 2-forms to W_8 may be described as a linear combination of germs of vector fields: $X_0 = E,\ X_1 = x_3 E,\ X_2 = x_2 E,\ X_3 = x_1 E,\ X_4 = x_3^2 E,\ X_5 = x_2 x_3 E,\ X_6 = x_2^2 E,\ X_7 = x_1 x_3 E$, where E is the Euler vector field

$$E = 6x_1 \frac{\partial}{\partial x_1} + 5x_2 \frac{\partial}{\partial x_2} + 4x_3 \frac{\partial}{\partial x_3}. \tag{5.3}$$

Proposition 5.12. *The infinitesimal action of germs of quasi-homogeneous vector fields tangent to $N \in (W_8)$ on the basis of the vector space of algebraic restrictions of closed 2-forms to N is presented in Table 7.*

$\mathcal{L}_{X_i}[\theta_j]$	$[\theta_1]$	$[\theta_2]$	$[\theta_3]$	$[\theta_4]$	$[\theta_5]$	$[\theta_6]$	$[\theta_7]$	$[\theta_8]$
$X_0=E$	$9[\theta_1]$	$10[\theta_2]$	$11[\theta_3]$	$13[\theta_4]$	$14[\theta_5]$	$15[\theta_6]$	$17[\theta_7]$	$19[\theta_8]$
$X_1=x_3E$	$13[\theta_4]$	$-28[\theta_5]$	$5[\theta_6]$	$17[\theta_7]$	$[0]$	$-57[\theta_8]$	$[0]$	$[0]$
$X_2=x_2E$	$14[\theta_5]$	$10[\theta_6]$	$[0]$	$[0]$	$19[\theta_8]$	$[0]$	$[0]$	$[0]$
$X_3=x_1E$	$5[\theta_6]$	$[0]$	$\frac{51}{2}[\theta_7]$	$-19[\theta_8]$	$[0]$	$[0]$	$[0]$	$[0]$
$X_4=x_3^2E$	$17[\theta_7]$	$[0]$	$-19[\theta_8]$	$[0]$	$[0]$	$[0]$	$[0]$	$[0]$
$X_5=x_2x_3E$	$[0]$	$-38[\theta_8]$	$[0]$	$[0]$	$[0]$	$[0]$	$[0]$	$[0]$
$X_6=x_2^2E$	$19[\theta_8]$	$[0]$	$[0]$	$[0]$	$[0]$	$[0]$	$[0]$	$[0]$
$X_7=x_1x_3E$	$-19[\theta_8]$	$[0]$	$[0]$	$[0]$	$[0]$	$[0]$	$[0]$	$[0]$

TABLE 7. Infinitesimal actions on algebraic restrictions of closed 2-forms to W_8. (E is defined as in (5.3.))

Let $\mathcal{A} = [c_1\theta_1 + c_2\theta_2 + c_3\theta_3 + c_4\theta_4 + c_5\theta_5 + c_6\theta_6 + c_7\theta_7 + c_8\theta_8]_{W_8}$ be the algebraic restriction of a symplectic form ω.

The first statement of Theorem 5.11 follows from the following lemmas.

Lemma 5.13. *If $c_1 \neq 0$ then the algebraic restriction $\mathcal{A} = [\sum_{k=1}^{8} c_k\theta_k]_{W_8}$ can be reduced by a symmetry of W_8 to an algebraic restriction $[\theta_1 + \widetilde{c}_2\theta_2 + \widetilde{c}_3\theta_3]_{W_8}$.*

Proof. Using the data of Table 7, we can see that for any algebraic restriction $[\theta_k]_{W_8}$, where $k \in \{4, 5, \ldots, 8\}$ we can find a vector field V_k tangent to W_8 such that $\mathcal{L}_{V_k}[\theta_1]_{W_8} = [\theta_k]_{W_8}$. We deduce from Proposition 5.5 that the algebraic restriction $\mathcal{A}$ is diffeomorphic to $[c_1\theta_1 + c_2\theta_2 + c_3\theta_3]_{W_8}$.

By the condition $c_1 \neq 0$, we have a diffeomorphism $\Psi \in Symm(W_8)$ of the form

$$\Psi : (x_1, x_2, x_3) \mapsto (c_1^{-\frac{6}{9}}x_1, c_1^{-\frac{5}{9}}x_2, c_1^{-\frac{4}{9}}x_3) \tag{5.4}$$

and finally we obtain

$$\Psi^*([c_1\theta_1 + c_2\theta_2 + c_3\theta_3]_{W_8}) = [\theta_1 + c_2c_1^{-\frac{10}{9}}\theta_2 + c_3c_1^{-\frac{11}{9}}\theta_3]_{W_8} = [\theta_1 + \widetilde{c}_2\theta_2 + \widetilde{c}_3\theta_3]_{W_8}.$$

□

Lemma 5.14. *If $c_1 = 0$ and $c_2 \cdot c_3 \neq 0$ then the algebraic restriction $\mathcal{A}$ can be reduced by a symmetry of W_8 to an algebraic restriction $[\widetilde{c}_2\theta_2 + \theta_3 + \widetilde{c}_4\theta_4]_{W_8}$.*

Proof of Lemma 5.14. We use the homotopy method to prove that $\mathcal{A}$ is diffeomorphic to $[\widetilde{c}_2\theta_2 + \theta_3 + \widetilde{c}_4\theta_4]_{W_8}$.

Let $\mathcal{B}_t = [c_2\theta_2 + c_4\theta_3 + c_4\theta_4 + (1-t)c_5\theta_5 + (1-t)c_6\theta_6 + (1-t)c_7\theta_7 + (1-t)c_8\theta_8]_{W_8}$ for $t \in [0; 1]$. Then $\mathcal{B}_0 = \mathcal{A}$ and $\mathcal{B}_1 = [c_2\theta_2 + c_3\theta_3 + c_4\theta_4]_{W_8}$. We prove that there exists a family $\Phi_t \in Symm(W_8)$, $t \in [0; 1]$ such that

$$\Phi_t^*\mathcal{B}_t = \mathcal{B}_0,\ \Phi_0 = id. \tag{5.5}$$

Let V_t be a vector field defined by $\frac{d\Phi_t}{dt} = V_t(\Phi_t)$. Then, by differentiating (5.5) we obtain

$$\mathcal{L}_{V_t}\mathcal{B}_t = [c_5\theta_5 + c_6\theta_6 + c_7\theta_7 + c_8\theta_8]_{W_8}. \tag{5.6}$$

We are looking for V_t in the form $V_t = \sum_{k=1}^{5} b_k(t)X_k$ where the $b_k(t)$ for $k = 1, \ldots, 5$ are smooth functions $b_k : [0; 1] \to \mathbb{R}$. Then, by Proposition 5.12, equation (5.6) has the form

$$\begin{bmatrix} -28c_2 & 0 & 0 & 0 & 0 \\ 5c_3 & 10c_2 & 0 & 0 & 0 \\ 17c_4 & 0 & \frac{51}{2}c_3 & 0 & 0 \\ -57c_6(1-t) & 19c_5(1-t) & -19c_4 & -19c_3 & -38c_2 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix} = \begin{bmatrix} c_5 \\ c_6 \\ c_7 \\ c_8 \end{bmatrix}. \tag{5.7}$$

If $c_2 \cdot c_3 \neq 0$ we can solve (5.7) and Φ_t may be obtained as a flow of the vector field V_t. The family Φ_t preserves W_8, because V_t is tangent to W_8 and $\Phi_t^*\mathcal{B}_t = \mathcal{A}$. Using the homotopy arguments, we have that $\mathcal{A}$ is diffeomorphic to $\mathcal{B}_1 = [c_2\theta_2 + c_3\theta_3 + c_4\theta_4]_{W_8}$. By the condition $c_3 \neq 0$, we have a diffeomorphism $\Psi \in Symm(W_8)$ of the form

$$\Psi : (x_1, x_2, x_3) \mapsto (c_3^{-\frac{6}{11}}x_1, c_3^{-\frac{5}{11}}x_2, c_3^{-\frac{4}{11}}x_3), \tag{5.8}$$

and we obtain

$$\Psi^*(\mathcal{B}_1) = [c_2c_3^{-\frac{10}{11}}\theta_2 + \theta_3 + c_4c_3^{-\frac{13}{11}}\theta_3]_{W_8} = [\widetilde{c}_2\theta_2 + \theta_3 + \widetilde{c}_4\theta_4]_{W_8}.$$

□

Lemma 5.15. *If $c_1 = c_3 = 0$ and $c_2 \neq 0$ then the algebraic restriction $\mathcal{A}$ can be reduced by a symmetry of W_8 to an algebraic restriction $[\pm\theta_2 + \widetilde{c}_4\theta_4 + \widetilde{c}_7\theta_7]_{W_8}$.*

Proof. We can see from Table 7 that for any algebraic restriction $[\theta_k]_{W_8}$, where $k \in \{5, 6, 8\}$ there exists a vector field V_k tangent to W_8 such that $\mathcal{L}_{V_k}[\theta_2]_{W_8} = [\theta_k]_{W_8}$. Using Proposition 5.5 we obtain that $\mathcal{A}$ is diffeomorphic to $[c_2\theta_2 + c_4\theta_4 + \widehat{c}_7\theta_7]_{W_8}$ for some $\widehat{c}_7 \in \mathbb{R}$.

By the condition $c_2 \neq 0$ we can use a diffeomorphism $\Psi \in Symm(W_8)$ of the form

$$\Psi : (x_1, x_2, x_3) \mapsto (|c_2|^{-\frac{6}{10}}x_1, |c_2|^{-\frac{5}{10}}x_2, |c_2|^{-\frac{4}{10}}x_3) \tag{5.9}$$

and we obtain

$$\Psi^*([c_2\theta_2 + c_4\theta_4 + \widehat{c}_7\theta_7]_{W_8}) = [\frac{c_2}{|c_2|}\theta_2 + c_4|c_2|^{-\frac{13}{10}}\theta_4 + \widehat{c}_7|c_2|^{-\frac{17}{10}}\theta_7]_{W_8} = [\pm\theta_2 + \widetilde{c}_4\theta_4 + \widetilde{c}_7\theta_7]_{W_8}.$$

The algebraic restrictions $[\theta_2 + \widetilde{c}_4\theta_4 + \widetilde{c}_7\theta_7]_{W_8}$ and $[-\theta_2 + \widetilde{b}_4\theta_4 + \widetilde{b}_7\theta_7]_{W_8}$ are not diffeomorphic. Any diffeomorphism $\Phi = (\Phi_1, \ldots, \Phi_{2n})$ of $(\mathbb{R}^{2n}, 0)$ preserving W_8 has to preserve a curve $C(t) = (t^6, t^5, -t^4, 0, \ldots, 0)$ which means that

$\Phi_1(t^6, t^5, -t^4, 0, \ldots, 0) = (\psi(t))^6$,
$\Phi_2(t^6, t^5, -t^4, 0, \ldots, 0) = (\psi(t))^5$,
$\Phi_3(t^6, t^5, -t^4, 0, \ldots, 0) = -(\psi(t))^4$,
$\Phi_k(t^6, t^5, -t^4, 0, \ldots, 0) = 0$ for $k > 3$,

where $\psi(t) = a_1t + a_2t^2 + a_3t^3 + \ldots$ is a diffeomorphism of $(\mathbb{R}, 0)$.

Hence Φ has a linear part

$$(5.10)\qquad \begin{array}{lllllllllllll}
\Phi_1: & A^6x_1 & & + & & & & A_{14}x_4 & + & \cdots & + & A_{1,2n}x_{2n} \\
\Phi_2: & A_{2,1}x_1 & + & A^5x_2 & & + & & A_{24}x_4 & + & \cdots & + & A_{2,2n}x_{2n} \\
\Phi_3: & A_{3,1}x_1 & + & A_{3,2}x_2 & + & A^4x_3 & + & A_{34}x_4 & + & \cdots & + & A_{3,2n}x_{2n} \\
\Phi_4: & & & & & & & A_{44}x_4 & + & \cdots & + & A_{4,2n}x_{2n} \\
\vdots & & & & & & & \vdots & \vdots & \vdots & \vdots & \\
\Phi_{2n}: & & & & & & & A_{2n,4}x_4 & + & \cdots & + & A_{2n,2n}x_{2n},
\end{array}$$

where $A, A_{i,j} \in \mathbb{R}$.

If we assume that $\Phi^*([\theta_2 + \widetilde{c}_4\theta_4 + \widetilde{c}_7\theta_7]_{W_8}) = [-\theta_2 + \widetilde{b}_4\theta_4 + \widetilde{b}_7\theta_7]_{W_8}$, then $A^{10}dx_1 \wedge dx_3|_0 = -dx_1 \wedge dx_3|_0$, which is a contradiction. □

Lemma 5.16. *If $c_1 = c_2 = 0$ and $c_3 \neq 0$ then the algebraic restriction $\mathcal{A}$ can be reduced by a symmetry of W_8 to an algebraic restriction $[\theta_3 + \widetilde{c}_4\theta_4 + \widetilde{c}_5\theta_5]_{W_8}$.*

Lemma 5.17. *If $c_1 = c_2 = c_3 = 0$ and $c_4 \neq 0$ then the algebraic restriction $\mathcal{A}$ can be reduced by a symmetry of W_8 to an algebraic restriction $[\theta_4 + \widetilde{c}_5\theta_5 + \widetilde{c}_6\theta_6]_{W_8}$.*

Lemma 5.18. *If $c_1 = 0, \ldots, c_4 = 0$ and $c_5 \neq 0$, then the algebraic restriction $\mathcal{A}$ can be reduced by a symmetry of W_8 to an algebraic restriction $[\pm\theta_5 + \widetilde{c}_6\theta_6 + \widetilde{c}_7\theta_7]_{W_8}$.*

Lemma 5.19. *If $c_1 = 0, \ldots, c_5 = 0$ and $c_6 \neq 0$ then the algebraic restriction $\mathcal{A}$ can be reduced by a symmetry of W_8 to an algebraic restriction $[\theta_6 + \widetilde{c}_7\theta_7]_{W_8}$.*

Lemma 5.20. *If $c_1 = 0, \ldots, c_6 = 0$ and $c_7 \neq 0$ then the algebraic restriction $\mathcal{A}$ can be reduced by a symmetry of W_8 to an algebraic restriction $[\theta_7 + \widetilde{c}_8\theta_8]_{W_8}$.*

Lemma 5.21. *If $c_1 = 0, \ldots, c_7 = 0$ and $c_8 \neq 0$ then the algebraic restriction $\mathcal{A}$ can be reduced by a symmetry of W_8 to an algebraic restriction $[\theta_8]_{W_8}$.*

The proofs of Lemmas 5.16 – 5.21 are similar to the proofs of Lemmas 5.13 – 5.15 and are based on Table 7.

Statement (ii) of Theorem 5.11 follows from the conditions in the proof of part (i) (the codimension) and from Theorem 5.4 and Proposition 5.6 (the symplectic multiplicity) and Proposition 5.7 (the index of isotropy).

To prove statement (iii) of Theorem 5.11 we have to show that singularity classes corresponding to normal forms are disjoint. The singularity classes that can be distinguished by geometric conditions obviously are disjoint. From Theorem 3.5 we see that only classes $(W_8)^1$ and $(W_8)^{2,a}$ can not be distinguished by the geometric conditions but their symplectic multiplicities are distinct , hence the classes are disjoint.

To prove statement (iv) of Theorem 5.11 we have to show that the parameters c, c_1, c_2 are moduli in the normal forms. The proofs are very similar in all cases. We consider as an example the normal form with two parameters $[\theta_1 + c_1\theta_2 + c_2\theta_3]_{W_8}$. From Table 7 we see that the tangent space to the orbit of $[\theta_1 + c_1\theta_2 + c_2\theta_3]_{W_8}$ at $[\theta_1 + c_1\theta_2 + c_2\theta_3]_{W_8}$ is spanned by the linearly independent algebraic restrictions $[9\theta_1 + 10c_1\theta_2 + 11c_2\theta_3]_{W_8}$, $[\theta_4]_{W_8}, [\theta_5]_{W_8}, [\theta_6]_{W_8}, [\theta_7]_{W_8}$ and $[\theta_8]_{W_8}$. Hence, the algebraic restrictions $[\theta_2]_{W_8}$ and $[\theta_3]_{W_8}$ do not belong to it. Therefore, the parameters c_1 and c_2 are independent moduli in the normal form $[\theta_1 + c_1\theta_2 + c_2\theta_3]_{W_8}$.

5.3. **Proofs for W_9 singularity.**

5.3.1. *Algebraic restrictions to* W_9 *and their classification.*

One has the following relations for (W_9)-singularities

$$[d(x_1^2 + x_2x_3^2)]_{W_9} = [2x_1dx_1 + 2x_2x_3dx_3 + x_3^2dx_2]_{W_9} = 0, \tag{5.11}$$

$$[d(x_2^2 + x_1x_3)]_{W_9} = [2x_2dx_2 + x_3dx_1 + x_1dx_3]_{W_9} = 0. \tag{5.12}$$

Multiplying these relations by suitable 1-forms we obtain the relations in Table 8.

δ	relations	proof
11	$[x_3dx_1 \wedge dx_3]_N = -2[x_2dx_2 \wedge dx_3]_N$	(5.12)$\wedge dx_3$
12	$[x_1dx_2 \wedge dx_3]_N = [x_3dx_1 \wedge dx_2]_N$	(5.12)$\wedge dx_2$
13	$[x_3^2dx_2 \wedge dx_3]_N = -2[x_1dx_1 \wedge dx_3]_N = 4[x_2dx_1 \wedge dx_1]_N$	(5.11)$\wedge dx_3$ and (5.12)$\wedge dx_1$
14	$[x_3^2dx_1 \wedge dx_3]_N = -2[x_1dx_1 \wedge dx_2]_N = -2[x_2x_3dx_2 \wedge dx_3]_N$	(5.11)$\wedge dx_2$, (5.12)$\wedge x_3dx_3$
15	$[\alpha]_N = 0$ for all 2-forms α of quasi-degree 15	relations for $\delta \in \{11, 12\}$ and (5.11)$\wedge dx_1$ and $[x_2^2 + x_1x_3]_N = 0$
16	$[x_3^3dx_2 \wedge dx_3]_N = -2[x_1x_3dx_1 \wedge dx_3]_N = 4[x_2x_3dx_1 \wedge dx_2]_N$ $[x_1x_3dx_1 \wedge dx_3]_N = -2[x_1x_2dx_2 \wedge dx_3]_N = -[x_2^2dx_1 \wedge dx_3]_N$	relations for $\delta = 13$ relations for $\delta \in \{11, 12\}$ and $[x_2^2 + x_1x_3]_N = 0$
17	$[\alpha]_N = 0$ for all 2-forms α of quasi-degree 17	relations for $\delta \in \{12, 13, 14\}$ and $[x_1^2 + x_2x_3^2]_N = 0$ i $[x_2^2 + x_1x_3]_N = 0$
18	$[\alpha]_N = 0$ for all 2-forms α of quasi-degree 18	relations for $\delta \in \{13, 14, 15\}$ and $[x_1^2 + x_2x_3^2]_N = 0$
19	$[\alpha]_N = 0$ for all 2-forms α of quasi-degree 19	relations for $\delta \in \{14, 15, 16\}$ and $[x_1^2 + x_2x_3^2]_N = 0$
20	$[\alpha]_N = 0$ for all 2-forms α of quasi-degree 20	relations for $\delta \in \{15, 16, 17\}$
21	$[\alpha]_N = 0$ for all 2-forms α of quasi-degree 21	relations for $\delta \in \{16, 17, 18\}$
>21	$[\alpha]_N = 0$ for all 2-forms α of quasi-degree $\delta > 21$	relations for $\delta > 16$

TABLE 8. Relations towards calculating $[\Lambda^2(\mathbb{R}^{2n})]_N$ for $N = W_9$.

Using the method of algebraic restrictions and Table 8 we obtain the following proposition:

Proposition 5.22. *The space* $[\Lambda^2(\mathbb{R}^{2n})]_{W_9}$ *is a* 10*-dimensional vector space spanned by the algebraic restrictions to* W_9 *of the* 2*-forms*

$\theta_1 = dx_2 \wedge dx_3$, $\theta_2 = dx_1 \wedge dx_3$, $\theta_3 = dx_1 \wedge dx_2$,
$\theta_4 = x_3dx_2 \wedge dx_3$, $\theta_5 = x_3dx_1 \wedge dx_3$, $\sigma_1 = x_1dx_2 \wedge dx_3$, $\sigma_2 = x_2dx_1 \wedge dx_3$,
$\theta_7 = x_3^2dx_2 \wedge dx_3$, $\theta_8 = x_3^2dx_1 \wedge dx_3$, $\theta_9 = x_3^3dx_2 \wedge dx_3$.

Proposition 5.22 and results of Section 5.1 imply the following description of the space $[Z^2(\mathbb{R}^{2n})]_{W_9}$ and the manifold $[\mathrm{Symp}(\mathbb{R}^{2n})]_{W_9}$.

Theorem 5.23. *The space* $[Z^2(\mathbb{R}^{2n})]_{W_9}$ *is a* 9*-dimensional vector space spanned by the algebraic restrictions to* W_9 *of the quasi-homogeneous* 2*-forms* θ_i *of degree* δ_i

$\theta_1 = dx_2 \wedge dx_3$, $\delta_1 = 7$,

$\theta_2 = dx_1 \wedge dx_3$, $\delta_2 = 8$,

$\theta_3 = dx_1 \wedge dx_2$, $\delta_3 = 9$,

$\theta_4 = x_3 dx_2 \wedge dx_3, \quad \delta_4 = 10,$

$\theta_5 = x_3 dx_1 \wedge dx_3, \quad \delta_5 = 11,$

$\theta_6 = \sigma_1 + \sigma_2 = x_1 dx_2 \wedge dx_3 + x_2 dx_1 \wedge dx_3, \quad \delta_6 = 12,$

$\theta_7 = x_3^2 dx_2 \wedge dx_3, \quad \delta_7 = 13,$

$\theta_8 = x_3^2 dx_1 \wedge dx_3, \quad \delta_8 = 14,$

$\theta_9 = x_3^3 dx_2 \wedge dx_3, \quad \delta_8 = 16,$

If $n \geq 3$ then $[\mathrm{Symp}(\mathbb{R}^{2n})]_{W_9} = [Z^2(\mathbb{R}^{2n})]_{W_9}$. The manifold $[\mathrm{Symp}(\mathbb{R}^4)]_{W_9}$ is an open part of the 9-space $[Z^2(\mathbb{R}^4)]_{W_9}$ consisting of algebraic restrictions of the form $[c_1\theta_1 + \cdots + c_9\theta_9]_{W_9}$ such that $(c_1, c_2, c_3) \neq (0,0,0)$.

Theorem 5.24.

(i) Any algebraic restriction in $[Z^2(\mathbb{R}^{2n})]_{W_9}$ can be brought by a symmetry of W_9 to one of the normal forms $[W_9]^i$ given in the second column of Table 9.

(ii) The codimension in $[Z^2(\mathbb{R}^{2n})]_{W_9}$ of the singularity class corresponding to the normal form $[W_9]^i$ is equal to i, the symplectic multiplicity and the index of isotropy are given in the fourth and fifth columns of Table 9.

(iii) The singularity classes corresponding to the normal forms are disjoint.

(iv) The parameters c, c_1, c_2 of the normal forms $[W_9]^i$ are moduli.

symplectic class	normal forms for algebraic restrictions	cod	μ^{sym}	ind
$(W_9)^0 \quad (2n \geq 4)$	$[W_9]^0 : [\theta_1 + c_1\theta_2 + c_2\theta_3]_{W_9}$,	0	2	0
$(W_9)^1 \quad (2n \geq 4)$	$[W_9]^1 : [\pm\theta_2 + c_1\theta_3 + c_2\theta_4]_{W_9}$	1	3	0
$(W_9)^2 \quad (2n \geq 4)$	$[W_9]^2 : [\theta_3 + c_1\theta_4 + c_2\theta_5]_{W_9}$,	2	4	0
$(W_9)^3 \quad (2n \geq 6)$	$[W_9]^3 : [\pm\theta_4 + c_1\theta_5 + c_2\theta_6]_{W_9}$,	3	5	1
$(W_9)^4 \quad (2n \geq 6)$	$[W_9]^4 : [\theta_5 + c_1\theta_6 + c_2\theta_7]_{W_9}$	4	6	1
$(W_9)^5 \quad (2n \geq 6)$	$[W_9]^5 : [\pm\theta_6 + c_1\theta_7 + c_2\theta_8]_{W_9}$	5	7	1
$(W_9)^6 \quad (2n \geq 6)$	$[W_9]^6 : [\theta_7 + c\theta_8]_{W_9}$	6	7	2
$(W_9)^7 \quad (2n \geq 6)$	$[W_9]^7 : [\pm\theta_8 + c\theta_9]_{W_9}$	7	8	2
$(W_9)^8 \quad (2n \geq 6)$	$[W_9]^8 : [\pm\theta_9]_{W_9}$	8	8	3
$(W_9)^9 \quad (2n \geq 6)$	$[W_9]^9 : [0]_{W_9}$	9	9	∞

TABLE 9. Classification of symplectic W_9 singularities.
(*cod* – codimension of the classes; μ^{sym}– the symplectic multiplicity;
ind – the index of isotropy.)

In the first column of Table 9 by $(W_9)^i$ we denote a subclass of (W_9) consisting of $N \in (W_9)$ such that the algebraic restriction $[\omega]_N$ is diffeomorphic to some algebraic restriction of the normal form $[W_9]^i$.

The proof of Theorem 5.24 is presented in Section 5.3.3.

5.3.2. *Symplectic normal forms.*

Let us transfer the normal forms $[W_9]^i$ to symplectic normal forms. Fix a family ω^i of symplectic forms on $\mathbb{R}^{2n}$ realizing the family $[W_9]^i$ of algebraic restrictions. We can fix, for example

$\omega^0 = \theta_1 + c_1\theta_2 + c_2\theta_3 + dx_1 \wedge dx_4 + dx_5 \wedge dx_6 + \cdots + dx_{2n-1} \wedge dx_{2n}$;

$\omega^1 = \pm\theta_2 + c_1\theta_3 + c_2\theta_4 + dx_2 \wedge dx_4 + dx_5 \wedge dx_6 + \cdots + dx_{2n-1} \wedge dx_{2n}$;

$\omega^2 = \theta_3 + c_1\theta_4 + c_2\theta_5 + dx_3 \wedge dx_4 + dx_5 \wedge dx_6 + \cdots + dx_{2n-1} \wedge dx_{2n}$;

$\omega^3 = \pm\theta_4 + c_1\theta_5 + c_2\theta_6 + dx_1 \wedge dx_4 + dx_2 \wedge dx_5 + dx_3 \wedge dx_6 + dx_7 \wedge dx_8 + \cdots + dx_{2n-1} \wedge dx_{2n}$;

$\omega^4 = \theta_5 + c_1\theta_6 + c_2\theta_7 + dx_1 \wedge dx_4 + dx_2 \wedge dx_5 + dx_3 \wedge dx_6 + dx_7 \wedge dx_8 + \cdots + dx_{2n-1} \wedge dx_{2n}$;

$\omega^5 = \pm\theta_6 + c_1\theta_7 + c_2\theta_8 + dx_1 \wedge dx_4 + dx_2 \wedge dx_5 + dx_3 \wedge dx_6 + dx_7 \wedge dx_8 + \cdots + dx_{2n-1} \wedge dx_{2n}$;

$\omega^6 = \theta_7 + c\theta_8 + dx_1 \wedge dx_4 + dx_2 \wedge dx_5 + dx_3 \wedge dx_6 + dx_7 \wedge dx_8 + \cdots + dx_{2n-1} \wedge dx_{2n}$;

$\omega^7 = \pm\theta_8 + c\theta_9 + dx_1 \wedge dx_4 + dx_2 \wedge dx_5 + dx_3 \wedge dx_6 + dx_7 \wedge dx_8 + \cdots + dx_{2n-1} \wedge dx_{2n}$;

$\omega^8 = \pm\theta_9 + dx_1 \wedge dx_4 + dx_2 \wedge dx_5 + dx_3 \wedge dx_6 + dx_7 \wedge dx_8 + \cdots + dx_{2n-1} \wedge dx_{2n}$;

$\omega^9 = dx_1 \wedge dx_4 + dx_2 \wedge dx_5 + dx_3 \wedge dx_6 + dx_7 \wedge dx_8 + \cdots + dx_{2n-1} \wedge dx_{2n}$.

Let $\omega_0 = \sum_{i=1}^m dp_i \wedge dq_i$, where $(p_1, q_1, \cdots, p_n, q_n)$ is the coordinate system on $\mathbb{R}^{2n}, n \geq 3$ (resp. $n = 2$). Fix, for $i = 0, 1, \cdots, 9$ (resp. for $i = 0, 1, 2$) a family Φ^i of local diffeomorphisms which bring the family of symplectic forms ω^i to the symplectic form ω_0: $(\Phi^i)^*\omega^i = \omega_0$. Consider the families $W_9^i = (\Phi^i)^{-1}(W_8)$. Any stratified submanifold of the symplectic space $(\mathbb{R}^{2n}, \omega_0)$ which is diffeomorphic to W_9 is symplectically equivalent to one and only one of the normal forms $W_9^i, i = 0, 1, \cdots, 9$ (resp. $i = 0, 1, 2$) presented in Theorem 4.1. By Theorem 5.24 we obtain that parameters c, c_1, c_2 of the normal forms are moduli.

5.3.3. *Proof of Theorem 5.24.*

In our proof we use vector fields tangent to $N \in W_9$.
The germ of a vector field tangent to W_8 of non trivial action on algebraic restrictions of closed 2-forms to W_9 may be described as a linear combination of germs of the following vector fields: $X_0 = E,\ X_1 = x_3E,\ X_2 = x_2E,\ X_3 = x_1E,\ X_4 = x_3^2E, X_5 = x_2x_3E,\ X_6 = x_2^2E,\ X_7 = x_1x_3E,$ $X_8 = x_1x_2E,\ X_9 = x_3^3E,$
where E is the Euler vector field

$$E = 5x_1\frac{\partial}{\partial x_1} + 4x_2\frac{\partial}{\partial x_2} + 3x_3\frac{\partial}{\partial x_3}. \tag{5.13}$$

Proposition 5.25. *The infinitesimal action of germs of quasi-homogeneous vector fields tangent to $N \in (W_9)$ on the basis of the vector space of algebraic restrictions of closed 2-forms to N is presented in Table 10.*

$\mathcal{L}_{X_i}[\theta_j]$	$[\theta_1]$	$[\theta_2]$	$[\theta_3]$	$[\theta_4]$	$[\theta_5]$	$[\theta_6]$	$[\theta_7]$	$[\theta_8]$	$[\theta_9]$
$X_0 = E$	$7[\theta_1]$	$8[\theta_2]$	$9[\theta_3]$	$10[\theta_4]$	$11[\theta_5]$	$12[\theta_6]$	$13[\theta_7]$	$14[\theta_8]$	$16[\theta_9]$
$X_1 = x_3E$	$10[\theta_4]$	$11[\theta_5]$	$4[\theta_6]$	$13[\theta_7]$	$14[\theta_8]$	$[0]$	$16[\theta_9]$	$[0]$	$[0]$
$X_2 = x_2E$	$-\frac{11}{2}[\theta_5]$	$8[\theta_6]$	$\frac{13}{4}[\theta_7]$	$-7[\theta_8]$	$[0]$	$12[\theta_9]$	$[0]$	$[0]$	$[0]$
$X_3 = x_1E$	$4[\theta_6]$	$-\frac{13}{2}[\theta_7]$	$-7[\theta_8]$	$[0]$	$-8[\theta_9]$	$[0]$	$[0]$	$[0]$	$[0]$
$X_4 = x_3^2E$	$13[\theta_7]$	$14[\theta_8]$	$[0]$	$16[\theta_9]$	$[0]$	$[0]$	$[0]$	$[0]$	$[0]$
$X_5 = x_2x_3E$	$-7[\theta_8]$	$[0]$	$4[\theta_9]$	$[0]$	$[0]$	$[0]$	$[0]$	$[0]$	$[0]$
$X_6 = x_2^2E$	$[0]$	$8[\theta_9]$	$[0]$	$[0]$	$[0]$	$[0]$	$[0]$	$[0]$	$[0]$
$X_7 = x_1x_3E$	$[0]$	$-8[\theta_9]$	$[0]$	$[0]$	$[0]$	$[0]$	$[0]$	$[0]$	$[0]$
$X_8 = x_1x_2E$	$4[\theta_9]$	$[0]$	$[0]$	$[0]$	$[0]$	$[0]$	$[0]$	$[0]$	$[0]$
$X_9 = x_3^3E$	$16[\theta_9]$	$[0]$	$[0]$	$[0]$	$[0]$	$[0]$	$[0]$	$[0]$	$[0]$

TABLE 10. Infinitesimal actions on algebraic restrictions of closed 2-forms to W_9. (E is defined as in (5.13).)

Let $\mathcal{A} = [\sum_{k=1}^{9} \theta_k]_{W_9}$ be the algebraic restriction of a symplectic form ω.

The first statement of Theorem 5.24 follows from the following lemmas.

Lemma 5.26. *If $c_1 \neq 0$ then the algebraic restriction $\mathcal{A} = [\sum_{k=1}^{9} c_k\theta_k]_{W_9}$ can be reduced by a symmetry of W_9 to an algebraic restriction $[\theta_1 + \widetilde{c}_2\theta_2 + \widetilde{c}_3\theta_3]_{W_9}$.*

Lemma 5.27. *If $c_1 = 0$ and $c_2 \neq 0$ then the algebraic restriction $\mathcal{A}$ can be reduced by a symmetry of W_9 to an algebraic restriction $[\pm\theta_2 + \widetilde{c}_3\theta_3 + \widetilde{c}_4\theta_4]_{W_9}$.*

Lemma 5.28. *If $c_1 = c_2 = 0$ and $c_3 \neq 0$ then the algebraic restriction $\mathcal{A}$ can be reduced by a symmetry of W_9 to an algebraic restriction $[\theta_3 + \widetilde{c}_4\theta_4 + \widetilde{c}_5\theta_5]_{W_9}$.*

Lemma 5.29. *If $c_1 = c_2 = c_3 = 0$ and $c_4 \neq 0$ then the algebraic restriction $\mathcal{A}$ can be reduced by a symmetry of W_9 to an algebraic restriction $[\pm\theta_4 + \widetilde{c}_5\theta_5 + \widetilde{c}_6\theta_6]_{W_9}$.*

Lemma 5.30. *If $c_1 = \ldots = c_4 = 0$ and $c_5 \neq 0$ then the algebraic restriction $\mathcal{A}$ can be reduced by a symmetry of W_9 to an algebraic restriction $[\theta_5 + \widetilde{c}_6\theta_6 + \widetilde{c}_7\theta_7]_{W_9}$.*

Lemma 5.31. *If $c_1 = \ldots = c_5 = 0$ and $c_6 \neq 0$ then the algebraic restriction $\mathcal{A}$ can be reduced by a symmetry of W_9 to an algebraic restriction $[\pm\theta_6 + \widetilde{c}_7\theta_7 + \widetilde{c}_8\theta_8]_{W_9}$.*

Lemma 5.32. *If $c_1 = \ldots = c_6 = 0$ and $c_7 \neq 0$ then the algebraic restriction $\mathcal{A}$ can be reduced by a symmetry of W_9 to an algebraic restriction $[\theta_7 + \widetilde{c}_8\theta_8]_{W_9}$.*

Lemma 5.33. *If $c_1 = \ldots = c_7 = 0$ and $c_8 \neq 0$ then the algebraic restriction $\mathcal{A}$ can be reduced by a symmetry of W_9 to an algebraic restriction $[\pm\theta_8 + \widetilde{c}_9\theta_9]_{W_9}$.*

Lemma 5.34. *If $c_1 = \ldots = c_8 = 0$ and $c_9 \neq 0$ then the algebraic restriction $\mathcal{A}$ can be reduced by a symmetry of W_9 to an algebraic restriction $[\pm\theta_9]_{W_9}$.*

The proofs of Lemmas 5.26 – 5.34 are similar to the proofs of the lemmas for the W_8 singularity. As an example we give the proof of Lemma 5.27.

Proof of Lemma 5.27. We see from Table 10 that for any algebraic restriction $[\theta_k]_{W_9}$, where $k \in \{5,6,7,8,9\}$, there exists a vector field V_k tangent to W_9 such that $\mathcal{L}_{V_k}[\theta_2]_{W_9} = [\theta_k]_{W_9}$. Using Proposition 5.5, we obtain that $\mathcal{A}$ is diffeomorphic to $[c_2\theta_2 + c_3\theta_3 + c_4\theta_4]_{W_9}$.

By the condition $c_2 \neq 0$, we have a diffeomorphism $\Psi \in Symm(W_9)$ of the form

$$\Psi : (x_1, x_2, x_3) \mapsto (|c_2|^{-\frac{5}{8}} x_1, |c_2|^{-\frac{4}{8}} x_2, |c_2|^{-\frac{3}{8}} x_3) \tag{5.14}$$

and we obtain

$$\Psi^*([c_2\theta_2 + c_3\theta_3 + c_4\theta_4]_{W_9}) = [\frac{c_2}{|c_2|}\theta_2 + c_3|c_2|^{-\frac{9}{8}}\theta_3 + c_4|c_2|^{-\frac{10}{8}}\theta_4]_{W_9} = [\pm\theta_2 + \widetilde{c}_3\theta_3 + \widetilde{c}_4\theta_4]_{W_9}.$$

The algebraic restrictions $[\theta_2 + \widetilde{c}_3\theta_3 + \widetilde{c}_4\theta_4]_{W_9}$ and $[-\theta_2 + \widetilde{b}_3\theta_3 + \widetilde{b}_4\theta_4]_{W_9}$ are not diffeomorphic. Any diffeomorphism $\Phi = (\Phi_1, \ldots, \Phi_{2n})$ of $(\mathbb{R}^{2n}, 0)$ preserving W_9 has to preserve a curve $C_2(t) = (t^5, -t^4, -t^3, 0, \ldots, 0)$. Hence, Φ has a linear part

$$\begin{array}{lccccccccccc}
\Phi_1 : & A^5 x_1 & & & + & & & A_{14}x_4 & + & \cdots & + & A_{1,2n}x_{2n} \\
\Phi_2 : & A_{2,1}x_1 & + & A^4 x_2 & & + & & A_{24}x_4 & + & \cdots & + & A_{2,2n}x_{2n} \\
\Phi_3 : & A_{3,1}x_1 & + & A_{3,2}x_2 & + & A^3 x_3 & + & A_{34}x_4 & + & \cdots & + & A_{3,2n}x_{2n} \\
\Phi_4 : & & & & & & & A_{44}x_4 & + & \cdots & + & A_{4,2n}x_{2n} \\
\vdots & & & & & & & \vdots & \vdots & \vdots & \vdots & \\
\Phi_{2n} : & & & & & & & A_{2n,4}x_4 & + & \cdots & + & A_{2n,2n}x_{2n}
\end{array} \tag{5.15}$$

where $A, A_{i,j} \in \mathbb{R}$.
If we assume that $\Phi^*([\theta_2 + \widetilde{c}_3\theta_3 + \widetilde{c}_4\theta_4]_{W_9}) = [-\theta_2 + \widetilde{b}_3\theta_3 + \widetilde{b}_4\theta_4]_{W_9}$, then
$A^8 dx_1 \wedge dx_3|_0 = -dx_1 \wedge dx_3|_0$, which is a contradiction. □

Statement (ii) of Theorem 5.24 follows from the conditions in the proof of part (i) (the codimension) and from Theorem 5.4 and Proposition 5.6 (the symplectic multiplicity) and Proposition 5.7 (the index of isotropy).

Statement (iii) of Theorem 5.24 follows from Theorem 4.4. The singularity classes corresponding to normal forms are disjoint because they can be distinguished by the geometric conditions.

To prove statement (iv) of Theorem 5.24 we have to show that the parameters c, c_1, c_2 are moduli in the normal forms. The proofs are very similar in all cases. We consider as an example the normal form with two parameters $[\theta_1 + c_1\theta_2 + c_2\theta_3]_{W_9}$. From Table 10 we see that the tangent space to the orbit of $[\theta_1 + c_1\theta_2 + c_2\theta_3]_{W_9}$ at $[\theta_1 + c_1\theta_2 + c_2\theta_3]_{W_9}$ is spanned by the linearly independent algebraic restrictions $[7\theta_1 + 8c_1\theta_2 + 9c_2\theta_3]_{W_9}$, $[\theta_4]_{W_9}, [\theta_5]_{W_9}, [\theta_6]_{W_9}, [\theta_7]_{W_9}, [\theta_8]_{W_9}$ and $[\theta_9]_{W_9}$. Hence, the algebraic restrictions $[\theta_2]_{W_9}$ and $[\theta_3]_{W_9}$ do not belong to it. Therefore, the parameters c_1 and c_2 are independent moduli in the normal form $[\theta_1 + c_1\theta_2 + c_2\theta_3]_{W_9}$.

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ON BI-LIPSCHITZ STABILITY OF FAMILIES OF FUNCTIONS

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Abstract. We focus on the Lipschitz stability of families of functions. We introduce a stability notion, called fiberwise bi-Lipschitz equivalence, which preserves the metric structure of the level surfaces of functions and show that it does not admit continuous moduli in the framework of semialgebraic geometry. We trivialize semialgebraic families of Lipschitz functions by constructing triangulations of their generic fibers which contain information about the metric structure of the sets.

0. Introduction

We study the metric stability of semialgebraic families of functions. In [S1], M. Shiota showed that a semialgebraic family of continuous functions $f_t : \mathbb{R}^n \to \mathbb{R}$, $t \in \mathbb{R}^p$, is generically topologically trivial. It means that we can find a partition of $\mathbb{R}^p$ and two semialgebraic families of homeomorphisms ϕ_t and h_t such that $\phi_t^{-1} \circ f_t \circ h_t$ is constant with respect to t on every element of this partition (see also [C, S2]). The fibers f_t are then said topologically equivalent. The main result of this paper is a partial Lipschitz counterpart of this theorem (Theorem 6.4).

The study of metric stability of analytic sets was initiated by T. Mostowski in his fundamental paper [M]. It was then developed, mainly by A. Parusiński [P1, P2], L. Birbrair [B], and the author of the present paper [V1, V2, V3]. The description of the metric structure of singularities provides a more accurate information than the description of their topology, valuable for applications [V5, V4]. The Lipschitz category can be considered as an intermediate category in between the C^1 category, too restricted to investigate singularities (C^1 equivalence admits continuous moduli), and the C^0 category, which often provides too vague information on the singularity.

The notion of semialgebraic bi-Lipschitz triviality of functions (Definition 6.1) is defined in the same way as the notion of topological triviality above, except that ϕ_t and h_t are required to be bi-Lipschitz. If many results about the topology have their counterpart in the framework of Lipschitz geometry [M, P1, P2, V1], it is however known that bi-Lipschitz equivalence of functions admits continuous moduli, in the sense that semialgebraic families of functions are not always generically bi-Lipschitz trivial. A counterexample was found by J.-P. Henry and A. Parusiński [H-P] (example 6.3 below). It was however shown in [RV] that bi-Lipschitz $\mathcal{K}$-equivalence does not admit continuous moduli.

We show in this paper that a slightly weaker equivalence notion than bi-Lipschitz equivalence does not admit continuous moduli for semialgebraic families of Lipschitz functions. This notion is stronger than C^0 equivalence since it preserves the metric structure of the level surfaces of the functions. Studying the stability of families of functions amounts to investigate triviality of foliations since the levels of the functions provide a singular foliation. In our equivalence

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relation, called *fiberwise bi-Lipschitz triviality* (see Definition 6.1), the homeomorphism is bi-Lipschitz on every level surface of the function, with *the same Lipschitz constant.* The Lipschitz condition may only fail for two points of two different fibers. The trivialization has however to vary continuously when we pass from one level of the function to one another.

Topological triviality of families of functions is proved in [BCR, C] by triangulating the generic fibers of semialgebraic families of functions. Triangulating and trivializing are thus two very related problems. In [V1], the author introduces the notion of *Lipschitz triangulation.* These are triangulations which provide information not only on the topology of the considered object but also on its metric structure. The metric type of a singularity is thus enclosed in finitely many combinatoric data in the sense that two singularities having the same Lipschitz triangulation are bi-Lipschitz homeomorphic. This is very convenient to describe the metric properties of semialgebraic sets or to prove finiteness properties regarding the metric structure of semialgebraic singularities [V2, V3]. Henry and Parusiński's example nevertheless shows that it is impossible to construct a triangulation of a semialgebraic function which would be a Lipschitz triangulation in sense of [V1] (since this would entail that bi-Lipschitz triviality of families of functions holds for generic parameters).

We prove generic fiberwise bi-Lipschitz triviality (Theorem 6.4) by showing that we can triangulate the generic fiber of a semialgebraic family of Lipschitz functions (Theorem 2.4). The triangulation that we construct satisfies a condition similar to the one required in the definition of the Lipschitz triangulations introduced in [V1], but just on points lying in the same fiber.

Our triviality theorem is thus, as in [C], derived from a triangulation theorem. Doing so, we have to work in an arbitrary real closed field (rather than in $\mathbb{R}$), since the generic fiber of the considered family lies in an extension of $\mathbb{R}$. We wish to emphasize here that even the study of semialgebraic functions of $\mathbb{R}^p \times \mathbb{R}^n$ requires, if one wants to use this kind of technique, to deal with an arbitrary real closed field. This kind of technique is classical and, although not completely elementary, has the significant advantage to get rid of the parameters during the best part of the proof. It is also worthy of notice that in this way we get two theorems (one showing triangulability and a second establishing triviality), both of their own interest. Noteworthy, these two theorems provide semialgebraic homeomorphisms. Semialgebraic mappings have nice properties. For instance, M. Shiota and Yoccoi established in [SY] a version of the Hauptvermutung for these mappings (see also [S2]). Semialgebraic bi-Lipschitz mappings have also nice differentiability properties used by the author of the present paper in [V4, V5] so as to study differential forms.

Content of the paper. In the first section we recall the known results on C^0 stability. This is useful so as to emphasize the close interplay between triangulations and trivializations. Indeed, the proof of the main theorem (Theorem 6.4) will make use of the same argument as the one used in the proof of Theorem 1.6. In section 2, we recall the notion of Lipschitz triangulation and state our triangulation theorem for functions (Theorem 2.4). The next sections are devoted to the proof of this theorem. Section 3 recalls some required results of [V1] and proves a parameterized version of the main tool used there, constructing "families of regular systems of hypersurfaces" for one parameter families of semialgebraic sets. Section 5 proves Theorem 2.4. The last section introduces the notion of fiberwise bi-Lipschitz triviality and yields it for semialgebraic families of Lipschitz functions, for generic parameters.

Notations 0.1. We write $\mathbb{Q}_+$ for the positive rational numbers. Let R be a real closed field. Given $A \subset R^n$ we denote by $int(A)$ the interior of A, $cl(A)$ the closure of A, and by $\delta(A)$ the topological boundary of A, $cl(A) \setminus int(A)$. We shall write $|.|$ for the Euclidean norm and $B(\lambda, r)$ for the ball of radius r centered at λ (for all the considered metric spaces R^n, $S^n, \dots$).

We denote by $e_1, \dots, e_n$ the canonical basis of R^n and by $\mathbb{G}_n^k$ the Grassmanian of k-dimensional vector spaces of R^n. We set $\mathbb{G}_n := \cup_{k=1}^{n-1} \mathbb{G}_n^k$. We denote by $\tau(A)$ the closure in the Grassmanian of the set of all the tangent spaces to A_{reg}, where A_{reg} stands for the set constituted by the points near which the set A is a C^1 manifold (of dimension $\dim A$ or smaller).

We shall denote by $d(\cdot,\cdot)$ the Euclidean distance in R^n. Given $x \in R^n$ and $P \subset R^n$, we write $d(x,P)$ for the distance to the subset P (defined by $\inf_{y\in P} d(x,y)$). Given a subset C of $\mathbb{G}_n$ we also set $d(x,C) := \inf_{P\in C} d(x,P)$.

A **Lipschitz function** is a function $f : A \to R$ satisfying for some $L \in R$ and all x and x' in A

$$|f(x) - f(x')| \leq L|x - x'|.$$

The function may be said **L-Lipschitz** is one wants to specify the constant. It is said **$\mathbb{Q}$-Lipschitz** if it is L-Lipschitz with $L \in \mathbb{Q}$.

Given a couple of functions ξ_1 and ξ_2 on A, we write $\xi_1 \sim_K \xi_2$ if there exist C in K such that $\xi_1 \leq C\xi_2$ and $\xi_2 \leq C\xi_1$ (here $K \subset R$). We denote by $[\xi_1, \xi_2]$ the set $\{(x,y) \in A \times R : \xi_1(x) \leq y \leq \xi_2(x)\}$.

1. Topological stability

1.1. Triangulations of functions.

Let R be a real closed field.

Simplicial complexes will be finite and may have open simplices (and hence will not always be compact). An **open simplex** is a simplex from which the proper faces have been taken off.

We will denote by $\widetilde{R^n}$ the real spectrum of the polynomial ring $R[X_1,\dots,X_n]$ (see [BCR, C]). Given $\alpha \in \widetilde{R^n}$ we shall write $k(\alpha)$ for the corresponding extension of R.

Definition 1.1. Let X be a semialgebraic set. A **triangulation of** X is the data of a finite simplicial complex K, and a semialgebraic homeomorphism $h : |K| \to X$.

Let $f : X \to R$ be a semialgebraic function. A **triangulation of** f is the data of a triangulation h of X together with a homeomorphism $\varphi : R \to R$ such that $\varphi^{-1} \circ f \circ h$ is piecewise linear.

Theorem 1.2. [S1] *Every continuous bounded semialgebraic function admits a C^0-triangulation. The vertices of the simplicial complex may be assumed to have coordinates in $\mathbb{Q}$.*

Remark 1.3. If we do not require that the vertices lie in $\mathbb{Q}^n$ then the map φ (see definition 1.1) may be required to be the identity.

1.2. Topological triviality of semialgebraic families of functions.

Definition 1.4. A semialgebraic family of sets of $R^p \times R^n$ is a semialgebraic subset of $R^p \times R^n$, the first p variables being considered as parameters. Let X be a semialgebraic family of sets of $R^p \times R^n$. A **semialgebraic family of functions** is a semialgebraic mapping $f : X \to R^p \times R$, of type $X \ni (t,x) \mapsto (t, f_t(x))$, the first p variables being considered as parameters.

For a parameter t in R^p, we call the function f_t the **fiber** at t of this family. Given $\alpha \in \widetilde{R^p}$, we denote by f_α the generic fiber at α (see [BCR, C]).

Given a semialgebraic family of functions, it is a natural problem to compare the fibers f_t with each other.

Definition 1.5. We say that a semialgebraic family $f : X \to R^p \times R$ is **semialgebraically C^0 trivial along** $W \subset R^p$ if there exist two semialgebraic families of homeomorphisms $h : W \times R^n \to W \times R^n$ and $\phi : W \times R \to W \times R$ such that for any $t \in W$:

$$h_t(X_{t_0}) = X_t, \quad \phi_t \circ f_t \circ h_t = f_{t_0}, \quad t_0 \in W.$$

The fibers f_t are then said to be **semialgebraically C^0 equivalent**.

Semialgebraic families of functions are generically semialgebraically topologically trivial:

Theorem 1.6. *(Shiota) Let $f : X \to R^p \times R$ be a semialgebraic family of continuous functions. There exist a semialgebraic partition $V_1, \dots, V_m$ of R^p such that for every i, f is semialgebraically topologically trivial along V_i.*

Proof. We first check that we can assume, without loss of generality, that f_t is bounded on X. Indeed, the function $u(y) := \frac{y}{1+|y|}$ is a homeomorphism from R onto $(-1;1)$. If we prove the result for $\hat{f} := u \circ f$, we are done. Let us assume that f_t is bounded for any t without changing notations.

Let $\alpha \in \widetilde{R^p}$. By Theorem 2.3, there exist semialgebraic homeomorphisms $h : |K| \to X_\alpha$ and $\varphi : k(\alpha) \to k(\alpha)$, with K finite simplicial complex, such that $\varphi^{-1} \circ f_\alpha \circ h$ is a piecewise linear function on every simplex. The simplicial complex K may be assumed to have vertices in $\mathbb{Q}^n$. As a matter of fact, $|K|$ is indeed the generic fiber of a constant family $U \times |K|$, with $U \in \alpha$ (see [BCR, C] for more details).

The homeomorphisms h and φ respectively give rise to families of semialgebraic homeomorphisms:

$$\theta : U \times |K| \to U \times X,$$

and $\gamma : U \times R \to U \times R$.

As $\gamma_\alpha^{-1} \circ f_\alpha \circ \theta_\alpha$ is piecewise linear, $\gamma_t^{-1} \circ f_t \circ \theta_t$ is constant with respect to t. If we set $H_t := \theta_t \theta_{t_0}^{-1}$ and $\phi_t(x) := \gamma_t \gamma_{t_0}^{-1}$, we have $\phi_t^{-1} \circ f_t \circ H_t = f_{t_0}$. This shows that f is trivial along U. By compactness of $\widetilde{R^p}$, we have the desired finite covering. □

2. Lipschitz triangulations

2.1. **Lipschitz triangulation of semialgebraic sets.** We recall in this section the results proved in [V1]. We will adapt these techniques to families of functions.

Given a point $q \in R^n$, we write $q_1, \dots, q_n$ for the coordinates of q in the canonical basis and $\pi_i : R^n \to R^i$ for the canonical projection.

Definition 2.1. Let $\sigma \subset R^n$ be an open simplex. A **tame system of coordinates of** σ is a homeomorphism (onto its image) $(\psi_1, \dots, \psi_n) : \sigma \to R^n$ of the following form:

$$\psi_i(q) = \frac{q_i - \theta_i(\pi_{i-1}(q))}{\theta_i(\pi_{i-1}(q)) - \theta'_i(\pi_{i-1}(q))}, \tag{2.1}$$

(and 0 whenever $\theta_i \circ \pi_{i-1}(q) = \theta'_i \circ \pi_{i-1}(q)$) where θ_i and θ'_i are piecewise linear functions on R^{i-1}. A **standard simplicial function** on σ is a function given by finitely many iterations of sums, powers (possibly negative), and products of distances to faces.

Standard simplicial functions will sometimes be defined on $\sigma \times \sigma$ since they will be functions of two variables q and q', being given by finite iterations of sums, products, and powers of functions of type $q \mapsto d(q, \tau)$ and $q' \mapsto d(q', \tau)$ with τ face of σ.

Definition 2.2. A **Lipschitz triangulation of** R^n is the data of a finite simplicial complex K together with a semialgebraic homeomorphism $h : |K| \to R^n$, such that for every $\sigma \in K$ there exist $\varphi_{\sigma,1}, \dots, \varphi_{\sigma,n}$, standard simplicial functions over $\sigma \times \sigma$ satisfying for any q and q' in σ:

$$|h(q) - h(q')| \sim_R \sum_{i=1}^{n} \varphi_{\sigma,i}(q; q') \cdot |q_{i,\sigma} - q'_{i,\sigma}|, \tag{2.2}$$

where $(q_{1,\sigma}, \dots, q_{n,\sigma})$ is a tame system of coordinates of R^n. Let $A_1, \dots, A_k$ be subsets of R^n. A **Lipschitz triangulation of** $A_1, \dots, A_k$ is a Lipschitz triangulation of R^n such that each $h^{-1}(A_i)$ is a union of open simplices.

With this definition two semialgebraic subsets admitting the same simplicial complex as semialgebraic triangulation, with $\sim_R$ functions φ_σ and same tame systems of coordinates are semialgebraically bi-Lipschitz homeomorphic. As a matter of fact, simultaneous Lipschitz triangulations of the fibers of a family provide bi-Lipschitz trivializations.

Theorem 2.3. [V1] *Every finite collection of semialgebraic sets admits a Lipschitz triangulation.*

2.2. **Lipschitz triangulations of functions.** The theorem below gives a version of Theorem 2.3 for functions. Unfortunately, it is not possible to construct a triangulation of a function which would be a Lipschitz triangulation (see example 6.3). We prove something somewhat weaker: we show that we can triangulate every semialgebraic bounded Lipschitz function in such a way that (2.2) holds for couples of points of the same fiber (with a constant independent of the fiber).

Theorem 2.4. *Let $f : X \to R$ be a semialgebraic bounded Lipschitz function, $X \subset R^n$. There exists a triangulation (K, ϕ, ψ) of f, with $K \subset R^{n+1}$, such that on every open simplex σ of K, we can find standard simplicial functions $\varphi_{\sigma,1}, \dots, \varphi_{\sigma,n+1}$ with:*

$$|\psi(q) - \psi(q')| \sim_R \sum_{i=1}^{n+1} \varphi_{\sigma,i}(q; q') \cdot |q_{i,\sigma} - q'_{i,\sigma}|, \tag{2.3}$$

on the set

$$\{(q, q') \in \sigma \times \sigma : f(\psi(q)) = f(\psi(q'))\},$$

where $(q_{1,\sigma}, \dots, q_{n+1,\sigma})$ is a tame system of coordinates of σ. Moreover, the vertices of the simplicial complex K lie in $\mathbb{Q}^{n+1}$ and ϕ is bi-Lipschitz.

Furthermore, given finitely many semialgebraic subsets $A_1, ..., A_k$ of X, we may choose the triangulation in such a way that each A_i is a union of images of open simplices of K.

This theorem will be proved in section 5.

3. Regular lines

We recall that, given a subset C of $\mathbb{G}_n$, we have set $d(x, C) := \inf_{P \in C} d(x, P)$, where d stands for the Euclidian distance of R^n (see Notation 0.1).

Definition 3.1. Let A be a semialgebraic set of R^n. An element λ of S^{n-1} is said to be **regular for the set** A if there is $\alpha \in \mathbb{Q}_+$ such that:

$$d(\lambda; \tau(A)) \geq \alpha.$$

We say that $\lambda \in S^{n-1}$ is **regular for a semialgebraic family** X of $R \times R^n$ if there exists $\alpha \in \mathbb{Q}_+$ such that for any parameter $t \in R$:

$$d(\lambda; \tau(X_t)) \geq \alpha.$$

A **subset** $C \subset S^{n-1}$ **is regular** for a set (resp. family) X if all the elements of $cl(C)$ are regular for the set (resp. family) X.

Remark 3.2. Of course, if a line is regular for a family then it is regular for all the fibers of this family. But it is indeed much stronger since, when a line is regular for a family, the angle between this line and the tangent spaces to the fibers is bounded below away from zero by a constant α *independent of* t.

Proposition 3.3. [V1] *Let A be a semialgebraic subset of R^n of empty interior. There exists a semialgebraic $\mathbb{Q}$-bi-Lipschitz homeomorphism $h : R^n \to R^n$ such that $h(A)$ has a regular vector.*

We will need a parameterized version of this proposition. More precisely, we shall establish the following proposition.

Proposition 3.4. *Let A be a semialgebraic family of $R \times R^n$ such that A_t has empty interior for every $t \in R$. There exists a continuous semialgebraic family of mappings $h : R \times R^n \to R \times R^n$ and $C \in \mathbb{Q}$ such that:*

(1) *h_t C-bi-Lipschitz for any t.*
(2) *e_n is regular for the family $h(A)$.*

We will prove Proposition 3.4 by generalizing to families the techniques introduced in [V1] for subsets of R^n.

3.1. **Some preliminary lemmas.** We need to recall some results which were already used in [V1].

Lemma 3.5. [K] *Given $\nu \in \mathbb{N}$, there exists a strictly positive constant $\sigma \in \mathbb{Q}_+$ such that for any $P_1, \dots, P_\nu$ in $\mathbb{G}_n$ there exists $P \in S^{n-1}$ such that for any i we have:*

$$d(P; P_i) \geq \sigma.$$

The second lemma we need was proved by the author of the present paper in [V1].

Lemma 3.6. *There exists $\{\lambda_1, \dots, \lambda_N\} \subset S^{n-1}$ such that for any semialgebraic sets $A_1, \dots, A_m$ of R^n, there exists a cell decomposition $(C_i)_{i \in I}$ of R^n adapted to all the A_k's and such that for each open cell C_i, we may find $\lambda_{j(i)}$, $1 \leq j(i) \leq N$, regular for δC_i.*

Given $\lambda \in S^{n-1}$, we denote by π_λ the projection along the line generated by λ onto the vector space N_λ, normal to this line. Given $q \in R^n$, we write q_λ for the Euclidean inner product $< q, \lambda >$.

The third result we shall recall is the preparation theorem, so called because it can be considered as a Weierstrass preparation theorem for semialgebraic functions.

Theorem 3.7. *(Preparation Theorem)* [vDS, LR, V1, P3] *Let $\xi : R^{n+1} \to R$ be a semialgebraic function. Then there exists a finite semialgebraic partition $(V_i)_{i \in I}$ of R^{n+1} such that for any V_i there exist semialgebraic continuous functions a, $\theta : \pi_{e_{n+1}}(V_i) \to R$, and $r \in \mathbb{Q}$ such that for $q = (x; q_{n+1}) \in V_i$:*

$$\xi(q) \sim_{\mathbb{Q}} (q_{n+1} - \theta(x))^r \, a(x). \tag{3.4}$$

Definition 3.8. The subset $A \subset R^n$ is **the graph for** $\lambda \in S^{n-1}$ of the function $\xi : E \to R$, where $E \subset N_\lambda$, if

$$A = \{q \in \pi_\lambda^{-1}(E) : q_\lambda = \xi(\pi_\lambda(q))\}.$$

If A is the graph for λ of the function $\xi : N_\lambda \to R$, we denote by

$$E(A, \lambda) := \{q \in R^n : q_\lambda \leq \xi(\pi_\lambda(q))\}.$$

If A is a family of $R \times R^n$ such that A_t is the graph for λ of the function $\xi_t : N_\lambda \to R$ for every t, then $E(A_t, \lambda)$, $t \in R$, is a semialgebraic family of sets of $R \times R^n$. Indeed, since $S^{n-1} \subset S^n$, $E(A, \lambda)$ is also well defined, and is the semialgebraic family of sets whose fiber at t is $E(A_t, \lambda)$.

When dealing with families of $R \times R^n$, we will also write π_λ for the (constant) family of mappings $\pi_\lambda : R \times R^n \to R \times R^n$ given by $\pi_{\lambda,t}(x) := \pi_\lambda(x)$ for $(t, x) \in R \times R^n$.

The next proposition is a consequence of the preparation theorem that will be of service for us.

Proposition 3.9. [V1] *Let $\xi : R^n \to R$ be a nonnegative semialgebraic function. There exists a finite semialgebraic partition of R^n such that over each element of this partition, the function ξ is $\sim_R$ to a product of powers of distances to semialgebraic subsets of R^n.*

Proposition 3.10. [V1] *Let B be a connected subset of S^{n-1}, $\lambda_0 \in B$, and let $\xi : N_{\lambda_0} \to R$ be a continuous semialgebraic function. Let H be the graph of ξ for λ_0. Suppose that B is regular for H. Then, for any $\lambda \in B$ the set H is the graph of a function $\xi^\lambda : N_\lambda \to R$. Moreover the set $E(H;\lambda)$ is independent of $\lambda \in B$.*

We now formulate some elementary observations that we shall need and which are taken from [V1].

Observations. Let $\lambda \in S^{n-1}$ and $r \in \mathbb{Q}_+$.

(1) If A is a union of graphs for λ of some $\mathbb{Q}-$Lipschitz functions then there exists $r \in \mathbb{Q}_+$ such that $B(\lambda;r)$ is regular for A. Also, if $B(\lambda;r) \subseteq S^{n-1}$ is regular for the semialgebraic set $A \subseteq R^n$, then A is the union of the graphs for λ of some $\mathbb{Q}-$Lipschitz functions. Moreover, if A is the graph for λ of a Lipschitz function $\xi : N_\lambda \to R$ then ξ is C-Lipschitz with $C \le \frac{1}{d(\lambda;\tau(A))}$.
(2) Every semialgebraic $C-$Lipschitz function ξ defined over a subset A of R^n may be extended to a semialgebraic $C-$Lipschitz function $\widehat{\xi}$ defined over the whole of R^n.
(3) If A is the union of the graphs for λ of some semialgebraic functions $\theta_1, \dots, \theta_k$ over N_λ we may find an ordered family of semialgebraic functions $\xi_1 \le \cdots \le \xi_k$ such that A is the union of the graphs of these functions for λ.
(4) Given a family of Lipschitz functions $f_{1,t} \dots, f_{k,t}$, $t \in R$, defined over $R \times R^{n-1}$, we can find some Lipschitz families of functions $\xi_{1,t} \le \cdots \le \xi_{l,t}$, $t \in R$, and a cell decomposition $\mathcal{D}$ of $R \times R^{n-1}$ such that for every cell $D \in \mathcal{D}$, the functions $|q_n - f_{i,t}(x)|$ (where $q = (t, x; q_n)$) are comparable with each other (for relation $\le$) and comparable with the functions $f_{i,t} \circ \pi_{e_n}$ on the cell $[\xi_{i|D_t}; \xi_{i+1|D_t}]$.

3.2. **Regular systems of hypersurfaces.** We now adapt the techniques of [V1] to families in order to prove Theorem 3.4.

The main tool of the proof of Proposition 3.3 is the notion of regular systems of hypersurfaces. We shall generalize it to one parameter families, introducing the notion of *families of* regular systems of hypersurfaces.

Definition 3.11. A **family of regular systems of hypersurfaces** of $R \times R^n$ is a family $H = (H_k;\lambda_k)_{1 \le k \le b}$ with $b \in \mathbb{N}$, of semialgebraic families H_k of $R \times R^n$ together with elements of $\lambda_k \in S^{n-1}$ such that the following properties hold for each $k < b$:

(i) For every $t \in R$, the sets $H_{k,t}$ and $H_{k+1,t}$ are the respective graphs for λ_k of two functions $\xi_{k,t}$ and $\xi'_{k,t}$ such that $\xi_{k,t} \le \xi'_{k,t}$.
(ii) The functions $\xi_{k,t}$ and $\xi'_{k,t}$ are C-Lipschitz with $C \in \mathbb{Q}$ (independent of t) and vary continuously with respect to t.
(iii) For every t we have:

$$E(H_{k+1,t};\lambda_k) = E(H_{k+1,t};\lambda_{k+1})$$

Let A be a semialgebraic family of $R \times R^n$. We say that the family H is **compatible** with A, if $A \subset \bigcup_{k=1}^b H_k$. An **extension** of H is a family of regular systems of hypersurfaces H' compatible with the set $\bigcup_{k=1}^b H_k$.

Observe that H_k is by definition the graph of the function $(x,t) \mapsto \xi_{k,t}(x)$ for $\lambda_k \in S^{n-1} \subset S^n$. Hence, $E(H_{k,t};\lambda_k)$ is the fiber at t of the semialgebraic family $E(H_k;\lambda_k)$.

Given a positive integer $k < b$, we set:

$$G_k(H) := E(H_{k+1};\lambda_k) \setminus int(E(H_k;\lambda_k)).$$

We shall write $\Lambda_k(H)$ for the connected component of

$$\{\lambda \in S^{n-1} : \lambda \text{ is regular for the family } H_k \cup H_{k+1}\}$$

which contains λ_k. Note that by Proposition 3.10, the family $G_k(H)$ may be defined using any $\lambda \in \Lambda_k(H)$.

We will say that another family of regular systems H' **coincides with** H **outside** $G_k(H)$ if for each j either $H'_j \subset G_k(H)$ or there exists j' such that $H'_j = H_{j'}$.

Remark 3.12. It is always possible to assume that the $G_k(H)$'s are of nonempty interior. Indeed if $int(G_k(H)) = \emptyset$ then $H_k = H_{k+1}$ and in this case we may remove $(H_k; \lambda_k)$ from the sequence.

Given $\lambda \in S^n$, we define $\widetilde{\pi}_\lambda : S^n \setminus \{\pm\lambda\} \to S^n \cap N_\lambda$ by $\widetilde{\pi}_\lambda(u) := \frac{\pi_\lambda(u)}{|\pi_\lambda(u)|}$.

Remark 3.13. Suppose $B \subset S^{n-2}$ to be regular for a subset $A \subset R^{n-1}$. Then, for any $a \in \mathbb{Q}_+$ the set

$$\widetilde{\pi}_{e_n}^{-1}(B) \cap \{\lambda \in S^{n-1} : d(\lambda; \{\pm e_n\}) \geq a\}$$

is regular for $\pi_{e_n}^{-1}(A)$. Furthermore, if A is the graph of a $\mathbb{Q}-$Lipschitz function for $\lambda \in B$, and if B is connected, then $\pi_{e_n}^{-1}(A)$ is the graph of a $\mathbb{Q}-$Lipschitz function for any λ' in

$$\widetilde{\pi}_{e_n}^{-1}(B) \cap \{\lambda' \in S^{n-1} / d(\lambda'; \{\pm e_n\}) \geq a\},$$

for any $a \in \mathbb{Q}_+$ (by Proposition 3.10). Moreover, in this case the following holds:

$$E(\pi_{e_n}^{-1}(A); \lambda') = \pi_{e_n}^{-1}(E(A; \lambda)).$$

3.3. **Some preliminary Lemmas.** We want to prove that every semialgebraic one-parameter family $A \subset R \times R^n$ with $\dim A_t < n$ for every $t \in R$, admits a family of regular systems compatible with it (Proposition 3.19). For this purpose, we prove some lemmas.

The following lemma says that we will be able to assume that the interiors of the $G_k(H)$'s are connected.

Lemma 3.14. *Let H be a family of regular systems of hypersurfaces. There exists an extension $\widehat{H}$ of H such that all the sets $int(G_k(\widehat{H}))$ are connected.*

Proof. Let $1 \leq m \leq b-1$. Suppose that $int(G_m(H))$ is not connected. Let $A_1, \dots, A_\nu$ be the connected components of $int(G_m(H))$. Set $A'_i = \pi_{\lambda_m}(A_i)$. For $t \in R$, the fiber $A_{i,t}$ is of the form:

$$\{q \in A'_{i,t} \oplus \lambda_m \cdot R \;/\; \xi_{m,t}(\pi_{\lambda_m}(q)) < q_{\lambda_m} < \xi'_{m,t}(\pi_{\lambda_m}(q))\}.$$

Clearly $\xi_{m,t} = \xi'_{m,t}$ on the boundary of $A'_{i,t}$. We thus may define some Lipschitz functions η_i, $1 \leq i \leq \nu-1$, as follows. We set over $A'_{j,t}$, $\eta_{i,t} := \xi'_{m,t}$, when $1 \leq j \leq i$, and $\eta_{i,t} := \xi_{m,t}$ whenever $i < j$. Extend the function $\eta_{i,t}$ by setting $\eta_{i,t} := \xi_{m,t} = \xi'_{m,t}$ on $N_{\lambda_m} \setminus \pi_{\lambda_m}(int(G_m(H)))$.

Therefore, we have that $\eta_{1,t} \leq \dots \leq \eta_{(\nu-1),t}$. Now, it suffices to

- let $\widehat{H}_k := H_k$ and $\widehat{\lambda}_k := \lambda_k$ if $k \leq m$
- let $\widehat{H}_{k,t}$ be the graph of $\eta_{k-m,t}$ for λ_m (for every $t \in R$) and $\widehat{\lambda}_k := \lambda_m$ for $m+1 \leq k \leq m+\nu-1$
- let $\widehat{H}_k := H_{k-\nu+1}$ and $\widehat{\lambda}_k := \lambda_{k-\nu+1}$ if $m+\nu \leq k \leq b+\nu-1$.

This is clearly a family of regular systems of hypersurfaces. Note that the $int(G_k(\widehat{H}))$, $m \leq k < m+\nu$, are the connected components of $int(G_m(H))$. $\square$

Given a family of regular systems of hypersurfaces (of $R \times R^n$) H, it will be convenient to extend the notations in the following way. Set for any $t \in R$: $H_{0,t} := \{-\infty\}$ and $H_{b+1,t} := \{+\infty\}$. By convention, all the elements of S^{n-1} will be regular for these two families. We will also consider that these two families of sets as the respective graphs of two functions which take $-\infty$ and $+\infty$ as constant values (for any λ). Define also $\lambda_0 := \lambda_1$, $\lambda_{b+1} := \lambda_b$, as well as $E(H_0; \lambda_0) := \emptyset$, $G_0(H) := E(H_1; \lambda_1)$, $G_b(H) := R \times R^n \setminus int(E(H_b; \lambda_b))$, as well as $E(H_{b+1}, \lambda_{b+1}) = R \times R^n$. Remark that now $R \times R^n = \bigcup_{k=0}^{b} G_k(H)$.

Lemma 3.15. *Let $H = (H_k; \lambda_k)_{1 \leq k \leq b}$ be a family of regular systems of hypersurfaces and let $j \in \{0, \ldots, b\}$. Let X be a semialgebraic family of subsets of $G_j(H)$ such that λ_j is regular for X. Then H can be extended to a family of regular systems of hypersurfaces H' compatible with X, which coincides with H outside $G_j(H)$.*

Proof. By property (i) of Definition 3.11, for every t, the sets $H_{j,t}$ and $H_{j+1,t}$ are the respective graphs for λ_j of two functions $\xi_{j,t}$ and $\xi'_{j,t}$. By Observations (1) and (2), the sets X_t, $t \in R$, may be included in a finite number of graphs for λ_j of functions, say $\theta_{1,t}, \ldots, \theta_{\nu,t}$, continuous with respect to t and C-Lipschitz, with $C \in \mathbb{Q}$ independent of t. Furthermore, by Observation (3), these families of functions can be assumed to be ordered and satisfy $\xi_{j,t} \leq \theta_{i,t} \leq \xi'_{j,t}$, for every t. Now,

- let $H'_k := H_k$ and $\lambda'_k := \lambda_k$ whenever $1 \leq k \leq j$,
- let $H'_{k,t}$ be the graph of $\theta_{k-j,t}$ for λ_j and $\lambda'_k := \lambda_j$ for $j < k \leq j + \nu$, $t \in R$,
- let $H'_k := H_{k-\nu}$ and $\lambda'_k := \lambda_{k-\nu}$, whenever $j + 1 + \nu \leq k \leq b + \nu$.

Properties (i), (ii) and (iii) clearly hold by construction. □

Lemma 3.16. *Let $U_1, \ldots, U_m$ be semialgebraic families covering $R \times R^n$. There exist finitely many semialgebraic families $V_1, \ldots, V_p$ covering $R \times R^n$ such that:*

1. *For every $i \leq p$, there are j and j' such that $V_i \subset U_j \cup U_{j'}$.*
2. *For every $i \leq p$ and $t \in R$, the fiber $(\delta V_i)_t$ has empty interior in R^n (see Notations 0.1 for δ).*

Proof. Let $f : R \times R^n \to R$ be the projection onto the x_1-axis. Consider a C^0 triangulation $h : |K| \to R \times R^n$ of f such that the families $U_1, \ldots, U_m$ are unions of images of simplices (up to a homeomorphism we may assume that the domain of f is bounded). Let $\sigma \in K$ be of dimension $(n + 1)$. The set $\delta h(\sigma)$ is the union of the images of the faces of σ of dimension $< n + 1$. Thus, $\delta h(\sigma)_t$ is of dimension n if and only if a face τ of σ of dimension n lies in the fiber σ_t. In this case there must be another simplex $l(\sigma)$ of which τ is also a face. The face τ is clearly always unique.

If the fiber $(\delta h(\sigma))_t$ is of dimension less than n for any t then set $l(\sigma) := \sigma$. Let $V_\sigma := cl(h(\sigma) \cup h(l(\sigma)))$. The family V_σ, $\sigma \in K$, has the required properties. □

Lemma 3.17. *Let $A \subset R \times R^n$ be a semialgebraic family of sets with $int(A_t) = \emptyset$ for any $t \in R$. There exists an integer ν such that for any $\varepsilon > 0$ we can find a finite semialgebraic partition $(A_i)_{i \in I}$ of $R \times R^{n-1}$ such that for every i the set*

$$\cup_{t \in R} \; \tau(\pi_{e_n}^{-1}(A_{i,t}) \cap A_t)$$

is included in ν balls of radius ε (in $\mathbb{G}_n$).

Proof. We can cover the Grassmanian by finitely many balls of radius ε. This gives rise to a covering $U_1, \ldots, U_k$ of A (via the Gaussian mappings $A_{t,reg} \ni x \mapsto T_x A_{t,reg}$). Consider a cell decomposition of $R \times R^n$ compatible with $U_1, \ldots, U_k$. The images of the cells under the canonical projection onto $R \times R^{n-1}$ constitute a covering having the desired property. □

Remark 3.18. The integer ν is indeed bounded by the maximal number of connected components of the fibers of the restriction of π_{e_n} to A.

3.4. **Existence of regular families.** We are ready to associate a family of regular systems of hypersurfaces to every semialgebraic family of nowhere dense sets.

Theorem 3.19. *Given a semialgebraic family of sets A of $R \times R^n$ such that every fiber A_t is of empty interior, there exists a family of regular systems of hypersurfaces of $R \times R^n$ compatible with A.*

Proof. Actually, we are going to prove by induction on n that there exists a family of regular systems of hypersurfaces of $R \times R^n$ compatible with a given semialgebraic family A of $R \times R^n$ (whose fibers have positive codimension) such that all the λ_k's can be chosen in a given ball $B(\lambda; \eta)$ in S^{n-1}, for $\eta \in \mathbb{Q}_+$.

For $n = 1$ the result is clear. So, we assume that it is true for $(n-1)$. Let A be a semialgebraic family of $R \times R^n$ such that A_t has empty interior for every t and consider a ball $B(\lambda; \eta) \subset S^{n-1}$, $\eta \in \mathbb{Q}_+$. We split the induction step into several steps.

Step 1. There exists a family of regular systems of hypersurfaces $H = (H_k; \lambda_k)_{1 \leq k \leq b}$ with $\lambda_k \in B(\lambda; \frac{\eta}{2})$ and such that for every k the family $G_k(H) \cap A$ has a regular vector $P \in S^{n-1} \setminus B(\pm\lambda, \frac{\eta}{2})$.

Take $e \in S^{n-1}$ such that $\pm e \notin B(\lambda; \eta)$ (we may assume η small).

By Lemma 3.17, for any $\sigma \in \mathbb{Q}_+$, there exists a finite semialgebraic partition $(A_i)_{i \in I}$ of $R \times N_e$ such that, for each $i \in I$, the set $\bigcup_{t \in R} \tau(\pi_e^{-1}(cl(A_{i,t})) \cap A_t)$ is included in the union of ν balls in $\mathbb{G}_n$ of radius $\frac{\sigma}{2}$. Consider such a partition for the σ given by Lemma 3.5. By Lemma 3.16, we may assume that $(\delta A_i)_t$ has empty interior for every t. Changing η, we may assume that $\eta \leq \frac{\sigma}{4}$.

Choose $\eta' \in \mathbb{Q}_+$ such that we have in S^{n-2}:

$$B(\widetilde{\pi}_e(\lambda); \eta') \subset \widetilde{\pi}_e(B(\lambda; \frac{\eta}{2})), \tag{3.5}$$

Apply the induction hypothesis (identify $R \times N_e$ with $R \times R^{n-1}$) to the families δA_i to get a family of regular systems of $R \times R^{n-1}$, $\overline{H} = (\overline{H}_k; \overline{\lambda}_k)_{k \leq b}$, such that all the $\overline{\lambda}_k$'s belong to $B(\widetilde{\pi}_e(\lambda); \eta')$.

By lemma 3.14, up to a refinement, we may assume that each $int(G_k(\overline{H}))$ is connected. We may also assume it to be of nonempty interior (see remark 3.12).

We claim that for each j and k, either $int(G_k(\overline{H}))$ is disjoint from A_j or $int(G_k(\overline{H})) \subset A_j$. To see this, observe that, as $\overline{H}$ is compatible with the δA_j's, all the sets $A_j \cap int(G_k(\overline{H}))$ are open and of empty (topological) boundary in $int(G_k(\overline{H}))$. Hence, if nonempty, these are connected components of $int(G_k(\overline{H}))$. But, as $int(G_k(\overline{H}))$ is connected, this entails that $A_j \cap int(G_k(\overline{H}))$ is either the empty set or $int(G_k(\overline{H}))$ itself, as claimed.

We turn to define the family of regular systems H claimed in step 1. For $1 \leq k \leq b$, let:

$$H_k := \pi_e^{-1}(\overline{H}_k).$$

Since $\overline{\lambda}_k \in B(\widetilde{\pi}_e(\lambda); \eta')$, by (3.5), we have $\overline{\lambda}_k \in \widetilde{\pi}_e(B(\lambda; \frac{\eta}{2}))$. Choose some $\lambda_k \in \widetilde{\pi}_e^{-1}(\overline{\lambda}_k) \cap B(\lambda; \frac{\eta}{2})$.

As $\lambda_k \in B(\lambda; \frac{\eta}{2})$ and neither e nor $-e$ belongs to $B(\lambda; \eta)$ we have:

$$d(\lambda_k; \pm e) \geq \frac{\eta}{2}, \qquad \forall\, k \leq b.$$

So, by Remark 3.13 (identify again $R \times N_e$ with $R \times R^{n-1}$), the set $H_{k,t}$ is the graph of a semialgebraic Lipschitz function. Moreover, as $\overline{H}$ satisfies $(i - iii)$, again by Remark 3.13, conditions $(i - iii)$ are clearly fulfilled by $H := (H_k; \lambda_k)_{k \leq b}$.

By Lemma 3.5 and our choice of σ, for all m, the family $A \cap int(G_m(H))$ is the union of finitely many semialgebraic families having a common regular element $P \in S^{n-1}$ (since we have seen that each $int(G_m(\overline{H}))$ is included in A_j for some j). Moving slightly P, we may assume that $d(P, \pm\lambda) \geq \eta$ (we have assumed $\eta \leq \frac{\sigma}{4}$).

This completes the first step.

The flaw of the first step is that the regular vector that we get for $G_m(H) \cap A$ might not be in $\Lambda_m(H)$. If it belongs to this set, Lemma 3.15 is enough to conclude. The next step provides another system $\widehat{H}$. We will then have to find (in Step 3) a common refinement of H and $\widehat{H}$, obtained at step 1 and 2 respectively.

Step 2. Fix $m \leq b$. There exists a family of regular systems of hypersurfaces $\widehat{H}$ such that for every k, $\widehat{\lambda}_k$ belongs to $\Lambda_m(H)$ and is regular for $G_m(H) \cap G_k(\widehat{H}) \cap A$.

Note that as λ_m is regular for the semialgebraic family of sets $H_m \cup H_{m+1}$, there exists $r \in \mathbb{Q}_+$ such that $B(\lambda_m; r)$ is regular for $H_m \cup H_{m+1}$. Taking r smaller if necessary, we may assume that $r \leq \frac{\eta}{2}$.

Let $r' \in \mathbb{Q}_+$ be such that we have in S^{n-2}:

$$B(\widetilde{\pi}_P(\lambda_m); r') \subset \widetilde{\pi}_P(B(\lambda_m; \frac{r}{2})). \tag{3.6}$$

To complete the proof of step 2 we need a lemma.

Lemma 3.20. *Let l in S^{n-1}, $r \in \mathbb{Q}_+$ and $\mu \in \mathbb{N}$. Let C be a subset of $\mathbb{G}_n$ and $P \in S^{n-1}$ such that:*

$$d(P; C) \geq \sigma, \tag{3.7}$$

with $\sigma \in \mathbb{Q}_+$. There exists $\alpha \in \mathbb{Q}_+$ such that for any $P_1, \dots, P_\mu$ in C and any $y \in \widetilde{\pi}_P(B(l; \frac{r}{2}))$ there exists $\widehat{\lambda} \in B(l; r) \cap \widetilde{\pi}_P^{-1}(y)$ such that:

$$d(\widehat{\lambda}; \cup_{i=1}^{\mu} P_i) \geq \alpha.$$

The proof of this lemma is postponed. We first see why it is enough to carry out the proof of step 2. Let μ be the maximal number of points of a finite fiber of the restriction of π_P to $A \cap G_m(H)$. Applying this lemma with this integer μ, with $C = \cup_{t \in R} \tau(A_t \cap G_m(H))$ and $l = \lambda_m$, we get a positive constant α.

Applying Lemma 3.17 to $G_m(H) \cap A$ (identify π_P with π_{e_n}) provides a finite covering $(A'_i)_{i \in I'}$ of $R \times N_P$ such that for any $i \in I'$ and any t:

$$\tau(\pi_P^{-1}(A'_{i,t}) \cap G_m(H)_t \cap A_t) \subset \bigcup_{j=1}^{\mu} B(P_j; \frac{\alpha}{2}),$$

for some $P_1, \dots, P_\mu$ (depending on $i \in I'$) in $\tau(A \cap G_m(H))$. By Lemma 3.16, we may assume that $(\delta A'_i)_t$ has empty interior for every t and i.

By Lemma 3.20, for any $i \in I'$ and any $y \in \widetilde{\pi}_P(B(\lambda_m; \frac{r}{2}))$, there exists $\widehat{\lambda} \in B(\lambda_m; r) \cap \widetilde{\pi}_P^{-1}(y)$ such that for any $t \in R$:

$$d\big(\widehat{\lambda}\,;\, \tau(\pi_P^{-1}(A'_{i,t}) \cap G_m(H)_t \cap A_t)\big) \geq \frac{\alpha}{2}. \tag{3.8}$$

Apply the induction hypothesis to get a family of regular systems of hypersurfaces H'' of $R \times N_P$ (identify N_P with R^{n-1}) compatible with the $\delta A'_i$'s. Do it in such a way that all the associated lines λ''_k are elements of $B(\widetilde{\pi}_P(\lambda_m); r')$ (where r' is given by (3.6)).

Define now:

$$\widehat{H}_{k,t} := \pi_P^{-1}(H''_{k,t}). \tag{3.9}$$

The compatibility with the sets $\delta A'_i$ implies that every $int(G_k(H''))$ is included in A'_i for some i (by the same argument that the one we used in Step 1 for $G_k(H)$ and the partition $(A_i)_{i\in I}$).

As a matter of fact, according to (3.8) for $y = \lambda''_k$, we know that for every integer $k \le b''$ there exists $\widehat{\lambda}_k \in B(\lambda_m; r) \cap \widetilde{\pi}_P^{-1}(\lambda''_k)$ such that for any $t \in R$:

$$d\big(\widehat{\lambda}_k\,;\,\tau(\pi_P^{-1}(G_k(H'')_t) \cap G_m(H)_t \cap A_t)\big) \ge \alpha. \tag{3.10}$$

Let us check that $\widehat{H} := (\widehat{H}_k; \widehat{\lambda}_k)_{k\le\widehat{b}}$ (where $\widehat{b} := b''$) is the desired family of regular systems of hypersurfaces. For this purpose, observe that, since neither P nor $-P$ belongs to $B(\lambda;\eta)$, we have for each k (recall that $r \le \frac{\eta}{2}$):

$$d(\widehat{\lambda}_k; \pm P) \ge \frac{r}{2}.$$

By construction and Remark 3.13, as $\widehat{\lambda}_k \in \widetilde{\pi}_P^{-1}(\lambda''_k)$, this implies that the family $\widehat{H}$ fulfills the three conditions of definition 3.11.

Furthermore, as $B(\lambda_m; r) \subset B(\lambda;\eta)$ (since $r \le \frac{\eta}{2}$ and $\lambda_m \in B(\lambda, \frac{\eta}{2})$), all the $\widehat{\lambda}_k$'s belong to $B(\lambda;\eta)$. Note also that as $B(\lambda_m; r)$ is regular for $H_m \cup H_{m+1}$, the vector $\widehat{\lambda}_k$ belongs to $\Lambda_m(H)$. This completes the proof of the second step.

The inconvenient of Step 2 is that the provided vector is regular for the family $A \cap G_m(H) \cap G_k(\widehat{H})$ (instead of $A \cap G_k(\widehat{H})$). If $\widehat{H}$ were an extension of the family H constructed in Step 1, this would be no problem since in this case we would have $G_k(\widehat{H}) \subset G_m(H)$ (or $int(G_k(\widehat{H})) \cap int(G_m(H)) = \emptyset$). Thus, we will have to find a common extension $\widetilde{H}$ of H and $\widehat{H}$ given by steps 1 and 2 respectively. This is what is carried out in the third step.

Step 3. There exists an extension $\widetilde{H} = (\widetilde{H}_k, \widetilde{\lambda}_k)_{k\le\widetilde{b}}$ of H which coincides with H outside $G_m(H)$ and such that $\widetilde{\lambda}_k$ is regular for the family $A \cap G_k(\widetilde{H}) \cap G_m(H)$ for all k.

Let $k \le \widehat{b}$ be an integer. Since $\widehat{\lambda}_k \in \Lambda_m(H)$, by Proposition 3.10, the sets H_m and H_{m+1} are respectively the graphs for $\widehat{\lambda}_k$ of two functions μ_k and μ'_k. Moreover, the set $\widehat{H}_k$ is also the graph for $\widehat{\lambda}_k$ of a function $\widehat{\xi}_k$. Define:

$$\eta_k := \min(\max(\mu_k; \widehat{\xi}_k); \mu'_k)$$

in order to get a function whose graph is included in $G_m(H)$. Now we define the desired regular family $(\widetilde{H}_k; \widetilde{\lambda}_k)_{1\le k\le\widetilde{b}}$ as follows.

- Let $\widetilde{H}_k := H_k$ and $\widetilde{\lambda}_k := \lambda_k$ if $k < m$.
- Let $\widetilde{H}_m := H_m$ and $\widetilde{\lambda}_m := \widehat{\lambda}_1$.
- Let $\widetilde{H}_k$ be the graph of η_{k-m} for $\widehat{\lambda}_{k-m}$, and let $\widetilde{\lambda}_k := \widehat{\lambda}_{k-m}$, whenever $m+1 \le k \le m+\widehat{b}$.
- And finally let $\widetilde{H}_k := H_{k-\widehat{b}}$ and $\widetilde{\lambda}_k := \lambda_{k-\widehat{b}}$ if $m+\widehat{b}+1 \le k \le b+\widehat{b}$.

We shall check that the properties $(i - iii)$ hold for the family $\widetilde{H}$ in every case.

For $k < m-1$, or $k \ge m+\widehat{b}+1$, the result is clear since the family $\widetilde{H}$ is indeed the family H.

For $k = m-1$, properties $(i - iii)$ follow from $(i - iii)$ for H and Proposition 3.10 since we have assumed $\widehat{\lambda}_1 \in \Lambda_m(H)$.

It remains to check $(i - iii)$ for $\widetilde{H}_{k+m}$, with $0 < k \le \widehat{b}$. Let us check (i) in this case.

By (i) for $\widehat{H}$, the set $\widehat{H}_{k+1}$ is the graph for $\widehat{\lambda}_k$ of a function $\widehat{\xi}'_k$ such that $\widehat{\xi}_k \leq \widehat{\xi}'_k$. Define now:

$$\eta'_k = \min(\max(\mu_k; \widehat{\xi}'_k); \mu'_k).$$

Claim. The graph of η'_k for $\widehat{\lambda}_k$ is that of η_{k+1} for $\widehat{\lambda}_{k+1}$.

To see this, note that the graph of η'_k (resp. η_k) matches with $\widehat{H}_{k+1}$ over $E(H_{m+1}; \widehat{\lambda}_k) \setminus E(H_m; \widehat{\lambda}_k)$ (resp. $\widehat{\lambda}_{k+1}$). But, by Proposition 3.10, the sets $E(H_m; l)$ and $E(H_{m+1}; l)$ do not depend on $l \in \Lambda_m(H)$. As $\widehat{\lambda}_k$ and $\widehat{\lambda}_{k+1}$ both belong to $\Lambda_m(H)$, this already shows that the two graphs involved in the above claim match over $int(G_m(H))$.

The graph of η'_k (resp. η_{k+1}) for $\widehat{\lambda}_k$ (resp. $\widehat{\lambda}_{k+1}$) is also constituted by the points of $H_m \setminus int(E(\widehat{H}_{k+1}, \widehat{\lambda}_k))$ (resp. $\widehat{\lambda}_{k+1}$) on the one hand and by the points of $H_{m+1} \cap E(\widehat{H}_{k+1}, \widehat{\lambda}_k)$ (resp. $\widehat{\lambda}_{k+1}$) on the other hand. By (iii) for $\widehat{H}$, the claim ensues.

This claim proves that $\widetilde{H}_{m+k+1}$ is the graph of η'_k for $\widehat{\lambda}_k$. Therefore, to check $(i - iii)$, we just have to prove that $\eta_k \leq \eta'_k$. But, as $\widehat{\xi}_k \leq \widehat{\xi}'_k$, this immediately comes down from the respective definitions of η'_k and η_k. This establishes (i) and (ii) (for $\widetilde{H}_{k+m}$, $k \leq \widehat{b}$).

Let us check property (iii) for $\widetilde{H}_{k+m}$, $k \leq \widehat{b}$. If $k = \widehat{b}$ it is a consequence of Proposition 3.10 since we have assumed that $\widehat{\lambda}_k$ belongs to $\Lambda_m(H)$.

Let k be such that $0 \leq k \leq \widehat{b} - 1$. First note that by (iii) for $\widehat{H}$ we have:

$$E(\widehat{H}_{k+1}; \widehat{\lambda}_k) = E(\widehat{H}_{k+1}; \widehat{\lambda}_{k+1}).$$

But, $E(\widetilde{H}_{k+m+1}; \widehat{\lambda}_k)$ (resp. $\widehat{\lambda}_{k+1}$) coincides with $E(\widehat{H}_{k+1}; \widehat{\lambda}_k)$ (resp. $\widehat{\lambda}_{k+1}$) over $int(G_m(H))$. It is also constituted by the points of $E(H_m, \widehat{\lambda}_k)$ (resp. $\widehat{\lambda}_{k+1}$) and the points of $E(H_{m+1}, \widehat{\lambda}_k) \cap E(\widehat{H}_{k+1}; \widehat{\lambda}_k)$ (resp. $\widehat{\lambda}_{k+1}$). As $\widehat{\lambda}_{k+1}$ and $\widehat{\lambda}_k$ both belong to $\Lambda_m(H)$, this establishes (iii).

To complete the proof of Step 3, it remains to make sure that for every $k \leq \hat{b}$ the line $\widetilde{\lambda}_{k+m}$ is regular for $G_{k+m}(\widetilde{H}) \cap G_m(H) \cap A$. By construction we have $\widetilde{\lambda}_m = \widehat{\lambda}_1$, $\widetilde{\lambda}_{k+m} = \widehat{\lambda}_k$ and:

$$G_{k+m}(\widetilde{H}) \subset G_k(\widehat{H}) \cap G_m(H), \tag{3.11}$$

for each $0 \leq k \leq \widehat{b}$.

As for any k the vector $\widehat{\lambda}_k$ is regular for $A \cap G_k(\widehat{H}) \cap G_m(H)$, this implies that for each $k \leq \widehat{b}$, the vector $\widetilde{\lambda}_{k+m}$ is regular for $G_{k+m}(\widetilde{H}) \cap A$. This completes the third step.

Finally, let us show why Step 3 is enough to conclude. By Lemma 3.15 (applied to $\widetilde{H}$ of Step 3), we may extend $\widetilde{H}$ to a family compatible with the set

$$G_m(H) \cap \cup_{k=0}^{\widetilde{b}} G_k(\widetilde{H}) \cap A = G_m(H) \cap A.$$

Since all the extensions coincide with H outside $G_m(H)$, we may carry out the construction on all the $G_m(H)$'s successively. This provides the desired family. ☐

It remains to prove Lemma 3.20. The lemma below describes a property of $\widetilde{\pi}_P$ that we need for this purpose.

Lemma 3.21. *Let λ and P in S^{n-1}, $T \in \mathbb{G}_n$ and $x \in T \cap \widetilde{\pi}_P^{-1}(\lambda)$. Let v be a unit vector tangent at x to the curve $\widetilde{\pi}_P^{-1}(\lambda)$. Then:*

$$d(P; T) \leq d(v; S^{n-1} \cap T).$$

Proof. Let w be the vector of $S^{n-1} \cap T$ which realizes $d(v; S^{n-1} \cap T)$. Remark that the vectors x, P, and v are in the same two dimensional vector space. Moreover $(x; v)$ is an orthonormal basis of this plane. Let $P = \alpha x + \beta v$ with $\alpha^2 + \beta^2 = 1$. Then, as x and w both belong to T we have

$$d(P;T) \leq |P - (\alpha x + \beta w)| = |\beta| \cdot |v - w| \leq d(v; S^{n-1} \cap T).$$

□

Proof of Lemma 3.20. We will work up to a ("projective") coordinate system of S^{n-1} defined as follows. Let U_i^+ (resp. U_i^-) denote

$$\{x \in S^{n-1}/x_i \geq \epsilon\}$$

(resp. $x_i \leq -\epsilon$) with $\epsilon \in \mathbb{Q}_+$ small enough. Define then: $h_i : U_i \to R^{n-1}$ by $h_i(x_1; \ldots; x_n) = (\frac{x_1}{x_i}; \ldots; \widehat{\frac{x_i}{x_i}}; \ldots; \frac{x_n}{x_i})$. Note that h_i is a $\mathbb{Q}$-bi-Lipschitz homeomorphism.

Through such a chart, the set $S^{n-1} \cap N_P$ is a vector subspace and $\widetilde{\pi}_P$ becomes an orthogonal projection along a line, say Q. By Lemma 3.21, hypothesis (3.7) implies that there exists $u \in \mathbb{Q}_+$ such that:

$$d(Q;T) \geq u,$$

for any $T \in C \subset \mathbb{G}_{n-1}$.

It is then an easy exercise of elementary geometry to derive from this that for any $x \in Q$ and any $P_1, \ldots, P_\mu$ in C:

$$d\big(x; \cup_{i=1}^{\mu} P_i \cap Q\big) \leq \frac{1}{u} \cdot d\big(x\,; \cup_{i=1}^{\mu} P_i\big). \tag{3.12}$$

For any $y \in \widetilde{\pi}_P(B(l; \frac{r}{2}))$ the length of the line segment $\widetilde{\pi}_P^{-1}(y) \cap B(l;r)$ is bounded below away from zero by a strictly positive rational number α_0.

Let α be the rational number $\frac{\alpha_0 u}{4\mu}$. Then, using (3.12) one can easily see that if the conclusion of the lemma failed for some $y \in \widetilde{\pi}_P(B(l; \frac{r}{2}))$, we could cover the segment $\widetilde{\pi}_P^{-1}(y) \cap B(l;r)$ by μ segments of length less than $\frac{\alpha_0}{2\mu}$. This contradicts the fact that the length of this segment is not less than α_0. □

3.5. **Proof of Proposition 3.4.**

Proof. By Proposition 3.19 there exists a family of regular systems of hypersurfaces $H = (H_k; \lambda_k)_{1 \leq k \leq b}$ compatible with A. We shall define h over $E(H_k; \lambda_k)$, by induction on k, in such a way that $h(E(H_k; \lambda_k)) = E(F_k; e_n)$ (so that $h(H_k) = F_k$) where F_k is the graph of a function $\eta_k : R \times R^{n-1} \to R$ for e_n.

For $k = 1$ choose an orthonormal basis of N_{λ_1} and set $h(q) := (x_{\lambda_1}; q_{\lambda_1})$ where x_{λ_1} are the coordinates of $\pi_{\lambda_1}(q)$ in this basis. Let $k \geq 1$. By (i) of Definition 3.11, the sets H_k and H_{k+1} are the graphs for λ_k of two functions ξ_k and ξ_k'. For $q \in E(H_{k+1}; \lambda_k) \setminus E(H_k; \lambda_k)$ define $h(q)$ as the element:

$$h(\pi_{\lambda_k}(q) + \xi_k(\pi_{\lambda_k}(q)) \cdot e_n) + (q_{\lambda_k} - \xi_k(\pi_{\lambda_k}(q)))e_n.$$

Thanks to the property (iii) of Definition 3.11 we have $E(H_{k+1}; \lambda_{k+1}) = E(H_{k+1}; \lambda_k)$, and hence h is actually defined over $E(H_{k+1}; \lambda_{k+1})$. Since $\xi_{k,t}$ is C-Lipschitz with $C \in \mathbb{Q}$, h_t is a family of bi-Lipschitz homeomorphisms. Note also that the image is $E(F_{k+1}; e_n)$ where F_{k+1} is the graph (for e_n) of the family of Lipschitz functions on $R \times R^{n-1}$:

$$\eta_{k+1}(x) := \eta_k(x) + (\xi_k' - \xi_k) \circ \pi_{\lambda_k} \circ h^{-1}(x; \eta_k(x)).$$

This gives h over $E(H_b; \lambda_b)$. To extend h to the whole of $R \times R^n$ do it similarly as in the case $k = 1$. □

4. On families of semialgebraic functions

Let $k(0_+)$ be the extension of R corresponding to the ultrafilter 0_+, constituted by all the semialgebraic sets of R containing a right-hand-side neighborhood of the origin (see [BCR]). The field $k(0_+)$ is the real closure of the fraction field of the ring $R[Y]$ endowed with the order relation that makes the indeterminate Y smaller than any element of R. We shall denote by $Y(0_+)$ the indeterminate regarded in $k(0_+)$.

Lemma 4.1. *For any $u \in k(0_+)$ there is a rational number ν such that:*

$$u \sim_R Y(0_+)^\nu.$$

Proof. There exists a semialgebraic function $\xi : (0,\varepsilon) \to R$ such that $\xi(Y(0_+)) = u$ (see [BCR]). By the preparation theorem, there exist a and b in R and $\nu \in \mathbb{Q}$ such that:

$$\xi(x) \sim_{\mathbb{Q}} b \cdot (x-a)^\nu.$$

Thus, $\xi(Y(0_+)) \sim_R Y(0_+)^\nu$ if $a = 0$ and $\xi \sim_R 1$ otherwise. □

Proposition 4.2. *Let $\xi : k(0_+)^n \to k(0_+)$ be a nonnegative semialgebraic function. There exists a cell decomposition of $k(0_+)^n$ such that over every cell:*

$$\xi(x) \sim_R Y(0_+)^r \cdot d(x,W_1)^{r_1} \cdots d(x,W_k)^{r_k} \tag{4.13}$$

where the W_i's are semialgebraic subsets of $k(0_+)^n$ and r as well as the r_i's are rational numbers.

Proof. We prove it by induction on n. The case $n = 0$ follows from Lemma 4.1.

Assume that the lemma is true for $(n-1)$ and apply the preparation theorem to the function ξ. Let $n \geq 1$ and let $\lambda_1, \dots, \lambda_N$ be the elements of S^{n-1} given by Lemma 3.6. Applying the preparation theorem (Theorem 3.7) to $\xi \circ A_i$, where A_i is an orthogonal linear mapping of $k(0_+)^n$ sending the vector e_n onto λ_i for $i \in \{1, \dots, N\}$, and taking a common refinement of the images under the A_i^{-1} of all the obtained partitions we get a semialgebraic partition $(V_j)_{j \in J}$ of $k(0_+)^n$. Therefore, over each V_j and for each i we can find continuous functions $a, \theta : \pi_{\lambda_i}(V_j) \to k(0_+)$ and $r \in \mathbb{Q}$ such that:

$$\xi(q) \sim_{\mathbb{Q}} (q_{\lambda_i} - \theta(x_{\lambda_i}))^r \, a(x_{\lambda_i}), \tag{4.14}$$

for $q = x_{\lambda_i} + q_{\lambda_i}\lambda_i \in \pi_{\lambda_i}(V_j) \oplus k(0_+) \cdot \lambda_i$.

Apply Proposition 3.6 to the family constituted by all the sets of the partition $(V_j)_{j \in J}$ and the zero locus of ξ. This gives rise to a partition $(V'_j)_{j \in J'}$ such that each V'_j which is open is of the form

$$\{q \in \pi_{\lambda_i}(V'_j) \oplus k(0_+) \cdot \lambda_i : \quad \xi_1(\pi_{\lambda_i}(q)) < q_{\lambda_i} < \xi_2(\pi_{\lambda_i}(q))\},$$

for some $i \in \{1, \dots, N\}$, where $\xi_\nu : \pi_{\lambda_i}(V'_j) \to k(0_+)$, $\nu = 1, 2$, are $\mathbb{Q}$−Lipschitz functions, and such that the function ξ is of the form (4.14) on V'_j for each vector λ_i.

Thanks to the induction hypothesis (identify N_{λ_i} with $k(0_+)^{n-1}$) it is sufficient to prove the result for the function $|q_{\lambda_i} - \theta(\pi_{\lambda_i}(q))|$.

Fix $j \in J'$. Due to the compatibility of the partition with the zero locus of ξ, we have, for every x, either $\theta(x) \leq \xi_1(x)$ or $\theta(x) \geq \xi_2(x)$. Up to a subpartition we may assume that only one case occurs over V'_j, for instance $\theta \leq \xi_1$. Writing

$$q_{\lambda_i} - \theta(\pi_{\lambda_i}(q)) = (q_{\lambda_i} - \xi_1(\pi_{\lambda_i}(q)) + (\xi_1(\pi_{\lambda_i}(q)) - \theta(\pi_{\lambda_i}(q))),$$

we see that (up to a refinement we may assume that these functions are comparable) $|q_{\lambda_i} - \theta(\pi_{\lambda_i}(q))|$ is $\sim_{\mathbb{Q}}$ either to $|q_{\lambda_i} - \xi_1(\pi_{\lambda_i}(q))|$ or to $|\xi_1(\pi_{\lambda_i}(q)) - \theta(\pi_{\lambda_i}(q))|$. For the former function, since ξ_1 is Lipschitz, $|q_{\lambda_i} - \xi_1(x_{\lambda_i})|$ is $\sim_{\mathbb{Q}}$ to the distance to the graph of ξ_1 for λ_i. For the latter one, this is a consequence of the induction hypothesis. For the V'_j's having positive codimension, one may deduce the result from the induction hypothesis. □

5. Proof of Theorem 2.4

Proof. We first check that we can assume, without loss of generality, that the mapping $f : X \to R$ is the projection on the first coordinate. Indeed, if we replace X with

$$\hat{X} := \{(y, x) \in R \times X : y = f(x)\},$$

and prove the result for $\hat{f} : \hat{X} \to R$, defined by $\hat{f}(y, x) := y$, we are done.

We shall establish a stronger result, proving by induction on n the following facts:

$(\mathcal{P}_n)$. Let $f : [-M, M] \times R^n \to R$, $M > 0$, be defined by $f(y, x) := y$. Let $A_1, \dots, A_k$ be semialgebraic subfamilies of $[-M, M] \times R^n$ and let $\eta_1, \dots, \eta_l$ be semialgebraic families of nonnegative functions on $[-M, M] \times R^n$. There is a triangulation (K, ϕ, ψ) of f such that:

(1) (2.3) holds.
(2) The A_i's are union of images (by ψ) of simplices.
(3) The functions $\eta_i \circ \psi$ are $\sim_R$ to standard simplicial functions.

For $n = 0$ the result is clear. Assume that it is true for some $n \geq 0$ and let us check it for $(n + 1)$. We denote by $\pi : R \times R^{n+1} \to R \times R^n$ the canonical projection.

We claim that there is a cell decomposition of $R \times R^{n+1}$ such that for every cell C, we can find some semialgebraic families $W_1, \dots, W_c$ of $R \times R^{n+1}$ as well as, for each i, some rational numbers $r, r_1, \dots, r_c$, and $y_0 \in R$, such that for $(y, x) \in C \subset [-M, M] \times R^{n+1}$:

$$\eta_{i,y}(x) \sim_R |y - y_0|^r d(x, W_{1,y})^{r_1} \cdots d(x, W_{c,y})^{r_c}, \tag{5.15}$$

where the constants of this equivalence are independent of y (below all the constants will be independent of the parameter y).

Let $\alpha \in \widetilde{[-M; M]}$ and denote by $k(\alpha)$ the corresponding extension of R. If α has a specialization then, by Proposition 4.2, we can find $U \in \alpha$ such that (5.15) holds true for the restriction of the η_i's to $U \times R^{n+1}$. If α has no specialization then every element of $k(\alpha)$ is bounded by an element of R. Hence, in this case (5.15) follows from Proposition 3.9 (applied to $\eta_{i,\alpha}$). In any case we thus find an element $U \in \alpha$ along which the desired equivalence may be established. By compactness of the real spectrum, we may extract a finite covering of $[-M, M]$. Taking a common refinement of all the corresponding cell decompositions, we get a cell decomposition $\mathcal{E}$ having the required property (5.15). We may assume that this cell decomposition is compatible the A_i's.

By Proposition 3.4, up to a family of bi-Lipschitz maps (that we will identify with the identity), we may assume that all the cells of this cell decomposition which are graphs (i.e. which are not bands) as well as the (topological) boundaries of the $W_{j,y}$'s (see (5.15)) are included in the union of a finite number of graphs of families of Lipschitz functions $\theta_{1,y} \leq \cdots \leq \theta_{\mu,y}$ (continuous with respect to y).

Applying Observation (4) to the θ_i's and to the functions $(y, x) \mapsto d(x; \pi(\delta W_{i,y}))$, we see that there exist a cell decomposition $\mathcal{D}$ of $R \times R^n$ and finitely many families of Lipschitz functions $\xi_{1,y} \leq \cdots \leq \xi_{m,y}$ whose graphs contain the graphs for the $\theta_{i,y}$'s, such that for every $D \in \mathcal{D}$, all the functions $|q_{n+1} - \theta_{\nu,y}(\pi(q))|$ are comparable (for $\leq$) with each other and comparable with the functions $d(x; \pi(\delta W_{i,y} \cap \Gamma_{\theta_{\nu,y}}))$ on the set $[\xi_{i,y|D_y}; \xi_{i+1,y|D_y}]$.

Consider a semialgebraic cell decomposition of $R \times R^{n+1}$ adapted to the graphs of the families of functions ξ_i, the cells of $\mathcal{D}$ and $\mathcal{E}$, as well as the sets W_j. Let $X_1, \dots, X_s$ be the images of the cells under π. Refining this partition, we may assume that the functions $d(x; \pi(\delta W_{i,y} \cap \Gamma_{\theta_{\nu},y}))$ are comparable with respect to each other on the cells. Apply the induction hypothesis to get a triangulation (K, ϕ, ψ) of f (restricted to $[-M, M] \times R^n$) such that the X_i's are unions of images of open simplices. Moreover, by (3) of the induction hypothesis, we may do it in such a way that over each simplex, each function $|\xi_j - \theta_i| \circ \psi$ as well as all the functions

$(y,x) \mapsto d(\psi_y(x); \pi(\delta W_{j,y} \cap \Gamma_{\theta_{i,y}}))$, and $\eta_{k,y}(\psi_y(x), \xi_{i,y}(\psi_y(x)))$, are $\sim$ to standard simplicial functions.

Let $\zeta_1 \leq \cdots \leq \zeta_m$ be piecewise linear functions over $|K|$ such that $\zeta_i \equiv \zeta_{i+1}$ on the set $\{\xi_i \circ \psi = \xi_{i+1} \circ \psi\}$ (this set is a subcomplex of K). Let also $\zeta_0 := \zeta_1 - 1$ and $\zeta_{m+1} := \zeta_m + 1$. Let

$$N = \{(y, x, q_{n+1}) \in R \times R^n \times R \ : \ \zeta_{0,y}(x) \leq q_{n+1} \leq \zeta_{m+1,y}(x)\}.$$

We obtain a polyhedral decomposition of N by taking the respective inverse images by $\pi_{|N}$ of the simplices of K of dimension n on the one hand, and by taking all the images of the simplices of $|K|$ by the mappings $x \to (x; \zeta_i(x))$ on the other hand. After a barycentric subdivision of this polyhedra we get a simplicial complex L.

Let $\widetilde{K}$ be the union of the open simplices σ included in

$$\{(y, x, q_{n+1}) \in |K| \times R : \zeta_{0,y}(x) < q_{n+1} < \zeta_{m+1,y}(x)\}.$$

Define now for $y \in R$ over $\widetilde{K}_y$ the desired family of homeomorphisms $\widetilde{\psi}_y$ in the following way:

$$\widetilde{\psi}_y(x; t\,\zeta_{i,y}(x) + (1-t)\zeta_{i+1,y}(x)) = (\psi_y(x); t\,\xi_{i,y}(\psi_y(x)) + (1-t)\xi_{i+1,y}(\psi_y(x)))$$

for $1 \leq i \leq m-1$, $x \in R^n$ and $t \in [0;1]$. Define also:

$$\widetilde{\psi}_y(x; t\,\zeta_{0,y}(x) + (1-t)\,\zeta_{1,y}(x)) = (\psi_y(x); \xi_1(\psi_y(x)) - \frac{t}{1-t})$$

and

$$\widetilde{\psi}_y(x; t\,\zeta_{m+1,y}(x) + (1-t)\,\zeta_{m,y}(x)) = (\psi_y(x); \xi_{m,y}(\psi_y(x)) + \frac{t}{1-t})$$

for $t \in [0;1)$. This defines a family of homeomorphisms $\widetilde{\psi} : |\widetilde{K}| \to [-M, M] \times R^{n+1}$.

We shall check that over each simplex σ the mapping $\widetilde{\psi}$ fulfills (2.3). Let $\sigma \subset [\zeta_i, \zeta_{i+1}]$ be a simplex of $\widetilde{K}$, q and q' two points of σ_y, $y \in R$ fixed. The points q and q' may be expressed $q = (x; t\zeta_i(x) + (1-t)\zeta_{i+1}(x))$ and $q' = (x'; t'\zeta_i(x') + (1-t')\zeta_{i+1}(x'))$ for some $0 \leq i \leq m$ and some $(t;t')$ in $[0;1]^2$. Then define

$$q'' := (x; t'\zeta_i(x) + (1-t')\zeta_{i+1}(x)).$$

We begin with the case where $1 \leq i \leq m-1$. Let $p = \widetilde{\psi}_y(q)$, $p' = \widetilde{\psi}_y(q')$ and $p'' = \widetilde{\psi}_y(q'')$. We may consider x, x', p, p' and p'' as functions of q and q'. As $\xi_{i,y}$ and $\xi_{i+1,y}$ are Lipschitz functions we have over $\sigma \times \sigma$:

$$|p - p'| \sim |p - p''| + |\psi_y(x) - \psi_y(x')|. \tag{5.16}$$

Let σ' be the simplex of K containing $\pi(\sigma)$. Thanks to the induction hypothesis, we may find some functions $\varphi_{\sigma',1}, \ldots, \varphi_{\sigma',n}$ and a tame system of coordinates $(x_{1,\sigma'}; \ldots; x_{n,\sigma'})$ such that for any x and x' in σ'_y:

$$|\psi_y(x) - \psi_y(x')| \sim \sum_{l=1}^{n} \varphi_{\sigma',l}(x;x')|x_{l,\sigma'} - x'_{l,\sigma'}|. \tag{5.17}$$

The result is therefore clear if $\zeta_i = \zeta_{i+1}$ on σ'. Otherwise, as $\pi(q) = \pi(q'')$, by construction we have:

$$|p_{n+1} - p''_{n+1}| \sim |q_{n+1} - q''_{n+1}| \cdot \frac{\xi_{i+1,y}(\psi_y(x)) - \xi_{i,y}(\psi_y(x))}{\zeta_{i+1,y}(x) - \zeta_{i,y}(x)}.$$

Recall that we have constructed the triangulation (K, ϕ, ψ) in such a way that for every i, $(\xi_{i+1} - \xi_i) \circ \psi$ is $\sim$ to a standard simplicial function of K, say ω_i. The composite $\omega_i \circ \pi$ gives a standard simplicial function of $\widetilde{K}$. The functions ζ_i and ζ_{i+1} define a tame coordinate on R^{n+1} that we will denote by $q_{n+1,\sigma}$. By the preceding estimation, we have:

$$|p - p''| \sim |q_{n+1,\sigma} - q'_{n+1,\sigma}| \cdot \varphi_{\sigma,n+1}(q; q') \tag{5.18}$$

for a standard simplicial function $\varphi_{\sigma,n+1}$ (which here actually depends only on q).

Define for $j < n+1$:

$$\varphi_{\sigma,j}(q; q') = \varphi_{\sigma',j}(\pi(q); \pi(q')).$$

Then by (5.18), (5.17) and (5.16) we get the desired equivalence (in the case $1 \le i \le m-1$).

The case $i = 0$ and m are dealt in an analogous way (see [V1] for details). This proves that $\widetilde{\psi}_y$ satisfies (2.3). By construction, the A_j's are images of open simplices.

It remains to check that the functions $\eta_j \circ \widetilde{\psi}$ are $\sim$ to standard simplicial functions over any simplex σ. Let $\sigma \in \widetilde{K}$; if the set $\widetilde{\psi}(\sigma)$ is included in the graph of ξ_i for some i, the result follows by induction. So, assume that it sits in $]\xi_i; \xi_{i+1}[$, for some $1 \le i \le m-1$. By construction, on $\widetilde{\psi}(\sigma)$, the $\eta_{j,y}$'s are $\sim_R$ to a product of powers of distances to the $W_{j,y}$'s (see (5.15)).

Therefore, it suffices to show the result for the functions $q \mapsto d(\widetilde{\psi}_y(q); W_{j,y})$. As $(\widetilde{\psi}; \widetilde{K})$ is also a triangulation of the sets W_j, for each j, either $\widetilde{\psi}(\sigma)_y$ is included in $W_{j,y}$ or the distance to $W_{j,y}$ is $\sim$ to the distance to its boundary. In the former case the result is obvious since the function $q \mapsto d(\widetilde{\psi}_y(q); W_{j,y})$ is zero over σ. By construction, the boundary δW_j is included in the union of the $\Gamma_{\theta_{\nu,y}}$'s.

Moreover, we have for any $\nu \in \{1, \dots, \mu\}$:

$$d(q; \delta W_{i,y} \cap \Gamma_{\theta_{\nu,y}}) \sim |q_{n+1} - \theta_{\nu,y}(x)| + d(x; \pi(\delta W_{i,y} \cap \Gamma_{\theta_{\nu,y}})) \tag{5.19}$$

where $q = (x; q_{n+1})$ in $\widetilde{\psi}_y(\sigma_y) \subset R^n \times R$.

As both terms of the right-hand-side are positive, the sum is $\sim$ to the max of these two terms that is to say is $\sim$ to one of them since they are comparable over $\widetilde{\psi}(\sigma)$. Note that clearly $d(q; \delta W_{i,y}) = \min_{1 \le \nu \le \mu} d(q; \delta W_{i,y} \cap \Gamma_{\theta_{\nu,y}})$. But as by construction the functions $g_{\nu,y} := d(\pi(q); \pi(\delta W_{i,y} \cap \Gamma_{\theta_{\nu,y}}))$ are comparable with each other and comparable with all the functions $|q_{n+1} - \theta_{\nu,y}(x)|$, the function $d(q; \delta W_{i,y})$ is equivalent over $\tilde{\psi}_y(\sigma_y)$ to one of the functions $g_{\nu,y}$ or to some function $|q_{n+1} - \theta_{\nu,y}(x)|$.

Recall that we have required the triangulation $(\psi; K)$ to be such that

$$(y, x) \mapsto d(\psi_y(x); \pi(\delta W_{j,y} \cap \Gamma_{\theta_{\nu,y}}))$$

is $\sim$ to a standard simplicial function of K. Hence, by (5.19), it suffices to prove that the function $(y, q) \mapsto |\psi_{n+1,y}(q) - \theta_{\nu,y}(\pi(\psi_y(q)))|$ is $\sim$ over σ to a standard simplicial function of $\widetilde{K}$. Assume that $\sigma \subset [\zeta_i; \zeta_{i+1}]$. We may write for $p = (y, x, p_{n+1}) \in \sigma \subset R \times R^n \times R$:

$$|p_{n+1} - \theta_\nu \circ \psi| = p_{n+1} - \xi_i \circ \psi + (\xi_i \circ \psi - \theta_\nu \circ \psi)$$

if $\theta_\nu \le \xi_i$ on $\pi(\tilde{\psi}(\sigma))$, and

$$|p_{n+1} - \theta_\nu \circ \psi| = \xi_{i+1} \circ \psi - p_{n+1} + (\theta_\nu \circ \psi - \xi_{i+1} \circ \psi)$$

if $\theta_\nu \ge \xi_{i+1}$ (with the convention $\xi_0 = -\infty$, $\xi_{m+1} = \infty$). By (5.18), we have over σ for $q = \tilde{\psi}^{-1}(p) = (y, z, q_{n+1})$:

$$p_{n+1} - \xi_{i,y}(\psi_y(x)) \sim |q_{n+1} - \zeta_{i,y}(z)| \,.\, \varphi_{\sigma,n+1}(q; q').$$

The function $|q_{n+1} - \zeta_{i,y}(x)|$ is $\sim$ to a standard simplicial function. As all the $|\xi_{i,y} \circ \psi_y - \theta_{\nu,y} \circ \psi_y|$ have been assumed to be equivalent to standard simplicial functions, the theorem is proved. $\square$

6. bi-Lipschitz triviality of families of functions

Definition 6.1. We say that a semialgebraic family of functions $f : X \to \mathbb{R}^p \times \mathbb{R}$ is **fiberwise semialgebraically bi-Lipschitz trivial along** $W \subset \mathbb{R}^p$ if there exist two families of semialgebraic homeomorphisms $h : W \times \mathbb{R}^n \to W \times \mathbb{R}^n$ and $\phi : W \times \mathbb{R} \to W \times \mathbb{R}$, such that for any $t \in W$:

(1) $h_t(X_{t_0}) = X_t$ and $\phi_t^{-1} \circ f_t \circ h_t = f_{t_0}$, $t_0 \in W$.
(2) ϕ_t is bi-Lipschitz.
(3) There is a constant $C_t \in \mathbb{R}$ such that the restriction of h_t to every fiber $f_t^{-1}(y)$ is C_t-bi-Lipschitz.

In the case where h_t is bi-Lipschitz (i.e. not only the restriction to the fibers but h_t itself), we say that it is **semialgebraically bi-Lipschitz trivial** along W.

Remark 6.2. It is worthy of notice that, in the definition of fiberwise bi-Lipschitz triviality, the mapping h_t is not only assumed to be C-bi-Lipschitz on every fiber: it is a homeomorphism.

The flaw of bi-Lipschitz triviality of functions is that it admits continuous moduli: the Lipschitz counterpart of Theorem 1.6 is not true, even for families as simple as two variable polynomials. The counterexample is due to A. Parusiński and J.-P. Henry.

Example 6.3. In [H-P] J-P. Henry and A. Parusiński gave the following example: $f_t(x, y) := x^3 + y^6 + 3t^2xy^4$. They proved by exhibiting some metric invariants for functions that there is no interval W of R along which this family is semialgebraically bi-Lipschitz trivial. As bi-Lipschitz triviality can be derived from triangulability (see proofs of Theorems 1.6 and 6.4), this example shows that in Theorem 2.4 we could not require (2.3) to hold for all couples (q, q') (not necessarily in the same fiber).

Nevertheless, fiberwise bi-Lipschitz triviality does *not* admit continuous moduli. This is the main theorem of this article.

Theorem 6.4. *Given a semialgebraic family of Lipschitz functions $f : X \to \mathbb{R}^p \times \mathbb{R}$ there exists a semialgebraic partition $V_1, \dots, V_m$ of $\mathbb{R}^p$ such that for every i, f is fiberwise semialgebraically bi-Lipschitz trivial along V_i.*

Proof. We apply exactly the same argument as in the proof of Theorem 1.6, replacing Theorem 1.2 with Theorem 2.4. As in the proof of the latter theorem, possibly replacing f_t with $u \circ f_t$ where $u(y) := \frac{y}{1+|y|}$, we may assume that f is bounded (if $\phi : R \to R$ is bi-Lipschitz and $\phi([-1, 1]) = [-1, 1]$ then $u^{-1} \circ \phi \circ u$ is bi-Lipschitz). By (2.3), the homeomorphisms h_t (at the end of the proof of Theorem 1.6) are C_t-bi-Lipschitz on the fibers $f_t^{-1}(y)$ with C_t independent of y. □

Remark 6.5. In the above theorem, we could also require the homeomorphism h_t (see Definition 6.1) to satisfy

$$d(h_t(x), f_t^{-1}(0)) \sim d(x, f_{t_0}^{-1}(0)).$$

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