Journal *of* Singularities

Volume 4

www.journalofsing.org

Journal
of
Singularities

Volume 4
2012

ISSN: # 1949-2006

Journal *of* Singularities

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Journal *of* Singularities
Volume 4 (2012), 1-22

received 9 September 2009
in revised form 14 July 2011
DOI: 10.5427/jsing.2012.4a

SPACES OF LOCALLY CONVEX CURVES IN $\mathbb{S}^n$ AND COMBINATORICS OF THE GROUP B^+_{n+1}

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Dedicated to the memory of the one and only Vladimir Igorevich Arnold

Abstract. In the 1920's Marston Morse developed what is now known as Morse theory trying to study the topology of the space of closed curves on $\mathbb{S}^2$ ([7], [5]). We propose to attack a very similar problem, which 80 years later remains open, about the topology of the space of closed curves on $\mathbb{S}^2$ which are locally convex (i.e., without inflection points). One of the main difficulties is the absence of the covering homotopy principle for the map sending a non-closed locally convex curve to the Frenet frame at its endpoint.

In the present paper we study the spaces of locally convex curves in $\mathbb{S}^n$ with a given initial and final Frenet frames. Using combinatorics of $B^+_{n+1} = B_{n+1} \cap SO_{n+1}$, where $B_{n+1} \subset O_{n+1}$ is the usual Coxeter-Weyl group, we show that for any $n \geq 2$ these spaces fall in at most $\lceil \frac{n}{2} \rceil + 1$ equivalence classes up to homeomorphism. We also study this classification in the double cover $\mathrm{Spin}(n+1)$. For $n = 2$ our results complete the classification of the corresponding spaces into two topologically distinct classes, or three classes in the spin case.

1. Introduction and main results

In what follows we will study different spaces of curves $\gamma : [0,1] \to \mathbb{S}^n$ (or to $\mathbb{R}^{n+1}$); we start with some basic definitions. A smooth curve $\gamma : [0,1] \to \mathbb{R}^{n+1}$ is called *locally convex* if its Wronskian $W_\gamma(t) = \det\big(\gamma(t), \gamma'(t), \gamma''(t), \ldots, \gamma^{(n)}(t)\big)$ is non-vanishing for all $t \in [0,1]$ (see [1], [11], [12], [13]). A smooth curve γ is called *(globally) convex* if for any linear hyperplane $H \subset \mathbb{R}^{n+1}$ the intersection $H \cap \gamma$ consists of at most n points counting multiplicities; it is an easy exercise to check that global convexity implies local. Observe that if $\gamma : [0,1] \to \mathbb{R}^{n+1}$ is locally convex then so is its spherical projection $\gamma/|\gamma| : [0,1] \to \mathbb{S}^n \subset \mathbb{R}^{n+1}$. Notice that for $n = 2$, a curve $\gamma : [0,1] \to \mathbb{S}^n$ is locally convex if its geodesic curvature is never zero (and therefore has constant sign) and a closed curve $\gamma : [0,1] \to \mathbb{S}^n$ is globally convex if it is the boundary of the intersection of the sphere with a convex cone.

For various technical reasons, the space of smooth curves is too small and not the most adequate. The definition of local convexity makes sense for other spaces, such as the Banach spaces C^r, $r \geq n$ and the Sobolev spaces H^r, $r > n$. In Section 2 below we shall introduce an "official" topology for the spaces of locally convex curves: this turns out to be a Hilbert space containing all the above spaces. As with other questions concerning infinite dimensional topology, the choice of space actually has little consequence.

Locally convex curves in $\mathbb{R}^{n+1}$ are closely related to fundamental solutions of linear ordinary homogeneous differential equations of order $n+1$ on $[0,1]$ with real-valued coefficients. Namely,

2000 *Mathematics Subject Classification.* Primary 58B05, 53A04, Secondary 52A10, 55P15.
Key words and phrases. Locally convex curves, Weyl group, homotopy equivalence.

if $y_0, y_1, \dots, y_n$ are linearly independent solutions of an equation

$$y^{(n+1)} + a_n(t)y^{(n)} + \cdots + a_0(t)y = 0$$

with $a_i(t) \in C[0,1]$, $i = 0, \dots, n$, then $\gamma = (y_0, y_1, \dots, y_n)$ is locally convex. A locally convex γ is called *positive* if $W_\gamma(t) > 0$ and *negative* otherwise. From now on we mostly consider positive curves.

Given a smooth positive locally convex $\gamma : [0,1] \to \mathbb{R}^{n+1}$, define its Frenet frame $\mathfrak{F}_\gamma : [0,1] \to SO_{n+1}$ as the result of the Gram-Schmidt orthogonalization of its Wronski curve $\big(\gamma(t), \gamma'(t), \gamma''(t), \dots, \gamma^{(n)}(t)\big)$. In other words, $\mathfrak{F}_\gamma$ satisfies the relation

$$\Big(\gamma(t), \gamma'(t), \gamma''(t), \dots, \gamma^{(n)}(t)\Big) = \mathfrak{F}_\gamma(t)R(t),$$

where $R(t)$ is an upper triangular matrix with positive diagonal. Let $\mathcal{L}\mathbb{S}^n$ be the space of all positive locally convex curves $\gamma : [0,1] \to \mathbb{S}^n$ (in the appropriate space) with the standard initial frame $\mathfrak{F}_\gamma(0) = I$, where $I \in SO_{n+1}$ is the identity matrix of size $(n+1) \times (n+1)$. As we shall see (Lemma 2.3), the space $\mathcal{L}\mathbb{S}^n$ is a contractible Hilbert manifold and therefore diffeomorphic to Hilbert space.

Given $Q \in SO_{n+1}$, let $\mathcal{L}\mathbb{S}^n(Q) \subset \mathcal{L}\mathbb{S}^n$ be the set of positive locally convex curves on $\mathbb{S}^n$ with the standard initial and the prescribed final frame $\mathfrak{F}_\gamma(1) = Q$; one of the main difficulties is that the map $\mathcal{L}\mathbb{S}^n \to SO_{n+1}$ taking γ to $\mathfrak{F}_\gamma(1)$ is not a fibre bundle. Let $\Pi : \mathrm{Spin}_{n+1} \to SO_{n+1}$ ($n \geq 2$) be the universal cover (which is a double cover). Denote by $\mathbf{1} \in \mathrm{Spin}_{n+1}$ the identity element and by $-\mathbf{1} \in \mathrm{Spin}_{n+1}$ the unique nontrivial element with $\Pi(-\mathbf{1}) = I$. For $\gamma \in \mathcal{L}\mathbb{S}^n$, the map $\mathfrak{F}_\gamma : [0,1] \to SO_{n+1}$ can be uniquely lifted to $\tilde{\mathfrak{F}}_\gamma : [0,1] \to \mathrm{Spin}_{n+1}$, $\mathfrak{F}_\gamma = \Pi \circ \tilde{\mathfrak{F}}_\gamma$, $\tilde{\mathfrak{F}}_\gamma(0) = \mathbf{1}$. Given $z \in \mathrm{Spin}_{n+1}$, let $\mathcal{L}\mathbb{S}^n(z) \subset \mathcal{L}\mathbb{S}^n(\Pi(z))$ be the set of positive locally convex curves $\gamma \in \mathcal{L}\mathbb{S}^n(\Pi(z))$ with $\tilde{\mathfrak{F}}_\gamma(1) = z$. One can immediately observe that $\mathcal{L}\mathbb{S}^n(\Pi(z)) = \mathcal{L}\mathbb{S}^n(z) \sqcup \mathcal{L}\mathbb{S}^n(-z)$. The Hilbert manifolds $\mathcal{L}\mathbb{S}^n(Q)$ and $\mathcal{L}\mathbb{S}^n(z)$ for various $Q \in SO_{n+1}$ and $z \in \mathrm{Spin}_{n+1}$ are the main objects of study in this paper.

Some information about the topology of $\mathcal{L}\mathbb{S}^n(Q)$, mostly in the case $Q = I$ or in the case $n = 2$, was earlier obtained in [1], [6], [8], [9], [11], [12] and [13]. In particular, it was shown that the number of connected components of $\mathcal{L}\mathbb{S}^n(I)$ equals 3 for even n and 2 for odd $n > 1$, which is related to the existence of closed globally convex curves on all even-dimensional spheres. It was also shown in [1] that for n even the space of closed globally convex curves with a fixed initial frame is contractible. The first nontrivial information about the higher homology and homotopy groups of these components can be found in [8] and [9].

In this paper we leave aside the fascinating and widely open question about the topology of the spaces $\mathcal{L}\mathbb{S}^n(Q)$ and concentrate on the following.

Problem 1. *How many different (i.e., non-homeomorphic) spaces are there among $\mathcal{L}\mathbb{S}^n(Q)$, $Q \in SO_{n+1}$, $n \geq 2$? Analogously, how many different spaces are there among $\mathcal{L}\mathbb{S}^n(z)$, $z \in Spin_{n+1}$?*

To formulate our partial answer to the latter question we need to introduce the following set of matrices. For a positive integer m let

$$M_s^m = \mathrm{diag}(-1, \dots, -1, 1, \dots, 1), \qquad s \in \mathbb{Z}, \quad |s| \leq m, \quad s \equiv m \pmod 2,$$

be the diagonal $m \times m$ matrix whose first $(m-s)/2$ entries equal to -1 and the remaining $(m+s)/2$ entries equal to 1. Notice that s equals both the trace and the signature of M_s^m and that $M_s^m \in SO_m$ if and only if $s \equiv m \pmod 4$. In the latter case, let $\pm w_s^m \in \mathrm{Spin}_m$ be the two preimages of $M_s^m \in SO_m$.

Our first result is as follows.

Theorem 1. *For $n \geq 2$, any $Q \in SO_{n+1}$, and any $z \in \mathrm{Spin}_{n+1}$ one has:*

(1) *Each space $\mathcal{L}\mathbb{S}^n(Q)$ is homeomorphic to one of the subspaces $\mathcal{L}\mathbb{S}^n(M_s^{n+1})$, where $|s| \leq n+1$, $s \equiv n+1 \pmod 4$ (there are $\lceil \frac{n}{2} \rceil + 1$ such subspaces).*
(2) *For n even, each space $\mathcal{L}\mathbb{S}^n(z)$, $z \in \mathrm{Spin}_{n+1}$, is homeomorphic to one of the subspaces $\mathcal{L}\mathbb{S}^n(\mathbf{1})$, $\mathcal{L}\mathbb{S}^n(-\mathbf{1})$, $\mathcal{L}\mathbb{S}^n(w_{n-3}^{n+1})$, $\mathcal{L}\mathbb{S}^n(w_{n-7}^{n+1})$, $\ldots$, $\mathcal{L}\mathbb{S}^n(w_{-n+5}^{n+1})$, $\mathcal{L}\mathbb{S}^n(w_{-n+1}^{n+1})$.*
(3) *For n odd, each space $\mathcal{L}\mathbb{S}^n(z)$, $z \in \mathrm{Spin}_{n+1}$, is homeomorphic to one of the subspaces $\mathcal{L}\mathbb{S}^n(\mathbf{1})$, $\mathcal{L}\mathbb{S}^n(-\mathbf{1})$, $\mathcal{L}\mathbb{S}^n(w_{n-3}^{n+1})$, $\mathcal{L}\mathbb{S}^n(w_{n-7}^{n+1})$, $\ldots$, $\mathcal{L}\mathbb{S}^n(w_{-n+3}^{n+1})$, $\mathcal{L}\mathbb{S}^n(w_{-n-1}^{n+1})$, $\mathcal{L}\mathbb{S}^n(-w_{-n-1}^{n+1})$.*

Using Theorem 2 below and results proved elsewhere ([8], [9]) we check that for $n = 2$ the above spaces are pairwise non-homeomorphic. It is natural to ask whether they are likewise non-homeomorphic for $n \geq 3$; see discussions in the first subsection of the conclusion.

We might want to describe the topology of these spaces; the next result gives some partial answers. Let $\Omega SO_{n+1}(Q)$ (resp. $\Omega\,\mathrm{Spin}_{n+1}(z)$) be the space of all continuous curves $\alpha : [0,1] \to SO_{n+1}$ (resp. $\alpha : [0,1] \to \mathrm{Spin}_{n+1}$) with $\alpha(0) = I$ and $\alpha(1) = Q$ (resp. $\alpha(0) = \mathbf{1}$ and $\alpha(1) = z$). Using the Frenet frame we define *Frenet frame injections*:

$$\begin{array}{rcl} \mathfrak{F}_{[Q]} : \mathcal{L}\mathbb{S}^n(Q) & \to & \Omega SO_{n+1}(Q), \\ \gamma & \mapsto & \mathfrak{F}_\gamma \end{array} \qquad \begin{array}{rcl} \tilde{\mathfrak{F}}_{[z]} : \mathcal{L}\mathbb{S}^n(z) & \to & \Omega\,\mathrm{Spin}_{n+1}(z). \\ \gamma & \mapsto & \tilde{\mathfrak{F}}_\gamma \end{array}$$

It is a classical fact that the value of Q (resp. z) does not change the space $\Omega SO_{n+1}(Q)$ (resp. $\Omega\,\mathrm{Spin}_{n+1}(z)$) up to homeomorphism. Therefore, we usually omit Q (resp. z) and write ΩSO_{n+1} (resp. $\Omega\,\mathrm{Spin}_{n+1}$) instead.

Theorem 2. *For $n \geq 2$, consider the Frenet frame injections as above.*

(1) *For all $Q \in SO_{n+1}$ and for all $z \in \mathrm{Spin}_{n+1}$ the maps $\mathfrak{F}_{[Q]}$ and $\tilde{\mathfrak{F}}_{[z]}$ are weakly homotopically surjective.*
(2) *If $|s| \leq 1$ then the Frenet frame injections $\mathfrak{F}_{[M_s^{n+1}]}$ and $\tilde{\mathfrak{F}}_{[w_s^{n+1}]}$ are weak homotopy equivalences. In this case there exist homeomorphisms*

$$\mathcal{L}\mathbb{S}^n(M_s^{n+1}) \approx \Omega SO_{n+1}, \qquad \mathcal{L}\mathbb{S}^n(w_s^{n+1}) \approx \Omega\,\mathrm{Spin}_{n+1}\,.$$

Recall that a map $X \to Y$ is weakly homotopically surjective if the induced maps $\pi_k(X) \to \pi_k(Y)$ are surjective; also, a map $X \to Y$ is a weak homotopy equivalence if the induced maps $\pi_k(X) \to \pi_k(Y)$ are isomorphisms.

Notice that, in general, for arbitrary Q or z it is by no means true that the Frenet frame injection induces a homotopy equivalence: even the number of connected components can be different.

Versions of Theorems 1 and 2 also hold for the spaces $C^k \cap \mathcal{L}\mathbb{S}^n(Q)$ and $C^k \cap \mathcal{L}\mathbb{S}^n(z)$. These facts follow from our results together with Theorem 2 in [4]; alternatively, our proofs can be adapted (with some extra rather routine work).

Acknowledgements. The first named author gratefully acknowledges the support of CNPq, FAPERJ and CAPES (Brazil) and is sincerely grateful to the mathematics department of the Stockholm University for the kind hospitality during his visits to Sweden in 2005 and 2007.

2. Frenet frames and Jacobian curves

We collect in this section a few basic notions and facts. The logarithmic derivative of a curve $\Gamma : [0,1] \to SO_{n+1}$ is defined as $\Lambda(t) = (\Gamma(t))^{-1}\Gamma'(t)$. Notice that $\Lambda(t)$ belongs to the Lie algebra and is therefore automatically skew symmetric. Let $\mathfrak{T} \subset so_{n+1}$ be the set of tridiagonal skew symmetric matrices with positive subdiagonal entries, or skew Jacobi matrices, i.e., of matrices of the form

$$\begin{pmatrix} & -c_1 & & & \\ c_1 & & -c_2 & & \\ & c_2 & & \ddots & \\ & & \ddots & & -c_n \\ & & & c_n & \end{pmatrix}, \qquad c_i > 0.$$

A curve $\Gamma : I \to SO_{n+1}$ is called *Jacobian* if its logarithmic derivative Λ satisfies $\Lambda(t) \in \mathfrak{T}$ for all $t \in I$ (where $I \subset \mathbb{R}$ is an interval).

Lemma 2.1. *Let $\Gamma : [0,1] \to SO_{n+1}$ be a smooth curve with $\Gamma(0) = I$. The curve Γ is Jacobian if and only if there exists $\gamma \in \mathcal{L}\mathbb{S}^n$ with $\mathfrak{F}_\gamma = \Gamma$.*

Recall that a smooth curve $\gamma : [0,1] \to \mathbb{S}^n$ belongs to $\mathcal{L}\mathbb{S}^n$ if and only if $\mathfrak{F}_\gamma(0) = I$ and γ is (positive) locally convex:

$$\det\left(\gamma(t), \gamma'(t), \gamma''(t), \ldots, \gamma^{(n)}(t)\right) > 0.$$

Proof. Consider $\gamma \in \mathcal{L}\mathbb{S}^n$ and its Wronski curve

$$G(t) = \left(\gamma(t)\ \gamma'(t)\ \cdots\ \gamma^{(n)}(t)\right) = \mathfrak{F}_\gamma(t)R(t).$$

We have

$$G'(t) = \left(\gamma'(t)\ \gamma''(t)\ \cdots\ \gamma^{(n+1)}(t)\right) = G(t)H(t)$$

for $H(t)$ an upper Hessenberg matrix whose subdiagonal entries equal to 1:

$$H(t) = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & * \\ 1 & 0 & 0 & \cdots & 0 & * \\ 0 & 1 & 0 & \cdots & 0 & * \\ & \vdots & & \vdots & & \\ 0 & 0 & 0 & \cdots & 1 & * \end{pmatrix}.$$

Recall that H is upper Hessenberg if $(H)_{ij} = 0$ whenever $i > j+1$. Write $\Gamma = \mathfrak{F}_\gamma$ and substitute $\Gamma(t)R(t)$ for $G(t)$ in the equations above to obtain

$$\Gamma'(t)R(t) + \Gamma(t)R'(t) = \Gamma(t)R(t)H(t)$$

and therefore

$$\Lambda(t) = (\Gamma(t))^{-1}\Gamma'(t) = -R'(t)(R(t))^{-1} + R(t)H(t)(R(t))^{-1}$$

which is upper Hessenberg with positive subdiagonal entries (the first product is upper triangular, the second one is upper Hessenberg). Since we know that $\Lambda(t) \in so_{n+1}$, we have $\Lambda(t) \in \mathfrak{T}$, proving one implication.

For the other implication, consider $\Gamma : [0,1] \to SO_{n+1}$ such that $\Gamma(0) = I$, $\Gamma'(t) = \Gamma(t)\Lambda(t)$ and $\Lambda(t) \in \mathfrak{T}$ for all $t \in [0,1]$. Set $\gamma(t) = \Gamma(t)e_1$. We have $\gamma'(t) = \Gamma'(t)e_1 = \Gamma(t)\Lambda(t)e_1 = \Gamma(t)(\Lambda(t))_{21}e_2 = p_1(t)\Gamma(t)e_2$, $p_1(t) > 0$. Similarly,

$$\gamma''(t) = (p_1)'(t)\Gamma(t)e_2 + p_1(t)\Gamma(t)\Lambda(t)e_2 = p_2(t)\Gamma(t)e_3 + r_{22}(t)\Gamma(t)e_2 + r_{21}(t)\Gamma(t)e_1,$$

where $p_2(t) > 0$ and the values of $r_{ij}(t)$ are not important. In general

$$\gamma^{(j)}(t) = p_j(t)\Gamma(t)e_{j+1} + \sum_{i \le j} r_{ji}(t)\Gamma(t)e_i, \qquad p_j(t) > 0.$$

Thus applying Gram-Schmidt to the Wronski curve

$$\begin{pmatrix} \gamma(t) & \gamma'(t) & \cdots & \gamma^{(n)}(t) \end{pmatrix}$$

yields $\mathfrak{F}_\gamma(t) = \Gamma(t)$, completing the proof. $\square$

A smooth Jacobian curve $\Gamma : I \to SO_{n+1}$ is called *globally Jacobian* if $\gamma : I \to \mathbb{S}^n$, $\gamma(t) = \Gamma(t)e_1$, is globally convex.

Notice that given a smooth function $\Lambda : [0,1] \to \mathfrak{T} \subset so_{n+1}$ the initial value problem

$$\Gamma'(t) = \Gamma(t)\Lambda(t), \quad \Gamma(0) = I \tag{$*$}$$

yields Γ as in the lemma and therefore a smooth curve $\gamma \in \mathcal{L}\mathbb{S}^n$. This establishes a homeomorphism between the space of smooth curves $\gamma \in \mathcal{L}\mathbb{S}^n$ and the convex set of smooth functions $\Lambda : [0,1] \to \mathfrak{T}$. We will denote this correspondence by

$$\Lambda_\gamma(t) = (\mathfrak{F}_\gamma(t))^{-1}(\mathfrak{F}_\gamma)'(t).$$

It will be convenient to have examples of locally convex curves and corresponding Jacobian curves.

Lemma 2.2. *For $n+1 = 2k$ let c_i and a_i $(i = 1, \ldots, k)$ be positive parameters with a_i mutually distinct and $c_1^2 + \cdots + c_k^2 = 1$. Set*

$$\xi(t) = (c_1\cos(a_1t), c_1\sin(a_1t), \ldots, c_k\cos(a_kt), c_k\sin(a_kt)).$$

For $n+1 = 2k+1$ let c_0, c_i and a_i $(i = 1, \ldots, k)$ be positive parameters with a_i mutually distinct and $c_0^2 + c_1^2 + \cdots + c_k^2 = 1$. Set

$$\xi(t) = (c_0, c_1\cos(a_1t), c_1\sin(a_1t), \ldots, c_k\cos(a_kt), c_k\sin(a_kt)).$$

In both cases the curve $\xi : [0,1] \to \mathbb{S}^n$ is locally convex with constant Λ_ξ. Conversely, if $\tilde{\xi} : [0,1] \to \mathbb{S}^n$ is locally convex with $\Lambda_{\tilde{\xi}}$ constant then $\tilde{\xi} = Q\xi$ for some $Q \in SO_{n+1}$ and ξ as above (for appropriate c_i and a_i). Furthermore, assume $a_i/(4\pi) \in \mathbb{Z}$ and set $Q = (\mathfrak{F}_\xi(0))^{-1}$ and $\xi_1(t) = Q\xi(t)$: we have $\xi_1 \in \mathcal{L}\mathbb{S}^n(\mathbf{1})$.

For $n = 2$, ξ is a circle; for $n = 3$, ξ turns around in one plane while it turns around at a faster rate in another plane: for suitable values of a_i and c_i, ξ looks like a phone wire (see Figure 1).

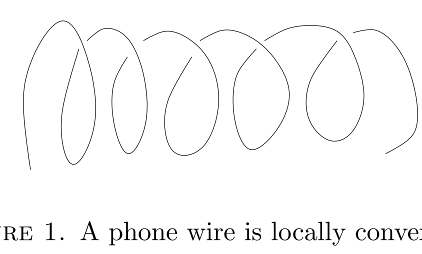

FIGURE 1. A phone wire is locally convex in $\mathbb{S}^3$

Proof. A straight-forward calculation gives, for $n+1=2k$,

$$\det(\xi(t),\xi'(t),\ldots,\xi^{(n)}(t)) = \left(\prod_i (c_i^2 a_i)\right)\left(\prod_{i<j}\left((a_i-a_j)^2(a_i+a_j)^2\right)\right) > 0$$

and for $n+1=2k+1$,

$$\det(\xi(t),\xi'(t),\ldots,\xi^{(n)}(t)) = c_0\left(\prod_i (c_i^2 a_i^3)\right)\left(\prod_{i<j}\left((a_i-a_j)^2(a_i+a_j)^2\right)\right) > 0.$$

Alternatively, we can compute $\Xi = \mathfrak{F}_\xi : [0,1] \to SO_{n+1}$ and its logarithmic derivative $\Lambda : [0,1] \to so_{n+1}$: it turns out to be a constant element of $\mathfrak{T} \subset so_{n+1}$. In general, if $Q \in SO_{n+1}$ and γ is locally convex then so is $Q\gamma$: thus, ξ_1 is locally convex.

Conversely, assume $\tilde{\xi} \in \mathcal{L}\mathbb{S}^n$ and that $\Lambda_{\tilde{\xi}}$ is constant equal to $B \in \mathfrak{T}$; we then have $\tilde{\Xi}(t) = \mathfrak{F}_{\tilde{\xi}}(t) = \exp(tB)$. The eigenvalues of B are all on the imaginary axis and therefore of the form $\pm a_j i$, $a_j > 0$, plus a 0 in case $n+1$ is odd. Thus there exists $Q \in SO_{n+1}$ such that

$$B = \begin{cases} Q \operatorname{diag}\left(\begin{pmatrix} 0 & -a_1 \\ a_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -a_2 \\ a_2 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & -a_k \\ a_k & 0 \end{pmatrix}\right) Q^T, & n = 2k-1, \\ Q \operatorname{diag}\left(0, \begin{pmatrix} 0 & -a_1 \\ a_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -a_2 \\ a_2 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & -a_k \\ a_k & 0 \end{pmatrix}\right) Q^T, & n = 2k. \end{cases}$$

Write

$$X(s) = \begin{pmatrix} \cos(s) & -\sin(s) \\ \sin(s) & \cos(s) \end{pmatrix} = \exp\left(s\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right).$$

Thus, according to parity we have:

$$\tilde{\Xi}(t) = \begin{cases} Q \operatorname{diag}\left(X(a_1 t), X(a_2 t), \ldots, X(a_k t)\right) Q^T, \\ Q \operatorname{diag}\left(0, X(a_1 t), X(a_2 t), \ldots, X(a_k t)\right) Q^T, \end{cases}$$

and therefore

$$\tilde{\xi}(t) = Q \operatorname{diag}\left([0,]X(a_1 t), X(a_2 t), \ldots, X(a_k t)\right) v_0, \quad v_0 = Q^T e_1.$$

Up to multiplication by a matrix of the form $\operatorname{diag}\left([1,]X(\theta_1), \ldots, X(\theta_k)\right)$, v_0 can be assumed to be of the form $v_0 = ([c_0,]c_1, 0, \ldots, c_k, 0)$ for $c_i \geq 0$ (with a corresponding change of Q). The formulas in the previous paragraph of this proof indicate that the parameters a_i and c_i must be positive and that the a_i's must be pairwise distinct for $\tilde{\xi}$ to be locally convex, as desired. The other claims are easy. □

The space of smooth curves is not the most convenient, however; we use the above correspondence to define our favorite space of curves: if we consider $\Lambda \in L^2([0,1],\mathfrak{T}) \subset L^2([0,1], so_{n+1})$ we can solve the initial value problem $(*)$ and determine $\Gamma : [0,1] \to SO_{n+1}$ and $\gamma(t) = \Gamma(t)e_1$. Notice that the curve γ constructed in this way from $\Lambda \in L^2$ belongs to $H^1([0,1],\mathbb{R}^{n+1})$ but the concept of local convexity does not make sense for all curves $\gamma : [0,1] \to \mathbb{S}^n$ with $\gamma \in H^1([0,1],\mathbb{R}^{n+1})$. A minor inconvenience is that $L^2([0,1],\mathfrak{T})$ is not a Hilbert manifold; we resolve this problem by defining a diffeomorphism $\phi : \mathfrak{T} \to \mathbb{R}^n$ with j-th coordinate $\phi_j(T) = g(T_{j+1,j})$, $g(x) = x - 1/x$. Given $\alpha \in L^2([0,1],\mathbb{R}^n)$ we set $\Lambda = \phi^{-1} \circ \alpha$ and Γ as above, thus defining a space $\hat{\mathcal{L}}\mathbb{S}^n$ of Jacobian curves and an explicit diffeomorphism $L^2([0,1],\mathbb{R}^n) \equiv \hat{\mathcal{L}}\mathbb{S}^n$. There are sometimes advantages in working with $\hat{\mathcal{L}}\mathbb{S}^n(z)$ rather than $\mathcal{L}\mathbb{S}^n(z)$: for instance, multiplication (in Spin_{n+1}) allows for a sort of superposition. This will be useful later in the paper.

Given $Q \in SO_{n+1}$ let $\hat{\mathcal{L}}\mathbb{S}^n(z) \subset \hat{\mathcal{L}}\mathbb{S}^n$ be the space of Jacobian curves $\Gamma : [0,1] \to SO_{n+1}$ with $\Gamma(0) = I$, $\Gamma(1) = Q$. Finally, we use the map $\tilde{\mathfrak{F}}$ to define the spaces $\mathcal{L}\mathbb{S}^n$ and $\mathcal{L}\mathbb{S}^n(Q)$, which are now Hilbert manifolds.

Recall also that two Hilbert manifolds are diffeomorphic if and only if they are homeomorphic which, in its turn, holds if and only if they are weakly homotopically equivalent ([3]). Besides the Hilbert manifold structure, the above definition of the spaces $\mathcal{L}\mathbb{S}^n(Q)$ (and other spaces) has the advantage of allowing discontinuities in Λ_γ thus bypassing the need for a roundabout smoothening process. The following result is now trivial.

Lemma 2.3. *The space $\mathcal{L}\mathbb{S}^n$ is contractible.*

3. Bruhat cells and the Coxeter-Weyl group B_{n+1}

As a first step, for any fixed dimension n we reduce Problem 1 to consideration of only finitely many different values of Q and z using well-known group actions. The key observation here is that if $\gamma_1 : [0,1] \to \mathbb{S}^n$ is positive locally convex and $A \in \mathbb{R}^{n\times n}$ has positive determinant than both $A\gamma_1 : [0,1] \to \mathbb{R}^{n+1}$ and $\gamma_2 : [0,1] \to \mathbb{S}^n$ with $\gamma_2(t) = \widehat{A\gamma_1(t)} = A\gamma_1(t)/|A\gamma_1(t)|$ are positive locally convex.

Let $\mathcal{U}_{n+1}^+$ be the group of real upper-triangular matrices with positive diagonal and $\mathcal{U}_{n+1}^{\mathbf{1}} \subset \mathcal{U}_{n+1}^+$ be the subgroup of matrices with diagonal entries equal to one. Consider the action of $\mathcal{U}_{n+1}^{\mathbf{1}}$ on $GL_{n+1}(\mathbb{R})$ by conjugation: in what follows we will refer to the action of $\mathcal{U}_{n+1}^{\mathbf{1}}$ on different spaces as the *Bruhat action*. This action induces the action of $\mathcal{U}_{n+1}^{\mathbf{1}}$ on SO_{n+1} as the postcomposition of the conjugation with the orthogonalization. In other words, $B(U,Q) = UQU'$ where U' is the only matrix in $\mathcal{U}_{n+1}^+$ such that $UQU' \in SO_{n+1}$; thus, $B(U,Q)$ is obtained from UQ by Gram-Schmidt. It is well-known that the Bruhat action on SO_{n+1} has finitely many orbits. These orbits are referred to as the *Bruhat cells* of SO_{n+1}: two orthogonal matrices $Q_1, Q_2 \in SO_{n+1}$ belong to the same Bruhat cell if and only if there exist upper triangular matrices U_1, U_2 with positive diagonal satisfying $U_1Q_1 = Q_2U_2$. We denote the Bruhat cell of $Q \in SO_{n+1}$ by $\mathrm{Bru}(Q) \subset SO_{n+1}$.

Let $B_{n+1} \subset O_{n+1}$ be the Coxeter-Weyl group of signed permutation matrices and let $B_{n+1}^+ = B_{n+1} \cap SO_{n+1}$. Let $\mathrm{Diag}_{n+1}^+ \subset B_{n+1}^+$ be the subgroup of diagonal matrices with entries ± 1 and determinant 1; thus Diag_{n+1}^+ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^n$. Each Bruhat cell contains exactly one element $Q_0 \in B_{n+1}^+$ and is diffeomorphic to a cell whose dimension equals the number of inversions of Q_0. In other words, for each $Q \in SO_{n+1}$ there is a unique $Q_0 \in B_{n+1}^+$ such that there exist $U_1, U_2 \in \mathcal{U}_{n+1}^+$ satisfying $Q = U_1Q_0U_2$.

We recall an algorithm producing Q_0 from a given Q; this algorithm will be used later, particularly in the study of chopping. Consider the first column of Q and look for the lowest non-zero entry, say Q_{i1}. We first multiply Q by a diagonal matrix $D \in \mathcal{U}_{n+1}^+$ to obtain a new matrix $\tilde{Q} = DQ$ for which $(\tilde{Q})_{i1} = \pm 1$; for simplicity, we may thus assume $Q_{i1} = \pm 1$. We next perform row operations on Q to clean the first column above row i: in other words, we obtain $U_1 \in \mathcal{U}_{n+1}^+$ such that $\tilde{Q} = U_1Q$ satisfies $\tilde{Q}e_1 = \pm e_i$; again assume from now on that $Qe_1 = \pm e_i$. Now perform column operations on Q to clean row i to the right of the first column, i.e., obtain $U_2 \in \mathcal{U}_{n+1}^+$ such that $\tilde{Q} = QU_2$ satisfies $e_i^T\tilde{Q} = \pm e_1^T$. Repeat the process for each column: at the end of the process we obtain $Q_0 = U_1QU_2$ $(U_1, U_2 \in \mathcal{U}_{n+1}^+)$ for which there exists a permutation π such that $Q_0e_i = \pm e_{\pi(i)}$. In other words, $Q_0 \in B_{n+1}$; since $\det(Q_0) = \det(U_1)\det(Q)\det(U_2) > 0$ we have $\det(Q_0) = 1$ and $Q_0 \in B_{n+1}^+$.

Recall that $\Pi : \mathrm{Spin}_{n+1} \to SO_{n+1}$ is a group homomorphism and the double cover of SO_{n+1} (for $n > 1$, this is the universal cover). Let

$$\tilde{B}^+_{n+1} = \Pi^{-1}(B^+_{n+1}) \subset \mathrm{Spin}_{n+1}, \qquad \widetilde{\mathrm{Diag}}^+_{n+1} = \Pi^{-1}(\mathrm{Diag}^+_{n+1}) \subset B^+_{n+1};$$

the groups $\tilde{B}^+_{n+1}$ and $\widetilde{\mathrm{Diag}}^+_{n+1}$ are $\mathbb{Z}/2\mathbb{Z}$-central extensions of B^+_{n+1} and Diag^+_{n+1}, respectively. Notice that

$$|B^+_{n+1}| = 2^n(n+1)!, \qquad |\tilde{B}^+_{n+1}| = 2^{n+1}(n+1)!, \qquad |\widetilde{\mathrm{Diag}}^+_{n+1}| = 2^{n+1};$$

for instance, $\widetilde{\mathrm{Diag}}^+_3$ is isomorphic to the quaternion group $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$.

The Bruhat cell decomposition can be lifted to Spin_{n+1} where each cell contains a unique element of $\tilde{B}^+_{n+1}$. Two elements of SO_{n+1} or Spin_{n+1} will be called *Bruhat equivalent* if they belong to the same cell in the corresponding Bruhat decomposition. We will also write $\mathrm{Bru}(z) \subset \mathrm{Spin}_{n+1}$ for the Bruhat cell of $z \in \mathrm{Spin}_{n+1}$.

The Bruhat action of $\mathcal{U}^{\mathbf{1}}_{n+1}$ on SO_{n+1} induces the Bruhat action of $\mathcal{U}^{\mathbf{1}}_{n+1}$ on the space $\mathcal{L}\mathbb{S}^n$ as follows: given $\gamma \in \mathcal{L}\mathbb{S}^n$ and $U \in \mathcal{U}^{\mathbf{1}}_{n+1}$, set $(B(U,\gamma))(t) = (B(U, \mathfrak{F}_\gamma(t)))e_1$ (where $e_1 = (1, 0, 0, \ldots, 0) \in \mathbb{R}^{n+1}$). Clearly, if $\gamma \in \mathcal{L}\mathbb{S}^n(z)$ then $B(U,\gamma) \in \mathcal{L}\mathbb{S}^n(B(U,z))$. The following lemma is now easy.

Lemma 3.1. *If $Q_1, Q_2 \in SO_{n+1}$ (resp. $z_1, z_2 \in \mathrm{Spin}_{n+1}$) are Bruhat equivalent then $\mathcal{L}\mathbb{S}^n(Q_1)$ and $\mathcal{L}\mathbb{S}^n(Q_2)$ (resp. $\mathcal{L}\mathbb{S}^n(z_1)$ and $\mathcal{L}\mathbb{S}^n(z_2)$) are homeomorphic.*

This explicit homeomorphism will be used again and we therefore introduce some notation. Let Q_1 and Q_2 be as in the lemma: there exists a matrix $U \in \mathcal{U}^{\mathbf{1}}_{n+1}$ with $B(U, Q_1) = Q_2$ and therefore $B(U^{-1}, Q_2) = Q_1$. Define $\mathbf{B}_{Q_1,U,Q_2} : \mathcal{L}\mathbb{S}^n(Q_1) \to \mathcal{L}\mathbb{S}^n(Q_2)$ by $\mathbf{B}_{Q_1,U,Q_2}(\gamma) = B(U,\gamma)$ (for $\gamma \in \mathcal{L}\mathbb{S}^n(Q_1)$). Similarly define $\mathbf{B}_{z_1,U,z_2} : \mathcal{L}\mathbb{S}^n(z_1) \to \mathcal{L}\mathbb{S}^n(z_2)$.

Proof. The map $\mathbf{B}_{Q_1,U,Q_2}$ is a homeomorphism with inverse $\mathbf{B}_{Q_2,U^{-1},Q_1}$; the spin case is similar. □

4. Time reversal

In this and the two following sections we introduce three natural operations acting on B^+_{n+1} and on the spaces of curves under consideration.

The naive idea here would be to consider the curve $t \mapsto \gamma(1-t)$; this curve however may be negative locally convex and has the wrong endpoints: we show how to fix these minor problems.

Let $J_+ = \mathrm{diag}(1, -1, 1, -1, \ldots) \in O_{n+1}$; notice that $\det(J_+) = (-1)^{n(n+1)/2}$. For $Q \in SO_{n+1}$, define $\mathbf{TR}(Q) = J_+ Q^T J_+$. The map $\mathbf{TR} : SO_{n+1} \to SO_{n+1}$ is an anti-automorphism which lifts to an anti-automorphism $\mathbf{TR} : \mathrm{Spin}_{n+1} \to \mathrm{Spin}_{n+1}$. Indeed, given $z \in \mathrm{Spin}_{n+1}$ consider a path $\tilde{\alpha} : [0,1] \to \mathrm{Spin}_{n+1}$ with $\tilde{\alpha}(0) = 1$, $\tilde{\alpha}(1) = z$; let $\alpha = \Pi \circ \tilde{\alpha}$ and $\beta : [0,1] \to SO_{n+1}$ with $\beta(t) = \mathbf{TR}(\alpha(t))$; lift β to define $\tilde{\beta} : [0,1] \to \mathrm{Spin}_{n+1}$ with $\tilde{\beta}(0) = 1$; define $\mathbf{TR}(z) = \tilde{\beta}(1)$. The map is well defined: two homotopic paths $\tilde{\alpha}_0$ and $\tilde{\alpha}_1$ yield homotopic paths α_0 and α_1; the paths β_0 and β_1 are also homotopic (apply $\mathbf{TR}$ to the homotopy) and therefore $\tilde{\beta}_0(1) = \tilde{\beta}_1(1)$.

These two anti-automorphisms preserve the subgroups $\mathrm{Diag}^+_{n+1} \subset B^+_{n+1} \subset SO_{n+1}$ and $\widetilde{\mathrm{Diag}}^+_{n+1} \subset \tilde{B}^+_{n+1} \subset \mathrm{Spin}_{n+1}$. In fact, the map $\mathbf{TR} : B^+_{n+1} \to B^+_{n+1}$ admits a simple combinatorial description: the matrix $\mathbf{TR}(Q)$ is obtained from Q by transposition and the change of sign of all entries with $i+j$ odd. We do not present a detailed combinatorial description of $\mathbf{TR}$ in $\tilde{B}_{n+1}^+$ but we record an observation for later use.

Lemma 4.1. *Let $s \in \mathbb{Z}$, $s \equiv n+1 \pmod 4$, $|s| < n+1$. Then there exists $z \in \widetilde{\mathrm{Diag}}_{n+1}^+$ with $\mathrm{trace}(z) = s$ and $\mathbf{TR}(z) = -z$.*

Proof. Let

$$Q = \mathrm{diag}(-1,-1,\ldots,-1,-1,-1,1,-1,1,1,\ldots,1,1)$$

and z with $\Pi(z) = Q$. We claim that z satisfies the claim; in order to perform this computation we construct paths in SO_{n+1} and lift them to Spin_{n+1}. Write

$$X(t) = \begin{pmatrix} \cos(\pi t) & -\sin(\pi t) \\ \sin(\pi t) & \cos(\pi t) \end{pmatrix}, \qquad Y(t) = \begin{pmatrix} \cos(\pi t) & 0 & -\sin(\pi t) \\ 0 & 1 & 0 \\ \sin(\pi t) & 0 & \cos(\pi t) \end{pmatrix}.$$

Take

$$\alpha(t) = \mathrm{diag}\left(X(t),\ldots,X(t),Y(t),1,\ldots,1\right),$$

with $((n+1-s)/4)-1$ small $X(t)$ blocks, one large $Y(t)$ block followed by $((n+1+s)/2)$ ones. Now lift the path α to $\tilde{\alpha} : [0,1] \to \mathrm{Spin}(n+1)$ with $\tilde{\alpha}(0) = 1$: without loss of generality, $z = \tilde{\alpha}(1)$. Clearly $\mathrm{trace}(z) = s$ and

$$\mathbf{TR}\alpha(t) = \mathrm{diag}\left(X(t),\ldots,X(t),Y(-t),1,\ldots,1\right).$$

Thus α and $\mathbf{TR}\alpha$ are only different in two of the coordinates of the Y block, where one makes a half-turn one way and the other makes a half-turn the other way. Thus $\mathbf{TR}(z) = -z$, as required. $\square$

Notice that the map $\mathbf{TR} : so_{n+1} \to so_{n+1}$ given by $\mathbf{TR}(X) = J_+ X^T J_+$ satisfies $\mathbf{TR}(X) = X$ for $X \in \mathfrak{T}$. For $\gamma \in \mathcal{L}\mathbb{S}^n(Q)$, define its *time reversal* by

$$\gamma^{\mathbf{TR}}(t) = J_+ Q^T \gamma(1-t)$$

where Q^T is the transpose of Q.

Lemma 4.2. *For any $\gamma \in \mathcal{L}\mathbb{S}^n(Q)$ we have $\gamma^{\mathbf{TR}} \in \mathcal{L}\mathbb{S}^n(\mathbf{TR}(Q))$. Furthermore,*

$$\mathfrak{F}_{\gamma^{\mathbf{TR}}}(t) = \mathbf{TR}(Q^T \mathfrak{F}_\gamma(1-t)); \quad \Lambda_{\gamma^{\mathbf{TR}}}(t) = \Lambda_\gamma(1-t).$$

In particular, if $\xi \in \mathcal{L}\mathbb{S}^n(I)$ is a locally convex curve for which Λ_ξ is constant then $\xi_1^{\mathbf{TR}} = \xi_1$.

Time reversal yields explicit homeomorphisms

$$\mathcal{L}\mathbb{S}^n(Q) \approx \mathcal{L}\mathbb{S}^n(\mathbf{TR}(Q)), \qquad \mathcal{L}\mathbb{S}^n(z) \approx \mathcal{L}\mathbb{S}^n(\mathbf{TR}(z)).$$

Proof. Consider a smooth curve $\gamma \in \mathcal{L}\mathbb{S}^n(Q)$; we must check that $\gamma^{\mathbf{TR}}$ is positive locally convex: we have

$$(\gamma^{\mathbf{TR}})^{(j)}(t) = (-1)^j J_+ Q^T \gamma^{(j)}(1-t)$$

and therefore

$$\det\Big((\gamma^{\mathbf{TR}})(t), (\gamma^{\mathbf{TR}})'(t), \ldots, (\gamma^{\mathbf{TR}})^{(n)}(t)\Big) =$$
$$= (-1)^{n(n+1)/2} \det(J_+) \det(Q^T) \det\Big(\gamma(1-t), \gamma'(1-t), \ldots, \gamma^{(n)}(1-t)\Big) =$$
$$= \det\Big(\gamma(1-t), \gamma'(1-t), \ldots, \gamma^{(n)}(1-t)\Big) > 0.$$

We must now check that $\mathfrak{F}_{\gamma^{\mathbf{TR}}}(0) = I$ and $\mathfrak{F}_{\gamma^{\mathbf{TR}}}(1) = \mathbf{TR}(Q)$. Recall that

$$\begin{aligned} \left(\gamma(t), \gamma'(t), \dots, \gamma^{(n)}(t)\right) &= \mathfrak{F}_\gamma(t) R_0(t), \\ \left((\gamma^{\mathbf{TR}})(t), (\gamma^{\mathbf{TR}})'(t), \dots, (\gamma^{\mathbf{TR}})^{(n)}(t)\right) &= \mathfrak{F}_{\gamma^{\mathbf{TR}}}(t) R_1(t) \end{aligned}$$

where R_0 and R_1 are upper triangular matrices with positive diagonal. We have

$$\left((\gamma^{\mathbf{TR}})(t), (\gamma^{\mathbf{TR}})'(t), \dots, (\gamma^{\mathbf{TR}})^{(n)}(t)\right) = J_+ Q^T \left(\gamma(t), \gamma'(t), \dots, \gamma^{(n)}(t)\right) J_+$$

and therefore

$$\begin{aligned} \mathfrak{F}_{\gamma^{\mathbf{TR}}}(t) &= J_+ Q^T \mathfrak{F}_\gamma(1-t) R_0(1-t) J_+ R_1^{-1}(t) = \\ &= (J_+ Q^T \mathfrak{F}_\gamma(1-t) J_+)(J_+ R_0(1-t) J_+ R_1^{-1}(t)). \end{aligned}$$

Since $(J_+ Q^T \mathfrak{F}_\gamma(1-t) J_+) \in SO_{n+1}$ and $(J_+ R_0(1-t) J_+ R_1^{-1}(t))$ is upper triangular with positive diagonal we have

$$\mathfrak{F}_{\gamma^{\mathbf{TR}}}(t) = J_+ Q^T \mathfrak{F}_\gamma(1-t) J_+;$$

in particular,

$$\begin{aligned} \mathfrak{F}_{\gamma^{\mathbf{TR}}}(0) &= J_+ Q^T \mathfrak{F}_\gamma(1) J_+ = I, \\ \mathfrak{F}_{\gamma^{\mathbf{TR}}}(1) &= J_+ Q^T \mathfrak{F}_\gamma(0) J_+ = \mathbf{TR}(Q). \end{aligned}$$

This completes the proof of the first claim and of the first identity for smooth γ; the second identity follows by taking derivatives and the final claim is now easy. The identities are extended to the general case (i.e., γ not necessarily smooth) by continuity, thus completing the proof. $\square$

5. Arnold duality

Let $A \in B^+_{n+1} \subset SO_{n+1}$ be the anti-diagonal matrix with entries $(A)_{i,n+2-i} = (-1)^{(i+1)}$; for instance, for $n = 2$ and $n = 3$ we have, respectively,

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix};$$

this matrix will appear in several places below. Define an automorphism $\mathbf{AD}$ of SO_{n+1} by $\mathbf{AD}(Q) = A^T Q A$; notice that the subgroup $B^+_{n+1} \subset SO_{n+1}$ is invariant under this automorphism. As before, lift this automorphism to define an automorphism (also called $\mathbf{AD}$) of Spin_{n+1} and $\tilde{B}^+_{n+1}$. The combinatorial description of $\mathbf{AD}$ on B^+_{n+1} is the following: rotate Q by a half-turn (meaning that the (i,j)-th entry of Q becomes the $(n-i+2, n-j+2)$-th entry of the new matrix) and change signs of all entries with $i+j$ odd. Notice that the map $\mathbf{AD} : so_{n+1} \to so_{n+1}$ given by $\mathbf{TR}(X) = A^T X A$ takes $\mathfrak{T}$ to itself (as a set), but reverts the order of the subdiagonal entries.

For $\gamma \in \mathcal{L}\mathbb{S}^n(Q)$, define its *Arnold dual* as

$$\gamma^{\mathbf{AD}}(t) = \mathbf{AD}(\mathfrak{F}_\gamma(t)) e_1.$$

It turns out that this operation is just the usual projective duality between oriented hyperplanes and unit vectors in disguise (comp [2]).

Lemma 5.1. *For any $\gamma \in \mathcal{L}\mathbb{S}^n(Q)$ one has that $\gamma^{\mathbf{AD}} \in \mathcal{L}\mathbb{S}^n(\mathbf{AD}(Q))$. Furthermore,*

$$\mathfrak{F}_{\gamma^{\mathbf{AD}}}(t) = \mathbf{AD}(\mathfrak{F}_\gamma(t)), \quad \Lambda_{\gamma^{\mathbf{AD}}}(t) = \mathbf{AD}(\Lambda_\gamma(t)).$$

Arnold duality gives explicit homeomorphisms

$$\mathcal{L}\mathbb{S}^n(Q) \approx \mathcal{L}\mathbb{S}^n(\mathbf{AD}(Q)), \quad \mathcal{L}\mathbb{S}^n(z) \approx \mathcal{L}\mathbb{S}^n(\mathbf{AD}(z)).$$

Proof. We must first check that if $\gamma \in \mathcal{L}\mathbb{S}^n$ is smooth then $\gamma^{\mathbf{AD}}$ as defined above also belongs to $\mathcal{L}\mathbb{S}^n$. Consider $\tilde{\Gamma} : [0,1] \to SO_{n+1}$ given by

$$\tilde{\Gamma}(t) = \mathbf{AD}(\mathfrak{F}_\gamma(t)) = A^T \mathfrak{F}_\gamma(t) A.$$

We have

$$(\tilde{\Gamma}(t))^{-1}\tilde{\Gamma}'(t) = A^T(\mathfrak{F}_\gamma(t))^{-1} A A^T \mathfrak{F}'_\gamma(t) A = A^T \Lambda_\gamma(t) A = \mathbf{AD}(\Lambda_\gamma(t)) \in \mathfrak{I};$$

by Lemma 2.1, $\tilde{\Gamma} = \mathfrak{F}_{\tilde{\gamma}}$ for $\tilde{\gamma} \in \mathcal{L}\mathbb{S}^n$: thus $\gamma^{\mathbf{AD}} = \tilde{\gamma} \in \mathcal{L}\mathbb{S}^n$, completing our first check. The formulas for $\mathfrak{F}_{\gamma^{\mathbf{AD}}}$ and $\Lambda_{\gamma^{\mathbf{AD}}}$ have also been proved for smooth γ and therefore, by continuity, for all $\gamma \in \mathcal{L}\mathbb{S}^n$. The formulas imply that if $\gamma \in \mathcal{L}\mathbb{S}^n(Q)$ then $\gamma^{\mathbf{AD}} \in \mathcal{L}\mathbb{S}^n(\mathbf{AD}(Q))$. The final claim is now easy. $\square$

6. Chopping operation

The first two operations corresponded to $\mathbb{Z}/2\mathbb{Z}$-symmetries in $\mathcal{L}\mathbb{S}^{n+1}$; our third operation is quite different, loosely corresponding to taking $\gamma \in \mathcal{L}\mathbb{S}^{n+1}$ and chopping off a small tip at the end. We again start with algebra and combinatorics.

For a signed permutation $Q \in B_{n+1}^+$ and a pair of indices (i,j) with $(Q)_{(i,j)} \neq 0$ define $\mathbf{NE}(Q,i,j)$ to be the number of pairs (i',j') with $i' < i$, $j' > j$ and $(Q)_{(i',j')} \neq 0$. In other words, $\mathbf{NE}(Q,i,j)$ is the number of nonzero entries of Q in the northeast quadrant. Also set

$$\mathbf{SW}(Q,i,j) = \mathbf{NE}(Q^T,j,i).$$

It is easy to check that for all Q one has $\mathbf{NE}(Q,i,j) - \mathbf{SW}(Q,i,j) = i - j$.

Using the above notation define

$$\delta_i(Q) = (Q)_{(i,j)}(-1)^{\mathbf{NE}(Q,i,j)}$$

where j is the only index for which $(Q)_{(i,j)} \neq 0$. Additionally, define

$$\Delta(Q) = \operatorname{diag}(\delta_1(Q), \delta_2(Q), \dots, \delta_{n+1}(Q)), \quad \text{and} \quad \operatorname{trd}(Q) = \operatorname{trace}(\Delta(Q)).$$

Lemma 6.1. $\det(Q) = \det(\Delta(Q))$.

Proof. Indeed, let π be the permutation such that $\pi(j) = i$ if j is the only index for which $(Q)_{(i,j)} \neq 0$. Then

$$\det(\Delta(Q)) = \prod_i \delta_i(Q) = \left(\prod_i (Q)_{(i,j)}\right)(-1)^{\sum_i \mathbf{NE}(Q,i,j)}$$
$$= \left(\prod_i (Q)_{(i,j)}\right)(-1)^{|\{(i,i');i'<i,\pi^{-1}(i')>\pi^{-1}(i)\}|} = \det(Q).$$

$\square$

Thus Δ is a function from B^+_{n+1} to $\mathrm{Diag}^+_{n+1} \subset B^+_{n+1}$. Notice that $\Delta(Q) = Q$ for any $Q \in \mathrm{Diag}^+_{n+1}$. We extend Δ to a function from SO_{n+1} to Diag^+_{n+1} by declaring that if Q and Q' are Bruhat equivalent then $\Delta(Q) = \Delta(Q')$; we similarly extend the function $\mathrm{trd}(Q)$ to SO_{n+1}. The map $\Delta : SO_{n+1} \to \mathrm{Diag}^+_{n+1}$ is a projection (in the sense that $\Delta(\Delta(Q)) = \Delta(Q)$) and therefore defines a partition of SO_{n+1} into 2^n classes of the from $\Delta^{-1}(Q)$, $Q \in \mathrm{Diag}^+_{n+1}$. Furthermore, if $Q \in \mathrm{Diag}^+_{n+1}$, we have $\Delta(QQ') = Q\Delta(Q')$ so that a class $\Delta^{-1}(Q)$ is a fundamental domain for the action of Diag^+_{n+1} on SO_{n+1} by multiplication.

Let A be the matrix used in the definition of Arnold duality. Notice that $\Delta(A) = I$ and therefore $\Delta(QA) = Q$ for all $Q \in \mathrm{Diag}^+_{n+1}$. For $Q \in SO_{n+1}$, its *chopping* is defined by $\mathbf{chop}(Q) = \Delta(Q)A$. Thus the Bruhat equivalence class of $\mathbf{chop}(Q)$ is an open set, dense in $\Delta^{-1}(\Delta(Q)) = \mathbf{chop}^{-1}(\mathbf{chop}(Q))$. The maps Δ and $\mathbf{chop}$ as well as the functon $\mathrm{trd} : B^+_{n+1} \to \mathbb{Z}$ will play a crucial role in our argument. (Notice that Δ is **not** a group homomorphism).

Let us present a geometric interpretation for Δ and $\mathbf{chop}$. For $\gamma \in \mathcal{L}\mathbb{S}^n(Q)$ and $\epsilon > 0$, we define the *naive chop* of γ by ϵ as

$$\mathbf{chop}_\epsilon(\gamma)(t) = \gamma((1-\epsilon)t).$$

A straightforward computation gives

$$\mathfrak{F}_{\mathbf{chop}_\epsilon(\gamma)}(t) = \mathfrak{F}_\gamma((1-\epsilon)t), \quad \Lambda_{\mathbf{chop}_\epsilon(\gamma)}(t) = (1-\epsilon)\Lambda_\gamma((1-\epsilon)t);$$

in particular,

$$\mathfrak{F}_{\mathbf{chop}_\epsilon(\gamma)}(1) = \mathfrak{F}_\gamma(1-\epsilon).$$

The inconvenience here is that if $\epsilon > 0$ is fixed and γ varies over the whole $\mathcal{L}\mathbb{S}^n(Q)$ we have no control of $\mathfrak{F}_{\mathbf{chop}_\epsilon(\gamma)}(1)$, the final frame of $\mathbf{chop}_\epsilon(\gamma)$. The situation improves if we adapt that the choice of ϵ depending on γ and focus on Bruhat cells instead of individual final frames.

Lemma 6.2. *For any $Q \in SO(n+1)$ and for any $\gamma \in \mathcal{L}\mathbb{S}^n(Q)$ there exists $\epsilon > 0$ such that for all $t \in (1-\epsilon, 1)$ we have that $\mathfrak{F}_\gamma(t) \in \mathrm{Bru}(\mathbf{chop}(Q))$.*

In other words, given $\gamma \in \mathcal{L}\mathbb{S}^n$ there exists $\tilde{\epsilon} > 0$ such that, for all $\epsilon \in (0, \tilde{\epsilon})$, $\mathfrak{F}_{\mathbf{chop}_\epsilon(\gamma)}(1)$ is Bruhat equivalent to $\mathbf{chop}(\mathfrak{F}_\gamma(1))$.

Before proving Lemma 6.2 we present an illustrative example for $n = 2$. Take

$$Q = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

and expand an arbitrary smooth curve $\gamma \in \mathcal{L}\mathbb{S}^2(Q)$ in a Taylor series near $t = 1$. Using $x = t-1$ we get $\gamma(x) \approx (x, -1, x^2/2)$ (up to higher order terms) so that, for $x \approx 0$,

$$\mathfrak{F}_\gamma(x) \approx \begin{pmatrix} x & 1 & 0 \\ -1 & 0 & 0 \\ x^2/2 & x & 1 \end{pmatrix}.$$

We now apply the above algorithm to find $Q_0 \in B^+_{n+1}$ which is Bruhat equivalent to $\mathfrak{F}_\gamma(x)$ when x is a negative number with a small absolute value (i.e., $Q_0 = U_1\mathfrak{F}_\gamma(x)U_2$). We start at the $(3,1)$-th entry $x^2/2$, which is positive. Thus, $(Q_0)_{3,1} = +1$. We now concentrate on the **SW** (i.e., bottom left) (2×2)-blocks of $\mathfrak{F}_\gamma(x)$ and Q_0: since $Q_0 = U_1\mathfrak{F}_\gamma(x)U_2$, the signs of the determinants of these two blocks should be equal; since its original value equals $-x > 0$, the $(2,2)$-th entry of Q_0 equals -1. Finally, the $(1,3)$-th entry must be set to 1 for the whole determinant to be positive. Summing up, if $\gamma \in \mathcal{L}\mathbb{S}^2(Q)$ then there exists $\epsilon > 0$ such that for any $t \in (1-\epsilon, 1)$ one has that $\mathfrak{F}_\gamma(t)$ is Bruhat equivalent to

$$\mathbf{chop}(Q) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

The general proof below follows the same idea.

Proof. As above, consider $Q \in B^+_{n+1}$ with associated permutation π, so that $Q_{i,j} \neq 0$ if and only if $\pi(j) = i$. Write a Taylor approximation $\mathfrak{F}_\gamma(x) \approx M(x)$ where

$$(M(x))_{i,j} = s_i g(\ell) x^\ell, \quad \pi^{-1}(i) = j + \ell, \quad (Q)_{i,j+\ell} = s_i,$$

where

$$g(\ell) = \begin{cases} 1/\ell!, & \ell \geq 0, \\ 0, & \ell < 0. \end{cases}$$

Let $M_k(x)$ be the **SW** $(k \times k)$-block of $M(x)$: from the algorithm, we must show that, for small negative x, the matrix $M_k(x)$ is invertible and compute the sign of its determinant.

Write $M(x) = EG^\pi X^\pi(x) \tilde{M} \tilde{X}(x)$ for

$$E = \operatorname{diag}(s_i), \quad G^\pi = \operatorname{diag}(g(\pi^{-1}(i) - 1)), \quad X^\pi(x) = \operatorname{diag}(x^{\pi^{-1}(i)-1}),$$
$$(\tilde{M})_{i,j} = (\pi^{-1}(i) - 1)^{\underline{j-1}}, \quad \tilde{X}(x) = \operatorname{diag}(x^{1-i}).$$

Here we use the notation $a^{\underline{b}} = a(a-1)\cdots(a-b+1)$. Let $E_k, G^\pi_k, X^\pi_k(x)$ be the **SE** $k \times k$-blocks of $E, G^\pi, X^\pi(x)$, respectively. Similarly, let $\tilde{M}_k$ and $\tilde{X}_k(x)$ be the **SW** and **NW** $k \times k$-blocks of $\tilde{M}$ and $\tilde{X}(x)$, respectively. We have $M_k(x) = E_k G^\pi_k X^\pi_k(x) \tilde{M}_k \tilde{X}_k(x)$ and therefore $\det(M_k(x))$ is the product of the determinants of these blocks. We must therefore determine the sign of the determinant of each block.

For real numbers a and b, we write $a \sim b$ if a and b have the same sign. We have $\det E_k = \prod_{j \geq n-k+2} s_i$, $\det G^\pi_k \sim 1$,

$$\det X^\pi_k(x) = \prod_{j \geq n-k+2} (x^{\pi^{-1}(i)-1}) = x^{\left(\sum_{j \geq n-k+1} (\pi^{-1}(i)-1)\right)}$$
$$(-1)^{\left(k + \sum_{j \geq n-k+2} \pi^{-1}(i)\right)}$$

and $\det \tilde{X}_k(x) = x^{-k(k-1)/2} \sim (-1)^{k(k-1)/2}$. In order to compute $\det \tilde{M}_k$, consider the Vandermonde matrix V^π with $(V^\pi)_{i,j} = (\pi^{-1}(i) - 1)^{j-1}$; notice that there exists $U \in \mathcal{U}^{\mathbf{1}}_{n+1}$ with $V^\pi = \tilde{M} U$. Let V^π_k be the **SW** $k \times k$-block of V^π, also a Vandermonde matrix. We have

$$\det \tilde{M}_k = \det V^\pi_k = \prod_{n-k+2 \leq j < j' \leq n+1} (\pi^{-1}(j') - \pi^{-1}(j)).$$

At this point we know that $\det M_k \neq 0$ (with the same sign for all small negative x) and therefore there exists a diagonal matrix $\hat{\Delta}(Q) \in B^+_{n+1}$ such that $M(x)$ and $\hat{M} = \hat{\Delta}(Q) A$ are Bruhat equivalent. Write $\hat{\Delta}(Q) = \operatorname{diag}(\hat{\delta}_i(Q))$; we must compute $\hat{\delta}_i(Q)$. Let $\hat{M}_k$ be the SW $k \times k$-block of $\hat{M}_k$: by Bruhat equivalence we have $\det \hat{M}_k \sim \det(M_k(x))$; by construction we have $\det \hat{M}_k = (-1)^{kn} \prod_{j \geq n-k+2} \hat{\delta}_j(Q)$. Thus $\hat{\delta}_{n-k+2}(Q) \sim (-1)^n \det(M_k(x)) \det(M_{k-1}(x))$.

We have

$$\begin{aligned}
\det E_k \det E_{k-1} &= s_{n-k+2} = Q_{n-k+2,\pi^{-1}(n-k+2)},\\
\det X_k^\pi(x) \det X_{k-1}^\pi(x) &\sim (-1)^{\pi^{-1}(n-k+2)-1},\\
\det \tilde{X}_k \det \tilde{X}_{k-1} &\sim (-1)^{k-1},\\
\det \tilde{M}_k \det \tilde{M}_{k-1} &\sim \prod_{n-k+2<j'\le n+1} (\pi^{-1}(j') - \pi^{-1}(n-k+2))\\
&\sim (-1)^{\mathbf{SW}(Q,n-k+2,\pi^{-1}(n-k+2))}
\end{aligned}$$

and therefore

$$\hat{\delta}_{n-k+2}(Q) \sim (-1)^n Q_{n-k+2,\pi^{-1}(n-k+2)} (-1)^{\mathbf{SW}(Q,n-k+2,\pi^{-1}(n-k+2))+k+\pi^{-1}(n-k+2)}.$$

Since both sides have absolute value 1 the latter relation is actually an equality; for $i = n-k+2$ and $j = \pi^{-1}(n-k+2)$ we then have

$$\hat{\delta}_i(Q) = Q_{i,j}(-1)^{\mathbf{SW}(Q,i,j)+i+j} = Q_{i,j}(-1)^{\mathbf{NE}(Q,i,j)} = \delta_i(Q).$$

Thus $\hat{\Delta}(Q) = \Delta(Q)$ and we are done. □

A geometric description of the situation is now more clear. The Bruhat cells of the form $\mathrm{Bru}(DA)$, $D \in \mathrm{Diag}^+_{n+1}$, are disjoint open sets and their union is dense in SO_{n+1}. The complement of this union is the disjoint union of Bruhat cells of lower dimension. Let $\Gamma : (-\epsilon,\epsilon) \to SO_{n+1}$ be a smooth Jacobian curve (i.e., with $\Lambda(t) = (\Gamma(t))^{-1}\Gamma'(t) \in \mathfrak{J}$ for all $t \in (-\epsilon,\epsilon)$): if $\Gamma(0)$ does not belong to a top-dimensional Bruhat cell then the function **chop** and Lemma 6.2 tell us in which cell $\Gamma(t)$ falls for $t < 0$, $|t|$ small. In other words, provided you follow a Jacobian curve you can only arrive at a given low-dimensional Bruhat cell from one of the adjacent top-dimensional cells.

As discussed above, the decomposition into Bruhat cells lifts of Spin_{n+1}. The above geometric characterization of **chop** thus also lifts to a map $\mathbf{chop} : \mathrm{Spin}_{n+1} \to \tilde{B}^+_{n+1}$. Let $\mathbf{a} = \mathbf{chop}(\mathbf{1})$ (so that $\Pi(\mathbf{a}) = A$) and define $\Delta : \mathrm{Spin}(n+1) \to \widetilde{\mathrm{Diag}}^+_{n+1}$ by $\mathbf{chop}(z) = \Delta(z)\mathbf{a}$. We shall not attempt to give a combinatorial description of Δ or **chop** in the spin groups.

We present yet another interpretation of the chopping operation. Let $\Gamma : (t_0 - c, t_0 + c) \to \mathrm{Spin}_{n+1}$ be a Jacobian curve. Notice that if Γ is Jacobian and $z \in \mathrm{Spin}_{n+1}$ then so is $z\Gamma$ (their logarithmic derivatives are equal). Thus, Lemma 6.2 can be extended to show that for all $z \in \mathrm{Spin}_{n+1}$ one has that $z\Gamma(t_0-\epsilon) \in \mathrm{Bru}(\mathbf{chop}(z\Gamma(t_0)))$ or $\Gamma(t_0-\epsilon) \in z^{-1}\mathrm{Bru}(\mathbf{chop}(z\Gamma(t_0)))$. In particular, taking $z = (\Gamma(t_0))^{-1}$, we have $\Gamma(t_0-\epsilon) \in \Gamma(t_0)\mathrm{Bru}(\mathbf{a})$. Conversely, given $Q_1 \in SO_{n+1}$ and $Q_0 \in Q_1\mathrm{Bru}(A)$ there exists a globally Jacobian curve $\Gamma : [0,1] \to SO_{n+1}$ with $\Gamma(0) = Q_0$, $\Gamma(1) = Q_1$ (so that $\gamma : [0,1] \to \mathbb{S}^n$, $\gamma(t) = \Gamma(t)e_1$, is globally convex). The following statement thus follows from Lemma 6.2.

Corollary 6.3. *Given $Q \in SO_{n+1}$ there exists an open set $U \subset SO_{n+1}$ with $Q \in U$ and $U \cap (Q\,\mathrm{Bru}(A)) \subseteq \mathrm{Bru}(\mathbf{chop}(Q))$. Similarly, given $z \in \mathrm{Spin}_{n+1}$ there exists an open set $U \subset \mathrm{Spin}_{n+1}$ with $z \in U$ and $U \cap (z\,\mathrm{Bru}(\mathbf{a})) \subseteq \mathrm{Bru}(\mathbf{chop}(z))$.*

Proof. The SO_{n+1} case follows from the remarks above together with Lemma 6.2; the Spin_{n+1} case is similar. □

The next statement is crucial in our consideration.

Proposition 6.4. *For any $z \in \tilde{B}^+_{n+1}$ there are homeomorphisms*

$$\mathcal{L}\mathbb{S}^{n+1}(z) \approx \mathcal{L}\mathbb{S}^{n+1}(\mathbf{chop}(z)) \approx \mathcal{L}\mathbb{S}^{n+1}(\Delta(z)).$$

We need a few preliminary constructions and results. Consider a Jacobian curve $\Gamma_0 : [0,1] \to SO_{n+1}$ with $\Gamma_0(0) = Q_0$ and $\Gamma_0(1) = Q_1$. Define $\mathbf{C}_{Q_0,\Gamma_0,Q_1} : \mathcal{L}\mathbb{S}^{n+1}(Q_0) \to \mathcal{L}\mathbb{S}^{n+1}(Q_1)$ by

$$(\mathbf{C}_{Q_0,\Gamma_0,Q_1}(\gamma))(t) = \begin{cases} \gamma(2t), & t \leq 1/2, \\ \Gamma_0(2t-1)e_1, & t \geq 1/2. \end{cases}$$

Lemma 6.5. *Consider a globally Jacobian curve $\Gamma_0 : [0,1] \to SO_{n+1}$ whose image is contained in a Bruhat cell. Let $Q_0 = \Gamma_0(0)$, $Q_1 = \Gamma_0(1)$ and $U \in \mathcal{U}^{\mathbf{1}}_{n+1}$ with $B(U,Q_0) = Q_1$. Then the maps $\mathbf{B}_{Q_0,U,Q_1}$ and $\mathbf{C}_{Q_0,\Gamma_0,Q_1}$ are homotopic. In particular, $\mathbf{C}_{Q_0,\Gamma_0,Q_1}$ is a homotopy equivalence.*

Proof. Since Q_0 and $\Gamma(s)$ are in the same Bruhat cell for all $s \in [0,1]$ we can define a continuous function $\mathbf{U} : [0,1] \to \mathcal{U}^{\mathbf{1}}_{n+1}$ with $B(\mathbf{U}(s), Q_0) = \Gamma(s)$, $\mathbf{U}(0) = I$, $\mathbf{U}(1) = U$. Define $H : \mathcal{L}\mathbb{S}^n(Q_0) \times [0,1] \to \mathcal{L}\mathbb{S}^n(Q_1)$ by

$$(H(\gamma,s))(t) = \begin{cases} \mathbf{B}_{Q_0,\mathbf{U}(s),\Gamma(s)}(2t/(1+s)), & t \leq (1+s)/2, \\ \Gamma_0(2t-1)e_1, & t \geq (1+s)/2. \end{cases}$$

The map H produces the desired homotopy from $\mathbf{C}_{Q_0,\Gamma_0,Q_1}$ to $\mathbf{B}_{Q_0,U,Q_1}$. □

To prove Proposition 6.4 we will also use the following previously known facts.

Fact 1 (comp. Lemma 5 in [13])**.** *For any $z \in \mathrm{Spin}_{n+1}$ the space $\mathcal{L}\mathbb{S}^n(z)$ has two connected components if and only if there exists a globally convex curve in $\mathcal{L}\mathbb{S}^n(z)$. One of these connected components is the set of all globally convex curves in $\mathcal{L}\mathbb{S}^n(z)$ and this connected component is contractible. If $\mathcal{L}\mathbb{S}^n(z)$ contains no globally convex curves then it is connected.*

Fact 2 (comp. Theorem 0.1 in [3])**.** *Let M and N be two topological Hilbert manifolds. Then any weak homotopy equivalence $f_0 : N \to M$ is homotopic to a homeomorphism $f_1 : N \to M$.*

Let $z_1 \in \tilde{B}^+_{n+1}$ and consider a smooth Jacobian curve $\Gamma_{\mathbf{aux}} : [-\epsilon, \epsilon] \to \mathrm{Spin}_{n+1}$ with $\Gamma_{\mathbf{aux}}(0) = z_1$. Choose ϵ sufficiently small so that the image of $\Gamma_{\mathbf{aux}}([-\epsilon, 0)) \subset \mathrm{Bru}(\mathbf{chop}(z_1)) \cap (z_1 \mathrm{Bru}(\mathbf{a}))$. Let $z_0 = \Gamma_{\mathbf{aux}}(-\epsilon)$, $\Gamma_0(t) = \Gamma_{\mathbf{aux}}(\epsilon(t-1))$. Proposition 6.4 now follows directly from the next lemma.

Lemma 6.6. *The map $\mathbf{C}_{z_0,\Gamma_0,z_1} : \mathcal{L}\mathbb{S}^n(z_0) \to \mathcal{L}\mathbb{S}^n(z_1)$ is a weak homotopy equivalence.*

Proof. For k a non-negative integer, let $\alpha : \mathbb{S}^k \to \mathcal{L}\mathbb{S}^{n+1}(z_1)$: we construct $\tilde{\alpha} : \mathbb{S}^k \to \mathcal{L}\mathbb{S}^{n+1}(z_0)$ and a homotopy $H : \mathbb{S}^k \times [0,1] \to \mathcal{L}\mathbb{S}^{n+1}(z_1)$ with $H(s,0) = \alpha(s)$, $H(s,1) = \mathbf{C}_{z_0,\Gamma_0,z_1}(\tilde{\alpha}(s))$. By compactness and continuity, there exists $\epsilon_1 > 0$ such that for all $s \in \mathbb{S}^k$ and for all $t \in [1-\epsilon_1, 1)$ we have $\mathfrak{F}_{\alpha(s)}(t) \in \mathrm{Bru}(\mathbf{chop}(z_1)) \cap (z_1 \mathrm{Bru}(\mathbf{a}))$. Again by compactness and continuity, there exists $\epsilon_2 > 0$, $\epsilon_2 < \epsilon_1/2$, such that for all $s \in \mathbb{S}^k$ we have $\mathfrak{F}_{\alpha(s)}(1-\epsilon_1) \in \mathrm{Bru}(\mathbf{chop}(z_1)) \cap (\Gamma_0(1-\epsilon_2)\mathrm{Bru}(\mathbf{a}))$. Thus, for each $s \in \mathbb{S}^k$, the space X_s of globally Jacobian curves $\Gamma_s : [1-\epsilon_1, 1]$ for which $\Gamma_s(1-\epsilon) = \mathfrak{F}_{\alpha(s)}(1-\epsilon_1)$ and $\Gamma_s(1) = z_1$ is non-empty (since $\alpha(s)|_{[1-\epsilon_1,1]} \in X_s$) and therefore, by Fact 1, a contractible space. Consider the subspace $Y_s \subset X_s$ of curves for which $\Gamma_s(t) = \Gamma_0(t)$ for $t \geq 1-\epsilon_2$; the condition $\mathfrak{F}_{\alpha(s)}(1-\epsilon_1) \in (\Gamma_0(1-\epsilon_2)\mathrm{Bru}(\mathbf{a}))$ implies that Y_s is non-empty and Fact 1 implies that is Y_s also contractible. We may therefore construct a homotopy $H_1 : \mathbb{S}^k \times [0,1] \to \mathcal{L}\mathbb{S}^{n+1}(z_1)$ with $H(s,0) = \alpha(s)$, $H(s,\tilde{s}) \in X_s$ and $H(s,1) \in Y_s$. In other words, we may assume without loss of generality that there exists $\epsilon_2 > 0$ such that $\alpha(s)(t) = \gamma_0(t)$ for all $s \in \mathbb{S}^k$ and $t > 1-\epsilon_2$.

Set $z_2 = \Gamma_0(1-\epsilon_2)$ and $\Gamma_2 : [0,1] \to \mathrm{Spin}_{n+1}$ with $\Gamma_2(t) = \Gamma_0((1-\epsilon_2)+\epsilon_2 t)$. We may reparameterize the curves so that, for all s, $\alpha(s)(1/2) = z_2$ and $\alpha(s)(t) = \Gamma_2(2t-1)$ for $t \geq 1/2$. In other words, we may assume that $\alpha(s) = \mathbf{C}_{z_2,\Gamma_2,z_1}\hat{\alpha}(s)$. Set $\Gamma_3(t) = \Gamma_0(t/(1-\epsilon_2))$; Lemma 6.5 tells us that $\mathbf{C}_{z_0,\Gamma_3,z_2} : \mathcal{L}\mathbb{S}^n(z_0) \to \mathcal{L}\mathbb{S}^n(z_2)$ is a homotopy equivalence: $\hat{\alpha}$ is therefore homotopic to $\mathbf{C}_{z_0,\Gamma_3,z_2} \circ \tilde{\alpha}$ for some $\tilde{\alpha} : \mathbb{S}^k \to \mathcal{L}\mathbb{S}^n(z_0)$, implying that α is homotopic to $\mathbf{C}_{z_0,\Gamma_0,z_1} \circ \tilde{\alpha}$, as desired. This completes the proof that $\pi_k(\mathbf{C}_{z_0,\Gamma_0,z_1}) : \pi_k(\mathcal{L}\mathbb{S}^n(z_0)) \to \pi_k(\mathcal{L}\mathbb{S}^n(z_1))$ is surjective.

The proof that this map is injective is similar. Let $\tilde{\alpha} : \mathbb{S}^k \to \mathcal{L}\mathbb{S}^n(z_0)$ and $\alpha = \mathbf{C}_{z_0,\Gamma_0,z_1} \circ \tilde{\alpha}$; assume that $\tilde{\alpha}$ is homotopically trivial, i.e., that there exists $H : \mathbb{B}^{k+1} \to \mathcal{L}\mathbb{S}^n(z_1)$ with $H|_{\mathbb{S}^k} = \alpha$: we need to prove that α is homotopically trivial. As above, change H so that $H(s)$ agrees with Γ_0 near 1, i.e., we may assume H to be of the form $H = \mathbf{C}_{z_2,\Gamma_2,z_1} \circ \hat{H}$. We therefore have that $\mathbf{C}_{z_0,\Gamma_3,z_2} \circ \tilde{\alpha}$ is homotopically trivial. Since $\mathbf{C}_{z_0,\Gamma_3,z_2} : \mathcal{L}\mathbb{S}^n(z_0) \to \mathcal{L}\mathbb{S}^n(z_2)$ is a homotopy equivalence we are done. $\square$

7. Proof of Theorem 1

First we reformulate Theorem 1 using the language of the prevous sections.

Theorem 3. *Let $Q_0, Q_1 \in SO_{n+1}$: if $\mathrm{trd}(Q_0) = \mathrm{trd}(Q_1)$ then $\mathcal{L}\mathbb{S}^n(Q_0)$ and $\mathcal{L}\mathbb{S}^n(Q_1)$ are homeomorphic.*

Let $z_0, z_1 \in \mathrm{Spin}_{n+1}$: if $\mathrm{trd}(z_0) = \mathrm{trd}(z_1)$ and $|\mathrm{trd}(z_0)| \neq n+1$ then $\mathcal{L}\mathbb{S}^n(z_0)$ and $\mathcal{L}\mathbb{S}^n(z_1)$ are homeomorphic.

Theorem 1 follows directly from Theorem 3. The condition $|\mathrm{trd}(z_0)| \neq n+1$ in the spin part is necessary: for $n = 2$ and $1, -1 \in \mathrm{Spin}_3$ the two central elements the spaces $\mathcal{L}\mathbb{S}^2(1)$ and $\mathcal{L}\mathbb{S}^2(-1)$ are not homeomorphic since they have different numbers of connected components.

Recall that from Lemmas 3.1 and 6.6 and Proposition 6.4 we already know that if $\Delta(Q_0) = \Delta(Q_1)$ then $\mathcal{L}\mathbb{S}^n(Q_0)$ and $\mathcal{L}\mathbb{S}^n(Q_1)$ (as well as $\mathcal{L}\mathbb{S}^n(\Delta(Q_0))$) are homeomorphic; we have a similar result for the spin group. We are therefore left to consider the spaces $\mathcal{L}\mathbb{S}^n(Q)$, $Q \in \mathrm{Diag}^+_{n+1}$, and their spin counterparts. A number of additional statements are required for the proof of Theorem 3.

Lemma 7.1. *Let $D_0, D_1 \in \mathrm{Diag}^+_{n+1}$ with $\mathrm{trd}(D_0) = \mathrm{trd}(D_1)$. Then there exists $Q \in B^+_{n+1}$ with $\Delta(Q) = D_0$, $\Delta(\mathbf{TR}(Q)) = D_1$. Thus $\mathcal{L}\mathbb{S}^n(D_0)$ and $\mathcal{L}\mathbb{S}^n(D_1)$ are homeomorphic.*

Proof: Let π be a permutation of $\{1, 2, \ldots, n+1\}$ with $(D_1)_{\pi(i),\pi(i)} = (D_0)_{i,i}$ for all i. Let P be a permutation matrix with $(P)_{(i,j)} = 1$ if and only if $j = \pi(i)$. Set $Q = D_0\Delta(P)P$: we have $\Delta(Q) = D_0\Delta(P)\Delta(P) = D_0$. On the other hand, if $\pi(i) = j$, we have $\delta_j(\mathbf{TR}(Q)) = \delta_i(Q)$ (from the proof of Lemma 7.3) and therefore $\delta_j(\mathbf{TR}(Q)) = (D_0)_{(i,i)}(\Delta(P))_{(i,i)}\delta_i(P) = (D_0)_{(i,i)} = (D_1)_{j,j}$ and $\Delta(\mathbf{TR}(Q)) = D_1$. The last claim follows from Proposition 6.4. $\square$

This completes the proof of Theorem 3 for the SO_{n+1} case: one judicious use of the equivalences proved in the previous section is enough. The spin case is slightly subtler: it turns out that a single instance of the equivalences is not enough, which can be readily checked by an exhaustive search in the case $n = 2$. A small chain of consecutive instances of the equivalences are therefore used.

Lemma 7.2. *Let $z_0, z_1 \in \widetilde{\mathrm{Diag}}^+_{n+1}$ with $\mathrm{trd}(z_0) = \mathrm{trd}(z_1) \neq \pm(n+1)$. Then there exist $w_0, w_1 \in \tilde{B}^+_{n+1}$ with $\Delta(w_0) = z_0$, $\Delta(\mathbf{TR}(w_1)) = z_1$ and either $\Delta(\mathbf{TR}(w_0)) = \Delta(w_1)$ or $\Delta(\mathbf{TR}(w_0)) = \mathbf{TR}(\Delta(w_1))$. Thus $\mathcal{L}\mathbb{S}^n(z_0)$ and $\mathcal{L}\mathbb{S}^n(z_1)$ are homeomorphic.*

Proof. Take $s = \operatorname{trd}(z_0)$ and apply Lemma 4.1 to obtain $z \in \widetilde{\operatorname{Diag}}^+_{n+1}$ with $\mathbf{TR}(z) = -z$. Let $Q_0 = \Pi(z_0)$, $Q_1 = \Pi(z_1)$, $Q = \Pi(z)$. By Lemma 7.1 there exist $P_0, P_1 \in B^+_{n+1}$ with $\Delta(P_0) = Q_0$, $\Delta(\mathbf{TR}(P_0)) = Q$, $\Delta(P_1) = Q$, $\Delta(\mathbf{TR}(P_1)) = Q_1$. Take $w_0, w_1 \in \tilde{B}^+_{n+1}$ with $\Pi(w_0) = P_0$, $\Pi(w_1) = P_1$, $\Delta(w_0) = z_0$, $\Delta(\mathbf{TR}(w_1)) = z_1$. We have $\Delta(\mathbf{TR}(w_0)) = \pm z$ and $\Delta(w_1) = \pm z$ and we are done. $\square$

Theorem 3 follows directly from Lemmas 7.1 and 7.2.

It is natural to ask whether Theorem 3 is the strongest possible such statement, i.e., if spaces which it does not declare homeomorphic are actually not homeomorphic. We do not know the answer to this question (see Problem 2 below) but the following proposition shows that it is the strongest result which follows (or follows directly) from the remarks of the previous sections.

Proposition 7.3. *For all $Q \in B^+_{n+1}$ we have* $\operatorname{trd}(\mathbf{AD}(Q)) = \operatorname{trd}(\mathbf{TR}(Q)) = \operatorname{trd}(Q)$.

Proof: Assume $(Q)_{(i,j)} \neq 0$. We have

$$\begin{aligned}\delta_{n+2-i}(\mathbf{AD}(Q)) &= (\mathbf{AD}(Q))_{(n+2-i,n+2-j)}(-1)^{\mathbf{NE}(\mathbf{AD}(Q),n+2-i,n+2-j)}\\ &= (-1)^{(i+j)}Q_{(i,j)}(-1)^{\mathbf{SW}(Q,i,j)} = Q_{(i,j)}(-1)^{\mathbf{NE}(Q,i,j)} = \delta_i(Q)\end{aligned}$$

and

$$\begin{aligned}\delta_j(\mathbf{TR}(Q)) &= (\mathbf{TR}(Q))_{(j,i)}(-1)^{\mathbf{NE}(\mathbf{TR}(Q),j,i)}\\ &= (-1)^{(i+j)}(Q)_{(i,j)}(-1)^{\mathbf{SW}(Q,i,j)} = Q_{(i,j)}(-1)^{\mathbf{NE}(Q,i,j)} = \delta_i(Q).\end{aligned}$$

The proposition now follows. $\square$

8. Proof of Theorem 2

Our nearest goal is to prove Theorem 2(i), i.e. the fact that the inclusion $\hat{\mathcal{L}}\mathbb{S}^n(z) \subset \Omega\operatorname{Spin}_{n+1}(z)$ is homotopically surjective for all z and then to settle Theorem 2(ii), i.e. that this inclusion is a homotopy equivalence if $\Pi(z) = \pm J_+$.

Recall that the group $SO_{n+1} \subset \mathbb{R}^{(n+1)\times(n+1)}$ has a natural Riemann metric and $\operatorname{Spin}_{n+1}$ inherits it via Π. With this metric, let $r_{n+1} > 0$ be the injectivity radius of the exponential map, i.e., r_{n+1} is such that if $z_0, z_1 \in \operatorname{Spin}_{n+1}$, $d(z_1, z_2) < r_{n+1}$, then there exists a unique shortest geodesic $g_{z_0,z_1} : [0,1] \to \operatorname{Spin}_{n+1}$ (parametrized by a constant multiple of arc length) joining z_0 and z_1 so that $g_{z_0,z_1}(i) = z_i$, $i = 0, 1$.

We will need another technical lemma.

Lemma 8.1. *Let K be a smooth compact manifold and $\alpha : K \times [0,1] \to \operatorname{Spin}_{n+1}$ be a smooth function and write $\alpha_s(t) = \alpha(s,t)$. Then there exists $\xi_\star \in \mathcal{L}\mathbb{S}^n(\mathbf{1})$ and corresponding $\Xi_\star \in \hat{\mathcal{L}}\mathbb{S}^n$ such that $\xi_\star^{\mathbf{TR}} = \xi_\star$ and the curves $\gamma_s(t) = \alpha_s(t)\xi_\star(t)$ are positive locally convex for all $s \in K$. Furthermore, given $\epsilon > 0$, $\epsilon < r_{n+1}$, we may assume that*

$$d(\tilde{\mathfrak{F}}_{\gamma_s}(t), \alpha_s(t)\Xi_\star(t)) < \epsilon$$

for all $s \in K$, $t \in [0,1]$.

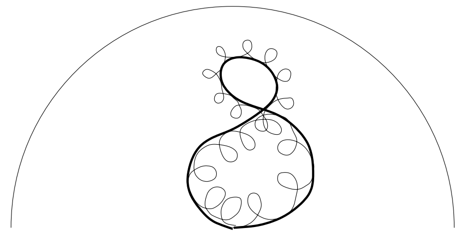

FIGURE 2. Approximating a curve by a locally convex curve

The intuitive picture here, at least for $n = 2$, is that an arbitrary curve $\gamma : \mathbb{S}^1 \to \mathbb{S}^n$ can be replaced by a phone wire, a locally convex curve which in some sense follows γ while quickly rotating in a transversal direction to guarantee local convexity (see Figure 2).

Proof: Take $\xi_1 \in \mathcal{L}\mathbb{S}^n(\mathbf{1})$ as in Lemma 2.2. We claim that $\xi_\star(t) = \xi_1(Nt)$ satisfies the lemma for a sufficiently large integer N. Notice that $\xi_\star^{(k)}(t) = N^k \xi_1^{(k)}(Nt)$ and

$$\begin{aligned}\gamma_s^{(k)}(t) &= N^k \left(\alpha_s(t)\xi_1^{(k)}(Nt) + \cdots + \frac{1}{N^j}\binom{k}{j}\alpha_s^{(j)}(t)\xi_1^{(k-j)}(Nt) + \cdots\right)\\ &= N^k(\alpha_s(t)\xi_1^{(k)}(Nt) + E_k(N,s,t))\end{aligned}$$

where $E_k(N,s,t)$ tends to 0 when $N \to \infty$. Since

$$\det(\alpha_s(t)\xi_1(Nt), \ldots, \alpha_s(t)\xi_1^{(n)}(Nt)) = \det(\xi_1(Nt), \ldots, \xi_1^{(n)}(Nt))$$

is positive and bounded away from 0 it follows that γ_s is indeed locally convex for sufficiently large N. Furthermore, the identities

$$\begin{gathered}\begin{pmatrix}\alpha_s(t)\xi_1(Nt) + E_0(N,s,t) & \cdots & \alpha_s(t)\xi_1^{(n)}(Nt) + E_n(N,s,t)\end{pmatrix} = \mathfrak{F}_{\gamma_s}(t)R_{\gamma_s}(t)\\ \begin{pmatrix}\alpha_s(t)\xi_1(Nt) & \cdots & \alpha_s(t)\xi_1^{(n)}(Nt)\end{pmatrix} = \alpha_s(t)\Xi_\star(t)R_{\xi_\star}(t),\end{gathered}$$

where $R_{\gamma_s}(t)$ and $R_{\xi_\star}(t)$ are upper triangular matrices with positive diagonals, show that $d(\tilde{\mathfrak{F}}_{\gamma_s}(t), \alpha_s(t)\Xi_\star(t))$ can be made arbitrarily small by choosing large N. $\square$

Proposition 8.2. *For any $z \in \mathrm{Spin}_{n+1}$ the inclusion $\hat{\mathcal{L}}\mathbb{S}^n(z) \subset \Omega\,\mathrm{Spin}_{n+1}(z)$ is homotopically surjective. In other words, given $\alpha_0 : \mathbb{S}^k \to \Omega\,\mathrm{Spin}_{n+1}(z)$ there exists a homotopy in $\Omega\,\mathrm{Spin}_{n+1}(z)$ from α_0 to $\alpha_1 : \mathbb{S}^k \to \hat{\mathcal{L}}\mathbb{S}^n(z) \subset \Omega\,\mathrm{Spin}_{n+1}(z)$.*

Proof. Write $\alpha_0(s;t) = \alpha_0(s)(t)$. Assume without loss of generality that α_0 is smooth if interpreted as $\alpha_0 : \mathbb{S}^k \times [0,1] \to \mathrm{Spin}_{n+1}$. Assume furthermore that α_0 is flat at both $t = 0$ and $t = 1$, i.e., that $(\alpha_0(s))^{(m)}(t) = 0$ for $t \in \{0,1\}$, for all $s \in \mathbb{S}^k$ and all $m > 0$. By Lemma 8.1, there exist $\xi_\star \in \mathcal{L}\mathbb{S}^n(\mathbf{1})$ and corresponding $\Xi_\star \in \hat{\mathcal{L}}\mathbb{S}^n(\mathbf{1})$ such that

$$d(\tilde{\mathfrak{F}}_{\gamma_s}(t), \alpha_0(s;t)\Xi_\star(t)) < \frac{r_{n+1}}{2}$$

for all $s \in \mathbb{S}^k$, $t \in [0,1]$; here, as in Lemma 8.1, $\gamma_s(t) = \alpha(s;t)\xi_\star(t)$. The flatness condition guarantees that

$$\tilde{\mathfrak{F}}_{\gamma_s}(0) = \alpha_0(s;0)\Xi_\star(0) = \mathbf{1}, \qquad \tilde{\mathfrak{F}}_{\gamma_s}(1) = \alpha_0(s;1)\Xi_\star(1) = z.$$

Recall that $\Xi_\star \in \Omega\,\mathrm{Spin}_{n+1}(\mathbf{1})$: let $H : [0,1] \to \Omega\,\mathrm{Spin}_{n+1}(\mathbf{1})$ be a homotopy between the constant path $H(0)(t) = \mathbf{1}$ and $H(1) = \Xi_\star$.

Take $\alpha_1(s;t) = \tilde{\mathfrak{F}}_{\gamma_s}(t)$ and $\alpha_{1/2}(s;t) = \alpha_0(s;t)\Xi_\star(t)$. Clearly, $\alpha_1 : \mathbb{S}^k \to \hat{\mathcal{L}}\mathbb{S}^n(z) \subset \Omega\operatorname{Spin}_{n+1}(z)$, as required. It suffices to construct homotopies between α_0 and $\alpha_{1/2}$ and between $\alpha_{1/2}$ and α_1. The homotopy between α_0 and $\alpha_{1/2}$ is given by

$$\alpha_\sigma(s;t) = \alpha_0(s;t)H(2\sigma)(t), \qquad \sigma \in [0, 1/2].$$

Recall that $d(\alpha_{1/2}(s;t), \alpha_1(s;t)) < r_{n+1}/2$: the homotopy between $\alpha_{1/2}$ and α_1 is defined by joining these two points of $\operatorname{Spin}_{n+1}$ by the uniquely defined shortest geodesic (parametrized by a constant multiple of arc length):

$$\alpha_\sigma(s;t) = g_{\alpha_{1/2}(s;t),\alpha_1(s;t)}(2\sigma - 1), \qquad \sigma \in [1/2, 1].$$

$\square$

Proposition 8.3. *Assume $\Pi(z) = \pm J_+$: then the inclusion $\hat{\mathcal{L}}\mathbb{S}^n(z) \subset \Omega\operatorname{Spin}_{n+1}(z)$ is a weak homotopy equivalence. In other words (given Proposition 8.2), if $\hat{H}_0 : \mathbb{B}^{k+1} \to \Omega\operatorname{Spin}_{n+1}(z)$ takes $\mathbb{S}^k \subset \mathbb{B}^{k+1}$ to $\hat{\mathcal{L}}\mathbb{S}^n(z) \subset \Omega\operatorname{Spin}_{n+1}(z)$ then there exist $H_1 : \mathbb{B}^{k+1} \to \mathcal{L}\mathbb{S}^n(z)$ and corresponding $\hat{H}_1 : \mathbb{B}^{k+1} \to \hat{\mathcal{L}}\mathbb{S}^n(z)$ with $\hat{H}_0|_{\mathbb{S}^k} = \hat{H}_1|_{\mathbb{S}^k}$.*

Clearly if $\Pi(z) = \pm J_+$ then $s(z)$ must be 0, 1 or -1. Theorem 2 therefore follows from Proposition 8.3 and Fact 2.

Proof. Assume without loss of generality that H_0 is smooth. Take $\xi_\star \in \mathcal{L}\mathbb{S}^n(\mathbf{1})$ as in Lemma 8.1 so that, for any $s \in \mathbb{B}^{k+1}$, $\gamma(s) = \hat{H}_0(s)\xi_\star \in \mathcal{L}\mathbb{S}^n(z)$. We may furthermore assume that the curves $\hat{H}_0(s)(C_1 t)\xi_\star(C_2 t + C_3)$ are locally convex for any $s \in \mathbb{B}^{k+1}$, for any $C_1, C_2 \in [1/10, 10]$ and for any $C_3 \in \mathbb{R}$. Recall that $\xi_\star(t) = \xi_1(Nt)$ for some large N: take N to be a multiple of 4 so that $\Xi_\star(1/4) = \Xi_\star(1/2) = \Xi_\star(3/4) = \mathbf{1}$, $\Xi_\star(t) = \mathbf{TR}(\Xi_\star(t)) = J_+(\Xi_\star(t))^{-1}J_+$ and $\Xi_\star(1-t) = (\Xi_\star(t))^{-1}$. Recall that $\Lambda_{\xi_\star}$ is constant: let $B = \Lambda_{\xi_\star}(t)$. Set

$$H_1(s)(t) = \gamma(2s)(t) = \hat{H}_0(s)(t)\xi_\star(t), \qquad |s| \le 1/2.$$

We now define H_1 in the two regions $|s| \in [1/2, 3/4]$ and $|s| \in [3/4, 1]$.

For $s \in [3/4, 1]$ we squeeze the function $\hat{H}_0(s/|s|)$ to a central interval $[1-|s|, |s|]$ and attach chunks of $\xi_\star(2t)$ outside the central interval. For

$$n = 2, \quad Q = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix} = \mathbf{chop}(-J_+),$$

the construction is illustrated in Figure 3: we can add chunks of a locally convex curve at the endpoints and translate in the sphere (i.e., rotate in $\mathbb{R}^3$) the central portion of the curve. Continue the process to add several closed circles at both endpoints.

The general construction is perhaps best stated in terms of Λ: for $|s| \in [3/4, 1]$

$$\Lambda_{H_1(s)}(t) = \begin{cases} 2B, & 0 \le t < 1 - |s|, \\ \frac{1}{2|s|-1}\Lambda_{\hat{H}_0(s/|s|)}\left(\frac{t-1+|s|}{2|s|-1}\right), & 1 - |s| \le t \le |s|, \\ 2B, & |s| < t \le 1. \end{cases}$$

Recall that Λ is only assumed to be of class L^2 and therefore the jump discontinuities are allowed. The curve $H_1(s)$ defined using the above Λ is by construction locally convex: we must verify

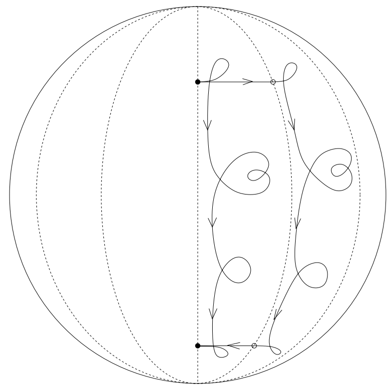

FIGURE 3. Approximating a curve by a locally convex curve

that $\mathfrak{F}_{H_1(s)}(1) = \hat{H}_1(s)(1) = z$. We have $\hat{H}_1(s)(1-|s|) = \Xi_\star(2(1-|s|))$; for $1-|s| \le t \le |s|$ we therefore have

$$\hat{H}_1(s)(t) = \Xi_\star(2(1-|s|))\hat{H}_0(s/|s|)\left(\frac{t-1+|s|}{2|s|-1}\right)$$

and $\hat{H}_1(s)(|s|) = \Xi_\star(2(1-|s|))z$; finally, at least in SO_{n+1} we have

$$\begin{aligned}\hat{H}_1(s)(1) &= \Xi_\star(2(1-|s|))z\Xi_\star(2(1-|s|))\\ &= \Xi_\star(2(1-|s|))(\pm J_+)\Xi_\star(2(1-|s|))(\pm J_+)z\\ &= \Xi_\star(2(1-|s|))(\Xi_\star(2(1-|s|)))^{-1}z = z\end{aligned}$$

(recall that $z = \pm J_+$ and that $J_+\Xi_\star(t)J_+ = (\Xi_\star(t))^{-1}$); by continuity we have $\hat{H}_1(s)(1) = z$ in Spin_{n+1} for all s with $|s| \in [3/4, 1]$.

The missing step is $|s| \in [1/2, 3/4]$. For $n = 2$, the circles which are concentrated at the endpoints for $|s| = 3/4$ must spread along the curve as s approaches $1/2$. More algebraically, notice that for both $|s| = 1/2$ and $|s| = 3/4$, we can write $H_1(s)(t) = (A_{|s|}(s/|s|)(t))(\beta_{|s|}(t))$, $A_{|s|}(s/|s|) : [0,1] \to SO_{n+1}$, $\beta_{|s|} : [0,1] \to \mathbb{S}^n$. From the constructions above we have

$$\begin{aligned}A_{\frac{1}{2}}(s/|s|)(t) &= \hat{H}_0(s/|s|)(t), & \beta_{\frac{1}{2}}(t) &= \xi_\star(t),\\ A_{\frac{3}{4}}(s/|s|)(t) &= \hat{H}_0(s/|s|)(g_{\frac{3}{4}}(t)), & \beta_{\frac{3}{4}}(t) &= \xi_\star(h_{\frac{3}{4}}(t))\end{aligned}$$

where

$$g_{\frac{3}{4}}(t) = \begin{cases}0, & 0 \le t \le \frac{1}{4},\\ 2t - \frac{1}{2}, & \frac{1}{4} \le t \le \frac{3}{4},\\ 1, & \frac{3}{4} \le t \le 1,\end{cases} \quad h_{\frac{3}{4}}(t) = \begin{cases}2t, & 0 \le t \le \frac{1}{4},\\ \frac{1}{2}, & \frac{1}{4} \le t \le \frac{3}{4},\\ 2t-1, & \frac{3}{4} \le t \le 1.\end{cases}$$

We complete the definition of H_1 with

$$\begin{aligned}H_1(s)(t) &= (A_{|s|}(s/|s|)(t))(\beta_{|s|}(t)), & 1/2 \le |s| \le 3/4\\ A_\sigma(s/|s|)(t) &= \hat{H}_0(s/|s|)(g_\sigma(t)), & \beta_\sigma(t) = \xi_\star(h_\sigma(t)),\end{aligned}$$

g_σ and h_σ as plotted in Figure 4 (notice that $g_{\frac{1}{2}}(t) = h_{\frac{1}{2}}(t) = t$).

We are left with proving that $H_1(s)$ is locally convex. For $t \in [0, |s| - \frac{1}{2}] \cup [\frac{3}{2} - |s|, 1]$, $H_1(s)$ is a reparametrization of $\xi_\star$ and therefore locally convex. For $t \in [|s| - \frac{1}{2}, 1 - |s|] \cup [|s|, \frac{3}{2} - |s|]$,

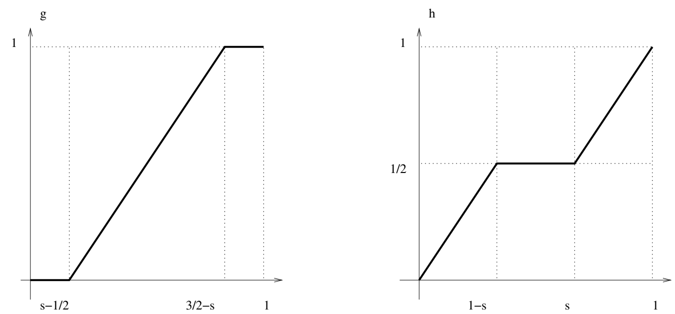

FIGURE 4. The functions g_σ and h_σ

locally convexity follows from Lemma 8.1 or, perhaps more precisely, from the choice of $\xi_\star$ as described at the beginning of the proof. Finally, for $t \in [1-|s|, |s|]$, $H_1(s)$ is a reparametrization of $H_0(s/|s|)$ and therefore again locally convex. This completes the construction of H_1 and the proof. $\square$

9. Final remarks and open problems

9.1. **Is Theorem 1 strong?** For $n = 2$, Theorems 1 and 2 imply that any space $\mathcal{L}\mathbb{S}^2(z)$ is homeomorphic to one of three spaces $\mathcal{L}\mathbb{S}^2(\mathbf{1})$, $\mathcal{L}\mathbb{S}^2(-\mathbf{1})$ or $\Omega\,\mathrm{Spin}(3) = \Omega\mathbb{S}^3$. From [6], we know that $\mathcal{L}\mathbb{S}^2(\mathbf{1})$ and $\mathcal{L}\mathbb{S}^2(-\mathbf{1})$ have 1 and 2 connected components, respectively, and $\Omega\mathbb{S}^3$ is clearly connected. From [8] and [9], we know that $\dim H^2(\mathcal{L}\mathbb{S}^2(\mathbf{1}); \mathbb{R}) = 2$, $\dim H^2(\mathcal{L}\mathbb{S}^2(-\mathbf{1}); \mathbb{R}) = 1$ and $\dim H^4(\mathcal{L}\mathbb{S}^2(-\mathbf{1}); \mathbb{R}) \geq 2$. Thus, these three spaces are not pairwise homeomorphic; also, the non-contractible connected component of $\mathcal{L}\mathbb{S}^2(-\mathbf{1})$ is not homeomorphic to either $\Omega\mathbb{S}^3$ or $\mathcal{L}\mathbb{S}^2(-\mathbf{1})$.

Unfortunately, similar information is unavailable for $n > 2$. We formulate the following question.

Problem 2. *Are the $\lceil \frac{n}{2} \rceil + 1$ subspaces $\mathcal{L}\mathbb{S}^n(M_s^{n+1})$ (and similar space of curves in Spin_{n+1}) appearing in Theorem 1 pairwise non-homeomorphic for $n > 2$?*

Our best guess is that the answer is positive.

9.2. **Bounded curvature.** A first natural generalization of the space of locally curves on $\mathbb{S}^2$ is the space of curves whose curvature κ at each point is bounded by two constants $m < \kappa < M$.

Problem 3. *Is it true that there are only finitely many topologically distinct spaces of curves whose curvature is bounded as above among the spaces of such curves with the fixed initial and variable finite frames?*

9.3. **Other Lie groups.** The space $\hat{\mathcal{L}}\mathbb{S}^n$ is a special instance of a more general construction on an arbitrary compact Lie group. Given a compact Lie group G, consider a non-holonomic subspace of its Lie algebra (i.e., this subspace generates the whole algebra). Consider some polytopal convex cone in this subspace. Take the left-invariant distribution of cones on G obtained by its left translation in the algebra. Finally, consider spaces of curves on G tangent to the obtained cone distribution which start at the unit element and end at some fixed point of G.

This generalization includes the scenario described in the previous subsection as a special case (G is SO_3 and the subspace consists of skew tridiagonal matrices, just as for our problem; the only difference is the cone).

Problem 4. *Is it true that there are only finitely many topologically distinct spaces of such curves with the fixed initial and variable finite point?*

This is likely to be too optimistic an attempt of generalization, but perhaps the finiteness condition holds true with some interesting additional hypothesis. For instance, our cone is the interior of the convex hull of a small set of rather special vectors: maybe some such condition is needed.

9.4. **The homotopy type of spaces of closed locally convex curves.** Finally, the most interesting problem in this context is to describe the homotopy type of the space of closed locally convex curves. The aim of [10] is to address this problem for $n = 2$; see partial results in [8], [9].

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Journal *of* Singularities
Volume 4 (2012), 23-34

received 14 October 2010
in revised form 19 December 2011
DOI: 10.5427/jsing.2012.4b

IRREDUCIBILITY CRITERION FOR QUASI-ORDINARY POLYNOMIALS

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Abstract. Using the notion of approximate roots and that of generalized Newton sets, we give a local criterion for a quasi ordinary polynomial to be irreducible. Such a criterion is useful in the study of singularities of quasi-ordinary hypersurfaces. It generalizes the criterion given by S.S. Abhyankar for algebraic plane curves.

Introduction

Let $\mathbf{K}$ be an algebraically closed field of characteristic zero, and let $\mathbf{R} = \mathbf{K}[[x_1, \ldots, x_e]] = \mathbf{K}[[\underline{x}]]$ be the ring of formal power series in $x_1, \ldots, x_e$ over $\mathbf{K}$. Let $F = y^n + a_1(\underline{x})y^{n-1} + \ldots + a_n(\underline{x})$ be a nonzero polynomial of $\mathbf{R}[y]$, and suppose that F is irreducible in $\mathbf{R}[y]$. Suppose that $e = 1$ and let g be a nonzero polynomial of $\mathbf{R}[y]$, then define the intersection multiplicity of F with g, denoted $\mathrm{int}(F, g)$, to be the x-order of the y resultant of F and g. The set of $\mathrm{int}(F, g), g \in \mathbf{R}[y]$, defines a semigroup, denoted $\Gamma(F)$. It is well known that a set of generators of $\Gamma(F)$ can be computed from polynomials having maximal contact with F (see [1]), namely, there exist $g_1, \ldots, g_h$ such that $n, \mathrm{int}(F, g_1), \ldots, \mathrm{int}(F, g_h)$ generate $\Gamma(F)$ and for all $1 \leq k \leq h$, the Newton-Puiseux expansion of g_k coincides with that of F up to a characteristic exponent of F. In [1], Abhyankar introduced a special set of polynomials called the approximate roots of F. These polynomials have the advantage that they can be calculated from the equation of F by using the Tschirnhausen transform. Suppose that $e \geq 2$ and that the y-discriminant of f, denoted by $D_y(F)$, is of the form $x_1^{N_1} \ldots . x_e^{N_e} .u(x_1, \ldots, x_e)$, where $N_1, \ldots, N_e \in \mathbf{N}$ and u is a unit in $\mathbf{R}$ (such a polynomial is called quasi-ordinary polynomial). By the Abhyankar-Jung Theorem (see [2]), the roots of $F(\underline{x}, y) = 0$ are all in $\mathbf{K}[[x_1^{\frac{1}{n}}, \ldots, x_e^{\frac{1}{n}}]]$, in particular there exists a power series $y(t_1, \ldots, t_e) = \sum_{p \in \mathbf{N}^e} c_p t_1^{p_1} \ldots . t_e^{p_e} \in \mathbf{K}[[t_1, \ldots, t_e]]$ such that $F(t_1^n, \ldots, t_e^n, y(t_1, \ldots, t_e)) = 0$ and the other roots of $F(t_1^n, \ldots, t_e^n, y) = 0$ are the conjugates of $y(t_1, \ldots, t_e)$ with respect to the nth roots of unity in $\mathbf{K}$. Given a polynomial g of $\mathbf{R}[y]$, we define the order of g to be the leading exponent with respect to the lexicographical order of the smallest homogeneous component of $g(t_1^n, \ldots, t_e^n, y(t_1, \ldots, t_e))$. The set of orders of polynomials of $\mathbf{R}[y]$ defines a semigroup, denoted $\Gamma(F)$. It turns out that, as in the curve case, there exists a set of approximate roots of F whose orders generate $\Gamma(F)$ (see [6], [8]). Furthermore,

(*) these approximate roots of F are quasi-ordinary and irreducible

In Section 4. we introduce the notion of generalized Newton set of a polynomial with respect to a set of polynomials and a set of elements of $\mathbb{N}^e$, and we define the notion of the straightness of such a set. It turns out that

(**) F is straight with respect to its set of approximate roots and the set of generators of its semigroup.

2000 *Mathematics Subject Classification.* 32S25, 32S70.

During the development of this work, the author visited the Department of Mathematics at the American University of Beirut. He would like to thank that institution for its hospitality and support. He also would like to thank the Center for Advanced Mathematical Sciences for offering access to many facilities.

The main result of the paper is that the two properties above, together with some numerical conditions, characterize irreducible quasi-ordinary polynomials (see Theorem 5.1.).

Note that if $e=1$, then any nonzero element of $\mathbf{K}[[x]][y]$ is quasi-ordinary, in particular our results generalize those of Abhyankar given in [3].
The paper is organized as follows: in Section 1 we discuss the main properties of an irreducible quasi-ordinary polynomial F. In Section 2 we introduce the notion of approximate roots of a polynomial in one variable over a commutative ring with unity. By [6], the orders of the approximate roots together with the canonical basis of $(n\mathbf{Z})^e$ give a set of generators of the semigroup of F. We recall this property in Section 3. Sections 4 and 5 are devoted to the irreducibility criterion: in Section 4 we introduce the notion of generalized Newton polygon, and we define the notion of straightness of a polynomial with respect to a set of polynomials, then we use these notions in Section 5 in order to decide if a given quasi-ordinary polynomial is irreducible. We finally end the paper with some examples in Section 6.

Acknowledgements. The author would like to thank the referee for helpful comments and suggestions on the original manuscript.

1. The semigroup of a quasi-ordinary polynomial

Let $\mathbf{K}$ be an algebraically closed field of characteristic zero, and let $\mathbf{R}=\mathbf{K}[[x_1,\ldots,x_e]]$ (denoted by $\mathbf{K}[[\underline{x}]]$) be the ring of formal power series in $x_1,\ldots,x_e$ over $\mathbf{K}$. Let $F=y^n+a_1(\underline{x})y^{n-1}+\ldots+a_n(\underline{x})$ be a nonzero polynomial of $\mathbf{R}[y]$ and assume, after a possible change of variables, that $a_1(\underline{x})=0$. Suppose that the discriminant of F is of the form $x_1^{N_1}.\ldots.x_e^{N_e}.u(x_1,\ldots,x_e)$, where $N_1,\ldots,N_e\in\mathbf{N}$ and $u(\underline{x})$ is a unit in $\mathbf{R}$. We call F a quasi-ordinary polynomial. It follows from the Abhyankar-Jung Theorem (see [2]) that there exists a formal power series $y(\underline{t})=y(t_1,\ldots,t_e)\in\mathbf{K}[[t_1,\ldots,t_e]]$ (denoted by $\mathbf{K}[[\underline{t}]]$) such that $F(t_1^n,\ldots,t_e^n,y(\underline{t}))=0$. Furthermore, if F is an irreducible polynomial, then we have:

$$F(t_1^n,\ldots,t_e^n,y)=\prod_{i=1}^{n}(y-y(w_1^it_1,\ldots,w_e^it_e))$$

where $(w_1^i,\ldots,w_e^i)_{1\le i\le n}$ are distinct elements of $(U_n)^e$, U_n being the group of nth roots of unity in $\mathbf{K}$.
Suppose that F is irreducible and let $y(\underline{t})$ be as above. Write $y(\underline{t})=\sum_p c_p\underline{t}^p$ and define the support of $y(\underline{t})$, denoted $\mathrm{Supp}(y(\underline{t}))$, to be the set $\{p|c_p\neq 0\}$. Obviously the support of $y(w_1t_1,\ldots,w_et_e)$ does not depend on $w_1,\ldots,w_e\in U_n$. We denote it by $\mathrm{Supp}(F)$ and we call it the support of F. Given $a,b\in\mathbb{N}^e$, we say that $a\le b$ (resp. $a<b$) if $a\le b$ coordinate-wise (resp. $a\le b$ coordinate-wise and $a\neq b$). By [9], there exists a finite sequence of elements in $\mathrm{Supp}(F)$, denoted $m_1,\ldots,m_h$, such that

i) $m_1<m_2<\ldots<m_h$. .
ii) If $p\in\mathrm{Supp}(F)$, then $p\in(n\mathbb{Z})^e+\sum_{p\in m_i+\mathbb{N}^e}m_i\mathbb{Z}$.
iii) $m_i\notin(n\mathbb{Z})^e+\sum_{j<i}m_j\mathbb{Z}$ for all $i=1,\ldots,h$.

The set of elements of this sequence is called the set of characteristic exponents of F, or the $\underline{m}$-sequence associated with F.

Let $glex$ be the well-ordering on $\mathbb{N}^e$ defined as follows: $\underline{\alpha}<_{glex}\underline{\beta}$ if and only if $|\alpha|=\sum_{i=1}^e\alpha_i<|\beta|=\sum_{i=1}^e\beta_i$ or $|\alpha|=|\beta|$ and $\alpha<_{lex}\beta$ (where lex denotes the lexicographical order).

Definition 1.1. Let $u=\sum_p c_p\underline{t}^p$ in $\mathbf{K}[[\underline{t}]]$ be a nonzero formal power series. We denote by $\mathrm{In}(u)$ the initial form of u: if $u=u_d+u_{d+1}+\ldots$ denotes the decomposition of u into a sum of homogeneous components, then $\mathrm{In}(u) = u_d$. We set $O_{\underline{t}}(u) = d$ and we call it the $\underline{t}$-order of u. We denote by $\exp_{glex}(u)$ the greatest exponent of u with respect to glex. We denote by $\mathrm{inco}_{glex}(u)$ the coefficient $c_{\exp_{glex}(u)}$, and we call it the initial coefficient of u. We finally set $\mathrm{M}_{glex}(u) = \mathrm{inco}_{glex}(u)\underline{t}^{exp_{glex}(u)}$, and we call it the initial monomial of u.

Remark 1.2. Let $u(\underline{t}) \in \mathbf{K}[[\underline{t}]]$ be a nonzero formal power series, and let $\mathrm{In}(u)$ be the initial form of u. Let $\prec$ be a well-ordering on $\mathbb{N}^e$ and define the leading exponent of u to be the leading exponent of $\mathrm{In}(u)$ with respect to $\prec$. If $\prec$ is not the lexicographical order, then we get a different notion of leading exponent (resp. initial coefficient, resp. initial monomial) of u. Note that if $\mathrm{In}(u)$ is a monomial, then these notions do not depend on the choice of $\prec$.

Denote by $\mathrm{Root}(f)$ the set of n roots of $F(t_1^n, \ldots, t_e^n, y) = 0$ introduced above and let $y(\underline{t})$ be an element of this set. We have the following:

Lemma 1.3. (See [9], paragraph 5.) $\mathrm{In}(y(\underline{t})-z(\underline{t}))$ is a monomial for all $z(\underline{t}) \in \mathrm{Root}(f)-\{y(\underline{t})\}$. Furthermore, $\{\exp_{glex}(y(\underline{t}) - z(\underline{t}))|z(\underline{t}) \in \mathrm{Root}(f) - \{y(\underline{t})\}\} = \{m_1, \ldots, m_h\}$.

Let g be a nonzero element of $\mathbf{R}[y]$. The order of g with respect to F, denoted $O_{glex}(F, g)$, is defined to be $\exp_{glex}(g(t_1^n, \ldots, t_e^n, y(\underline{t}))$. Note that it does not depend on the choice of the root $y(\underline{t})$ of $F(t_1^n, \ldots, t_e^n, y) = 0$. The set $\{O_{glex}(F, g)|g \in \mathbf{R}[y], g \notin (F)\}$ defines a subsemigroup of $\mathbf{Z}^e$. We call it the semigroup associated with F and we denote it by $\Gamma(F)$ (see [6], [8],[10], and [11] for the several definitions of the semigroup of F).

Let $\underline{m}_0 = (m_0^1, \ldots, m_0^e)$ be the canonical basis of $(n\mathbb{Z})^e$. Let I_e be the unit $e \times e$ matrix, and let $D_1 = n^e$ and for all $1 \leq i \leq h$, let D_{i+1} be the gcd of the (e, e) minors of the matrix $(nI_e, {m_1}^T, \ldots, {m_i}^T)$ (where T denotes the transpose of a matrix). Since $m_i \notin (n\mathbf{Z})^e + \sum_{j<i} m_j\mathbf{Z}$ for all $1 \leq i \leq h$, then $D_{i+1} < D_i$. We call $(D_1, \ldots, D_{h+1})$ the $\underline{D}$-sequence associated with F, and we denote it by $\mathrm{GCDM}(m_0^1, \ldots, m_0^e, m_1, \ldots, m_h)$. We define the sequence $(e_i)_{1\leq i\leq h}$ to be $e_i = \dfrac{D_i}{D_{i+1}}$ for all $1 \leq i \leq h$, and we call it the $\underline{e}$-sequence associated with F.

Let $F_0 = \mathbf{K}((\underline{x}))$ and let $\mathbf{F}_k = \mathbf{F}_{k-1}(x_1^{\frac{m_k^1}{n}} \ldots . x_e^{\frac{m_k^e}{n}})$ for all $k = 1, \ldots, h$. In particular we have:

$$\mathbf{F}_0 \subseteq \mathbf{F}_1 \subseteq \mathbf{F}_2 \subseteq \ldots \subseteq \mathbf{F}_h = \mathbf{F}_0(x_1^{\frac{m_1^1}{n}} \ldots . x_e^{\frac{m_1^e}{n}}, \ldots, x_1^{\frac{m_h^1}{n}} \ldots . x_e^{\frac{m_h^e}{n}})$$

Proposition 1.4. With the notations above, we have the following:

i) If $y(\underline{x})$ is a root of $F(\underline{x}, y) = 0$, then $F_h = \mathbf{K}((y(\underline{x})))$.

ii) For all $k = 1, \ldots, h$, $\mathbf{F}_k$ is an algebraic extension of degree e_k of $\mathbf{F}_{k-1}$.

iii) For all $k = 1, \ldots, h$, $\mathbf{F}_k$ is an algebraic extension of degree $e_k.e_{k-1}.\ldots.e_1$ of $\mathbf{F}_0$.

iv) $n = \deg_y(F) = e_1.\ldots.e_h = \dfrac{D_1}{D_{h+1}} = \dfrac{n^e}{D_{h+1}}$. In particular $D_{h+1} = n^{e-1}$.

Proof. . ii), iii), and iv) are obvious. For a proof of i) see [9], Paragraph 5. □

Remark 1.5. (see [9]) Conversely, let $N \in \mathbb{N}^*$ and let $Y(\underline{t}) = \sum_p c_p\underline{t}^p \in \mathbf{K}[[\underline{t}]]$, and suppose that there exists a finite sequence of elements $m'_1, \ldots, m'_{h'}$ of $\mathrm{Supp}(Y(\underline{t}))$ such that the following holds:

i) $m'_1 < m'_2 < \ldots < m'_{h'}$.
ii) If $p \in \mathrm{Supp}(Y(\underline{t}))$, then $p \in (N\mathbb{Z})^e + \sum_{p\in m'_i+\mathbb{N}^e} m'_i\mathbb{Z}$.
iii) $m_i \notin (N\mathbb{Z})^e + \sum_{j<i} m'_j\mathbb{Z}$ for all $i = 1, \ldots, h'$.

Let $\bar{F}(\underline{x},y)$ be the minimal polynomial of $Y(\underline{x}^{\frac{1}{N}})=\sum_p c_p\underline{x}^{\frac{p}{N}}$. If $\deg_y(\bar{F})=N$, then $\mathbf{F}_0(Y(\underline{x}^{\frac{1}{N}}))=\mathbf{F}_0(x_1^{\frac{m'^1_1}{N}}\ldots\ldots x_e^{\frac{m'^e_1}{N}},\ldots,x_1^{\frac{m'^1_{h'}}{N}}\ldots\ldots x_e^{\frac{m'^e_{h'}}{N}})$. In particular, for all $Z(\underline{t})\in\mathrm{Root}(\bar{F})$, $\mathrm{In}(Y(\underline{t})-Z(\underline{t}))=\tilde{a'}.\underline{t}^{\tilde{m_k}}$, where $\tilde{a'}\in\mathbf{K}^*$ and $1\leq k\leq h'$. This implies that $D_y(\bar{F})=a.\underline{x}^{\alpha}(1+u(\underline{x}))$, where $a\in\mathbf{K}^*$ and $u(\underline{0})=0$, i.e. $\bar{F}$ is a quasi-ordinary polynomial.

The result of Proposition 1.4. has also the following interpretation: let $M_0=(n\mathbf{Z})^e$ and let $M_i=(n\mathbf{Z})^e+\sum_{j=1}^i m_j\mathbf{Z}$ for all $1\leq i\leq h$. Then $M_0\subseteq M_1\subseteq\cdots\subseteq M_h\subseteq\mathbb{Z}^e$. In particular, since M_0 and $\mathbb{Z}^e$ are free abelian groups of rank e, then for all $1\leq i\leq h$, M_i is a free abelian group of rank e. Furthermore, e_i is the index of the lattice M_{i-1} in M_i.
Let $1\leq i\leq h$ and let $v_1,\cdots,v_e$ be a basis of M_i, and recall that D_{i+1} is the determinant of the matrix $(v_1^T,\cdots,v_e^T)$. We have the following:

Proposition 1.6. Let v be a nonzero element of $\mathbb{Z}^e$ and let $\tilde{D}$ be the gcd of the (e,e) minors of the matrix $(v_1^T,\ldots,v_e^T,v^T)$. Then $\tilde{D}$ is also the gcd of the (e,e) minors of the matrix $(nI_e,m_1^T,\cdots,m_i^T,v^T)$. With these notations, we have the following:

i) $v\in M_i$ if and only if $\tilde{D}=D_{i+1}$.

ii) $\dfrac{D_{i+1}}{\tilde{D}}.v\in M_i$ and if $D_{i+1}>\tilde{D}$ then for all $1\leq k<\dfrac{D_{i+1}}{\tilde{D}}$, $k.v\notin M_i$.

In particular, since $m_{i+1}\notin M_i$, then $D_{i+2}>D_{i+1}$, $e_{i+1}m_{i+1}\in M_i$, and $km_{i+1}\notin M_i$ for all $1\leq k<e_{i+1}$.

Proof. . i) For all $k=1,\ldots,e$, let $\tilde{D}_k$ be the determinant of the matrix $(v_1^T,\ldots,v_{k-1}^T,v^T,v_{k+1}^T,\ldots,v_e^T)$. If $\tilde{D}=D_{i+1}$ then D_{i+1} divides $\tilde{D}_k$. In particular the Cramer system $\lambda_1v_1+\ldots+\lambda_ev_e=v$ has the unique solution $\lambda_k=\dfrac{\tilde{D}_k}{D_{i+1}}\in\mathbb{Z}$. Conversely, if $v\in M_i$, then there exist unique integers $\lambda_1,\ldots,\lambda_e$ such that $v=\lambda_1v_1+\ldots+\lambda_ev_e$, but $(\lambda_1,\ldots,\lambda_e)$ is the unique solution to the (e,e) system $a_1v_1+\ldots+a_ev_e=v$, in particular $\lambda_k=\dfrac{\tilde{D}_k}{D_{i+1}}$ for all $k=1,\ldots,e$. This proves that $\tilde{D}=D_{i+1}$.

ii) Let the notations be as in i) and let $1\leq k\leq\dfrac{D_{i+1}}{\tilde{D}}$. Let $\bar{D}$ be the gcd of the (e,e) minors of the matrix $[v_1^T,\cdots,v_e^T,(k.v)^T]$. Clearly $\bar{D}=\gcd(k\tilde{D}_1,\cdots,k\tilde{D}_e,D_{i+1})$. If $k=\dfrac{D_{i+1}}{\tilde{D}}$, then $\bar{D}=\gcd(D_{i+1}\dfrac{\tilde{D}_1}{\tilde{D}},\cdots,D_{i+1}\dfrac{\tilde{D}_e}{\tilde{D}},D_{i+1})=D_{i+1}$, which implies by i) that $k.v\in M_i$. Suppose that $D_{i+1}>\tilde{D}$ and that $1\leq k<\dfrac{D_{i+1}}{\tilde{D}}$. If $k.v\in M_i$, then $\bar{D}=D_{i+1}$, which implies that D_{i+1} divides $\gcd(k\tilde{D}_1,\cdots,k\tilde{D}_e,kD_{i+1})=k.\tilde{D}$. This is a contradiction because $k.\tilde{D}<D_{i+1}$.

□

The following result will be used later in the paper:

Corollary 1.7. Let the notations be as in Remark 1.5., i.e. $N\in\mathbb{N}^*$, $Y(\underline{t})=\sum_p c_p\underline{t}^p\in\mathbf{K}[[\underline{t}]]$, and there exists a finite sequence of elements $m'_1,\ldots,m'_{h'}$ of $\mathrm{Supp}(Y(\underline{t}))$ such that the following holds:

i) $m'_1<m'_2<\ldots<m'_{h'}$.

ii) If $p\in\mathrm{Supp}(Y(\underline{t}))$ then $p\in(N\mathbb{Z})^e+\sum_{p\in m'_i+\mathbb{N}^e}m'_i\mathbb{Z}$.

iii) $m'_i\notin(N\mathbb{Z})^e+\sum_{j<i}m'_j\mathbb{Z}$ for all $i=1,\ldots,h'$.

Let $F(\underline{x},y)$ be the minimal polynomial of $Y(\underline{x}^{\frac{1}{N}})=\sum_p c_p\underline{x}^{\frac{p}{N}}$ over $\mathbf{K}((\underline{x}))$ and suppose that $\deg_y F=N$. Let $m\in\mathbb{N}^e, m'_{h'}<_{glex}m$, and let $\bar{Y}(\underline{t})=Y(\underline{t})+c_m\underline{t}^m, c_m\in\mathbf{K}^*$. Let finally $\bar{F}(\underline{x},y)$ be the minimal polynomial of $\bar{Y}(\underline{x}^{\frac{1}{N}})$ over $\mathbf{K}((\underline{x}))$. We have the following:

1) $\deg_y(\bar{F})\geq N$ and $\deg_y(\bar{F})=N$ if and only if $m\in M_{h'}=(N\mathbb{Z})^e+\sum_{i=1}^{h'}m'_i\mathbb{Z}$.

2) If $m\in m'_{h'}+\mathbb{N}^e$, then $\bar{F}$ is quasi-ordinary.

Proof. . 1) Let $(D_1=N^e,\ldots,D_{h'+1}=N^{e-1})$ be the $\underline{D}$-sequence associated with F. We have $\deg_y\bar{F}\geq N.[\mathbf{F}_0(\bar{Y}(\underline{x}^{\frac{1}{N}})),\mathbf{F}_{h'}]\geq N$, and $m\in M_{h'}$ if and only if $\mathbf{F}_{h'}=\mathbf{F}_0(\bar{Y}(\underline{x}^{\frac{1}{N}}))$, and this holds if and only if $\deg_y(\bar{F})=N$.

2) If $m\in M_{h'}$ (resp. $m\notin M_{h'}$), then $\bar{Y}(\underline{x})$ and $m'_1,\ldots,m'_{h'}$ (resp. $\bar{Y}(\underline{x})$ and $m'_1,\ldots,m'_{h'},m'_{h'+1}=m$) satisfy the conditions of Remark 1.5., and $\bar{F}$ is quasi-ordinary. □

Let $d_i=\dfrac{D_i}{D_{h+1}}$ for all $1\leq i\leq h+1$. In particular $d_1=n$ and $d_{h+1}=1$. The sequence $(d_1,d_2,\ldots,d_{h+1})$ is called the gcd-sequence of F or the $\underline{d}$-sequence associated with F. Let $(r_0^1,\cdots,r_0^e)=(m_0^1,\cdots,m_0^e)$ be the canonical basis of $(n\mathbb{Z})^e$ and define the sequence $(r_k)_{1\leq k\leq h}$ by $r_1=m_1$ and:

$$r_{k+1}=e_kr_k+m_{k+1}-m_k$$

for all $1\leq k\leq h-1$. We call $(r_0^1,\cdots,r_0^e,r_1,\cdots,r_h)$ the $\underline{r}$-sequence associated with F. Note that each of the sequences $(m_k)_{1\leq k\leq h}$ and $(r_k)_{1\leq k\leq h}$ determines the other. More precisely $m_1=r_1$ and $r_kd_k=m_1d_1+\sum_{j=2}^k(m_j-m_{j-1})d_j$ (resp. $m_k=r_k-\sum_{j=1}^{k-1}(e_j-1)r_j$) for all $2\leq k\leq h$. In particular $M_k=(n\mathbb{Z})^e+\sum_{j=1}^k m_j\mathbb{Z}=(n\mathbb{Z})^e+\sum_{j=1}^k r_j\mathbb{Z}$ for all $k=1,\ldots,h$. It also follows that $\mathrm{GCDM}(r_0^1,\cdots,r_0^e,r_1,\cdots,r_h)=\mathrm{GCDM}(m_0^1,\cdots,m_0^e,m_1,\cdots,m_h)$, in particular, the results of Proposition 1.6. hold if we replace $(m_1,\cdots,m_h)$ by $(r_1,\cdots,r_h)$.

Corollary 1.8. (see also [6], Lemma 3.3.) Let $(r_0^1,\cdots,r_0^e,r_1,\cdots,r_h)$ be the $\underline{r}$-sequence associated with F. For all $1\leq k\leq h-1$, we have:

i) $r_kd_k<r_{k+1}d_{k+1}$.

ii) $e_kr_k\in M_{k-1}$.

iii) For all $1\leq i<e_k, ir_k\notin M_{k-1}$.

Proof. . This results from Proposition 1.6. and the equalities above. □

Let $\phi(\underline{t})=(t_1^p,\ldots,t_e^p,Y(\underline{t}))$ and $\psi(\underline{t})=(t_1^q,\ldots,t_e^q,Z(\underline{t}))$ be two nonzero elements of $\mathbf{K}[[\underline{t}]]^{e+1}$. We define the contact between ϕ and ψ, denoted $\mathrm{c}_{glex}(\phi,\psi)$, to be the element $\dfrac{1}{pq}\exp_{glex}(Y(t_1^q,\ldots,t_e^q)-Z(t_1^p,\ldots,t_e^p))$.

We define the contact between F and ϕ, denoted $\mathrm{c}_{glex}(F,\phi)$, to be the maximal element of

$$\{\mathrm{c}_{glex}(\phi,(t_1^n,\ldots,t_e^n,y(\underline{t})))|y(\underline{t})\in\mathrm{Root}(f)\}.$$

Let $g=y^m+b_1(\underline{x})y^{m-1}+\ldots+b_m(\underline{x})$ be a nonzero polynomial of $\mathbf{R}[y]$. Suppose that g is an irreducible quasi-ordinary polynomial and let $\psi(\underline{t})=(t_1^m,\ldots,t_e^m,Z(\underline{t}))$ be a root of $g(t_1^m,\ldots,t_e^m,y)=0$. We define the contact between F and g, denoted $\mathrm{c}_{glex}(F,g)$, to be the contact between F and ψ. Note that this definition does not depend on the choice of the root $Z(\underline{t})$ of g, and that if $F.g$ is a quasi-ordinary polynomial, then $\mathrm{In}(F(\psi(\underline{t}))=M_{glex}(F(\psi(\underline{t})))$. In this case, the contact $c_{glex}(F,g)$ coincides with the notion of contact introduced in [4] and [12]. The following Proposition generalizes a well known result for plane curves. It calculates the order $O_{glex}(F,g)$ in terms of the contact $\mathrm{c}_{glex}(F,g)$ and the characteristic sequences of F. When $F.g$ is quasi-ordinary, this result has been proved in [12], Proposition 2.14 and Proposition 5.9.

Proposition 1.9. Let $g = y^m + b_1(\underline{x})y^{m-1} + \ldots + b_m(\underline{x})$ be an irreducible quasi-ordinary polynomial of $\mathbf{R}[y]$ and suppose that $m \leq n$. If $c = c_{glex}(F, g)$ then we have the following:

i) If $nc <_{glex} m_1$, then $O_{glex}(F, g) = nmc$.

ii) Otherwise, let $1 \leq q \leq h-1$ be the smallest integer such that $m_q \leq_{glex} nc <_{glex} m_{q+1}$, then $O_{glex}(F, g) = (r_q d_q + (nc - m_q)d_{q+1}) \cdot \dfrac{m}{n}$. In particular $O_{glex}(F, g) <_{glex} r_{q+1}h_{q+1} \cdot \dfrac{m}{n}$.

Proof. . The proof is technical. It uses the same arguments as in the case of plane curves (see also [12], Proposition 5.9.). We shall consequently omit the details.

□

2. G-adic expansions

Let $\mathbf{S}$ be a commutative ring with unity and let $\mathbf{S}[y]$ be the ring of polynomials in y with coefficients in $\mathbf{S}$. Let $f = y^n + a_1 y^{n-1} + \ldots + a_n$ be a monic polynomial of $\mathbf{S}[y]$ of degree $n > 0$ in y. Let $d \in \mathbf{N}$ and suppose that d divides n. Let g be a monic polynomial of $\mathbf{S}[y]$ of degree $\dfrac{n}{d}$ in y. There exist unique polynomials $a_1(y), \ldots, a_d(y) \in \mathbf{S}[y]$ such that:

$$f = g^d + \sum_{i=1}^{d} a_i(y).g^{d-i}$$

and for all $1 \leq i \leq d$, $\deg_y(a_i(y)) < \dfrac{n}{d} = \deg_y g$ (where $\deg_y$ denotes the y-degree). The equation above is called the g-adic expansion of f. Assume that d is a unit in $\mathbf{S}$. The Tschirnhausen transform of f with respect to g, denoted $\tau_f(g)$, is defined to be $\tau_f(g) = g + d^{-1}a_1$. Note that $\tau_f(g) = g$ if and only if $a_1 = 0$. By [1], $\tau_f(g) = g$ if and only if $\deg_y(f - g^d) < n - \dfrac{n}{d}$. If one of these equivalent conditions is satisfied, then the polynomial g is called a d-th approximate root of f. By [1], there exists a unique d-th approximate root of f. We denote it by $\mathrm{App}_d(f)$.

Let $n = d_1 > d_2 > \cdots > d_h$ be a sequence of integers such that d_{i+1} divides d_i for all $1 \leq i \leq h-1$, and set $e_i = \dfrac{d_i}{d_{i+1}}$, $1 \leq i \leq h-1$ and $e_h = +\infty$. For all $1 \leq i \leq h$, let g_i be a monic polynomial of $\mathbf{S}[y]$ of degree $\dfrac{n}{d_i}$ in y. Set $\underline{G} = (g_1, \ldots, g_h)$ and let $B = \{\underline{\theta} = (\theta_1, \ldots, \theta_h) \in \mathbb{N}^h, 0 \leq \theta_i < e_i$ for all $1 \leq i \leq h\}$. Then f can be uniquely written as $f = \sum_{\underline{\theta} \in B} a_{\underline{\theta}}.\underline{g}^{\underline{\theta}}$ where $\underline{g}^{\underline{\theta}} = g_1^{\theta_1}.\ldots.g_h^{\theta_h}$ and $a_{\underline{\theta}} \in \mathbf{S}$ for all $\underline{\theta} \in B$. We call this expansion the $\underline{G}$-adic expansion of f.

3. Generators of the semigroup of F

Let the notations be as in Sections 1. and 2., in particular $F = y^n + a_2(\underline{x})y^{n-2} + \ldots + a_n(\underline{x})$ is an irreducible quasi-ordinary polynomial of $\mathbf{R}[y] = \mathbf{K}[[\underline{x}]][y]$. We have the following:

Theorem 3.1. (see [6], [8]) Let the notations be as above, and let $d_1 = n, \ldots, d_h, d_{h+1} = 1$ be the gcd-sequence of F. The d_k-th approximate root $\mathrm{App}_{d_k}(F)$ is an irreducible quasi-ordinary polynomial for all $k = 1, \ldots, h$. Furthermore, $c_{glex}(F, \mathrm{App}_{d_k}(F)) = \dfrac{m_k}{n}$ and $O_{glex}(F, \mathrm{App}_{d_k}(F)) = r_k$.

Let $\underline{G} = (g_1, \ldots, g_h, g_{h+1})$ be the d_k-th approximate roots of F, $1 \leq k \leq h+1$, and recall that $g_1 = y$, $g_{h+1} = F$. Let $B(\underline{G}) = \{\underline{\theta} = (\theta_1, \ldots, \theta_h, \theta_{h+1}) \in \mathbf{N}^{h+1} | \theta_{h+1} < +\infty$ and $0 \leq \theta_k < e_k$ for all $1 \leq k \leq h\}$.

Lemma 3.2. (see [8], (2.3)) Given two elements $\underline{\theta}^1, \underline{\theta}^2 \in B(\underline{G})$ and two elements $\underline{\gamma}^1, \underline{\gamma}^2 \in \mathbf{N}^e$, if $\theta^1_{h+1} = \theta^2_{h+1}$ and $\underline{\theta}^1 \neq \underline{\theta}^2$, then $\sum_{i=1}^e \gamma_i^1 r_0^i + \sum_{k=1}^h \theta^1_k r_k \neq \sum_{i=1}^e \gamma_i^2 r_0^i + \sum_{k=1}^h \theta^2_k r_k$.

Let $\bar{F}(\underline{x}, y)$ be a monic polynomial of $\mathbf{R}[y]$ and let

$$\bar{F} = \sum_{\underline{\theta} \in B(\underline{G})} c_{\underline{\theta}}(\underline{x}) g_1^{\theta_1} \cdot \ldots \cdot g_h^{\theta_h} . g_{h+1}^{\theta_{h+1}}$$

be the $\underline{G}$-adic expansion of $\bar{F}$. Let $\mathrm{Supp}_{\underline{G}}(\bar{F}) = \{\underline{\theta} \in B(\underline{G}) | c_\theta \neq 0\}$ and let $B'(\underline{G}) = \{\underline{\theta} \in \mathrm{Supp}_{\underline{G}}(\bar{F}) | \theta_{h+1} = 0\}$. Clearly F divides $\bar{F}$ if and only if $B'(\underline{G}) = \emptyset$. Otherwise, by Lemma 3.2., there is a unique $\underline{\theta}_0 \in \mathrm{Supp}_{\underline{G}}(\bar{F})$ such that $O_{glex}(F, \bar{F}) = O_{glex}(F, M(c_{\underline{\theta}_0}(\underline{x})) g_1^{\theta_0^1} \cdot \ldots \cdot g_h^{\theta_0^h}) = O_{glex}(F, M(c_{\underline{\theta}_0}(\underline{x}))) + \sum_{i=1}^h \theta_0^i r_i$. We set $M_{\underline{G}}(\bar{F}) = M_{glex}(c_{\underline{\theta}_0}(\underline{x})) g_1^{\theta_0^1} \cdot \ldots \cdot g_h^{\theta_0^h}$ and we call it the $\underline{G}$-initial monomial of $\bar{F}$. This leds to the following proposition:

Proposition 3.3. (see also [6], [8])With the notations above, $r_0^1, \ldots, r_0^e, r_1, \ldots, r_h$ generate $\Gamma(F)$.

Lemma 3.4. (see also [6], Prop. 2.3. or [11], Lemmas 7.4. and 7.5.) Let $\bar{F}$ be a non zero polynomial of $\mathbf{R}[y]$. If $\deg_y(\bar{F}) < \dfrac{n}{d_k}$ for some $1 \leq k \leq h$, then $O_{glex}(F, \bar{F}) \in < r_0^1, \ldots, r_0^e, r_1, \ldots, r_{k-1} >$. More precisely, there are unique $\theta_0^1, \cdots, \theta_0^e, \theta_1, \cdots, \theta_{k-1} \in \mathbb{N}$ such that $O_{glex}(F, \bar{F}) = \sum_{i=1}^e \theta_0^i r_0^i + \sum_{j=1}^{k-1} \theta_j r_j$ where $0 \leq \theta_j < e_j$ for all $1 \leq j \leq k-1$.

Proof. . Let the notations be as above, and let

$$\bar{F} = \sum_{\underline{\theta} \in B(\underline{G})} c_\theta(\underline{x}) g_1^{\theta_1} \cdot \ldots \cdot g_h^{\theta_h} . g_{h+1}^{\theta_{h+1}}$$

be the $\underline{G}$-adic expansion of $\bar{F}$. Since $\deg_y(\bar{F}) < \dfrac{n}{d_k}$, then for all $\underline{\theta} \in \mathrm{Supp}_{\underline{G}}(\bar{F})$, $\theta_k = \cdots = \theta_h = 0$. This implies the result. □

4. Generalized Newton sets

Let $n \in \mathbb{N}, n > 1$ and let $\underline{r}_0 = (r_0^1, \ldots, r_0^e)$ be the canonical basis of $(n\mathbb{Z})^e$. Let $r_1 < \ldots < r_h$ be a sequence of elements of $\mathbb{N}^e$. Set $D_1 = n^e$ and for all $1 \leq k \leq h$, let D_{k+1} be the GCD of the $e \times e$ minors of the $e \times (e+k)$ matrix $(nI_e, (r_1)^T, \ldots, (r_k)^T)$. Suppose that n^{e-1} divides D_k for all $1 \leq k \leq h+1$ and that $D_{h+1} = n^{e-1}$, and also that $D_1 > D_2 > \ldots > D_{h+1}$, in such a way that if we set $d_1 = n$ and $d_k = \dfrac{D_k}{n^{e-1}}$ for all $2 \leq k \leq h+1$, then $d_1 = n > d_2 > \ldots > d_{h+1} = 1$.

For all $1 \leq k \leq h+1$, let g_k be a monic polynomial of degree $\dfrac{n}{d_k}$ in y and set $\underline{G} = (g_1, \ldots, g_h, g_{h+1})$, $\underline{r} = (r_1, \ldots, r_h)$. Let H be a nonzero polynomial of $\mathbf{R}[y]$, and let

$$H = \sum_{\underline{\theta} \in B(\underline{G})} c_{\underline{\theta}}(\underline{x}) g_1^{\theta_1} \cdot \ldots \cdot g_h^{\theta_h} g_{h+1}^{\theta_{h+1}}$$

where $B(\underline{G}) = \{\underline{\theta} = (\theta_1, \ldots, \theta_h, \theta_{h+1}) | \theta_{h+1} < +\infty \text{ and } 0 \leq \theta_i < \dfrac{d_i}{d_{i+1}} \forall 1 \leq i \leq h\}$, be the $\underline{G}$-adic expansion of H. Let $\mathrm{Supp}_{\underline{G}}(H) = \{\underline{\theta} \in B(\underline{G}) | c_{\underline{\theta}} \neq 0\}$ and let $B'(\underline{G}) = \{\underline{\theta} \in \mathrm{Supp}_{\underline{G}}(H) | \theta_{h+1} = 0\}$. Suppose that $B'(\underline{G}) \neq \emptyset$. Given $\underline{\theta} \in B'(\underline{G})$, if $\underline{\gamma}_\theta = \exp_{glex}(c_{\underline{\theta}}(\underline{x}))$, we shall associate with the monomial $c_{\underline{\theta}}(\underline{x}) g_1^{\theta_1} \cdot \ldots \cdot g_h^{\theta_h}$ the e-tuple

$$< (\underline{\gamma}_{\underline{\theta}}, \underline{\theta}), (\underline{r}_0, \underline{r}) >= \sum_{i=1}^{e} \gamma_{\theta_i} r_0^i + \sum_{j=1}^{h} \theta_j r_j.$$

We set $N_{\underline{G}}(H) = \{< (\underline{\gamma}_{\underline{\theta}}, \underline{\theta}), (\underline{r}_0, \underline{r}) >, \underline{\theta} \in B'(\underline{G})\}$, and we call it the $\underline{G}$-Newton set of H. By Lemma 3.2., there is a unique $\underline{\theta}^0 \in B'(\underline{G})$ such that if $\underline{\gamma}_{\underline{\theta}^0} = \exp_{glex}(c_{\underline{\theta}^0}(\underline{x}))$, then:

$$< (\underline{\gamma}_{\underline{\theta}^0}, \underline{\theta}^0), (\underline{r}_0, \underline{r}) >= \min_{glex}(N_{\underline{G}}(H))$$

where $\min_{glex}$ means the minimal element with respect to the well-ordering $glex$. We set $\mathrm{fO}(\underline{r}, \underline{G}, H) = < (\underline{\gamma}_{\underline{\theta}^0}, \underline{\theta}^0), (\underline{r}_0, \underline{r}) >$ and we call it the formal order of H with respect to $(\underline{r}, \underline{G})$. We also set $M_{\underline{G}}(H) = M_{glex}(c_{\underline{\theta}^0}(\underline{x})).g_1^{\theta_1^0}.\dots.g_h^{\theta_h^0}$ and we call it the initial monomial of H with respect to $(\underline{r}, \underline{G})$. If $B'(\underline{G}) = \emptyset$, then we set $\mathrm{fO}(\underline{r}, \underline{G}, H) = (+\infty, \dots, +\infty)$. Note that this holds if and only if g_{h+1} divides H.

Let $f = y^n + a_1(\underline{x})y^{n-1} + \dots + a_n(\underline{x})$ be a quasi-ordinary polynomial of $\mathbf{R}[y]$ and let $d \in \mathbb{N}, d > 1$ be a divisor of n. Let g be a monic polynomial of $\mathbf{R}[y]$ of degree $\frac{n}{d}$ in y and let $f = g^d + \alpha_1(\underline{x}, y)g^{d-1} + \dots + \alpha_d(\underline{x}, y)$ be the g-adic expansion of f. We associate with f the set of points:

$$\{(\mathrm{fO}(\underline{r}, \underline{G}, \alpha_k), (d-k)\mathrm{fO}(\underline{r}, \underline{G}, g)), k = 0, \dots, d\} \subseteq \mathbb{N}^e \times \mathbb{N}^e$$

We denote this set by $\mathrm{GNS}(f, \underline{r}, \underline{G}, g)$ and we call it the generalized Newton set of f with respect to $(\underline{r}, \underline{G}, g)$ (note that, since $\alpha_0 = 1$, then $\mathrm{fO}(\underline{r}, \underline{G}, \alpha_0) = \underline{0} \in \mathbb{N}^e$).

Definition 4.1. We say that f is straight with respect to $(\underline{r}, \underline{G}, g)$ if the following holds:

i) $\mathrm{fO}(\underline{r}, \underline{G}, \alpha_d) = d.\mathrm{fO}(\underline{r}, \underline{G}, g)$ and $\mathrm{fO}(\underline{r}, \underline{G}, \alpha_d) << (\underline{\gamma}_{\underline{\theta}}, \underline{\theta}), (\underline{r}_0, \underline{r}) >$ for all $\underline{\theta} \in N_{\underline{G}}(\alpha_d - M_{\underline{G}}(\alpha_d))$.

ii) For all $1 \leq k \leq d-1$, and for all $\underline{\theta} \in N_{\underline{G}}(\alpha_k)$, $k.\mathrm{fO}(\underline{r}, \underline{G}, g) \leq< (\underline{\gamma}_{\underline{\theta}}, \underline{\theta}), (\underline{r}_0, \underline{r}) >$.

We say that f is strictly straight with respect to $(\underline{r}, \underline{G}, g)$ if the inequality in ii) is a strict inequality.

Example 4.2. i) Let $f = (y^2 - x^3)^2 - x^5 y + x^{10} \in \mathbf{K}[[x]][y]$, and let $r_0 = 4, r_1 = 6, r_2 = 13$, $\underline{G} = (g_1 = y, g_2 = y^2 - x^3, g_3 = f)$, $\underline{r} = (r_1, r_2)$: $f = g_2^2 - x^5 g_1$ is the g_2-expansion of f. Furthermore, $\mathrm{fO}(\underline{r}, \underline{G}, g_2) = r_2 = 13$, $\mathrm{fO}(\underline{r}, \underline{G}, x^5 g_1 + x^{10}) = 5r_0 + r_1 = 26 < 10r_0 = 40$. In particular, $\mathrm{GNS}(f, \underline{r}, \underline{G}, g_2) = \{(0, 26), (26, 0)\}$, and f is strictly straight with respect to $(\underline{r}, \underline{G}, g_2)$. Note that f is irreducible, and that $\Gamma(f) =< 4, 6, 13 >$.

ii) Let f be as in i), and let $r_0 = 4, r_1 = 10, r_2 = 13$. If $\underline{G} = (g_1 = y, g_2 = y^2 - x^3, g_3 = f)$ and $\underline{r} = (10, 13)$, then $\mathrm{GNS}(f, \underline{r}, \underline{G}, g_2) = \{(0, 26), (30 = 5r_0 + r_1, 0)\}$, in particular, f is not straight with respect to $(\underline{r}, \underline{G}, g_2)$.

5. **The criterion**

Let $f = y^n + a_1(x)y^{n-1} + \dots + a_n(x)$ be a nonzero quasi-ordinary polynomial of $\mathbf{R}[y]$ and assume, after possibly a change of variables, that $a_1(\underline{x}) = 0$. Let $\underline{r}_0 = (r_0^1, \dots, r_0^e)$ be the canonical basis of $(n\mathbb{Z})^e$ and let $D_1 = n^e$, $d_1 = n$. Let $g_1 = y$ be the d_1-th approximate root of f and set $r_1 = \exp_{glex}(a_n(\underline{x}))$. Let D_2 be the gcd of the (e, e) minors of the $e \times (e+1)$ matrix $(nI_e, {r_1}^T)$. We set $d_2 = \frac{D_2}{n^{e-1}}$, $g_2 = \mathrm{App}_{d_2}(f)$, and $e_2 = \frac{d_1}{d_2} = \frac{n}{d_2}$. Similarly we shall construct $r_k, g_k, d_{k+1}, e_k,\ k \geq 2$ as follows: given $(r_1, \ldots, r_{k-1})$ and $(d_1, \ldots, d_k)$, let g_k be the d_k-th approximate root of f, and let

$$f = g_k^{d_k} + \beta_2^k g_k^{d_k-2} + \ldots + \beta_{d_k}^k$$

be the g_k-adic expansion of f. We set $r_k = \mathrm{fO}(\underline{r}^k, \underline{G}^k, \beta_{d_k}^k)$, where $(\frac{r_0^1}{d_k}, \ldots, \frac{r_0^e}{d_k})$ denotes the canonical basis of $(\frac{n}{d_k}\mathbb{Z})^e$, $\underline{r}^k = (\frac{r_1}{d_k}, \ldots, \frac{r_{k-1}}{d_k})$ and $\underline{G}^k = (g_1, \ldots, g_{k-1})$. We also set $D_{k+1} =$ the gcd of the (e,e) minors of the matrix $[nI_e, r_1^T, \ldots, r_k^T]$, $d_{k+1} = \frac{D_{k+1}}{n^{e-1}}$, and $e_k = \frac{d_k}{d_{k+1}}$. With these notations we have the following:

Theorem 5.1. The quasi-ordinary polynomial f is irreducible if and only if the following holds:

i) There is an integer h such that $d_{h+1} = 1$.
ii) $g_1, \cdots, g_h$ are irreducible quasi-ordinary polynomials.
iii) For all $1 \leq k \leq h-1, r_k d_k < r_{k+1} d_{k+1}$.
iv) For all $2 \leq k \leq h+1$, g_k is strictly straight with respect to $(\underline{r}^k, \underline{G}^k, g_{k-1})$.

We shall first prove the following results:

Lemma 5.2. Let $c \in \mathbf{K}^*$. The quasi-ordinary polynomial $F = y^N - cx_1^{\alpha_1} \ldots . x_e^{\alpha_e}$ is irreducible in $\mathbf{R}[y]$ if and only if $\gcd(N, \alpha_1, \ldots, \alpha_e) = 1$, or equivalently if and only if the gcd of the (e,e) minors of the matrix $(NI_e, (\alpha_1, \ldots, \alpha_e)^T)$ is N^{e-1}.

Proof. . Let $\tilde{c}$ be an N-th root of c in $\mathbf{K}$ and let $d = \gcd(n, \alpha_1, \ldots, \alpha_e)$. If $d > 1$, then $F = \prod_{w^d=1}(y^{\frac{N}{d}} - w\tilde{c}x_1^{\frac{\alpha_1}{d}} \ldots . x_e^{\frac{\alpha_e}{d}})$, which is a contradiction. Conversely, let $Y = \tilde{c}x_1^{\frac{\alpha_1}{N}} \ldots . x_e^{\frac{\alpha_e}{N}} \in \mathbf{K}((x_1^{\frac{1}{N}}, \ldots, x_e^{\frac{1}{N}}))$. Then F is the minimal polynomial of Y over $\mathbf{K}((\underline{x}))$. In particular it is irreducible. □

Proposition 5.3. Let $F = y^N + b_2(\underline{x})y^{N-2} + \ldots + b_N(\underline{x})$ be an irreducible quasi-ordinary polynomial of degree N in y, and let $(m'_k)_{1 \leq k \leq h'}$ be the set of characteristic exponents of F. Let also $(d'_k)_{1 \leq k \leq h'+1}$ (resp. $(r'_k)_{1 \leq k \leq h'}$) be the $\underline{d}$-sequence (resp. the $\underline{r}$-sequence) of F. Let F' be a quasi-ordinary polynomial of $\mathbf{R}[y]$ and assume that F' is monic of degree N in y. If $r'_{h'} d'_{h'} <_{glex} \mathrm{O}_{glex}(F, F')$, then F' is irreducible in $\mathbf{R}[y]$.

Proof. . Assume that F' is not irreducible and let $\tilde{F'}$ be an irreducible component of F' in $\mathbf{R}[y]$. Let $C = c_{glex}(F, \tilde{F'})$ be the contact of F with $\tilde{F'}$. If $m'_{h'} <_{glex} NC$, then $\deg_y(\tilde{F'}) \geq N = \deg_y(F')$ (see Corollary 1.7.), which is a contradiction. Finally $NC \leq_{glex} m'_{h'}$, in particular, by Proposition 1.9., $O_{glex}(F, \tilde{F'}) \leq_{glex} r'_{h'} d'_{h'} \frac{\deg_y(\tilde{F'})}{N}$. Since this is true for all irreducible components of F', then $O_{glex}(F,' F) \leq_{glex} r'_{h'} d'_{h'} \frac{\deg_y(F')}{N} = r'_{h'} d'_{h'}$, which contradicts the hypothesis. □

Proof of Theorem 5.1. Suppose first that f is irreducible. Condition i) follows from the results of Section 1, condition ii) follows from Theorem 3.1., and condition iii) is nothing but Corollary 1.8.,i). Now for all $1 \leq k \leq h+1$, g_k is an irreducible quasi-ordinary polynomial and $g_1, \ldots, g_{k-1}$ are the approximate roots of g_k. In particular, to prove iv), it suffices to prove that $f = g_{h+1}$ is strictly straight with respect to $(\underline{r}, \underline{G}, g_h)$. Let

$$f = g_h^{d_h} + \beta_2^h g_h^{d_h-2} + \ldots + \beta_{d_h}^h$$

be the g_h-adic expansion of f and let $\Gamma^{h-1}(f)$ be the semigroup generated by $r^0_1, \ldots, r^0_e, r_1, \ldots, r_{h-1}$. We have the following:

- For all $2 \leq i \leq h-1$, $O_{glex}(\beta^h_i, f) \in \Gamma^{h-1}(f)$ (by Lemma 3.4.).
- For all $0 < a < d_h = e_h, a.r_h \notin \Gamma^{h-1}(f)$ (by Corollary 1.8.).

It follows that for all $2 \leq i \leq h-1, O_{glex}(\beta^h_i, f) \neq i.r_h$ and for all $2 \leq i \neq j \leq d_h - 1$, $O_{glex}(\beta^h_i, f) + (d_h - i)r_h \neq O_{glex}(\beta^h_j, f) + (d_h - j)r_h$. Since $O_{glex}(g_h^{d_h}, f) = r_h d_h$, then $O_{glex}(\beta^h_{d_h}, f) = r_h d_h$ and $i.r_h < O_{glex}(\beta^h_i, f)$ for all $2 \leq i \leq d_h - 1$. The other assertions follow by a similar argument.

Conversely suppose that f satisfies the conditions i), ii), iii), and iv). We shall prove by induction on h that f is irreducible. Suppose that $h = 1$, then $f = y^n + a_2(\underline{x})y^{n-2} + \ldots + a_n(\underline{x})$, $\underline{G} = (y, f)$, and $\underline{r} = r_1 = \exp_{glex}(a_n(x))$. Now condition iv) implies that $i.\exp_{glex}(a_n(\underline{x})) < \exp_{glex}(a_i(\underline{x}))$ for all $2 \leq i \leq n-1$. Furthermore, $D_2 = n^{e-1}$ by condition i). In particular $F = y^n + M_{glex}(a_n(\underline{x}))$ is irreducible by Lemma 5.2. Since $r_1 d_1 < O_{glex}(F, f) = O_{glex}(f - F, f)$, then f is irreducible by Proposition 5.3.

Let $h > 1$ and assume that g_k is an irreducible quasi-ordinary polynomial for all $1 \leq k \leq h$. Let $m^1_0 = r^1_0, \cdots, m^e_0 = r^e_0, m_1 = r_1$ and for all $2 \leq i \leq h$, let:

$$m_i = r_i - \sum_{k=1}^{i-1}(e_k - 1)r_k$$

Let $f = g_h^{d_h} + \beta^h_2 g_h^{d_h - 2} + \ldots + \beta^h_{d_h}$ be the g_h-adic expansion of f and let $Y(\underline{t}) = \sum_p Y_p \underline{t}^p$ be a root of $g_h(t_1^{\frac{n}{d_h}}, \ldots, t_e^{\frac{n}{d_h}}, y) = 0$. Since the quasi-ordinary polynomial g_h is irreducible, then the $\underline{m}$-sequence associated with g_h is $(\frac{m^1_0}{d_h}, \ldots, \frac{m^e_0}{d_h}, \frac{m_1}{d_h}, \cdots, \frac{m_{h-1}}{d_h})$. In particular,

$$\mathrm{GCDM}(\frac{m^1_0}{d_h}, \ldots, \frac{m^e_0}{d_h}, \frac{m_1}{d_h}, \cdots, \frac{m_{h-1}}{d_h}) = ((\frac{n}{d_h})^e, \frac{d_2}{d_h}(\frac{n}{d_h})^{e-1}, \cdots, \frac{d_{h-1}}{d_h}(\frac{n}{d_h})^{e-1}, (\frac{n}{d_h})^{e-1}).$$

Note that, by Corollary 1.7., since $\deg_y g_h < n$, then $Y_{\frac{m_h}{d_h}} = 0$.

Let λ be an indeterminate and let

$$y(\underline{t}, \lambda) = \sum_p Y_p \underline{t}^{d_h \cdot p} + \lambda \underline{t}^{m_h} = Y(\underline{t}^{d_h}) + \lambda \underline{t}^{m_h}.$$

Let $F(\underline{x}, y, \lambda)$ be the minimal polynomial of $y(\underline{x}^{\frac{1}{n}}, \lambda)$ over $\mathbf{K}(\lambda)((\underline{x}))$. Conditions i) and iii) imply that the polynomial F is an irreducible quasi-ordinary polynomial of $\mathbf{K}(\lambda)[[\underline{x}]][y]$, of degree n in y. Furthermore, the $\underline{m}$-sequence (resp. the $\underline{r}$-sequence) associated with F is $(m^1_0, \ldots, m^e_0, m_1, \cdots, m_h)$ (resp. $(r^1_0, \ldots, r^e_0, r_1, \cdots, r_h)$), and

$$\mathrm{GCDM}(m^1_0, \ldots, m^e_0, m_1, \cdots, m_{h-1}, m_h) = (n^e, d_2 n^{e-1}, \cdots, d_{h-1} n^{e-1}, d_h n^{e-1}, n^{e-1}).$$

Now an easy calculation shows that $c_{glex}(F, g_h) = \dfrac{m_h}{n}$, hence $O_{glex}(F, g_h) = r_h$. Furthermore, if we denote by $Y_1(\underline{t}) = Y(\underline{t}), Y_2(\underline{t}), \cdots, Y_{\frac{n}{d_h}}(\underline{t})$ the set of roots of $g_h(t_1^{\frac{n}{d_h}}, \cdots, t_e^{\frac{n}{d_h}}, y) = 0$, then we have:

$$M_{glex}(y(\underline{t}, \lambda) - Y_k(t_1^{d_h}, \cdots, t_e^{d_h})) = \begin{cases} \lambda t^{m_h} & \text{if } k = 1 \\ a_k t^{d_h \exp_{glex}(Y_1 - Y_k)}, a_k \neq 0 & \text{if } k > 1. \end{cases}$$

In particular, $\exp_{glex}(g_h(t_1^n,\cdots,t_e^n,y(\underline{t},\lambda))=m_h+d_h\exp_{glex}(D_y(g_h))=m_h+\sum_{k=1}^{h-1}(e_k-1)r_k=r_h$, finally, if $a=a_2\cdots a_{\frac{n}{d_h}}$, then:

$$g_h(t_1^n,\cdots,t_e^n,y(\underline{t},\lambda))=a.\lambda t^{r_h}.u(\underline{t},\lambda)$$

where $u(\underline{t},\lambda)$ is a unit in $\mathbf{K}(\lambda)[[\underline{t}]]$. Let $M_{\underline{G}^h}(\beta^h_{d_h})=c.\underline{x}^{\underline{\theta}_0}.g_1^{\theta_1}.\dots.g_{h-1}^{\theta_{h-1}}$, where $\underline{G}^h=(g_1,\dots,g_h)$ and $c\in\mathbf{K}^*$. We have:

$$O_{glex}(M_{\underline{G}^h}(\beta^h_{d_h}),F)=\sum_{i=1}^{e}\theta_0^ir_0^i+\sum_{k=1}^{h-1}\theta_kr_k$$

which is r_hd_h by condition iv). By the same condition, the following hold:

- $\beta_{d_h}(t_1^n,\cdots,t_e^n,Y(\underline{t},\lambda))=\bar{c}\underline{t}^{r_hd_h}(1+\bar{u}(\underline{t},\lambda))$, where $\bar{u}(\underline{0},\lambda)=0$ and $\bar{c}\neq 0$.
- $r_hd_h<\exp_{glex}(\beta_ig_i^{d_h-i}(t_1^n,\cdots,t_e^n,Y(\underline{t},\lambda)))$.

In particular $f(t_1^n,\cdots,t_e^n,y(\underline{t},\lambda))=(\bar{c}+\lambda)t^{r_hd_h}.u_1(\underline{t},\lambda)$, where $u_1(\underline{t},\lambda)$ is a unit in $\mathbf{K}(\lambda)[[\underline{t}]]$. Finally $r_hd_h<\mathrm{O}_{glex}(F(\underline{x},y,-\bar{c}),f)$, which implies by Proposition 5.3. that f is irreducible.

6. Examples

Example 1: Let $f=y^8-2x_1x_2y^4+x_1^2x_2^2-x_1^3x_2^2\in\mathbf{K}[[x_1,x_2]][y]$. Then we have:

- $D_1=n^2=8^2=64, d_1=n=8$, $r_0^1=(8,0), r_0^2=(0,8)$, $g_1=$App$_{d_1}(f)=y$, and $r_1=O(f,g_1)=(2,2)$.
- D_2 is the gcd of the 2×2 minors of the matrix $(8\cdot I_e,(2,2)^T)$, then $D_2=16=8.2$, in particular $d_2=2$. Since $f=(y^4-x_1x_2)^2-x_1^3x_2^2$, then g_2=App$_{d_2}(f)=y^4-x_1x_2$. Let $\underline{r}^2=(\frac{r_0^1}{d_2},\frac{r_0^2}{d_2},\frac{r_1}{d_2})=((4,0),(0,4),(1,1))$ and $\underline{G}^2=(g_1)$, then $r_2=$fO$(\underline{r}^2,\underline{G}^2,x_1^3x_2^2)=3(4,0)+2(0,4)=(12,8)$.
- D_3 is the gcd of the 2×2 minors of the matrix $(8I_2,(2,2)^T,(12,8)^T)$, then $D_3=8$, in particular $d_3=1$.
- Now GNP$(g_2,\underline{r}^2,\underline{G}^2)=\{((0,0),4.(1,1)),((4,4),(0,0))\}$ and GNP$(f,\underline{r}^3=(r_0^1,r_0^2,r_1,r_2),\underline{G}^3=(g_1,g_2))=\{((0,0),2.(12,8)),((24,16),(0,0))\}$, then the strict straightness condition is verified. Since $g_1=y$ is irreducible, then so is g_2, but g_2 is quasi-ordinary and $r_1d_1<r_2d_2$, then f is irreducible. Note that $m_2=r_2-(\frac{d_1}{d_2}-1)r_1=(12,8)-3(2,2)=(6,2))$ is the second characteristic exponent of f.

Example 2: Let $f=y^8-2x_1x_2y^4+x_1^2x_2^2-x_1^4x_2^2-x_1^5x_2^3\in\mathbf{K}[[x_1,x_2]][y]$. Then we have:

- $D_1=n^2=8^2=64, d_1=n=8$, $r_0^1=(8,0), r_0^2=(0,8)$, $g_1=$App$_{d_1}(f)=y$, and $r_1=(2,2)$.
- D_2 is the gcd of the 2×2 minors of the matrix $(8I_2,(2,2)^T)$, then $D_2=16=8.2$, in particular $d_2=2$. Since $f=(y^4-x_1x_2)^2-x_1^4x_2^2-x_1^5x_2^3$, then g_2=App$_{d_2}(f)=y^4-x_1x_2$. Let $\underline{r}^2=(\frac{r_0^1}{d_2},\frac{r_0^2}{d_2},\frac{r_1}{d_2})=((4,0),(0,4),(1,1))$ and $\underline{G}^2=(g_1)$, then $r_2=$fO$(\underline{r}^2,\underline{G}^2,x_1^4x_2^2)=4(4,0)+2(0,4)=(16,8)$.
- D_3 is the gcd of the 2×2 minors of the matrix $(8\cdot I_2,(2,2)^T,(16,8)^T)$, then $D_3=16$, in particular $d_3=d_2=2$. In particular f is not irreducible. Note that in this example the strict straightness condition is verified for f and g_2.

Example 3: Let $f=y^8-2x_1x_2y^4+x_1^3x_2^2-x_1y^5\in\mathbf{K}[[x_1,x_2]][y]$. Then we have:

- $D_1=n^2=8^2=64, d_1=n=8$, $r_0^1=(8,0), r_0^2=(0,8)$, $g_1=$App$_{d_1}(f)=y$, and $r_1=(3,2)$.
- D_2 is the gcd of the 2×2 minors of the matrix $(8I_2,(3,2)^T)$, then $D_2=8$, in particular $d_2=1$.
- GNP$(f,\underline{r}^2=(r_0^1,r_0^2,r_1),\underline{G}^2=(g_1))=\{((0,0),8.(3,2)),((8,0),5.(3,2)),((8,0)+(0,8),4.(3,2)),(3.(8,0)+2.(0,8),(0,0))\}=\{((0,0),(24,16)),((8,0),(15,10)),((8,8),(12,8)),((24,16),(0,0))\}$. Here the strict straightness is not verified, then f is not irreducible.

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Journal *of* Singularities
Volume 4 (2012), 35-67

received 3 October 2011
in revised form 9 January 2012
DOI: 10.5427/jsing.2012.4c

Fronts of Whitney umbrella – a differential geometric approach via blowing up

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Abstract

We investigate the differential geometric ingredients for Whitney umbrella, which is known as the only stable singularity of surface to 3-dimensional Euclidean space. We obtain several criteria of the singularity types of fronts of Whitney umbrella in terms of differential geometric language we discuss.

Contents

1 Introduction

H. Whitney [28] has found Whitney umbrella (also known as the cross-cap) as singularities which are not avoidable by small perturbation. This is very important singularity type, since it is the only singularity of a map of surface to 3-dimensional Euclidean space which is stable under small deformations. This singularity is fundamental in the context of differential topology but it does not seems that Whitney umbrella is a subject of differential geometry,

Mathematics Subject Classification (2010). Primary 53A05; Secondary 58K20.
Keywords and phrases. Whitney umbrella, Cross-cap, Singularities, Distance squared function, Blowing up, Front.

at least before C. Gutierrez and J. Sotomayor 's paper [10]. Motivated by Darboux's classification ([5]), they aimed to determine the configuration of lines of curvature near Whitney umbrella, and complete it in [9]. J. W. Bruce and J. M. West [3] investigated functions on Whitney umbrella using singularity theory. In [8], we show that a unified treatment for differential geometric properties for regular and singular maps $g : (\boldsymbol{R}^2, \mathbf{0}) \to (\boldsymbol{R}^3, \mathbf{0})$ and show that Whitney umbrella is not a bad singularity ([8, proposition 4.2]) from the view point of investigating singularities of distance squared functions. In other words, Whitney's umbrella is as good as (or as bad as) Darbouxian umbilics (i.e., both are characterised by the condition that $\operatorname{rank} R(g, 0) = 4$ in the notation of [8]). Several authors continue to investigate the configuration of the solution curves of particular binary differential equations (i.e. lines of curvature, asymptotic and characteristic curves) in [20, 24]. In [19], the classification of parabolic lines of Whitney umbrella is used to investigate the projections of smooth surface in $\boldsymbol{R}^4$ to 3-spaces.

When we consider parallel surfaces of a regular surface, we are not able to avoid singularities. These singularities are often called front and this subject is investigated by M. Kokubu, W. Rossman, K. Saji, M. Umehara and K. Yamada [16]. They mean by a front a map $g : (\boldsymbol{R}^2, \mathbf{0}) \to \boldsymbol{R}^3$ such that there exists a well-defined normal $\mathbf{n} : (\boldsymbol{R}^2, \mathbf{0}) \to S^2 \subset \boldsymbol{R}^3$ so that $(g, \mathbf{n}) : (\boldsymbol{R}^2, \mathbf{0}) \to (\boldsymbol{R}^3, \mathbf{0}) \times S^2$ is an immersion. Cuspidal edges and swallowtails are typical singularity types of fronts. In [16], criteria for the these singularities are given. Furthermore, criteria for the cuspidal lips and the cuspidal beaks are given in [15], criterion for the cuspidal butterfly is given in [14], and criteria for the D_4-singularities are given in [23]. Whitney umbrella is not a front in their sense, since any unit normals defined at regular points near the singular point cannot extend continuously to the singular point.

Physically, the wave propagation is described by Huygens's principle: every point to which a luminous disturbance reached becomes a source of a spherical wave, and the sum of these secondary waves determines the form of the wave front at any subsequent time. We remark that this does not require the notion of unit normal vectors. Mathematically, a wave front is the envelope of the spherical waves, and this requires us to investigate the singularities of the members of the family of functions:

$$\Phi : (\boldsymbol{R}^2, \mathbf{0}) \times \boldsymbol{R}^3 \to \boldsymbol{R}, \qquad (u, v) \times (x, y, z) \mapsto -\frac{1}{2}(\|(x, y, z) - g(u, v)\|^2 - {t_0}^2) \tag{1.1}$$

where t_0 is a constant. The family Φ is an unfolding of the distance squared function $\varphi(u, v) = \Phi(u, v, x_0, y_0, z_0)$ where (x_0, y_0, z_0) is a point in $\boldsymbol{R}^3$, and the discriminant set $\mathcal{D}(\Phi)$ of Φ is a (wave) front of g at distance $|t_0|$ where

$$\mathcal{D}(\Phi) = \{(x, y, z) \in \boldsymbol{R}^3 \ ; \ \Phi = \Phi_u = \Phi_v = 0 \text{ for some } (u, v) \in (\boldsymbol{R}^2, \mathbf{0})\}.$$

For regular surfaces, we investigate the distance squared unfolding Φ and show several criteria for singularity types of parallel surfaces in terms of differential geometric language (principal curvatures, ridge points, sub-parabolic points, etc.) in [7]. In this paper, we investigate singularities of the distance squared unfolding for Whitney's umbrella, and show similar criteria for versality (Theorems 3.7). To investigate the singularities of the distance squared unfolding for Whitney's umbrella, we need several differential geometric languages of Whitney umbrella.

We also investigate the focal sets (caustics) of Whitney umbrella, since the bifurcation set $\mathcal{B}(\Phi)$ of Φ represents the focal set where

$$\mathcal{B}(\Phi) = \{(x,y,z) \in \boldsymbol{R}^3 \ ; \ \Phi_u = \Phi_v = \Phi_{uu}\Phi_{vv} - {\Phi_{uv}}^2 = 0 \text{ for some } (u,v) \in (\boldsymbol{R}^2, \boldsymbol{0})\}.$$

In Section 2, we introduce some differential geometric ingredients (principal curvatures, ridge, sub-parabolic points, etc.) for Whitney umbrella. For regular surfaces in Euclidean 3-space, several authors investigate ridge points and sub-parabolic points; see for example [2], [4], [18], [21], and [22]. The ridge points were first studied in details by I. Porteous [21] in terms of singularities of distance squared functions. The ridge line is the locus of points where one principal curvature has an extremal value along lines of the same principal curvature. The sub-parabolic points were studied in details by J. W. Bruce and T. C. Wilkinson [4] in terms of singularities of folding maps. The sub-parabolic line is the locus of points where one principal curvature has an extremal value along lines of the other principal curvature. Recently, in the case of the hyperbolic space, the analogous notion to the ridge point of hypersurfaces is introduced in [13], and the analogous notion to the sub-parabolic point of smooth surfaces is introduced in [12]. We develop the differential geometric ingredients over Whitney umbrella, which seem to be missing pieces of knowledge of the people who work on singularity theory and differential geometry. Since Whitney umbrella is a singularity of rank one, the tangent planes of nearby point degenerate to the tangent line at the singularity, and the normal lines are developed to the normal plane at the singular point. This means that we have a chance to have a bounded normal curvature in one direction at singular point.

In Section 2.1, we first show that for Whitney umbrella there is a well-defined unit normal via the double oriented blowing-up ([11, example (a) in p. 221]):

$$\tilde{\pi} : \boldsymbol{R} \times S^1 \to \boldsymbol{R}^2, \qquad (r,\theta) \mapsto (r\cos\theta, r\sin\theta). \tag{1.2}$$

Then we are able to talk about principal curvatures and principal directions via $\tilde{\pi}$, and we discuss their asymptotic behaviours in Section 2.3.

Let $\mathcal{M}$ denote the quotient space of $\boldsymbol{R} \times S^1$ with identification $(r,\theta) \sim (-r, \theta+\pi)$. Then we obtain a natural map

$$\pi : \mathcal{M} \to \boldsymbol{R}^2, \qquad [(r,\theta)] \mapsto (r\cos\theta, r\sin\theta), \tag{1.3}$$

which we usually call a blow up. We remark that $\mathcal{M}$ is topologically a Möbius strip. It is a natural problem to ask configurations of the parabolic line, ridge lines, sub-parabolic lines etc. on $\mathcal{M}$ near the exceptional set $X = \pi^{-1}(0,0)$. We show

(1) The parabolic line intersects with X in at most two points (Proposition 2.3),

(2) Along an arc which reaches the singularity of Whitney umbrella, one principal curvature κ_1 is bounded if the arc is not tangent to the double point locus, and the other principal curvature κ_2 tends to infinity (Lemma 2.2),

(3) The ridge line with respect to κ_1 intersects with X in at most four points (Lemma 2.6, see Lemma 2.7 also) in generic context,

(4) The ridge line with respect to κ_2 intersects with X at two points (Proposition 2.10),

(5) The sub-parabolic line with respect to κ_1 intersects with X in at most three points (Lemma 2.11), and

(6) A constant principal curvature (CPC) line intersects with X in at most four points (Proposition 2.16).

In Section 3, we investigate singularities of the distance squared unfolding Φ defined by (1.1). We define the focal conic in the normal plane as a counterpart of focal points, and discuss versality of the unfolding Φ of φ, which is one of the fundamental notion in singularity theory. As a consequence, we are able to determine singularity types of caustics and fronts of Whitney umbrella in Section 4. We summarise our results as follows:

(1) If (x_0, y_0, z_0) is on the focal conic, then φ is at least A_2-singularity.

(2) If $(x_0, y_0, z_0) \neq g(0, 0)$ does not correspond to the ridge over Whitney umbrella, then φ has an A_2-singularity and Φ is an $\mathcal{R}^+$-versal unfolding (and a $\mathcal{K}$-versal unfolding). The caustic is nonsingular at (x_0, y_0, z_0), and the front has the cuspidal edge at (x_0, y_0, z_0).

(3) If $(x_0, y_0, z_0) \neq g(0, 0)$ corresponds to the first-order ridge over Whitney umbrella, then φ has an A_3-singularity and Φ is $\mathcal{R}^+$-versal. Thus the caustic has the cuspidal edge at (x_0, y_0, z_0). Moreover if (x_0, y_0, z_0) does not correspond to the sub-parabolic point over Whitney umbrella, then Φ is a $\mathcal{K}$-versal unfolding. We thus conclude that the front has the swallowtail at (x_0, y_0, z_0)

(4) If $(x_0, y_0, z_0) \neq g(0, 0)$ corresponds to the first-order ridge and the sub-parabolic over Whitney umbrella, and the CPC line has definite (resp. indefinite) Morse singularity on X, then the front is the cuspidal lips (resp. cuspidal beaks) at (x_0, y_0, z_0).

(5) If $(x_0, y_0, z_0) \neq g(0, 0)$ corresponds to the second-order ridge and does not correspond to the sub-parabolic point over Whitney umbrella, then the front is the cuspidal butterfly at (x_0, y_0, y_0).

See Theorem 3.7 and Theorem 4.3 for a precise statement. We remark that there are no D_4-singularities (or worse) for distance squared function φ at Whitney umbrella.

2 Differential geometry for Whitney umbrella

We consider a smooth map $g : U \to \boldsymbol{R}^3$ given by $g(u, v) = (g_1(u, v), g_2(u, v), g_3(u, v))$ which defines a surface in $\boldsymbol{R}^3$, where $U \subset \boldsymbol{R}^2$ is an open subset. The map g possibly has singularities. The map $g : (\boldsymbol{R}^2, \mathbf{0}) \to (\boldsymbol{R}^3, \mathbf{0})$ has a *Whiney umbrella* singularity at $(0, 0)$ if it is $\mathcal{A}$-equivalent to the map germ:

$$(\boldsymbol{R}^2, \mathbf{0}) \to (\boldsymbol{R}^3, \mathbf{0}), \quad (u, v) \mapsto (u, uv, v^2).$$

We remark that some authors distinguish between Whitney umbrellas and cross-caps as follows: the Whitney umbrella is the zero-set of the function $x^2 z - y^2 = 0$; the cross-cap is the image of the map that is $\mathcal{A}$-equivalent to $(u, v) \mapsto (u, uv, v^2)$ (see, for example, [3] and [24]). But the authors prefer to use the word "Whitney umbrella" for map germs with respect for H. Whitney's work.

Away from singularities a unit normal vector is defined by $\mathbf{n} = (g_u \times g_v)/\|g_u \times g_v\|$, and the first and second fundamental forms for g are given by

$$\mathrm{I} = E\,du^2 + 2F\,du\,dv + G\,dv^2, \quad \mathrm{II} = L\,du^2 + 2M\,du\,dv + N\,dv^2,$$

respectively, where

$$E = \langle g_u, g_u \rangle, \quad F = \langle g_u, g_v \rangle, \quad G = \langle g_v, g_v \rangle, \quad L = \langle g_{uu}, \mathbf{n} \rangle, \quad M = \langle g_{uv}, \mathbf{n} \rangle, \quad N = \langle g_{vv}, \mathbf{n} \rangle.$$

The principal curvatures κ_1 and κ_2 are the roots of the equation

$$\begin{vmatrix} L - \kappa_i E & M - \kappa_i F \\ M - \kappa_i F & N - \kappa_i G \end{vmatrix} = 0.$$

If a non-zero vector $\mathbf{v}_i = (\xi_i, \eta_i)$ $(i = 1, 2)$ is the principal vector with principal curvature κ_i, then

$$\begin{pmatrix} L - \kappa_i E & M - \kappa_i F \\ M - \kappa_i F & N - \kappa_i G \end{pmatrix} \begin{pmatrix} \xi_i \\ \eta_i \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.1}$$

We can choose (ξ_i, η_i) so that the tangent vector $\xi_i g_u + \eta_i g_v$ is of unit length.

We investigate the asymptotic behaviour of these ingredients near Whitney umbrella.

2.1 The unit normal vectors

Now we suppose that g has a rank one singularity at $(0, 0)$. Take the image of dg_0 to be the x-axis. Then we may write g as

$$g(u, v) = \left(u,\ \frac{1}{2}(a_{02}u^2 + 2a_{11}uv + a_{02}v^2) + O(u, v)^3,\ \frac{1}{2}(b_{20}u^2 + 2b_{11}uv + b_{02}v^2) + O(u, v)^3 \right).$$

We consider the unit normal vector $\tilde{\mathbf{n}} = \mathbf{n} \circ \tilde{\pi}$ in the coordinates (r, θ), where $\tilde{\pi}$ is as in (1.2). By a straightforward calculation we show that the unit normal vector $\tilde{\mathbf{n}}$ is expressed as follows:

$$\tilde{\mathbf{n}}(r, \theta) = \frac{(0 + O(r),\ -b_{11}\cos\theta - b_{02}\sin\theta + O(r),\ a_{11}\cos\theta + a_{02}\sin\theta + O(r))}{\sqrt{({a_{11}}^2 + {b_{11}}^2)\cos^2\theta + 2(a_{11}a_{02} + b_{11}b_{02})\cos\theta\sin\theta + ({a_{02}}^2 + {b_{02}}^2)\sin^2\theta}}.$$

If the singular point of g is a Whitney umbrella, then

$$\begin{vmatrix} a_{11} & a_{02} \\ b_{11} & b_{02} \end{vmatrix} \neq 0.$$

We thus conclude the unit normal vector $\tilde{\mathbf{n}}$ is well-defined on $\{r = 0\}$, since

$$(a_{11}a_{02} + b_{11}b_{02})^2 - ({a_{11}}^2 + {b_{11}}^2)({a_{02}}^2 + {b_{02}}^2) = -(a_{11}b_{02} - a_{02}b_{11})^2.$$

2.2 Normal form of Whitney umbrella

For a regular surface, we can take the z-axis as the normal line, and, after suitable rotation if necessary, we can express the surface in the Monge normal form:

$$(u,v) \mapsto \big(u,\ v,\ \frac{1}{2}(k_1u^2+k_2v^2)+O(u,v)^3\big)$$

For Whitney umbrella we can perform similar computations and obtain the following normal form theorem of Whitney umbrella.

Proposition 2.1. *Let $g:(\boldsymbol{R}^2,\mathbf{0})\to(\boldsymbol{R}^3,\mathbf{0})$ be a smooth map with a Whitney umbrella at $(0,0)$. Then there are a rotation $T:\boldsymbol{R}^3\to\boldsymbol{R}^3$ and a diffeomorphism $\phi:(\boldsymbol{R}^2,\mathbf{0})\to(\boldsymbol{R}^2,\mathbf{0})$ so that*

$$T\circ g\circ\phi(u,v)=\Big(u,\ uv+B(v)+O(u,v)^{k+1},\ \sum_{j=2}^{k}A_j(u,v)+O(u,v)^{k+1}\Big)\quad(k\geq 3),$$

where

$$B(v)=\sum_{i=3}^{k}\frac{b_i}{i!}v^i,\quad \text{and}\quad A_j(u,v)=\sum_{i=0}^{j}\frac{a_{i,j-i}}{i!(j-i)!}u^iv^{j-i}\quad \text{with}\quad a_{02}\neq 0.$$

The result was first proved in [27], but we repeat the proof for completeness.

Proof. Take the image of dg_0 to be the x-axis. Then we may write g as

$$g(u,v)=\Big(u,\ \sum_{i+j=2}\frac{b^*_{ij}}{i!j!}u^iv^j+O(u,v)^3,\ \sum_{i+j=2}\frac{a^*_{ij}}{i!j!}u^iv^j+O(u,v)^3\Big)\quad\text{with}\quad \begin{vmatrix} b^*_{11} & b^*_{02}\\ a^*_{11} & a^*_{02}\end{vmatrix}\neq 0.$$

Take θ so that $(\cos\theta,\sin\theta)=(a^*_{02},b^*_{02})/\sqrt{{a^*_{02}}^2+{b^*_{02}}^2}$, and set a rotation of $\boldsymbol{R}^3$

$$T=\begin{pmatrix} 1 & 0 & 0\\ 0 & \cos\theta & -\sin\theta\\ 0 & \sin\theta & \cos\theta\end{pmatrix}$$

and a change of coordinates

$$\psi(u,v)=\left(u,\ \frac{1}{\begin{vmatrix} b^*_{11} & b^*_{02}\\ a^*_{11} & a^*_{02}\end{vmatrix}}\left(-\frac{1}{2}\begin{vmatrix} b^*_{20} & b^*_{02}\\ a^*_{20} & a^*_{02}\end{vmatrix}u+\sqrt{{a^*_{02}}^2+{b^*_{02}}^2}\,v\right)\right).$$

Then we have

$$T\circ g\circ\psi(u,v)=\Big(u,uv+\sum_{i+j=3}\frac{\beta_{ij}}{i!j!}u^iv^j+O(u,v)^4,\ \sum_{i+j=2}\frac{\alpha_{ij}}{i!j!}u^iv^j+O(u,v)^3\Big)$$

for some constants α_{ij} and β_{ij}. Setting $B_k = \sum_{i+j=k} b_{ij} u^i v^j/(i!j!)$ $(k \geq 3)$ and replacing v by $v + \sum_{i+j=k-1} c_{ij} u^i v^j/(i!j!)$, we have

$$uv + B_k + O(u,v)^{k+1} = uv + \sum_{i+j=k} \frac{ic_{i-1,j} + b_{ij}}{i!j!} u^i v^j + O(u,v)^{k+1}.$$

For a suitable choice of c_{ij} $(i+j = k-1)$, we can reduce this to $b_{(0,k)} v^k/k! + O(u,v)^{k+1}$. Hence, we obtain the result. $\square$

2.3 Principal curvatures and principal directions

Coefficients of the first and second fundamental forms. Throughout the rest of the paper, we suppose that g is given in the normal form of Whitney umbrella:

$$g(u,v) = \left(u,\ uv + B(v) + O(u,v)^5,\ \sum_{j=2}^{4} A_j(u,v) + O(u,v)^5\right), \tag{2.2}$$

where

$$B(v) = \sum_{i=3}^{4} \frac{b_i}{i!} v^i, \quad \text{and} \quad A_j(u,v) = \sum_{i=0}^{j} \frac{a_{i,j-i}}{i!(j-i)!} u^i v^{j-i} \quad \text{with} \quad a_{02} \neq 0.$$

Then we have

$$g_u = \left(1,\ v + O(u,v)^4,\ \sum_{j=2}^{4} (A_j)_u + O(u,v)^4\right),$$
$$g_v = \left(0,\ u + B_v + O(u,v)^4,\ \sum_{j=2}^{4} (A_j)_v + O(u,v)^4\right)$$

and thus have

$$\begin{aligned} E &= 1 + v^2 + (A_{2u})^2 + 2A_{3u}A_{2u} + O(u,v)^4, \\ F &= uv + A_{2u}A_{2v} + A_{3u}A_{2v} + A_{3v}A_{2u} + \frac{1}{2}b_3 v^3 + O(u,v)^4, \\ G &= u^2 + (A_{2v})^2 + 2A_{3v}A_{2v} + b_3 uv^2 + O(u,v)^4. \end{aligned} \tag{2.3}$$

Since

$$g_u \times g_v = \left(\sum_{j=2}^{3} \left(v\,(A_j)_v - u\,(A_j)_u\right) - \frac{b_3}{2} v^2 A_{2u} + O(u,v)^4,\right.$$
$$\left. -\sum_{j=2}^{4} (A_j)_v + O(u,v)^4,\ u + \frac{1}{2}b_3 v^2 + \frac{1}{6} b_4 v^3 + O(u,v)^4 \right),$$

we have

$$\|g_u \times g_v\|^2 = \lambda_2 + \lambda_3 + \lambda_4 + O(u,v)^5,$$

where

$$\begin{aligned} &\lambda_2 = u^2 + A_{2v}^2, \qquad \lambda_3 = 2A_{3v}A_{2v} + b_3uv^2, \\ &\lambda_4 = 2A_{4v}A_{2v} + A_{3v}^2 + (uA_{2u} - vA_{2v})^2 + \frac{1}{3}b_4uv^3 + \frac{1}{4}{b_3}^2v^4. \end{aligned} \tag{2.4}$$

It follows that

$$\tilde{\mathbf{n}}(0,\theta) = \frac{1}{\mathcal{A}}(0,\ -a_{11}\cos\theta - a_{02}\sin\theta,\ \cos\theta), \tag{2.5}$$

where $\mathcal{A} = \mathcal{A}(\theta) = \sqrt{\cos^2\theta + (a_{11}\cos\theta + a_{02}\sin\theta)^2}$. Since $a_{02} \neq 0$, $\tilde{\mathbf{n}}(0,\theta)$ defines an isomorphism from real projective line $P^1(\boldsymbol{R})$

$$P^1(\boldsymbol{R}) \to P^1(\boldsymbol{R}), \qquad \theta \mapsto \tilde{\mathbf{n}}(0,\theta).$$

We set

$$l = \langle g_{uu}, g_u \times g_v \rangle, \qquad m = \langle g_{uv}, g_u \times g_v \rangle, \qquad n = \langle g_{vv}, g_u \times g_v \rangle.$$

Since

$$\begin{aligned} g_{uu} &= \left(0,\ 0,\ \sum_{j=2}^{4}(A_j)_{uu} + O(u,v)^3\right), \\ g_{uv} &= \left(0,\ 1 + O(u,v)^3,\ \sum_{j=2}^{4}(A_j)_{uv} + O(u,v)^3\right), \\ g_{vv} &= \left(0,\ B_{vv} + O(u,v)^3, \sum_{j=2}^{4}(A_j)_{vv} + O(u,v)^3\right), \end{aligned}$$

l, m, and n are expressed as follows:

$$l = l_1 + l_2 + l_3 + O(u,v)^4, \quad m = m_1 + m_2 + m_3 + O(u,v)^4, \quad n = n_1 + n_2 + n_3 + O(u,v)^4,$$

where

$$\begin{aligned} l_1 &= a_{20}u, \quad l_2 = uA_{3uu} + \frac{1}{2}a_{20}b_3v^2, \quad l_3 = uA_{4uu} + \frac{1}{2}b_3A_{3uu}v^2 + \frac{1}{6}a_{20}b_4v^3 \\ m_1 &= -a_{02}v, \quad m_2 = uA_{3uv} - A_{3v} + \frac{1}{2}a_{11}b_3v^2, \\ m_3 &= uA_{4uv} - A_{4uv} + \frac{1}{2}b_3A_{3uv} + \frac{1}{6}a_{11}b_4v^3 \\ n_1 &= a_{02}u, \quad n_2 = uA_{3vv} + \frac{1}{2}b_3(a_{02}v^2 - 2vA_{2v}), \\ n_3 &= uA_{4vv} + \frac{1}{2}b_3v(vA_{3vv} - 2A_{3v}) + \frac{1}{6}b_4v^2(a_{02}v - 3A_{2v}). \end{aligned} \tag{2.6}$$

By using the Taylor series for $L(ru, rv)$, $M(ru, rv)$, and $N(ru, rv)$ in r we obtain

$$
\begin{aligned}
L &= \frac{a_{20}u}{\sqrt{\lambda_2}} + \frac{2l_2\lambda_2 - l_1\lambda_3}{2\lambda_2^{3/2}}r + \frac{8l_3{\lambda_2}^2 - 4l_2\lambda_2\lambda_3 - 4l_1\lambda_2\lambda_4 + l_1\lambda_3}{8\lambda_2^{5/2}}r^2 + O(r^3),\\
M &= \frac{-a_{02}v}{\sqrt{\lambda_2}} + \frac{2m_2\lambda_2 - m_1\lambda_3}{2\lambda_2^{3/2}}r + \frac{8m_3{\lambda_2}^2 - 4m_2\lambda_2\lambda_3 - 4m_1\lambda_2\lambda_4 + m_1\lambda_3}{8\lambda_2^{5/2}}r^2 + O(r^3),\\
N &= \frac{a_{02}u}{\sqrt{\lambda_2}} + \frac{2n_2\lambda_2 - n_1\lambda_3}{2\lambda_2^{3/2}}r + \frac{8n_3{\lambda_2}^2 - 4n_2\lambda_2\lambda_3 - 4n_1\lambda_2\lambda_4 + n_1\lambda_3}{8\lambda_2^{5/2}}r^2 + O(r^3).
\end{aligned}
\tag{2.7}
$$

Principal curvatures.

Lemma 2.2. *The principal curvatures $\tilde{\kappa}_i = \kappa_i \circ \tilde{\pi}$ are expressed as follows:*

$$
\begin{aligned}
\tilde{\kappa}_1(r,\theta) &= k_{10}(\theta) + k_{11}(\theta)r + k_{12}(\theta)r^2 + O(r^3),\\
\tilde{\kappa}_2(r,\theta) &= \frac{1}{r^2}\left[k_{20}(\theta) + k_{21}(\theta)r + O(r^2)\right],
\end{aligned}
\tag{2.8}
$$

where

$$
k_{10} = \frac{A_2^* \sec\theta}{\mathcal{A}}, \tag{2.9}
$$

$$
k_{11} = \frac{1}{\mathcal{A}^3}\begin{pmatrix} 6a_{02}\tilde{A}_3\tilde{A}_{2v}\tan\theta + 2\tilde{A}_{3u}(a_{11}\tilde{A}_{2v} + \cos\theta)\cos\theta \\ -\tilde{A}_{3v}\tilde{A}_{2v}(a_{20}\cos^2\theta + a_{02}\sin^2\theta)\sec\theta \end{pmatrix} + \frac{b_3A_2^*\tilde{A}_{2v}^2\tan^2\theta}{2\mathcal{A}^3}, \tag{2.10}
$$

$$
\begin{aligned}
k_{12} = \frac{1}{2a_{02}\mathcal{A}^5}\Big[&24a_{02}\mathcal{A}^4\tilde{A}_4\sec\theta - 12a_{02}\mathcal{A}^2\tilde{A}_3\tilde{A}_{3v}\tilde{A}_{2v}\sec\theta\\
&- 2a_{02}\mathcal{A}^2\tilde{A}_{4v}(\tilde{A}_{2u}\tilde{A}_{2v} + \mathcal{A}^2\tan\theta + \cos\theta\sin\theta)\\
&- \tilde{A}_{3v}^2\begin{pmatrix} 2\tilde{A}_{2v}^2({a_{11}}^2 - a_{20}a_{02}) + 4a_{11}\tilde{A}_{2v}\cos\theta \\ +(a_{20}a_{02}+2)\cos^2\theta - a_{02}\sin^2\theta \end{pmatrix}\cos\theta\\
&- 8a_{02}\mathcal{A}^4A_2^*\tilde{A}_2^2\sec^3\theta - 8a_{02}\mathcal{A}^4A_2^*\sin\theta\tan\theta - a_{02}\mathcal{A}^2{A_2^*}^3\sec\theta\Big],
\end{aligned}
\tag{2.11}
$$

$$
k_{20} = \frac{a_{02}\cos\theta}{\mathcal{A}^3}, \tag{2.12}
$$

$$
\begin{aligned}
k_{21} = &\frac{1}{\mathcal{A}^5}\left[(-3a_{02}\tilde{A}_{3v}\tilde{A}_{2v} + \mathcal{A}^2\tilde{A}_{3vv})\cos\theta\right]\\
&+ \frac{1}{2\mathcal{A}^5}\left[b_3(-2\mathcal{A}^2\tilde{A}_{2v} + a_{02}\mathcal{A}^2\sin\theta - 3a_{02}\cos^2\theta\sin\theta)\sin\theta\right].
\end{aligned}
\tag{2.13}
$$

Here $A_2^* = a_{20}\cos^2\theta - a_{02}\sin^2\theta$, $\tilde{A}_{2v} = A_{2v}\circ\tilde{\pi}\mid_{r=1}$, $\tilde{A}_{3v} = A_{3v}\circ\tilde{\pi}\mid_{r=1}$, and so on.

Proof. The principal curvatures κ_i are the roots of the equation

$$
(EG - F^2)k^2 - (EN - 2FM + GL)k + (LN - M^2) = 0.
$$

From (2.3), (2.4), (2.6), and (2.7), it follows that

$$
(EG - F^2)\circ\tilde{\pi} = a_2r^2 + a_3r^3 + a_4r^4 + O(r^5),
$$

$$
\begin{aligned}
-(EN-2FM+GL)\circ\tilde{\pi} &= b_0+b_1r+b_2r^2+O(r^3),\\
(LN-M^2)\circ\tilde{\pi} &= c_0+c_1r+c_2r^2+O(r^3),
\end{aligned}
$$

where

$$
a_2=\mathcal{A}^2,\quad b_0=-\frac{a_{02}\cos\theta}{\mathcal{A}},\quad c_0=\frac{a_{02}A_2^*}{\mathcal{A}^2},
$$

and the coefficients a_3, a_4, b_1, b_2, c_1, and c_2 are the trigonometric polynomials in the coefficients appearing in the terms of degree four or less in the normal form of Whitney umbrella. Therefore, we obtain

$$
\begin{aligned}
\tilde{\kappa}_1 &= -\frac{c_0}{b_0}+\frac{b_1c_0-b_0c_1}{{b_0}^2}r+\frac{{b_1}^2c_0-b_0b_2c_0+a_2{c_0}^2-b_0b_1c_1+{b_0}^2c_2}{{b_0}^3}r^2+O(r^3),\\
\tilde{\kappa}_2 &= \frac{1}{r^2}\Big(-\frac{b_0}{a_2}+\frac{a_3b_0-a_2b_1}{{a_2}^3}r+O(r^2)\Big).
\end{aligned}
$$

We thus obtain (2.9)–(2.13) by a straightforward calculation. □

Gaussian curvature. Since the Gaussian curvature K is the product of the principal curvatures, it does not depend on the choice of the unit normal vector. From (2.9) and (2.12), the Gaussian curvature is expressed as follows:

$$
\tilde{K}(r,\theta)=K\circ\pi(r,\theta)=\frac{1}{r^2}\left(\frac{a_{02}(a_{20}\cos^2\theta-a_{02}\sin^2\theta)}{{\mathcal{A}(\theta)}^4}+O(r)\right),
$$

where π is as in (1.3). By this expression, we say that a point $(0,\theta_0)$ on the Möbius strip $\mathcal{M}$ is *elliptic*, *hyperbolic*, or *parabolic point over Whitney umbrella* if $r^2\tilde{K}(0,\theta_0)$ is positive, negative, or zero, respectively. We often omit the phrase "over Whitney umbrella" if no confusion is possible from the context.

We immediately have the following proposition.

Proposition 2.3. (1) *There is no parabolic point over Whitney umbrella if and only if* $a_{20}a_{02}<0$.

(2) *There is one parabolic point over Whitney umbrella if and only if* $a_{20}=0$.

(3) *There are two parabolic points over Whitney umbrella if and only if* $a_{20}a_{02}>0$.

Furthermore, in terms of the parabolic line in a domain, Whitney umbrella is classified into three types. We say that Whitney umbrella is *hyperbolic*, *elliptic*, or *parabolic* if the parabolic line has an A_1^+-singularity (an isolated point), A_1^--singularity (a pair of smooth curves intersecting transversally), or A_2-singularity (a cusp), respectively. In the case of the hyperbolic Whitney umbrella all non-singular points near the Whitney umbrella singularity are hyperbolic. In the case of the elliptic Whitney umbrella the parabolic line divides the surface into hyperbolic and elliptic regions (see [27] for details).

From (2.6), the parabolic line in the Möbius strip $\mathcal{M}$ is expressed by the equation

$$
a_{02}A_2^*+\big[(a_{20}\tilde{A}_{3vv}+a_{02}\tilde{A}_{3uu})\cos^2\theta+2a_{02}(\tilde{A}_{3uv}\cos\theta-\tilde{A}_{3v})\sin\theta-a_{11}b_3A_2^*\sin\theta\big]r+O(r^2)=0.
$$

If $a_{20}a_{02} < 0$, then the parabolic line dose not meet with the exceptional set $X = \pi^{-1}(0,0)$ on $\mathcal{M}$, in which case the surface is the hyperbolic Whitney umbrella. If $a_{20}a_{02} > 0$, then the parabolic line meets with X at two parabolic point over Whitney umbrella, in which case the surface is the elliptic Whitney umbrella. If $a_{20} = 0$, then the parabolic line meets with X at one parabolic point $(r,\theta) = (0,0)$, which is the sub-parabolic point over Whitney umbrella (See Section 2.4). Calculating the tangent vector of the parabolic line at $(0,0)$ on $\mathcal{M}$, we show that the parabolic line meets tangentially with X at $(0,0)$ if and only if $a_{30} \neq 0$. In this case, we have the parabolic Whitney umbrella (This classification according to the coefficients of the normal form of Whitney umbrella is also obtained in [20]). Moreover, the parabolic point $(0,0)$ is the singular point of the parabolic line if and only if $a_{30} = 0$, equivalently this point is the ridge point over Whitney umbrella (See Section 2.4). In this case, this point is of Morse type if and only if $3{a_{21}}^2 + 2a_{40}a_{02} \neq 0$.

Principal directions.

Lemma 2.4. *The unit principal vectors $\tilde{\mathbf{v}}_i$ in the coordinates (r,θ) are expressed as follows:*

$$\tilde{\mathbf{v}}_1 = (\sec\theta + O(r))\frac{\partial}{\partial r} + \left(\frac{-2\tilde{A}_{3v} + b_3\tilde{A}_{2v}\sin\theta\tan\theta}{2a_{02}} + O(r)\right)\frac{\partial}{\partial\theta},$$
$$\tilde{\mathbf{v}}_2 = \frac{1}{r^2}\left[\left(\frac{\sin\theta}{\mathcal{A}}r + O(r^2)\right)\frac{\partial}{\partial r} + \left(\frac{\cos\theta}{\mathcal{A}} + O(r)\right)\frac{\partial}{\partial\theta}\right].$$

Proof. From the equation (2.1), one of the vectors along the principal vectors $\mathbf{v}_i$ in the coordinates (u,v) are given by

$$\xi_i\frac{\partial}{\partial u} + \eta_i\frac{\partial}{\partial v} = (N - \kappa_i G)\frac{\partial}{\partial u} + (-M + \kappa_i F)\frac{\partial}{\partial v}. \tag{2.14}$$

Since

$$\frac{\partial}{\partial u} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta} \quad \text{and} \quad \frac{\partial}{\partial v} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta},$$

the vector (2.14) can be lifted by π and we obtain

$$\begin{aligned}\tilde{\xi}_i\frac{\partial}{\partial r} + \tilde{\eta}_i\frac{\partial}{\partial\theta} = &\Big[(N\circ\tilde{\pi} - \tilde{\kappa}_i G\circ\tilde{\pi})\cos\theta + (\tilde{\kappa}_i F\circ\tilde{\pi} - M\circ\tilde{\pi})\sin\theta\Big]\frac{\partial}{\partial r}\\ &+\frac{1}{r}\Big[(\tilde{\kappa}_i F\circ\tilde{\pi} - M\circ\tilde{\pi})\cos\theta + (\tilde{\kappa}_i G\circ\tilde{\pi} - N\circ\tilde{\pi})\sin\theta\Big]\frac{\partial}{\partial\theta}.\end{aligned}$$

From (2.3), (2.4), (2.6), and (2.7), we have

$$\begin{aligned} F\circ\tilde{\pi} &= F_2r^2 + F_3r^3 + O(r^4), & G\circ\tilde{\pi} &= G_2r^2 + G_3r^3O(r^4),\\ M\circ\tilde{\pi} &= M_0 + M_1r + M_2r^2 + O(r^3), & N\circ\tilde{\pi} &= N_0 + N_1r + N_2r^2 + O(r^3),\end{aligned}$$

where

$$\begin{aligned} &F_2 = \cos\theta\sin\theta + \tilde{A}_{2u}\tilde{A}_{2v}, \quad F_3 = \tilde{A}_{3u}\tilde{A}_{2v} + \tilde{A}_{3v}\tilde{A}_{2u} + \frac{1}{2}b_3\sin^3\theta,\\ &G_2 = \cos^2\theta + \tilde{A}_{2v}^2, \quad G_3 = 2\tilde{A}_{3v}\tilde{A}_{2v} + b_3\cos\theta\sin^2\theta,\\ &M_0 = -\frac{a_{02}\sin\theta}{\mathcal{A}}, \quad N_0 = \frac{a_{02}\cos\theta}{\mathcal{A}},\end{aligned}$$

and coefficients L_1, M_1, N_1, L_2, M_2, and N_2 are the trigonometric polynomials in the coefficients appearing in the terms of degree four or less in the normal form of Whitney umbrella. It follows that $\tilde{\xi}_i$ and $\tilde{\eta}_i$ are expressed as follows:

$$\begin{aligned}
\tilde{\xi}_1 &= N_0 \cos\theta - M_0 \sin\theta + (N_1 \cos\theta - M_1 \sin\theta) r + O(r^2),\\
\tilde{\eta}_1 &= -M_1 \cos\theta - N_1 \sin\theta + \left[(F_2 k_{10} - M_2)\cos\theta + (G_2 k_{10} - N_2)\sin\theta\right] r + O(r^2),\\
\tilde{\xi}_2 &= \big(-G_2 k_{20} + N_0\big)\cos\theta + \big(F_2 k_{20} - M_0\big)\sin\theta\\
&\quad + \left[(-G_3 k_{20} - G_2 k_{21} + N_1)\cos\theta + (F_3 k_{20} + F_2 k_{21} - M_1)\sin\theta\right] r + O(r^2),\\
\tilde{\eta}_2 &= \frac{1}{r}\left[k_{20}(F_2\cos\theta + G_2\sin\theta)\right.\\
&\quad \left. + \ ((F_3 k_{20} + F_2 k_{21} - M_1)\cos\theta + (G_3 k_{20} + G_2 k_{21} - N_1)\sin\theta) r + O(r^2)\right].
\end{aligned}$$

After a long calculation, it follows that $\tilde{\xi}_i$ and $\tilde{\eta}_i$ are expressed as follows:

$$\begin{aligned}
\tilde{\xi}_1 &= \tilde{\xi}_{10} + \tilde{\xi}_{11} r + O(r^2), & \tilde{\eta}_1 &= \tilde{\eta}_{10} + \tilde{\eta}_{11} r + O(r^2),\\
\tilde{\xi}_2 &= \tilde{\xi}_{20} + \tilde{\xi}_{21} r + O(r^2), & \tilde{\eta}_2 &= \frac{1}{r}\left[\tilde{\eta}_{20} + \tilde{\eta}_{21} r + O(r^2)\right],
\end{aligned}$$

where

$$\begin{aligned}
\tilde{\xi}_{10} &= \frac{a_{02}}{\mathcal{A}}, \quad \tilde{\eta}_{10} = \frac{1}{2\mathcal{A}}\big(-2\tilde{A}_{3v}\cos\theta + b_3 \tilde{A}_{2v}\sin^2\theta\big),\\
\tilde{\xi}_{11} &= \frac{1}{2\mathcal{A}^3}\Big[2\big(\tilde{A}_{3v}(a_{02}\tilde{A}_{2v}\cos\theta - a_{11}\tilde{A}_{2v}\sin\theta - \cos\theta\sin\theta)\\
&\quad - a_{02}\tilde{A}_{3uv}\tilde{A}_{2v} + \tilde{A}_{3vv}(a_{11}\tilde{A}_{2v} + \cos\theta)\big)\\
&\quad - b_3\tilde{A}_{2v}(a_{11}\tilde{A}_{2v} + \mathcal{A}^2\cos^2\theta + \cos\theta)\sin\theta\Big],\\
\tilde{\eta}_{11} &= \frac{1}{12\mathcal{A}^3}\Big[-24\mathcal{A}^2\tilde{A}_{4v}\cos\theta + 12\tilde{A}_{3v}^2\tilde{A}_{2v}\cos\theta + 24\tilde{A}_2\tilde{A}_{2v}A_2^*\cos\theta\\
&\quad + 12a_{02}\tilde{A}_{2u}^2(3a_{11}\tilde{A}_{2v}\cos\theta + {a_{02}}^2\sin^2\theta)\cos\theta\sin\theta - 24a_{11}a_{02}\tilde{A}_{2v}^3\sin^3\theta\\
&\quad + 12{a_{02}}^3\tilde{A}_{2u}^2\cos\theta\sin^3\theta + 12a_{20}{a_{11}}^3\tilde{A}_{2u}\cos^5\theta + 60{a_{11}}^2{a_{02}}^2\tilde{A}_{2v}\cos\theta\sin^4\theta\\
&\quad + 24\mathcal{A}^2A_2^*\cos\theta + 12a_{20}{a_{11}}^4\cos^5\theta - {a_{02}}^5\sin^5\theta\tan\theta + 4b_4\mathcal{A}^2\tilde{A}_{2v}\sin^3\theta\\
&\quad -3{b_3}^2\tilde{A}_{2v}\cos\theta\sin^4\theta - 6b_3\tilde{A}_{3v}(\tilde{A}_{2v}^2 - \cos^2\theta)\sin^2\theta\Big],\\
\tilde{\xi}_{20} &= \frac{2a_{02}}{\mathcal{A}^3}(\tilde{A}_2\tilde{A}_{2v} + \cos^2\theta\sin\theta)\sin\theta, \quad \tilde{\eta}_{20} = \tilde{\xi}_{20}\cot\theta, \quad \tilde{\eta}_{21} = \tilde{\xi}_{21}\cot\theta,\\
\tilde{\xi}_{21} &= \frac{1}{\mathcal{A}^5}\Big[3a_{11}a_{02}\mathcal{A}^2\tilde{A}_3 + {a_{02}}^2\mathcal{A}^2\tilde{A}_{3u}\sin\theta\\
&\quad + \tilde{A}_{3v}(2a_{02}\tilde{A}_2\tilde{A}_{2v}^2 - a_{11}\mathcal{A}^2 - \mathcal{A}^2 - 8a_{11}a_{02}\cos\theta\sin\theta)\cos\theta\\
&\quad - a_{02}\tilde{A}_{3uv}(2\tilde{A}_2\tilde{A}_{2v}^2 + a_{11}\tilde{A}_{2v}^2\cos\theta\sin\theta + 3a_{02}\cos^2\theta\sin^2\theta)\\
&\quad + \ \tilde{A}_{3vv}(2\tilde{A}_2\tilde{A}_{2v} + a_{11}\tilde{A}_{2u}\tilde{A}_{2v}^2 + a_{11}\tilde{A}_{2v}^2\sin\theta + 2\mathcal{A}^2\sin\theta - 3{a_{02}}^2\sin^3\theta)\Big]\cos\theta\sin\theta\\
&\quad - \frac{b_3}{2\mathcal{A}^5}\Big[2\tilde{A}_2\tilde{A}_{2v}(a_{11}\tilde{A}_{2v}^2 + \tilde{A}_{2v}\cos\theta - 2a_{11}\cos^2\theta) + 3\tilde{A}_{2u}\tilde{A}_{2v}^2\cos^2\theta
\end{aligned}$$

$$+ 5a_{11}\tilde{A}_{2v}^2 \cos^2\theta \sin\theta + 3\tilde{A}_{2v}\cos^3\theta\sin\theta + a_{02}\cos^3\theta\sin^2\theta\Big]\sin^2\theta.$$

The unit principal vectors $\tilde{\mathbf{v}}_i$ are given by

$$\tilde{\mathbf{v}}_i = \frac{1}{\sqrt{\tilde{E}\tilde{\xi}_i^2 + 2\tilde{F}\tilde{\xi}_i\tilde{\eta}_i + \tilde{G}\tilde{\eta}_i^2}}\left(\tilde{\xi}_i\frac{\partial}{\partial r} + \tilde{\eta}_i\frac{\partial}{\partial\theta}\right), \tag{2.15}$$

where $\tilde{E}$, $\tilde{F}$, and $\tilde{G}$ are the coefficients of the first fundamental form of g in the coordinates (r,θ). We calculate that

$$\begin{aligned}
\tilde{E} &= \cos^2\theta + 4\left(\cos^2\theta\sin^2\theta + \tilde{A}_2^2\right)r^2 + O(r^3),\\
\tilde{F} &= -r\cos\theta\sin\theta\\
&\qquad + 2\left[(\cos^2\theta - \sin^2\theta)\cos\theta\sin\theta + \tilde{A}_2\big(\tilde{A}_{2v}\cos\theta - \tilde{A}_{2u}\sin\theta\big)\right]r^3 + O(r^4),\\
\tilde{G} &= r^2\sin^2\theta + \left[(\cos^2\theta - \sin^2\theta)^2 + \big(\tilde{A}_{2v}\cos\theta - \tilde{A}_{2u}\sin\theta\big)^2\right]r^4 + O(r^5).
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
\tilde{E}\tilde{\xi}_1^2 + 2\tilde{F}\tilde{\xi}_1\tilde{\eta}_1 + \tilde{G}\tilde{\eta}_1^2 &= \frac{{a_{02}}^2\cos^2\theta}{\mathcal{A}^2} + O(r),\\
\tilde{E}\tilde{\xi}_2^2 + 2\tilde{F}\tilde{\xi}_2\tilde{\eta}_2 + \tilde{G}\tilde{\eta}_2^2 &= \frac{4{a_{02}}^2(\tilde{A}_2\tilde{A}_{2v} + \cos^2\theta\sin\theta)^2}{\mathcal{A}^4}r^2 + O(r^3).
\end{aligned}$$

This completes proof together with (2.15). $\square$

Remark 2.5. The (unit) principal vector $\tilde{\mathbf{v}}_1$ is extendible on $\{(r,\theta)\,;\, r\neq 0 \text{ or } \cos\theta\neq 0\}$ and thus the principal field defined by $\mathbf{v}_1$ is extendible on the Möbius strip $\mathcal{M}$ except on the set $\{(r,\theta)\,;\, r=0,\, \cos\theta = 0\}$. The principal curvature vector $r^2\tilde{\mathbf{v}}_2$ is extendible over $\boldsymbol{R}\times S^1$ even though $\tilde{\mathbf{v}}_2$ is not. So the principal field defined by $\mathbf{v}_i$ is extendible over $\mathcal{M}$.

2.4 Ridge points and sub-parabolic points over Whitney umbrella

Ridge points. By the computation in the previous subsection, we can express $\tilde{\mathbf{v}}_i\tilde{\kappa}_i$ as follows:

$$\tilde{\mathbf{v}}_1\tilde{\kappa}_1(r,\theta) = R_{110}(\theta) + R_{111}(\theta)r + \cdots, \qquad \tilde{\mathbf{v}}_2\tilde{\kappa}_2(r,\theta) = \frac{1}{r^4}(R_{210}(\theta) + R_{211}(\theta)r + \cdots,$$

where $\tilde{\mathbf{v}}_i\tilde{\kappa}_i$ denotes the directional derivative of the principal curvature $\tilde{\kappa}_i$ in the principal vector $\tilde{\mathbf{v}}_i$. We say that a point (r_0,θ_0) is a *ridge point relative to the principal vector* $\tilde{\mathbf{v}}_1$ (resp. $\tilde{\mathbf{v}}_2$) if $\tilde{\mathbf{v}}_1\tilde{\kappa}_1(r_0,\theta_0) = 0$ (resp. $r^4\tilde{\mathbf{v}}_2\tilde{\kappa}_2(r_0,\theta_0) = 0$). If the ridge point (r_0,θ_0) is over Whitney umbrella (that is, $r_0 = 0$) this is equivalent that $R_{110}(\theta_0) = 0$ (resp. $R_{210}(\theta_0) = 0$). It is possible that $R_{i10}(\theta)$ has multiple roots. We say that $(0,\theta_0)$ is a *multiple ridge point relative to* $\tilde{\mathbf{v}}_i$ if θ_0 is a multiple root of $R_{i10}(\theta)$. We say that a point $(0,\theta_0)$ is a *n-th order ridge point relative to* $\tilde{\mathbf{v}}_1$ *(resp.* $\tilde{\mathbf{v}}_2$*) over Whitney umbrella* if $R_{im0}(\theta_0) = 0$ $(1\leqq m\leqq n)$ and $R_{i,n+1,0}(\theta_0)\neq 0$, where

$$\tilde{\mathbf{v}}_1^{(m)}\tilde{\kappa}_1(r,\theta) = R_{1m0}(\theta) + R_{1m1}(\theta)r + \cdots, \quad \tilde{\mathbf{v}}_2^{(m)}\tilde{\kappa}_2(r,\theta) = \frac{1}{r^{2+2m}}(R_{2m0}(\theta) + R_{2m1}(\theta)r + \cdots.$$

Here, $\tilde{\mathbf{v}}_i^{(n)}\tilde{\kappa}_i$ denotes the n-th time directional derivative of $\tilde{\kappa}_i$ in the direction $\tilde{\mathbf{v}}_i$. The ridge line relative to $\tilde{\mathbf{v}}_i$ near the exceptional set $X = \pi^{-1}(0,0)$ is expressed by the equation:

$$R_{i10}(\theta) + R_{i11}(\theta)r + \cdots = 0.$$

In terms of the normal form of Whitney umbrella, we have $\tilde{\kappa}_1(0,\theta)$ tends to infinity as θ approaches $\pm\pi/2$, by (2.9), and, after some calculations, we obtain

$$\tilde{\mathbf{v}}_1\tilde{\kappa}_1(r,\theta) = \frac{\Gamma_3(\theta)\sec^3\theta}{\mathcal{A}(\theta)} + O(r), \tag{2.16}$$

$$\tilde{\mathbf{v}}_2\tilde{\kappa}_1(r,\theta) = \frac{1}{r^2}\left(-\frac{a_{02}\Gamma_3^*(\theta)\sec\theta}{\mathcal{A}(\theta)^4} + O(r)\right), \tag{2.17}$$

$$\tilde{\mathbf{v}}_2\tilde{\kappa}_2(r,\theta) = \frac{1}{r^4}\left(-\frac{3{a_{02}}^2(a_{11}\cos\theta + a_{02}\sin\theta)\cos\theta}{\mathcal{A}(\theta)^6} + O(r)\right), \tag{2.18}$$

$$\tilde{\mathbf{v}}_1\tilde{\kappa}_2(r,\theta) = \frac{1}{r^3}\left(-\frac{2a_{02}}{\mathcal{A}(\theta)^3} + O(r)\right), \tag{2.19}$$

where

$$\Gamma_3(\theta) = 6\tilde{A}_3\cos\theta - b_3\tilde{A}_{2v}\sin^3\theta, \quad \Gamma_3^*(\theta) = 2\tilde{A}_2\tilde{A}_{2v} + 2\cos^2\theta\sin\theta.$$

Lemma 2.6. *A point $(0,\theta_0)$ with $\cos\theta_0 \neq 0$ is a ridge point relative to $\tilde{\mathbf{v}}_1$ if and only if $\Gamma_3(\theta_0) = 0$. Moreover, the point $(0,\theta_0)$ is a first order ridge point relative to $\tilde{\mathbf{v}}_1$ if and only if $\Gamma_4(\theta_0) \neq 0$, where*

$$\begin{aligned}\Gamma_4(\theta) =\,& 24a_{02}\tilde{A}_4\cos^2\theta - 12\tilde{A}_{3v}^2\cos^2\theta - 12a_{02}A_2^*\tilde{A}_2^2 - 12a_{02}A_2^*\cos^2\theta\sin^2\theta \\ &- a_{02}b_4\tilde{A}_{2v}\cos\theta\sin^4\theta + 12{b_3}^2\tilde{A}_{3v}\tilde{A}_{2v}\cos\theta\sin^2\theta - 3b_3\tilde{A}_{2v}^2\sin^4\theta.\end{aligned}$$

Proof. The expansion (2.16) implies the first assertion. Since $\cos\theta_0 \neq 0$, the condition $\Gamma_3(\theta_0) = 0$ is equivalent to

$$\begin{aligned}a_{30} =& -\big[\cos\theta_0(3a_{21}\cos^2\theta_0\sin\theta_0 + 3a_{12}\cos\theta_0\sin^2\theta_0 + a_{03}\sin^3\theta_0) \\ &- b_3\sin^3\theta_0(a_{11}\cos\theta_0 + a_{02}\sin\theta_0)\big]\sec^4\theta_0.\end{aligned}$$

Using the above relation, we can reduce $\tilde{\mathbf{v}}_1^2\tilde{\kappa}_1(0,\theta_0)$ to

$$\tilde{\mathbf{v}}_1^2\kappa_1(0,\theta_0) = \frac{\Gamma_4(\theta_0)\sec^5\theta_0}{a_{02}\mathcal{A}(\theta_0)^3},$$

and the proof is complete. □

Lemma 2.7. *If $b_3 \neq 0$, then there are at most four ridge points relative to $\tilde{\mathbf{v}}_1$ over Whitney umbrella.*

Proof. The ridge points relative to $\tilde{\mathbf{v}}_1$ over Whitney umbrella are given by $\Gamma_3(\theta) = 0$; equivalently,

$$a_{30} + 3a_{21}\tan\theta + 3a_{12}\tan^2\theta + (a_{30} - a_{11}b_3)\tan^3\theta - a_{02}b_3\tan^4\theta = 0.$$

This implies assertions. $\square$

Remark 2.8. When $b_3 = a_{03} = a_{12} = a_{21} = a_{30} = 0$, the multiplicity of ridge points is not defined but the order is defined. In fact, we have $\Gamma_4(\theta) \not\equiv 0$.

Proposition 2.9. *Suppose that a point $(0, \theta_0)$ is a ridge point relative to $\tilde{\mathbf{v}}_1$ over Whitney umbrella, and that the ridge line relative to $\tilde{\mathbf{v}}_1$ is non-singular at $(0, \theta_0)$. Then the ridge line is tangent to X at $(0, \theta_0)$ if and only if $(0, \theta_0)$ is the multiple ridge point relative to $\tilde{\mathbf{v}}_1$.*

Proof. It follows form (2.16) that if and only if $\Gamma_3'(\theta_0) = 0$, the ridge line relative to $\tilde{\mathbf{v}}_1$ is tangent to X at $(0, \theta_0)$. Hence, we have proved the proposition. $\square$

From (2.18), we have the following proposition.

Proposition 2.10. *There are two simple ridge points relative to $\tilde{\mathbf{v}}_2$ over Whitney umbrella. That is, the ridge line relative to $\tilde{\mathbf{v}}_2$ is not tangent to X.*

Sub-parabolic points. By the computation in the previous subsection, we can express $\tilde{\mathbf{v}}_i\tilde{\kappa}_j$ $(i \neq j)$ as follows:

$$\tilde{\mathbf{v}}_1\tilde{\kappa}_2(r, \theta) = \frac{1}{r^3}(P_{10}(\theta) + P_{11}(\theta)r + \cdots), \quad \tilde{\mathbf{v}}_2\tilde{\kappa}_1(r, \theta) = \frac{1}{r^2}(P_{20}(\theta) + P_{21}(\theta)r + \cdots).$$

We say that a point (r_0, θ_0) is a *sub-parabolic point relative to the principal vector* $\tilde{\mathbf{v}}_1$ (resp. $\tilde{\mathbf{v}}_2$) if $r^3\tilde{\mathbf{v}}_1\tilde{\kappa}_2(r_0, \theta_0) = 0$ (resp. $r^2\tilde{\mathbf{v}}_2\tilde{\kappa}_1(r_0, \theta_0) = 0$). When the sub-parabolic point is over Whitney umbrella (that is, $r_0 = 0$), we obtain $P_{i0}(\theta_0) = 0$. A point $(0, \theta_0)$ is said to be a *multiple sub-parabolic point relative to $\mathbf{v}_i$ over Whitney umbrella* if θ_0 is a multiple root of $P_{i0}(\theta) = 0$. The sub-parabolic line relative to $\mathbf{v}_i$ near X is expressed by the equation:

$$P_{i0}(\theta) + P_{i1}(\theta)r + \cdots = 0.$$

From (2.17) we have the following lemma.

Lemma 2.11. *A point $(0, \theta_0)$ is a sub-parabolic point relative to $\tilde{\mathbf{v}}_2$ if and only if $\Gamma_3^*(\theta_0) = 0$ holds.*

From (2.19), we have the following proposition.

Proposition 2.12. *There is no sub-parabolic point relative to $\tilde{\mathbf{v}}_1$ over Whitney umbrella.*

Lemma 2.13. *There is at least one and at most three sub-parabolic points relative to $\tilde{\mathbf{v}}_2$ over Whitney umbrella.*

Proof. The sub-parabolic points relative to $\tilde{\mathbf{v}}_2$ over Whitney umbrella are given by $\Gamma_3^*(\theta) = 0$; equivalently,

$$a_{20}a_{11} + (2 + a_{20}a_{02} + 2{a_{11}}^2)\tan\theta + 3a_{11}a_{02}\tan^2\theta + {a_{02}}^2\tan^3\theta = 0.$$

Since $a_{02} \neq 0$, the equation has at least one and at most three roots and we have completed the proof of the Lemma. $\square$

It follows from (2.17) that we obtain the following proposition.

Proposition 2.14. *Suppose that a point $(0, \theta_0)$ is a sub-parabolic point relative to $\tilde{\mathbf{v}}_2$ over Whitney umbrella, and that the sub-parabolic line relative to $\tilde{\mathbf{v}}_2$ is non-singular at $(0, \theta_0)$. Then the sub-parabolic line is tangent to X at $(0, \theta_0)$ if and only if $(0, \theta_0)$ is the multiple sub-parabolic point relative to $\tilde{\mathbf{v}}_2$.*

Example 2.15. We set $(a_{20}, a_{11}, a_{02}) = (-3, 0, 1)$ in the normal form of Whitney umbrella, then we have

$$\Gamma_3^*(\theta) = -\cos^2\theta\sin\theta + \sin^3\theta.$$

The roots of $\Gamma_3^*(\theta) = 0$ are $\theta = \pm\pi/4$ and $\theta = 0$. Hence, we have the distinct three simple sub-parabolic points $(0, \pm\pi/4)$ and $(0, 0)$ relative to $\tilde{\mathbf{v}}_2$ over Whitney umbrella.

(1) We take $(a_{30}, a_{21}, a_{12}, a_{03}, b_3) = (-1, 0, 10/9, 0, 1)$. Then we have

$$\Gamma_3(\theta) = -\cos^4\theta + \frac{10}{3}\cos^2\theta\sin^2\theta - \sin^4\theta.$$

The roots of $\Gamma_3(\theta) = 0$ are $\theta = \pm\pi/3$ and $\theta = \pm\pi/6$. Hence, we have four distinct simple ridge points $(0, \pm\pi/3)$ and $(0, \pm\pi/6)$ relative to $\tilde{\mathbf{v}}_1$ over Whitney umbrella.

(2) We take $(a_{30}, a_{21}, a_{12}, a_{03}, b_3) = (-3, 0, 4/3, 0, 1)$. Then we have

$$\Gamma_3(\theta) = -3\cos^4\theta + 4\cos^2\theta\sin^2\theta - \sin^4\theta.$$

The roots of $\Gamma_3(\theta) = 0$ are $\theta = \pm\pi/3$ and $\theta = \pm\pi/4$. Hence, we have four distinct simple ridge points $(0, \pm\pi/3)$ and $(0, \pm\pi/4)$ relative to $\tilde{\mathbf{v}}_1$ over Whitney umbrella. Remark that the points $(0, \pm\pi/4)$ are ridge relative to $\tilde{\mathbf{v}}_1$ and sub-parabolic relative to $\tilde{\mathbf{v}}_2$.

2.5 Constant principal curvature lines

We set Σ_k^1 (resp. Σ_k^2) $(k \geq 0)$ as the image of

$$\{(r, \theta) \in \boldsymbol{R} \times S^1\,;\, \tilde{\kappa}_1(r, \theta) = \pm k\} \quad (\text{resp. } \{(r, \theta) \in \boldsymbol{R} \times S^1\,;\, \tilde{\kappa}_2(r, \theta) = \pm k\})$$

by the double covering $\boldsymbol{R} \times S^1 \to \mathcal{M}$. We call $\Sigma_k = \Sigma_k^1 \cup \Sigma_k^2$ by the *constant principal curvature (CPC) line with a constant value of k*. Remark that $X \cap \Sigma_k^2 = \emptyset$. Remark also that Σ_0 is nothing but the parabolic line, which we already described in Proposition 2.3. Remember that $\mathcal{M}$ is non-orientable and the induced image of $\tilde{\mathbf{n}}$, by $\tilde{\pi} : \boldsymbol{R} \times S^1 \to \boldsymbol{R}^2$, covers "all possible unit normals". We remark that Σ_k $(k > 0)$ is the singular set of the front of g at distance $1/k$.

Proposition 2.16. *A CPC line Σ_k $(k \neq 0)$ intersects with X in at most four points.*

Proof. The number of the intersection points of Σ_k and X equals the number of roots of the equation $|\tilde{\kappa}_1(0,\theta)| = k$. From (2.9), we obtain the equation

$$\left| \frac{(a_{20}\cos^2\theta - a_{02}\sin^2\theta)\sec\theta}{\sqrt{\cos^2\theta + (a_{11}\cos\theta + a_{02}\sin\theta)^2}} \right| = k.$$

Squaring both sides and setting the equation to 0, we get

$$\left[{a_{20}}^2 - k^2({a_{11}}^2+1)\right]\cos^4\theta - 2a_{02}a_{11}k^2\cos^3\theta\sin\theta - a_{02}(2a_{20}-a_{02}k)\cos^2\theta\sin^2\theta + {a_{02}}^2\sin^4\theta = 0.$$

Dividing both sides by $\cos^4\theta$, we obtain

$${a_{20}}^2 - k^2({a_{11}}^2+1) - 2a_{02}a_{11}k^2\tan\theta - a_{02}(2a_{20} - a_{02}k^2)\tan^2\theta + {a_{02}}^2\tan^4\theta = 0.$$

Since $a_{02} \neq 0$, this equation is quartic in $\tan\theta$ and we have thus completed the proof. $\square$

Setting $C(r,\theta) = \tilde{\kappa}_1(r,\theta)^2 - k^2$ $(k \neq 0)$ and by (2.8), we have

$$C(r,\theta) = -k^2 + k_{10}(\theta)^2 + 2k_{10}(\theta)k_{11}(\theta)r + \big(2k_{10}(\theta)k_{12}(\theta) + k_{11}(\theta)^2\big)r^2 + \cdots. \tag{2.20}$$

From (2.16) and (2.18), the principal vectors $\tilde{\mathbf{v}}_i$ can be written in

$$\begin{aligned}
\tilde{\mathbf{v}}_1(r,\theta) &= (x_{10}(\theta) + x_{11}(\theta)r + \cdots)\frac{\partial}{\partial r} + (y_{10}(\theta) + y_{11}(\theta)r + \cdots)\frac{\partial}{\partial\theta},\\
\tilde{\mathbf{v}}_2(r,\theta) &= \frac{1}{r^2}\Big[\big(x_{21}(\theta)r + x_{22}(\theta)r^2 + \cdots\big)\frac{\partial}{\partial r} + (y_{20}(\theta) + y_{21}(\theta)r + \cdots)\frac{\partial}{\partial\theta}\Big].
\end{aligned}$$

Note that $x_{10}(\theta) \neq 0$ and $y_{20}(\theta) \neq 0$. Therefore, the directional derivatives of $\tilde{\kappa}_1(r,\theta)$ by $\tilde{\mathbf{v}}_i$ are expressed as follows:

$$\begin{aligned}
\tilde{\mathbf{v}}_1\tilde{\kappa}_1(r,\theta) =\ & x_{10}(\theta)k_{11}(\theta) + y_{10}(\theta)k'_{10}(\theta)\\
& + (2x_{10}(\theta)k_{12}(\theta) + x_{11}(\theta)k_{11}(\theta) + y_{10}(\theta)k'_{11}(\theta) + y_{11}(\theta)k'_{10}(\theta))r + \cdots,\\
\tilde{\mathbf{v}}_2\tilde{\kappa}_1(r,\theta) =\ & \frac{1}{r^2}\big[y_{20}(\theta)k'_{10}(\theta) + (x_{21}(\theta)k_{11}(\theta) + y_{20}(\theta)k'_{11}(\theta) + y_{21}(\theta)k'_{10}(\theta))r + \cdots\big].
\end{aligned} \tag{2.21}$$

The following lemma provides the criterion for the singularity of the CPC line intersecting with X in terms of the configurations of the ridge line and the sub-parabolic line.

Lemma 2.17. *Suppose that a point $(0,\theta_0)$ is not parabolic and that the CPC line Σ_k meets X at $(0,\theta_0)$. Then the CPC line Σ_k is singular at $(0,\theta_0)$ if and only if $(0,\theta_0)$ is the ridge point relative to $\tilde{\mathbf{v}}_1$ and the sub-parabolic point relative to $\tilde{\mathbf{v}}_2$. In this case, the singularity is of Morse type if and only if the ridge line relative to $\tilde{\mathbf{v}}_1$ and the sub-parabolic line relative to $\tilde{\mathbf{v}}_2$ intersect transversely at $(0,\theta_0)$.*

Proof. Let us use expansions (2.20) and (2.21). Now we have $k_{10}(\theta_0) \neq 0$. The CPC line Σ_k is singular at $(0,\theta_0)$ if and only if $C_r(0,\theta_0) = C_\theta(0,\theta_0) = 0$. By computation, we have

$$C_r(0,\theta_0) = 2k_{10}(\theta_0)k_{11}(\theta_0) \quad \text{and} \quad C_\theta(0,\theta_0) = 2k_{10}(\theta_0)k'_{10}(\theta_0).$$

It follows that Σ_k has singularity at $(0,\theta_0)$ if and only if $k'_{10}(\theta_0) = k_{11}(\theta_0) = 0$. From (2.21), we deduce that $(0,\theta_0)$ is the ridge point relative to $\tilde{\mathbf{v}}_1$ and the sub-parabolic point relative to $\tilde{\mathbf{v}}_2$ if and only if $k'_{10}(\theta_0) = k_{11}(\theta_0) = 0$. This completes the proof of the first assertion.

We show the second assertion. Assume that $(0,\theta_0)$ is a singularity of Σ_k. Then $(0,\theta_0)$ is a Morse singularity if and only if

$$\begin{vmatrix} C_{rr}(0,\theta_0) & C_{r\theta}(0,\theta_0) \\ C_{r\theta}(0,\theta_0) & C_{\theta\theta}(0,\theta_0) \end{vmatrix} \neq 0.$$

This is equivalent to

$$2k''_{10}(\theta_0)k_{12}(\theta) - k'_{11}(\theta_0)^2 \neq 0.$$

It follows from (2.21) that the ridge line relative to $\tilde{\mathbf{v}}_1$ and the sub-parabolic line relative to $\tilde{\mathbf{v}}_2$ intersect transversely at $(0,\theta_0)$ if and only if

$$\begin{vmatrix} 2x_{10}(\theta_0)k_{12}(\theta_0) + y_{10}(\theta_0)k'_{11}(\theta_0) & x_{10}(\theta_0)k'_{11}(\theta_0) + y_{10}(\theta_0)k''_{10}(\theta) \\ y_{20}(\theta_0)k'_{11}(\theta_0) & y_{20}(\theta_0)k''_{10}(\theta_0) \end{vmatrix} \neq 0;$$

equivalently,

$$x_{10}(\theta_0)y_{20}(\theta_0)(2k''_{10}(\theta_0)k_{12}(\theta) - k'_{11}(\theta_0)^2) \neq 0.$$

We thus have completed of the proof of the second assertion. □

Lemma 2.18. *Suppose that a point $(0,\theta_0)$ is not parabolic and that the CPC line Σ_k meets X at $(0,\theta_0)$. Then the CPC line Σ_k is tangent to X at $(0,\theta_0)$ if and only if $(0,\theta_0)$ is a sub-parabolic point relative to $\tilde{\mathbf{v}}_2$ which is not a ridge point relative to $\tilde{\mathbf{v}}_1$.*

Proof. Let us consider expansions (2.20) and (2.21). From (2.20), the equation of the tangent line of Σ_k at $(0,\theta_0)$ is then

$$k_{11}(\theta_0)r + k'_{10}(\theta_0)(\theta - \theta_0) = 0.$$

It follows that the CPC line Σ_k is tangent to X if and only if $k'_{10}(\theta_0) = 0$ and $k_{11}(\theta_0) \neq 0$. From (2.21), we show that the point $(0,\theta_0)$ is a sub-parabolic point relative to $\tilde{\mathbf{v}}_2$ which is not a ridge point relative to $\tilde{\mathbf{v}}_1$ if and only if $k'_{10}(\theta_0) = 0$ and $k_{11}(\theta_0) \neq 0$. □

Lemma 2.19. *Assume that the CPC line Σ_k and the ridge line relative to $\tilde{\mathbf{v}}_1$ meet at a point $(0,\theta_0)$ which is not parabolic over Whitney umbrella. Then*

(1) *These two curves intersect transversely at the point $(0,\theta_0)$ if and only if the point $(0,\theta_0)$ is a first order ridge point relative to $\tilde{\mathbf{v}}_1$.*

(2) *These two curves are tangent at the point $(0,\theta_0)$ if and only if the point $(0,\theta_0)$ is a second or higher order ridge point relative to $\tilde{\mathbf{v}}_1$.*

Proof. Let us consider expansions (2.20) and (2.21). Remark that $k_{10}(\theta_0) \neq 0$. The ridge line relative to $\tilde{\mathbf{v}}_1$ passes through $(0,\theta_0)$, that is, $(0,\theta_0)$ is a ridge point relative to $\tilde{\mathbf{v}}_1$ over Whitney umbrella if and only if

$$x_{10}(\theta_0)k_{11}(\theta_0) + y_{10}(\theta_0)k'_{10}(\theta_0) = 0.$$

Since $x_{10} \neq 0$, this is equivalent to

$$k_{11}(\theta_0) = -\frac{y_{10}(\theta_0)k'_{10}(\theta_0)}{x_{10}(\theta_0)}. \tag{2.22}$$

The CPC line Σ_k and the ridge line relative to $\tilde{\mathbf{v}}_1$ intersect transversely at $(0, \theta_0)$ if and only if the determinant

$$\begin{vmatrix} k_{11}(\theta_0) & 2x_{10}(\theta_0)k_{12}(\theta_0) + x_{11}(\theta_0)k_{11}(\theta_0) + y_{10}(\theta_0)k'_{11}(\theta_0) + y_{11}(\theta_0)k'_{10}(\theta_0) \\ k'_{10}(\theta_0) & x_{10}(\theta_0)k'_{11}(\theta_0) + x'_{10}(\theta_0)k_{11}(\theta_0) + y_{10}(\theta_0)k''_{10}(\theta_0) + y'_{10}(\theta_0)k'_{10}(\theta_0) \end{vmatrix} \tag{2.23}$$

is not zero. Otherwise, these two curves are tangent at $(0, \theta_0)$ if and only if the determinant (2.23) is zero. By using (2.22), the determinant (2.23) is expanded as

$$\begin{aligned} &-\frac{1}{x_{10}(\theta_0)^2}k'_{10}(\theta_0)\big[2x_{10}(\theta_0)^3k_{12}(\theta_0) + 2x_{10}(\theta_0)^2y_{10}(\theta_0)k'_{11}(\theta_0) \\ &\quad + x_{10}(\theta_0)^2y_{11}(\theta_0)k'_{10}(\theta_0) + x_{10}(\theta_0)y_{10}(\theta_0)^2k''_{10}(\theta_0) - x_{10}(\theta_0)x_{11}(\theta_0)y_{10}(\theta_0)k'_{10}(\theta_0) \\ &\qquad\qquad + x_{10}(\theta_0)y_{10}(\theta_0)^2k'_{10}(\theta_0) - y_{10}(\theta_0)^2x'_{10}(\theta_0)k'_{10}(\theta_0)\big], \end{aligned}$$

and we obtain

$$\begin{aligned} \tilde{\mathbf{v}}_1^2\tilde{\kappa}_1(0, \theta_0) = \frac{1}{x_{10}(\theta_0)}\Big[&2x_{10}(\theta_0)^3k_{12}(\theta_0) + 2x_{10}(\theta_0)^2y_{10}(\theta_0)k'_{11}(\theta_0) + x_{10}(\theta_0)^2y_{11}(\theta_0)k'_{10}(\theta_0) \\ &+ x_{10}(\theta_0)y_{10}(\theta_0)^2k''_{10}(\theta_0) - x_{10}(\theta_0)x_{11}(\theta_0)y_{10}(\theta_0)k'_{10}(\theta_0) \\ &+ x_{10}(\theta_0)y_{10}(\theta_0)^2k'_{10}(\theta_0) - y_{10}(\theta_0)^2x'_{10}(\theta_0)k'_{10}(\theta_0)\Big]. \end{aligned}$$

Hence, we conclude that the determinant (2.23) is zero (resp. non-zero) if and only if $(0, \theta_0)$ is a first order (resp. second or higher order) ridge point relative to $\tilde{\mathbf{v}}_1$, and we have completed the proof. □

3 Singularities of the distance squared unfolding

We assume that $g : (\boldsymbol{R}^2, \mathbf{0}) \to (\boldsymbol{R}^3, \mathbf{0})$ is given by (2.2). In this section, we investigate the singularities of the members of the family of the distance squared function:

$$\Phi : (\boldsymbol{R}^2, \mathbf{0}) \times \boldsymbol{R}^3 \to \boldsymbol{R}, \qquad (u, v) \times (x, y, z) \mapsto -\frac{1}{2}(\|(x, y, z) - g(u, v)\|^2 - {t_0}^2) \tag{3.1}$$

where t_0 is a constant. We set $\varphi(u, v) = \Phi(u, v, x_0, y_0, z_0)$ where (x_0, y_0, z_0) is a point in $\boldsymbol{R}^3$, and take t_0 so that $\varphi(0, 0) = 0$, that is, ${t_0}^2 = {x_0}^2 + {y_0}^2 + {z_0}^2$.

Now we recall the definition of the normal plane. When the map $g : (\boldsymbol{R}^2, \mathbf{0}) \to (\boldsymbol{R}^3, \mathbf{0})$ has Whitney umbrella at $(0, 0)$, the image of $dg_{(0,0)}$ is a line in $\boldsymbol{R}^3$. We call the plane perpendicular to this line the *normal plane at Whitney umbrella.*

Proposition 3.1. *The following conditions are equivalent:*

(1) *The function φ has at least an A_1-singularity at* $(0, 0)$*;*

(2) *The point* (x_0, y_0, z_0) *is on the normal plane at Whitney umbrella;*

(3) *There exist* $\rho_0 \in \boldsymbol{R}$ *and* θ_0 *with* $\theta_0 \in [-\pi/2, \pi/2]$ *such that* $(x_0, y_0, z_0) = \rho_0 \tilde{\mathbf{n}}(0, \theta_0)$.

Proof. The unfolding Φ can be expressed as follows:

$$\Phi(u, v, x, y, z) = c_{00}(x, y, z) + xu + \sum_{i+j=2}^{4} \frac{1}{i!j!} c_{ij}(x, y, z) u^i v^j + O(u, v)^5. \tag{3.2}$$

We have

$$\begin{aligned}
&c_{00} = \frac{1}{2}({t_0}^2 - x^2 - y^2 - z^2), \quad c_{20} = a_{20}z - 1, \quad c_{11} = y + a_{11}z, \quad c_{02} = a_{02}z, \\
&c_{30} = a_{30}z, \quad c_{21} = a_{21}z, \quad c_{12} = a_{12}z, \quad c_{03} = a_{03}z + b_3y, \\
&c_{40} = -3{a_{20}}^2 + a_{40}z, \quad c_{31} = -3a_{20}a_{11} + a_{31}z, \quad c_{22} = -2 - 2{a_{11}}^2 - a_{20}a_{02} + a_{22}z, \\
&c_{13} = -a_{11}a_{02} + a_{13}z, \quad c_{04} = -3{a_{02}}^2 + b_4y + a_{04}z.
\end{aligned}$$

Then φ can be written in the form

$$\varphi(u, v) = x_0u + \sum_{i+j=2}^{4} \frac{1}{i!j!} c_{ij}^0 u^i v^j + O(u, v)^5, \tag{3.3}$$

where $c_{ij}^0 = c_{ij}(x_0, y_0, z_0)$. It follows that φ has at least an A_1-singularity at $(0, 0)$ if and only if $x_0 = 0$. Directly from the definition of the normal form we obtain that the image of $dg_{(0,0)}$ is the x-axis. Hence, the normal plane is the yz-plane. Thus (1) and (2) are equivalent.

Next, suppose (2) holds. Since $a_{02} \neq 0$,

$$\tilde{\mathbf{n}}(0, \theta) = \frac{(0, -a_{11}\cos\theta - a_{02}\sin\theta, \cos\theta)}{\sqrt{\cos^2\theta + (a_{11}\cos\theta + a_{02}\sin\theta)^2}}$$

expands in all direction in the yz-plane. Hence, there exist $\rho_0 \in \boldsymbol{R}$ and θ_0 with $\theta_0 \in [-\pi/2, \pi/2]$ such that $(x_0, y_0, z_0) = \rho_0 \tilde{\mathbf{n}}(0, \theta_0)$.

Finally, suppose (3) holds. Then we have $x_0 = 0$. Thus (3) implies (1). □

3.1 Focal conics

Proposition 3.2. *The points* (x_0, y_0, z_0) *at which* φ *has at least* A_2*-singularity (valid for the rest of the paper "an* A_k*-singularity") at* $(0, 0)$ *form a conic in the normal plane.*

Proof. Assume that φ has at least an A_1-singularity at $(0, 0)$, that is, $x_0 = 0$. Then the determinant of the Hessian of φ at $(0, 0)$ is given by

$$\begin{vmatrix} \varphi_{uu}(0,0) & \varphi_{uv}(0,0) \\ \varphi_{uv}(0,0) & \varphi_{vv}(0,0) \end{vmatrix} = \begin{vmatrix} c_{20}^0 & c_{11}^0 \\ c_{11}^0 & c_{02}^0 \end{vmatrix} = -{y_0}^2 - 2a_{11}y_0z_0 + (a_{20}a_{02} - {a_{11}}^2){z_0}^2 - a_{02}z_0.$$

Therefore, the locus of the equation

$$-y^2 - 2a_{11}yz + (a_{20}a_{02} - {a_{11}}^2)z^2 - a_{02}z = 0$$

is the set of the points at which φ is an A_k-singularity at $(0, 0)$. Thus we complete the proof. □

Lemma 3.3. *The function φ has an A_k-singularity at $(0,0)$ if and only if $(x_0, y_0, z_0) = (0,0,0)$ or*

$$(x_0, y_0, z_0) = \frac{1}{\tilde{\kappa}_1(0,\theta_0)}\tilde{\mathbf{n}}(0,\theta_0) \quad \textit{with} \quad \tilde{\kappa}_1(0,\theta_0) \neq 0, \quad \textit{where} \quad \theta_0 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

Proof. Suppose that φ is at least an A_1-singularity at $(0,0)$. By Proposition 3.1, we have $(x_0, y_0, z_0) = \rho_0\tilde{\mathbf{n}}(0,\theta_0)$ where $\rho_0 \in \boldsymbol{R}$ and $\theta_0 \in [-\pi/2, \pi/2]$. Substituting this into the equation $-{y_0}^2 - 2a_{11}y_0z_0 + (a_{20}a_{02} - {a_{11}}^2){z_0}^2 - a_{02}z_0 = 0$, we obtain

$$\rho_0 = 0, \qquad \text{or} \qquad \rho_0 = \frac{\sqrt{\cos^2\theta_0 + (a_{11}\cos\theta_0 + a_{02}\sin\theta_0)^2}}{(a_{20}\cos^2\theta_0 - a_{02}\sin^2\theta_0)\sec\theta_0}.$$

When $\rho_0 = 0$, the point (x_0, y_0, z_0) coincides with $(0,0,0)$. In the later case, ρ_0 coincides with the principal radius $1/\tilde{\kappa}_1(0,\theta_0)$. □

For this reason, we call the set of the points (x, y, z) at which φ has an A_k-singularity at $(0,0)$ the *focal conic of Whitney umbrella*. Focal conics are classified into three types as shown in Figure 1. The following proposition provides a classification of focal conics.

Proposition 3.4. (1) *The focal conic is an ellipse if and only if $a_{20}a_{02} < 0$.*

(2) *The focal conic is a hyperbola if and only if $a_{20}a_{02} > 0$, in which case its asymptotes are parallel to $y + (a_{11} \pm \sqrt{a_{20}a_{02}})z = 0$.*

(3) *The focal conic is a parabola if and only if $a_{20} = 0$, in which case its axis of symmetry is parallel to $y + a_{11}z = 0$.*

Proof. Remark that the focal conic is the zero locus of

$$FC(y,z) = -y^2 - 2a_{11}yz + (a_{20}a_{02} - {a_{11}}^2)z^2 - a_{02}z.$$

Firstly, we assume that $a_{20} \neq 0$. Replacing y and z by $y - a_{11}/(2a_{20})$ and $z + 1/(2a_{20})$, respectively. Then the equation $FC(y,z) = 0$ has the form

$$-\frac{a_{02}}{4a_{20}} - (y + a_{11}z)^2 + a_{20}a_{02}z^2 = 0.$$

This form implies the assertion (1) and (2).

Next, we assume that $a_{20} = 0$. Then the equation $FC(y,z) = 0$ reduces to

$$-(y + a_{11}z)^2 - a_{02}z = 0.$$

This implies the assertion (3). □

The following proposition provides properties of the focal conic. It is easy to verify this proposition and we omit its proof.

Proposition 3.5. *Let g be given in the normal form of Whitney umbrella and let C_k denote the circle centred at the origin with radius $1/k$ in the normal plane.*

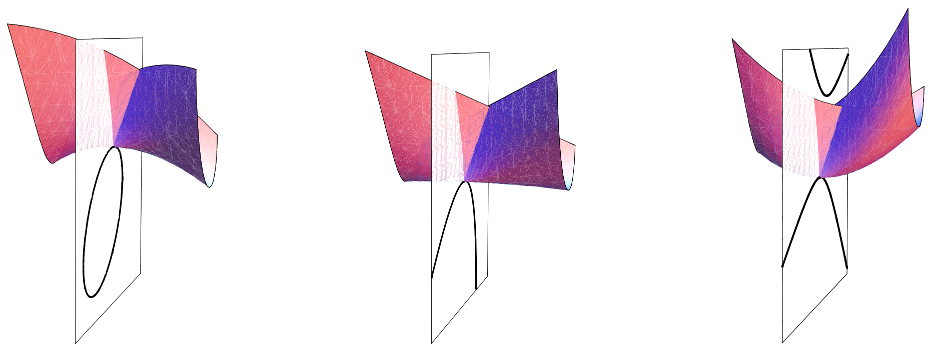


Figure. 1: The three types of focal conics: focal ellipse left, focal parabola center, focal hyperbola right.

(1) *The y-axis is tangent to the focal conic at the origin.*

(2) *The circle C_k is a subset of the front at distance $1/k$.*

(3) *For $k > 0$ we have $\#\Sigma_k \cap X = \#C_k \cap \{(y,z)\ ;\ FC(y,z) = 0\}$ where X is the exceptional set $X = \pi^{-1}(0,0)$ on the Möbius strip $\mathcal{M}$. Since $\#C_k \cap \{(y,z)\ ;\ FC(y,z) = 0\}$ is at most four, $\#\Sigma_k \cap X$ is also at most four* (*cf.* *Proposition* 2.16). *Moreover, we have*

$$C_k \cap \{(y,z)\ ;\ FC(y,z) = 0\} = \bigcup_{\theta:\tilde{\kappa}_1(0,\theta)=k} \boldsymbol{R}\tilde{n}(0,\theta) \cap \{(y,z)\ ;\ FC(y,z) = 0\} \setminus \{(0,0)\}$$
$$= \bigcup_{\theta:\tilde{\kappa}_1(0,\theta)=k} \frac{1}{k}\tilde{\mathbf{n}}(0,\theta).$$

If $\tilde{\mathbf{n}}(0,\theta)$ is parallel to the axis of symmetry of the focal parabola or the asymptotes of the focal hyperbola, then $\boldsymbol{R}\tilde{n}(0,\theta) \cap \{(y,z)\ ;\ FC(y,z) = 0\} \setminus \{(0,0)\} = \emptyset$ and $\tilde{\kappa}_1(0,\theta) = 0$, that is, $(0,\theta)$ is a parabolic point over Whitney umbrella.

(4) *The circle C_k is tangent to the focal conic at $\tilde{\mathbf{n}}(0,\theta)/k$ if and only if $(0,\theta)$ a is sub-parabolic point relative to $\tilde{\mathbf{v}}_2$ over Whitney umbrella. For any focal conic, there exists at least one and at most three values of k such that C_k is tangent to the focal conic. This implies that the number of the sub-parabolic points relative to $\tilde{\mathbf{v}}_2$ over Whitney umbrella is at least one and at most three* (*cf.* *Lemma* 2.13).

(5) *The origin of the normal plane is the vertex of the focal conic if and only if $a_{11} = 0$. In this case, we have $\tilde{\kappa}_1(0,-\theta) = \tilde{\kappa}_1(0,\theta)$, and $(r,\theta) = (0,0)$ is a sub-parabolic point relative to $\tilde{\mathbf{v}}_2$.*

(6) *When the focal conic is a parabola, $(r,\theta) = (0,0)$ is parabolic* (*in fact, $\tilde{\kappa}_1(0,0) = 0$*) *and sub-parabolic relative to $\tilde{\mathbf{v}}_2$ over Whitney umbrella* (*i.e.*, $\Gamma_3^*(0) = 0$).

We obtain the following corollary by Propositions 2.3 and 3.4.

Corollary 3.6. (1) *There is no parabolic point over Whitney umbrella, that is, the parabolic line dose not meet with X, if and only if the focal conic is an ellipse, in which case we have the hyperbolic Whitney umbrella.*

(2) *There is one parabolic point over Whitney umbrella, that is, the parabolic line meets with X at one point, if and only if the focal conic is a parabola. In this case the intersection point of the parabolic line and X is a sub-parabolic point relative to $\tilde{\mathbf{v}}_2$. Furthermore, the parabolic line is tangent to X at the point if and only if the point is not a ridge point relative to $\tilde{\mathbf{v}}_1$. In this case, we have the parabolic Whitney umbrella.*

(3) *There are two parabolic points, that is, the parabolic line meets with X at two points, if and only if the focal conic is a hyperbola, in which case we have the elliptic Whitney umbrella.*

3.2 Versality of distance squared unfolding

We do not repeat here the definition of versal unfolding, which is fundamental in singularity theory. Please refer to [1] for elegant explanation, [17] for elementary introduction, and [25] for carefully prepared survey. The notation in [25] becomes the standard in singularity theory.

Theorem 3.7. *Suppose that $g : (\boldsymbol{R}^2, \mathbf{0}) \to (\boldsymbol{R}^3, \mathbf{0})$ is given in the normal form of Whitney umbrella. Assume that $\Phi : (\boldsymbol{R}^2, \mathbf{0}) \times \boldsymbol{R}^3 \to \boldsymbol{R}$ is the distance squared function defined by* (3.1) *and that $\varphi(u, v) = \Phi(u, v, x_0, y_0, z_0)$ where (x_0, y_0, z_0) is a point in $\boldsymbol{R}^3$.*

(1) *Suppose that $(x_0, y_0, z_0) = \tilde{\mathbf{n}}(0, \theta_0)/\tilde{\kappa}_1(0, \theta_0) \neq (0, 0, 0)$ with $\tilde{\kappa}_1(0, \theta_0) \neq 0$ and ${t_0}^2 = {x_0}^2 + {y_0}^2 + {z_0}^2$, where $\theta_0 \in (-\pi/2, \pi/2)$.*

  (a) *The function $\varphi(u, v)$ has an A_2-singularity at $(0, 0)$ if and only if $(0, \theta_0)$ is not a ridge point relative to $\tilde{\mathbf{v}}_1$ over Whitney umbrella. In this case, Φ is $\mathcal{R}^+$ and $\mathcal{K}$-versal unfolding of φ.*

  (b) *The function φ has an A_3-singularity at $(0, 0)$ if and only if $(0, \theta_0)$ is a first order ridge point relative to $\tilde{\mathbf{v}}_1$ over Whitney umbrella. In this case, Φ is an $\mathcal{R}^+$-versal unfolding of φ. Moreover, Φ is a $\mathcal{K}$-versal unfolding of φ if and only if $(0, \theta_0)$ is not a sub-parabolic point relative to $\tilde{\mathbf{v}}_2$ over Whitney umbrella.*

(2) *Suppose that $(x_0, y_0, z_0) = (0, 0, 0)$ and $t_0 = 0$. Then φ has an A_3-singularity at $(0, 0)$. In this case, Φ is neither an $\mathcal{R}^+$-versal nor a $\mathcal{K}$-versal unfolding of φ.*

Proof. Let us use expansions of Φ and φ as in (3.2) and (3.3), respectively. We remark that the coefficient a_{02} appearing in the normal form of Whitney umbrella is not zero.

(1) We first prove the condition for the point $(0, 0)$ to be an A_2 or A_3-singularity of φ. Lemma 3.3 now shows that φ has an A_k-singularity at $(0, 0)$. By (2.5) and (2.9), we have

$$(x_0, y_0, z_0) = \left(0,\ -\frac{(a_{11}\cos\theta_0 + a_{02}\sin\theta_0)\cos\theta_0}{a_{20}\cos^2\theta - a_{02}\sin^2\theta_0},\ \frac{\cos^2\theta_0}{a_{20}\cos^2\theta_0 - a_{02}\sin^2\theta_0}\right) \neq (0, 0, 0).$$

Simple calculations show that

$$\begin{pmatrix} c^0_{20} & c^0_{11} \\ c^0_{11} & c^0_{02} \end{pmatrix} = \frac{a_{02}}{a_{20}\cos^2\theta_0 - a_{02}\sin^2\theta_0} \begin{pmatrix} \sin^2\theta_0 & -\cos\theta_0\sin\theta_0 \\ -\cos\theta_0\sin\theta_0 & \cos^2\theta_0 \end{pmatrix}.$$

Taking s and (ξ, η) so that

$$s\begin{pmatrix} \eta^2 & -\xi\eta \\ -\xi\eta & \xi^2 \end{pmatrix} = \begin{pmatrix} c_{20}^0 & c_{11}^0 \\ c_{11}^0 & c_{02}^0 \end{pmatrix},$$

we obtain

$$s = \frac{a_{02}}{a_{20}\cos^2\theta_0 - a_{02}\sin^2\theta_0} \quad \text{and} \quad (\xi, \eta) = (\cos\theta_0, \sin\theta_0).$$

Setting $c = 2s(c_{21}^0\xi^2 + 2c_{12}^0\xi\eta + c_{03}^0\eta^2)/\xi^4$ and replacing v by $v + (\eta/\xi)u - cu^2$, we have

$$\Phi = c_{00}(x, y, z) + xu + \sum_{i+j=2}^{4} \frac{1}{i!j!}\hat{c}_{ij}(x, y, z)u^i v^j + O(u, v)^5,$$

where

$$\begin{aligned}
\hat{c}_{20} &= \frac{1}{\xi^2}(c_{20}\xi^2 + 2c_{11}\xi\eta + c_{02}\eta^2), \quad \hat{c}_{11} = \frac{1}{\xi}(c_{11}\xi + c_{02}\eta), \quad \hat{c}_{02} = c_{02}, \\
\hat{c}_{30} &= \frac{1}{s\xi^5}\Big(s\xi^2(c_{30}\xi^3 + 3c_{21}\xi^2\eta + 3c_{12}\xi\eta^2 + c_{03}\eta^3) - 3(c_{11}\xi + c_{02}\eta)(c_{21}^0\xi^2 + 2c_{12}^0\xi\eta + c_{02}^0\eta^2)\Big) \\
\hat{c}_{21} &= \frac{1}{s\xi^4}\left[s\xi^2(c_{21}\xi^2 + 2c_{12}\xi\eta + c_{02}\eta^2) - c_{02}(c_{21}^0\xi^2 + 2c_{12}^0\xi\eta + c_{02}^0\eta^2)\right], \\
\hat{c}_{12} &= \frac{1}{\xi}(c_{21}\xi + c_{03}\eta), \quad \hat{c}_{03} = c_{03}, \\
\hat{c}_{40} &= \frac{1}{\xi^4}(c_{40}^0\xi^4 + 4c_{31}^0\xi^3\eta + 6c_{22}^0\xi^2\eta^2 + 4c_{13}^0\xi^3\eta + c_{04}^0\eta^4) \\
&\quad - \frac{3}{s\xi^6}(c_{21}^0\xi^2 + 2c_{12}^0\xi\eta + c_{03}^0\eta^2)\left[2(c_{21}\xi^2 + 2c_{12}\xi\eta + c_{02}\eta^2) - (c_{21}^0\xi^2 + 2c_{12}^0\xi\eta + c_{03}^0\eta^2)\right].
\end{aligned}$$

Therefore, φ is expressed as

$$\varphi = \frac{1}{2}\hat{c}_{02}^0 v^2 + \frac{1}{6}(\hat{c}_{30}^0 u^3 + 3\hat{c}_{12}^0 uv^2 + \hat{c}_{03}^0 v^3) + \sum_{i+j=4} \frac{1}{i!j!}\hat{c}_{ij}^0 u^i v^j + O(u, v)^5,$$

where $\hat{c}_{ij}^0 = \hat{c}_{ij}(x_0, y_0, z_0)$. By this form, φ does not have D_4 or worse singularities. This form also shows that φ has an A_2 or A_3 singularity for (x_0, y_0, z_0) at $(0, 0)$ if and only if $\hat{c}_{30}^0 \neq 0$, or $\hat{c}_{30}^0 = 0$ and $\hat{c}_{40}^0 \neq 0$, respectively. After some computations, we extract that

$$\hat{c}_{30}^0 = \frac{\Gamma_3(\theta_0)\sec^2\theta_0}{a_{20}\cos^2\theta_0 - a_{02}\sin^2\theta_0} \quad \text{and} \quad \hat{c}_{40}^0 = \frac{\Gamma_4(\theta_0)\sec^4\theta_0}{a_{02}(a_{20}\cos^2\theta_0 - a_{02}\sin^2\theta_0)}.$$

Therefore, from Lemma 2.6 it follows that φ has an A_2 or A_3-singularity at $(0, 0)$ if and only if $(0, \theta_0)$ is not a ridge point relative to $\tilde{\mathbf{v}}_1$ or a first order ridge point relative to $\tilde{\mathbf{v}}_1$, respectively.

We now turn to the versality of Φ. Firstly, we prove Case (a). Suppose that $(0, 0)$ is an A_2-singularity. We remark that A_2-singularity is 3-determined. To show the $\mathcal{R}^+$-versality and the $\mathcal{K}$-versality of Φ, it is enough to verify that, respectively,

$$\mathcal{E}_2 = \langle \varphi_u, \varphi_v \rangle_{\mathcal{E}_2} + \langle \Phi_x|_{\boldsymbol{R}^2\times p_0}, \Phi_y|_{\boldsymbol{R}^2\times p_0}, \Phi_z|_{\boldsymbol{R}^2\times p_0} \rangle_{\boldsymbol{R}} + \langle 1 \rangle_{\boldsymbol{R}} + \langle u, v \rangle_{\mathcal{E}_2}^4, \quad \text{and} \tag{3.4}$$

$$\mathcal{E}_2 = \langle \varphi_u, \varphi_v, \varphi \rangle_{\mathcal{E}_2} + \langle \Phi_x|_{\boldsymbol{R}^2\times p_0}, \Phi_y|_{\boldsymbol{R}^2\times p_0}, \Phi_z|_{\boldsymbol{R}^2\times p_0} \rangle_{\boldsymbol{R}} + \langle u, v \rangle^4_{\mathcal{E}_2}, \tag{3.5}$$

where $p_0 = (x_0, y_0, z_0)$. The coefficients of $u^i v^j$ of functions appearing in (3.4) and (3.5) are given by the following table:

	1	u	v	u^2	uv	v^2	u^3	u^2v	uv^2	v^3
Φ_x	0	$\boxed{1}$	0	0	0	0	0	0	0	0
Φ_y	$-y_0$	0	0	α_{20}	1	0	α_{30}	α_{21}	α_{12}	α_{03}
Φ_z	$\boxed{-z_0}$	0	0	β_{20}	β_{11}	β_{02}	β_{30}	β_{21}	β_{12}	β_{03}
φ_u	0	0	0	$\boxed{\frac{1}{2}\hat{c}^0_{30}}$	0	$\frac{1}{2}\hat{c}^0_{12}$	$\frac{1}{6}\hat{c}^0_{40}$	$\frac{1}{2}\hat{c}^0_{31}$	$\frac{1}{2}\hat{c}^0_{22}$	$\frac{1}{6}\hat{c}^0_{13}$
φ_v	0	0	$\boxed{\hat{c}^0_{02}}$	0	$\hat{c}^0_{12}$	$\frac{1}{2}\hat{c}^0_{03}$	$\frac{1}{6}\hat{c}^0_{31}$	$\frac{1}{2}\hat{c}^0_{22}$	$\frac{1}{2}\hat{c}^0_{13}$	$\frac{1}{6}\hat{c}^0_{04}$
φ	0	0	0	0	0	$\frac{1}{2}\hat{c}^0_{02}$	$\frac{1}{6}\hat{c}^0_{30}$	0	$\frac{1}{2}\hat{c}^0_{12}$	$\frac{1}{6}\hat{c}^0_{03}$
$u\varphi_u$	0	0	0	0	0	0	$\boxed{\frac{1}{2}\hat{c}^0_{30}}$	0	$\frac{1}{2}\hat{c}^0_{12}$	0
$v\varphi_u$	0	0	0	0	0	0	0	$\frac{1}{2}\hat{c}^0_{30}$	0	$\frac{1}{2}\hat{c}^0_{12}$
$u\varphi_v$	0	0	0	0	$\boxed{\hat{c}^0_{02}}$	0	0	$\hat{c}^0_{12}$	$\frac{1}{2}\hat{c}^0_{03}$	0
$v\varphi_v$	0	0	0	0	0	$\boxed{\hat{c}^0_{02}}$	0	0	$\hat{c}^0_{12}$	$\frac{1}{2}\hat{c}^0_{03}$
$u^2\varphi_v$	0	0	0	0	0	0	0	$\boxed{\hat{c}^0_{02}}$	0	0
$uv\varphi_v$	0	0	0	0	0	0	0	0	$\boxed{\hat{c}^0_{02}}$	0
$v^2\varphi_v$	0	0	0	0	0	0	0	0	0	$\boxed{\hat{c}^0_{02}}$

Here,

$$\alpha_{ij} = \frac{\partial \hat{c}_{ij}}{\partial y}(x_0, y_0, z_0), \quad \beta_{ij} = \frac{\partial \hat{c}_{ij}}{\partial z}(x_0, y_0, z_0),$$

and boxed elements are non-zero. Hence, Gauss's elimination method using boxed elements as pivots leads to that the matrix presented by this table is full rank. Thus the equality (3.5) holds. The case of (3.4) is similar.

Next, we consider Case (b). Suppose that $(0,0)$ is an A_3-singularity. We remark that A_3-singularity is 4-determined. To show the $\mathcal{R}^+$-versality and the $\mathcal{K}$-versality of Φ, it is enough to verify that

$$\mathcal{E}_2 = \langle \varphi_u, \varphi_v \rangle_{\mathcal{E}_2} + \langle \Phi_x|_{\boldsymbol{R}^2\times p_0}, \Phi_y|_{\boldsymbol{R}^2\times p_0}, \Phi_z|_{\boldsymbol{R}^2\times p_0} \rangle_{\boldsymbol{R}} + \langle 1 \rangle_{\boldsymbol{R}} + \langle u, v \rangle^5, \quad \text{and} \tag{3.6}$$

$$\mathcal{E}_2 = \langle \varphi_u, \varphi_v, \varphi \rangle_{\mathcal{E}_2} + \langle \Phi_x|_{\boldsymbol{R}^2\times p_0}, \Phi_y|_{\boldsymbol{R}^2\times p_0}, \Phi_z|_{\boldsymbol{R}^2\times p_0} \rangle_{\boldsymbol{R}} + \langle u, v \rangle^5, \tag{3.7}$$

respectively. The coefficients of $u^i v^j$ of functions appearing in (3.6) and (3.7) are given by the following table:

	1	u	v	u^2	uv	v^2	u^3	u^2v	uv^2	v^3	u^4
Φ_x	0	$\boxed{1}$	0	0	0	0	0	0	0	0	0
Φ_y	$-y_0$	0	0	α_{20}	1	0	α_{30}	α_{21}	α_{12}	α_{03}	α_{40}
Φ_z	$-z_0$	0	0	β_{20}	β_{11}	β_{02}	β_{30}	β_{21}	β_{12}	β_{03}	β_{40}
φ_u	0	0	0	0	0	$\frac{1}{2}\hat{c}^0_{12}$	$\boxed{\frac{1}{6}\hat{c}^0_{40}}$	$\frac{1}{2}\hat{c}^0_{31}$	$\frac{1}{2}\hat{c}^0_{22}$	$\frac{1}{6}\hat{c}^0_{13}$	$\frac{1}{24}\hat{c}^0_{50}$
φ_v	0	0	$\boxed{\hat{c}^0_{02}}$	0	$\hat{c}^0_{12}$	$\frac{1}{2}\hat{c}^0_{03}$	$\frac{1}{6}\hat{c}^0_{31}$	$\frac{1}{2}\hat{c}^0_{22}$	$\frac{1}{2}\hat{c}^0_{13}$	$\frac{1}{6}\hat{c}^0_{04}$	$\frac{1}{24}\hat{c}^0_{41}$
φ	0	0	0	0	0	$\frac{1}{2}\hat{c}^0_{02}$	0	0	$\frac{1}{2}\hat{c}^0_{12}$	$\frac{1}{6}\hat{c}^0_{03}$	$\frac{1}{24}\hat{c}^0_{40}$
$u\varphi_u$	0	0	0	0	0	0	0	0	$\frac{1}{2}\hat{c}^0_{12}$	0	$\boxed{\frac{1}{6}\hat{c}^0_{40}}$
$v\varphi_u$	0	0	0	0	0	0	0	0	0	$\frac{1}{2}\hat{c}^0_{12}$	0
$u\varphi_v$	0	0	0	0	$\boxed{\hat{c}^0_{02}}$	0	0	$\hat{c}^0_{12}$	$\frac{1}{2}\hat{c}^0_{03}$	0	$\frac{1}{6}\hat{c}^0_{31}$
$v\varphi_v$	0	0	0	0	0	$\boxed{\hat{c}^0_{02}}$	0	0	$\hat{c}^0_{12}$	$\frac{1}{2}\hat{c}^0_{03}$	0
$u^2\varphi_v$	0	0	0	0	0	0	0	$\boxed{\hat{c}^0_{02}}$	0	0	0
$uv\varphi_v$	0	0	0	0	0	0	0	0	$\boxed{\hat{c}^0_{02}}$	0	0
$v^2\varphi_v$	0	0	0	0	0	0	0	0	0	$\boxed{\hat{c}^0_{02}}$	0

	$u^iv^j\ (i+j\leq 3)$	u^4	u^3v	u^2v^2	uv^3	v^4
$u^3\varphi_u$	0	0	$\boxed{\hat{c}^0_{02}}$	0	0	0
$u^2v\varphi_u$	0	0	0	$\boxed{\hat{c}^0_{02}}$	0	0
$uv^2\varphi_u$	0	0	0	0	$\boxed{\hat{c}^0_{02}}$	0
$v^3\varphi_u$	0	0	0	0	0	$\boxed{\hat{c}^0_{02}}$

The equality (3.6) holds if and only if the matrix presented by this table except the first column is full rank. This requires that α_{20} or β_{20} is non-zero. Similarly, (3.7) holds if and only if

$$\begin{vmatrix} y_0 & \alpha_{20} \\ z_0 & \beta_{20} \end{vmatrix} \neq 0. \tag{3.8}$$

Some calculations show that

$$\alpha_{20} = 2\tan\theta_0 \quad \text{and} \quad \beta_{20} = a_{20} + 2a_{11}\tan\theta_0 + a_{02}\tan^2\theta_0.$$

Now we assume that $(\alpha_{20}, \beta_{20}) = (0,0)$. Then we have $\theta_0 = 0$ and $a_{20} = 0$. Hence, $\tilde{\kappa}_1(0,\theta_0) = 0$. Since this opposes the assumption $\tilde{\kappa}_1(0,\theta_0) \neq 0$, the equality (3.6) holds.

We now turn to (3.7). A Calculation shows that

$$\begin{vmatrix} y_0 & \alpha_{20} \\ z_0 & \beta_{20} \end{vmatrix} = \frac{2\Gamma_2^*(\theta_0)\sec\theta_0}{A_2^*(\theta_0)}.$$

By Lemma 2.11, it follows that (3.8) holds if and only if $(0, \theta_0)$ is not a sub-parabolic point relative to $\tilde{\mathbf{v}}_2$ over Whitney umbrella.

(2) Now we can reduce φ to

$$\begin{aligned}\varphi = &-\frac{1}{2}u^2 - \frac{1}{8}{a_{20}}^2u^4 - \frac{1}{2}a_{20}a_{11}u^3v \\ &- \frac{1}{4}(2 + 2{a_{11}}^2 + a_{20}a_{02})u^2v^2 - \frac{1}{2}a_{11}a_{02}uv^3 - \frac{1}{8}{a_{02}}^2v^4 + \cdots .\end{aligned}$$

It follows that φ has an A_3-singularity at $(0,0)$. Next, we show that Φ is not $\mathcal{R}^+$-versal. Since A_3-singularity is 4-determined, Φ is an $\mathcal{R}^+$-versal unfolding of φ if and only if the following equality holds.

$$\mathcal{E}_2 = \langle \varphi_u, \varphi_v \rangle_{\mathcal{E}_2} + \langle \Phi_x|_{\boldsymbol{R}^2\times\{\mathbf{0}\}}, \Phi_y|_{\boldsymbol{R}^2\times\{\mathbf{0}\}}, \Phi_z|_{\boldsymbol{R}^2\times\{\mathbf{0}\}} \rangle_{\boldsymbol{R}} + \langle 1 \rangle_{\boldsymbol{R}} + \langle u, v \rangle^5.$$

Since

$$\begin{aligned}&\Phi_x = u + \cdots, \quad \Phi_y = uv + \cdots, \quad \Phi_z = \frac{1}{2}(a_{20}u^2 + 2a_{11}uv + a_{02}v^2) + \cdots, \\ &\varphi_u = -u + \cdots, \\ &\varphi_v = -\frac{1}{2}\left[a_{20}a_{11}u^3 + (2 + a_{20}a_{02} + 2{a_{11}}^2)u^2v + a_{11}a_{02}uv^2 + {a_{02}}^2v^3\right] + \cdots,\end{aligned}$$

the sum of two ideals $\langle \varphi_u, \varphi_v \rangle_{\mathcal{E}_2} + \langle \Phi_x|_{\boldsymbol{R}^2\times\{\mathbf{0}\}}, \Phi_y|_{\boldsymbol{R}^2\times\{\mathbf{0}\}}, \Phi_z|_{\boldsymbol{R}^2\times\{\mathbf{0}\}} \rangle_{\boldsymbol{R}}$ does not contain v, and Φ is not an $\mathcal{R}^+$-versal unfolding of φ. In a similar way, we can prove that Φ is not $\mathcal{K}$-versal. □

4 Singularities of caustics and fronts of Whitney umbrella

If a smooth function germ $f : (\boldsymbol{R}^2, \mathbf{0}) \to (\boldsymbol{R}, 0)$ is right equivalent to A_2-singularity, then the discriminant set of a $\mathcal{K}$-versal unfolding $F : (\boldsymbol{R}^2 \times \boldsymbol{R}^3, \mathbf{0}) \to (\boldsymbol{R}, 0)$ of f is locally diffeomorphic to the discriminant set of the following unfolding:

$$G(u, v, x, y, z) = u^3 \pm v^2 + x + yu.$$

The singularity of the discriminant set of G is the cuspidal edge. Here, the *cuspidal edge* is the image of a map germ $\mathcal{A}$-equivalent to $(u, v) \mapsto (u, v^2, v^3)$ at the origin. The picture of the cuspidal edge is shown in Figure 2 (i). Similarly, if a smooth function f is right equivalent to A_3-singularity, then the discriminant (resp. bifurcation) set of a $\mathcal{K}$-versal (resp. $\mathcal{R}^+$) unfolding F is locally diffeomorphic to the discriminant (resp. bifurcation) set of the following unfolding:

$$G(u, v, x, y, z) = u^4 \pm v^2 + x + yu + zu^2 \quad (\text{resp. } \hat{G}(u, v, x, y, z) = u^4 \pm v^2 + xu^2 + yu).$$

The singularity of the discriminant set of G (resp. bifurcation set of $\hat{G}$) is the swallowtail (resp. cuspidal edge). Here, the *swallowtail* is the image of a map germ $\mathcal{A}$-equivalent to $(u, v) \mapsto (u, 3v^4 + uv^2, 4v^3 + 2uv)$ at the origin. The picture of the swallowtail is shown in Figure 2 (ii). Therefore, Theorem 3.7 leads to the following.

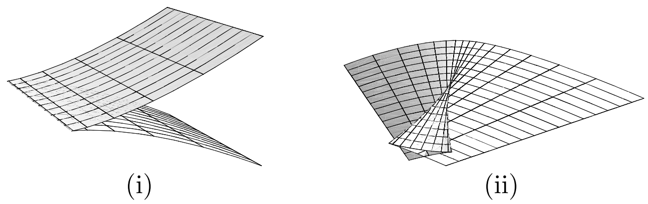


Figure. 2: (i) Cuspidal edge; (ii) swallowtail.

Theorem 4.1. *Let $g : (\boldsymbol{R}^2, \mathbf{0}) \to (\boldsymbol{R}^3, \mathbf{0})$ be given in the normal form of Whitney umbrella. Suppose that $(0, \theta_0)$ is not a parabolic point over Whitney umbrella where $\theta_0 \in (-\pi/2, \pi/2)$.*

(1) *Suppose that $(0, \theta_0)$ is not a ridge point relative to $\tilde{\mathbf{v}}_1$ over Whitney umbrella. Then the front of g at distance $1/|\tilde{\kappa}_1(0, \theta_0)|$ is locally diffeomorphic to a cuspidal edge at $\tilde{\mathbf{n}}(0, \theta_0)/\tilde{\kappa}_1(0, \theta_0)$.*

(2) *Suppose that $(0, \theta_0)$ is a first order ridge point relative to $\tilde{\mathbf{v}}_1$ over Whitney umbrella. Then the caustic of g is locally diffeomorphic to a cuspidal edge at $\tilde{\mathbf{n}}(0, \theta_0)/\tilde{\kappa}_1(0, \theta_0)$. Additionally, if $(0, \theta_0)$ is not a sub-parabolic point relative to $\tilde{\mathbf{v}}_2$ over Whitney umbrella, then the front of g at distance $1/|\tilde{\kappa}_1(0, \theta_0)|$ is locally diffeomorphic to a swallowtail at $\tilde{\mathbf{n}}(0, \theta_0)/\tilde{\kappa}_1(0, \theta_0)$.*

Theorem 4.1 (1) and Proposition 2.16 imply that the front of g has at most four cuspidal edge singularities on C_k. Similarly, Theorem 4.1 (2) and Lemma 2.7 imply that the front of g has at most four swallowtail singularities on C_k. If for example g is given in the normal form of Whitney umbrella determined by coefficients

$$(a_{20}, a_{11}, a_{02}, a_{30}, a_{21}, a_{12}, a_{03}, b_3) = (3, 0, 1, -7, 0, 8/3, 0, 1),$$

then the front of g at distance $1/\sqrt{2}$ has four swallowtail singularities on $C_{\sqrt{2}}$.

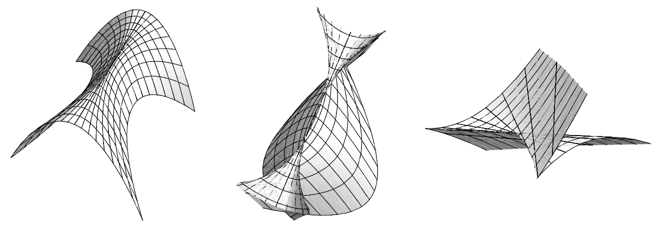
In Theorem 4.3 below, we give the criteria for the cuspidal lips, the cuspidal beaks and the cuspidal butterfly of fronts of Whitney umbrella. To prove Theorem 4.3, we use the criteria for these singularities of parallel surfaces of regular surfaces, which shown in the authors' previous work [7]. We present these criteria as the following:

Theorem 4.2 ([[7], theorem 5·3])**.** *Suppose that $g : U \to \boldsymbol{R}^3$ is a smooth map which defines a regular surface in $\boldsymbol{R}^3$ and that g^t denotes the parallel surface of g at distance t. Assume that $\kappa_i(p) \neq 0$.*

(1) *Assume that $g(p)$ is a first order ridge point relative to the principal vector $\mathbf{v}_i$ and a sub-parabolic point relative to the other principal vector, and* $\det(\text{Hess } \kappa_i(p)) > 0$ *(resp. < 0), where* Hess κ_i *denotes the Hessian matrix of κ_i. Then the parallel surface g^t at distance $t = 1/\kappa_i(p)$ is locally diffeomorphic to a cuspidal lips (resp. cuspidal beaks) at $g^t(p)$.*

(2) *Assume that $g(p)$ is a second order ridge point relative to the principal vector $\mathbf{v}_i$ and not a sub-parabolic point relative to the other principal vector. Then the parallel surface g^t at distance $t = 1/\kappa_i(p)$ is locally diffeomorphic to a cuspidal butterfly at $g^t(p)$.*

Here, the *cuspidal lips* is the image of a map germ $\mathcal{A}$-equivalent to $(u, v) \mapsto (3u^4 + 2u^2v^2, u^3 + uv^2, v)$ at the origin, the *cuspidal beaks* is the image of a map germ $\mathcal{A}$-equivalent to $(u, v) \mapsto (3u^4 - 2u^2v^2, u^3 - uv^2, v)$ at the origin, and the *cuspidal butterfly* is the image of a map germ $\mathcal{A}$-equivalent to $(u, v) \mapsto (4u^5 + u^2v, 5u^4 + 2uv, v)$ at the origin. The pictures of these singularities are shown in Figure 3.



(i) (ii) (iii)

Figure. 3: (i) Cuspidal lips; (ii) cuspidal beaks; (iii) cuspidal butterfly.

Theorem 4.3. *Let $g : (\boldsymbol{R}^2, \mathbf{0}) \to (\boldsymbol{R}^3, \mathbf{0})$ be given in the normal of Whitney umbrella. Suppose that $(0, \theta_0)$ is not a parabolic point over Whitney umbrella, where $\theta_0 \in (-\pi/2, \pi/2)$.*

(1) *Assume that $(0, \theta_0)$ is a first order ridge point relative to $\tilde{\mathbf{v}}_1$ and sub-parabolic point relative to $\tilde{\mathbf{v}}_2$ over Whitney umbrella, and that* $\det(\mathrm{Hess}\,\tilde{\kappa}_1(0, \theta_0)) > 0$ (*resp.* < 0). *Then the front of g at distance $1/|\tilde{\kappa}_1(0, \theta_0)|$ is locally diffeomorphic to a cuspidal lips* (*resp. cuspidal beaks*) *at $\tilde{\mathbf{n}}(0, \theta_0)/\tilde{\kappa}_1(0, \theta_0)$.*

(2) *Assume that $(0, \theta_0)$ is a second order ridge point relative to $\tilde{\mathbf{v}}_1$ and not a sub-parabolic point relative to $\tilde{\mathbf{v}}_2$ over Whitney umbrella. Then the front of g at distance $1/|\tilde{\kappa}_1(0, \theta_0)|$ is locally diffeomorphic to a cuspidal butterfly at $\tilde{\mathbf{n}}(0, \theta_0)/\tilde{\kappa}_1(0, \theta_0)$.*

Proof. We shall prove the assertion (1). The proof of the assertion (2) is similar and we omit the detail. We set $\tilde{g} = g \circ \tilde{\pi} : \boldsymbol{R} \times S^1 \to \boldsymbol{R}^3$ and $\tilde{g}^t(r, \theta) = \tilde{g}(r, \theta) + t\tilde{\mathbf{n}}(r, \theta)$ $(t \neq 0)$. Then $\tilde{g}^t$ is the parallel surface of $\tilde{g}$ at distance t, whose image is the front of g at distance $|t|$. The principal radii of $\tilde{g}^t$ are given by

$$\frac{1}{\tilde{\kappa}_i^t} = \frac{1}{\tilde{\kappa}_i} - t. \tag{4.1}$$

We consider two parallel surfaces of $\tilde{g}$: one is at distance $t_0 = 1/\tilde{\kappa}_1(0, \theta_0)$ by $\tilde{g}^{t_0}$, and the other is at distance $t_1 \neq t_0$ by $\tilde{g}^{t_1}$. We remark that $\tilde{g}^{t_0}$ is the parallel surface of $\tilde{g}^{t_1}$ at distance $1/\tilde{\kappa}_1(0, \theta_0) - t_1 = 1/\tilde{\kappa}_1^{t_1}(0, \theta_0)$. Now suppose that $\tilde{g}^{t_1}(0, \theta_0)$ is a first order ridge point relative to $\tilde{\mathbf{v}}_1^{t_1}$ and a sub-parabolic point relative to $\tilde{\mathbf{v}}_2^{t_1}$, that is,

$$\tilde{\mathbf{v}}_1^{t_1}\tilde{\kappa}_1^{t_1}(0, \theta_0) = 0, \quad (\tilde{\mathbf{v}}_1^{t_1})^2\tilde{\kappa}_1^{t_1}(0, \theta_0) \neq 0, \quad \text{and} \quad \tilde{\mathbf{v}}_2^{t_1}\tilde{\kappa}_1^{t_1}(0, \theta_0) = 0, \tag{4.2}$$

and that $\det(\mathrm{Hess}\ \tilde{\kappa}_1^{t_1}(0, \theta_0)) > 0$ (resp. < 0). Then it follows from Theorem 4.2 that $\tilde{g}^{t_0}$ is locally diffeomorphic to a cuspidal lips (resp. cuspidal beaks) at $\tilde{g}^{t_0}(0, \theta_0)$.

Since the lines of curvature on all parallel surfaces correspond to one another (cf. [6] p. 121, see also [26] p. 159), it follows that $\tilde{\mathbf{v}}_1^{t_1}$ (resp. $\tilde{\mathbf{v}}_2^{t_1}$) is parallel to $\tilde{\mathbf{v}}_1$ (resp. $r^2\tilde{\mathbf{v}}_2$) at $(0,\theta_0)$. This and (4.1) show that the condition (4.2) holds if and only if

$$\tilde{\mathbf{v}}_1\tilde{\kappa}_1(0,\theta_0)=0, \quad \tilde{\mathbf{v}}_1^2\tilde{\kappa}_1(0,\theta_0)\neq 0, \quad \text{and} \quad r^2\tilde{\mathbf{v}}_2\tilde{\kappa}_1(0,\theta_0)=0.$$

These conditions are equivalent to that $(0,\theta_0)$ is a first order ridge point relative to $\tilde{\mathbf{v}}_1$ and a sub-parabolic point relative to $\tilde{\mathbf{v}}_2$ over Whitney umbrella. Moreover, it follows from (4.1) that the sign of $\det(\text{Hess }\tilde{\kappa}_1^{t_1}(0,\theta_0))$ is the same as the sign of $\det(\text{Hess }\tilde{\kappa}_1(0,\theta_0))$. Thus we have completed the proof. □

We obtain criteria for the cuspidal edge, the swallowtail, the cuspidal lips, the cuspidal beaks, and the cuspidal butterfly of fronts of Whitney umbrella, and the criterion for the cuspidal edge of caustics of Whitney umbrella except at singular point of Whitney umbrella (Table 1). Finding a normal form of the caustic there is an open problem.

Table 1: Criteria for singularities of caustics and fronts of Whitney umbrella.

		caustic	front
no ridges	—	non singular	cuspidal edge
1-ridges	not sub-parabolic	cuspidal edge	swallowtail
	sub-parabolic	—	cuspidal lips or beaks if CPC is Morse
2-ridges	not sub-parabolic	—	cuspidal butterfly

Example 4.4. Let g be given in the normal form of Whitney umbrella determined by coefficients

$$(a_{20},a_{11},a_{02},a_{30},a_{21},a_{12},a_{03},b_3)=(0,0,1,0,1,-1,0,0).$$

Then we obtain

$$\begin{aligned}
\tilde{\mathbf{n}}(0,\theta)&=(0,-\sin\theta,\cos\theta),\\
\tilde{\kappa}_1(0,\theta)&=-\sin\theta\tan\theta,\\
\Gamma_3(\theta)&=3\cos^3\theta\sin\theta-3\cos^2\theta\sin^2\theta,\\
\Gamma_3^*(\theta)&=2\cos^2\theta\sin\theta+\sin^3\theta,\\
\Gamma_4(\theta)&=-3\cos^6\theta+3\cos^5\theta\sin\theta-3\cos^4\theta\sin^2\theta+12\cos^2\theta\sin^4\theta+3\sin^6\theta.
\end{aligned}$$

We set $k=1/\sqrt{2}$. The CPC line Σ_k and the exceptional set $X=\pi^{-1}(0,0)$ meet at two points $(r,\theta)=(0,\pm\pi/4)$. Therefore, the front of g at distance $\sqrt{2}$ has two singular points on C_k at

$$\frac{\tilde{\mathbf{n}}(0,\pi/4)}{\tilde{\kappa}_1(0,\pi/4)}=(0,1,-1) \quad \text{and} \quad \frac{\tilde{\mathbf{n}}(0,-\pi/4)}{\tilde{\kappa}_1(0,-\pi/4)}=(0,-1,1).$$

Conditions Γ_3, Γ_3^*, and Γ_4 are shown in Table 2. From Table 2, it follows that $(0,\pi/4)$ is the first order ridge point relative to $\tilde{\mathbf{v}}_1$ and not sub-parabolic point relative to $\tilde{\mathbf{v}}_2$ over Whitney umbrella, and $(0,-\pi/4)$ is neither the ridge point relative to $\tilde{\mathbf{v}}_1$ nor sub-parabolic

Table 2: Conditions for points to be the ridge or sub-parabolic point.

	$\Gamma_3(\theta)$	$\Gamma_3^*(\theta)$	$\Gamma_4(\theta)$
$\theta = \pi/4$	0	$3/2$	$3/2$
$\theta = -\pi/4$	$-3/2$	$3/4$	$-$

point relative to $\tilde{\mathbf{v}}_2$ over Whitney umbrella. Hence, the front of g at distance $\sqrt{2}$ is locally diffeomorphic to the swallowtail at $(0, 1, -1)$. Moreover, this front is locally diffeomorphic to the cuspidal edge at $(0, -1, -1)$. The picture of this front is shown in Figure 4. The thick circle in Figure 4 is C_k.

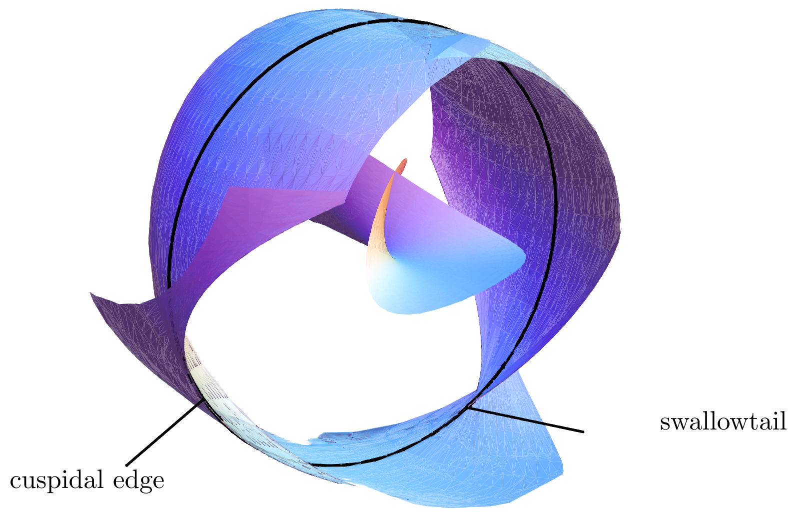


Figure. 4: The front of Whitney umbrella g as in Example 4.4 at distance $\sqrt{2}$.

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Journal *of* Singularities
Volume 4 (2012), 68-73

received 4 January 2011
in revised form 11 February 2012
DOI: 10.5427/jsing.2012.4d

KAHN'S CORRESPONDENCE AND COHEN–MACAULAY MODULES OVER ABSTRACT SURFACE AND CURVE SINGULARITIES

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Abstract. We generalize Kahn's correspondence between Cohen–Macaulay modules over normal surface singularities over an algebraically closed field and vector bundles over some projective curves to abstract surface singularities, which need not be algebras over a field. As a consequence, we also generalize to the abstract case the Drozd–Greuel criterion for tameness of curve singularities [4].

1. Introduction

In [7] Kahn established a connection between the categories of (maximal) Cohen–Macaulay modules over normal surface singularities and vector bundles over a special divisor on the exceptional locus of the resolution of this singularity. In the case of arbitrary normal surface singularity Kahn's conditions from [7, Theorem 1.4] are quite cumbersome and difficult to deal with. However, in the case of minimally elliptic singularities there is a simple and explicit description of this connection. Kahn himself [7] and Drozd, Greuel and Kashuba [6] used this result to obtain a classification of Cohen–Macaulay modules over simple elliptic and cusp singularities, to establish a tame-wild criterion for minimally elliptic singularities, as well to study Cohen–Macaulay modules over some curve singularities. In particular, in [6] it is obtained a classification of Cohen–Macaulay modules over the curve singularities of type T_{pq}, which gave a new proof that they are Cohen–Macaulay tame. Previously this result was obtained quite implicitly using deformations and semicontinuity theorems [4].

In this paper we extend these results to a more general situation. Namely, we consider *abstract surface and curve singularities*, that is complete noetherian rings of Krull dimension 2 or 1, which need not be algebras over an algebraically closed field or over a field at all. We establish that Kahn's results hold for this situation too, though some details of the proof change. Using them, we also generalize the tame-wild criteria from [4, 6] to the abstract situation. Unfortunately, the case when the residue field is of characteristic 2 remains unconsidered, since we have to use the suspension trick of Knörrer [10] which does not work in this case. We also have to change the definition of T_{pq} singularities using a parametrization description instead of equations. It is necessary since the classification of such singularities in positive characteristic, moreover, in the abstract situation is still inaccessible, though some important results have been obtained in [2]. Note that in the abstract case one cannot use the deformation arguments from [4], since there are no appropriate results about semicontinuity (in the paper [3] only the case of algebras over an algebraically closed field is considered and this restriction is unavoidable there).

2. The result of Kahn

We are going to generalize the results of Kahn [7] in the following situation. Let (X, x) be a spectrum of a local complete and normal noetherian ring $(R, \mathfrak{m})$ of Krull dimension 2 with maximal ideal $\mathfrak{m}$ and residue field k. We call such schema an (*abstract*) *normal surface singularity*, since in general the ring R is not supposed to be an algebra over the field. Such singularity is *isolated*, that means its closed point x, which corresponds to $\mathfrak{m}$, is a unique singular point of X. It is known from [11] that there exists a *resolution* of (X, x), that is a birational projective morphism $\pi : (\widetilde{X}, E) \to (X, x)$, where $\widetilde{X}$ is smooth and π induces an isomorphism $\widetilde{X} \setminus E \simeq X \setminus \{x\}$, where $E = \pi^{-1}(x)_{\mathrm{red}}$ is an exceptional locus. The resolution is obtained by finite sequence $\overline{X} = X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow ... \leftarrow X_n$ where $\overline{X}$ is normalization of X and each X_{i+1} is obtained from X_i by blowing up all the singular points of X_i and normalizing the resulting surface $\widetilde{X} = \overline{X}_n$. Put $E = \bigcup_{i=1}^{l} E_i$, where E_i are the irreducible components of E (all of them are projective curves over k). Note that $\widetilde{X}$ is minimal resolution if and only if there is no component E_i which is a smooth rational curve with self-intersection index $E_i \cdot E_i = -1$. Recall some necessary definitions we need (for details cf. [7]).

Definition 1. A module M over R is called *reflexive* if $M^{\vee\vee} \simeq M$, where $M^{\vee} = \mathrm{Hom}_R(M, R)$. In our case such modules coincide with maximal Cohen-Macaulay modules over R and we denote by $\mathrm{MCM}(X)$ the category of reflexive R-modules.

A locally free sheaf $\mathcal{F}$ on $\widetilde{X}$ is called *full* if $\mathcal{F} \simeq (\pi^* M)^{\vee\vee}$ for some $M \in \mathrm{MCM}(X)$.

An effective divisor $Z > 0$ on $\widetilde{X}$ is called a *reduction cycle* if

(i) $\mathcal{O}_Z(-Z)$ is generically generated by global sections (i.e. generated outside a finite set);

(ii) $\mathrm{H}^1(E, \mathcal{O}_Z(-Z)) = 0$;

(iii) $\omega_Z^{\vee}$ is generically generated by global sections, where $\omega_Z = \omega_{\widetilde{X}}(Z) \otimes \mathcal{O}_Z$.

For a reduction cycle Z the functor $R_Z : \mathrm{MCM}(X) \to \mathrm{VB}(Z)$ from category of reflexive modules to category of locally free sheaves on Z is defined by $R_Z(M) = (\pi^* M)^{\vee\vee} \otimes \mathcal{O}_Z$.

Theorem 2. [7, Theorem 1.4] *Let $\pi : (\widetilde{X}, E) \to (X, x)$ be a resolution of a normal surface singularity, and let Z be a reduction cycle on $\widetilde{X}$. Then the functor R_Z maps non-isomorphic objects from* $\mathrm{MCM}(X)$ *to non-isomorphic ones from* $\mathrm{VB}(Z)$ *and a vector bundle $F \in \mathrm{VB}(Z)$ is isomorphic to $R_Z M$ for some reflexive module M if and only if it is generically generated by global sections and there is an extension of F to a vector bundle F_2 on $2Z$ such that the exact sequence*

$$0 \longrightarrow F(-Z) \longrightarrow F_2 \longrightarrow F \longrightarrow 0$$

induces a monomorphism $\mathrm{H}^0(E, F(Z)) \to \mathrm{H}^1(E, F)$.

The proof of this theorem is divided into the following two propositions. Note that in our case the proof of Proposition 3 is absolutely analogous to the original one, so we can omit it here.

Proposition 3. [7, Proposition 1.6] *Let $Z > 0$ be a cycle on $\widetilde{X}$ such that $\mathcal{O}_Z(-Z)$ is generically generated by global sections and* $\mathrm{H}^1(E, \mathcal{O}_Z(-Z)) = 0$. *Assume that $\mathcal{F}$ is a locally free sheaf on $\widetilde{X}$ and denote by $F = \mathcal{F} \otimes \mathcal{O}_Z$ its restriction to Z. Then, $\mathcal{F}$ is full if and only if*

(i) F is generically generated by global sections;

(ii) The coboundary map $\mathrm{H}^0(E, F(Z)) \to \mathrm{H}^1(E, F)$ *is injective.*

Proposition 4. [7, Proposition 1.9] *Let Z be a reduction cycle on $\widetilde{X}$ and assume that F is a locally free sheaf on Z such that*

(i) F is generically generated by global sections;

(ii) There is an extension of F to a locally free sheaf over $\mathcal{O}_{2Z}$ such that $\mathrm{H}^0(E, F(Z)) \to \mathrm{H}^1(E, F)$ *is injective.*

Then there is a full sheaf $\mathcal{F}$ on $\widetilde{X}$ with $\mathcal{F} \otimes \mathcal{O}_Z \cong F$. The extension of F to a full sheaf on $\widetilde{X}$ is unique up to isomorphism.

Proof. We follow the proof of Kahn with some obvious changes, which are necessary due to the fact that an underlying field may not exist, or if existing it may not be algebraically closed. We omit most of details that shall not be changed in our case. Fix an extension F_2 of sheaf F. The sheaf $\mathcal{F}$ from Proposition 3 is constructed as the projective limit $\varprojlim F_n$ of sheaves F_n, where sheaf F_{n+1} is such locally free extension of the sheaf F_n over $\mathcal{O}_{(n+1)Z}$ that the following sequence is exact

$$0 \to F(-nZ) \to F_{n+1} \to F_n \to 0.$$

Note that these extensions exist because the obstructions for the existence belong to $\mathrm{H}^2(E, F^\vee \otimes F(-nZ))$, which is zero by dimensional reasons.

To prove the uniqueness it is enough to show that at every step of the construction given above the next sheaf is defined uniquely up to isomorphism.

We can consider an extension of the sheaf F_n to a locally-free sheaf F_{n+1} over $\mathcal{O}_{(n+1)Z}$ as an element of the group $\mathrm{Ext}^1_{\mathcal{O}_{(n+1)Z}}(F_n, F(-nZ))$. Choosing an element e from this space such that e is locally-free over $\mathcal{O}_{(n+1)Z}$, we obtain that all locally-free extensions of F_n correspond to the points of the coset $e + \mathrm{Ext}^1_{\mathcal{O}_{nZ}}(F_n, F(-nZ))$. There is an obvious action of the group of automorphisms of the sheaf F_n on the group $\mathrm{Ext}^1_{\mathcal{O}_{(n+1)Z}}(F_n, F(-nZ))$. Namely, if e is an extension of F_n by $F(-nZ)$ and φ is an automorphism of F_n then by the product $\varphi^* e$ we mean the pull-back of e with respect to φ. Then it is enough to show that an arbitrary locally-free extension is of the form $\varphi^* e$ for some $\varphi \in \mathrm{Aut}(F_n)$. Put $\varphi = \mathrm{id} + h$, where $h \in \mathrm{End}(F_n)$. Then $\varphi^* e = e + h^* e$ and we can consider the product $h^* e$ as the element of $\mathrm{H}^1(E, F^\vee \otimes F(-nZ)) \cong \mathrm{Ext}^1_{\mathcal{O}_{nZ}}(F_n, F(-nZ))$. We have that $h^* e = \delta(h)$, where the map $\delta : \mathrm{H}^0(E, F_n^\vee \otimes F_n) \to \mathrm{H}^1(F^\vee \otimes F(-nZ))$ is induced by the exact sequence $0 \to F^\vee \otimes F(-nZ)) \to F_{n+1}^\vee \otimes F_{n+1} \to F_n^\vee \otimes F_n \to 0$. It is possible to consider only those h that belong to the set $\mathrm{H}^0(E, F^\vee \otimes F(-(n-1)Z))$, which is contained in the radical of $\mathrm{End}(F_n)$ in the case $n > 1$. Note that the natural restriction of the coboundary map to $\delta' : \mathrm{H}^0(E, F^\vee \otimes F(-(n-1)Z)) \to \mathrm{H}^1(E, F^\vee \otimes F(-nZ))$ is surjective by [7, Claim, p.148]. This proves the statement for $n > 1$.

Let us consider the case $n = 1$. The reduction cycle Z is a scheme over the ring $\overline{R} = R/\mathfrak{m}^l$ for some $l > 0$. Put $B = \mathrm{End}_{\overline{R}}(F)$. It is necessary to note that not every element $h \in \mathrm{H}^0(E, F^\vee \otimes F)$ defines an invertible morphism $\varphi = \mathrm{id} + h$, because $\mathrm{H}^0(E, F^\vee \otimes F) \cong B$ which is not a vector space and there exists locally-free extensions of F which are not full. Consider at first the situation when the underlying field exists (this means $l = 1$) and it is infinite. The set of those $h \in B$ for which $\mathrm{id} + h$ is invertible form an open and dense subset in B. Since δ' is linear and surjective, the $\mathrm{Aut}(F)$-orbit of e is open and dense in $e + \mathrm{Ext}^1_{O_Z}(F, F(-Z))$ and contains only full extensions of F. Therefore, any two such orbits have common points, hence, coincide.

Consider now the case when the field k is finite. Consider the sheaf F over $\mathcal{O}_Z$ and its locally free extension F_2 over $\mathcal{O}_{2Z}$. Consider the schema $Y = X \otimes_k K$ and its resolution $\widetilde{Y} = \widetilde{X} \otimes_k K$, where K is an algebraic closure of k. Reduction cycle on $\widetilde{Y}$ is $\overline{Z} = Z \otimes_k K$. The sheaf F corresponds then to the sheaf $\overline{F}$ over $\mathcal{O}_{\overline{Z}}$ such that for every open U on Z $\overline{F}(\overline{U}) \cong F(U) \otimes_k K$. In the same way the sheaf $\overline{F}_2$ corresponds to F_2 and it is a locally free extension of $\overline{F}$ over $\mathcal{O}_{2\overline{Z}}$. For the scheme $\widetilde{Y}$ and the sheaf $\overline{F}$ Proposition 4 holds. In particular if F_2' is a locally free extension of F over $\mathcal{O}_{2Z}$, not equal to F_2, then $\overline{F}_2 \cong \overline{F}_2'$. Denote by $E(F)$ the set of isomorphism classes of locally free extension of F over $\mathcal{O}_{2Z}$. Fix some sheaf $F_2 \in E(F)$. The group $G = \mathrm{Gal}(K/k)$ acts on $\mathrm{Aut}_K(\overline{F}_2)$ in an obvious way and $E(F)$ is the set of all K/k-forms of sheaf F_2 in the sense of [14]. It is easy to see (analogous to [14]) that there exists an injective map of sets $E(F) \longrightarrow \mathrm{H}^1(G, \mathrm{Aut}_K(\overline{F}_2))$. At last, we are going to show that $\mathrm{H}^1(G, \mathrm{Aut}_K(\overline{F}_2)) = \{1\}$. Put $\overline{B} = \mathrm{End}_K(\overline{F}_2)$. We have that $\overline{B} = B \otimes_k K$, $\overline{B}^\times = \mathrm{Aut}_K(\overline{F}_2)$ and $\overline{B}^\times/(1 + \mathrm{rad}\,\overline{B}) \cong \prod \mathrm{GL}_{n_i}(K)$.

On the other hand there exists a chain of subgroups $1 + \operatorname{rad} \overline{B} = G_0 \supset G_1 \supset \ldots \supset G_m = 0$ such that $G_i/G_{i+1} \cong K^+$. From the fact that $\mathrm{H}^1(G, \mathrm{GL}_n(K)) = \mathrm{H}^1(G, K^+) = \{1\}$ and the exact sequence for Galois cohomology we are done.

Suppose now that there is no underlying field. Let e_1 be some full extension. Then, by the result obtained above $e_1 \equiv e_0 + h^* e_0 (\mathrm{mod}\, \mathfrak{m})$ for some $h \in B$. This means that $e_1 = e_0 + h^* e_0 + u$ for some $u \in \mathfrak{m} \operatorname{Ext}^1_{O_Z}(F, F(-Z))$. Since δ' is surjection, $u = \delta'(v) = v^* e_0$ for some $v \in \mathfrak{m}B$, so$e_1 = e_0 + h^* e_0 + v^* e_0 = e_0 + (h+v)^* e_0 = \varphi^* e_0$ for the invertible $\varphi = \mathrm{id} + (h+v)$. □

For minimally elliptic singularities we fix some notations. We define a *minimally elliptic singularity* such that R be a Gorenstein ring, i.e. $\omega_X \simeq \mathcal{O}_X$ and $\mathrm{H}^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) \simeq K$, where K is the residue field of $\mathrm{H}^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}})$. As in the original work of Kahn we can choose in a role of a reduction cycle Z on $\widetilde{X}$ the fundamental cycle, for which we have $\mathcal{O}_Z \simeq \omega_Z$. The restriction char $K = 0$ (which allows to use the Grauert-Riemenschneider vanishing theorem) can be omitted since $\mathrm{H}^1(E, \mathcal{O}_Z(-Z))$ by Serre's duality is dual to $\mathrm{H}^0(E, \omega_Z(Z)) = \mathrm{H}^0(E, \mathcal{O}_Z(Z)) = 0$, because $\deg \mathcal{O}_Z(Z) = Z \cdot Z < 0$.

Note that in the minimally elliptic case Z coincides with the *cohomological cycle* [12, p. 99], that is the unique cycle Z_1 on $\widetilde{X}$ such that Z_1 is smallest among all divisors supported on E and the length of the module $\mathrm{H}^1(E, \mathcal{O}_{Z_1})$ takes the maximal value. From isomorphisms $R^1\pi_* \mathcal{O}_{\widetilde{X}} \simeq \mathrm{H}^1(E, \mathcal{O}_Z)$, $R^1\pi_*\mathcal{O}_{\widetilde{X}} \simeq \mathrm{H}^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}})$ we have that $\mathrm{H}^1(E, \mathcal{O}_Z) \simeq \mathrm{H}^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}})$ and by Serre's duality $\mathrm{H}^1(E, \mathcal{O}_Z)$ is dual to $\mathrm{H}^0(E, \mathcal{O}_Z)$, so $\mathrm{H}^0(E, \mathcal{O}_Z) \simeq K$. For a full sheaf $\mathcal{F}$ on $\widetilde{X}$ and the locally free sheaf F of $\mathcal{O}_Z$-modules define h^i as the dimension over K of the i-th cohomology.

Theorem 5. [7, Theorem 2.1] *Let $\pi : (\widetilde{X}, E) \to (X, x)$ be a resolution of a minimally elliptic singularity and let Z be the fundamental cycle on $\widetilde{X}$. Then there is a bijective correspondence, induced by R_Z, between isomorphism classes of non-projective indecomposable reflexive R-modules and isomorphism classes of locally free sheaves of $\mathcal{O}_Z$-modules of the form $n\mathcal{O}_Z \oplus G$ with G indecomposable and*

(i) G is generically generated by global sections;

(ii) $\mathrm{H}^1(E, G) = 0$;

(iii) $n = \mathrm{h}^0(E, G(Z))$.

The proof is analogous to the Kahn's situation.

3. Curve singularities

During this section we suppose that A is a reduced, complete, local and noetherian ring of Krull dimensional 1 with maximal ideal $\mathfrak{m}$ and residue field k. Such a ring A is said to be a *curve singularity*. By Cohen's structure theorem A is a finite algebra over a discrete valuation ring $\mathcal{O}$ with residue field equal to k. We assume $\mathcal{O}$ and k are of characteristic not equal to 2 and k is perfect (another possible assumption for k is that char $k \neq 2, 3$). For uniformizing parameter $x \in \mathcal{O}$, $p, q \in \mathbb{N}$ such that $\frac{1}{p} + \frac{1}{q} \leqslant \frac{1}{2}$ and $\alpha \in \mathcal{O}^\times$ define the singularities T_{pq} by the parametrization in the following way:

if p and q are both odd, T_{pq} is isomorphic to the subalgebra of $\mathcal{O}^2$ generated by (x^2, x^{q-2}) and $(\alpha x^{p-2}, x^2)$;

if p is odd and q is even, T_{pq} is isomorphic to the subalgebra of $\mathcal{O}^3$ generated by $(x, x, \alpha x^{p-2})$ and $(0, x^{q/2-1}, x^2)$;

if p and q are both even, T_{pq} is isomorphic to the subalgebra of $\mathcal{O}^4$ generated by $(x, x, \alpha x^{p/2-1}, 0)$ and $(x^{q/2-1}, 0, x, x)$ if $(p, q) \neq (4, 4)$ and by $(x, 0, x, x)$ and $(0, x, x, \alpha x)$ with $\alpha \neq 0, 1$ if $(p, q) = (4, 4)$.

From the parametrization given above we can represent singularity T_{pq} as the quotient $\mathcal{O}[[y]]/(f)$, where $f = (x^2 - y^{q-2})(x^{p-2} - \alpha^2 y^2)$ and note that it is isolated. Also note that in the case when $\mathcal{O}$ is the ring of power series over a field of characteristic zero or char k doesn't divide p or q (by [2]) this definition of T_{pq} coincides with the standard one from the Arnold's classification list [1]. We are going to prove that the following theorem still holds:

Theorem 6. [4, Theorem 1] *Let A be a curve singularity of infinite Cohen-Macaulay type. Then it is of tame type if and only if it dominates one of the singularities T_{pq}.*

For proving sufficiency we must prove the tameness of T_{pq}. We can do it in the following way.

Define the singularity T_{pq2} as $\mathcal{O}[[y,z]]/(z^2 + f)$. The minimal resolution of T_{pq2} is given for instance in [8] and [9], where it is obtained by blowing-up and normalizations and proved that the exceptional locus of a minimal resolution of T_{pq2} is either elliptic curve or a so-called *cyclic configuration*. The description of the vector bundles on cyclic configurations is given in [5] and according to our case this description implies that indecomposable vector bundles on E are in one-to-one correspondence to triples of the form $(\mathbf{d}, m, p(t))$, where $\mathbf{d}$ is some finite sequence of integers called *aperiodic* [6], m is positive integer and $p(t) \in k[t] \setminus \{0\}$. Applying the result of Kahn we immediately obtain that T_{pq2} is of tame type.

The tameness of T_{pq} then follows from the tameness of T_{pq2} by the results of Knörrer. Namely, there exist functors from the categories of matrix factorizations (we keep the notations of paper [10]) $G : \mathcal{MF}(f) \to \mathcal{MF}(f+z^2)$ and $\mathrm{Rest} : \mathcal{MF}(f+z^2) \to \mathcal{MF}(f)$ such that every $X \in \mathcal{MF}(f)$ is a direct summand in $\mathrm{Rest}\circ G(X)$. This implies that one-parameter families of Cohen–Macaulay modules over T_{pq2} exhaust the category of Cohen–Macaulay modules over T_{pq}, so T_{pq} is of tame type.

For proving necessity we produce similar observations as in [4]. Let $\bar{A}$ be the normalization of A in its full ring of fractions and $\bar{A} = \prod_{i=1}^{s} A_i$ be its decomposition into a product of discrete valuation rings. By t_i we denote the uniformizing element of the ring A_i. For a ring B such that $A \subset B \subset \bar{A}$ let $B/\mathfrak{m}B \simeq B_1 \times ... \times B_m$, where each B_i is a local algebras of dimension d_i. Set $d(B) = d_1 + ... + d_m$. Let e_i be the idempotent of $\bar{A}$ such that $A_i = e_i A$, $t = (t_1, ..., t_s)$ and $\theta \in \mathfrak{m}$ such that $\bar{A}\mathfrak{m} = \theta\bar{A}$. Put $A' = t\bar{A} + A$, $A'' = \theta t\bar{A} + A$ and $A'_i = A' + \mathcal{O}e_i$. In this situation the overring conditions of the [4, Theorem 3] rewrites in the following form

Theorem 7. [4, Theorem 3] *Let A be a curve singularity of infinite CM type. The following condition are necessary and sufficient for A to be of tame CM type:*

(O1) $d(\bar{A}) \leqslant 4$ *and* $t^2\bar{A} \subset \mathfrak{m}$,

(O2) $d(A') \leqslant 3$ *and* A'_i *has no local 3-dimensional factor,*

(O3) *if* $d(\bar{A}) = 3$, *then* $d(A'') \leqslant 2$.

The conditions (O1-O3) are invariant under separable field extensions of k and under our assumptions for k we can choose separable extension K of k such that the residue fields of all algebras above are equal to K. Then we obtain the original conditions from [4, Theorem 3]. The proof of the theorem is analogous to the original one. Namely, necessity follows from geometric observation just as in [4]. Then using the parametrization of T_{pq} we prove that if A satisfies the conditions (O1-O3) of the previous theorem then it dominates some of T_{pq}.

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Journal *of* Singularities
Volume 4 (2012), 74-103

received 6 June 2011
in revised form 15 February 2012
DOI: 10.5427/jsing.2012.4e

INFLECTION POINTS OF REAL AND TROPICAL PLANE CURVES

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Abstract. We prove that Viro's patchworking produces real algebraic curves with the maximal number of real inflection points. In particular this implies that maximally inflected real algebraic M-curves realize many isotopy types. The strategy we adopt in this paper is tropical: we study tropical limits of inflection points of classical plane algebraic curves. The main tropical tool we use to understand these tropical inflection points are tropical modifications.

1. Introduction

Let k be any field of characteristic 0, and consider a plane algebraic curve X in kP^2 given by the homogeneous equation $P(z,w,u) = 0$. The Hessian of the polynomial $P(z,w,u)$, denoted by $Hess_P(z,w,u)$, is the homogeneous polynomial defined as

$$Hess_P(z,w,u) = \det \begin{pmatrix} \frac{\partial^2 P}{\partial^2 z} & \frac{\partial^2 P}{\partial z \partial w} & \frac{\partial^2 P}{\partial z \partial u} \\ \frac{\partial^2 P}{\partial z \partial w} & \frac{\partial^2 P}{\partial^2 w} & \frac{\partial^2 P}{\partial w \partial u} \\ \frac{\partial^2 P}{\partial z \partial u} & \frac{\partial^2 P}{\partial w \partial u} & \frac{\partial^2 P}{\partial^2 u} \end{pmatrix}.$$

If $Hess_P(z,w,u)$ is not the null polynomial, it defines a curve $Hess_X$ called the **Hessian** of X. Note that $Hess_X$ only depends on X, and is invariant under projective transformations of kP^2. An **inflection point** of the curve X is by definition a point p in $X \cap Hess_X$, of multiplicity m if $(X \circ Hess_X)_p = m$.

A plane algebraic curve has two kinds of inflection points: its singular points, and non-singular points having a contact of order $l \geq 3$ with their tangent line. In this latter case, the multiplicity of such an inflection point is exactly $l-2$.

If k is algebraically closed, Bézout's Theorem implies that an algebraic curve X in kP^2 of degree $d \geq 2$ which is reduced and does not contain any line has exactly $3d(d-2)$ inflection points (counted with multiplicity). Moreover, a non-singular generic curve X has exactly $3d(d-2)$ inflection points, all of them of multiplicity 1.

When k is not algebraically closed, the situation becomes more subtle. First, the number of inflection points of an algebraic curve in kP^2 depends not only on its degree, but also on the coefficients of its equation. In the case $k = \mathbb{R}$, it has been known for a long time that a non-singular real cubic has only 3 real points among its 9 inflection points. More generally, Klein proved that at most one third of the complex inflection points of a non-singular real algebraic curve can actually be real.

Theorem 1.1 (Klein [Kle76a], see also [Ron98], [Sch04], and [Vir88])**.** *A non-singular real algebraic curve in $\mathbb{R}P^2$ of degree $d \geq 3$ cannot have more than $d(d-2)$ real inflection points.*

Key words and phrases. Tropical geometry, Patchworking, Inflection points, Tropical modifications, Real algebraic curves.

Klein also proved that this upper bound is sharp. Following [KS03], we say that a non-singular real algebraic curve of degree d in $\mathbb{R}P^2$ is **maximally inflected** if it has $d(d-2)$ distinct real inflexion points. If a real algebraic curve has a node p with two real branches such that each branch is locally strictly convex around p, then any smoothing of p produces two real inflection points. Applying Hilbert's method of construction, the previous observation implies immediately the existence of maximally inflected curves in any degree at least 2. However, real inflection points of maximally inflected curves remains quite mysterious. For example, which rigid-isotopy classes of real algebraic curves contain a maximally inflected curve? How real inflection points can be distributed among the connected component of a maximally inflected curve?

The first step to answer questions of this sort is of course to find a systematic way to construct maximally real inflected curves. Invented by Viro at the end of the seventies (see [Vir82]), the patchworking technique turned out to be one of the most powerful method to construct real algebraic curves with controlled topology. One of the main contribution of this paper is to prove that patchworking also provides a systematic method to construct maximally inflected real curves.

For the sake of shortness we do not recall this technique here, we refer instead to the tropical presentation made in [Vir01], [Mik04], or [Bru09]. In non-tropical terms, Theorem 1 states that any real primitive T-curve, under a mild assumption on the corresponding convex function, is maximally inflected. Note that this result is of the same flavor as the fact that T-curves have asymptotically maximal total curvature (see [Lop06] and [Ri]). We denote by T_d the triangle in $\mathbb{R}^2$ with vertices $(0,0)$, $(d,0)$ and $(0,d)$. All precise definitions needed in Theorem 1 are given in section 3.

Theorem 1. *Let C be a non-singular tropical curve in $\mathbb{R}^2$ defined by the tropical polynomial "$\sum_{i,j} a_{i,j}x^iy^j$" with Newton polygon the triangle T_d with $d \geq 2$. Suppose that if v is a vertex of C dual to T_1, then its 3 adjacent edges have 3 different length. Then the real algebraic curve defined by the polynomial $P(z,w) = \sum_{i,j} \alpha_{i,j}t^{-a_{i,j}}z^iw^j$ with $\alpha_{i,j} \in \mathbb{R}$ has exactly $d(d-2)$ inflection points in $\mathbb{R}P^2$ for $t > 0$ small enough.*

As an example of application of Theorem 1, combined with classical results in topology of real algebraic curves (see [Vir82] and [Vir84] for example), we get the following corollary.

Corollary 1. *Any rigid isotopy class of non-singular real algebraic curves of degree at most 6 with non-empty real part contains a maximally inflected curve.*

Any real scheme with non-empty real part realized by a non-singular real algebraic curve of degree 7 is realized by a maximally inflected curve of degree 7.

Theorem 1 is a weak version of Theorem 5.7: the polynomials $P(z,w)$ are in fact polynomials over the field of generalized Puiseux series, and we give in addition the distribution of real inflection points among the connected components of a real algebraic curve obtained by patchworking. See Figures 17 and 18 from Example 5.8, as well as section 7 for some examples of such patchworkings.

A plane **tropical curve** C can be thought as a combinatorial encoding of a 1-parametric degeneration of plane complex algebraic curves $X(t)$ (see section 3 for definitions). The main part in the proof of Theorem 1 is then to understand which points of C represent a limit of inflection points of the algebraic curves $X(t)$. Since plane tropical curves are piecewise linear objects, the location of these tropical intersection points is not obvious at first sight, and we need to refine the tropical limit process. **Tropical modifications**, introduced by Mikhalkin in [Mik06], allow such a refinement.

It follows from Kapranov's Theorem that the tropicalization C of a family of plane complex algebraic curves $X(t)$ only depends on the first order term in t of the coefficients of the equation of $X(t)$. As rough as it may seem, the curve C keeps track of a non-negligible amount of information about the family $(X(t))$. For example, if C is non-singular, the genus of $X(t)$ is equal to the first Betti number of C. However, some information depending on more than just first order terms might be lost when passing from $(X(t))$ to C. Tropical modifications refine the tropicalization process, and allows one to recover some information about $(X(t))$ sensitive to higher order terms.

By means of these tropical modifications, we identify a finite number of **inflection components** on any non-singular tropical curve C (Proposition 5.2). These inflection components are the tropical analogues of inflection points. Using further tropical modifications, we prove that the **multiplicity** $\mu(\mathcal{E})$ of such a component $\mathcal{E}$ (i.e. the number of inflection points of $X(t)$ which tropicalize in $\mathcal{E}$) only depends on the combinatoric of C (Theorem 5.6). Now suppose that $X(t)$ is a family of *real* algebraic curves. As an immediate consequence, we get that the number of real inflection points of $X(t)$ which tropicalize in $\mathcal{E}$ has the same parity

as $\mu(\mathcal{E})$. In Theorem 5.7, we establish that a generic tropical curve has exactly $d(d-2)$ inflection components with odd multiplicity. Hence Theorem 5.7 together with Klein Theorem imply that $X(t)$ has exactly $d(d-2)$ real inflection points when t is small enough.

At several places in the text, we will see that tropical modifications can also be used to **localize** a problem. For example, relation between classical and tropical intersections (Proposition 4.5), or intersections between a curve and its Hessian (Theorem 5.7), are reduced to easy local considerations after a suitable tropical modification.

Contents

Acknowledgment: We are grateful to Viatcheslav Kharlamov for pointing us the fact that any smoothing of a real node with two local real convex branches produces two real inflection points, as well as for his simplification of the proof of Proposition 7.1. We are also indebted to Grigory Mikhalkin for many useful discussions about tropical modifications.

Finally, we thank Jean-Jacques Risler and Frank Sottile for their encouragements, as well as Cristhian Garay and the unkonwn referee for many useful comments on preliminary versions of this paper.

A major part of this work has been done during the visit of E.B. at the Universidad Nacional Autónoma de México (Instituo de Matemáticas. Unidad Cuernavaca), and during the Tropical Geometry semester at MSRI in Fall 2009. We thank these institutions for their support and excellent working conditions they offered us. Both authors were partially founded by UNAM-PAPIIT IN-117110 and CONACyT 55084. E.B. is also partially supported by the ANR-09-BLAN-0039-01 and ANR-09-JCJC-0097-01. L.L. is also partially supported by UNAM-DGAPA and Laboratorio Internacional Solomon Lefschetz (LAISLA), asociated to the CNRS (France) and CONACYT (Mexico).

2. Convention

Here we pose once for all some notations and conventions we will use throughout the paper. Almost all of them are commonly used in the literature.

An integer convex polytope in $\mathbb{R}^n$ is a convex polytope with vertices in $\mathbb{Z}^n$. The **integer volume** is the Euclidean volume normalized so that the standard simplex with vertices 0, $(1, 0, \ldots, 0)$, $(0, 1, 0, \ldots, 0)$, $\ldots$, $(0, \ldots, 0, 1)$ has volume 1. That is to say, the integer volume in $\mathbb{R}^n$ is $n!$ times the Euclidean volume in $\mathbb{R}^n$. In this paper, we only consider integer volumes. A simplex Δ in $\mathbb{R}^n$ will be called **primitive** if it has volume 1. Equivalently, Δ is primitive if and only if it is the image of the standard simplex under an element of $GL_n(\mathbb{Z})$ composed with a translation.

Given $d \geq 1$, we denote by T_d the integer triangle in $\mathbb{R}^2$ with vertices $(0, 0)$, $(d, 0)$, and $(0, d)$.

A **facet** of a polyhedral complex is a face of maximal dimension.

The letter k denotes an arbitrary field of characteristic 0. Given $P(z)$ a polynomial in n variables over k, we denote by $V(P)$ the hypersurface of $(k^*)^n$ defined by $P(z)$. We write $P(z) = \sum a_i z^i$ with $i = (i_1, \ldots, i_n)$, $z = (z_n, \ldots, z_n)$, and $a_i z^i = a_i z_1^{i_1} \ldots z_n^{i_n}$. The Newton polytope of $P(z)$ is denoted by $\Delta(P) \subset \mathbb{R}^n$, and given a subset Δ' of $\Delta(P)$, we define the **restriction** of $P(z)$ along Δ', by

$$P^{\Delta'}(z) := \sum_{i \in \Delta' \cap \mathbb{Z}^n} a_i z^i.$$

If X and X' are two algebraic curves in the projective plane kP^2, the intersection multiplicity of X and X' at a point $p \in kP^2$ is denoted by $(X \circ X')_p$.

3. Standard tropical geometry

In this section we review briefly some standard facts about tropical geometry, and we fix the notations used in this paper. For a more educational exposition, we refer, for example, to [Mik06], [IMS07], [RGST05], and [BPS08]. There exist several non-equivalent definitions of tropical varieties in the literature. In this paper, we have chosen for practical reasons to present them via non-archimedean amoebas.

3.1. **Non-archimedean amoebas.** A **locally convergent generalized Puiseux series** is a formal series of the form

$$a(t) = \sum_{r \in R} \alpha_r t^r$$

where $R \subset \mathbb{R}$ is a well-ordered set, $\alpha_r \in \mathbb{C}$, and the series is convergent for $t > 0$ small enough. We denote by $\mathbb{K}$ the set of all locally convergent generalized Puiseux series. It is naturally a field of characteristic 0, which turns out to be algebraically closed. An element $a(t) = \sum_{r\in R} \alpha_r t^r$ of $\mathbb{K}$ is said to be **real** if $\alpha_r \in \mathbb{R}$ for all r, and we denote by $\mathbb{R}\mathbb{K}$ the subfield of $\mathbb{K}$ composed of real series.

Since elements of $\mathbb{K}$ are convergent for $t > 0$ small enough, an algebraic variety over $\mathbb{K}$ (resp. $\mathbb{R}\mathbb{K}$) can be seen as a one parametric family of algebraic varieties over $\mathbb{C}$ (resp. $\mathbb{R}$).

The field $\mathbb{K}$ has a natural non-archimedean valuation defined as follows:

$$\begin{array}{cccc} val: & \mathbb{K} & \longrightarrow & \mathbb{R}\cup\{-\infty\} \\ & 0 & \longmapsto & -\infty \\ & \sum_{r\in R}\alpha_r t^r \neq 0 & \longmapsto & -min\{r \mid \alpha_r \neq 0\}. \end{array}$$

Note that we call val a valuation, although it is rather the opposite of a valuation for classical litterature. This valuation extends naturally to a map $Val : \mathbb{K}^n \to (\mathbb{R}\cup\{-\infty\})^n$ by evaluating val coordinate-wise, i.e. $Val(z_1, \ldots, z_n) = (val(z_1), \ldots, val(z_n))$. If $X \subset (\mathbb{K}^*)^n$ is an algebraic variety, $Val(X) \subset \mathbb{R}^n$ is called the **non-archimedean amoeba** of X.

Example 3.1. An integer matrix $M \in \mathcal{M}_{n,m}(\mathbb{Z})$ defines a multiplicative map $\Phi_M : (\mathbb{K}^*)^m \to (\mathbb{K}^*)^n$. The non-archimedean amoeba of $\Phi_M((\mathbb{K}^*)^m)$ is the vector subspace of $\mathbb{R}^n$ spanned by the columns of M.

Let X be an irreducible algebraic variety of dimension m. In this case, Bieri and Groves proved in [BG84] that $Val(X)$ is a finite rational polyhedral complex of pure dimension m (rational means that each of its faces has a direction defined over $\mathbb{Q}$). Given a facet F of $Val(X)$, we associate a positive integer number $w(F)$, called **the weight** of F, as follows: pick a point $(p_1, \ldots, p_n)$ in the relative interior of F, and choose a basis $(e_1, \ldots, e_n)$ of $\mathbb{Z}^n \subset \mathbb{R}^n$ such that $(e_1, \ldots, e_m)$ is a basis of the direction of F; denote by $Y_F \subset (\mathbb{K}^*)^n$ the multiplicative translation of $\Phi_{(e_{m+1},\ldots,e_n)}((\mathbb{K}^*)^{n-m})$ along $(t^{p_1}, \ldots, t^{p_n})$, and define $w(F)$ as the number (counted with multiplicity) of intersection points of X and Y_F with valuation $(p_1, \ldots, p_n)$. Note that $w(F)$ does not depend on the choice of the point $(p_1, \ldots, p_n)$.

Example 3.2. A matrix $M \in \mathcal{M}_{n,m}(\mathbb{Z})$ with $Ker\ M = \{0\}$ maps the lattice $\mathbb{Z}^m$ to a sub-lattice Λ' of $\Lambda = \mathbb{Z}^n \cap Im\ M$. The weight of the non-archimedean amoeba of $\Phi_M((\mathbb{K}^*)^m)$ is the index of Λ' in Λ.

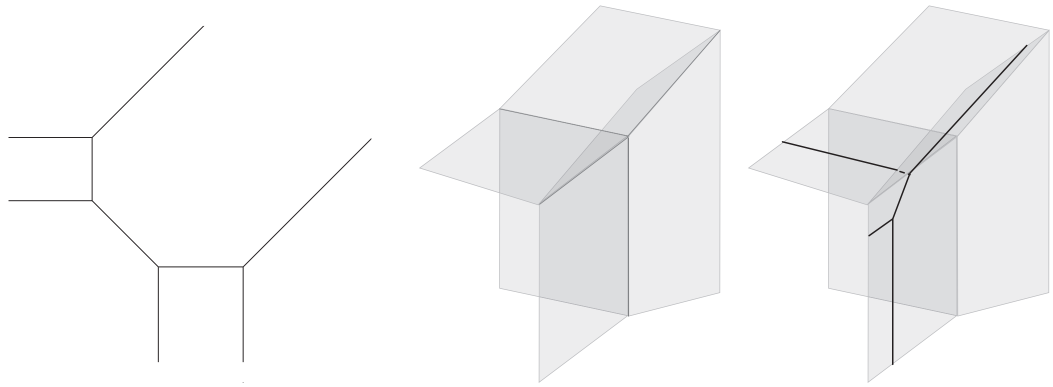
Definition 3.3. *The non-archimedean amoeba of X equipped with the weight function on its facets is called the tropicalization of X, and is denoted by $Trop(X)$.*

The notion of tropicalization extends naturally to any algebraic subvariety of $(\mathbb{K}^*)^n$, not necessarily of pure dimension. In this paper, a **tropical variety** is a finite rational polyhedral complex in $\mathbb{R}^n$ equipped with a weight function, and which is the tropicalization of some algebraic subvariety of $(\mathbb{K}^*)^n$.

Example 3.4. A plane tropical curve, a tropical plane in $\mathbb{R}^3$, and a tropical curve contained in this tropical plane are depicted in Figures 1a, 1b and 1c.

Definition 3.5. *Let V be a tropical variety in $\mathbb{R}^n$, and X be an algebraic subvariety of $(\mathbb{K}^*)^n$. We say that X realizes V if $V = Trop(X)$. If $X = V(P)$ for some polynomial $P(z)$, we say that $P(z)$ realizes V.*

Tropical varieties satisfy the so-called **balancing condition**. We give here this property only for tropical curves, since this is anyway the only case we need in this paper and makes the exposition easier. We refer to [Mik06] for the general case.



a) b) c)

FIGURE 1. Examples of tropical varieties. In these cases all the weights are equal to 1.

Let $C \subset \mathbb{R}^n$ be a tropical curve, and let v be a vertex of C. Let $e_1, \ldots, e_l$ be the edges of C adjacent to v. Since C is a rational graph, each edge e_i has a primitive integer direction. If in addition we ask that the orientation of e_i defined by this vector points away from v, then this primitive integer vector is unique. Let us denote by u_{v,e_i} this vector.

Proposition 3.6 (Balancing condition)**.** *For any vertex v, one has*

$$\sum_{i=1}^{l} w(e_i) u_{v,e_i} = 0.$$

If C is a tropical curve in $\mathbb{R}^n$, any of its bounded edge e has a **length** $l(e)$ defined as follows:

$$l(e) = \frac{||v_1 v_2||}{w(e)||u_{v_1,e}||}$$

where v_1 and v_2 are its adjacent vertices, and $||v_1 v_2||$ (resp. $||u_{v_1,e}||$) denotes the Euclidean length of the vector $v_1 v_2$ (resp. $u_{v_1,e}$).

3.2. **Tropical hypersurfaces.** Let us now study closer tropical hypersurfaces, i.e. tropical varieties in $\mathbb{R}^n$ of pure dimension $n-1$. These particular tropical varieties can easily be described as algebraic varieties over the **tropical semi-field** $(\mathbb{T}, \text{“}+\text{”}, \text{“}\times\text{”})$. Recall that $\mathbb{T} = \mathbb{R} \cup \{-\infty\}$ and that for any two elements a and b in $\mathbb{T}$, one has

$$\text{“}a + b\text{”} = \max(a, b) \text{ and } \text{“}a \times b\text{”} = a + b.$$

As usual, we abbreviate $a \times b$ in ab, and $(\mathbb{T}, \text{“}+\text{”}, \text{“}\times\text{”})$ in $\mathbb{T}$, and we use the convention that $\max(-\infty, a) = a$ and $-\infty + a = -\infty$. Note that $\mathbb{T}^* = \mathbb{R}$.

Since $\mathbb{T}$ is a semi-field, we have a natural notion of tropical polynomials, i.e. polynomials over $\mathbb{T}$. Such a polynomial $P(x) = \text{“}\sum a_i x^i\text{”}$ induces a function

$$\begin{array}{cccc} P: & \mathbb{T}^n & \longrightarrow & \mathbb{T} \\ & x & \longmapsto & \max(\langle x, i\rangle + a_i) \end{array}$$

where $x = (x_1, \ldots, x_n) \in \mathbb{T}^n$, $i = (i_1, \ldots, i_n) \in \mathbb{N}^n$, $x^i = \text{“}x_1^{i_1} \ldots, x_n^{i_n}\text{”}$, and $\langle\ ,\ \rangle$ denotes the standard Euclidean product on $\mathbb{R}^n$.

We denote by $\overset{\circ}{V}(P)$ the set of points x in $\mathbb{R}^n$ for which the value of $P(x)$ is given by at least 2 monomials. This is a finite rational polyhedral complex, which induces a subdivision Θ of $\mathbb{R}^n$. Given F a face of Θ and x a point in the relative interior of F, the set $\{i \in \Delta(P) \mid P(x) = \text{“}a_i x^i\text{”}\}$ does not depend on x. We denote its convex hull by Δ_F. All together, the polyhedrons Δ_F form a subdivision of $\Delta(P)$, called the **dual subdivision** of $P(x)$. The polyhedron Δ_F is called the **dual cell** of F, and $\dim \Delta_F = n - \dim F$. In particular, if F is a facet of $\overset{\circ}{V}(P)$ then Δ_F is a segment, and we define the **weight** of F by $w(F) = Card(\Delta_F \cap \mathbb{Z}^n) - 1$. We denote by $V(P)$ the polyhedral complex $\overset{\circ}{V}(P)$ equipped with the map w on its facets. $V(P)$ is called the **tropical hypersurface** defined by $P(x)$.

The Newton polygon of $P(x)$ and its dual subdivision are entirely determined, up to translation, by $V(P)$. A tropical hypersurface is said to be **non-singular** if all the maximal cells of its dual subdivision are primitive simplices. In particular, any facet of a non-singular tropical hypersurface has weight 1.

Note that we have used the same notations as in section 3.1. This is justified by the following fundamental Theorem, due to Kapranov.

Theorem 3.7 (Kapranov [Kap00])**.** *Let $P(z) = \sum a_i z^i$ be a polynomial over $\mathbb{K}$. If we define $P_{trop}(x) = \text{“}\sum val(a_i) x^i\text{”}$, then we have*

$$Trop(V(P)) = V(P_{trop}).$$

Example 3.8. The tropical planar curve and the tropical plane in Figure 1a and 1b, are given respectively by the tropical polynomials

$P(x,y) = \text{“}x^2 + y^2 + 2x + 2y + 3xy + 3\text{”}$ and $Q(x,y,z) = \text{“}x + y + z + 1\text{”}$.

Let $P(z)$ be a polynomial over $\mathbb{K}$ realizing a tropical hypersurface V in $\mathbb{R}^n$. To each face F of V dual to the polyhedron Δ_F, we associate below a complex polynomial $P_{\mathbb{C},F}(z)$. Let $i_1, \ldots, i_l$ be the vertices of Δ_F.

Let us first suppose that Δ_F has dimension n. In this case, the points $(i_1, -val(a_{i_1})), \ldots, (i_l, -val(a_{i_l}))$ lie on the same hyperplane in $\mathbb{R}^n \times \mathbb{R}$. Hence we have $-val(a_{i_j}) = \lambda_{\Delta_F}(i_j)$ where $\lambda_{\Delta_F} : \mathbb{R}^n \to \mathbb{R}$ is a linear-affine map. The maps λ_{Δ_F} glue along faces of codimension 1 to produce a convex piecewise-affine map $\lambda : \Delta(P) \to \mathbb{R}$. Note that the cells of dimension n of the dual subdivision of V correspond exactly to the domains of linearity of λ, and that $-val(a_i) \geq \lambda(i)$ for any $i \in \Delta(P) \cap \mathbb{Z}^n$.

Let us go back to the case when Δ_F may have any dimension between 0 and n. According to the preceding paragraph, there exists a linear-affine function $\lambda_{\Delta_F} : \mathbb{R}^n \to \mathbb{R}$ such that $-val(a_{i_j}) = \lambda_{\Delta_F}(i_j)$ for all $j = 1 \ldots l$, and $-val(a_i) > \lambda_{\Delta_F}(i)$ for any i not in Δ_F. If Δ_F has dimension n, then λ_{Δ_F} is unique and is precisely the map we defined above. Write $\lambda_{\Delta_F}(i) = \sum \gamma_j i_j + \alpha$, and define $\widetilde{P}(z) = t^\alpha P(t^{\gamma_1} z_1, \ldots, t^{\gamma_n} z_n)$. If we write $\widetilde{P}(z) = \sum \widetilde{a}_i z^i$, then $-val(\widetilde{a}_i) \geq 0$ for $i \in \Delta_F$, and $-val(\widetilde{a}_i) > 0$ for $i \notin \Delta_F$. Hence, if we plug $t = 0$ in $\widetilde{P}(z)$, we obtain a well defined complex polynomial $P_{\mathbb{C},F}(z)$ with Newton polygon Δ_F. Note that if $P(z)$ is defined over $\mathbb{R}\mathbb{K}$, then all the polynomials $P_{\mathbb{C},F}(z)$ are real.

3.3. **Tropical intersection.** Let $P_1(x,y)$ and $P_2(x,y)$ be two tropical polynomials defining respectively the tropical curves C_1 and C_2 in $\mathbb{R}^2$. Then, the polynomial $P_3(x,y) = \text{“}P_1(x,y)P_2(x,y)\text{”}$ defines a tropical curve C_3, whose underlying set is the union of $C_2 \cup C_3$. A vertex of C_3 which is in the set-theoretic intersection $C_1 \cap C_2$ is called a **tropical intersection point**[1] of C_1 and C_2. The set of tropical intersection points of C_1 and C_2 is denoted by $C_1 \cap_{\mathbb{T}} C_2$.

[1]Such points are also called **stable intersection points** in the literature, we refer to [RGST05] for a justification of this terminology.

Two tropical curves might have an infinite set-theoretic intersection, however they always have a finite number of tropical intersection points. Now we assign a multiplicity $(C_1 \circ_{\mathbb{T}} C_2)_v$ to each tropical intersection point v of C_1 and C_2 as follows

$$(C_1 \circ_{\mathbb{T}} C_2)_v = \frac{1}{2}\left(Area(\Delta_v) - \delta_v\right)$$

where

- $\delta_v = 0$ if v is an isolated intersection point of two edges of C_1 and C_2;
- $\delta_v = Area(\Delta_{v'})$ if v is a vertex v' of C_1 (resp. C_2) but not of C_2 (resp. C_1);
- $\delta_v = Area(\Delta_{v'}) + Area(\Delta_{v''})$ if v is a vertex v' of C_1, but also a vertex v'' of C_2;

Note that $(C_1 \circ_{\mathbb{T}} C_2)_v$ only depends on C_1 and C_2, and neither on $P_1(x,y)$ nor $P_2(x,y)$.

A **component** of $C_1 \cap C_2$ is a connected component of this set. Such a component E has a multiplicity defined as

$$(C_1 \circ_{\mathbb{T}} C_2)_E = \sum_{v \in C_1 \cap_{\mathbb{T},E} C_2} (C_1 \circ_{\mathbb{T}} C_2)_v$$

where $C_1 \cap_{\mathbb{T},E} C_2$ is the set of tropical intersection points of C_1 and C_2 contained in E.

Example 3.9. A transverse and a non-transverse intersection of planar tropical lines are depicted respectively in Figures 2a and 2b.

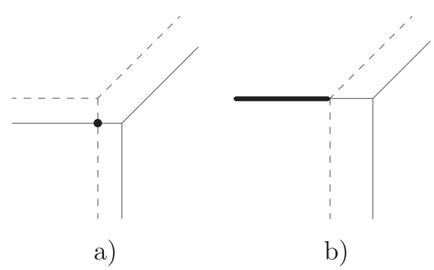


FIGURE 2. Transverse and non-transverse intersections. In both cases, $\mu(E) = 1$.

Example 3.10. Let C_1 and C_2 be two non-singular tropical curves in $\mathbb{R}^2$ with a component E of $C_1 \cap C_2$ not reduced to a point. Suppose that E contains a boundary point p which is not a vertex of both C_1 and C_2 (see Figure 3). Then $(C_1 \circ_{\mathbb{T}} C_2)_p = 1$.

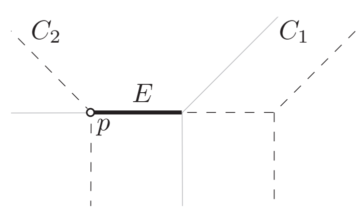


FIGURE 3. $(C_1 \circ_{\mathbb{T}} C_2)_p = 1$ and $(C_1 \circ_{\mathbb{T}} C_2)_E = 2$.

Intersection in $(\mathbb{K}^*)^2$ and tropical intersection are related by the following Proposition. When the set-theoretic intersection is infinite, we use tropical modifications to reduce the problem to local computations. Hence we postpone the proof of Proposition 3.11 to section 4 (see Proposition 4.5). Note that Rabinoff also gave in [R] a proof of Proposition 3.11 using Berkovich spaces.

Proposition 3.11. *Let X_1 and X_2 be two algebraic curves in $(\mathbb{K}^*)^2$ intersecting in a finite number of points, and let E be a component of the intersection of $C_1 = Trop(X_1)$ and $C_2 = Trop(X_2)$. Then, the number of intersection point (counted with multiplicity) of X_1 and X_2 with valuation in E is at most $(C_1 \circ_{\mathbb{T}} C_2)_E$, with equality if E is compact.*

Next we prove some easy lemmas we will use later in this paper. Lemma 3.12 is probably already known, however we couldn't find it explicitely in the litterature.

Lemma 3.12. *A polynomial in one variable with l monomials cannot have a root of order l other than 0.*

Proof. We prove the Lemma by induction on l. The Lemma is obviously true if $l = 1$. Suppose now that the Lemma is true for some $l \geq 1$, and let $P(z)$ be a polynomial in one variable with $l+1$ monomials. Since we are looking at roots $z \neq 0$, we may suppose that the constant term of $P(z)$ is non null. In particular, the derivative $P'(z)$, of $P(z)$ has l monomials. So $P(z)$ cannot have a root $z \neq 0$ of order bigger than $l+1$ since otherwise it would be a root of order bigger than l of $P'(z)$. □

Lemma 3.13. *Let X_1 and X_2 be two algebraic curves in $(\mathbb{K}^*)^2$, and suppose that there exists a tropical intersection point p of $C_1 = Trop(X_1)$ and $C_2 = Trop(X_2)$ which is the isolated intersection of an edge e_1 of C_1 and an edge e_2 of C_2 (see Figure 4a). Suppose in addition that p is neither a vertex of C_1 nor of C_2, and that $w(e_1) = w(e_2) = 1$. Then, any intersection point of X_1 and X_2 with valuation p is transverse.*

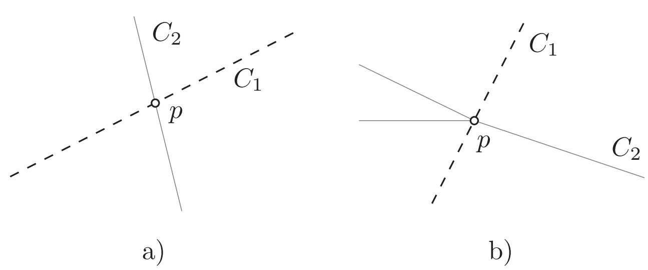


FIGURE 4. In this two cases, if $Trop(X_i) = C_i$ and C_i is non-singular, no intersection point of X_1 and X_2 with valuation p has multiplicity bigger that 2.

Proof. Suppose that X_i is defined by a polynomial $P_i(z,w)$, and suppose that there exists an intersection point of X_1 and X_2 with valuation p with multiplicity at least 2. Without loss of generality, we may suppose that $P_1^{\Delta_{e_1}}(z,w) = z - 1$, and that the two coefficients of $P_2(z,w)$ corresponding to Δ_{e_2} have valuation 0. In particular, $p = (0,0)$. Then, the algebraic varieties $X_1(t)$ and $X_2(t)$ have an intersection point of multiplicity at least 2 which converges in $(\mathbb{C}^*)^2$ when $t \to 0$. This implies that the two curves $V(z-1)$ and $V(P_{2,\mathbb{C},e_2})$ have an intersection point of multiplicity at least 2 in $(\mathbb{C}^*)^2$. Since these intersection points are solution in $\mathbb{C}^*$ of the equation $P_{2,\mathbb{C},e_2}(1,w) = 0$ which is a binomial equation, this is impossible by Lemma 3.12. □

Lemma 3.14. *Let X_1 and X_2 be two algebraic curves in $(\mathbb{K}^*)^2$, and suppose that there exists a tropical intersection point p of $C_1 = Trop(X_1)$ and $C_2 = Trop(X_2)$ such that p is a vertex v of C_2 but not of C_1 (see Figure 4b). Suppose in addition that Δ_v is primitive, and that $w(e) = 1$ where e is the edge of C_1 containing p. Then, any intersection point of X_1 and X_2 with valuation p is of multiplicity at most 2.*

Proof. As in the proof Lemma 3.13, we may suppose that X_1 is the line with equation $z = 1$ and that $P_2(z, w)$ is a trinomial. Once again, the result follows from Lemma 3.12. □

For a deeper study of simple tropical tangencies, we refer the interested reader to the forthcoming papers [BBM] and [BM].

Lemma 3.15. *Let l be a positive integer, let X_1 be an algebraic curve in $(\mathbb{K}^*)^2$ with Newton polygon the triangle with vertices $(0, 0)$, $(0, 1)$ and $(1, l)$, and let X_2 be a line in $(\mathbb{K}^*)^2$. Suppose that v is a vertex of both $Trop(X_1)$ and $Trop(X_2)$ (see Figure 5). Then, counting with multiplicity, at least $l - 1$ intersection points of X_1 and X_2 have valuation v (note that $(Trop(X_1) \circ_{\mathbb{T}} Trop(X_2))_v = l + 1$).*

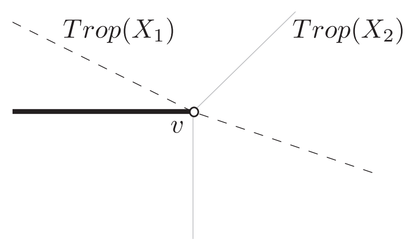


FIGURE 5. At least $l - 1$ intersection points of X_1 and X_2 have valuation v.

Proof. Without lost of generality, we may suppose that X_1 is defined by the polynomial $P(z, w) = 1 + w + zw^l$, and X_2 by $Q(z, w) = a + bz - w$ with $val(a) = val(b) = 0$. In particular, $v = (0, 0)$. The intersection points of X_1 and X_2 are the points $(z, a+bz)$ where z is a root of the polynomial $\widetilde{P}(z) = P(z, a + bz)$. We have

$$\begin{aligned} \widetilde{P}(z) &= (1 + a) + (b + a^l)z + \textstyle\sum_{j=1}^{l} \binom{l}{j} a^{l-j} b^j z^{j+1} \\ &= \textstyle\sum_{j=0}^{l+1} c_j z^j. \end{aligned}$$

Since $val(a) = val(b) = 0$, we have $val(c_0) \leq 0$, $val(c_1) \leq 0$ and $val(c_j) = 0$ for $j \geq 2$. Hence, 0 is a tropical root of order at least $l - 1$ of $\widetilde{P}_{trop}$. □

4. TROPICAL MODIFICATIONS

The tropicalization of an algebraic variety X in $(\mathbb{K}^*)^n$ defined by an ideal I only depends on the first order term of elements of I. For hypersurfaces this follows immediately from Kapranov's Theorem; in the general situation one can refer to [BJS+07] or [AN]. As rough as it may seem, the tropicalization process keeps track of a non-negligible amount of information about original algebraic varieties, e.g. intersection multiplicities. However, some information depending on more than just first order terms might be lost when passing from X to $Trop(X)$. Tropical modifications, introduced by Mikhalkin in [Mik06], can be seen as a refinement of the tropicalization process, and allows one to recover some information about X sensitive to higher order terms.

4.1. **Example.** Let us start with a simple example illustrating our approach. Consider the two lines X_1 and X_2 in $(\mathbb{K}^*)^2$ with equation

$$X_1: \ P_1(z,w) = (1+t^2) + z + w = 0 \quad \text{and} \quad X_2: \ (1+t) + z + t^{-1}w = 0.$$

It is not hard to compute that these two lines intersect at the point $p = (-1, -t^2)$ which has valuation $(0,-2)$. Suppose now that we want to compute the valuation of p just using tropical geometry, i.e. looking at $Trop(X_1)$ and $Trop(X_2)$. As depicted in Figure 6a, the set $Trop(X_1) \cap Trop(X_2)$ is infinite, and it is not clear at all which point on $Trop(X_1) \cap Trop(X_2)$ corresponds to $Val(p)$. Proposition 3.11 and the stable intersection point $(0,-1)$ of $Trop(X_1)$ and $Trop(X_2)$ tell us that X_1 and X_2 intersect in at most 1 point, but turn out to be useless in the exact determination of $Val(p)$.

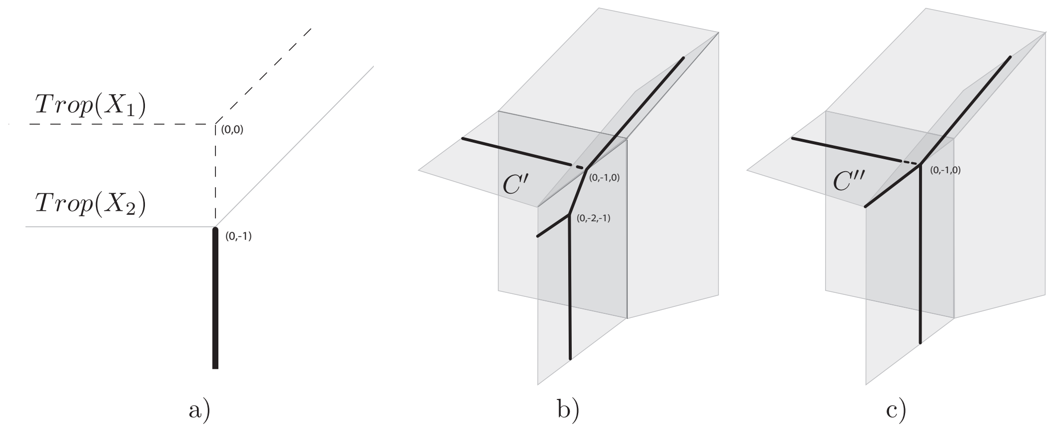


FIGURE 6. Two tropical modifications of $Trop(X_2)$

To resolve the infinite set-theoretic intersection $Trop(X_1) \cap Trop(X_2)$, we use one of the two lines, say X_1, to embed our plane in $(\mathbb{K}^*)^3$.

Let us denote $Y = (\mathbb{K}^*)^2 \setminus X_1$ and consider the following map

$$\begin{array}{rccc} \Phi: & Y & \longrightarrow & (\mathbb{K}^*)^3 \\ & (z,w) & \longmapsto & (z,w, \ P_1(z,w)). \end{array}$$

The map Φ restricts to $X_2 \setminus \{p\}$, and $Trop(\Phi(X_2 \setminus \{p\}))$ is a tropical curve in $\mathbb{R}^3$ with an edge starting at $(0,-2,-1)$ and unbounded in the direction $(0,0,-1)$ (see Figure 6b). Clearly, this edge corresponds to p, and tells us that $Val(p) = (0,-2)$.

Next section is devoted to generalizing the method used in this example.

4.2. **General method.** Let $P(z)$ be a polynomial in n variables over $\mathbb{K}$ and denote $Y = (\mathbb{K}^*)^n \setminus V(P)$. As in the preceding section, this polynomial defines the following embedding of Y to $(\mathbb{K}^*)^{n+1}$

$$\begin{array}{rccc} \Phi: & Y & \longrightarrow & (\mathbb{K}^*)^{n+1} \\ & z & \longmapsto & (z, P(z)). \end{array}$$

The tropical variety $W = Trop(\Phi(Y))$ is called the **tropical modification** of $\mathbb{R}^n$ defined by $P(z)$. Since $\Phi(Y)$ has equation $z_{n+1} - P(z_1, \dots, z_n) = 0$, it follows from Kapranov's Theorem that W is given by the tropical polynomial "$x_{n+1} + P_{trop}(x_1, \dots, x_n)$". If $\pi^{\mathbb{K}}_{n+1} : (\mathbb{K}^*)^{n+1} \to (\mathbb{K}^*)^n$ (resp. $\pi_{n+1} : \mathbb{R}^{n+1} \to \mathbb{R}^n$) denotes the projection forgetting the last coordinate, we obviously have $Trop \circ \pi^{\mathbb{K}}_{n+1} = \pi_{n+1} \circ Trop$.

Since W is a tropical hypersurface, many combinatorial properties of W are straightforward: the map π_{n+1} restrict to a surjective map $\pi_W : W \to \mathbb{R}^n$, one-to-one above $\mathbb{R}^n \setminus V(P_{trop})$; if $p \in V(P_{trop})$, then $\pi_W^{-1}(p)$ is a half ray, unbounded in the direction $(0, \dots, 0, -1)$; the weight of a facet F' of W is equal to $w(F)$ if $\pi_W(F') = F$ is a facet of $V(P_{trop})$, and is equal to 1 otherwise.

Example 4.1. The tropical plane in Figure 1b is the tropical modification of $\mathbb{R}^2$ defined by a polynomial of degree 1.

More generally, if X is an algebraic variety in $(\mathbb{K}^*)^n$ with no component contained in $V(P)$, the polynomial $P(z)$ defines a divisor $D(P)$ on X, and the map Φ defines an embedding of $X' = X \setminus V(P)$ in $(\mathbb{K}^*)^{n+1}$. The tropical variety $V' = Trop(X')$ is called the tropical modification of $V = Trop(X)$ defined by $P(z)$.

Unlike in the case of a tropical modification of $\mathbb{R}^n$, the tropical variety V' does not depend only on first order terms of X and $P(z)$. The map π_{n+1} still restricts to a surjective map $\pi_{V'} : V' \to V$, but this map is one-to-one only above $V \setminus V(P_{trop})$, i.e. $\pi_{V'}$ could be not injective not only on $\pi_{V'}^{-1}(Trop(V(P) \cap X))$ but also on the (potentially strictly) bigger set $\pi_{V'}^{-1}(V(P_{trop}) \cap V)$. Hence very few combinatorial properties of $\pi_{V'}^{-1}(V(P_{trop}))$ can be deduced in general only from those of V and $V(P_{trop})$: the set $\pi_{V'}^{-1}(p)$ is a bounded set if $p \in V \setminus Trop(V(P) \cap X)$, and unbounded in the direction $(0, \dots, 0, -1)$ otherwise; the weight of a facet F' of V', unbounded in the direction $(0, \dots, 0, -1)$ and such that $\pi_{V'}(F') = F$ is a facet of $Trop(D(P))$, is equal to $w(F)$ (recall that by definition, each facet of $Trop(D(P))$ has a weight); the weight of a facet F' of V' not contained in $\pi_{V'}^{-1}(V(P_{trop}) \cap V))$ is equal to the weight of the facet of V containing $\pi_{V'}(F')$.

Example 4.2. Let us illustrate the dependency of tropical modifications on higher order terms by going on with the example of preceding section. Recall that X_1 and X_2 are the two lines in $\mathbb{R}^2$ given by

$$X_1 : \ (1 + t^2) + z + w = 0 \quad \text{and} \quad X_2 : \ (1 + t) + z + t^{-1}w = 0.$$

We have already seen that the tropical modification of $Trop(X_2)$ along X_1 is a tropical line in $\mathbb{R}^3$ with two 3-valent vertices. Consider now the line X_3 defined by the equation $2 + z + w = 0$. Note that $Trop(X_1) = Trop(X_3)$. Since X_2 and X_3 intersect at $(-2 - t, t)$ which has valuation $(0, -1)$, the tropical modification of $Trop(X_2)$ along X_3 is a tropical line in $\mathbb{R}^3$ with one 4-valent vertex (see Figure 6c).

Example 4.3. Consider the curve X in $(\mathbb{K}^*)^2$ given by the equation

$$Q(z, w) = (t^{-9} + t^{-5} + 1) + (2t^{-4} + 1)z + tz^2 + (1 + t^{-5})w + 2zw + t^5z^2w + t^3w^2.$$

The tropical curve $C = Trop(X)$ and its tropical modification C' given by the polynomial $P(z, w) = z + t^{-5}$ are depicted in Figure 7. In particular C is a singular tropical curve, and the map $\pi_{C'}$ is not injective on a strictly bigger set than $\pi_{C'}^{-1}(Trop(V(P) \cap X))$. To see that C' is as depicted in Figure 7, just notice that the projection $\mathbb{R}^3 \to \mathbb{R}^2$ forgetting the first coordinate sends C' to $C'' = Trop(V(Q''))$ where $Q''(z, w) = Q(z - t^{-5}, w)$, and compute easily

$$Q''(z, w) = 1 + z + w + tz^2 + t^3w^2 + t^5wz^2.$$

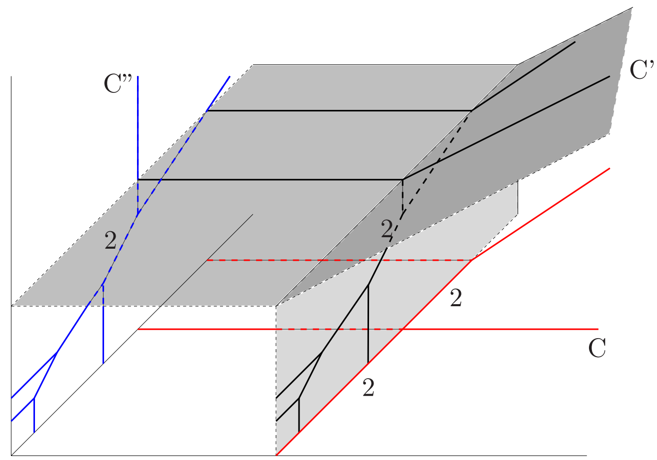


FIGURE 7. The tropical curves C and $V(P_{trop})$ do not determine C'.

4.3. **The case of plane curves.** In the case of plane curves, the situation is much simpler than the general case discussed above. In particular, we will see that the tropical modification of a non-singular plane tropical curve depends on very few combinatorial data. An edge unbounded in the direction $(0, 0, -1)$ of a tropical curve C will be called a **vertical end** of C.

Let $P_1(z, w)$ and $P_2(z, w)$ be two polynomials defining respectively the curves X_1 and X_2 in $(\mathbb{K}^*)^2$, such that X_1 and X_2 have no irreducible component in common. We denote $C_i = Trop(X_i)$, and by C_1' the tropical modification of C_1 given by $P_2(z, w)$.

Next Lemma is a restatement in the particular case of plane curves of the material we discussed in section 4.2.

Lemma 4.4. *If e is a vertical end of C_1', then $\pi_{C_1'}(e) \in Trop(X_1 \cap X_2)$.*

Conversely, if $p \in Trop(X_1 \cap X_2)$, then $\pi_{C_1'}^{-1}(p)$ contains a vertical end e of C_1', and

$$w(e) = \sum_{q \in X_1 \cap X_2 \cap Val^{-1}(p)} (X_1 \circ X_2)_q.$$

Tropical modifications allow us to relate easily intersection in $(\mathbb{K}^*)^2$ and tropical intersection.

Proposition 4.5. *Let E be a component of $C_1 \cap C_2$, and let m be the sum of the weight of all vertical ends in $\pi_{C_1'}^{-1}(E)$. Then*

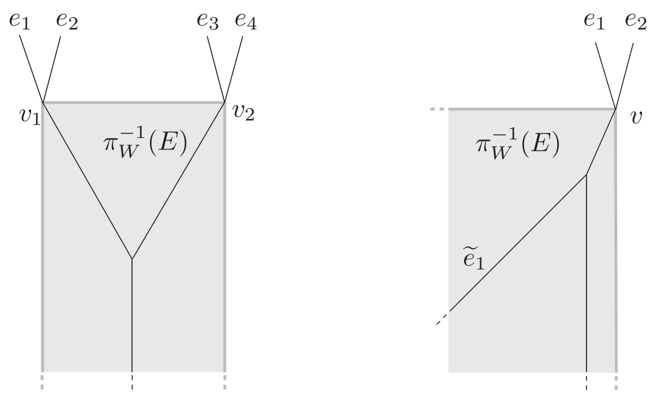
$$m \leq (C_1 \circ_{\mathbb{T}} C_2)_E$$

and equality holds if E is compact.

Proof. Let $e_1, \ldots, e_r$ (resp. $\widetilde{e}_1, \ldots, \widetilde{e}_s$) be the edges of C_1' which are not contained in $\pi_{C_1'}^{-1}(E)$ but adjacent to a vertex v_i in $\pi_{C_1'}^{-1}(E)$ (resp. which are unbounded but not vertical, and contained in $\pi_{C_1'}^{-1}(E)$). See Figure 8.

Let (x_i, y_i, z_i) (resp. $(\widetilde{x}_i, \widetilde{y}_i, \widetilde{z}_i)$) be the primitive integer direction of e_i (resp. $\widetilde{e}_i$) pointing away from v_i (resp. pointing to infinity). Then, it follows from the balancing condition that

$$m = \sum_{i=1}^{r} w(e_i) z_i + \sum_{i=1}^{s} w(\widetilde{e}_i) \widetilde{z}_i.$$



a) The compact case ($s = 0$). b) The non compact case ($s > 0$).

FIGURE 8.

Note that the balancing condition implies that if E is compact (i.e. when $s = 0$), then the integer m depends only on C_1 and C_2. In the case where E is not compact, we define an integer m' as follows.

We denote by W the tropical modification of $\mathbb{R}^2$ given by $P_2(z, w)$.

For any $1 \leq i \leq s$, we denote by $\widehat{e}_i$ the non vertical edge of W such that $\pi_W(\widehat{e}_i) \cap \pi_W(\widetilde{e}_i) \neq \emptyset$, and by $(\widehat{x}_i, \widehat{y}_i, \widehat{z}_i)$ the primitive vector of $\widehat{e}_i$ pointing to infinity. Then, $(\widetilde{x}_i, \widetilde{y}_i) = \lambda(\widehat{x}_i, \widehat{y}_i)$ with λ a positive rational number. Since $\widetilde{e}_i$ is contained in $\pi_W^{-1}(E)$, the slope of $\widetilde{e}_i$ is bounded by the slope of $\widehat{e}_i$. Hence, we necessarily have $\widetilde{z}_i \leq \lambda \widehat{z}_i$ with equality if and only if $\widetilde{e}_i$ and $\widehat{e}_i$ are parallel (see Figure 9).

Let us define

$$m' = m + \sum_{i=1}^{s} w(\widetilde{e}_i)\,(\lambda \widehat{z}_i - \widetilde{z}_i)\,.$$

Hence we have $m \leq m'$. In particular, $m = m'$ if and only if all the unbounded edges of $\pi_{C_1'}^{-1}(E)$ are vertical ends or parallels to the corresponding edge of W.

Now the balancing condition implies that the integer m' only depends on C_1 and C_2, and not any more on X_1 and X_2.

Hence to conclude, it remains to prove that $m' = (C_1 \circ_{\mathbb{T}} C_2)_E$. To do it, it is sufficient to prove that there exists a balanced graph Γ with rational slopes in W (i.e. a tropical 1-cycle in the terminology of [Mik06]) such that $\pi(\Gamma) = C_1$, and such that $\pi_\Gamma^{-1}(p)$ is a vertical end of weight $\frac{1}{2}(Area(\Delta_p) - \delta_p)$ if $p \in C_1 \circ_{\mathbb{T},E} C_2$, and $\pi_\Gamma^{-1}(p)$ is a point otherwise. The existence of Γ is clear if E is the isolated intersection of two edges of C_1 and C_2. The general case reduces to the latter case via stable intersections (see [RGST05]). $\square$

Note that Proposition 4.5 and its proof do not depend of the algebraically closed ground field $\mathbb{K}$, and generalize to intersections of tropical varieties of higher dimensions. The existence of Γ can also be established using the fact that tropical intersections are tropical varieties (see [OP]).

Using a more involved tropical intersection theory (see for example [Mik06], [S], or [A]), the proof of Proposition 4.5 should generalize easily when replacing the ambient space $\mathbb{R}^n$ by any smooth realizable tropical variety.

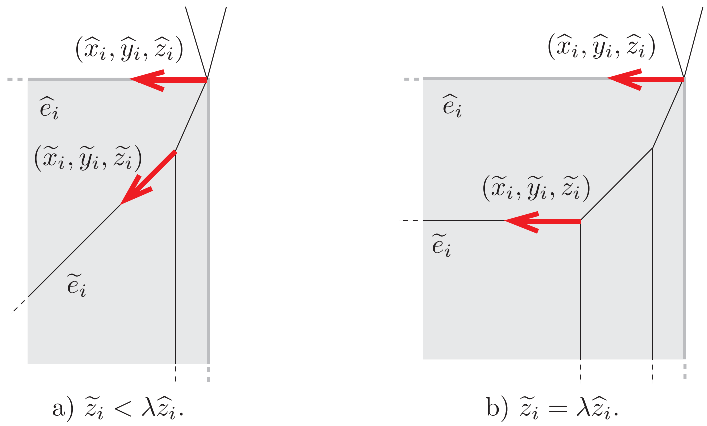


FIGURE 9. The slope of $\widetilde{e}_i$ is bounded by the slope of $\widehat{e}_i$.

Lemma 4.6. *Let p be a point on an edge e of C_1 such that $\pi_{C_1'}^{-1}(p)$ does not contain any vertex of C_1', and denote by $e_1, \ldots, e_l$ the edges of C_1' containing a point of $\pi_{C_1'}^{-1}(p)$. Then*

$$\sum_{i=1}^{l} w(e_i) \leq w(e).$$

Proof. By assumption, the set $\pi_{C_1'}^{-1}(p)$ is finite. Without loss of generality, we may assume that there exists a tropical line L given by "$y + a$", with $a \in \mathbb{T}$, having an isolated intersection point with e at p. Let C_3 be the non-singular tropical curve in $\mathbb{R}^2$ containing e and defined by a binomial polynomial (in particular C_3 is a classical line). If we denote by m_p the number of intersection points of X_1 with the line of equation $w + t^{-a} = 0$, then we have

$$m_p = (C_3 \circ_{\mathbb{T}} L)_p w(e).$$

Now the result follows from the fact that for each i, at least $(C_3 \circ_{\mathbb{T}} L)_p w(e_i)$ points in $X_1' \cap \{w + t^{-a} = 0\} \subset (\mathbb{K}^*)^3$ have valuation contained in e_i. □

Corollary 4.7. *If C_1 is a non-singular tropical curve with no component contained in C_2, then C_1' is entirely determined by C_1, C_2, and $Trop(X_1 \cap X_2)$. More precisely, we have*

- *$\pi_{C_1'}$ is one-to-one above $C_1 \setminus Trop(X_1 \cap X_2)$;*
- *for any $p \in Trop(X_1 \cap X_2)$, the set $\pi_{C_1'}^{-1}(p)$ is a vertical end of weight $w(p)$ (recall that by definition, each point in $Trop(X_1 \cap X_2)$ comes with a multiplicity);*
- *any edge e of C_1' which is not a vertical end is of weight 1.*

Proof. Let us denote by W the tropical modification of $\mathbb{R}^2$ given by $P_2(z, w)$. According to Lemmas 4.5 and 4.6, C_1' is entirely determined by the knowledge of $Trop(X_1 \cap X_2)$, the direction of one edge of C_1' and one point of C_1'. By hypothesis, there exists a point p in $C_1 \setminus C_2$ on an e edge of C_1. Since π_W is one to one over $\mathbb{R}^2 \setminus C_2$, the point $\pi_{C_1'}^{-1}(p) = \pi_W^{-1}(p) \in W$ is fixed, as well as the direction of the edges of C_1' passing through $\pi_{C_1'}^{-1}(p)$. □

5. Tropicalization of inflection points

Now we come to the core of this paper. Namely, given a non-singular tropical curve C, we study the possible tropicalizations for the inflection points of a realization of C. Our main result is that for almost all tropical curves, there exists a finite number of such points p on C and that the number of inflection points which tropicalize to p only depends on C, and not on the chosen realization.

Before going into the details, let us give an outline of our strategy. Let X be a realization of C, and T be a tangent line to X at an inflection point p. First of all we prove in Proposition 5.1 that the vertex of $L = Trop(T)$ has to be a vertex of C, which leaves only finitely many possibilities for L. In a second step, we refine in Proposition 5.2 the possible locations of $Val(p)$ by studying the tropical modifications of C and $\mathbb{R}^2$ defined by T. In particular we identify finitely many subsets of C, independant of X and called **inflection components** of C, which may possibly contain $Val(p)$. Note that these inflection components are often reduced to a point. Finally, we prove in Theorem 5.6 that the number of inflection points of X with valuation in a given inflection component $\mathcal{E}$ of C only depends on $\mathcal{E}$. We call this number the **multiplicity** of $\mathcal{E}$. The proof of Theorem 5.6 is postponed to section 6, and goes by the study of the tropical modification of C and $\mathbb{R}^2$ defined by $Hess_X$.

5.1. **Inflection points of curves in** $(k^*)^2$**.** Let k be any field of characteristic 0. Given a (non-necessarily homogeneous) polynomial in two variables $P(z, w)$, we denote by $P^{hom}(z, w, u)$ its homegeneization. Inflection points of the curve $V(P)$ (recall that by definition $V(P) \subset (k^*)^2$) are defined as the inflection points in $(k^*)^2 = \{[z : w : u] \in kP^2 \mid zwu \neq 0\}$ of the projective curve defined by $P^{hom}(z, w, u)$. Note that inflection points of $V(P)$ are invariant under the transformations $(i, j) \in \mathbb{N}^2 \mapsto z^i w^j P(z, w)$ but are **not** invariant in general under invertible monomial transformations of $(k^*)^2$ (i.e. automorphisms of $(k^*)^2$).

Since the torus $(k^*)^2$ is not compact, the number of inflection points of $V(P)$ may depend on the coefficients of $P(z, w)$. Indeed, some inflection points could escape from $(k^*)^2$ for some specific values of these coefficients. However, we will see in Proposition 6.1 that this can happen only if $\Delta(P)$ contains edges parallel to an edge of the simplex T_1.

5.2. **Location.** In the whole section, X is an algebraic curve in $(\mathbb{K}^*)^2$ which is not a line, and whose tropicalization is a non-singular tropical curve C. As we will see with Lemma 6.6 , it is hopeless to locate the tropicalization of inflection points of X looking at $Trop(X) \cap Trop(Hess_X)$, this intersection being highly non-transverse. However, an inflection point q of X comes together with its tangent T, which has intersection at least 3 with X at q. It turns out that the determination of all possible $Trop(T)$ is a much easier task.

Hence let q be an inflection point of X, with tangent line T. We denote by L the tropical line $Trop(T)$, by p the point $Val(q)$, by v the vertex of L, and by E the component of $C \cap L$ containing p.

Proposition 5.1. *The vertex v is a common vertex of C and L, and is contained in E. Moreover, E is one of the following (see Figure 10):*

- *reduced to v;*
- *an edge of C;*
- *three adjacent edges of C, at least 2 of them being bounded.*

Proof. According to Proposition 3.11, if E contains a tropical inflection point, then $(C \cap_{\mathbb{T}} L)_E \geq 3$. Since C and L are both non-singular, if E does not fulfill the conclusion of the Proposition, then E is one of the following (see Figure 11):

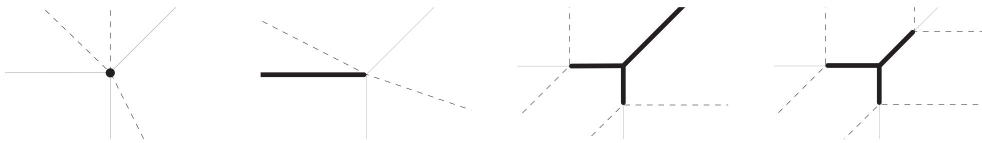


FIGURE 10. Possible tropicalizations of third-order tangent lines. The intersection of L and C is in bold.

- reduced to a point which is not a common vertex of both C and L;
- an unbounded edge of C or L but not of both;
- a bounded segment which is not an edge of C neither of L;
- three adjacent edges of C with only one of them being bounded;

The conclusion in the first case follows from Lemmas 3.13 and 3.14. In the last three cases we have

$$(C \cap_{\mathbb{T}} L)_E < 3.$$

Hence we excluded all cases not listed in the proposition. □

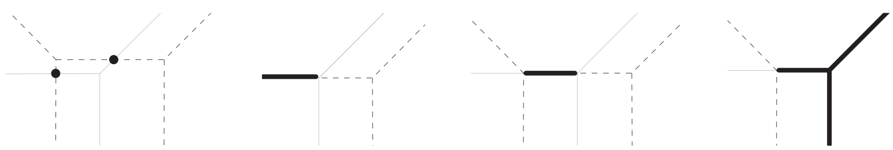


FIGURE 11. Impossible tropicalizations of third-order tangent lines.

An immediate and important consequence of Proposition 5.1 is that since the vertex of L must be a vertex of C, there are only finitely many possibilities for L. For each of these possibilities, we use tropical modifications to get a refinement on the possible locations of p. If E contains a bounded edge e, then we denote by v_e the other vertex of C adjacent to e. Recall that $l(e)$ is the length of e as an edge of C. We define the subset $\mathfrak{I}_L$ of E as follows :

- if $E = \{v\}$ or E is an unbounded edge of L, then $\mathfrak{I}_L = \{v\}$ (see Figure 12a and 12b);
- if E is a bounded edge e of C, then $\mathfrak{I}_L = \{v, p_e\}$ where p_e is the point on e at distance $\frac{l(e)}{3}$ from v (see Figure 12c);
- if E is the union of 2 bounded edges e_1, e_2, and one unbounded edge e_3, then
  - if $l(e_1) > l(e_2)$, then $\mathfrak{I}_L = \{p_{e_1}\}$ where p_{e_1} is the point on e_1 at distance $\frac{l(e_1)-l(e_2)}{3}$ from v (see Figure 12d);
  - if $l(e_1) = l(e_2)$, then $\mathfrak{I}_L$ is the whole edge e_3 (see Figure 12e);
- if E is the union of 3 bounded edges e_1, e_2, and e_3, then
  - if $l(e_1) \geq l(e_2) > l(e_3)$, then $\mathfrak{I}_L = \{p_{e_1}, p_{e_2}\}$ where p_{e_i} is the point on e_i at distance $\frac{l(e_i)-l(e_3)}{3}$ from v (see Figure 12f);
  - if $l(e_1) > l(e_2) = l(e_3)$, then $\mathfrak{I}_L$ is the whole segment $[v; p_{e_1}]$ where p_{e_1} is the point on e_1 at distance $\frac{l(e_1)-l(e_2)}{3}$ from v (see Figure 12g);
  - if $l(e_1) = l(e_2) = l(e_3)$, then $\mathfrak{I}_L = \{v\}$ (see Figure 12h).

Note that except in two cases, the set $\mathfrak{I}_L$ is finite.

Proposition 5.2. *The point p is in $\mathfrak{I}_L$.*

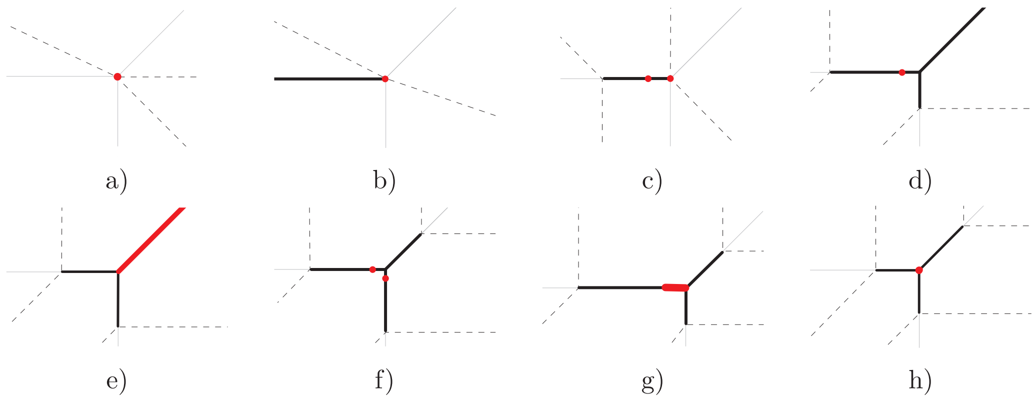


FIGURE 12. Description of $\mathfrak{I}_L$ in all the possibles cases.

Proof. Without loss of generality, we may assume that $v = (0,0)$ and that T is given by the equation $1 + z + w = 0$. Let W (resp. C') be the tropical modification of $\mathbb{R}^2$ (resp. C) given by the polynomial $1 + z + w$. Recall that the tropical hypersurface W has been described in Examples 4.1. We denote $p = (x_p, y_p)$. Given a vertex v_1 of C, we denote by v_1' the vertex of C' such that $\pi_{C'}(v_1') = v_1$. According to Lemma 4.4, the tropical curve C' must have a vertical end e_p with $w(e_p) \geq 3$ such that $\pi_{C'}(e_p) = p$. Let us prove the Proposition case by case.

Case 1: $E = \{v\}$**.** This case is trivial.

Case 2: E **is an unbounded edge** e **of** C**.** We may assume that E is a horizontal edge. Then the proposition follows from Lemma 3.15.

Case 3: E **is a bounded edge** e **of** C**.** We may assume that E is a horizontal edge. Since $C' \subset W$, we have $v' = (0,0,0)$ and $v_e' = (-l(e), 0, 0)$ (see Figure 13). If $p = v$ there is nothing to prove, so suppose now that $p \neq v$. We have $(C \cap_{\mathbb{T}} L)_E = (C \cap_{\mathbb{T}} L)_v + 1$. Hence according to Corollary 4.5 and Lemma 3.15, the edge e_p has weight exactly 3 and is the only vertical end of C' above $e \setminus \{v\}$. According to Corollary 4.7 and the balancing condition, C' has exactly 3 edges above $e \setminus \{v\}$: e_p, an edge with primitive integer direction $(1, 0, -1)$ adjacent to v_1', and an edge with primitive integer direction $(1, 0, 2)$ adjacent to v'. Hence, we have $(l(e) + x_p) + 2x_p = 0$ which reduces to $x_p = -\frac{l(e)}{3}$.

Case 4: E **is the union of 2 bounded edges** e_1**,** e_2**, and one unbounded edge** e_3**.** We may assume that e_1 is horizontal, e_2 is vertical, and that $l(e_1) \geq l(e_2)$. Since $C' \subset W$, we have

$$v_{e_1}' = (-l(e_1), 0, 0), \quad v_{e_2}' = (0, -l(e_2), 0), \quad \text{and} \quad v' = (0, 0, -a) \text{ with } a \geq 0.$$

In this case $(C \cap_{\mathbb{T}} L)_E = 3$. Then, the edge e_p has weight exactly 3 and is the only vertical end of C' above E. Moreover, the curve C' is completely determined once p is known.

If $p \in e_1$ (i.e. $y_p = 0$), then the fact that v' is a vertex of C' gives us the equations $l(e_2) = a$ and $(l(e_1) + x_p) + 2x_p = a$ which reduces to $x_p = -\frac{l(e_1) - l(e_2)}{3}$ (see Figure 14a).

If $p \in e_2$, then $y_p = -\frac{l(e_1) - l(e_2)}{3}$. Since in this case y_p is non-positive, this is possible only if $l(e_1) = l(e_2)$ and $y_p = 0$ (see Figure 14b).

If $p \in e_3$, then the vertex v' imposes the condition $l(e_1) = l(e_2)$. Hence, as soon as $l(e_1) = l(e_2)$, the point p may be anywhere on e_3 (see Figure 14c).

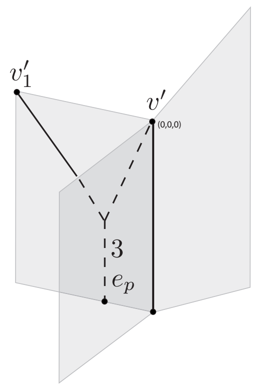
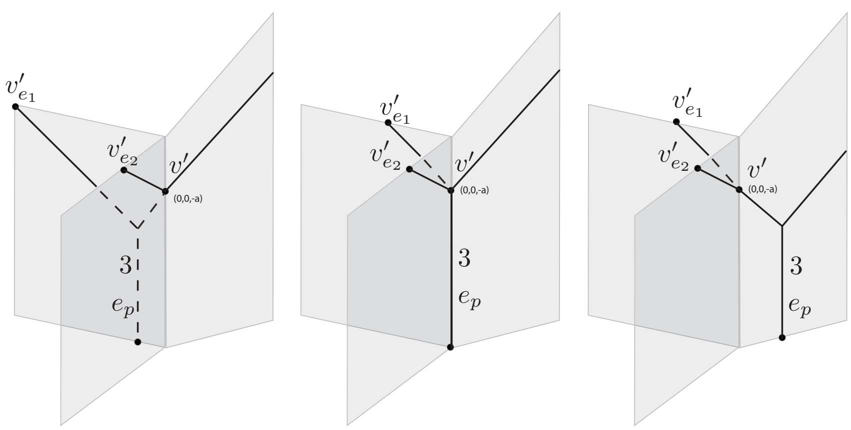


FIGURE 13. Case 3: E is a bounded edge.



a) $l(e_1) > l(e_2)$. b) $l(e_1) = l(e_2)$. c) $l(e_1) = l(e_2)$.

FIGURE 14. Case 4: E is the union of 3 edges, 2 of them bounded.

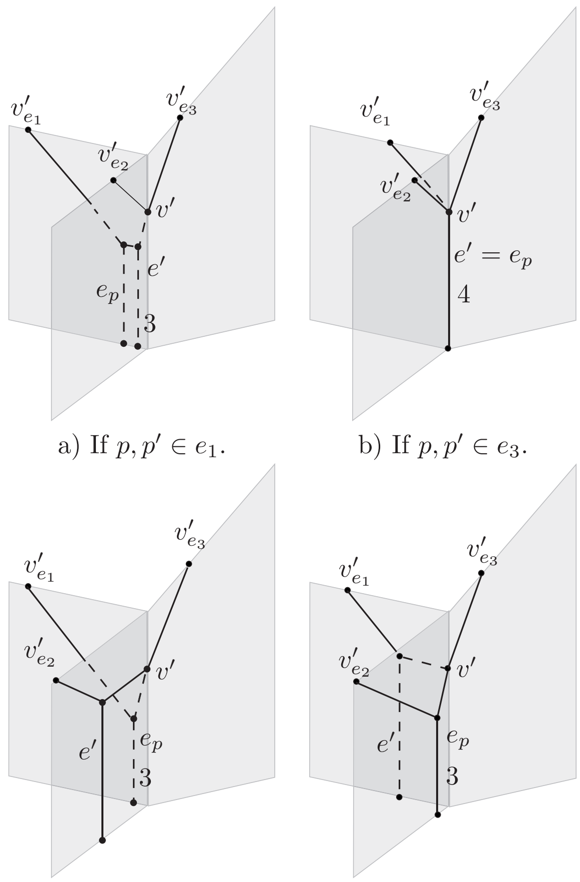
Case 5: E is the union of 3 bounded edges e_1, e_2, and e_3. We may assume that e_1 is horizontal, e_2 vertical, and that $l(e_1) \geq l(e_2) \geq l(e_3)$. Since $C' \subset W$, we have (see Figure 15)

$$v'_{e_1} = (-l(e_1), 0, 0),\ v'_{e_2} = (0, -l(e_2), 0),\ v'_{e_3} = (l(e_3), l(e_3), l(e_3)),\ \text{ and } v' = (0, 0, -a) \text{ with } a \geq 0.$$

We deduce from $(C \cap_{\mathbb{T}} L)_E = 4$ that the edge e_p may have weight 3 or 4. If $w(e_p) = 4$, then e_p is the only vertical end of C' above E. If $w(e_p) = 3$, there exist exactly two vertical ends of C' above E, e_p and e'. Note that $w(e') = 1$, and that we may assume that $w(e_p) = 3$ since the case $w(e_p) = 4$ corresponds to the case $e_p = e'$. Moreover, the curve C' is completely determined once p and $p' = \pi_{C'}(e') = (x', y')$ are known.

If $p, p' \in e_1$, then the equations given by the vertex v' of C' reduce to $l(e_2) = l(e_3) = a$ and $x_p = -\frac{l(e_1)-l(e_2)-x'}{3}$. So this is possible only if $l(e_2) = l(e_3)$, and in this case the point p may be anywhere on e_1 as long as it is at distance at most $\frac{l(e_1)-l(e_2)}{3}$ from v (See Figure 15a).

In the same way, the points p and p' can be both either on e_2 or on e_3 if and only if $l(e_1) = l(e_3)$, and in this case $p = p' = v$. (See Figure 15b).



a) If $p, p' \in e_1$. b) If $p, p' \in e_3$.

c) If $p \in e_1$ and $p' \in e_2$. d) If $p \in e_2$ and $p' \in e_1$.

FIGURE 15. Case 5: E is the union of 3 bounded edges.

If $p \in e_1$ and $p' \in e_2$, then we get $x_p = -\frac{l(e_1)-l(e_3)}{3}$ (See Figure 15c).
If $p \in e_2$ and $p' \in e_1$, then we get $y_p = -\frac{l(e_2)-l(e_3)}{3}$ (See Figure 15d).
If p is in e_3, then we get $a = l(e_1)$ or $a = l(e_2)$, and $a \leq l(e_3) + 2x_p$ which is possible only if $x_p = 0$.
In the same way, if $p' \in e_3$, then $p' = v$. □

5.3. **Multiplicities.** In the preceding section, we have seen that if C is a non-singular tropical curve in $\mathbb{R}^2$, then the inflection points of any realization X of C tropicalize in a simple subset I_C of C, which depends only on C. Namely, given a tropical line L whose vertex v is also a vertex of C and such that v is contained in a component of $C \cap L$ of multiplicity at least 3, we define the set $\mathfrak{I}_L$ as in section 5.2. Then we define $I_C = \bigcup \mathfrak{I}_L$ where L ranges over all such tropical lines. We also define $\mathfrak{I}_C$ as the set of all connected components of I_C. In this section we prove that given any element $\mathcal{E}$ of $\mathfrak{I}_C$, the number of inflection points of X which tropicalize in $\mathcal{E}$ only depends on $\mathcal{E}$.

Let $\Delta \subset \mathbb{R}^2$ be an integer convex polygon, and let δ be an edge of Δ. If δ is not parallel to any edge of T_1, then we set $r_\delta = 0$; if δ is supported on the line with equation $i = a$ (resp. $j = a$, $i + j = a$) and Δ is contained in the half-plane defined by $i \leq a$ (resp. $j \leq a$, $i + j \geq a$), then we set $r_\delta = Card(\delta \cap \mathbb{Z}^2) - 1$; otherwise we set $r_\delta = 2(Card(\delta \cap \mathbb{Z}^2) - 1)$; finally, we define

$$i_\Delta = 3Area(\Delta) - \sum_{\delta \text{ edge of } \Delta} r_\delta.$$

Note that $i_\Delta < 0$ if and only if Δ is equal to T_1 or one of its edges.

Definition 5.3. *An element of $\mathfrak{I}_C$ is called an inflection component of C. The multiplicity of an inflection component $\mathcal{E}$, denoted by $\mu_\mathcal{E}$, is defined as follows*

- *if $\mathcal{E}$ is a vertex of C dual to the primitive triangle $\Delta \neq T_1$, then*
$$\mu_\mathcal{E} = i_\Delta;$$
- *if $\mathcal{E}$ is bounded and contains a vertex of C dual to the primitive triangle T_1, then*
$$\mu_\mathcal{E} = 6;$$
- *in all other cases,*
$$\mu_\mathcal{E} = 3.$$

Example 5.4. We depicted in Figure 16 some honeycomb tropical curves together with their inflection components. Each one of these components is a point of multiplicity 3.

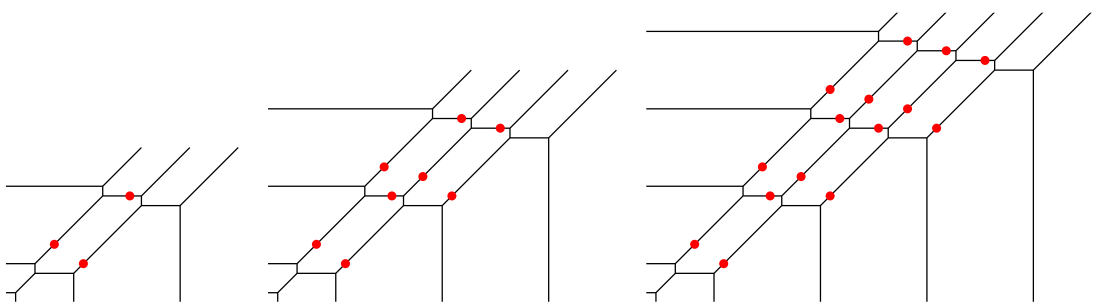


FIGURE 16. Some honeycomb tropical curves and their inflection points

Proposition 5.5. *For any non-singular tropical curve C in $\mathbb{R}^2$ with Newton polygon T_d, we have*

$$\sum_{\mathcal{E} \in \mathfrak{I}_C} \mu_\mathcal{E} = 3d(d-2).$$

Proof. Let us first introduce some terminology. Let Δ be a polygon of the dual subdivision of C, and let δ be one of its edges. The edge δ is said to be bounded if the edge of C dual to δ is bounded. The edge δ is said to have Δ-degree 1 if δ is supported either on the line $\{i = a\}$, or $\{j = a\}$, or $\{i + j = a\}$, and Δ is contained in the half plane defined respectively by $\{i \geq a\}$, or $\{j \geq a\}$, or $\{i + j \leq a\}$. The number of bounded Δ-degree 1 edges of Δ is denoted by γ_Δ. Finally, let α be the number of bounded edges of the dual subdivision of C which are parallel to an edge of T_1.

From section 5.2 and the definition of $\mu_{\mathcal{E}}$, it follows immediately that for any vertex v of C, we have

$$\sum_{\mathcal{E}\in\mathfrak{I}_L} \mu_{\mathcal{E}} = i_{\Delta_v} + 3\gamma_{\Delta_v}$$

where L is the line with vertex v. Hence we deduce that

$$\begin{aligned}\textstyle\sum_{\mathcal{E}\in\mathfrak{I}_C} \mu_{\mathcal{E}} &= 3Area(T_d) - 2Card(\partial T_d \cap \mathbb{Z}^2) - 3\alpha + 3\alpha \\ &= 3d(d-2)\end{aligned}$$

as announced. □

To get a genuine correspondence between inflection points of an algebraic curve and the inflection components of its tropicalization, we actually need to pass to projective curves. It is well known that the compactification process we are going to describe now can be adapted to construct general non-singular tropical toric varieties. However, since we will just need to deal with plane projective curves, we restrict ourselves to the construction of tropical projective spaces (see [BJS+07]).

As in classical geometry, the **tropical projective space** $\mathbb{T}P^n$ of dimension $n \geq 1$ is defined as the quotient of the space $\mathbb{T}^{n+1} \setminus \{(-\infty, \ldots, -\infty)\}$ by the equivalence relation

$$v \sim \text{“}\lambda v\text{”} = v + \lambda(1, \ldots, 1).$$

that is

$$\mathbb{T}P^n = (\mathbb{T}^{n+1} \setminus \{(-\infty, \ldots, -\infty)\})/(1, \ldots, 1)$$

Topologically, the space $\mathbb{T}P^n$ is a simplex of dimension n, in particular it is a triangle when $n = 2$.

The coordinate system $(x_1, \ldots, x_{n+1})$ on $\mathbb{T}^{n+1}$ induces a **tropical homogeneous coordinate system** $[x_1 : \ldots, x_{n+1}]$ on $\mathbb{T}P^n$, and we have the natural embedding

$$\begin{array}{ccc} \mathbb{R}^n & \longrightarrow & \mathbb{T}P^n \\ (x_1, \ldots, x_n) & \longmapsto & [x_1 : \ldots, x_n : 0]. \end{array}$$

Hence any tropical variety V in $\mathbb{R}^n$ has a natural compactification $\overline{V}$ in $\mathbb{T}P^n$. Also, any non-compact inflection component of a non-singular tropical curve C in $\mathbb{R}^2$ compactifies in an inflection component of $\overline{C}$. The map $Val : \mathbb{K}^{n+1} \to \mathbb{T}^{n+1}$ induces a map $Val : \mathbb{K}P^n \to \mathbb{T}P^n$, and if X is an algebraic variety in $(\mathbb{K}^*)^n$ with closure $\overline{X}$ in $\mathbb{K}P^n$, we have

$$Trop(\overline{X}) = \overline{Trop(X)}.$$

Theorem 5.6. *Let C be a non-singular tropical curve in $\mathbb{R}^2$ with Newton polygon the triangle T_d with $d \geq 2$, and let X be any realization of C. Then for any inflection point p of $\overline{X}$, the point $Val(p)$ is contained in an inflection component of $\overline{C}$, and for any inflection component $\mathcal{E}$ of $\overline{C}$, exactly $\mu_{\mathcal{E}}$ inflection points of $\overline{X}$ have valuation in $\mathcal{E}$.*

We postpone the proof of Theorem 5.6 to section 6. The fact that any inflection point of $\overline{X}$ tropicalizes in some inflection component of C has already been proved in Proposition 5.2. The fact that exactly $\mu_{\mathcal{E}}$ inflection points of $\overline{X}$ have valuation in $\mathcal{E}$ for any inflection component $\mathcal{E}$ follows from Lemmas 6.4, 6.7, 6.8, and 6.9.

5.4. **Application to real algebraic geometry.** Here we give a real version of Theorem 5.6, which implies immediately Theorem 1. Given $\mathcal{E}$ an inflection component of a non-singular tropical curve C, we define its real multiplicity $\mu_{\mathcal{E}}^{\mathbb{R}}$ by

$$\mu_{\mathcal{E}}^{\mathbb{R}} = 0 \text{ if } \mu_{\mathcal{E}} \text{ is even,} \qquad \mu_{\mathcal{E}}^{\mathbb{R}} = 1 \text{ if } \mu_{\mathcal{E}} \text{ is odd.}$$

Theorem 5.7. *Let C be a non-singular tropical curve in $\mathbb{R}^2$ with Newton polygon the triangle T_d with $d \geq 2$. Suppose that if v is a vertex of C adjacent to 3 bounded edges and such that $\Delta_v = T_1$, then these edges have 3 different length. Then given any realization X of C over $\mathbb{RK}$ and given any inflection component $\mathcal{E}$ of $\overline{C}$, exactly $\mu_{\mathcal{E}}^{\mathbb{R}}$ inflection points of $\overline{X}$ have valuation in $\mathcal{E}$.*

In particular, the curve $\overline{X}$ has exactly $d(d-2)$ inflection points in $\mathbb{RK}P^2$, and the curve $\overline{X}(t)$ has also exactly $d(d-2)$ inflection points in $\mathbb{R}P^2$ for $t > 0$ small enough.

Proof. Since $\overline{X}$ is defined over $\mathbb{RK}$, its inflection points are either in $\mathbb{RK}P^2$ or they come in pairs of conjugated points. Hence, for each inflection component $\mathcal{E}$ of C, at least $\mu_{\mathcal{E}}^{\mathbb{R}}$ inflection points of $\overline{X}$ are real and have valuation in $\mathcal{E}$.

If C satisfies the hypothesis of the theorem, any of its inflection component $\mathcal{E}$ has multiplicity at most 3. Let us prove that the number of inflection points of C of multiplicity 1 is equal to the number of inflection points of C of multiplicity 2: this is obviously true when C is a honeycomb tropical curve (i.e. all its edges have direction $(1,0)$, $(0,1)$, or $(1,1)$); one checks easily that if this is true for C, then this is also true for any tropical curve whose dual subdivision is obtained from the one of C by a flip; in conclusion this is true for any non-singular tropical curve since any two primitive regular integer triangulations of T_d can be obtained one from the other by a finite sequence of flips.

As a consequence, we deduce

$$\begin{aligned} \textstyle\sum_{\mathcal{E}\in\mathfrak{I}_C} \mu_{\mathcal{E}}^{\mathbb{R}} &= \tfrac{1}{3}\textstyle\sum_{\mathcal{E}\in\mathfrak{I}_C} \mu_{\mathcal{E}} \\ &= d(d-2). \end{aligned}$$

As a consequence the algebraic curve $\overline{X}$ has at least $d(d-2)$ inflection points in $\mathbb{RK}P^2$. Since $\overline{X}$ is defined over $\mathbb{RK}$, it cannot have more according to Theorem 1.1. Hence the curve $\overline{X}$ has exactly $d(d-2)$ inflection points in $\mathbb{RK}P^2$, and exactly $\mu_{\mathcal{E}}^{\mathbb{R}}$ inflection points of $\overline{X} \cap \mathbb{RK}P^2$ have valuation in $\mathcal{E}$ for any inflection component $\mathcal{E}$ of C. □

Example 5.8. In Figures 17 and 18, we depicted one possible patchworking of real curve together with its real inflection points for each honeycomb tropical curve depicted in Figure 16.

6. End of the proof of Theorem 5.6

Here we prove that the multiplicity of an inflection component $\mathcal{E}$ of $\overline{C}$ corresponds to the number of inflection points of $\overline{X}$ which tropicalizes in $\mathcal{E}$. Thanks to tropical modifications, all computations are reduced to elementary local considerations.

Our first task is to study inflection points of algebraic curves in the torus.

6.1. **Hessian of a primitive polynomial.** Given a polynomial $P(z,w)$ in $k[z,w]$, we define $P^h(z,w,u) = z^2w^2u^2P^{hom}(z,w,u)$, and $H_P(z,w) = Hess_{P^h}(z,w,1)$. Clearly, the curves $V(P)$ and $V(P^h)$ have the same inflection points in $(k^*)^2$.

Proposition 6.1. *Let $P(z,w)$ be a polynomial in 2 variables over k. If the curve $V(P)$ is reduced and does not contain any line, then the number of inflection points of $V(P)$ is at most $i_{\Delta(P)}$ (recall that $i_{\Delta(P)}$ has been defined in section 5.3). Moreover, this number is exactly equal*

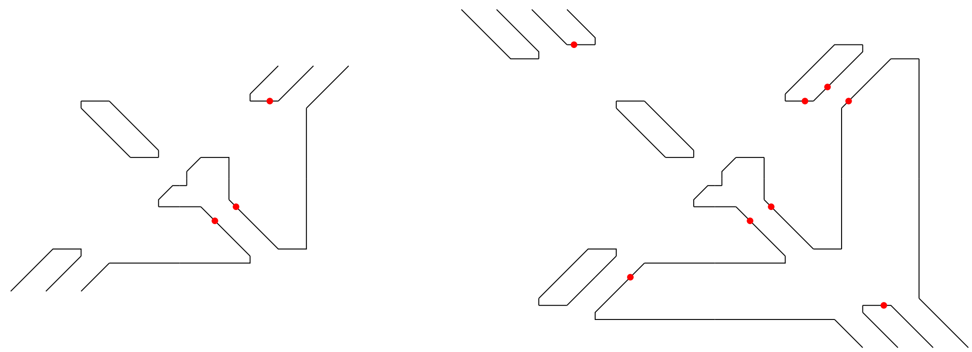


FIGURE 17. A Harnack curve of degree 3 and 4

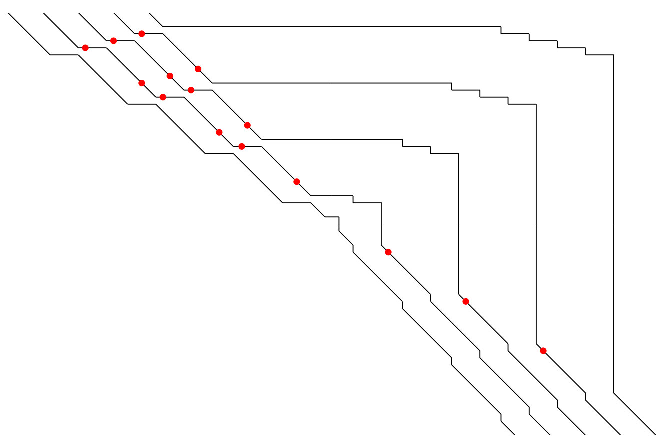


FIGURE 18. A hyperbolic quintic

to $i_{\Delta(P)}$ *if* k *is algebraically closed, and* $r_\delta = 0$ *and* P^δ *has no multiple roots in* $(\mathbb{C}^*)^2$ *for all edges* δ *of* Δ*.*

Proof. To prove the Proposition, we may suppose that k is algebraically closed. By assumption on $V(P)$, it has finitely many inflection points. The Newton polygon of H_P is, up to translation, $3\Delta(P)$. Given δ an edge of $\Delta(P)$, we denote by δ_{he} the corresponding edge of $\Delta(H_P)$, by n_δ the number of common roots of the polynomials P^δ and $H_P^{\delta_{he}}$, and we define

$$i_P = \sum_{\delta \text{ edge of } \Delta(P)} n_\delta.$$

Hence, according to Bernstein Theorem, the number of inflection points of $V(P)$ is at most

$$\frac{1}{2}\left(Area(4\Delta(P)) - Area(3\Delta(P)) - Area(\Delta(P))\right) - i_P = 3Area(\Delta(P)) - i_P$$

with equality if $i_P = 0$.

Hence, we are left to the study of n_δ when δ ranges over all edges of $\Delta(P)$. Since $P^h(z,w,u)$ is divisible by $z^2w^2u^2$, we have $H_{P^\delta} = (H_P)^{\delta_{he}}$ for any edge δ of $\Delta(P)$. Hence we may suppose that $\Delta(P)$ is reduced to the edge δ. In this case $P^h(z,w,u)$ splits into the product of $Card(\delta\cap\mathbb{Z}^2)-1$ binomials, so we may further assume that $Card(\delta\cap\mathbb{Z}^2)=2$. Then the curve $V(P)$ is non-singular in $(k^*)^2$, so the only possibility for n_δ to be equal to 1, is for $V(P)$ to be a line. That is, δ must be parallel to an edge of T_1.

In the case when $\Delta(P)$ has an edge δ supported on the line with equation $i=a$ (resp. $j=a$, $i+j=a$) and Δ is contained in the half-plane defined by $i\geq a$ (resp. $j\geq a$, $i+j\leq a$), we can refine the upper bound. Without loss of generality, we may assume that δ is supported on the line with equation $i=0$, and we define $Q(z,w,u)=w^2u^2P^{hom}(z,w,u)$. The polygon $\Delta(Hess_Q)$ is contained, up to translation, in the polygon $3\Delta(P)\cap\{i\geq 2\}$, so Bernstein Theorem implies that the number of inflection points of $V(P)$ is at most

$$\frac{1}{2}\left(Area(4\Delta(P)\cap\{i\geq 2\}) - Area(3\Delta(P)\cap\{i\geq 2\}) - Area(\Delta(P))\right) - \sum_{\delta'\neq\delta \text{ edge of } \Delta(Q)} n_{\delta'}.$$

Since

$$Area(m\Delta(P)\cap\{i\leq 2\}) = Area(\Delta(P)\cap\{i\leq 2\}) + 4(m-1)(Card(\delta\cap\mathbb{Z}^2)-1),$$

we see that we can in fact substract $2(Card(\delta\cap\mathbb{Z}^2)-1)$ to $3Area(\Delta(P))$ in the upper bound for the number of inflection points of $V(P)$. □

Example 6.2. Any curve over an algebraically closed field, whose Newton polygon is a primitive triangle without any edge parallel to an edge of T_1, has exactly 3 inflection points in $(k^*)^2$. Indeed, such a curve is non-singular in $(k^*)^2$ and does not contain any line.

Example 6.3. Proposition 6.1 might not be sharp, even if k is algebraically closed. Indeed, let X be a cubic curve in kP^2. It is classical (see for example [Sha94]) that a line passing through two inflection points of X also passes through a third inflection point of X. Hence, any algebraic curve with Newton polygon the triangle Δ with vertices $(0,0)$, $(1,2)$, and $(2,1)$, cannot have more than 6 inflection points in $(k^*)^2$, although in this case $i_\Delta = 7$.

However, Proposition 6.1 is sharp for curves with primitive Newton polygon.

Lemma 6.4. *If $\Delta(P)$ is primitive and distinct from T_1, then the curve $V(P)$ has exactly $i_{\Delta(P)}$ inflection points in $(k^*)^2$.*

Proof. According to Example 6.2, it remains to check by hand the lemma in the following two particular cases (all coefficients may be chosen equal to 1 since $\Delta(P)$ is primitive):

- $P(z,w)=w+z^{l-1}+z^l$ with $l\geq 2$: inflection points are given by the roots in k^* of the second derivative of the polynomial $z^{l-1}+z^l$. We have 1 (resp. no) such root when $l\geq 3$ (resp. $l=2$).
- $P(z,w)=w+zw+z^l$ with $l\geq 0$: inflection points are given by the roots in k^* of the second derivative of the function $f(z)=-\frac{z^l}{1+z}$. We have 2 (resp. no) such roots when $l\geq 3$ (resp. $l\leq 2$).

□

Let us fix a polynomial $P(z,w)$ in $\mathbb{C}[z,w]\subset\mathbb{K}[z,w]$ such that $\Delta(P)$ is primitive and different from T_1, and define $C=Trop(V(P))$. Recall that both tropical curves C and $Trop(V(H_P))$ have the same underlying set. Let H'_C (resp. W) be the tropical modification of $Trop(V(H_P))$ (resp. $\mathbb{R}^2$) given by $P(z,w)$, and e_1, e_2, and e_3 be the three edges of W such that $\pi(e_i)$ is an edge of C. Let (x_i,y_i,z_i) be the primitive integer direction of e_i which points to infinity, and by $\widetilde{e}_{i,1}, \dots \widetilde{e}_{i,s_i}$ the ends of H'_C such that $\pi(e_{i,j}) = \pi(e_i)$. Finally we denote by $(x_i, y_i, \widetilde{z}_{i,j})$ the primitive integer direction of $\widetilde{e}_{i,j}$ pointing to infinity, and we define

$$z_{H'_C,i} = \sum_{j=1}^{s_i} w(\widetilde{e}_{i,j})\widetilde{z}_{i,j}.$$

Lemma 6.5. *The tropical curve H'_C has a unique vertex, which is also the vertex of W. Moreover H'_C has a vertical end with weight $i_{\Delta(P)}$, and for all $i = 1, 2, 3$ we have*

- $z_{H'_C,i} = 3z_i$ *if* $(x_i, y_i) \neq \pm(1, 0), \pm(0, 1)$ *and* $\pm(1, 1)$;
- $z_{H'_C,i} = 3z_i - 1$ *if* $(x_i, y_i) = (1, 0), (0, 1)$ *or* $(-1, -1)$;
- $z_{H'_C,i} = 3z_i - 2$ *if* $(x_i, y_i) = (-1, 0), (0, -1)$ *or* $(1, 1)$.

Proof. The only thing which does not follow straightforwardly from Proposition 6.1 and Lemma 6.4 is the difference $3z_i - z_{H'_C,i}$. However, this difference corresponds exactly to the common roots of the truncation of $P(z, w)$ and $H_P(z, w)$ along the corresponding edge of $\Delta(P)$ and $\Delta(H_P)$, which have been computed in the proof of Proposition 6.1 and Lemma 6.4. □

6.2. **Localization.** In this whole section C is non-singular tropical curve in $\mathbb{R}^2$ with Newton polygon the triangle T_d, and $P(z, w)$ is a polynomial of degree d in $\mathbb{K}[z, w]$ such that $Trop(V(P)) = C$.

The proof of next Lemma is the same as the one of [BB10, Proposition 2.1].

Lemma 6.6. *Let F be a cell of the dual subdivision of C. Then the Newton polygon F' of $H_{P_{\mathbb{C},F}}(z, w)$ is a cell of the dual subdivision of the tropical curve $Trop(V(H_P))$, and $(H_P)_{\mathbb{C},F'}(z, w) = H_{P_{\mathbb{C},F}}(z, w)$.*

It follows easily from Lemma 6.6 that the tropical curve $Trop(V(H_P))$ has the same underlying set as C, and all its edges are of weight 3.

Let H'_C (resp. W) be the tropical modification of $Trop(V(H_P))$ (resp. $\mathbb{R}^2$) given by $z^2w^2P(z, w)$. It follows from Lemma 6.6 that H'_C lies entirely in $\pi^{-1}_{|W}(C)$. Following Lemma 4.4, Theorem 5.6 reduces to estimating the weight and the direction of all vertical ends of H'_C.

Lemma 6.7. *Let v be a vertex of C with $\Delta_v \neq T_1$. Then H'_C has a vertex v' with $\pi(v') = v$ and which is also a vertex of W. Moreover if B is a ball centered in v' of radius ε small enough, then $B \cap H'_C$ is equal to a translation of $B' \cap H'_{C_v}$, where H'_{C_v} is the tropical modification of $Trop(V(H_{P_{\mathbb{C},v}}))$ given by $P_{\mathbb{C},v}(z, w)$ and B' is a ball centered in the origin of radius ε.*

Proof. The tropical curve H'_C is the tropicalization of the curve in $(\mathbb{K}^*)^3$ given by the system of equations

$$\left\{ \begin{array}{l} H_P(z, w) = 0 \\ \\ u - P(z, w) = 0 \end{array} \right. \tag{1}$$

(the coordinates in $(\mathbb{K}^*)^3$ are z, w, and u). Without loss of generality, we may assume that $v = (0, 0)$, that the point $v'' = (0, 0, 0)$ is a vertex of W, and that the coefficients of the monomials of P corresponding to the vertex of Δ_v have valuation 0. Hence if B is a small ball centered in v'', we have that $B \cap H'_C$ is given by the tropicalization of the curve obtained by plugging $t = 0$ in the system (1). According to Lemma 6.6, this tropical curve is exactly the tropical modification of $Trop(V(H_{P_{\mathbb{C},v}}))$ given by $P_{\mathbb{C},v}(z, w)$. □

Lemma 6.7 implies that given v a vertex of C such that $\Delta_v \neq T_1$, if $B_v \subset \mathbb{R}^2$ is a ball small enough centered in v, then the tropical curve H'_C is completely determined in $B_v \times \mathbb{R}$ by the tropical modification of $V(H_{P_{\mathbb{C},v}})$ given by $P_{\mathbb{C},v}(z,w)$. Since this modification is given in Lemma 6.5, we see that the curve C determines the curve H'_C in $\cup_v B_v \times \mathbb{R}$. Hence it remains to study H'_C in $\mathbb{R}^3 \setminus (\cup_v B_v \times \mathbb{R})$.

Let v be a vertex of C, and L be a tropical line with vertex v and such that $C \cap L$ contains an inflection component of multiplicity at least 3 which is not reduced to v. We denote by E the connected component of $C \cap L$ containing v. If $\Delta_v \neq T_1$ we define $E' = \overline{E} \setminus \{v\}$, and we define $E' = \overline{E}$ otherwise.

Lemma 6.8. *The number of inflection points of $\overline{X}$ with valuation in E' is 6 if E is made of 3 bounded edges of C, and 3 otherwise.*

Proof. Let $e_1, \ldots, e_r$ (resp. $\widetilde{e}_1, \ldots, \widetilde{e}_s$) be the vertical ends of H'_C (resp. unbounded edges of H'_C which are contained in $\pi_{C'}^{-1}(E)$ and not vertical), and let $(\widetilde{x}_i, \widetilde{y}_i, \widetilde{z}_i)$ be the primitive integer direction of $\widetilde{e}_i$ pointing to infinity. If $s > 0$, then there exists a unique unbounded edge e of E such that $\pi_W^{-1}(e)$ contains all edges $\widetilde{e}_1, \ldots, \widetilde{e}_s$. Let (x_e, y_e, z_e) be the primitive integer direction pointing to infinity of e. The number of inflection points of $\overline{X}$ with valuation in E' is equal to

$$\sum_{i=1}^{r} w(e_i) + 3z_e - \sum_{i=1}^{s} w(\widetilde{e}_i)\widetilde{z}_i.$$

According to Proposition 5.2, the balancing condition, Lemma 6.5, and Lemma 6.7, this sum is precisely equal to 6 if E is made of 3 bounded edges of C, and to 3 otherwise. $\square$

So far we have proved Theorem 5.6 in all cases except when E is made of three bounded edges and contains exactly 2 inflection components in its relative interior.

Lemma 6.9. *Suppose that E is made of three bounded edges and contains exactly 2 inflection components p_1 and p_2 in its relative interior. Then $\overline{X}$ has exactly 3 inflection points with valuation in p_i, $i = 1, 2$.*

Proof. The number of inflection points we are looking for is the weight of the vertical end e_i (if any) with $\pi(e_i) = p_i$. According to Proposition 5.2, the balancing condition, Lemma 6.5, and Lemma 6.7, one sees easily that there is no other possibility for this weight other than to be equal to 3. $\square$

7. Examples

In this section we use Theorem 5.7 to construct some examples of maximally inflected real curves. In Proposition 7.1 we classify all possible distributions of real inflection points among the connected components of a real quartic. In Proposition 7.2, we construct maximally inflected curves with many connected components and all real inflection points on only one of them.

Before stating Proposition 7.1, let us introduce the following notation: we say that a non-singular real algebraic curve $\mathbb{R}X$ in $\mathbb{R}P^2$ has inflection type $(n_1, \ldots, n_k)$ if $\mathbb{R}X$ has exactly k ovals $O_1, \ldots, O_k$ and O_i contains exactly n_i real inflection points. Note that a maximally inflected real quartic made of two nested ovals automatically has inflection type $(8, 0)$.

Proposition 7.1. *The inflection types realized by maximally inflected quartics in $\mathbb{R}P^2$ are exactly*

$$(8),\ (8,0),\ (6,2),\ (4,4),\ (8,0,0),\ (6,2,0),\ (4,2,2),\ (4,4,0),$$

$$(6,2,0,0),\ (4,2,2,0),\ (4,4,0,0),\ (2,2,2,2).$$

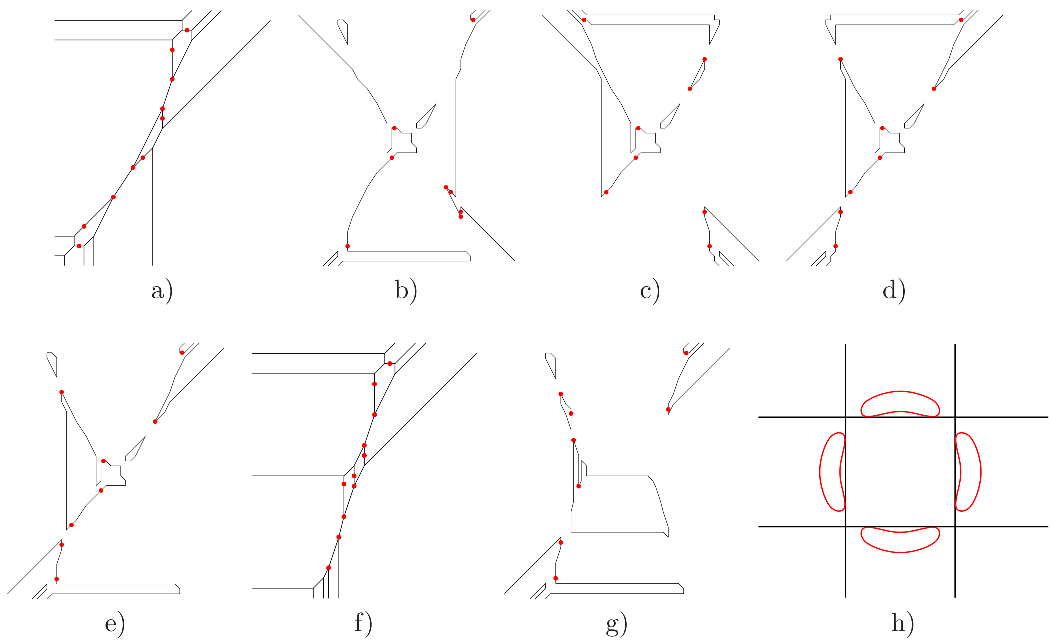


FIGURE 19. Maximally inflected quartics

Proof. The inflection types $(8), (8, 0), (6, 2), (4, 4), (4, 2, 2)$ and $(2, 2, 2, 2)$ are realized by perturbing the union of two ellipses intersecting in 4 real points. The inflection type $(6, 2, 2, 0)$ is realized by the Harnack quartic constructed in Figure 17. The inflection types $(8, 0, 0)$, $(6, 2, 0)$, $(4, 4, 0)$, and $(4, 4, 0, 0)$ are realized out of the tropical curve depicted in Figure 19a by the patchworkings depicted respectively in Figures 19b,c, d, and e. The inflection type $(4, 2, 2, 0)$ is realized out of the tropical curve depicted in Figure 19f by the patchworking depicted in Figure 19g. Note that some of these inflection types can also be realized by smoothing maximally inflected rational quartics from [KS03].

Hence it remains to prove that the inflection type $(8, 0, 0, 0)$ is not realizable by any quartic. The following argument is due to Kharlamov and simplifies considerably our original proof of this fact. It is a Theorem by Klein ([Kle76b]) that the rigid isotopy class of a non-singular real quartic curve is determined by its isotopy type in $\mathbb{R}P^2$. Moreover, it is easy to see that this isotopy type also determines the number of real bitangents to the quartic. In the case of real quartics with 4 ovals, one sees by perturbing the union of two conics (see Figure 19h) that all the 28 complex bitangents to these curves are in fact real: 24 bitangents tangent to two distinct ovals, and 4 remaining bitangents. These latter subdivide $\mathbb{R}P^2$ into 3 quadrangles and 4 triangles, each of these triangles containing exactly one oval of the quartic (see Figure 19h). In particular, no oval has 4 bitangents, which implies that no oval contains 8 real inflection points. $\square$

Proposition 7.2. *Given $k \geq 1$, there exists a maximally inflected real algebraic curve of degree $2k$ with one oval containing all real inflection points and $(k-1)^2$ other convex ovals, and there exists a maximally inflected real algebraic curve of degree $2k+1$ with the pseudo-line containing all real inflection points and k^2 other convex ovals.*

Proof. Let us consider a non-singular tropical curve of degree $2k+1$ (resp. $2k$) which contains k^2 (resp. $(k-1)^2$) copies of the fragment $\mathcal{F}$ depicted in Figure 20a (see Figure 20b in the case

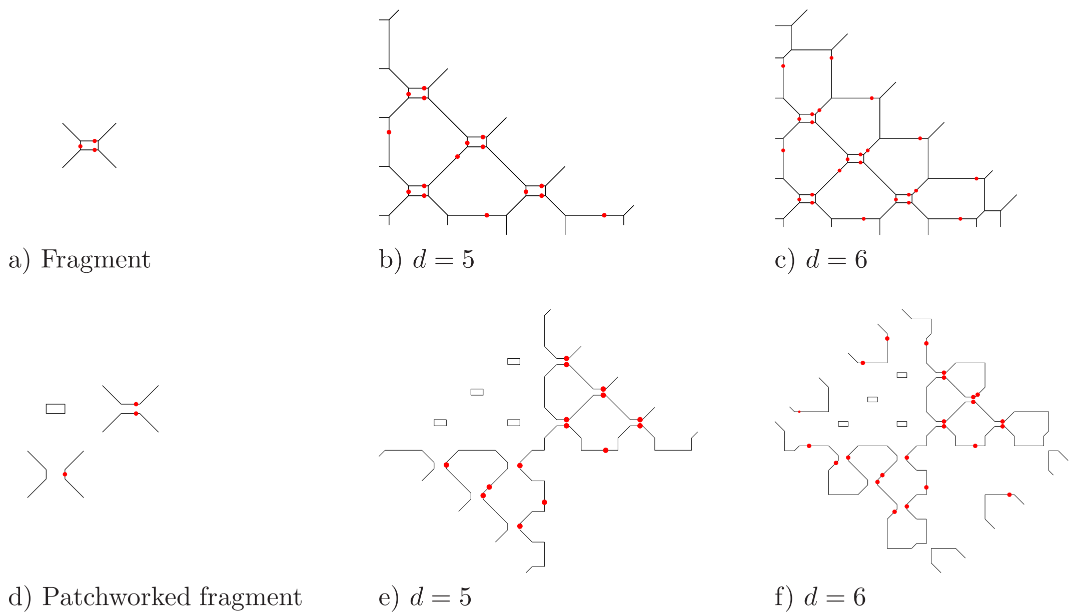


FIGURE 20. Maximally inflected curves with all inflection points on a single component

$2k + 1 = 5$, and Figure 20c in the case $2k = 6$). The curves whose existence is claim in the proposition can easily be constructed by patchworking all fragments $\mathcal{F}$ as depicted in Figure 20d (see Figure 20e in the case $2k + 1 = 5$, and Figure 20f in the case $2k = 6$). □

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Journal *of* Singularities
Volume 4 (2012), 104-113

received: 29 April 2011
in revised form: 15 March 2012
DOI: 10.5427/jsing.2012.4f

A VOLUME-PRESERVING NORMAL FORM FOR A REDUCED NORMAL CROSSING FUNCTION GERM

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Abstract. We will derive a volume-preserving normal form for holomorphic function germs that are right-equivalent to the product of all coordinates.

1. Introduction and Statement of Result

The complex version of the Morse lemma asserts that a holomorphic critical germ $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$, whose Hessian determinant is nonzero at the origin, is right equivalent to $x_1^2 + \ldots + x_n^2$. If one tightens the notion of right equivalence by stipulating that the coordinate change has to be volume-preserving, then one gets the classical theorem by J. Vey ([Vey77]), asserting that there is a volume-preserving coordinate transformation mapping f to $\Psi(x_1^2 + \ldots + x_n^2)$ where $\Psi \in \mathbb{C}\{t\}$. There is another proof of this result by M. Garay ([Gar04]) even of a much more general statement, see the third section for further explanation. And there is a third proof by J.-P. Françoise in [Fra78]. His idea was the following. Assume that you already have the desired relation $f \circ \Phi(\mathbf{x}) = \Psi(x_1^2 + \ldots + x_n^2)$ with $\Psi(t) = t + o(t)$, say. Putting $\Psi(t) = tu(t)^2$ for some u with $u(0) \neq 0$ one rewrites the relation as $f \circ \Psi(\mathbf{x}) = [x_1 u(x_1^2 + \ldots + x_n^2)]^2 + \ldots + [x_n u(x_1^2 + \ldots + x_n^2)]^2$. Then the map $(x_1, \ldots, x_n) \mapsto (x_1 u(x_1^2 + \ldots + x_n^2), \ldots, x_n u(x_1^2 + \ldots + x_n^2))$ is a coordinate transformation and once it is applied, we can reduce the problem to a problem on the Brieskorn module. It is interesting to note that both Françoise and Garay use this module.
In this article we generalize the approach by Françoise to quasihomogeneous polynomials P instead of the $x_1^2 + \ldots + x_n^2$ in the lemmas 2.1, 2.2 and 3.1. They deal with the above-mentioned coordinate change which was only roughly sketched in Françoise's paper. Having established this, we can use a nonisolated version of the Brieskorn module which was already considered in [Fra82] to deduce a normal form for $P(x_1, \ldots, x_n) = x_1 \cdots x_n$:

Theorem 1.1. *Consider a holomorphic germ $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ that is right equivalent to the product of all coordinates: $f \sim x_1 \cdots x_n$. Then there exists a volume-preserving automorphism $\Phi\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ and an automorphism $\Psi\colon (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ such that*

$$f(\Phi(\mathbf{x})) = \Psi(x_1 \cdots x_n).$$

Ψ is uniquely determined by f up to a sign.

The uniqueness of Ψ is established in the fourth section by the technique of integrating over the fibre of f. In the final section we make several comments regarding the search for volume-preserving normal forms in general.

As usual, $\mathcal{O}_{\mathbb{C}^n,0}$ denotes the ring of germs of holomorphic functions at the origin in $\mathbb{C}^n$ and $\mathfrak{m}_{\mathbb{C}^n,0}$ its maximal ideal. Writing $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is equivalent to $f \in \mathfrak{m}_{\mathbb{C}^n,0}$. The group of biholomorphisms between sufficiently small neighbourhoods of the origin in $\mathbb{C}^n$ is denoted by

Key words and phrases. nonisolated singularity, volume-preserving normal form.

$\mathrm{Aut}(\mathbb{C}^n, 0)$. An element of this group provides a right-equivalence. Such an element is volume-preserving if the Jacobian determinant of the automorphism is equal to the constant function one in a neighbourhood of the origin.

2. Main Lemma

For the proof of the main lemma we need the following fact from linear algebra which is easily proved by looking at the eigenvalues of the matrix vw^t.

Lemma 2.1. *For $v, w \in \mathbb{C}^n$ (written as column vectors) and $a, b \in \mathbb{C}$ we have*

$$\det(aI + bvw^t) = a^{n-1}(a + bv^t w).$$

Let $w_1, \ldots, w_n$ and N be positive integers. A polynomial $P \in \mathbb{C}[x_1, \ldots, x_n]$ is called quasihomogeneous of type $(w_1, \ldots, w_n; N)$ if for all $\mathbf{x} \in \mathbb{C}^n$ and all $\lambda \in \mathbb{C}$ the relation

$$P(\lambda^{w_1} x_1, \ldots, \lambda^{w_n} x_n) = \lambda^N P(\mathbf{x})$$

holds.

Lemma 2.2 (Main Lemma)**.**
Let P be quasihomogeneous of type $(w_1, \ldots, w_n; N)$. Let $u \in \mathcal{O}_{\mathbb{C},0}$ with $u(0) \neq 0$. Then the map

$$A \colon (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0), \quad \mathbf{x} \mapsto (u(P(\mathbf{x}))^{w_1} x_1, \ldots, u(P(\mathbf{x}))^{w_n} x_n)$$

defines an automorphism of $(\mathbb{C}^n, 0)$ with the following properties:

a) There exists a unique $v \in \mathcal{O}_{\mathbb{C},0}$ such that the inverse map A^{-1} is given by

$$\mathbf{z} \mapsto (v(P(\mathbf{z}))^{w_1} z_1, \ldots, v(P(\mathbf{z}))^{w_n} z_n).$$

Furthermore $v(0) \neq 0$.

b) With this v, the Jacobian determinant of A^{-1} is given by

$$\det(DA^{-1}(\mathbf{z})) = \left(v(P)^w + \frac{N}{w} P \frac{d}{dP} v(P)^w \right)\Bigg|_{P = P(\mathbf{z})}.$$

Here we have put $w := w_1 + \ldots + w_n$.

c) If we denote the assignment $u \mapsto v$ by $E \colon \mathrm{Units}(\mathcal{O}_{\mathbb{C},0}) \to \mathrm{Units}(\mathcal{O}_{\mathbb{C},0})$, then $E \circ E = \mathrm{id}$.

Proof. The assignment

$$A \colon \mathbf{x} \mapsto \mathbf{z} := (u(P(\mathbf{x}))^{w_1} x_1, \ldots, u(P(\mathbf{x}))^{w_n} x_n)$$

is an automorphism of $(\mathbb{C}^n, 0)$ since its Jacobian at the origin is regular:

$$DA(0) = \begin{pmatrix} u(0)^{w_1} & 0 & \ldots \\ & \ldots & \\ \ldots & 0 & u(0)^{w_n} \end{pmatrix}.$$

It is clear that the inverse of A is of the form $A^{-1} \colon \mathbf{z} \to (\tilde{v}_1(\mathbf{z}) z_1, \ldots, \tilde{v}_n(\mathbf{z}) z_n)$ for some $\tilde{v} \in \mathcal{O}_{\mathbb{C}^n,0}$. (In fact, if we write $x_i = x_i(\mathbf{z})$ for the components of $A^{-1}(\mathbf{z})$, then $z_i = u(P(\mathbf{x}(\mathbf{z})))^{w_i} x_i(\mathbf{z})$, so that z_i must divide $x_i(\mathbf{z})$.)

In the sequel we are going to show that it is even of the form

$$\mathbf{z} \mapsto (v(P(\mathbf{z}))^{w_1} z_1, \ldots, v(P(\mathbf{z}))^{w_n} z_n)$$

for some $v \in \mathcal{O}_{\mathbb{C},0}$! We also show that v is uniquely determined by u and that $v(0) \neq 0$.

Let $\mathbf{z} = A(\mathbf{x})$. From

$$\begin{aligned}\mathbf{z} &= (u(P(\mathbf{x}))^{w_1} x_1, \ldots, u(P(\mathbf{x}))^{w_n} x_n) \\ &= (u(P(\mathbf{x}))^{w_1} \tilde{v}_1(\mathbf{z}) z_1, \ldots, u(P(\mathbf{x}))^{w_n} \tilde{v}_n(\mathbf{z}) z_n)\end{aligned}$$

we conclude that

$$1 = u(P(\mathbf{x}))^{w_i} \tilde{v}_i(\mathbf{z}) \text{ for } i = 1, \ldots, n.$$

Hence for the function $\hat{v} \in \mathcal{O}_{\mathbb{C}^n,0}$ defined by

$$\hat{v}(\mathbf{z}) := \frac{1}{u(P(A^{-1}(\mathbf{z})))},$$

we have

$$\tilde{v}_i(\mathbf{z}) = \hat{v}(\mathbf{z})^{w_i} \text{ for all } i.$$

Now let us show that the function $\hat{v}$ factors through $P(\mathbf{z})$. First we rewrite its defining equation

$$\begin{aligned}1 &= u(P(\mathbf{x}))\, \hat{v}(\mathbf{z}) \\ &= u(P(\tilde{v}_1(\mathbf{z}) z_1, \ldots, \tilde{v}_n(\mathbf{z}) z_n))\, \hat{v}(\mathbf{z}) \\ &= u(P(\hat{v}(\mathbf{z})^{w_1} z_1, \ldots, \hat{v}(\mathbf{z})^{w_n} z_n))\, \hat{v}(\mathbf{z}) \\ &= u(P(z_1, \ldots, z_n) \hat{v}(\mathbf{z})^N)\, \hat{v}(\mathbf{z}). \qquad (2.1)\end{aligned}$$

To see factorization through P, we apply twice the implicit function theorem as follows.

(1) The implicit equation $u(v^N t)v = 1$ for v has a unique local solution $v = v(t)\colon (\mathbb{C}, 0) \to (\mathbb{C}, 1/u(0))$. Indeed, the point $(t = 0, v = 1/u(0))$ is a solution and the derivative after v is nonzero at this point:

$$\partial_v \; (u(0)v)|_{v=1/u(0)} = u(0) \neq 0.$$

(2) The implicit equation $u(V^N P(\mathbf{z}))V = 1$ for V has a unique local solution $V = V(\mathbf{z})\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 1/u(0))$. Indeed, the point $(\mathbf{z} = 0, V = 1/u(0))$ is a solution and the derivative after V at this point is nonzero:

$$\partial_V \; (u(0)V)|_{V=1/u(0)} = u(0) \neq 0.$$

Now by the first item (only existence is used), $v(P(\mathbf{z}))$ fulfils $v(P(0)) = 1/u(0)$ and solves the equation $u(v(P(\mathbf{z}))^N P(\mathbf{z}))v(P(\mathbf{z})) = 1$. Comparing this result and equation (2.1) we can deduce from the second item (only uniqueness is used) that

$$v(P(\mathbf{z})) = \hat{v}(\mathbf{z}). \qquad (2.2)$$

Hence $\tilde{v}_i(\mathbf{z}) = \hat{v}(\mathbf{z})^{w_i} = v(P(\mathbf{z}))^{w_i}$, hence A^{-1} is of the desired form as stated in part a) of the assertion. Note that $\hat{v}(0) = 1/u(0)$ by its definition and therefore also $v(0) = 1/u(0)$.

The proof of part a) is not yet quite complete. What about the uniqueness of v when we have just given u? By its very definition, $\hat{v}$ is uniquely determined by u (and P). Since $v(P(\mathbf{z})) = \hat{v}(\mathbf{z})$ and since P is surjective onto a neighbourhood of zero, also v is uniquely determined by u.

For part c) of the assertion we note that the operator E which asigns to u the function v is given by solving the implicit equation $u(v^N t)v = 1$ with $v(0) = 1/u(0)$. So let $E(u) = v$

and $E(v) = w$. Then we also have $v(w(s)^N s)w(s) = 1$ for all $s \in (\mathbb{C}, 0)$. If in the equation $u(v(t)^N t)v(t) = 1$ we substitute $t = w(s)^N s$, we get

$$\begin{aligned} u\left[v(w(s)^N s)^N w(s)^N s\right] v(w^N s) &= 1 \\ u[1^N s] \cdot 1/w(s) &= 1 \\ u(s) &= w(s). \end{aligned}$$

This shows part c).

It remains to prove part b). The (i, j)th entry in the Jacobian matrix of the transformation

$$A^{-1}\colon \quad \mathbf{z} \mapsto (v(P(\mathbf{z}))^{w_1} z_1, \ldots, v(P(\mathbf{z}))^{w_n} z_n)$$

is given by

$$\begin{aligned} \partial_i(v(P(\mathbf{z}))^{w_j} z_j) &= w_j (v(P(\mathbf{z})))^{w_j - 1} v'(P(\mathbf{z}))\partial_i P(\mathbf{z}) z_j + v(P(\mathbf{z})^{w_j} \delta_{ij} \\ &= (v(P(\mathbf{z})))^{w_j - 1} \left[w_j v'(P(\mathbf{z}))\partial_i P(\mathbf{z}) z_j + v(P(\mathbf{z}))\delta_{ij}\right] \end{aligned}$$

In order to compute its determinant we use lemma 2.1 from above. This together with the Euler relation for weighted homogeneous polynomials yields

$$\begin{aligned} &\det(DA^{-1}(\mathbf{z})) \\ &= \prod_{j=1}^{n} (v(P(\mathbf{z})))^{w_j - 1} \cdot \det(v'(P(\mathbf{z}))\partial_i P(\mathbf{z}) w_j z_j + v(P(\mathbf{z}))\delta_{ij}) \\ &= \prod_{j=1}^{n} (v(P(\mathbf{z})))^{w_j - 1} \cdot (v(P(\mathbf{z})))^{n-1} \left[v(P(\mathbf{z})) + v'(P(\mathbf{z})) \sum_{j=1}^{n} w_j z_j \partial_j P(\mathbf{z})\right] \\ &= v(P(\mathbf{z}))^{w_1 + \ldots + w_n - n + n - 1} \cdot [v(P(\mathbf{z})) + v'(P(\mathbf{z})) N P(\mathbf{z})] \\ &= \left(v(P)^w + \frac{N}{w} P \frac{d}{dP} v(P)^w \right)\Big|_{P = P(\mathbf{z})}, \end{aligned}$$

where we used the abbreviation $w = w_1 + \ldots + w_n$. □

Given u, we get the map A of the lemma which we also denote by A_u. Then we have

$$A_{E(u)} \circ A_u = \mathrm{id}\,.$$

We make a remark which however will not be used elsewhere in the paper. Assume that instead of u we have just given the map A (of the form A_u with an unspecified u). Of course the $\tilde{v}_i z_i$ which are the component functions of A^{-1} are uniquely determined by A. Then from $\tilde{v}_i(\mathbf{z}) = \hat{v}(\mathbf{z})^{w_i}$ we infer that the function $\hat{v}$ is uniquely determined up to the multiplication with some number $\xi \in \mathbb{C}$ which fulfills $\xi^{w_i} = 1$ for all i. If we demand that the greatest common divisor of the $w_1, \ldots, w_n$ is equal to one, then $\xi = 1$ and so $\hat{v}$ and also v are uniquely determined by A. Applying this argument to A^{-1} we see that given a map A (of the form A_u with some unknown $u \in \mathrm{Units}(\mathcal{O}_{\mathbb{C},0})$) the u is uniquely determined if $\gcd(w_1, \ldots, w_n) = 1$.

3. Existence of the Normal Form

By a germ of a volume form at the origin in $\mathbb{C}^n$ we understand a germ of a holomorphic n-form which does not vanish at the origin. Let (f_0, Ω_0) be a pair consisting of a germ of a function $f_0 \in \mathcal{O}_{\mathbb{C}^n,0}$ which vanishes at the origin and a germ of a volume form $\Omega_0 \in \Omega^n_{\mathbb{C}^n,0}$. Then the group $\mathrm{Aut}(\mathbb{C}^n, 0)$ acts on the set of such pairs by the usual pulling back of functions resp. forms. A normal form for a pair (f_0, Ω_0) should then be a nicely chosen pair in the same orbit. One way to achieve this is to look only at pairs in the orbit of (f_0, Ω_0) with the same $f = f_0$. Another way would be to consider only those pairs in the orbit of (f_0, Ω_0) with the same $\Omega = \Omega_0$. The latter would give us an Ω_0-preserving normal form for functions which are right equivalent to f_0. That these two approaches are interchangeable when the right normal form is chosen is the content of the following lemma which we will later only use in the direction $(ii) \Rightarrow (i)$.

Lemma 3.1 (Exchange Lemma)**.**
Let P be quasihomogeneous of type $(w_1, \dots, w_n; N)$. For a holomorphic function germ $f = f(\mathbf{y})\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ the following statements are equivalent:

i) There exist an automorphism $\Phi \in \mathrm{Aut}(\mathbb{C}^n, 0), \mathbf{y} \mapsto \mathbf{x}$ and an automorphism $\Psi \in \mathrm{Aut}(\mathbb{C}, 0)$ such that
$$f(\Phi^{-1}(\mathbf{x})) = \Psi(P(\mathbf{x})) \text{ and } (\Phi^{-1})^* d^n\mathbf{y} = d^n\mathbf{x}.$$
ii) There exist an automorphism $\phi \in \mathrm{Aut}(\mathbb{C}^n, 0), \mathbf{z} \mapsto \mathbf{y}$ and a function $\psi \in \mathcal{O}_{\mathbb{C},0}$ with $\psi(0) \neq 0$ such that
$$f(\phi(\mathbf{z})) = P(\mathbf{z}) \text{ and } \phi^* d^n\mathbf{y} = \psi(P(\mathbf{z}))d^n\mathbf{z}.$$

Proof. We start with the implication $(i) \Rightarrow (ii)$. Since $\Psi'(0) \neq 0$ there is a germ $u \in \mathcal{O}_{\mathbb{C},0}, u(0) \neq 0$ with $\Psi(t) = tu(t)^N$. From the quasihomogeneity of P we get

$$\begin{aligned}\Psi(P(\mathbf{x})) &= P(\mathbf{x})u(P(\mathbf{x}))^N \\ &= P(u(P(\mathbf{x}))^{w_1}x_1, \dots, u(P(\mathbf{x}))^{w_n}x_n).\end{aligned}$$

If we define the map

$$A\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0), \mathbf{x} \mapsto \mathbf{z} := (u(P(\mathbf{x}))^{w_1}x_1, \dots, u(P(\mathbf{x}))^{w_n}x_n)$$

then $\Psi(P(\mathbf{x})) = P(A(\mathbf{x}))$. The first part of item (i), $f(\Phi^{-1}(\mathbf{x})) = \Psi(P(\mathbf{x}))$ can therefore be rewritten as $f(\Phi^{-1}(\mathbf{x})) = P(A(\mathbf{x}))$. Since by lemma 2.2 the map A is an automorphism of $(\mathbb{C}^n, 0)$, we can rewrite this again: we let $\phi \in \mathrm{Aut}(\mathbb{C}^n, 0), \mathbf{z} \mapsto \mathbf{y}$ with $\phi := \Phi^{-1} \circ A^{-1}$, then it follows $f(\phi(\mathbf{z})) = P(\mathbf{z})$. This is the first assertion of item (ii).

Again by lemma 2.2 there is $v \in \mathcal{O}_{\mathbb{C},0}, v(0) \neq 0$ with

$$\det(DA^{-1}(\mathbf{z})) = \left.\left(v(P)^w + \frac{N}{w}P\frac{d}{dP}v(P)^w\right)\right|_{P=P(\mathbf{z})}.$$

If we define $\psi\colon (\mathbb{C}, 0) \to \mathbb{C}$ by this bracket, i.e.

$$\psi(t) := v(t)^w + \frac{N}{w}t\frac{d}{dt}\left(v(t)^w\right),$$

then $\psi(0) \neq 0$ and we can write the pullback of the volume form as

$$\begin{aligned}\phi^* d^n\mathbf{y} &= (A^{-1})^*(\Phi^{-1})^* d^n\mathbf{y} \\ &= (A^{-1})^* d^n\mathbf{x} \\ &= \psi(P(\mathbf{z}))d^n\mathbf{z}.\end{aligned}$$

This is the second assertion of item (*ii*).

Now we prove the converse direction. So let us assume (*ii*) is valid. First we seek a solution $v\colon (\mathbb{C},0) \to \mathbb{C}, v(0) \neq 0$ of the equation

$$\left(v(t)^w + \frac{N}{w} t \frac{d}{dt} v(t)^w\right) = \psi(t) \tag{3.1}$$

where ψ is the function as given in statement (ii), i.e. $\psi\colon (\mathbb{C},0) \to \mathbb{C}, \psi(0) \neq 0$. A solution can be obtained from a power series ansatz, namely if $\psi(t) = \sum a_i t^i$ and $v^w = \sum b_i t^i$, then comparison of the coefficients shows that the stipulation

$$b_i := \frac{a_i}{1 + (Ni/w)}$$

will provide a solution v^w of the differential equation. Since $\psi(0)$ is nonzero so is $v^w(0)$. Hence, taking some wth root v of v^w will give us v.

Now we define $u \in \mathcal{O}_{\mathbb{C},0}$ as $u = E^{-1}(v)$, cf. lemma 2.2. Then $\det(DA_u^{-1}(\mathbf{z})) = \det(DA_v(\mathbf{z})) = \psi(P(\mathbf{z}))$ by that lemma and the definition of v. Now define $\Phi := A_u^{-1}\phi^{-1}$. Then

$$\begin{aligned}
(\Phi^{-1})^* d^n\mathbf{y} &= (\phi \circ A_u)^* d^n\mathbf{y} \\
&= A_u^* \phi^* d^n\mathbf{y} \\
&= A_u^*(\psi(P(\mathbf{z})) d^n\mathbf{z}) \\
&= \psi(P(A_u(\mathbf{x}))) \det DA_u(\mathbf{x}) d^n\mathbf{x} \\
&= \psi(P(\mathbf{z})) \det DA_u(\mathbf{x}) d^n\mathbf{x} \\
&= d^n\mathbf{x}
\end{aligned}$$

Finally when we insert into the given relation $f(\phi(\mathbf{z})) = P(\mathbf{z})$ the expression $\mathbf{z} = A_u(\mathbf{x})$ we can rewrite it as

$$\begin{aligned}
f(\Phi^{-1}(\mathbf{x})) &= P(A_u(\mathbf{x})) \\
&= P(u(P(\mathbf{x}))^{w_1} x_1, \ldots, u(P(\mathbf{x}))^{w_n} x_n) \\
&= P(\mathbf{x}) u(P(\mathbf{x}))^N.
\end{aligned}$$

So letting $\Psi(t) := tu(t)^N$ we have the statement $f \circ \Phi^{-1}(\mathbf{x}) = \Psi(P(\mathbf{x}))$ of our assertion. We note $\Psi'(0) = u(0)^N = 1/v(0)^N \neq 0$, so Ψ is an automorphism of $(\mathbb{C},0)$. This completes the proof. □

We now show that part (*ii*) in lemma 3.1 is true for $P = x_1 \cdots x_n$. Prior to this a digression on the Brieskorn modules is neccessary.

In the seminal paper [Bri70] Brieskorn has introduced different $\mathbb{C}\{t\}$-modules for the investigation of the monodromy of an isolated singularity. One of these modules is given for an isolated singularity $f\colon (\mathbb{C}^n,0) \to (\mathbb{C},0)$ by

$$H_f'' = \frac{\Omega^n_{\mathbb{C}^n,0}}{df \wedge d\Omega^{n-2}_{\mathbb{C}^n,0}}.$$

Here $\Omega^k_{\mathbb{C}^n,0}$ denotes the vector space of germs of holomorphic k-forms at the origin in $\mathbb{C}^n$. The $\mathbb{C}\{t\}$-module structure of this module comes from multiplication with f. It is shown in the cited paper together with Sebastiani's paper [Seb70], see also Malgrange [Mal74], that this is a free module with rank equal to the Milnor number $\mu(f,0)$ of f at the origin. This classical Brieskorn module was extended to apply for isolated complete intersection singularities by Greuel in [Gre75]. It is this "parametrized version" of the Brieskorn module which allowed Garay in [Gar04] to proof his volume-preserving versal unfolding theorem from which one can deduce the theorem of Vey. The former theorem roughly states that there are $\mu(f,0)$ holomorphic moduli for volume-preserving right equivalence. One can ask if it possible to gain similar results for nonisolated singularities. Following analogy we face the problem of choosing the right nonisolated version of the Brieskorn module. Such nonisolated versions were e.g. looked at in the paper by van Straten [vSt87]. But also Françoise in his study of normal forms was already considering

$$F_f := \frac{\Omega^n_{\mathbb{C}^n,0}}{\{d\eta|\ df \wedge \eta = 0\}},$$

which is again a $\mathbb{C}\{t\}$-module. For isolated singularities F_f equals H''_f by the de Rham lemma. But for arbitrary singularities not much is known. At least for $n = 2$ Barlet has shown (cf. [BS07]) that this module is free of finite rank. However, in more than two dimensions freeness of F_f is in general not given ([BS07]). If $P(\mathbf{x}) = x_1^{m_1} \dots x_n^{m_n}$ then F_P has $\gcd(m_1, \dots, m_n)$ generators which are given explicitely in [Fra82]. For $P = x_1 \cdots x_n$, F_P is generated by the single form $d^n\mathbf{x} = dx_1 \wedge \dots \wedge dx_n$.

Now let f be right equivalent to this P. Choosing $\Phi_1 \in \mathrm{Aut}(\mathbb{C}^n, 0)$ with $\Phi_1^* f = P$ there exists $\psi \in \mathbb{C}\{t\}$ and η with $dP \wedge \eta = 0$ such that $\Phi_1^* d^n\mathbf{x} = \psi \circ P\, d^n\mathbf{x} + d\eta$. Now it is important to note - as shown in the proof by Françoise - that among the power series terms on the left-hand side only the constant term, i.e. $\det(D\Phi_1)(0)$, will contribute to the constant term of ψ and they are *equal*. In particular $\psi(0) \neq 0$. Finally we note that $\eta(0) = 0$.

We now make use of the

Lemma 3.2. *Let $g \in \mathfrak{m}_{\mathbb{C}^n,0}$ and Ω_1, Ω_2 two n-forms on $(\mathbb{C}^n, 0)$ with the* same nonzero value *at the origin. If there is an $(n-1)$-form η with $\Omega_1 - \Omega_2 = d\eta$ and $\eta(0) = 0$ such that $dg \wedge \eta = 0$, then there exists $\Phi_2 \in \mathrm{Aut}(\mathbb{C}^n, 0)$ with $\Phi_2^* g = g$ and $\Phi_2^* \Omega_1 = \Omega_2$.*

The proof is based on the path method and can be found in [Fra82].

Applying it to $\Omega_1 := \Phi_1^* d^n\mathbf{x}, \Omega_2 := \psi \circ P d^n\mathbf{x}$ and $g := P$ we get an automorphism Φ_2 with $\Phi_2^* \Phi_1^* d^n\mathbf{x} = \psi \circ P\, d^n\mathbf{x}$ and $\Phi_2^* P = P$. So if we put $\phi := \Phi_1 \circ \Phi_2$ we have

$$\phi^* d^n\mathbf{x} = \psi \circ P\, d^n\mathbf{x} \quad \text{and} \quad \phi^* f = P.$$

This is item (ii) of lemma 3.1. The implication $(ii) \Rightarrow (i)$ thus yields the existence of the normal form.

4. Uniqueness of the Normal Form

We now address the question of unicity of Ψ. The equation $f \circ \Phi(y) = \Psi(P(y))$ can be written as a commutative diagram

$$\begin{tikzcd}
(\mathbb{C}^n, 0) \arrow[d, "f"'] & (\mathbb{C}^n, 0) \arrow[l, "\Phi"'] \arrow[d, "P"] \\
(\mathbb{C}, 0) & (\mathbb{C}, 0) \arrow[l, "\Psi"]
\end{tikzcd}$$

For sufficiently small $\epsilon > 0$ and for all sufficiently small $0 < \delta \ll \epsilon$ we have the Milnor-Lê fibration ([Lê77]) $f \colon B_\epsilon \cap f^{-1}(D_\delta^*) \to D_\delta^*$ where B_ϵ is the open ϵ-ball around $0 \in \mathbb{C}^n$ and D_δ^* is the open δ-ball around the origin in $\mathbb{C}$ minus this point. The general fibre is called the Milnor fibre $\mathrm{Mil}_{f,0}$ of f. For a quasihomogeneous P we can compute the Milnor fibre as the general fibre of the global affine fibration $P\colon \mathbb{C}^n \setminus P^{-1}(0) \to \mathbb{C}^*$, see ([Dim92], p. 68 - 72). Hence the Milnor fibre of P over $s \in D_\delta^*$

$$\mathrm{Mil}_{P,0}(s) = \{\mathbf{x} \in B_\epsilon |\, x_1 \cdots x_n = s\}$$

is diffeomorphic to

$$\{(x_1, \dots, x_n) \in \mathbb{C}^n |\, x_1 \cdots x_n = 1\} \cong \{(x_2, \dots, x_n) \in (\mathbb{C}^*)^{n-1}\}.$$

Similar statements hold if replace the standard ball B_ϵ by a ball defined by a rug function ($B_\epsilon(\rho) = \{\mathbf{x} \in \mathbb{C}^n | \rho(\mathbf{x}) < \epsilon\}$ where $\rho\colon (\mathbb{C}^n, 0) \to \mathbb{R}_{\geq 0}$ is real analytic such that $\rho^{-1}(0) = \{0\}$.) So $H_{n-1}(\mathrm{Mil}_{P,0}(s); \mathbb{Z}) \cong \mathbb{Z}$ with generator $\gamma(P, s)$ given by the product of $(n-1)$ circles. In fact for s real and $s < (\epsilon/\sqrt{n})^n$ we have a map

$$\underbrace{S^1 \times \ldots \times S^1}_{n-1} \hookrightarrow \mathrm{Mil}_{P,0}(s),\ (z_2, \dots, z_n) \mapsto (s^{1/n}/(z_2 \ldots z_n), z_2 s^{1/n}, \dots, z_n s^{1/n}),$$

which is easily checked to be an embedding. Along this cycle we can integrate any holomorphic $(n-1)$-form λ and if we choose λ as a holomorphic primitive of $d^n\mathbf{x}$, e.g. $\lambda = x_1 dx_2 \wedge \ldots \wedge dx_n$, then we evaluate the integral of λ over one of the generators of $H_{n-1}(\mathrm{Mil}_{P,0}(s); \mathbb{Z})$ as

$$\int_{\gamma(P,s)} \lambda = \int_{S^1 \times \ldots \times S^1} s \frac{dz_2}{z_2} \ldots \frac{dz_n}{z_n} = \pm (2\pi i)^{n-1} s.$$

(Of course, if we had chosen the canonical orientation of $\mathrm{Mil}_{P,0}(s)$ as a complex manifold we would get a plus sign, but it is not important here.)

Finally let $\gamma(f, \cdot)$ be a locally constant section of the $(n-1)$st homological fibration of f, obtained by parallel translating one of the two homology generators of a single reference fibre. Then we get an a priori multivalued holomorphic function germ $t \mapsto \int_{\gamma(f,t)} \lambda$. From the commutativity of the above diagram it follows that an integral homology generator of $\mathrm{Mil}_{P,0}(\Psi^{-1}(t))$ is sent via Φ_* to one of the two generators of $H_{n-1}(\mathrm{Mil}_{f,0}(t); \mathbb{Z})$ and so we obtain

$$\int_{\gamma(f,t)} \lambda = \pm \int_{\Phi_* \gamma(P, \Psi^{-1}(t))} \lambda.$$

Now Φ being volume-preserving, it preserves λ up to a differential, so the right-hand side becomes

$$\begin{aligned} &\int_{\gamma(P,\Psi^{-1}(t))} \Phi^* \lambda \\ &= \int_{\gamma(P,\Psi^{-1}(t))} \lambda \\ &= \pm (2\pi i)^{n-1} \Psi^{-1}(t). \end{aligned}$$

Hence $\Psi^{-1}(t) = \pm \left(\frac{1}{2\pi i}\right)^{n-1} \int_{\gamma(f,t)} \lambda$ so that Ψ is uniquely determined by f, possibly up to a sign.

And indeed we show that the alleged ambiguity in the choice of Ψ's sign cannot be eliminated: Take any permutation matrix $S \in \mathbb{C}^{n \times n}$ with determinant -1 and let c be any number with $c^n = -1$. Then the linear map $\mathbf{x} \mapsto \Phi(\mathbf{x}) := cS\mathbf{x}$ is volume-preserving and transforms $x_1 \cdots x_n$ to $(cx_1) \cdots (cx_n) = -x_1 \cdots x_n$.

5. Comments

Stokes theorem in the real two-dimensional plane asserts that $\oint_C xdy$ computes the area of the interior that is surrounded by the simple closed curve C. When we think of x, y as complex variables and the curve to be a cycle lying in some smooth fibre of a function $f \in \mathfrak{m}_{\mathbb{C}^2,0}$, then one is led to believe that such an integral should be significant for the study of volume-preserving equivalence. And indeed it is, as we have seen for example in section four. Recalling that Garay's unfolding theorem can roughly be interpreted that an isolated singularity f has $\mu(f,0)$ continuous moduli for volume-preserving equivalence, it seems natural to expect that when $\mu(f,0) = 1$ there is only one continuous obstruction. This obstruction should then be the aforementioned integral. And in fact it is the function Ψ from Vey's statement. It is natural to conjecture that we can find volume-preserving normal forms for nonisolated singularities f as well, as long as $H_{n-1}(\mathrm{Mil}_{f,0})$ has rank one. This has been done in this article when f is right equivalent to $x_1 \cdots x_n$. What about other cases? For a singularity f in two variables $H_{n-1}(\mathrm{Mil}_{f,0})$ has rank one if and only if f is right equivalent to $x^a y^b$ with $\gcd(a,b) = 1$. (This should be well-known; it follows e.g. if we compare the homotopy exact sequences of the Milnor fibrations associated to f itself and its reduced version f_{red}. For details the reader is sent to [Sza12].) For $a = b = 1$ we have Vey's lemma, but for other values of a and b, we cannot use the above methods anymore: Instead of $[1 \cdot dx \wedge dy]$, according to Françoise, $[x^{a-1}y^{b-1}\, dx \wedge dy]$ if a generator of F_f but then lemma 3.2 has to be applied e.g. to $\Omega_1 = \Phi_1^* dx \wedge dy$ and $\Omega_2 = dx \wedge dy + \psi \circ P \cdot x^{a-1}y^{b-1}\, dx \wedge dy$ which will however not yield the statement of lemma 3.1(ii).

Finally we can check that the integral of $\lambda = x\, dy$ over a generator of $H_1(\mathrm{Mil}_{f,0})$ is zero: Choose real numbers $0 < s \ll \epsilon \ll 1$ such that $\mathrm{Mil}_f(s) = \{(x,y) \in B_\epsilon(0) |\ x^a y^b = s\}$ is the Milnor fibre of $f(x,y) = x^a y^b$ where $a, b \in \mathbb{N}$ are coprime integers. Then we can embed S^1 into the Milnor fibre over s using the map

$$S^1 \ni z \mapsto (x(t), y(t)) = (z^b s^{1/a+b}, z^{-a} s^{1/a+b}).$$

In fact this map is an injective immersion of a compact space, hence an embedding. (The injectivity follows from $\gcd(a,b) = 1$.) We now integrate the form $x^m y^n dy$ along this cycle:

$$\begin{aligned}\int_{S^1} x^m y^n\, dy &= \int_{S^1} z^{mb} s^{m/(a+b)} z^{-an} s^{n/(a+b)} (-a) z^{-a-1} s^{1/(a+b)} dz \\ &= -a s^{(m+n+1)/(a+b)} \int_{S^1} z^{mb-an-a-1} dz.\end{aligned}$$

This integral is nonzero if and only if $mb - an - a = 0$. Of course there are choices of m, n where this is achieved. Thus, the embedded circle is homologically nontrivial, i.e. represents a generator of $H_1(\mathrm{Mil}_{f,0}; \mathbb{C}) \cong \mathbb{C}$. Now we let $m = 1$ and $n = 0$, so that $\lambda = x\, dy$ is a primitive of the volume form. Its integral is nonzero if and only if $b - a = 0$. But since $\gcd(a,b) = 1$ this holds only if $a = b = 1$.

So a normal form for functions which are right equivalent to $x^a y^b$ with coprime a, b is unlikely to exist in the simple form $f \circ \Phi = \Psi(x^a y^b)$ with Φ volume-preserving. But at least we believe that it might exist in more complicated form.

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Journal *of* Singularities
Volume 4 (2012), 114-130

received: 10 January 2012
in revised form: 13 April 2012
DOI: 10.5427/jsing.2012.4g

A POLYNOMIAL GENERALIZATION OF THE EULER CHARACTERISTIC FOR ALGEBRAIC SETS.

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WITH AN APPENDIX BY J. V. RENNEMO

Abstract. We present a method to compute the Euler characteristic of an algebraic subset of $\mathbb{C}^n$. This method relies on classical tools such as Gröbner basis and primary decomposition. The existence of this method allows us to define a new invariant for such varieties. This invariant is related to the problem of counting rational points over finite fields. In an appendix, Jørgen Vold Rennemo proves the relation between this invariant and the Chern-Schwartz-MacPherson class of the variety.

1. Introduction

One of the main invariants of a topological space is its Euler characteristic. It was initially defined for cell complexes, but several extensions have been defined to more general classes of spaces. In the setting of complex algebraic varieties, the natural extension is the Euler characteristic with compact support. In [8] Szafraniec gives a method to compute the Euler characteristic of a complex algebraic set by using methods from the real geometry. Aluffi gave in [1] another method based on the computation of Chern-Schwartz-MacPherson classes. In this paper, we present another method, that only makes use of the basic properties of the Euler characteristic, and classical results on algebraic sets. This way of computing the Euler characteristic gives naturally a stronger invariant, which we define.

The method works as follows. Consider $V \subseteq \mathbb{C}^n$ an irreducible algebraic set of dimension d and degree g. Take a generic linear projection $\pi : \mathbb{C}^n \to \mathbb{C}^d$. If we consider π restricted to V, it is a $g : 1$ branched cover. The branching locus Δ and its preimage $\pi |_V^{-1} (\Delta)$ can be computed. From the additivity and the multiplicativity for covers of the Euler characteristic, we have the following formula:

$$\chi(V) = g \cdot \chi(\mathbb{C}^d) - g \cdot \chi(\Delta) + \chi(\pi |_V^{-1} (\Delta)).$$

So the computation of $\chi(V)$ is reduced to the computation of the Euler characteristic of algebraic sets of lower dimension, allowing us to use a recursion process.

In the previous method, we make use of the fact that $\chi(\mathbb{C}^d) = 1$. If instead of making this substitution, we keep track of $\chi(\mathbb{C}^d)$ as a formal symbol, we obtain a stronger invariant $F(V)$. This invariant is defined as a polynomial in $\mathbb{Z}[L]$, which turns out to coincide with the counting polynomial defined by Plesken in [6], and has some interesting properties: the dimension, degree, and Euler characteristic of an algebraic set can be computed from this polynomial. It also gives information on the number of points on some varieties over finite fields. This relation with finite varieties could be used to compute this invariant by counting points.

In Sections 2 and 3 we show the preliminary results that prove the correctness of the method to compute the Euler characteristic, and describe the algorithm. Section 4 is devoted to the generalization of this method to the new invariant, which is also defined and some of its properties

Partially supported by the ERC Starting Grant project TGASS, by Spanish Contract MTM2010-21740-C02-01, and by research group "Geometra" E15 of the government of Aragn.

are shown. This invariant is defined after both the variety and the used projection, although we conjecture that, assuming general position, every projection would give the same result, making the invariant independent of the projection. After the first version of this paper was made public, Jørgen Vold Rennemo contacted the author proposing a way to prove the main conjecture. His proof consists on showing that this invariant is directly related to the Chern-Schwartz-MacPherson class of the variety. This proof is included in the appendix.

The extension of this invariant to projective varieties is discussed in Section 5. In Section 6 we include some examples. In particular, we show that in the case of hyperplane arrangements this invariant coincides with the characteristic polynomial. Finally, the relationship of the invariant with the number of points over finite fields is shown in Section 7. As an annex, Section 8 includes an implementation of the algorithms in Sage together with some timings.

2. Theoretical justification

Let $V = V(I) \subseteq \mathbb{C}^n$ be the algebraic set determined by a radical ideal I. Without loss of generality, we can assume that it is in general position (in the sense that we will precise later). By computing the associated primes of I we obtain the decomposition in irreducible components $V = V_1 \cup \cdots \cup V_c$. The Euler characteristic $\chi(V)$ can be expressed as $\chi(V_1) - \chi((V_1) \cap (V_2 \cup \cdots \cup V_c)) + \chi(V_2 \cup \cdots \cup V_c)$. The variety $(V_1) \cap (V_2 \cup \cdots \cup V_c)$ is an algebraic set of lower dimension. So, by a double induction argument (over the dimension and over the number of irreducible components), we may reduce the problem of computing $\chi(V)$ to the case where V is either zero-dimensional or irreducible.

If V is zero-dimensional, it consists of a number of isolated points, and its Euler characteristic equals the number of points. This number of points can be computed as the degree of the homogenization of the radical of I (which can be computed via the Hilbert polynomial, see [3, Chapter 5] for example).

For the case of an irreducible variety $V = V(I) \subseteq \mathbb{C}^n$ being $I \trianglelefteq \mathbb{C}[x_1, \ldots, x_n]$ a radical ideal of Krull dimension d, we will distinguish the homogeneous case from the non homogeneous.

If I is a homogeneous ideal, the variety V has a conic structure (it is formed by a union of lines that go through the origin). It means that V is contractible and hence its Euler characteristic is 1.

For the non homogeneous case, consider the projection

$$\begin{array}{rcl} \pi : \mathbb{C}^n & \to & \mathbb{C}^d \\ (x_1, \ldots, x_n) & \mapsto & (x_1, \ldots x_d) \end{array}$$

We may assume (applying a generic linear change of coordinates if necessary) that the following condition is satisfied:

Definition 1. Consider $I_h \trianglelefteq \mathbb{C}[x_0, x_1, \ldots, x_n]$ the homogenization of I. We will say that I is in **general position** if $\sqrt{I_h + (x_0, x_1, \ldots, x_d)} \supseteq (x_0, x_1, \ldots, x_n)$.

Theorem 2. *The previous condition is satisfied by any ideal I after a generic linear change of variables. Moreover, when this condition is satisfied the map π restricted to V is surjective.*

Proof. If we consider the projectivization $\bar{V} \subseteq \mathbb{CP}^n$, the projection π consists of taking as a center the $n-d-1$ dimensional subspace $S = \{[x_0 : x_1 : \cdots : x_n] \mid x_0 = x_1 = \cdots = x_d = 0\}$. Since the dimension of $\bar{V}$ is d, the intersection $\bar{V} \cap S$ is generically empty. We may hence assume that after a generic linear change of variables, $S \cap \bar{V} = \emptyset$. This intersection is given precisely by the ideal $\sqrt{I_h + (x_0, x_1, \ldots, x_d)}$, which is homogeneous. The condition of $\bar{V} \cap S$ being empty is equivalent to the ideal I being in general position.

In this situation the preimage by $\pi\mid_V$ of a point $[1 : x_1 : \cdots : x_d]$ is given by the intersection of the subspace $\{[y_0 : y_1 : \cdots : y_n] \mid y_1 = y_0 x_1, \ldots, y_d = y_0 x_d\}$ with $\bar{V}$. By the genericity assumption, this intersection does not have points in the infinity. By dimension arguments, this intersection cannot be empty, and must be contained in the affine part of $\bar{V}$. We have then proved that π restricted to V is surjective.

□

The intersection of a generic linear subspace of dimension $n-d$ with $\bar{V}$ is a union of g distinct points, g being the degree of I_h. This degree can be computed through the Hilbert polynomial. Since I is in general position, all of the intersections of $\bar{V}$ with the fibers of π will happen in the affine part. This means that π restricted to V is a branched cover of degree g. We will see now that the branching locus of this cover is contained in a subvariety of $\mathbb{C}^d$ that can be computed.

Assume $I = (f_1, \ldots, f_s)$ is in general position. Consider the matrix

$$M := \begin{pmatrix} \frac{\partial f_1}{\partial x_{d+1}} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_s}{\partial x_{d+1}} & \cdots & \frac{\partial f_s}{\partial x_n} \end{pmatrix}$$

and the ideal J generated by its $(n-d)\times(n-d)$ minors.

Theorem 3. *The branching locus of $\pi\mid_V$ is contained in the zero locus of the elimination ideal $(I+J)\cap\mathbb{C}[x_1,\ldots,x_d]$.*

Proof. Consider a point $p = (x_1, \ldots, x_n) \in V$. If the linear space $\pi^{-1}(\pi(p))$ intersects V at p transversely, then there is no ramification at p, since it means that at a neighborhood of p the map $\pi\mid_V$ is a diffeomorphism. This condition of transversality can be expressed as follows: the complex normal space of V in p and the complex normal space of $\pi^{-1}(\pi(p))$ generate the complex tangent space of $\mathbb{C}^n$ in p. We can identify in a natural way the complex tangent space of $\mathbb{C}^n$ in p with the vector space $\mathbb{C}^n$ itself. The complex normal space of V in p is generated by the rows of the matrix

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1}(p) & \cdots & \frac{\partial f_1}{\partial x_n}(p) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_s}{\partial x_1}(p) & \cdots & \frac{\partial f_s}{\partial x_n}(p) \end{pmatrix}.$$

The complex normal space of $\pi^{-1}(\pi(p))$ is generated by the first d vectors of the canonical basis. A Gaussian elimination argument tells us that these two spaces generate the whole space if and only if the matrix M has rank $(n-d)$. So the set of points of V where $\pi\mid_V$ ramifies is contained in the set S of zeros of $I+J$.

The zero set of the elimination ideal $\mathbb{C}[x_1, \ldots x_d]\cap(I+J)$ is the Zariski closure of $\pi(S)$.

□

Since $V \setminus \pi^{-1}(\pi(S))$ is a cover over $\mathbb{C}^d \setminus \pi(S)$ of degree g, we have that

$$\chi(V) = \chi(V \setminus \pi^{-1}(\pi(S))) + \chi(V \cap \pi^{-1}(\pi(S))) = g\cdot\chi(\mathbb{C}^d \setminus \pi(S)) + \chi(V \cap \pi^{-1}(\pi(S))) =$$

$$= g\cdot(\chi(\mathbb{C}^d) - \chi(\pi(S))) + \chi(V \cap \pi^{-1}(\pi(S))) = g - g\cdot\chi(\pi(S)) + \chi(V \cap \pi^{-1}(\pi(S))).$$

Both $\pi(S)$ and $V \cap \pi^{-1}(\pi(S))$ are varieties of dimension smaller than V, so, by induction hypothesis, we can compute their Euler characteristic in the same form.

3. Description of the algorithm

Now we will describe, step by step, an algorithm to compute the Euler characteristic of the zero set of an ideal $I = (f_1, \dots, f_s)$.

Algorithm 1. (Compute the Euler characteristic of the algebraic set defined by the ideal I):

(1) Check if I is homogeneous. If it is, return 1.
(2) Compute the associated primes $(I_1, \dots, I_m)$ of I (see [3, Chapter 4]).
(3) If there is more than one associated prime, we have that

$$\chi(V(I)) = \chi(V(I_1)) + \chi(V(I_2 \cap \dots \cap I_m)) - \chi(V(I_1 + (I_2 \cap \dots \cap I_m))).$$

by recursion, each summand can be computed with this algorithm. The following parts of this algorithm consider only the irreducible case, since we have already computed the associated primes, we will assume that I_1 is prime.
(4) Compute the dimension d and the degree g of $V(I)$. If d is zero, return g.
(5) Check that I is in general position. This can be done by computing a Gröbner basis of $\sqrt{I_h + (x_0, x_1, \dots, x_d)}$ (where I_h is the homogenization of I) and using it to check that $x_{d+1}, \dots, x_n$ are in it. If it is not in general position, apply a generic linear change of variables and start again the algorithm.
(6) Construct the ideal J generated by the $(n-d) \times (n-d)$ minors of the matrix

$$M := \begin{pmatrix} \frac{\partial f_1}{\partial x_{d+1}} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_s}{\partial x_{d+1}} & \cdots & \frac{\partial f_s}{\partial x_n} \end{pmatrix}.$$

(7) Compute the elimination ideal $K = (I+J) \cap \mathbb{C}[x_1, \cdots, x_d]$ (this can be done by computing a Gröbner basis w.r.t. an elimination ordering).
(8) Compute by recursion $\chi(V(K))$ and $\chi(V(I+K))$. Return the number $g - g \cdot \chi(V(K)) + \chi(V(I+K))$.

4. A finer invariant

The previous method essentially consists in decomposing our variety V in pieces, each of which is compared to $\mathbb{C}^i$ through linear maps that are unbranched covers. At the end of the day, it gives us a linear combination (with integer coefficients), of the Euler characteristic of $\mathbb{C}^i$.

Now we will show that we can actually keep the information in this linear combination, defining a slightly different invariant. This information will be kept in a polynomial $F_\pi(V) \in \mathbb{Z}[L]$, where L^i will play the role of $\chi(\mathbb{C}^i)$

We follow the same method as before but with two differences:

- If the ideal I is homogeneous, we don't end returning a 1. Instead, we continue the algorithm, taking as I_h the ideal generated by I inside $\mathbb{C}[x_0, \dots, x_n]$.
- In the final step, we return $g \cdot L^d - g \cdot F_\pi(V(K)) + F_\pi(V(I+K)))$ instead of $g - g \cdot \chi(V(K)) + \chi(V(I+K))$.

So the algorithm results like this:

Algorithm 2. (Compute the polynomial $F_\pi(V)$ associated to an algebraic set $V(I)$ in general position).

(1) Compute the associated primes $(I_1, \dots, I_m)$ of I.
(2) If there is more than one associated prime, we have that

$$F_\pi(V(I)) = F_\pi(V(I_1)) + F_\pi(V(I_2 \cap \dots \cap I_m)) - F_\pi(V(I_1 + (I_2 \cap \dots \cap I_m))).$$

by recursion, each summand can be computed with this algorithm. The following parts of this algorithm consider only the irreducible case, since we have already computed the associated primes, we will assume that I_1 is prime.

(3) Compute the dimension d and the degree g of $V(I)$. If d is zero, return g.
(4) Construct the ideal J generated by the $(n-d)\times(n-d)$ minors of the matrix

$$M := \begin{pmatrix} \frac{\partial f_1}{\partial x_{d+1}} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_s}{\partial x_{d+1}} & \cdots & \frac{\partial f_s}{\partial x_n} \end{pmatrix}.$$

(5) Compute the elimination ideal $K = (I+J)\cap \mathbb{C}[x_1,\cdots,x_d]$.
(6) Compute by recursion $F_\pi(V(K))$ and $F_\pi(V(I+K))$. Return the polynomial $gL^d - g\cdot F_\pi(V(K)) + F_\pi(V(I+K))$.

Note that both algorithms 1 and 2 can run differently if we apply a linear change of coordinates to I (which would change the projection π). The topological properties of the Euler characteristic tells us that the final result of the algorithm 1 will coincide with the Euler characteristic regardless of this linear change of coordinates. But in the case of $F_\pi(V)$ we cannot ensure such a result. Nevertheless, for two sufficiently generic projections, algorithm 2 will follow the same exact steps, so we can define $F(V)$ as the polynomial obtained by the algorithm 2 for generic projections.

More precisely, there must exist a Zariski open set $T\subseteq GL(n,\mathbb{C})$ such that, the polynomial $F_\pi(\sigma(V(I)))$ is the same for every linear change of coordinates $\sigma\in T$.

Definition 4. Given an ideal $I \trianglelefteq \mathbb{C}[x_1,\ldots,x_n]$, we define the polynomial $F(V(I))$ as the polynomial $F_\pi(\sigma(V(I))$ for any $\sigma\in T$.

We will say that I or $V(I)$ are in **generic position**, or that we are in **generic coordinates** if $F_\pi(V(I)) = F(V(I))$.

Since so far we have no algorithmic criterion to determine if a projection is generic enough or not, the generic case can be computed by introducing the parameters of the projection, and computing the Gröbner basis with those parameters. Experimental evidence suggested the following conjecture:

Conjecture 1. *If an ideal is in general position, it is also in generic position.*

Remark 5. Note that this conjecture is equivalent to saying that, assuming general position, the polynomial $F(V)$ is an invariant of V. That is: it does not depend on the projection.

Shortly after the first version of this paper was made public, Jørgen Vold Rennemo proved that the polynomial $F(V)$ contains essentially the same information as the Chern-Schwartz-MacPherson class. This result proves the conjecture, since the CSM class is an invariant of the variety. His proof is included in the appendix at the end of this document.

Some partial results in this direction are easy to show.

Lemma 6. *If V in in general position, the leading term of $F_\pi(V)$ coincides with the leading term of $F(V)$.*

Proof. It is immediate to check that the degree of $F_\pi(V)$ coincides with the dimension of V, and that the leading coefficient of $F_\pi(V)$ coincide with the degree of V, regardless of the projection used to compute it. □

Remark 7. The value of $F_\pi(V)$ at $L=1$ equals $\chi(V)$, independently of the choices of projections made for its computation, as long as we are in general position.

These two results actually show that $F(V)$ is independent of the projection for the case of curves (since in this case it is a degree 1 polynomial whose leading term and value at 1 are fixed).

We will now show that the invariant F_π behaves well with respect to the product of varieties:

Proposition 8. *Let $I_1 \trianglelefteq \mathbb{C}[x_1, \dots, x_n]$ and $I_2 \trianglelefteq \mathbb{C}[y_1, \dots, y_m]$ be two ideals on polynomial rings with separated variables, and let $V_1 \subseteq \mathbb{C}^n$ and $V_2 \subseteq \mathbb{C}^m$ be their corresponding algebraic sets of dimensions d_1 and d_2 respectively. Consider the ideal*

$$I := I_1 + I_2 \trianglelefteq \mathbb{C}[x_1 \dots, x_{d_1}, y_1, \dots, y_{d_2}, x_{d_1+1}, \dots, x_n, y_{d_2+1}, \dots, y_m].$$

Its corresponding algebraic set is $V = V_1 \times V_2 \subseteq \mathbb{C}^n \times \mathbb{C}^m = \mathbb{C}^{n+m}$. Denote by $\pi_1 : V_1 \to \mathbb{C}^{d_1}$ the map defined by the projection on the first coordinates. Analogously, denote $\pi_2 : V_2 \to \mathbb{C}^{d_2}$ and $\pi : V \to \mathbb{C}^{d_1+d_2}$. Then $F_\pi(V) = F_{\pi_1}(V_1) \cdot F_{\pi_2}(V_2)$.

Proof. Without loss of generality, we can assume that we are in the irreducible case. We will work on induction over the dimension. If V_1 or V_2 are zero dimensional, the result is immediate.

Consider g_1, g_2 the degrees of V_1 and V_2, and g the degree of V. It is easy to check that $g = g_1 \cdot g_2$. Is is also immediate to check that, if $V_2 = \mathbb{C}^m$, the statement holds (that is, $F_\pi(V_1 \times \mathbb{C}^m) = F_\pi(V_1) \cdot L^m$). Consider also Δ_1,Δ_2 and Δ the branching loci of π_1,π_2 and π respectively.

Now we will show that $\Delta = (\Delta_1 \times \mathbb{C}^{d_2}) \cup (\mathbb{C}^{d_1} \times \Delta_2)$. Let

$$p = (x_1, \dots, x_{d_1}, y_1, \dots, y_{d_2}) \in \mathbb{C}^{d_1} \times \mathbb{C}^{d_2}.$$

The set of points in V that project on p is the product of the set of points in V_1 that project in $(x_1 \dots, x_{d_1})$ and the set of points in V_2 that project in $(y_1, \dots, y_{d_2})$. This set has less than $g_1 \cdot g_2$ points if and only if $(x_1, \dots, x_{d_1}) \in \Delta_1$ or $(y_1, \dots, y_{d_1}) \in \Delta_2$. It is immediate also that $(\Delta_1 \times \mathbb{C}^{d_2}) \cap (\mathbb{C}^{d_1} \times \Delta_2) = \Delta_1 \times \Delta_2$. By induction hypothesis, we have that

$$F_\pi(\Delta) = L^{d_2} \cdot F_{\pi_1}(\Delta_1) + L^{d_1} \cdot F_{\pi_2}(\Delta_2) - F_{\pi_1}(\Delta_1) \cdot F_{\pi_2}(\Delta_2).$$

Reasoning analogously, we can conclude that

$$F_\pi(\pi^{-1}(\Delta)) = F_{\pi_1}({\pi_1}^{-1}(\Delta_1)) \cdot F_{\pi_2}(V_2) + F_{\pi_2}({\pi_2}^{-1}(\Delta_2)) \cdot F_{\pi_1}(V_1) - F_{\pi_1}({\pi_1}^{-1}(\Delta_1)) \cdot F_{\pi_2}({\pi_2}^{-1}(\Delta_2)).$$

So, summarizing, we have that

$$\begin{array}{rcl} F_\pi(V) & = & g_1 g_2 (L^{d_1+d_2} - F_\pi(\Delta)) + F_\pi(\pi^{-1}(\Delta)) \\ F_{\pi_1}(V_1) & = & g_1(L^{d_1} - F_{\pi_1}(\Delta_1)) + F_{\pi_1}({\pi_1}^{-1}(\Delta_1)) \\ F_{\pi_2}(V_2) & = & g_2(L^{d_2} - F_{\pi_2}(\Delta_2)) + F_{\pi_2}({\pi_2}^{-1}(\Delta_2)). \end{array}$$

Using all the previous formulas one can easily check that $F_\pi(V) = F_{\pi_1}(V_1) \cdot F_{\pi_2}(V_2)$. □

Since conjecture 1 is true, the same result is true for $F(V)$. In fact, a weaker condition is enough: if the product of two generic projections is generic, then the invariant F is multiplicative. This could be useful, for example, to give a criterion to check if an algebraic set can be the product of two nontrivial algebraic sets. If $F(V)$ is irreducible in $\mathbb{Z}[L]$, then V couldn't be a product.

5. The projective case

To compute the Euler characteristic of the projective variety $\bar{V}$ defined by a homogeneous ideal $I_h \trianglelefteq \mathbb{C}[x_0, \dots, x_n]$ we can also use algorithm 1. In order to do so, we will consider the hyperplane H "at infinity" given by the equation $x_0 = 0$. This allows us to decompose $\bar{V}$ as its affine part $V := \bar{V} \setminus H$ and its part at infinity $\bar{V}^\infty := \bar{V} \cap H$. It is clear that $\chi(\bar{V}) = \chi(V) + \chi(\bar{V}^\infty)$.

The affine part V is an affine variety defined by the ideal obtained by substituting $x_0 = 1$ in the generators of I_h, whose Euler characteristic can be computed as seen before.

The part at infinity $\bar{V}^\infty$ is a projective variety embedded in a projective space of less dimension. The homogeneous ideal that defines it is obtained by substituting $X_0 = 0$ in the generators of I_h. Its Euler characteristic can be computed by recursion. If we are in the case of $\mathbb{CP}^1$, $\bar{V}$ will consist on a finite number of points, which can be computed as the degree of $\sqrt{I_h}$.

This same idea of decomposing a projective variety as a disjoint union of affine pieces can be used to extend the definition of F_π to projective varieties. The invariant of a projective variety is defined as the sum of the invariant of its affine pieces.

Now we will show a different way to compute the Euler characteristic of a projective variety using the polynomial $F(V)$.

Theorem 9. *Let $I = \mathbb{C}[x_0, \ldots, x_n] \cdot (f_1, \ldots, f_s)$ be a homogeneous ideal in generic position. Assume that the generators $f_1, \ldots, f_s$ are homogeneous. Denote by $I_0 := (I + \mathbb{C}[x_0, \ldots, x_n] \cdot (x_0)) \cap \mathbb{C}[x_1, \ldots, x_n]$, and $I_1 := (I + \mathbb{C}[x_0, \ldots, x_n] \cdot (x_0 - 1)) \cap \mathbb{C}[x_1, \ldots, x_n]$. That is, the ideals that represent the intersection of $V(I)$ with the hyperplanes $\{x_0 = 0\}$ and $\{x_0 = 1\}$ respectively, seeing the two hyperplanes as ambient spaces. Then the following formula holds:*

$$F(V(I)) = (L - 1) \cdot F(V(I_1)) + F(V(I_0))$$

Proof. By induction on the dimension of $V(I)$. If the dimension is zero, $V(I)$ must consist only on the origin, since I is homogeneous. In this case, $V(I_1)$ is empty, and $V(I_0)$ is also the origin. We have that

$$1 = F(V(I)) = (L - 1) \cdot 0 + 1 = (L - 1) \cdot F(V(I_1)) + F(V(I_0)).$$

If the dimension d of $V(I)$ is positive, consider the ideals J, K and $H = I + K$ as before. Construct also H_0, H_1, K_0 and K_1 in the same way as I_0 and I_1. Note that, since we are in generic position, the ideals H_0' and K_0' needed to compute $F(V(I_0))$ are precisely H_0 and K_0 (that is, specializing $x_0 = 0$ and then computing the minors of the matrix M and the elimination ideal is the same as computing the minors of the matrix and the elimination and then specializing). The same happens with J_1 and K_1.

By induction hypothesis, we have that

$$F(V(H)) = (L - 1) \cdot F(V(H_1)) + F(V(H_0))$$

and

$$F(V(K)) = (L - 1) \cdot F(V(K_1)) + F(V(K_0)).$$

Now we have that

$$\begin{gathered} F(V(I)) \;=\; g \cdot L^d - g \cdot F(V(K)) + F(V(H)) \;= \\ g \cdot L^d - g \cdot ((L - 1) \cdot F(V(K_1)) + F(V(K_0))) + (L - 1) \cdot F(V(H_1)) + F(V(H_0)) \;= \\ (L - 1) \cdot (g \cdot L^{d-1} - g \cdot F(V(K_1)) + F(V(H_1))) + g \cdot L^{d-1} - g \cdot F(V(K_0)) + F(V(H_0)) = \\ (L - 1) \cdot F(V(I_1)) + F(V(I_0)), \end{gathered}$$

and this proves the result. $\square$

This theorem allows us to relate the invariant F of the affine algebraic set defined by a homogeneous ideal, and the invariant F of the projective variety defined by the same ideal as follows:

Corollary 10. *Let I be a homogeneous ideal in $\mathbb{C}[x_1,\dots,x_n]$ in generic position. Let V be the algebraic set defined by I, and V' the projective variety defined by the same ideal. Then $F(V) = (L-1)\cdot F(V') + 1$.*

Proof. Applying the previous result recursively, we have that $F(V(I))$ equals

$$(L-1)\cdot F(V(I_1)) + (L-1)\cdot F(V((I_0)_1)) + \cdots$$
$$+(L-1)\cdot F(V((\cdots(I_0)\cdots)_0)_1)) + F(V((\cdots(I_0)\cdots)_0)).$$

Note that, if we compute $F(V')$ through the sum of the affine pieces, we obtain precisely $F(V(I_1)) + F(V((I_0)_1)) + \cdots + F(V((\cdots(I_0)\cdots)_0)_1))$. Since $V((\cdots(I_0)\cdots)_0))$ consists only of the origin, we have the result.

This last result can be interpreted as the fact that V is the complex cone over V'. That is, $V\setminus\{0\}$ is the product $V'\times\mathbb{C}^*$.

□

6. Examples

Both the polynomial $F(V)$ and the Hilbert polynomial P_I have the same degree, and the leading term is determined by the degree of V. This would point in the direction of considering that they contain the same information. The following example shows this is not the case:

Example 11. Consider the conics $C_1, C_2 \in \mathbb{C}^2$ given by $C_1 := V(x^2+y^2-1)$ and $C_2 := V(x^2+y^2)$. If we compute the Hilbert polynomial of the corresponding homogeneous ideals in $\mathbb{C}[x,y,z]$ we get that $P_{(x^2+y^2+z^2)} = P_{(x^2+y^2)} = 2\cdot t + 1$.

Let's now compute the polynomial $V(C_1)$ using the canonical projection $\pi : \mathbb{C}^2 \to \mathbb{C}$ to the first component. This projection is a $2:1$ cover of $\mathbb{C}$ branched along the points ± 1. Over each of these points, there is only one preimage. So finally we have that $F(C_1) = 2(L-2)+1+1 = 2L-2$.

On the other hand, the curve C_2 also projects $2:1$ over $\mathbb{C}$, but now there is only one branching point $(x=0)$. So the result is $F(C_2) = 2(L-1)+1 = 2L-1$.

That is, this example shows that the polynomial $F(V)$ contains information that is not contained in the Hilbert polynomial.

An important example of algebraic sets is the case of hyperplane arrangements. We will now recall some related notions (see [5, Chapter II]).

Definition 12. Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{C}^n$. Its **intersection lattice** $L(\mathcal{A})$ is the set of all its intersections ordered by reverse inclusion, with the convention that the intersection of the empty set is $\mathbb{C}^n$ itself.

The Mbius function is the only function $\mu : L \to \mathbb{Z}$ satisfying that:

$$\begin{array}{l} \mu(\mathbb{C}^n) = 1 \\ \sum_{Y\le X}\mu(Y) = 0 \quad \forall X \in L\setminus\{\mathbb{C}^n\}. \end{array}$$

Definition 13. The **characteristic polynomial** of $\mathcal{A}$ is defined as

$$\chi(\mathcal{A}, L) := \sum_{X\in L(\mathcal{A})} \mu(X)\cdot L^{dim(X)}.$$

Theorem 14 (Deletion-Restriction)**.** *Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{C}^n$, and H a hyperplane of $\mathcal{A}$. Let $\mathcal{A}'$ be the arrangement resulting from eliminating H from $\mathcal{A}$, and let $\mathcal{A}''$ be the hyperplane arrangement inside H induced by the intersection with $\mathcal{A}'$. Then the following formula holds:*

$$\chi(\mathcal{A}, L) = \chi(\mathcal{A}', L) - \chi(\mathcal{A}'', L).$$

Now we will see how this characteristic polynomial relates to the polynomial $F(V)$.

Theorem 15. *Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{C}^n$. The following holds:*

$$F(\mathcal{A}) = L^n - \chi(\mathcal{A}, L).$$

Proof. By induction on the number of hyperplanes. The case of only one hyperplane is immediate.

If there are more than one hyperplane, take one hyperplane H of $\mathcal{A}$. Since $\mathcal{A} = \mathcal{A}' \cup H$, we have, by additivity, that

$$F(\mathcal{A}) = F(\mathcal{A}') + F(H) - F(\mathcal{A}' \cap H) = F(\mathcal{A}') + L^{n-1} - F(\mathcal{A}'').$$

Both $\mathcal{A}'$ and $\mathcal{A}''$ are hyperplane arrangements with less hyperplanes than $\mathcal{A}$, so, by induction hypothesis the following formulas hold:

$$\begin{gathered} F(\mathcal{A}') = L^n - \chi(\mathcal{A}', L) \\ F(\mathcal{A}'') = L^{n-1} - \chi(\mathcal{A}'', L). \end{gathered}$$

Substituting these formulas in the previous one, and using the deletion-restriction theorem we get the result. $\square$

This result tells us that the characteristic polynomial of $\mathcal{A}$ can be thought of as the polynomial F of its complement. If we consider a projective arrangement $\mathcal{A}$ and its affine cone $\hat{\mathcal{A}}$, we can use Corollary 10 to relate $\chi(\hat{\mathcal{A}}, L)$ and $F(\mathcal{A})$. The result in the appendix then relates $F(\mathcal{A})$ with the Chern-Schwartz-MacPherson class of its complement. This way we obtain a new proof of the formula in [2, Theorem 3.1].

7. Counting points over finite fields

In this section we will see how the polynomial $F(V)$ can be related to the number of points of the variety considered over a finite field. Let's illustrate this fact with an example.

Example 16. Consider the conic given by the equation

$$x_1^2 + x_2^2 - 1$$

in the affine plane over the field of 5 elements $\mathbb{F}_5$.

The set of rational points is the following:

$$(0,1), (0,4), (1,0), (4,0)$$

If we project it to the x_1 axis, we see that over the point (0), we have two preimages, as expected by the degree. Over the points (1) and (4), we have just one point of the curve, since the cover ramifies there. But over the points (2) and (3) we have no points of the curve. The reason for this is that the equations $2^2 + x_2^2 - 1$ and $3^2 + x_2^2 - 1$ have no roots over $\mathbb{F}_5$. However, they do have all their solutions over a quadratic extension $\mathbb{F}_{25}$ of degree 2. In particular, if we look at the points of $\mathbb{F}_5 \times \mathbb{F}_{25}$ that satisfy the equation, we obtain the set

$$\{(0,1), (0,4), (1,0), (2, a+2), (2, 4a+3), (3, a+2), (3, 4a+3), (4,0)\},$$

where a is an element of $\mathbb{F}_{25}$ that has minimal polynomial $x^2 + 3$ over $\mathbb{F}_5$.

It might seem strange to consider the set of points in $\mathbb{F}_5 \times \mathbb{F}_{25}$ that satisfy a given equation. But this set is in fact an algebraic set. Indeed, it can be expressed as the set of points of the affine plane over $\mathbb{F}_{25}$ that satisfy the equations

$$\begin{gathered} x_1^2 + x_2^2 - 1 \\ x_1^5 - x_1 \end{gathered}$$

Recall that the polynomial $F(V)$ for such a conic was $f = 2L - 2$. In this case, we have obtained 8 points in total, which is precisely the value of f for $L = 5$. A quick look at the points shows why this happens: there are two points over each value of $\mathbb{F}_5$, with the exception of the two branching points, where there is only one.

Note that both algorithms 1 and 2 can be run over finite fields in the same way as they run over the rationals. With a small exception, though: to ensure the existence of a change of coordinates that puts the ideal in general position, we might need to work in a finite field extension. Once done that, both algorithms would mimic the steps given by the algorithm run over $\mathbb{Q}$, except if some leading coefficient becomes zero. But that would happen only for a finite number of prime numbers p.

Notation 17. We can then define the polynomial $F_p(V)$ as the result of running algorithm 2 in a field of characteristic p.

As we have seen before, $F_p(V) = F(V)$ for almost every prime number p.

Notation 18. Given a polynomial

$$f = a_0 + a_1 L + a_2 L^2 + \cdots + a_n L^n \in \mathbb{Z}[L]$$

and a list of numbers $(d_1, \ldots, d_s)$ with $s \geq n$, we will denote by $f(d_1, \ldots, d_s)$ the number

$$f(d_1, \ldots, d_s) = a_0 + a_1 d_1 + a_2 d_1 d_2 + \cdots + a_n d_1 d_2 \cdots d_n$$

Another difference between the way the algorithms would run over $\mathbb{Q}$ and over finite fields lies in the primary decomposition. But then again, this will only happen for a finite number of primes.

Theorem 19. *Given an ideal $I \trianglelefteq \mathbb{F}_p[x_1, \ldots, x_n]$, and the corresponding $f := F_p(V)$ of degree s, there exists a list of numbers $(d_1 \leq \cdots \leq d_n)$ such that the number of points in*

$$\left(\mathbb{F}_{p^{d_1}} \times \cdots \times \mathbb{F}_{p^{d_n}}\right) \cap V(I)$$

equals the number $f(p^{d_1}, \ldots, p^{d_n})$.

Moreover, for any such list $(d_1 \leq \cdots \leq d_n)$ and any d'_i multiple of d_i there exists another list $(d_1 \leq \cdots \leq d_{i-1} \leq d'_i \leq d'_{i+1} \leq \cdots \leq d'_n)$ satisfying the same property.

Proof. By induction over the dimension. If I is zero dimensional, $V(I)$ consists of $F_p(V) = deg(I)$ distinct points, whose coordinates lie in a sufficiently big extension $\mathbb{F}_{p^{d_1}}$. Any sequence starting with d_1 would satisfy the theorem.

If I has dimension $d > 0$ and degree g, we have that $F_p(V(I)) = g(L^d - F_p(V(K))) + F_p(V(I + K))$. By induction, we can assume that both K and $I + K$ satisfy the result. Let $(d_1^1 \leq \cdots \leq d_d^1)$ and $(d_1^2 \leq \cdots \leq d_n^2)$ be the corresponding sequences. Take $d_1 = lcm(d_1^1, d_1^2)$. There exist two sequences $(d_1 \leq {d_2^1}' \leq \cdots \leq {d_d^1}')$ and $(d_1 \leq {d_2^2}' \leq \cdots \leq {d_n^2}')$ that are valid for K and $I + K$ respectively, and coinciding in the first term. Repeating this reasoning we can obtain two sequences $(d_1 \leq \cdots \leq d_d)$ and $(d_1 \leq \cdots \leq d_d \leq d_{d+1} \leq \cdots \leq d_n)$ that are valid for K and $I + K$ respectively.

Note that, since both $F_p(V(K))$ and $F_p(V(I + K))$ are of dimension at most $d - 1$, the terms $d_d, \ldots, d_n$ can be changed arbitrarily and still the sequence would be valid for K and $I + K$.

Now take any point $q := (q_1, \ldots, q_d) \in (\mathbb{F}_{p^{d_1}} \times \cdots \times \mathbb{F}_{p^{d_d}}) \setminus V(K)$. Taking an appropriate field $\mathbb{F}_{p^{n_q}}$, we can ensure that there are exactly g points in $\{(x_1, \ldots, x_n) \in (\mathbb{F}_{p^{n_q}})^n \mid x_1 = q_1, \ldots, x_d = q_d\} \cap V$. We can do the same for every point q, and take a common field extension $\mathbb{F}_{p^s}$ of all the different $\mathbb{F}_{p^{n_q}}$. This way, we have that there are exactly g points of $V \cap (\mathbb{F}_{p^{d_1}} \times \cdots \times \mathbb{F}_{p^{d_d}} \times \mathbb{F}_{p^s} \times \cdots \times \mathbb{F}_{p^s})$ over each point of $(\mathbb{F}_{p^{d_1}} \times \cdots \times \mathbb{F}_{p^{d_d}}) \setminus V(K)$.

By definition, we have that

$$F_p(V) = g(L^d - F_p(V(K))) + F_p(V(I+K)).$$

Now if we restrict ourselves to the points in $\mathbb{F}_{p^{d_1}} \times \cdots \times \mathbb{F}_{p^{d_d}} \times \mathbb{F}_{p^s} \times \cdots \times \mathbb{F}_{p^s}$, we have that

$$\#V = g \cdot (p^{d_1} p^{d_2} \cdots p^{d_d} - \#V(K)) + \#V(I+K).$$

Making use of the induction hypothesis, and the fact that $F_p(V(K))$ and $F_p(V(I+K))$ are of dimension less than d, the result follows easily.

□

Corollary 20. *Let V be an algebraic set in $\mathbb{C}^n$ defined by an ideal $I \trianglelefteq \mathbb{K}[x_1, \ldots, x_n]$, where $\mathbb{K}$ is an algebraic extension of $\mathbb{Q}$. Then for almost every prime p there exists a list of positive integers $(d_{1,p}, \ldots, d_{n,p})$ such that the number of points in*

$$\left(\mathbb{F}_{p^{d_{1,p}}} \times \cdots \times \mathbb{F}_{p^{d_{n,p}}}\right)$$

that satisfy the equations of I equals the number $F(V(I))(p^{d_{1,p}}, \ldots, p^{d_{n,p}})$.

This corollary provides a different way to compute the polynomial $F(V(I))$ by counting points over finite fields. If we know the value of $F(V(I))(S)$ for a sufficient number of such sequences S, recovering the coefficients of the polynomial is a simple linear algebra problem.

8. Code and timings

Here we show an implementation in Sage ([7]) of the two algorithms. This same code can be found at http://www.sagenb.org/home/pub/3961.

8.1. **Implementation of Algorithm 1.**

```
def Euler_characteristic(I):
    R=I.ring()
    if I.is_one():
        return 0
    if R.ngens()==1:
        return sum([j[0].degree() for j in I.gen().factor()])
    J1=I.radical()
    if J1.is_homogeneous():
        return 1
    primdec=J1.associated_primes()
    J1=primdec[0]
    m=len(primdec)
    if m>1:
        J2=R.ideal(1)
        for j in [1..m-1]:
            J2=J2.intersection(primdec[j])
        return Euler_characteristic(J1)+Euler_characteristic(J2)-Euler_characteristic(J1+J2)
    P=J1.homogenize().hilbert_polynomial()
    if P.is_zero():
        deg=0
    else:
        deg=P.leading_coefficient()*P.degree().factorial()
    if deg==1:
        return 1
```

```
    dim=J1.dimension()
    n=R.ngens()
    vars1=R.gens()[0:n-dim]
    vars2=R.gens()[n-dim:n]
    varpiv=vars1[-1]
    IH=J1.homogenize()
    S=IH.ring()
    JH=IH+S.ideal(S.gens()[n-dim:])
    if JH.dimension()>0:
        det=0
        while det==0:
            MH=random_matrix(R.base_ring(),n)
            det=MH.determinant()
        L=list(MH*vector(list(R.gens())))
        return Euler_characteristic(R.hom(L)(J1))
    if dim==0:
        return deg
    M=matrix([[f.derivative(v) for v in vars1] for f in J1.gens()])
    J=R.ideal(M.minors(n-dim))
    K=(J+J1).elimination_ideal(vars1)
    S=PolynomialRing(R.base_ring(),vars2)
    H=R.hom([S(0) for j in vars1]+[S(j) for j in vars2])
    C=deg-deg*Euler_characteristic(H(K))+Euler_characteristic(K+J1)
    return C
```

This algorithm may be very slow (since it involves several Gröbner basis computations), but in several interesting cases, it gives a useful answer in reasonable time. Let's show here some examples.

In the case of curves and surfaces, the result is often reasonably fast; but it may vary a lot if a random change of variables has to be applied. Here we show a few examples. These tests have been run on a Dual-Core AMD Opteron 8220.

Three examples of plane curves:

```
sage: R.<x,y>=QQ[]
sage: time Euler_characteristic(R.ideal(x^5+1))
5
Time: CPU 0.16 s, Wall: 0.17 s
sage: time Euler_characteristic(R.ideal(y^4+x^3-1))
-5
Time: CPU 0.19 s, Wall: 0.20 s
time Euler_characteristic(R.ideal(x^2+y^2-5*x^2*y^4+x*y-1))
-8
Time: CPU 17.82 s, Wall: 17.82 s
```

A curve and a surface in $\mathbb{C}^3$:

```
S.<x,y,z>=QQ[]
time Euler_characteristic(S.ideal(x^5+y^2+2*x*y+1,3*x-5*y*x+y^2+1))
10
Time: CPU 0.17 s, Wall: 0.18 s
time Euler_characteristic(S.ideal(x^5+y^2+2*x*y+1))
-3
```

```
Time: CPU 0.49 s, Wall: 0.49 s
```

8.2. **Implementation of Algorithm 2.**

```
@parallel(7)
@cached_function
def FV(I,var='L'):
    FS=PolynomialRing(ZZ,var)
    L=FS.gen()
    R=I.ring()
    if R.ngens()==0:
        return 0
    if I.is_zero():
        return L^R.ngens()
    if I.is_one():
        return 0
    if R.ngens()==1:
        return FS(sum([j[0].degree() for j in I.gen().factor()]))
    J1=I.radical()
    if J1.is_homogeneous():
        S1=PolynomialRing(R.base_ring(),R.gens()[0:-1])
        H1=R.hom(list(S1.gens())+[S1(1)])
        H2=R.hom(list(S1.gens())+[S1(0)])
        resulp=FV([H1(J1),H2(J1)])
        d=dict([[a[0][0][0],a[1]] for a in resulp])
        [i1,i2]=[d[H1(J1)],d[H2(J1)]]
        return (L-1)*(i1)+i2
    primdec=J1.associated_primes()
    J1=primdec[0]
    m=len(primdec)
    if m>1:
        J2=R.ideal(1)
        for j in [1..m-1]:
            J2=J2.intersection(primdec[j])
            resulp=FV([J1,J2,J1+J2])
            d=dict([[a[0][0][0],a[1]] for a in resulp])
            [i1,i2,i3]=[d[J1],d[J2],d[J1+J2]]
        return i1+i2-i3
    P=J1.homogenize().hilbert_polynomial()
    if P.is_zero():
        deg=0
    else:
        deg=P.leading_coefficient()*P.degree().factorial()
    dim=J1.dimension()
    if deg==1:
        return FS(L^dim)
    n=R.ngens()
    vars1=R.gens()[0:n-dim]
    vars2=R.gens()[n-dim:n]
    varpiv=vars1[-1]
```

```
    IH=J1.homogenize()
    S=IH.ring()
    JH=IH+S.ideal(S.gens()[n-dim:])
    if JH.dimension()>0:
        det=0
        while det==0:
            MH=random_matrix(R.base_ring(),n)
            det=MH.determinant()
        L=list(MH*vector(list(R.gens())))
        return FV(R.hom(L)(J1))
    if dim==0:
        return FS(deg)
    M=matrix([[f.derivative(v) for v in vars1] for f in J1.gens()])
    J=R.ideal(M.minors(n-dim))
    K=(J+J1).elimination_ideal(vars1)
    S=PolynomialRing(R.base_ring(),vars2)
    H=R.hom([S(0) for j in vars1]+[S(j) for j in vars2])
    d=dict([[a[0][0][0],a[1]] for a in FV([H(K),K+J1])])
    [i1,i2]=[d[H(K)],d[K+J1]]
    C=deg*FS(L^dim-i1)+i2
    return C
```

This implementation makes use of the Sage framework for parallel computations, allowing to use several processor cores at the same time to compute the intermediate steps. It also caches the already computed results in case they would be needed later.

Note that the implementation assumes that the ideal is in generic position. Since Conjecture 1 is true, it is enough to check that it is in general position (recall that, to do so, one has to compute the ideal $\sqrt{I_h + (x_0, \ldots, x_d)}$ and see if it contains all the monomials x_i).

There is another way to check if the ideal is in generic position without using Conjecture 1. Considering a ring that contains the parameters of the possible linear transformations, and write the matrix $(a_{i,j})$ whose entries are these parameters. This matrix represents a generic linear change of coordinates, so we can apply it to our ideal and run the computation over this ring with parameters. The Gröbner basis computations in this ring would mimic the ones in the original one, for generic values of the parameters. Unluckily, the methods to compute the primary decomposition do not work over rings with parameters. One way to proceed is to compute the primary decomposition over the original ring without parameters. Then compute the discriminant with parameters, and then choose a value for the parameters in the open part of the Gröbner cover (see [4] for a definition and algorithm).

Again, this method can be very slow, but in some cases it is fast enough to be useful. Here we present some of those examples:

The complex 2-sphere:

```
sage: R.<x,y,z>=QQ[]
sage: time FV(R.ideal(x^2+y^2+z^2-1))
2*L^2 - 2*L + 2
Time: CPU 0.07 s, Wall: 0.32 s
```

Another surface:

```
sage: time FV(R.ideal(x^3+y^3+z^3-1))
3*L^2 - 6*L + 12
Time: CPU 0.07 s, Wall: 0.29 s
```

The intersection of the two:

```
sage: time FV(R.ideal(x^3+y^3+z^3-1,x^2+y^2+z^2-1))
6*L - 15
Time: CPU 0.08 s, Wall: 43.01 s
```

and their union (the timing is done after cleaning the cache of the function):

```
sage: time FV(R.ideal((x^3+y^3+z^3-1)*(x^2+y^2+z^2-1)))
5*L^2 - 14*L + 29
Time: CPU 0.08 s, Wall: 43.02 s
```

notice the additivity of the polynomial.

The Whitney umbrella:

```
sage: time FV(R.ideal(x*y^2-z^2))
3*L^2 - 4*L + 2
Time: CPU 0.13 s, Wall: 3.44 s
```

The 3-sphere:

```
sage: S.<x,y,z,t>=QQ[]
sage: time FV(S.ideal(x^2+y^2+z^2+t^2-1))
2*L^3 - 2*L^2 + 2*L - 2
Time: CPU 0.09 s, Wall: 0.43 s
```

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APPENDIX: THE POLYNOMIAL INVARIANT AS A CHERN-SCHWARTZ-MACPHERSON CLASS

JøRGEN VOLD RENNEMO

ABSTRACT: After a change of variables, the polynomial $F(V(I))$ defined in the paper equals the Chern-Schwartz-MacPherson class of $V(I)$ considered as a subset of $\mathbb{P}^n$.

Let X be a reduced and irreducible variety. The Chern-Schwartz-MacPherson class (abbreviated CSM class) is a construction which to any constructible set $Y \subseteq X$ assigns a class $c_{SM}(Y) \in H_*(X, \mathbb{Z})$. For our purposes, the essential properties of the CSM class are the following.

(1) If Y is complete and nonsingular and $i : Y \to X$ denotes the embedding, then $c_{\mathrm{SM}}(Y) = i_*(c_\bullet(T_Y) \cap [Y])$.
(2) If Y is a disjoint union of constructible sets Y_1, Y_2, then $c_{\mathrm{SM}}(Y) = c_{\mathrm{SM}}(Y_1) + c_{\mathrm{SM}}(Y_2)$.
(3) Let $f : X \to Z$ be a proper morphism of reduced and irreducible varieties, and assume that restricted to Y the morphism f is set-theoretically g-to-1. Then the relation $f_*(c_{\mathrm{SM}}(Y)) = g \cdot c_{\mathrm{SM}}(f(Y))$ holds.

For the construction and general properties of CSM classes, see [Fu, Ex 19.1.7] or [Mac]. What we call $c_{\mathrm{SM}}(Y)$ corresponds in the notation of these references to $c_*(1_Y)$, where $1_Y : X \to \mathbb{Z}$ is the characteristic function of Y.

Identify the homology group $H_*(\mathbb{P}^n, \mathbb{Z})$ with $\mathbb{Z}[Q]/(Q^{n+1})$, letting Q^d correspond to the class of a linear d-dimensional subspace $\mathbb{P}^d \subseteq \mathbb{P}^n$. Note that this is strictly an isomorphism of abelian groups and is not compatible with any of the usual multiplicative structures on the groups. We can now state the result.

Proposition. *Let I be an ideal as in Definition 4 of the paper, and consider $V(I)$ as a locally closed subset of $\mathbb{P}^n$ via the standard compactification $\mathbb{C}^n \subset \mathbb{P}^n$. After the change of variables $L \to 1 + Q$, the equation $F(V(I)) = c_{SM}(V(I))$ holds.*

Proof. Let $V = V(I)$. As $F(V)$ is defined by Algorithm 2 of the paper, it suffices to show that after the variable change $L \to (1 + Q)$ and replacing F_π with c_{SM} everywhere, this algorithm gives a computation of $c_{SM}(V)$. The only steps that need justification in this setting are Steps 2, 3 and 6. The second listed property of the CSM class implies that the inclusion-exclusion principle holds for c_{SM}, and so the relation in Step 2 is valid. The CSM class of a point is the ordinary class of the point, showing the validity of Step 3.

It remains to justify Step 6, in other words we must verify that

$$c_{\mathrm{SM}}(V) = g(1 + Q)^d - g\, c_{\mathrm{SM}}(V(K)) + c_{\mathrm{SM}}(V(I + K)).$$

In what follows we shall consider algebraic sets which can be considered as subsets of more than one variety. In order to compare the CSM classes with respect to the different inclusions, we introduce the following notation: If Y is an algebraic subset of X, denote by $c_{SM}^X(Y)$ the CSM class of Y considered as a subset of X, i.e. so that $c_{\mathrm{SM}}^X(Y)$ is an element of $H_*(X)$.

Let $\mathbb{C}^d \subset \mathbb{P}^n$ be a linear space, and compute $c_{\mathrm{SM}}^{\mathbb{P}^n}(\mathbb{C}^d)$ by

$$c_{\mathrm{SM}}^{\mathbb{P}^n}(\mathbb{C}^d) = c_{\mathrm{SM}}^{\mathbb{P}^n}(\mathbb{P}^d) - c_{\mathrm{SM}}^{\mathbb{P}^n}(\mathbb{P}^{d-1}) = (1 + Q)^d,$$

where the last step follows from the first property of the CSM class.

Now, let $\pi : V \to \mathbb{C}^d$ be the restriction of a generic projection $\pi' : \mathbb{C}^n \to \mathbb{C}^d$, and let $U \subseteq \mathbb{C}^d$ be the complement of $V(K)$ as defined in the algorithm. Theorem 3 of the paper shows that over U the map π is g-to-1. In view of the computation of $c_{\mathrm{SM}}^{\mathbb{P}^n}(\mathbb{C}^d)$, the justification of Step 6 now amounts to showing the relation

$$c_{\mathrm{SM}}^{\mathbb{P}^n}(V) = g(c_{\mathrm{SM}}^{\mathbb{P}^n}(\mathbb{C}^d) - c_{\mathrm{SM}}^{\mathbb{P}^n}(\mathbb{C}^d \setminus U)) + c_{\mathrm{SM}}^{\mathbb{P}^n}(\pi^{-1}(\mathbb{C}^d \setminus U)),$$

which is equivalent to

$$c_{\mathrm{SM}}^{\mathbb{P}^n}(V) = g\, c_{\mathrm{SM}}^{\mathbb{P}^n}(U) + c_{\mathrm{SM}}^{\mathbb{P}^n}(\pi^{-1}(\mathbb{C}^d \setminus U)). \tag{8.1}$$

Let X be the blow up of $\mathbb{P}^n$ along the $\mathbb{P}^{n-d-1}$ that is the centre of the projection $\pi' : \mathbb{C}^n \to \mathbb{C}^d$, and let $p : X \to \mathbb{P}^n$ be the blowup morphism. Since π' is generic, for dimension reasons we may assume that the centre of the projection is disjoint from the closure $\overline{V}$ of V in $\mathbb{P}^n$. Let $\pi'' : X \to \mathbb{P}^d$ be the unique extension of π' to X. We thus have the following diagram.

$$\begin{tikzcd}
V \arrow[r, hook] \arrow[dr, "\pi"'] & \mathbb{C}^n \arrow[r, hook] \arrow[d, "\pi'"'] & X \arrow[r, "p"] \arrow[d, "\pi''"'] & \mathbb{P}^n \\
 & \mathbb{C}^d \arrow[r, hook] & \mathbb{P}^d \arrow[ur, hook, "i"'] &
\end{tikzcd}$$

The left hand triangle and the square are commutative, and the right hand triangle is not. However, when we restrict to the subgroup of $H_*(X, \mathbb{Z})$ generated by cycles with support disjoint from the exceptional divisor of p, the pushforward maps on homology satisfy $i_* \circ \pi''_* = p_*$.

We now have

$$c_{\mathrm{SM}}^{\mathbb{P}^n}(V) = p_*(c_{\mathrm{SM}}^{X}(V)) = i_* \pi''_*(c_{\mathrm{SM}}^{X}(V)).$$

The last equality follows from the fact that $c_{\mathrm{SM}}^{X}(V)$ is the pushforward from $\overline{V}$ to X of the class $c_{\mathrm{SM}}^{\overline{V}}(V)$, and hence it is represented by a cycle that is disjoint from the exceptional divisor in X.

We now compute

$$\begin{aligned}
i_*\pi''_*(c_{\mathrm{SM}}^{X}(V)) &= i_*\pi''_*(c_{\mathrm{SM}}^{X}(\pi^{-1}(U))) + i_*\pi''_*(c_{\mathrm{SM}}^{X}(\pi^{-1}(\mathbb{C}^d \setminus U))) \\
&= i_*(g\, c_{\mathrm{SM}}^{\mathbb{P}^d}(U)) + p_*(c_{\mathrm{SM}}^{X}(\pi^{-1}(\mathbb{C}^d \setminus U))) \\
&= g\, c_{\mathrm{SM}}^{\mathbb{P}^n}(U) + c_{\mathrm{SM}}^{\mathbb{P}^n}(\pi^{-1}(\mathbb{C}^d \setminus U)),
\end{aligned}$$

which proves (8.1). □

I thank the referee for pointing out an important mistake in the submitted version of this appendix.

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Journal *of* Singularities
Volume 4 (2012), 131-153

received: 3 September 2010
in revised form: 11 July 2012
DOI: 10.5427/jsing.2012.4h

ON THE CLASSIFICATION OF QUASIHOMOGENEOUS SINGULARITIES

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Abstract. The motivations for this paper are computer calculations of complete lists of weight systems of quasihomogeneous polynomials with isolated singularity at 0 up to rather large Milnor numbers. We review combinatorial characterizations of such weight systems for any number of variables. This leads to certain types and graphs of such weight systems. Using them, we prove an upper bound for the common denominator (and the order of the monodromy) by the Milnor number, and we show surprising consequences if the Milnor number is a prime number.

Contents

1. Introduction

Several people have (re)discovered characterizations of those weight systems which admit quasihomogeneous polynomials with isolated singularity at 0. Section 2 collects and compares these characterizations and gives all references which we found. The results of this section are not new. But the references are not well known and for several reasons, it is not so easy to extract the results from them. Also, we will need part of the characterizations for a good control of such weight systems in the later sections.

In section 3 a part of the conditions is used to associate after a choice a type and a graph to a quasihomogeneous singularity. The idea for this is contained in [Ar][AGV], and there it is carried out for 2 and 3 variables. The general case is carried out in [OR1], but that part of [OR1] was never published. As we will need the graphs in the sections 4 and 6, we rewrite the general case. Section 3 also makes the classification in the case of 4 variables in [YS] more precise, showing how necessary and sufficient conditions are obtained.

Section 4 gives an estimate $d \leq \text{const}(n) \cdot \mu$ for the weighted degree d of a reduced weight system of a quasihomogeneous singularity from above by the Milnor number μ. The calculations start with the well known formula for μ in terms of the weights, but refine this formula using a graph and a type of the singularity. The estimate is useful for a computer calculation of

2000 *Mathematics Subject Classification.* 32S25, 14J17, 58K40.

Key words and phrases. Quasihomogeneous polynomial, weight system, classification of quasihomogeneous singularities.

all reduced weight systems of quasihomogeneous singularities up to a given Milnor number. We carried out such computer calculations for 2, 3 and 4 variables and $\mu \leq 9000$, 9000 and 2000. The long tables are available on the homepage [HK]. Some observations from them are formulated in section 5.

Section 6 proves a surprising fact which we found looking at these tables. If the Milnor number of a quasihomogeneous singularity is a prime number, then the only type which one can associate to it is the chain type (up to adding or removing squares from the singularity), and furthermore, all eigenvalues of the monodromy have multiplicity one. The proof further refines the formula for the Milnor number from section 4.

We thank Sasha Aleksandrov for translating [Ko2] and for the reference [YS] and Wolfgang Ebeling and Atsushi Takahashi for discussions related to lemma 3.5 and [ET]. This paper was written during a stay at the Tokyo Metropolitan University. We thank the TMU and Martin Guest for hospitality.

2. Combinatorial characterizations of weight systems of quasihomogeneous singularities

We note $\mathbb{N}_0 = \{0, 1, 2, ...\} \supset \mathbb{N} = \{1, 2, ...\}$. The support of a polynomial

$$f = \sum_{\alpha \in \mathbb{N}_0^n} a_\alpha \cdot x^\alpha \in \mathbb{C}[x_1, ..., x_n] \quad \text{where} \quad x^\alpha = x_1^{\alpha_1} ... x_n^{\alpha_n}$$

is $\operatorname{supp} f = \{\alpha \in \mathbb{N}_0^n \mid a_\alpha \neq 0\}$. The polynomial is called *quasihomogeneous* with weight system $(w_1, ..., w_n, d) \in \mathbb{R}_{>0}^{n+1}$ if

$$\sum_{i=1}^n \alpha_i \cdot w_i = d \quad \text{for any } \alpha \in \operatorname{supp} f.$$

Here $w_1, ..., w_n$ are the weights and d is the weighted degree. If a polynomial is quasihomogeneous with some weight system it is also quasihomogeneous with a weight system $(w_1, ..., w_n, d) \in \mathbb{Q}_{>0}^{n+1}$. If a quasihomogeneous polynomial has an isolated singularity at 0, that is, if the $\frac{\partial f}{\partial x_i}$ vanish simultaneously precisely at 0, then $w_i < d$ for all i. Therefore, from now on throughout the whole paper we consider only weight systems with

$$(w_1, ..., w_n, d) \in \mathbb{Q}_{>0}^{n+1} \qquad \text{and} \qquad w_i < d \text{ for all } i.$$

Furthermore, from now on we reserve the letters $v_1, ..., v_n$ for weights of weight systems $(v_1, ..., v_n, d) \in \mathbb{N}^{n+1}$, and the letters $w_1, ..., w_n$ for weights of normalized weight systems $(w_1, ..., w_n, 1) \in \mathbb{Q}_{>0}^{n+1}$, that is, with weighted degree 1.

A weight system $(v_1, ..., v_n, d) \in \mathbb{N}^{n+1}$ is called *reduced* if $\gcd(v_1, ..., v_n, d) = 1$. In later chapters, but not in this one, we will also use a result in [Sa1] and restrict to weight systems with $v_i \leq \frac{d}{2}$ and $w_i \leq \frac{1}{2}$.

Fix $n \in \mathbb{N}$ and denote $N := \{1, ..., n\}$ and $e_i := (\delta_{ij})_{j=1,...,n} \in \mathbb{N}_0^n$. For $J \subset N$ and a weight system $(v_1, ..., v_n, d) \in \mathbb{N}^{n+1}$ (with $v_i < d$) and $k \in \mathbb{N}_0$ denote

$$\begin{aligned} \mathbb{N}_0^J &:= \{\alpha \in \mathbb{N}_0^n \mid \alpha_i = 0 \text{ for } i \notin J\}, \\ (\mathbb{N}_0^n)_k &:= \{\alpha \in \mathbb{N}_0^n \mid \sum_i \alpha_i \cdot v_i = k\}, \\ (\mathbb{N}_0^J)_k &:= \mathbb{N}_0^J \cap (\mathbb{N}_0^n)_k. \end{aligned}$$

The following combinatorial lemma will help to compare in theorem 2.2 several characterizations of weight systems which admit quasihomogeneous polynomials with isolated singularities. A discussion of the history and references will be given after theorem 2.2.

Lemma 2.1. *Fix a weight system* $(v_1, ..., v_n, d) \in \mathbb{N}^{n+1}$ *with* $v_i < d$ *and a subset* $R \subset (\mathbb{N}_0^n)_d$. *For any* $k \in N$ *define*

$$R_k := \{\alpha \in (\mathbb{N}_0^n)_{d-v_k} \mid \alpha + e_k \in R\}.$$

The following five conditions (C1), (C1)', (C2), (C2)' and (C3) are equivalent.

(C1): $\forall\ J \subset N$ with $J \neq \emptyset$
$\exists\ \alpha \in R \cap \mathbb{N}_0^J$
or $\exists\ K \subset N - J$ with $|K| = |J|$
and $\forall\ k \in K\ \exists\ \alpha \in R_k \cap \mathbb{N}_0^J$.

(C1)': As (C1), but only J *with* $|J| \leq \frac{n+1}{2}$.

(C2): $\forall\ J \subset N$ with $J \neq \emptyset$
$\exists\ K \subset N$ with $|K| = |J|$
and $\forall\ k \in K\ \exists\ \alpha \in R_k \cap \mathbb{N}_0^J$.

(C2)': As (C2), but only J *with* $|J| \leq \frac{n+1}{2}$.

(C3): $\forall\ I,\ J \subset N$ with $|I| < |J|$
$\exists\ k \in N - I$ and $\exists\ \alpha \in R_k \cap \mathbb{N}_0^J$.

Proof: (C1)⇒(C1)' and (C2)⇒(C2)' are trivial.

(C1)'⇒(C1): Consider $J \subset N$ with $|J| > \frac{n+1}{2}$ and $I \subset J$ with $|I| = \left[\frac{n+1}{2}\right]$. If there exists $\alpha \in R \cap \mathbb{N}_0^I$, then also $\alpha \in R \cap \mathbb{N}_0^J$. If not, then there exists $K \subset N - I$ with $|K| = |I|$ and $\forall\ k \in K\ \exists\ \alpha \in R_k \cap \mathbb{N}_0^I$. Then n is even and $K = N - I$, and $K \cap J \neq \emptyset$, and for $k \in K \cap J$ and $\alpha \in R_k \cap \mathbb{N}_0^I$ one finds $\alpha + e_k \in R \cap \mathbb{N}_0^J$.

(C2)'⇒(C1)': Consider $J \subset N$ with $0 < |J| \leq \frac{n+1}{2}$ and $K \subset N$ such that J and K satisfy (C2)'.

1st case: $K \subset N - J$. Then J and K satisfy (C1)'.

2nd case: $K \cap J \neq \emptyset$. Then for $k \in K \cap J$ and $\alpha \in R_k \cap \mathbb{N}_0^J$ one obtains $\alpha + e_k \in R \cap \mathbb{N}_0^J$, so J satisfies (C1)'.

(C3)⇒(C2): Consider $J \subset N$ with $J \neq \emptyset$. Construct elements $k_1, ..., k_{|J|} \in N$ and subsets $I_j = \{k_1, ..., k_j\}$ for $0 \leq j \leq |J| - 1$ and $K := \{k_1, ..., k_{|J|}\}$ as follows. (C3) gives for J and I_j an element $k_{j+1} \in N - I_j$ with $R_{k_{j+1}} \cap \mathbb{N}_0^J \neq \emptyset$. Obviously $|K| = |J|$, and J and K satisfy (C2).

(C1)⇒(C3): Consider $I, J \subset N$ with $|I| < |J|$. Then $J \neq \emptyset$.

1st case, J and some set K satisfy (C1): Because of $|I| < |J| = |K|$ there is a $k \in (N - I) \cap K$ with $R_k \cap \mathbb{N}_0^J \neq \emptyset$.

2nd case, $R \cap \mathbb{N}_0^J \neq \emptyset$: If $R \cap \mathbb{N}_0^J \neq R \cap \mathbb{N}_0^{J \cap I}$ then there exists an $\alpha \in R \cap \mathbb{N}_0^J - R \cap \mathbb{N}_0^{J \cap I}$ and a $k \in J - J \cap I$ with $\alpha_k > 0$. Then $k \in N - I$ and $\alpha - e_k \in R_k \cap \mathbb{N}_0^J$.

So suppose $R \cap \mathbb{N}_0^J = R \cap \mathbb{N}_0^{J \cap I}$. Then $J_1 := J - J \cap I \neq \emptyset$ because of $|I| < |J|$, and $R \cap \mathbb{N}_0^{J_1} = \emptyset$, so there exists a $K_1 \subset N - J_1$ such that J_1 and K_1 satisfy (C1). If $K_1 \cap J \neq \emptyset$ then for $k \in K_1 \cap J$ and $\alpha \in R_k \cap \mathbb{N}_0^{J_1}$ one has $\alpha + e_k \in R \cap \mathbb{N}_0^J - R \cap \mathbb{N}_0^{J \cap I}$, a contradiction. Thus $K_1 \cap J = \emptyset$.

This and $|K_1| = |J_1| > |I - (J \cap I)|$ give $K_1 - I = K_1 - (I - (J \cap I)) \neq \emptyset$. Any $k \in K_1 - I$ satisfies $\emptyset \neq R_k \cap \mathbb{N}_0^{J_1} \subset R_k \cap \mathbb{N}_0^J$. □

Theorem 2.2. *Let* $(v_1, ..., v_n, d) \in \mathbb{N}^{n+1}$ *with* $v_i < d$ *be a weight system.*

(a) Let $f \in \mathbb{C}[x_0, ..., x_n]$ *be a quasihomogeneous polynomial. The condition*

(IS1): f *has an isolated singularity at* 0,

implies that $R := \operatorname{supp} f \subset (\mathbb{N}_0^n)_d$ *satisfies (C1) to (C3).*

(b) Let R be a subset of $(\mathbb{N}_0^n)_d$. The following conditions are equivalent.

(IS2): There exists a quasihomogeneous polynomial f with $\operatorname{supp} f \subset R$ *and an isolated singularity at 0.*

(IS2)': A generic quasihomogeneous polynomial with $\operatorname{supp} f \subset R$ *has an isolated singularity at 0.*

(C1) to (C3): R satisfies (C1) to (C3).

(c) In the case $R = (\mathbb{N}_0^n)_d$ obviously $R_k = (\mathbb{N}_0^n)_{d-v_k}$. The following conditions are equivalent.

(IS3): There exists a quasihomogeneous polynomial f with an isolated singularity at 0.

(IS3)': A generic quasihomogeneous polynomial has an isolated singularity at 0.

(C1) to (C3): $R = (\mathbb{N}_0^n)_d$ satisfies (C1) to (C3).

Remarks 2.3. Several people (re)discovered parts of this theorem. We will not reprove it here, but comment on the history and the references.

(i) Of course, (IS2) $\iff$ (IS2)' and (IS3) $\iff$ (IS3)' and (b)$\Rightarrow$(c) and (a)$\Rightarrow$ $\big($(IS2) $\Rightarrow$ (C1) to (C3)$\big)$.

(ii) Part (a) is quite elementary, for example (IS1)$\Rightarrow$(C1) is contained in K. Saito's paper [Sa1, Lemma 1.5], and it can also be extracted from [Sh, Remark 3].

(iii) (IS2) $\iff$ (C2) is part of an equivalence for more general functions in [Ko1, Remarque 1.13 (ii)], but there Kouchnirenko did not carry out the proof in detail. He gave a short proof of the refined version (IS2) $\iff$ (C2)' in [Ko2, Theorem 1]. This reference [Ko2] seems to have been cited up to now only in [Sh], it seems to have been almost completely ignored.

(iv) Around the same time as Kouchnirenko, Orlik and Randell proved (IS3) $\iff$ (C3) in the preprint [OR1, Theorem 2.12], but the published part [OR2] of it does not contain this result. It seems that they have not published this result.

(v) O.P. Shcherbak stated a more general result [Sh, Theorem 1] from which one can extract (IS2) $\iff$ (C1), but he did not provide a proof. That was done by Wall [Wa, Ch. 5], who also stated explicitly (IS2) $\iff$ (C1) and (IS3) $\iff$ (C1), they are Theorem 5-1 and Theorem 5-3 in [Wa] for the hypersurface case (explicit in (5-7)). But as he covers a much more general case, his proof is long.

(vi) A short proof of (IS3) $\iff$ (C1) is given by Kreuzer and Skarke [KS, proof of Theorem 1], though it requires some work to see that the condition stated in [KS, Theorem 1] is equivalent to (C1).

(vii) In theorem 2.2 (c) conditions $(\mathbb{N}_0^J)_k \neq \emptyset$ for some $k \in \mathbb{N}_0$ arise. For $k \in \mathbb{Z}$ denote $\mathbb{Z}^J$, $(\mathbb{Z}^n)_k$ and $(\mathbb{Z}^J)_k$ analogously to $\mathbb{N}_0^J$, $(\mathbb{N}_0^n)_k$ and $(\mathbb{N}_0^J)_k$. Then $(\mathbb{Z}^J)_k \neq \emptyset$ is equivalent to $\gcd(v_j \,|\, j \in J) \,|\, k$. But $(\mathbb{N}_0^J)_k \neq \emptyset$ (for $k \geq 0$) is more delicate. In the case $J = \{1, 2\}$ sufficient conditions are $\gcd(v_1, v_2) \,|\, k$ and $\operatorname{lcm}(v_1, v_2) - v_1 - v_2 + 1 \leq k$, because then

$$\left(\frac{\operatorname{lcm}(v_1, v_2)}{v_1} - 1\right) \cdot v_1 + (-1) \cdot v_2 = (-1) \cdot v_1 + \left(\frac{\operatorname{lcm}(v_1, v_2)}{v_2} - 1\right) \cdot v_2$$

is the largest multiple of $\gcd(v_1, v_2)$ missing in $\mathbb{N}_0 \cdot v_1 + \mathbb{N}_0 \cdot v_2$.

For any weight system $(v_1, ..., v_n, d) \in \mathbb{N}^{n+1}$ with $v_i < d$ define the rational function

$$\rho_{(v,d)}(t) := \prod_{i=1}^{n} (t^d - t^{v_i})(t^{v_i} - 1)^{-1}.$$

It is well known that $\rho_{(v,d)}(t) \in \mathbb{N}_0[t]$ if a quasihomogeneous polynomial with isolated singularity at 0 exists.

The conditions $\rho_{(v,d)}(t) \in \mathbb{Z}[t]$ and $\rho_{(v,d)}(t) \in \mathbb{N}_0[t]$ are in general weaker than (C1) to (C3), but $\rho_{(v,d)}(t) \in \mathbb{Z}[t]$ is equivalent to a surprisingly similar statement. Denote by $\overline{(C1)}$ and $\overline{(C2)}$

the conditions obtained from (C1) and (C2) in lemma 2.1 with $\mathbb{N}_0$ replaced by $\mathbb{Z}$ in (C1) and (C2) and in the definitions of R and R_k.

Lemma 2.4. *Fix a weight system $(v_1, ..., v_n, d) \in \mathbb{N}^{n+1}$ with $v_i < d$. The following conditions are equivalent.*

$\overline{(IS3)}$: $\rho_{(v,d)}(t) \in \mathbb{Z}[t]$.
(GCD): $\forall\ J \subset N$ *the* $\gcd(v_j \mid j \in J)$ *divides at least* $|J|$ *of the numbers* $d - v_k$.
$\overline{(C2)}$: *for* $R = (\mathbb{Z}^n)_d$.
$\overline{(C1)}$: *for* $R = (\mathbb{Z}^n)_d$.

Proof: $\overline{(IS3)}$ means that all zeros of $\prod_{i=1}^n (t^{v_i} - 1)$ are zeros of $\prod_{i=1}^n (t^{d-v_i} - 1)$ with at least the same multiplicity. This shows $\overline{(IS3)} \iff$ (GCD). The equivalence (GCD) $\iff \overline{(C2)}$ is trivial. The equivalence $\overline{(C2)} \iff \overline{(C1)}$ follows as in lemma 2.1. □

Lemma 2.5. *Fix a weight system $(v_1, ..., v_n, d) \in \mathbb{N}^{n+1}$ with $v_i < d$. If $n \leq 3$ then (IS3) $\iff$ $\overline{(IS3)}$.*

Proof: We restrict to the case $n = 3$. It is sufficient to show $\overline{(C1)} \Rightarrow$(C1) for $R = (\mathbb{N}_0^n)_d$. The ($|J| = 1$)-parts of $\overline{(C1)}$ and (C1) coincide.

Consider $J = \{1, 2\}$, $J_1 = \{1\}$, $J_2 = \{2\}$. Then J satisfies (C1) if and only if $(\mathbb{N}_0^J)_d \neq \emptyset$. Now consider the different possibilities how J_1 and J_2 can satisfy (C1). The only case where $(\mathbb{N}_0^J)_d \neq \emptyset$ is not obvious is the case when $J_1(= \{1\})\&K_1 = \{3\}$ and $J_2(= \{2\})\&K_2 = \{3\}$ satisfy (C1), that is, when $v_1|(d - v_3)$ and $v_2|(d - v_3)$. Of course, then also $\mathrm{lcm}(v_1, v_2)|(d - v_3)$ and $\mathrm{lcm}(v_1, v_2) \leq d - v_3$.

$\overline{(C1)}$ for J gives $(\mathbb{Z}^J)_d \neq \emptyset$, that is, $\gcd(v_1, v_2)|d$.

The conditions $\gcd(v_1, v_2)|d$ and $\mathrm{lcm}(v_1, v_2) \leq d - v_3$ imply $(\mathbb{N}_0^J)_d \neq \emptyset$ by remark 2.3 (vii), so J satisfies (C1).

The ($|J| = 3$)-part of (C1) follows from the ($|J| = 2$)-part. □

Remarks 2.6. (i) Lemma 2.5 is Theorem 3 in [Sa2]. It is also stated in [Ar, remark after cor. 4.13] and [AGV, 2nd remark in 12.3].

(ii) For $n \geq 4$ $\overline{(IS3)}$ is weaker than (IS3). [AGV, 12.3] contains the example $(v_1, v_2, v_3, v_4, d) = (1, 33, 58, 24, 265)$ of Ivlev. Here $\rho_{(v,d)}(t) \in \mathbb{N}_0[t]$, but (C1) fails for $J = \{2, 4\}$.

(iii) The equivalence $\overline{(IS3)} \iff \overline{(C1)}$ in lemma 2.4 is (up to rewriting their condition as $\overline{(C1)}$) Lemma 1 in [KS].

(iv) Chapter 3 in [Wa] contains results and short proofs for 0-dimensional quasihomogeneous complete intersections which are very close to theorem 2.2 (b)+(c), lemma 2.4 and lemma 2.5.

3. Types and graphs of quasihomogeneous singularities

Here a classification of quasihomogeneous polynomials with isolated singularity at 0 by certain *types*, which are encoded in certain *graphs*, will be given. For $n \in \{2, 3\}$ this is treated in [Ar][AGV], the general case is carried out in a part of [OR1] which is not published in [OR2].

The type will come from some choice. Often several choices are possible, and they may lead to different types or the same type, so, often there are several types for one quasihomogeneous polynomial.

Now consider $n \in \mathbb{N}$, $N = \{1, ..., n\}$, a weight system $(v_1, ..., v_n, d) \in \mathbb{N}^{n+1}$ with $v_i < d$ and a quasihomogeneous polynomial $f \in \mathbb{C}[x_1, ..., x_n]$ with an isolated singularity at 0. Then $\operatorname{supp} f \subset (\mathbb{N}_0^n)_d$ satisfies (C2) by theorem 2.2 (a).

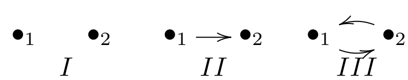
The choice is a map $\kappa : N \to N$ such that for any $j \in N$ the sets $J = \{j\}$ and $K = \{\kappa(j)\}$ satisfy (C2) with $R = \operatorname{supp} f$, that is, f contains a summand $b \cdot x_j^a \cdot x_{\kappa(j)}$ for some $b \in \mathbb{C}^*$, $a \in \mathbb{N}$. The *type* is the conjugacy class of this map κ with respect to the symmetric group S_n. The graph which encodes the map κ is the ordered graph with n vertices with numbers $1, ..., n$ and an arrow from j to $\kappa(j)$ for any $j \in N$ with $j \neq \kappa(j)$. The ordered graph without the numbering of the vertices obviously encodes the type.

In order to describe the graphs, an oriented tree is called *globally oriented* if each vertex except one has exactly one outgoing arrow. Then the exceptional vertex has only incoming arrows and is called *root*. Starting at any vertex and following the arrows one arrives at the root.

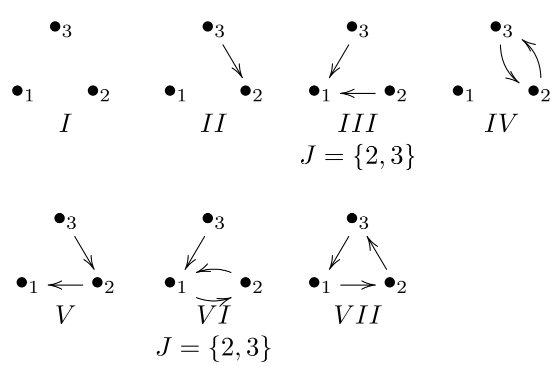
An oriented cycle is called *globally oriented* if each vertex has one incoming and one outgoing arrow. Following the arrows one runs around the cycle. The following lemma is obvious.

Lemma 3.1. *Exactly those graphs occur as graphs of maps $\kappa : N \to N$ whose components either are globally oriented trees or consist of one globally oriented cycle and finitely many globally oriented trees whose roots are on the cycle.*

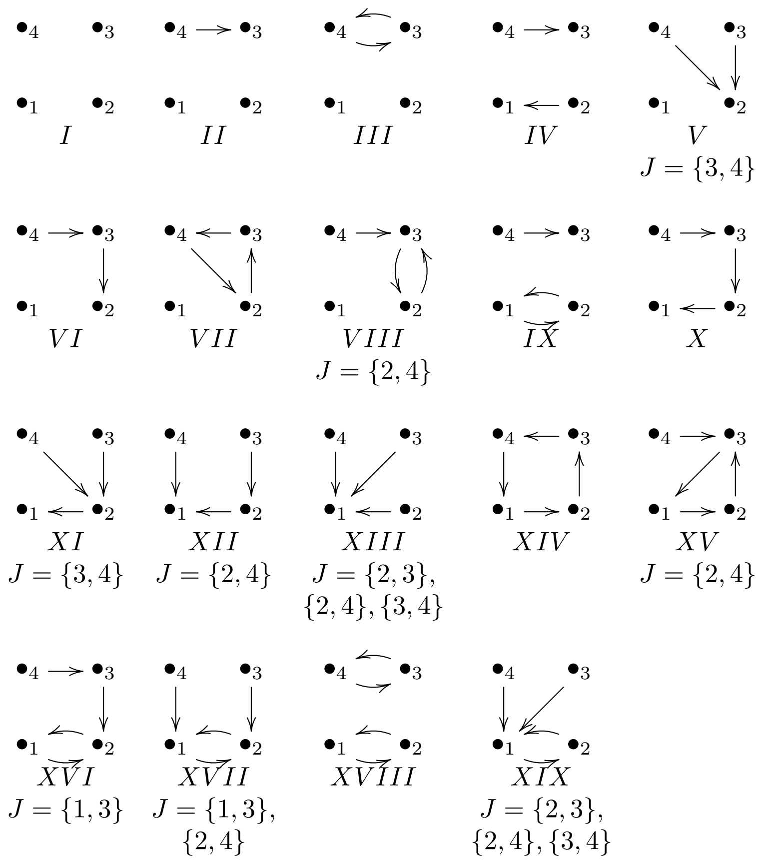
Examples 3.2. (i) $n = 2$: [Ar][AGV] 3 types,



(ii) $n = 3$: [Ar][AGV] 7 types. The sets J under the graphs III and VI are explained in example 3.6.



(iii) $n = 4$: [OR1] and [YS] 19 types. We follow the numbering in [YS, Proposition 3.5]. The sets J under 9 of the 19 graphs are explained in example 3.6.



(iv) $n = 5$: 47 types.
(v) $n = 6$: 128 types.

Remark 3.3. Fix a weight system $(v_1, ..., v_n, d) \in \mathbb{N}^{n+1}$ with $v_i < d$, a quasihomogeneous polynomial f and a map $\kappa : N \to N$ as above. Then for any $j \in N$ the sets $J = \{j\}$ and $K = \{\kappa(j)\}$ satisfy (C2) with $R = \operatorname{supp} f$ *in a unique way*: There is a unique $a_j \in \mathbb{N}$ with $\alpha := a_j e_j \in R_{\kappa(j)} \cap \mathbb{N}_0^J$, that is, there is a unique monomial $x_j^{a_j} x_{\kappa(j)}$ with exponent $a_j e_j + e_{\kappa(j)}$ in the support of f.

Now we forget $(v_1, ..., v_n, d)$ and f and start anew with such a tuple of monomials. We fix $n \in \mathbb{N}$, $N = \{1, ..., n\}$, a map $\kappa : N \to N$, numbers $a_1, ..., a_n \in \mathbb{N}$ and the set $R := \{a_j e_j + e_{\kappa(j)} \mid j \in N\} \subset \mathbb{N}_0^n$ of exponents of the monomials $x_j^{a_j} x_{\kappa(j)}$.

Always $|R| \leq n$, and most often $|R| = n$. The difference $n - |R|$ is the number of 2-cycles in the graph of κ with vertices j_1 and j_2 and numbers $a_{j_1} = a_{j_2} = 1$.

Lemma 3.4. *A weight system $(v_1, ..., v_n, d) \in \mathbb{N}^{n+1}$ with $v_i < d$ and $R \subset (\mathbb{N}_0^n)_d$ exists if and only if any even cycle with vertices $j_1, ..., j_l$ (l even) satisfies either*

(EC1) neither $a_{j_1} = a_{j_3} = ... = a_{j_{l-1}} = 1$
nor $a_{j_2} = a_{j_4} = ... = a_{j_l} = 1$,

or

(EC2) $a_{j_1} = a_{j_2} = ... = a_{j_l} = 1$

(here EC stands for Even Cycle). If such a weight system exists it is unique up to rescaling if and only if all even cycles satisfy (EC1).

Proof: We work with a normalized weight system $(w_1, ..., w_n, 1) \in (\mathbb{Q} \cap (0,1))^n \times \{1\}$. It is a solution of the system of linear equations $a_j w_j + w_{\kappa(j)} = 1$, $j \in N$. We discuss in this order (1) roots of trees not on a cycle, (2) vertices on a cycle, (3) vertices on trees different from the roots.

(1) If j is the root of a tree and is not on a cycle then $\kappa(j) = j$ and $w_j = \frac{1}{a_j+1} \in \mathbb{Q} \cap (0,1)$.

(2) The restriction of the equations $a_j w_j + w_{\kappa(j)} = 1$, $j \in N$, to the vertices $j_1, ..., j_l$ of a cycle with $\kappa(j_l) = j_1$ and $\kappa(j_i) = j_{i+1}$ for $1 \leq i \leq l-1$ has a unique solution $(w_{j_1}, ..., w_{j_l}) \in \mathbb{Q}^l$ if and only if

$$0 \neq \det \begin{pmatrix} a_{j_1} & 1 & & \\ & a_{j_2} & & \\ & & \ddots & 1 \\ 1 & & & a_{j_l} \end{pmatrix} = a_{j_1} \cdot ... \cdot a_{j_l} - (-1)^l,$$

that is, if the cycle is odd or does not satisfy (EC2).

In that case one calculates easily that the solution is

$$w_{j_i} = \frac{\rho(a_{j_{i+1}}, a_{j_{i+2}}, ..., a_{j_l}, a_{j_1}, ..., a_{j_{i-1}})}{a_{j_1} \cdot ... \cdot a_{j_l} - (-1)^l} \tag{3.1}$$

where

$$\rho : \bigcup_{k=0}^{\infty} \mathbb{Z}^k \quad \to \quad \mathbb{Z}, \tag{3.2}$$

$$\rho(x_1, ..., x_k) \quad = \quad x_1 ... x_k - x_2 ... x_k + ... + (-1)^{k-1} x_k + (-1)^k. \tag{3.3}$$

If all $x_i \geq 1$ then $\rho(x_1, ..., x_k) \geq 0$, and then $\rho(x_1, ..., x_k) = 0$ if and only if k is odd and $x_1 = x_3 = ... = x_k = 1$. Therefore in the case $a_{j_1} \cdot ... \cdot a_{j_l} - (-1)^l \neq 0$ all $w_{j_i} > 0$ if and only if the cycle is odd or is even and satisfies (EC1). In that case the inequalities $w_{j_i} > 0$ and $w_{j_{i+1}} > 0$ and the equation $a_{j_i} w_{j_i} + w_{j_{i+1}} = 1$ show also $0 < w_{j_i}$ and $0 < w_{j_{i+1}}$.

In the case $a_{j_1} \cdot ... \cdot a_{j_l} - (-1)^l = 0$ the cycle is even and satisfies (EC2), and the equations $a_{j_i} w_{j_i} + w_{j_{i+1}} = 1$ give only

$$w_{j_1} = w_{j_3} = ... = w_{j_{l-1}} \quad \text{and} \quad w_{j_2} = w_{j_4} = ... = w_{j_l} = 1 - w_{j_1}.$$

Any choice $w_{j_1} \in \mathbb{Q} \cap (0,1)$ works.

(3) The weights of vertices on the trees different from the roots are successively determined by

$$w_j = \frac{1 - w_{\kappa(j)}}{a_j}$$

and automatically satisfy $0 < w_j < 1$. □

The following lemma 3.5 is related to the notion of *invertible polynomial* [ET] and is known to some specialists. We keep the situation after remark 3.3. We need some notations.

The map $\kappa : N \to N$ is of *Fermat type* if $\kappa = \text{id}$, that is, if its graph has no arrows. It is of *cycle type* if its graph is a cycle. It is of *chain type* if it has the vertices $j_1, ..., j_n$ and the $n-1$ arrows from j_i to j_{i+1} for $1 \leq i \leq n-1$. The type of κ is a *sum of Fermat type, cycle types and chain types* if its graph is a union of the corresponding graphs.

Lemma 3.5. *Let $n \in \mathbb{N}$, $N = \{1, ..., n\}$, $\kappa : N \to N$, $a_1, ..., a_n \in \mathbb{N}$ and $R = \{a_j e_j + e_{\kappa(j)} \,|\, j \in N\}$ be as above such that any even cycle in the graph of κ satisfies (EC1) or (EC2) (in lemma 3.4). Let $(v_1, ..., v_n, d) \in \mathbb{N}^{n+1}$ be a weight system with $v_i < d$ and $R \subset (\mathbb{N}_0^n)_d$ (it exists by lemma 3.4). Then the following 2 conditions are equivalent:*

(IS4) A generic linear combination of the (at most n) monomials $x_j^{a_j} x_{\kappa(j)}$, $j \in N$, is a quasihomogeneous polynomial with an isolated singularity at 0.

(FCC) The type of κ is a sum of Fermat type, cycle types and chain types.

Proof: By theorem 2.2 (b), (IS4) is equivalent to (C2) for R as above. The implication (FCC)⇒(IS4) is well known, also a direct proof of (FCC)⇒(C2) is easy.

The other implication (C2)⇒(FCC) will be proved indirectly: Suppose that (FCC) does not hold. Then there are two indices $j_1, j_2 \in N$ with $j_1 \neq j_2$ and $\kappa(j_1) = \kappa(j_2)$. The set $J := \{j_1, j_2\}$ does not satisfy (C2) for R as above. □

Examples 3.6. We return to the examples 3.2.

(i) For a fixed map $\kappa : N \to N$ and numbers $a_1, ..., a_n \in \mathbb{N}$, the conditions (EC1) and (EC2) in lemma 3.4 are not empty if the graph of κ contains an even cycle, that is type III for $n = 2$, the types IV and VI for $n = 3$ and the types III, VIII, IX, XIV, XVI, XVII, XVIII and XIX for $n = 4$.

(ii) $n = 2$: Type I is Fermat type, type II is chain type, type III is cycle type. In type III one must avoid $a_1 = 1,\ a_2 > 1$ and $a_1 > 1,\ a_2 = 1$. Apart from that (IS4) holds for arbitrary $a_1, a_2 \in \mathbb{N}$.

(iii) $n = 3$ and $n = 4$: 5 of the 7 types with $n = 3$ and 10 of the 19 types with $n = 4$ are sums of Fermat type, cycle types and chain types. There (IS4) and (IS3) (in theorem 2.2 (c)) hold for almost arbitrary $a_1, ..., a_n \in \mathbb{N}$, with the only constraints from (EC1) or (EC2) in lemma 3.4.

For the other types, the sets J which fail to satisfy (C1)' for $R = \{a_j e_j + e_{\kappa(j)} \mid j \in N\}$ are indicated under the graphs in example 3.2 (ii) and (iii). For these types one needs more monomials than those with exponents in R in order to satisfy (C1)'. This leads to further constraints on the numbers $a_1, ..., a_n$.

(iv) $n = 3$: In both cases, III and VI, the failing set is $J = \{2, 3\}$. Suppose that a weight system $(v_1, ..., v_n, d)$ as in lemma 3.4 is determined from $a_1, a_2, a_3 \in \mathbb{N}$ (uniquely except for $a_1 = a_2 = 1$ in type VI). For (IS3) to hold one needs $(\mathbb{N}_0^J)_d \neq \emptyset$. By lemma 2.5 this is equivalent to $(\mathbb{Z}^J)_d \neq \emptyset$ and to $\gcd(v_1, v_2) \,|\, d$. This condition is made explicit in [Ar][AGV, 13.2].

(v) $n = 4$: Suppose that a weight system (v_1, v_2, v_3, v_4, d) as in lemma 3.4 is determined from $a_1, a_2, a_3, a_4 \in \mathbb{N}$. Consider in each of the 9 cases which are not sums of Fermat type, cycle types and chain types a failing set $J = \{j_1, j_2\}$, that is, with $\kappa(j_1) = \kappa(j_2) = j_3$ and $\{1, 2, 3, 4\} = \{j_1, j_2, j_3, j_4\}$. For (IS3) to hold one needs $(\mathbb{N}_0^J)_d \neq \emptyset$ or $(\mathbb{N}_0^J)_{d - v_{j_4}} \neq \emptyset$. As in lemma 2.5, the condition $(\mathbb{N}_0^J)_d \neq \emptyset$ is equivalent to $(\mathbb{Z}^j)_d \neq \emptyset$ and to $\gcd(v_{j_1}, v_{j_2}) \,|\, d$. But the condition $(\mathbb{N}_0^J)_{d - v_{j_4}} \neq \emptyset$ may be stronger than $(\mathbb{Z}^J)_{d - v_{j_4}} \neq \emptyset$ and $\gcd(v_{j_1}, v_{j_2}) \,|\, d - v_{j_4}$.

(vi) We consider the case XII with $n = 4$ in detail. There one starts with arbitrary $a_1, a_2, a_3, a_4 \in \mathbb{N}$ and with the monomials $x_1^{a_1+1}$, $x_2^{a_2} x_1$, $x_3^{a_3} x_2$, $x_4^{a_4} x_1$. The weight system

$$\begin{aligned}&(v_1, v_2, v_3, v_4, d)\\ &= (a_2 a_3 a_4, a_1 a_3 a_4, ((a_1 + 1)(a_2 - 1) + 1) a_4, a_1 a_2 a_3, (a_1 + 1) a_2 a_3 a_4)\end{aligned}$$

is unique up to rescaling. The only failing set is $J = \{2, 4\}$, and $\kappa(2) = \kappa(4) = 1$, so $(\mathbb{N}_0^J)_{d - v_1} \neq \emptyset$. One needs $(\mathbb{N}_0^J)_d \neq \emptyset$ or $(\mathbb{N}_0^J)_{d - v_3} \neq \emptyset$ for (IS3) to hold. Now

$$\begin{aligned}(\mathbb{N}_0^J)_d \neq \emptyset \quad &\iff \quad (\mathbb{Z}^J)_d \neq \emptyset\\ &\iff \quad \gcd(v_2, v_4) \,|\, d \iff a_1 \mid \operatorname{lcm}(a_2, a_4).\end{aligned}$$

And

$$\begin{aligned}(\mathbb{N}_0^J)_{d-v_3} \neq \emptyset \quad &\Longrightarrow \quad (\mathbb{Z}^J)_{d-v_3} \neq \emptyset \\ &\Longleftrightarrow \quad \gcd(v_2, v_4) \,|\, d - v_3 \\ &\Longleftrightarrow \quad a_1 a_3 \,|\, \frac{a_4}{\gcd(a_2, a_4)}(((a_1+1)(a_2a_3 - a_2 + 1) - 1).\end{aligned}$$

(vii) Ivlev's example (remark 2.6 (ii), [AGV, 12.3]) $(v_1, v_2, v_3, v_4, d) = (1, 33, 58, 24, 265)$ is of type XII with the monomials x_1^{265}, $x_2^8 x_1$, $x_3^4 x_2$, $x_4^{11} x_1$, so $(a_1, a_2, a_3, a_4) = (264, 8, 4, 11)$. Here $(\mathbb{N}_0^J)_d = \emptyset$ and $(\mathbb{N}_0^J)_{d-v_3} = \emptyset$, so (IS3) does not hold, but $(\mathbb{Z}^J)_{d-v_3} \neq \emptyset$, so $\overline{(IS3)}$ holds and $\rho_{\mathbf{v},d}(t) \in \mathbb{Z}[t]$, even $\in \mathbb{N}_0[t]$.

Two function germs $f_1, f_2 \in \mathcal{O}_{\mathbb{C}^n,0}$ are right equivalent if there is a local coordinate change $\varphi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ such that $f_1 \circ \varphi = f_2$. Often in one right equivalence class of functions with an isolated singularity at 0, there are several quasihomogeneous functions with different weight systems. For example $x_1^{a_1} x_2 + x_2^{a_2} x_3 + x_3 x_1$ with weight system $(v_1, v_2, v_3, d) = (a_2, 1, a_1a_2 - a_2 + 1, a_1a_2 + 1)$ and $x_1^{a_1a_2+1} + x_2^2 + x_3^2$ with weight system $(v_1', v_2', v_3', v_3', d') = (2, a_1a_2+1, a_1a_2+1, 2a_1a_2+2)$ are in the same right equivalence class of $A_{a_1a_2}$-singularities [ET]. The ambiguity was analysed in [Sa1].

Theorem 3.7. [Sa1] *Let $f \in \mathcal{O}_{\mathbb{C}^n,0}$ be a function germ with an isolated singularity at 0.*

(a) f is right equivalent to a quasihomogeneous polynomial if and only if

$$f \in J_f := \left(\frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n}\right) \subset \mathcal{O}_{\mathbb{C}^n,0}.$$

(b) If f is quasihomogeneous with normalized weight system $(w_1, ..., w_n, 1)$ with $0 < w_1 \leq ... \leq w_n < 1$ and if $f \in \mathbf{m}^3_{\mathbb{C}^n,0}$, then the weight system is unique and $0 < w_1 \leq ... \leq w_n < \frac{1}{2}$.

(c) If $f \in J_f$ then f is right equivalent to a quasihomogeneous polynomial $g(x_1, ..., x_k) + x_{k+1}^2 + ... + x_n^2$ with $g \in \mathbf{m}^3_{\mathbb{C}^k,0}$. Especially, its normalized weight system satisfies $0 < w_1 \leq ... \leq w_k < w_{k+1} = ... = w_n = \frac{1}{2}$.

(d) If f and $\widetilde{f} \in \mathcal{O}_{\mathbb{C}^n,0}$ are right equivalent and quasihomogeneous with normalized weight systems $(w_1, ..., w_n, 1)$ and $(\widetilde{w}_1, ..., \widetilde{w}_n, 1)$ with $w_1 \leq ... \leq w_n \leq \frac{1}{2}$ and $\widetilde{w}_1 \leq ... \leq \widetilde{w}_n \leq \frac{1}{2}$ then $w_i = \widetilde{w}_i$.

Remarks 3.8. (i) Part (b) can be proved with the arguments in the proof of lemma 3.4. The condition $f \in \mathbf{m}^3_{\mathbb{C}^n,0}$ is equivalent to all $a_2 \geq 2$. The implication $w_j < \frac{1}{2}$ is nontrivial only in case (2) in the proof of lemma 3.4.

(ii) Part (c) follows from (a) and the splitting lemma and (b).

(iii) An argument for part (d) different from the proof in [Sa1] is as follows. If f is quasihomogeneous with some weight system $(v_1, ..., v_n, d) \in \mathbb{N}^{n+1}$ then $\rho_{(\mathbf{v},d)}(t) \in \mathbb{N}_0[t]$, so

$$\rho_{(\mathbf{v},d)}(t^{1/d}) = \sum_{j=1}^{\mu} t^{\alpha_j}$$

for certain numbers $\alpha_1, ..., \alpha_\mu \in \frac{1}{d}\mathbb{N}$. These numbers and $\rho_{(\mathbf{v},d)}(t^{1/d})$ are invariants of the right equivalence class of f. This is well known and follows essentially from calculations in [Br]. The numbers $\alpha_1, ..., \alpha_\mu$ are the *exponents* of the right equivalence class of f. By part (c) there exists a weight system $(\widetilde{v}_1, ...\widetilde{v}_n, \widetilde{d})$ with $\widetilde{v}_i \leq \frac{\widetilde{d}}{2}$ and

$$\sum_{j=1}^{\mu} t^{\alpha_j} = \rho_{(\widetilde{\mathbf{v}},\widetilde{d})}(t^{1/\widetilde{d}}).$$

It is easy to see that one can recover the normalized weight system $\frac{1}{\widetilde{d}}(\widetilde{v}_1, ..., \widetilde{v}_n, \widetilde{d})$ from the exponents and this equation. Therefore this normalized weight system is unique.

4. Milnor number versus weighted degree

Let p_i, $i \in \mathbb{N}$, be the i-th prime number, so $(p_1, p_2) = (2, 3)$. Define

$$l(n) := \prod_{i=1}^{n} \frac{p_i}{p_i - 1},$$

so $(l(1), l(2), l(3), l(4), l(5)) = (2, 3, \frac{15}{4}, \frac{35}{8}, \frac{77}{16})$. The prime number theorem in the form $p_n = n \log n \cdot (1 + o(1))$ [HW, Theorem 8] and Mertens' theorem

$$\prod_{prime\ numbers\ p \leq x} \frac{p}{p-1} = e^{\gamma} \cdot \log x \cdot (1 + o(1))$$

with $\gamma =$ Euler's constant [HW, Theorem 429] imply

$$l(n) = e^{\gamma} \cdot \log n \cdot (1 + o(1)).$$

Theorem 4.1. *(a) Let $f \in \mathbb{C}[x_1, ..., x_n]$ be a quasihomogeneous polynomial with an isolated singularity at 0 and reduced weight system $(v_1, ..., v_n, d) \in \mathbb{N}^{n+1}$ with $v_i \leq \frac{d}{2}$ for all i (reduced: $\gcd(v_1, ..., v_n, d) = 1$). Then*

$$d \leq l(n) \cdot \mu.$$

(b) If $v_i < \frac{d}{2}$ for all i and $n \geq 2$ then

$$d \leq l(n-1) \cdot \mu.$$

These estimates rely only on the conditions for J with $|J| = 1$ in (C1)-(C3) for $R = (\mathbb{N}_0^n)_d$, the conditions for $|J| \geq 2$ are not needed. Theorem 4.3 formulates this more general case. Both theorems are proved after stating theorem 4.3.

Remarks 4.2. (i) These estimates are useful for a classification of such weight systems using computer, for a fixed number of variables and with Milnor numbers up to a chosen bound. See section 5.

(ii) Calculations in [Br] show that for a quasihomogeneous polynomial f as in theorem 4.1 the monodromy on the Milnor lattice is semisimple with eigenvalues $e^{-2\pi i \alpha_1}$, ..., $e^{-2\pi i \alpha_\mu}$, where α_1,..., α_μ are the exponents considered in remark 3.8 (iii). For $f \in \mathbf{m}^3_{\mathbb{C}^n,0}$ the procedure mentioned in remark 3.8 (iii), which recovers the normalized weights $(w_1, ..., w_n)$ from the exponents, shows that the tuples $(w_1, ..., w_n)$ and $(\alpha_1, ..., \alpha_\mu)$ have the same common denominator d. Therefore in the case $f \in \mathbf{m}^3_{\mathbb{C}^n,0}$ the order of the monodromy is d. Adding squares $x^2_{n+1} + ... x^2_{n+m}$ changes the eigenvalues by the factor $(-1)^m$ and replaces d by $\widetilde{d}$ with $\widetilde{d} = 2d$ for odd d and $\widetilde{d} = d$ for even d. Then the order of the monodromy is $\widetilde{d}$ or $\frac{\widetilde{d}}{2}$.

Theorem 4.3. *Fix $n \in \mathbb{N}$, $N = \{1, ..., n\}$, a map $\kappa : N \to N$, numbers $a_1, ..., a_n \in \mathbb{N}$ and the set $R = \{a_j e_j + e_{\kappa(j)} \mid j \in N\}$ of exponents of the monomials $x_j^{a_j} x_{\kappa(j)}$. Suppose that $a_j \geq 2$ for all $j \in N$ which lie in components C of the graph of κ with $|C| \geq 2$.*

(a) By lemma 3.4 there is a unique reduced weight system $(v_1, ..., v_n, d) \in \mathbb{N}^{n+1}$ with $R \subset (\mathbb{N}_0^n)_d$. It satisfies $v_j < \frac{d}{2}$ for $a_j \geq 2$ and $v_j = \frac{d}{2}$ for $a_j = 1$. Define

$$\mu := \prod_{j=1}^{n} \left(\frac{d}{v_j} - 1 \right).$$

(b)

$$d \leq l(n) \cdot \mu.$$

(c) If all $a_j \geq 2$ and $n \geq 2$ then
$$d \leq l(n-1) \cdot \mu.$$
(d) If $n = 1$ then $d = a_1 + 1$ and $\mu = a_1$.

Proof of theorem 4.1: Suppose $v_1 \leq ... \leq v_k < v_{k+1} = ... = v_n = \frac{1}{2}$ for some k with $0 \leq k \leq n$. By theorem 3.7 f is right equivalent to a quasihomogeneous polynomial $g(x_1, ..., x_k) + x_{k+1}^2 + ... + x_n^2$ with $g \in \mathbf{m}^3_{\mathbb{C}^k,0}$ with an isolated singularity at 0 and the same weight system $(v_1, ..., v_n, d)$.

Choose a map $\kappa : N \to N$ for $g + x_{k+1}^2 + ... + x_n^2$ as in section 3. By remark 3.3 there are unique numbers $a_1, ..., a_n \in \mathbb{N}$ such that $a_j e_j + e_{\kappa(j)}$ are in $\operatorname{supp}(g + x_{k+1}^2 + ... + x_n^2)$. The hypotheses in theorem 4.3 are satisfied. Theorem 4.3 (b) and (c) give theorem 4.1 (a) and (b). $\square$

Proof of theorem 4.3: (a) The first part follows from lemma 3.4. If $a_j = 1$ then j is itself a component of the graph of κ, so $(a_j + 1)v_j = d$, so $v_j = \frac{d}{2}$. If $a_j \geq 2$ then j lies in a component C of the graph of κ with $a_i \geq 2$ for all $i \in C$. Then $v_j < \frac{d}{2}$ follows as in remark 3.8 (i) with the arguments in the proof of lemma 3.4.

(b) and (c) Write $\frac{v_j}{d} = w_j = \frac{s_j}{t_j}$ with $w_j \in \mathbb{Q} \cap (0, \frac{1}{2}]$ and $s_j, t_j \in \mathbb{N}$, $\gcd(s_j, t_j) = 1$. An elementary, but important observation is
$$j \neq \kappa(j) \Longrightarrow t_j = t_{\kappa(j)} \cdot \beta_j \text{ for some } \beta_j \in \mathbb{N} \text{ with } \beta_j \,|\, a_j. \tag{4.1}$$
This follows from
$$\frac{s_j}{t_j} = w_j = \frac{1 - w_{\kappa(j)}}{a_j} = \frac{t_{\kappa(j)} - s_{\kappa(j)}}{t_{\kappa(j)} \cdot a_j} \quad \text{and} \quad \gcd(t_{\kappa(j)}, t_{\kappa(j)} - s_{\kappa(j)}) = 1.$$
For any subset $C \subset N$ define
$$\begin{aligned} \mu(C) &:= \prod_{j \in C} (\frac{1}{w_j} - 1), \quad \text{especially } \mu(\emptyset) = 1,\ \mu(N) = \mu, \\ d(C) &:= \operatorname{lcm}(t_j \,|\, j \in C), \quad \text{especially } d(\emptyset) = 1,\ d(N) = d. \end{aligned}$$
Let C_{Fermat} be the union of all components C of the graph of κ with $|C| = 1$. For $j \in C_{Fermat}$ $w_j = \frac{1}{a_j + 1}$, so
$$\mu(C_{Fermat}) = \prod_{j \in C_{Fermat}} a_j, \tag{4.2}$$
$$d(C_{Fermat}) = \operatorname{lcm}(a_j + 1 \,|\, j \in C_{Fermat}). \tag{4.3}$$
Now we will study $\mu(C)$ and $d(C)$ for a component C of the graph of κ with $|C| \geq 2$. By hypothesis $a_j \geq 2$ for $j \in C$.

Case 1, C is a cycle: Suppose $C = \{1, ..., m\}$ with $\kappa(j) = j - 1$ for $2 \leq j \leq m$ and $\kappa(1) = m$. (4.1) gives immediately $t_1 = t_2 = ... = t_m = d(C)$. (3.1) shows (with ρ as in (3.2))
$$d(C) = t_1 = ... = t_m = \frac{1}{\gamma} \cdot (a_1 ... a_m - (-1)^m) \tag{4.4}$$
$$\text{where } \gamma = \gcd(a_1 ... a_m - (-1)^m, \rho(a_{j-1}, ..., a_1, a_m, ..., a_{j+1})) \tag{4.5}$$
for any $j \in \{1, ..., m\}$. Define here $\widetilde{d}(C) := \gamma \cdot d(C) = a_1 ... a_m - (-1)^m$.

One calculates

$$\begin{aligned}\mu(C) &= \prod_{j=1}^{m} \frac{d-v_j}{v_j} = \prod_{j=1}^{m} \frac{a_1...a_m - (-1)^m - \rho(a_{j-1},...,a_1,a_m,...a_{j+1})}{\rho(a_{j-1},...,a_1,a_m,...,a_{j+1})} \\ &= \prod_{j=1}^{m} \frac{a_{j+1}\cdot\rho(a_j,...,a_1,a_m,...,a_{j+2})}{\rho(a_{j-1},...,a_1,a_m,...,a_{j+1})} = a_1\cdot...\cdot a_m. \end{aligned} \tag{4.6}$$

Case 2, C is not a cycle: Then C is either a tree or a cycle with one or several attached trees. If C is a tree suppose $C_1 = \{1\} \subset C$ is the root, and define $m := 1$. If C is a cycle with attached trees suppose $C_1 = \{1,...,m\}$ is the cycle, and $\kappa(j) = j-1$ for $2 \le j \le m$, $\kappa(1) = m$. In both cases the *set of leaves* is the subset $C_2 \subset C - C_1$ of vertices with no incoming arrows. For any leaf $j \in C_2$ denote by $C(j)$ the set of vertices on the path from j to C_1, excluding the vertex in C_1, so

$$C(j) = \{j, \kappa(j), ..., \kappa^{l(j)}(j)\} \subset C - C_1 \text{ with } \kappa^{l(j)+1}(j) \in C_1.$$

Then with $\gamma := 1$ if $m = 1$ and γ as in (4.5) if $m \ge 2$ one has

$$d(C_1) = \frac{1}{\gamma}\cdot(a_1...a_m - (-1)^m).$$

With (4.1) and β_i as defined in (4.1) one finds

$$t_j = d(C_1)\cdot\prod_{i\in C(j)}\beta_i \qquad \text{for } j \in C_2, \tag{4.7}$$

$$\begin{aligned} d(C) &= \operatorname{lcm}(t_j \,|\, t_j \in C_2) \\ &= d(C_1)\cdot\operatorname{lcm}\Big(\prod_{i\in C(j)}\beta_i \,|\, j\in C_2\Big). \end{aligned}\tag{4.8}$$

We will estimate $d(C)$ by $\widetilde{d}(C)$ with $d(C)\,|\,\widetilde{d}(C)$ and

$$\widetilde{d}(C) := (a_1...a_m - (-1)^m)\cdot\left(\prod_{j\in C-(C_1\cup C_2)} a_j\right)\cdot\operatorname{lcm}(a_j\,|\,j\in C_2). \tag{4.9}$$

In order to estimate $\mu(C)$ from above, we choose a decomposition of $C - C_1$ into a disjoint union

$$C - C_1 = \dot{\bigcup_{j\in C_2}} \widetilde{C}(j)$$

with $\widetilde{C}(j) \subset C(j)$ being a suitable sub-chain of $C(j)$,

$$\widetilde{C}(j) = \{j, \kappa(j), ..., \kappa^{\widetilde{l}(j)}(j)\} \qquad \text{for some } \widetilde{l}(j) \le l(j).$$

To simplify notations suppose for a moment that one such sub-chain $\widetilde{C}(j)$ takes the form $\widetilde{C}(j) = \{j, j-1, ..., k\}$ with $\kappa(i) = i-1$ for $k \le i \le j$. Using $w_l = \frac{1-w_{\kappa(l)}}{a_l}$ repeatedly one finds by an easy induction for $k \le i \le j$

$$w_i = \frac{\rho(a_{i-1},...,a_{k+1},a_k) + (-1)^{i-1-k}w_{k-1}}{a_k a_{k+1}...a_{i-1}a_i}. \tag{4.10}$$

Therefore

$$\begin{aligned}\mu(\widetilde{C}(j)) &= \prod_{i\in\widetilde{C}(j)} \frac{1-w_i}{w_i} = \prod_{i\in\widetilde{C}(j)} \frac{\rho(a_i,...,a_{k+1},a_k)+(-1)^{i-k}w_{k-1}}{\rho(a_{i-1},...,a_{k+1},a_k)+(-1)^{i-1-k}w_{k-1}} \\ &= \frac{\rho(a_j,...,a_{k+1},a_k)+(-1)^{j-k}w_{k-1}}{1-w_{k-1}}. \end{aligned} \tag{4.11}$$

Because all $a_i \geq 2$ for $i \in C$, one can estimate

$$\begin{aligned}\rho(a_j,...,a_{k+1},a_k)+(-1)^{j-k}w_{k-1} &> a_k...a_{j-1}\cdot(a_j-1), \\ \mu(\widetilde{C}(j)) > \frac{a_k...a_{j-1}\cdot(a_j-1)}{1-w_{k-1}} &> a_k...a_{j-1}\cdot(a_j-1). \end{aligned} \tag{4.12}$$

The following additional estimate is relevant only for odd m. But it holds for all m, and it will be smoother to treat even and odd m simultaneously. For $k-1 \in C_1$

$$\begin{aligned} & \mu(C_1)\cdot\frac{1}{1-w_{k-1}} \\ = \;& a_1...a_m\cdot\frac{a_1...a_m-(-1)^m}{a_1...a_m-(-1)^m-\rho(a_{k-2},...,a_1,a_m,...,a_k)} \\ \geq \;& a_1...a_m-(-1)^m. \end{aligned} \tag{4.13}$$

Now we put together the pieces and estimate $\mu(C)$ from above. There is (at least) one leaf $j_0 \in C_2$ with $\widetilde{C}(j_0) = C(j_0)$, so $k-1 := \kappa^{\widetilde{l}(j_0)+1}(j) \in C_1$. For this leaf j_0 we use the finer estimate in (4.12)

$$\mu(\widetilde{C}(j_0)) > \frac{1}{1-w_{k-1}}\cdot(a_{j_0}-1)\cdot\prod_{i\in C(j_0)-\{j_0\}} a_i.$$

Together with (4.12) for all other leaves $j \in C_2$ and (4.13) we obtain

$$\begin{aligned} & \mu(C) = \mu(C_1)\cdot\prod_{j\in C_2}\mu(\widetilde{C}(j)) \\ \geq \;& (a_1...a_m-(-1)^m)\cdot\left(\prod_{j\in C-(C_1\cup C_2)} a_j\right)\cdot\left(\prod_{j\in C_2}(a_j-1)\right). \end{aligned} \tag{4.14}$$

Now case 2 is finished. We can estimate d and μ and their quotient. $C_{Leaf} \subset N - C_{Fermat}$ denotes the union of the leaves of all components C with $|C| \geq 2$. For any such C the notations of case 2 are preserved, C_1 is the root or the cycle in it, and C_2 is the set of leaves in it. If C is a cycle then $C = C_1$.

$$\begin{aligned} d &= \mathrm{lcm}\,(d(C_{Fermat});\ d(C) \text{ for all components } C \text{ with } |C|\geq 2) \\ &\leq \mathrm{lcm}\Big(d(C_{Fermat});\ \widetilde{d}(C) \text{ for } C \text{ with } |C|\geq 2\Big) \\ &\leq \prod_{C \text{ with } |C|\geq 2,\, C \text{ not an odd cycle}} \left(\left(\prod_{j\in C_1} a_j - (-1)^{|C_1|}\right)\cdot\prod_{j\in C-(C_1\cup C_2)} a_j\right)\cdot \\ & \quad\ \mathrm{lcm}\Big(a_j+1 \text{ for } j\in C_{Fermat}; \\ & \quad\ \prod_{j\in C} a_j+1 \text{ for } C \text{ an odd cycle };\ a_j \text{ for } j\in C_{Leaf}\Big). \end{aligned} \tag{4.15}$$

$$\begin{aligned}
\mu &= \mu(C_{Fermat}) \cdot \prod_{C \text{ a cycle}} \mu(C) \cdot \prod_{C \text{ not a cycle}, |C| \geq 2} \mu(C) \\
&\geq \prod_{j \in C_{Fermat}} a_j \cdot \prod_{C \text{ a cycle}} \left(\prod_{j \in C} a_j \right) \cdot \qquad (4.16) \\
&\quad \prod_{C \text{ not a cycle}, |C| \geq 2} \left(\left(\prod_{j \in C_1} a_j - (-1)^{|C_1|} \right) \cdot \prod_{j \in C - (C_1 \cup C_2)} a_j \cdot \prod_{j \in C_2} (a_j - 1) \right).
\end{aligned}$$

$$\frac{d}{\mu} \leq \frac{\operatorname{lcm}\left(\begin{array}{c} a_j + 1 \text{ for } j \in C_{Fermat}; \\ \prod_{j \in C} a_j + 1 \text{ for } C \text{ an odd cycle; } a_j \text{ for } j \in C_{Leaf} \end{array} \right)}{\prod_{j \in C_{Fermat}} a_j \cdot \prod_{C \text{ an odd cycle}} \left(\prod_{j \in C} a_j \right) \cdot \prod_{j \in C_{Leaf}} (a_j - 1).} \qquad (4.17)$$

In lemma 4.4 two numbers $l_1(n)$ and $l_2(n) \in \mathbb{Q}_{>0}$ are defined. Obviously $\frac{d}{\mu} \leq l_1(n)$, and if all $a_j \geq 2$ and $n \geq 2$ then $\frac{d}{\mu} \leq \max(l_2(n), l_1(n-1))$. The parts (b) and (c) of theorem 4.3 follow now with lemma 4.4. Part (d) is trivial. □

Lemma 4.4. *For $n \in \mathbb{N}$ define*

$$\begin{aligned}
l_1(n) &= \max \left(\frac{\operatorname{lcm}(b_1, ..., b_n)}{(b_1 - 1) \cdot ... \cdot (b_n - 1)} \mid b_1, ..., b_n \in \mathbb{N} - \{1\} \right), \\
l_2(n) &= \max \left(\frac{\operatorname{lcm}(b_1, ..., b_n)}{(b_1 - 1) \cdot ... \cdot (b_n - 1)} \mid b_1, ..., b_n \in \mathbb{N} - \{1, 2\} \right).
\end{aligned}$$

Then

$$l_1(n) = l(n) := \prod_{i=1}^{n} \frac{p_i}{p_i - 1} \geq l_2(n+1),$$

here p_i is the i-th prime number.

Proof: First, $l_1(n) = l(n)$ will be proved. Choose $b_1, ..., b_n \in \mathbb{N}$ arbitrarily. Write $\operatorname{lcm}(b_1, ..., b_n) = \prod_{i \in I} p_i^{r_i}$ with $I \subset \mathbb{N}$ finite, $r_i \geq 1$ for $i \in I$. For any $i \in I$ choose $\beta(i) \in N$ with $p_i^{r_i} \mid b_{\beta(i)}$. Define

$$\widetilde{b}_j := \prod_{i \text{ with } \beta(i) = j} p_i^{r_i}.$$

For any j with $\widetilde{b}_j > 1$ let $i(j)$ be the minimal i with $\beta(i) = j$. Then

$$\begin{aligned}
\operatorname{lcm}(b_1, ..., b_n) &= \operatorname{lcm}(\widetilde{b}_j \mid \widetilde{b}_j > 1) = \prod_{j \text{ with } \widetilde{b}_j > 1} \widetilde{b}_j, \\
\frac{\operatorname{lcm}(b_1, ..., b_n)}{(b_1 - 1) \cdot ... \cdot (b_n - 1)} &\leq \prod_{j \text{ with } \widetilde{b}_j > 1} \frac{\widetilde{b}_j}{\widetilde{b}_j - 1} \\
&\leq \prod_{j \text{ with } \widetilde{b}_j > 1} \frac{p_{i(j)}}{p_{i(j)} - 1} \leq \prod_{i=1}^{n} \frac{p_i}{p_i - 1}.
\end{aligned}$$

This proves $l_1(n) \leq l(n)$. The choice $b_i = p_i$ proves $l_1(n) \geq l(n)$.

Analogously one shows for $n \geq 2$

$$l_2(n) = \frac{3}{3-1} \cdot \frac{4}{4-1} \cdot \prod_{i=3}^{n} \frac{p_i}{p_i - 1}.$$

$l_2(2) = 2 = l(1)$. For $n \geq 2$ the estimate $l_2(n+1) \leq l(n)$ follows from

$$\frac{4}{4-1} \cdot \frac{p_{n+1}}{p_{n+1}-1} \leq \frac{4}{3} \cdot \frac{p_3}{p_3 - 1} = \frac{5}{3} < \frac{2}{2-1}.$$

□

5. Computer calculations

Theorem 2.2 (c) gives combinatorial characterizations (C1)-(C3) of those reduced weight systems $(v_1, ..., v_n, d) \in \mathbb{N}^{n+1}$ for which quasihomogeneous polynomials with an isolated singularity at 0 exist. These characterizations can be used in computer programs to find all such weight systems with Milnor number up to some chosen bound. Because of theorem 3.7 for most purposes it is sufficient to restrict to weight systems with $v_i < \frac{d}{2}$. Theorem 4.1 (b) gives then the bound $d \leq l(n-1) \cdot \mu$ for d if $n \geq 2$.

The second author carried out such computer calculations for $n = 2, 3, 4$. The following table lists for $n = 2, 3, 4$ the number of reduced weight systems $(v_1, ..., v_n, d)$ (up to reordering of $v_1, ..., v_n$) with $v_i < \frac{d}{2}$ which satisfy (C1)-(C3) for $R = (\mathbb{N}_0^n)_d$ and whose Milnor number is less or equal than the number μ in the left column.

μ	$n=1$	$n=2$	$n=3$	$n=4$
50	50	187	217	100
100	100	493	806	590
150	150	847	1627	1442
200	200	1242	2623	2678
300	300	2083	5027	6059
400	400	2998	7832	10459
500	500	3957	10931	15634
1000	1000	9246	30241	52761
1500	1500	15058	53698	103841
2000	2000	21194	80055	165624
3000	3000	34177	139343	?
4000	4000	47833	205191	?
5000	5000	62012	276169	?
6000	6000	76545	351335	?
7000	7000	91439	430009	?
8000	8000	106616	512141	?
9000	9000	122040	596879	?

On the homepage [HK] tables with all these weight systems and the characteristic polynomials of the monodromy are available. Of course for $n = 1$ one has just the A_μ-singularities $x_1^{\mu+1}$ with $(v_1, d) = (1, \mu + 1)$ for $\mu \geq 1$. The A_1-singularity is taken into account in the column for $n = 1$ despite $v_1 = \frac{d}{2}$ in that case.

For example, the total number of reduced weight systems for $n = 4$ with $v_i \leq \frac{d}{2}$ and (C1)-(C3) and $\mu \leq 50$ is $50 + 187 + 217 + 100$.

The weight system $(\frac{v_1}{d}, \frac{v_2}{d}, \frac{v_3}{d}, \frac{v_4}{d})$ with $\frac{v_i}{d} < \frac{1}{2}$ and the largest d within $\mu \leq 500$ is $(\frac{1}{58}, \frac{1}{5}, \frac{1}{3}, \frac{57}{116})$ with $\mu = 473$, $d = 1740$, $l(3) \cdot \mu = 1773,75$. This indicates that the estimate in theorem 4.1 (b) cannot be improved much.

For any n the weight system with $v_i < \frac{d}{2}$ with the smallest Milnor number is $(1,...,1,3)$ with $d=3$ and $\mu = 2^n$. This follows from [KS, Lemma 2]. This lemma says that there is an injective map

$$\nu : \{i \,|\, v_i > \frac{1}{3}\} \to \{i \,|\, v_i < \frac{1}{3}\} \quad \text{with} \quad v_{\nu(i)} = d - 2v_i.$$

Then

$$(\frac{d}{v_i} - 1)(\frac{d}{v_{\nu(i)}} - 1) > 4.$$

For $n = 2$ weight systems with $v_i < \frac{d}{2}$ exist for any $\mu \geq 4$, because of the D_μ-singularities $x_1^{\mu-1} + x_2^2 x_1$. But for $n = 3$ and $n = 4$ there are some gaps, some numbers $> 2^n$ which are not Milnor numbers of any quasihomogeneous singularities $f \in \mathbf{m}^3_{\mathbb{C}^n,0}$. We list all gaps up to $\mu = 1000$ for $n = 3$ and up to $\mu = 500$ for $n = 4$.

$$\begin{aligned} n=3: \quad \mu = \quad & 9, 13, 37, 61, 73, 157, 193, 277, 313, 397, 421, \\ & 457, 541, 613, 661, 673, 733, 757, 877, 997. \\ n=4: \quad \mu = \quad & 17, 18, 19, 23, 27, 47, 59, 74, 83, 107, 167, 179, \\ & 219, 227, 263, 314, 347, 359, 383, 467, 479. \end{aligned}$$

Corollary 6.3 will give an explanation of the majority of these gaps in terms of Sophie Germain prime numbers and similar prime numbers.

Yonemura [Yo] had classified all reduced weight systems (v_1, v_2, v_3, v_4, d) with $\sum_i v_i = d$ and (C1)-(C3) for $R = (\mathbb{N}_0^n)_d$. Using our lists, we recovered his 95 weight systems. 48 are in our list for $n = 3$ with $\sum_{i=1}^3 v_i = \frac{d}{2}$, with Milnor numbers ranging between 125 ((1,1,1,6) and 492 ((1,6,14,42)). 47 are in our list for $n = 4$, with Milnor numbers ranging between 81 ((1,1,1,1,4)) and 264 ((1,3,7,10,21)).

6. The case Milnor number = prime number

The computer calculations mentioned in section 5 led us to expect the following result. This section is devoted to its proof.

Theorem 6.1. *Let $f \in \mathbb{C}[x_1, ..., x_n]$ be a quasihomogeneous polynomial with an isolated singularity at 0 and normalized weight system $(w_1, ..., w_n, 1) \in (\mathbb{Q} \cap (0, \frac{1}{2}))^n \times \{1\}$ such that its Milnor number μ is a prime number.*

(a) There are numbers $a_1, ..., a_n \in \mathbb{N} - \{1\}$ and $c_1, ..., c_n \in \mathbb{C}^*$ such that

$$f = c_1 x_1^{a_1+1} + c_2 x_2^{a_2} x_1 + ... + c_n x_n^{a_n} x_{n-1}.$$

Therefore f is of chain type by the map $\kappa : N \to N$ with $\kappa(1) = 1$, $\kappa(j) = j - 1$ for $2 \leq j \leq n$. And this is the only possible map κ as in section 3. Also, by rescaling of $x_1, ..., x_n$ one can arrange $c_1 = ... = c_n = 1$. So, f is unique up to right equivalence.

(b) Write $w_i = \frac{s_i}{t_i}$ with $s_i, t_i \in \mathbb{N}$, $\gcd(s_i, t_i) = 1$. Then

$$\begin{aligned} t_i &= a_i ... a_2 \cdot (a_1 + 1), \quad d = t_n, \\ s_i &= \rho(a_{i-1}, ..., a_2, a_1 + 1) \quad (\text{with } \rho \text{ as in (3.2)}), \\ s_1 &= 1, \quad s_{i+1} = t_i - s_i = t_i - t_{i-1} + t_{i-2} - ... + (-1)^i, \\ \mu &= \rho(a_n, ..., a_2, a_1 + 1). \end{aligned}$$

(c) The characteristic polynomial of the monodromy on the Milnor lattice of f is $\prod_{m:(6.1)} \Phi_m$, here Φ_m is the cyclotomic polynomial of the m-th primitive unit roots, and (6.1) is the condition

$$m \,|\, a_n ... a_2(a_1 + 1), \quad \min(i \mid m \,|\, a_i ... a_2(a_1 + 1)) \equiv n \mod 2. \tag{6.1}$$

Especially, all eigenvalues have multiplicity 1.

Examples 6.2. For $n = 2, 3$ all tuples $(a_1, ..., a_n)$ as in theorem 6.1 with $\mu \leq 23$ are listed below, for $n = 4$ all tuples with $\mu \leq 31$.

μ	$n=2$	$n=3$	$n=4$
5	(3,2)	-	-
7	(5,2), (2,3)	-	-
11	(9,2), (4,3)	(3,2,2), (2,3,2)	-
13	(11,2), (5,3), (3,4), (2,5)	-	-
17	(15,2), (7,3), (3,5)	(5,2,2), (2,5,2)	-
19	(17,2), (8,3), (5,4), (2,7)	(4,3,2), (3,4,2), (3,2,3)	-
23	(21,2), (10,3)	(7,2,2), (5,3,2), (3,5,2), (2,7,2)	-
29	4 tuples	6 tuples	(3,2,3,2)
31	6 tuples	2 tuples	(5,2,2,2)

Proof of theorem 6.1: Let $\kappa : N \to N$ be a map as in section 3, so for any $j \in N$ the sets $J = \{j\}$ and $K = \{\kappa(j)\}$ satisfy (C2) for $R = \operatorname{supp} f$.

The proof proceeds in 4 steps: Step 1 extends some notations and formulas from the proof of theorem 4.3. Step 2 shows that κ is of chain type. Step 3 shows all remaining statements in (a) and (b). Step 4 proves part (c).

Step 1. We consider the graph of κ. The union of components C with $|C| = 1$ is called C_{Fermat}. For a component C let $C_1 \subset C$ be the root of C if C is a tree, the cycle in C if C contains a cycle, and $C_1 = C$ if $|C| = 1$.

For a component C with $|C| \geq 2$ let $C_2 \subset C - C_1$ be the set of leaves, that is, the vertices without incoming arrows, and let $C_3 \subset C - C_2$ be the set of branch points, that is, the vertices with ≥ 2 incoming arrows. The multiplicity $r(j) \in \mathbb{N}$ of a branch point $j \in C_3$ is the number of incoming arrows minus 1. If C is not a cycle then $C_3 \neq \emptyset$, $C_3 \cap C_1 \neq \emptyset$ and $\sum_{c \in C_3} r(c) = |C_2|$.

The union of all leaves is called C_{Leaf}, the union of all branch points is called C_{Branch}.

For a component C with $|C| \geq 2$ and for $j \in C$ let

$$\widehat{C}(j) = (j, \kappa(j), ..., \kappa^{\widehat{l}(j)}(j))$$

be the longest tuple witout repetition: If C is a tree then $\kappa^{\widehat{l}(j)}(j)$ is the root and $\kappa^{\widehat{l}(j)-1}(j)$ is not the root. If C contains a cycle, $\widehat{C}(j)$ hits the cycle and runs around it almost once, so it hits the cycle in $\kappa^{\widehat{l}(j)+1}(j)$. If $k \in \widehat{C}(j)$ let $C(j,k)$ be the tuple from j to k, $C(j,k) = (j, \kappa(j), ..., k)$.

The definition of $C(j)$ in the proof of theorem 4.3 is slightly changed here: For $j \in C - C_1$ (not only $j \in C_2$), let $C(j) = (j, \kappa(j), ..., \kappa^{l(j)}(j))$ be the sub-tuple of $\widehat{C}(j)$ which stops just before reaching C_1, so $\kappa^{l(j)}(j) \notin C_1$, $\kappa^{l(j)+1}(j) \in C_1$. For $j \in C_1$ define $C(j) := \emptyset$.

For any $C(j,k)$ define with ρ as in (3.2)

$$\widehat{\rho}(C(j,k)) := \rho(j, \kappa(j), ..., k)$$

and define $\widehat{\rho}(\emptyset) := 1$.

Now formula (3.1) for the weight w_j of a vertex $j \in C_1$ on a cycle can be rephrased as

$$w_j = \frac{\widehat{\rho}(\widehat{C}(j) - \{j\})}{\prod_{k \in C_1} a_k - (-1)^{|C_1|}}. \tag{6.2}$$

And formula (4.11) generalizes to

$$\mu(C(j,k)) = \prod_{i \in C(j,k)} \left(\frac{1}{w_i} - 1\right) = \frac{\widehat{\rho}(C(j,k)) + (-1)^{|C(j,k)|+1} w_{\kappa(k)}}{1 - w_{\kappa(k)}}. \tag{6.3}$$

For $j,k\in C-C_1$ and $k\in C(j)$ the tuple $\widehat{C}(j)$ contains the tuple $\widehat{C}(\kappa(k))$, they hit the cycle or root C_1 at the same vertex $l_1\in C_1$ and end at the same vertex $l_2\in C_1$, with $\kappa(l_2)=l_1$. For such j and k one calculates with (6.2) and (6.3)

$$\begin{aligned}
&\mu(C(j,k))=\frac{\mu(\widehat{C}(j))}{\mu(\widehat{C}(\kappa(k)))}\\
=&\quad\frac{\widehat{\rho}(\widehat{C}(j))+(-1)^{|\widehat{C}(j)|+1}w_{l_1}}{\widehat{\rho}(\widehat{C}(\kappa(k)))+(-1)^{|\widehat{C}(\kappa(k))|+1}w_{l_1}}\\
=&\quad\frac{\left(\prod_{l\in C_1}a_l-(-1)^{|C_1|}\right)\widehat{\rho}(\widehat{C}(j))+(-1)^{|\widehat{C}(j)|+1}\widehat{\rho}(\widehat{C}(l_1)-\{l_1\})}{\left(\prod_{l\in C_1}a_l-(-1)^{|C_1|}\right)\widehat{\rho}(\widehat{C}(\kappa(k)))+(-1)^{|\widehat{C}(\kappa(k))|+1}\widehat{\rho}(\widehat{C}(l_1)-\{l_1\})}\\
=&\quad\frac{\left(\prod_{l\in C_1}a_l\right)\widehat{\rho}(\widehat{C}(j))+(-1)^{|C_1|+1}\left(\prod_{l\in C_1}a_l\right)\widehat{\rho}(C(j))}{\left(\prod_{l\in C_1}a_l\right)\widehat{\rho}(\widehat{C}(\kappa(k)))+(-1)^{|C_1|+1}\left(\prod_{l\in C_1}a_l\right)\widehat{\rho}(C(\kappa(k)))}\\
=&\quad\frac{\widehat{\rho}(\widehat{C}(j))+(-1)^{|C_1|+1}\widehat{\rho}(C(j))}{\widehat{\rho}(\widehat{C}(\kappa(k)))+(-1)^{|C_1|+1}\widehat{\rho}(C(\kappa(k))}
\end{aligned}\tag{6.4}$$

A component C with $|C|\geq 2$ which is not a cycle is a tree or a cycle with attached trees. One can choose a map $\beta: C_2\to C_3$ from the leaves to the branch points such that $k\in C_3$ is the image of $r(k)$ leaves and $\beta(j)\in\widehat{C}(j)$ for any leaf j. Then $C-C_1$ is the disjoint union $\bigcup_{j\in C_2}(C(j)-C(\beta(j)))$, here the sets underlying the tuples are meant. Therefore

$$\begin{aligned}
\mu(C)\quad&=\quad\prod_{j\in C_1}a_j\cdot\prod_{j\in C_2}\mu(C(j))\cdot\prod_{j\in C_3}\mu(C(j))^{-r(j)}\\
&=\quad\prod_{j\in C_1}a_j\cdot\prod_{j\in C_2}\left(\widehat{\rho}(\widehat{C}(j))+(-1)^{|C_1|+1}\widehat{\rho}(C(j))\right)\\
&\qquad\cdot\prod_{j\in C_3}\left(\widehat{\rho}(\widehat{C}(j))+(-1)^{|C_1|+1}\widehat{\rho}(C(j))\right)^{-r(j)}
\end{aligned}\tag{6.5}$$

Step 2. If $C_{Leaf}=\emptyset$ then the graph of κ is a union of points and cycles, and

$$\mu=\prod_{j\in C_{Fermat}}(a_j+1)\cdot\prod_{C\ cycle}\prod_{j\in C}a_j.$$

Then $\mu=$ prime number and all $a_j\geq 2$ imply $n=1$.

So suppose $C_{Leaf}\neq\emptyset$. Then there is a leaf $j_0\in C_{Leaf}$ such that compared to all leaves $j\in C_{Leaf}$ the number $\widehat{\rho}(\widehat{C}(j))+(-1)^{|C_1|+1}\widehat{\rho}(C(j))$ is maximal for $j=j_0$. Here and later by a slight abuse of notation we denote for any $j\in C-C_{Fermat}$ the cycle or root in the component of j by C_1. Now choose a map $\beta: C_{Leaf}\to C_{Branch}$ as at the end of step 1 and with the additional property $\beta(j_0)\in C_1$, so $\widehat{C}(j_0)$ hits C_1 in $\beta(j_0)$. This is possible. Define the following natural numbers

$$\begin{aligned}
A_0 &:= \widehat{\rho}(\widehat{C}(j_0)) + (-1)^{|C_1|+1}\widehat{\rho}(C(j_0)),\\
A_j &:= a_j + 1 \qquad \text{for } j \in C_{Fermat},\\
A_j &:= a_j \qquad \text{for } j \in \bigcup_{cycles\ C} C,\\
B_{j_0} &:= \mu(C_1) = \prod_{j\in C_1} a_j \qquad (C_1 \text{ for } j_0),\\
B_j &:= \widehat{\rho}(\widehat{C}(j)) + (-1)^{|C_1|+1}\widehat{\rho}(C(j)) \qquad \text{for } j \in C_{Leaf} - \{j_0\},\\
D_j &:= \widehat{\rho}(\widehat{C}(\beta(j))) + (-1)^{|C_1|+1}\widehat{\rho}(C(\beta(j))) \qquad \text{for } j \in C_{Leaf}.
\end{aligned}$$

Then

$$\mu = A_0 \cdot \prod_{j\in C_{Fermat}\cup(\text{all cycles})} A_j \cdot \prod_{j\in C_{Leaf}} \frac{B_j}{D_j}, \tag{6.6}$$

and

$$\begin{aligned}
& \text{all } A_j \geq 2,\\
& A_0 \geq B_j \qquad \text{for } j \in C_{Leaf} - \{j_0\},\\
\frac{B_j}{D_j} &= \mu(C(j,\beta(j)) - \{\beta(j)\}) > 1 \qquad \text{for } j \in C_{Leaf} - \{j_0\}.
\end{aligned}$$

For $j = j_0$ the map β was chosen with $\beta(j_0) \in C_1$, so $C(\beta(j_0)) = \emptyset$, $\widehat{\rho}(C(\beta_0)) = 1$, and

$$\frac{B_{j_0}}{D_{j_0}} = \frac{\prod_{j\in C_1} a_j}{\widehat{\rho}(\widehat{C}(\beta(j_0))) + (-1)^{|C_1|+1}\widehat{\rho}(C(\beta(j_0)))} \begin{cases} = 1 \text{ if } |C_1| = 1\\ > 1 \text{ if } |C_1| > 1\end{cases}$$

And

$$\begin{aligned}
A_0 &= \Big(B_{j_0} - (-1)^{|C_1|}\Big)\,\widehat{\rho}(C(j_0)) + (-1)^{|C(j_0)|+1}\widehat{\rho}(\widehat{C}(\beta(j_0) - \{\beta(j_0)\})),\\
1 &\leq \widehat{\rho}(\widehat{C}(\beta(j_0)) - \{\beta(j_0)\}) < B_{j_0},\\
1 &\leq \widehat{\rho}(C(j_0)) \qquad \text{and}\\
3 &\leq \widehat{\rho}(C(j_0)) \text{ if } |C(j_0)| \geq 2,
\end{aligned}$$

so always

$$A_0 \geq B_{j_0}.$$

Summarizing, we obtain

$$\begin{aligned}
& A_j < \mu \text{ for } j \neq 0, \quad B_j \leq A_0, \quad A_0 \leq \mu, \qquad (6.7)\\
A_0 = \mu \iff & C_{Fermat} \cup (cycles) = \emptyset, \quad C_{Leaf} = \{j_0\}, \quad |C_1| = 1\\
\iff & \kappa \text{ is of chain type with the chain } \widehat{C}(j_0). \qquad (6.8)
\end{aligned}$$

μ is a prime number by assumption. It must divide one of the factors A_j or B_j in (6.6). Because of (6.7) this forces $A_0 = \mu$. Because of (6.8) κ is of chain type with the chain $\widehat{C}(j_0)$.

Step 3. After renumbering of the vertices of its graph, $\kappa : N \to N$ is the map with $\kappa(1) = 1$, $\kappa(i) = i-1$ for $2 \leq i \leq n$. Then f contains the monomials $x_1^{a_1+1}, x_2^{a_2}x_1, ..., x_n^{a_n}x_{n-1}$. The Milnor number is

$$\mu = A_0 = \widehat{\rho}(\widehat{C}(n)) + \widehat{\rho}(C(n)) = \rho(a_n, ..., a_2, a_1 + 1).$$

The weights w_i and the numbers $s_i, t_i \in \mathbb{N}$ with $gcd(s_i, t_i) = 1$, $w_i = \frac{s_i}{t_i}$ are determined recursively by $w_1 = \frac{1}{a_1+1}$, $s_1 = 1$, $t_1 = a_1 + 1$,

$$\begin{aligned} \frac{s_{i+1}}{t_{i+1}} &= w_{i+1} = \frac{1-w_i}{a_{i+1}} = \frac{t_i - s_i}{t_i \cdot a_{i+1}}, \\ s_{i+1} &= \frac{t_i - s_i}{\gamma_i},\ t_{i+1} = \beta_i t_i, \\ \text{where} \quad & a_{i+1} = \beta_i \gamma_i,\ \gamma_i = \gcd(a_{i+1}, t_i - s_i). \end{aligned}$$

Thus

$$\mu = \prod_{i=1}^{n} \left(\frac{1}{w_i} - 1 \right) = \prod_{i=1}^{n} \frac{t_i - s_i}{s_i} = \gamma_1 \cdot ... \cdot \gamma_{n-1} \cdot (t_n - s_n).$$

μ being a prime number forces $\gamma_i = 1$, $\beta_i = a_{i+1}$, $s_{i+1} = t_i - s_i$ and

$$\begin{aligned} t_i &= a_i t_{i-1} = a_i ... a_2 \cdot (a_1 + 1), \\ s_i &= \rho(a_{i-1}, ..., a_2, a_1 + 1) = t_{i-1} - t_{i-2} + ... + (-1)^{i-1}. \end{aligned}$$

Finally we show that the only monomials of weighted degree d are $x_1^{a_1+1}, x_2^{a_2} x_1, ..., x_n^{a_n} x_{n-1}$. Then f is as claimed in (a). Let $\sum_{i=1}^n \delta_i e_i \in (\mathbb{N}_0^n)_d$. Let j be maximal with $\delta_j > 0$. Then

$$\delta_j \cdot \frac{s_j}{t_j} = 1 - \sum_{i<j} \delta_i \cdot \frac{s_i}{t_i}.$$

The denominator of the rational number on the right hand side is a divisor of t_{j-1}, and $t_j = a_j t_{j-1}$. Therefore $\delta_j = a_j \varepsilon$ for some $\varepsilon \in \mathbb{N}$. But

$$a_j w_j + w_{j-1} = 1, \quad \text{so } 2a_j w_j > 1, \quad \text{so } \varepsilon = 1, \quad \text{so } \sum_{i<j} \delta_i w_i = w_{j-1}.$$

Then $\delta_{j-1} = 1$, $\delta_i = 0$ for $i < j-1$, so $\sum_i \delta_i e_i = a_j e_j + e_{j-1}$.

Step 4. Following [MO], we define the divisor $\operatorname{div} p(t)$ of a unitary polynomial $p(t) = \prod_{i=1}^k (t - \lambda_i)$ with zeros $\lambda_i \in S^1$ as the element

$$\operatorname{div} p(t) := \sum_{i=1}^{k} \langle \lambda_j \rangle \in \mathbb{Q}[S^1]$$

in the group ring $\mathbb{Q}(S^1)$. Denote $\Lambda_k := \operatorname{div}(t^k - 1)$. Then $1 = \Lambda_1$ is a unit element and $\Lambda_a \cdot \Lambda_b = \gcd(a, b) \cdot \Lambda_{lcm(a,b)}$.

By [MO, Theorem 4] the divisor of the characteristic polynomial $\Delta(t)$ of the monodromy of f is

$$\operatorname{div} \Delta(t) = \prod_{i=1}^{n} \left(\frac{1}{s_i} \Lambda_{t_i} - 1 \right).$$

Using $s_{i+1} = t_i - t_{i-1} + ... + (-1)^i$ and $\Lambda_{t_i} \cdot \Lambda_{t_j} = t_i \cdot \Lambda_{t_j}$ for $i \leq j$, we calculate

$$\begin{aligned} \operatorname{div} \Delta(t) &= (\Lambda_{t_1} - 1) \left(\frac{1}{s_2} \Lambda_{t_2} - 1 \right) \cdot ... \\ &= \left(\frac{t_1 - 1}{s_2} \Lambda_{t_2} - \Lambda_{t_1} + 1 \right) \cdot ... = (\Lambda_2 - \Lambda_1 + 1) \cdot ... \\ &= ... = \Lambda_{t_n} - \Lambda_{t_{n-1}} + ... + (-1)^{n-1} \Lambda_{t_1} + (-1)^n. \end{aligned}$$

This shows part (c) of theorem 6.1 $\square$

For a fixed $n \in \mathbb{N}$ a natural number $\mu > 2^n$ is called an *n-gap* if there does not exist a quasihomogeneous polynomial $f \in \mathbf{m}^3_{\mathbb{C}^n,0}$ with an isolated singularity at 0 and Milnor number μ.

Corollary 6.3. *For $n \geq 3$ the set of n-gaps contains the set*

$$\{2p + (-1)^n \,|\, p \text{ and } 2p + (-1)^n \text{ are prime numbers}, 2p + (-1)^n > 2^n\}.$$

Proof: Consider a $p \in \mathbb{N}$ such that $\mu = 2p + (-1)^n$ is bigger than 2^n and is a prime number, but not an n-gap. Then by theorem 6.1 there exist $a_1, ..., a_n \in \mathbb{N} - \{1\}$ with

$$\begin{aligned} 2p + (-1)^n &= \rho(a_n, ..., a_2, a_1 + 1) \\ \text{thus} \quad 2p &= (a_1 + 1)(\rho(a_n, ..., a_2) + (-1)^{n-1}). \end{aligned}$$

But $a_1 + 1 \geq 3$ and $\rho(a_n, ..., a_2) + (-1)^{n-1} \geq 3$ if $n \geq 3$, thus p cannot be a prime number. $\square$

Remarks 6.4. (i) [Ri] A natural number p such that p and $2p + 1$ are prime numbers is called a *Sophie Germain prime number*. There are conjectures of Dickson (1904) (and a generalization called *hypothesis H* of Schinzel (1956)) and of Hardy and Littlewood (1923) which would imply that the set of Sophie Germain prime numbers as well as the set $\{p \,|\, p \text{ and } 2p - 1 \text{ are prime numbers}\}$ are infinite. But the infinity of both sets seems to be unknown.

(ii) It is also interesting to ask how many other n-gaps exist for $n \geq 3$. There are 20 3-gaps with $8 < \mu \leq 1000$, 19 of them are of the type $2p - 1$ with p and $2p - 1$ being prime numbers, 9 is the only other gap. There are 21 4-gaps with $16 < \mu \leq 500$, 14 of them are of the type $2p + 1$ with p a Sophie Germain prime number, the other ones are 17, 18, 19, 27, 74, 219, 314.

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Journal *of* Singularities
Volume 4 (2012), 154-170

received 25 July 2012
in revised form 14 October 2012
DOI: 10.5427/jsing.2012.4i

PICARD GROUPS OF NORMAL SURFACES

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Abstract. We study the fixed singularities imposed on members of a linear system of surfaces in $\mathbb{P}^3_{\mathbb{C}}$ by its base locus Z. For a 1-dimensional subscheme $Z \subset \mathbb{P}^3$ with finitely many points p_i of embedding dimension three and $d \gg 0$, we determine the nature of the singularities $p_i \in S$ for general $S \in |H^0(\mathbb{P}^3, I_Z(d))|$ and give a method to compute the kernel of the restriction map $\operatorname{Cl} S \to \operatorname{Cl} \mathcal{O}_{S,p_i}$. One tool developed is an algorithm to identify the type of an $\mathbf{A}_n$ singularity via its local equation. We illustrate the method for representative Z and use Noether-Lefschetz theory to compute $\operatorname{Pic} S$.

1. Introduction

The problem of computing Picard groups of surfaces $S \subset \mathbb{P}^3_{\mathbb{C}}$ has a long history. The solution for smooth quadric and cubic surfaces was known in the 1800s in terms of lines on these surfaces. In the 1880s Noether suggested what happens in higher degree, but it was not until the 1920s that Lefschetz proved the famous theorem bearing their names: the very general surface S of degree $d > 3$ has Picard group $\operatorname{Pic} S \cong \mathbb{Z}$, generated by the hyperplane section H. Here *very general* refers to a countable intersection of nonempty Zariski open subsets. To produce typical families of surfaces S with $\operatorname{Pic} S$ not generated by H, Lopez proved that very general surfaces S of high degree containing a smooth connected curve Z have Picard group freely generated by H and Z [15, II, Thm. 3.1], a geometrically pleasing result with many applications [4, 5, 6, 7].

Recently we extended these results, proving that the class group $\operatorname{Cl} S$ of the very general surface S containing an arbitrary 1-dimensional subscheme Z with at most finitely many points of embedding dimension three [1] is freely generated by H and the supports of the irreducible curve components of Z [2, Thm. 1.1]. This allows access to the Picard group via the exact sequence of Jaffe [11, Prop. 3.2] (see also [9, Prop. 2.15])

$$0 \to \operatorname{Pic} S \to \operatorname{Cl} S \to \bigoplus_{p \in \operatorname{Sing} S} \operatorname{Cl} \mathcal{O}_{S,p} \tag{1}$$

provided we can find the kernels of the restriction maps $\operatorname{Cl} S \to \operatorname{Cl} \mathcal{O}_{S,p}$ at the singular points $p \in S$, where $\operatorname{Cl} \mathcal{O}_{S,p}$ is the divisor class group of the local ring. The answer being known at singular points of S where Z has embedding dimension ≤ 2 [2, Prop. 2.2], our motivating question becomes:

Problem 1.1. For $Z \subset \mathbb{P}^3$ and $p \in Z$ a point of embedding dimension three, find the kernel of the restriction map $\operatorname{Cl} S \to \operatorname{Cl} \mathcal{O}_{S,p}$.

A general solution to Problem 1.1 is out of reach because one would need to classify all embedding dimension three points p on curves Z to state an answer. Instead we give a method of attack on the problem:

Method 1.2. The kernel of the restriction $\operatorname{Cl} S \to \operatorname{Cl} \mathcal{O}_{S,p}$ can be computed as follows.

2000 *Mathematics Subject Classification.* Primary: 14B07, 14H10, 14H50.

[1]This is the weakest condition allowing S to be a normal surface, so that $\operatorname{Cl} S$ is defined.

Step 1. The natural map $\operatorname{Cl}\mathcal{O}_{S,p} \to \operatorname{Cl}\widehat{\mathcal{O}}_{S,p}$ being injective, we consider the composite map $\operatorname{Cl} S \to \operatorname{Cl}\mathcal{O}_{S,p} \hookrightarrow \operatorname{Cl}\widehat{\mathcal{O}}_{S,p}$, where power series tools are available.

Step 2. Working in $\widehat{\mathcal{O}}_{S,p}$, use analytic coordinate changes to recognize the form of the singularity and compute the local class group $\operatorname{Cl}\widehat{\mathcal{O}}_{S,p}$ when possible.

Step 3. Since $\operatorname{Cl} S$ is freely generated by H and the supports of the curve components of Z [2, Thm 1.1], it is enough to find the images of those supports for irreducible curve components of Z passing through p in $\operatorname{Cl}\widehat{\mathcal{O}}_{S,p}$ (the rest map to zero).

Remark 1.3. Method 1.2 can *always* be carried out if Z is locally contained in two smooth surfaces meeting transversely at p. This is because the analytic local equation of S at p contains an xy term and we can employ our *recognition theorem*: Theorem 2.6 gives an inductive algorithm that recognizes the type of an $\mathbf{A}_n$ singularity in at most n steps, but finishes in just 1 step with probability 1. While most of this paper is devoted to illustrations of Method 1.2, Theorem 2.6 may be the most useful general result presented here.

Remark 1.4. Regarding Step 3, the images of the supports of the curve components of Z containing p generate $\operatorname{Cl} O_{S,p}$ as a subgroup of $\operatorname{Cl}\widehat{O}_{S,p}$ for very general S [3, Prop. 2.3]. This gives a geometric way to see the class group of a ring in its completion.

(a) In particular, the map $\operatorname{Cl} S \to \operatorname{Cl}\mathcal{O}_{S,p}$ is zero if p is an isolated point of Z, since Z has no curve components passing through p. Therefore only the 1-dimensional part of Z contributes to the answer.

(b) Srinivas has asked [20, Ques. 3.1] which subgroups appear as $\operatorname{Cl} B \subset \operatorname{Cl} A$ where B is a local $\mathbb{C}$-algebra with $A = \widehat{B}$. We proved that for complete local rings A corresponding to the rational double points $\mathbf{A}_n, \mathbf{D}_n, \mathbf{E}_6, \mathbf{E}_7, \mathbf{E}_8$, the answer is *every* subgroup [3, Thm. 1.3]. We constructed the rings B as the geometric local rings $\mathcal{O}_{S,p}$ arising from general surfaces $S \subset \mathbb{P}^3$ containing a fixed base locus forcing the singularity at p. This poses a stark contrast to results of Kumar [13], who showed that if B has fraction field $\mathbb{C}(x, y)$ and the singularity type is $\mathbf{E}_6, \mathbf{E}_7$ or $\mathbf{A}_n$ with $n \neq 7, 8$, then B is determined by A and hence $\operatorname{Cl} B = \operatorname{Cl} A$.

We illustrate Method 1.2 by giving complete answers for the following base loci Z:

(1) Unions of two multiplicity structures near p which are locally contained in smooth surfaces with distinct tangent spaces at p.

(2) Multiplicity structures on a smooth curve of multiplicity ≤ 4 near p.

Regarding organization, we review $\mathbf{A}_n$ singularities and their analytic equations in Section 2, proving the recognition theorem, Theorem 2.6. In Sections 3 - 4 we solve Problem 1.1 in the cases (1) and (2) listed above. Finally in Section 5 we prove Theorem 5.1, which shows how to compute $\operatorname{Pic} S$ and give examples.

2. Analytic expressions for rational double points

In this section we briefly review rational double points of type $\mathbf{A}_n$ and some results about analytic change of coordinates.

2.1. **$\mathbf{A}_n$ singularities.** An $\mathbf{A}_n$ surface singularity has local analytic equation $xy - z^{n+1}$, thus it is analytically isomorphic to $\operatorname{Spec}(R)$ with $R = k[[x, y, z]]/(xy - z^{n+1})$. The resolution of this singularity is well known [10, 5.2]: an $\mathbf{A}_1$ resolves in a single blow-up with one rational exceptional curve having self-intersection -2; an $\mathbf{A}_2$ resolves in one blow-up but with two (-2)-curves meeting at a point. For $n \geq 3$, blowing up with new variables $x_1 = x/z, y_1 = y/z$ gives two exceptional curves, namely E_{x_1} defined by (x_1, z) and E_{y_1} defined by (y_1, z), meeting transversely at an $\mathbf{A}_{n-2}$ at the origin. Blowing up and continuing inductively, the singularity unfolds and we obtain a resolution with exceptional divisors forming a chain of n rational (-2)-curves meeting pairwise transversely. We will adopt the convention, identifying a curve with its strict transform, that $E_1 = E_{x_1}, E_2 = E_{x_2}, \dots, E_n = E_{y_1}$.

To calculate $\mathrm{Cl}\, R$, identify a curve C with the sequence $(\tilde{C}.E_1, \tilde{C}.E_2 \dots)$ of intersection numbers of its strict transform with the exceptional curves. Then $\mathrm{Cl}\, R$ is the quotient of the free abelian group on the exceptional curves with relations given by the fact that the exceptional curves themselves correspond to the trivial class [14, §14 and §17]. Let (u_j) be the ordered basis for the free group; then the relations for an $\mathbf{A}_n$ singularity are

$$-2u_1 + u_2, u_1 - 2u_2 + u_3, \dots, u_{n-1} - 2u_n$$

so that $\mathrm{Cl}\, R \cong \mathbb{Z}/(n+1)\mathbb{Z}$ generated by u_1 and satisfying $u_j = ju_1$ for all j.

Example 2.1. Let $R = k[[x, y, z]]/(xy - z^{n+1})$ be the complete local ring of an $\mathbf{A}_n$ surface singularity. Under the identification $\mathrm{Cl}(R) \cong \mathbb{Z}/(n+1)\mathbb{Z}$, we identify the following classes:

(a) The class of the curve D_1 given by (x, z) is 1.
(b) The class of the curve D_2 given by (y, z) is -1.
(c) For $1 \le r \le n$, the class of the curve $(x - z^{n-r+1}, y - z^r)$ is r.

Parts (a) and (b) are contained in [10, Prop. 5.2] and part (c) is [10, Rem. 5.2.1], where the class considered is $(x - az^{n-r+1}, y - a^{-1}z^r)$ for a unit a. To save a change of coordinates at the end of a calculation, we will often apply part (c) to the curve $(x - uz^{n-r+1}, y - vz^r)$ in the ring $k[[x, y, z]]/(xy - uvz^{n+1})$ with u, v units.

2.2. **Analytic coordinate changes.** We will use coordinate changes in $k[[x, y, z]]$ to recognize the structure of surface singularities. Let $R = k[[x_1, \dots, x_n]]$ be the ring of formal powers series over a field k with maximal ideal $\mathfrak{m}$. A *change of variables* for R is an assignment $x_i \mapsto x_i' \in \mathfrak{m}$ inducing an automorphism of R. An assignment $x_i \mapsto x_i'$ induces an automorphism if and only if induced maps $\mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{m}/\mathfrak{m}^2$ is an isomorphism if and only if the matrix A of coefficients of linear terms in the x_i' is nonsingular (this is noted by Jaffe [12, Prop. 3.2] when $n = 2$). Examples include multiplication of variables by units and translations of variables by elements in $\mathfrak{m}^2$.

The following lemma allows us to take roots in power series rings.

Lemma 2.2. *Let $(R, \mathfrak{m})$ be a complete local domain, and let n be a positive integer that is a unit in R. If $a_0 \in R$ is a unit and $u \equiv a_0^n \bmod \mathfrak{m}^k$ for some fixed $k > 0$, then there exists $a \in R$ such that $a^n = u$ and $a \equiv a_0 \bmod \mathfrak{m}^k$.*

Proof. Using Hensel's method, we construct a sequence $\{a_i\}$ with $a_i \equiv a_{i+1} \bmod \mathfrak{m}^{(i+1)k}$ and $a_i^n \equiv u \bmod \mathfrak{m}^{(i+1)k}$. Write $u^{(i)} = u - a_i^n \in \mathfrak{m}^{(i+1)k}$ and let $a_{i+1} = a_i + \frac{u^{(i)}}{na_i^{n-1}}$; then we have $a_{i+1} \equiv a_i \bmod \mathrm{m}^{(i+1)k}$ and $a_{i+1}^n = a_i^n + u^{(i)} + \tilde{u}, \tilde{u} \in \mathfrak{m}^{2(i+1)k} \Rightarrow a_{i+1}^n \equiv u \bmod \mathfrak{m}^{(i+2)k}$. □

Lemma 2.3. *Let $R = k[[x, y]]$ with maximal ideal $\mathfrak{m} \subset R$. For $f \in \mathfrak{m}^3$, there is a change of coordinates X, Y such that*

$$xy + f = XY$$

and X, Y may be chosen so that $x \equiv X \bmod \mathfrak{m}^2$ and $y \equiv Y \bmod \mathfrak{m}^2$.

Proof. (*Cf.* [8, I, Ex. 5.6.3]) Since $f \in \mathfrak{m}^3$, we may write $f = xh_1 + yg_1$ with $h_1, g_1 \in \mathfrak{m}^2$. Now write $x_1 = x + g_1, y_1 = y + h_1$ so that $xy + f = x_1y_1 - g_1h_1$, where $g_1h_1 \in \mathfrak{m}^4$. Continue the process inductively, constructing a sequence of coordinate changes that converge to X, Y and for which $f = XY$. □

We frequently encounter the complete local rings considered in the following Proposition.

Proposition 2.4. *Let $R = k[[x,y,z]]/(xy - uyz^t - vx^n)$, where $u, v \in k[[x,y,z]]$ are units and $t \geq 1, n \geq 3$ are integers. Then* $\operatorname{Spec} R$ *is an $\mathbf{A}_{tn-1}$ singularity and the class of the curve (x,z) (resp. (x,y)) maps to 1 (resp. $tn - t$) via the isomorphism* $\operatorname{Cl} R \cong \mathbb{Z}/tn\mathbb{Z}$.

Proof. In setting $X = x - uz^t$, the expression $xy - uyz^t - vx^n$ becomes

$$Xy - v(X + uz^t)^n = Xy - vX\underbrace{(X^{n-1} + nX^{n-2}uz^t + \cdots + nu^{n-1}z^{t\cdot(n-1)})}_{\alpha} - vu^n z^{tn}. \tag{2}$$

For α as shown, set $Y = y - v\alpha$ to obtain $XY - vu^n z^{tn}$. Absorbing vu^n into either X or Y brings the expression to the standard form for an $\mathbf{A}_{tn-1}$ singularity, hence we have $\operatorname{Cl} R \cong \mathbb{Z}/tn\mathbb{Z}$ from the previous section. Moreover $(x,z) = (X + uz^t, z) = (X, z)$ gives the canonical generator 1 in $\operatorname{Cl} R = \mathbb{Z}/tn\mathbb{Z}$ by Example 2.1 (a).

For the second curve, write $(x,y) = (X + uz^t, y) = (X + uz^t, Y + v\alpha)$. Then

$$(X + uz^t)^n = X\alpha + (uz^t)^n = (X + uz^t)\alpha - uz^t\alpha + (uz^t)^n$$

by definition of α so that $(uz^t)\alpha \equiv (uz^t)^n \bmod (X + uz^t)$. Since the quotient ring modulo $X + uz^t$ is an integral domain in which uz^t is nonzero, we see that $\alpha \equiv (uz^t)^{n-1} \bmod (X + uz^t)$ and therefore $(x,y) = (X + uz^t, Y + vu^{n-1}z^{tn-t})$ which corresponds to the class $tn - t \in \mathbb{Z}/tn\mathbb{Z}$ by Example 2.1 (c).

□

2.3. **Recognizing $\mathbf{A}_n$ Singularities.** We develop an algorithm to identify the $\mathbf{A}_n$-singularity type defined by a power series F in three variables defining a double point with nondegenerate tangent cone, so that the degree-2 part is not a square. In this case it is known [10, Thm. 4.5] that F defines an $\mathbf{A}_n$ singularity for some n or F factors, which we interpret as $n = \infty$. Our goal is to identify the answer by inspection if possible. Such F can be written

$$F = \sum_{i+j+k>1} c_{i,j,k} x^i y^j z^k \in \mathfrak{m}^2 \subset k[[x,y,z]]$$

with $c_{1,1,0} = 1, c_{2,0,0} = c_{0,2,0} = 0$.

Let A be the sum of all terms satisfying $i, j > 0$ and $i + j + k > 2$. Then $A = xyB$ with $B \in \mathfrak{m}$ and the remaining terms of F fall into three categories: (a) $i = j = 0$, which we write as $h(z) \in k[[z]]$, (b) $i = 0, j > 0$, which we can write as $\sum_{j=1} y^j g_j(z)$ with $g_j \in k[[z]]$ and (c) $j = 0, i > 0$ which can be written as $\sum_{i=1} x^i f_i(z)$ with $f_i \in k[[z]]$. With these choices F becomes

$$F = xy + h(z) + \sum_{i=1}^{\infty} x^i f_i + \sum_{j=1}^{\infty} y^j g_j + xyB \tag{3}$$

where $h, f_i, g_j \in k[[z]]$, $B \in \mathfrak{m}$ and $\operatorname{ord} g_2 > 0$. Letting $u = 1 + B$ we can drop the last term at the expense of multiplying the xy term by the unit u: now let $X = ux$ and replace f_i with $(u^{-1})^i f_i$ to obtain

$$F = xy + h(z) + \sum_{i=1}^{\infty} x^i f_i + \sum_{j=1}^{\infty} y^j g_j \tag{4}$$

with $g_j \in k[[z]]$ and $f_i(z) = z^{r_i} u_i$ with u_i a unit. To determine the singularity type, we may assume $r_1 < \infty$ or $\operatorname{ord} g_1 < \infty$, since $f_1 = g_1 = 0$ gives an $\mathbf{A}_{h-1}$ with $h = \operatorname{ord} h(z)$. We make one more simplification. Set $X = x + g_1$ to obtain

$$F = Xy + h(z) + \sum_{i=1}^{\infty} (X - g_1)^i f_i + \sum_{j=2}^{\infty} y^j g_j.$$

Regrouping the f_i by powers of X after expanding the powers of $(X - g_1)$ we arrive at

$$F = xy + h(z) + \sum_{i=1}^{\infty} x^i f_i + \sum_{j=2}^{\infty} y^j g_j. \tag{5}$$

where the f_i, g_j are equal to a power of z times a unit, $0 < \operatorname{ord} f_1 < \infty$ and $0 < \operatorname{ord} g_2$. Since we only make variable changes which fix z, the crux of the matter is to understand the case when $h = 0$.

Lemma 2.5. *Consider the power series*

$$F = xy + \sum_{i=1}^{\infty} x^i f_i + \sum_{j=2}^{\infty} y^j g_j \tag{6}$$

where f_i, g_j are powers of z up to units, (a) $0 < \operatorname{ord} f_1 < \infty$ and (b) $\operatorname{ord} f_2 > 0$ or $\operatorname{ord} g_2 > 0$. Set $m = \min\{\operatorname{ord} f_1^j g_j\}$ and write $\sum_{j=2}^{\infty} (-1)^j f_1^j g_j = z^m \cdot \delta$. Then the change of variables $X = x, Y = y + f_1$ yields

$$F = z^m \cdot \delta + XY + \sum_{i=2}^{\infty} X^i F_i + \sum_{j=1}^{\infty} Y^j G_j \tag{7}$$

where F_i, G_j are powers of z up to units such that

(a) $0 < \operatorname{ord} G_1 < \infty$*;*
(b) $\operatorname{ord} F_2 > 0$ *or* $\operatorname{ord} G_2 > 0$*;*
(c) $M = \min\{\operatorname{ord} G_1^i F_i\} > m$*; and*
(d) $\operatorname{ord} G_1 \geq m - \operatorname{ord} f_1$.

Proof. Setting $Y = y + f_1$ we have

$$F = xY + \sum_{i=2}^{\infty} x^i f_i + \sum_{j=2}^{\infty} (Y - f_1)^j g_j.$$

The part of the last sum with degree 0 in Y is $\sum_{j=2}^{\infty} (-1)^j f_1^j g_j = z^m \cdot \delta$ by definition of δ.

We take $F_i = f_i$ for $i \geq 2$ and calculate G_j by gathering terms with like powers of Y:

$$G_1 = -2f_1 g_2 + 3f_1^2 g_3 - \cdots = \sum_{k=2}^{\infty} (-1)^{k-1} k f_1^{k-1} g_k \tag{8}$$

and for $j \geq 2$,

$$G_j = \sum_{k=j}^{\infty} (-1)^{k-j} \binom{k}{j} f_1^{k-j} g_k.$$

Thus F takes the form of equation (7) and it remains to show that $M = \min\{\operatorname{ord} G_1^i F_i\} > m$. When expanded, each term in $G_1^i F_i = G_1^i f_i$ has the form

$$c f_i f_1^{k_1 + k_2 + \cdots + k_i - i} g_{k_1} g_{k_2} \cdots g_{k_i}$$

where c is a constant and the $k_\ell \geq 2$ are not necessarily distinct. The order of this term is strictly greater than $\operatorname{ord} f_1^{k_1} g_{k_1} \geq m$, unless $i = 2, k_1 = k_2 = 2$. In the case $i = 2, k_1 = k_2 = 2$ we would like to see that $\operatorname{ord} f_2 f_1^2 g_2^2 > \operatorname{ord} f_1^2 g_2$, but this follows from the condition that $\operatorname{ord} f_2 > 0$ or $\operatorname{ord} g_2 > 0$. Thus $\operatorname{ord} G_1^i f_i > m$ for all i and $M > m$.

For (d), the order of the k^{th} term in sum (8) is $(k-1) \operatorname{ord} f_1 + \operatorname{ord} g_k \geq m - \operatorname{ord} f_1$. □

Theorem 2.6. *For F as in equation (5), let $m = \min\{j r_1 + \operatorname{ord} g_j\}$ and set $\mu = \min\{\operatorname{ord} h, m\}$. Let $\delta(\mu)$ be the coefficient of z^μ in $H = h + \sum_{j=2}^{\infty} (-1)^j f_1^j g_j$. Then*

(a) *F defines an $\mathbf{A}_n$ singularity with $n \geq \mu - 1$ ($n = \infty$ is possible).*
(b) *F defines an $\mathbf{A}_{\mu-1}$ singularity if $\delta(\mu) \neq 0$.*

Proof. Apply the lemma to $F - h(z)$ and then add $h(z)$ back in to obtain the form

$$F = H + \sum_{i=2}^{\infty} X^i F_i + \sum_{j=1}^{\infty} Y^j G_j;$$

then relabel and repeat. By Lemma 2.5 (c), m is strictly increasing. Note that after each change of variables $Y = y + f_1$(or $X = x + g_1$), the new variables (x, Y, z) still form a regular system of parameters at the origin. Now, consider two iterations of the algorithm; start with x, y and f_i, g_j and m-value m; then change to x, Y with f_i, G_j and m-value $M > m$, and next to X, Y with F_i, G_j. Then

$$\operatorname{ord} F_1 \geq M - \operatorname{ord} G_1 > m - (m - \operatorname{ord} f_1) = \operatorname{ord} f_1$$

by Lemma 2.5(d), so $\operatorname{ord} f_1$ increases with every change of x-variable; and similarly for $\operatorname{ord} g_1$. Thus the sequence of variable changes forms a Cauchy sequence and moreover in the limit the terms f_1 and g_1 both vanish. Therefore the expression becomes

$$XY + H(z) + \sum_{i=2}^{\infty} X^i F_i + \sum_{j=2}^{\infty} Y^j G_j,$$

and applying [10, Prop. 4.4] after subtracting $H(z)$ brings us to the form

$$F = XY + H(z).$$

If some $\delta(\mu) \neq 0$, H retains a term of order μ in every subsequent change of variables because each only involves terms of order $\geq m > \mu$, so μ stabilizes. Therefore in this case the form of F is $XY +$ unit $\cdot z^{\mu}$, an $\mathbf{A}_{\mu-1}$ singularity. Otherwise $\delta(\mu) = 0$ for every μ and $H \to 0$ as $\mu \to \infty$, so F factors. ☐

Remark 2.7. The inductive procedure given in Theorem 2.6 and Lemma 2.5 recognizes an $\mathbf{A}_n$ singularity in at most n steps. However condition (b) in Theorem 2.6 is an open condition among equations of fixed degree, so the algorithm terminates after only *one* step with probability 1.

Example 2.8. We illustrate the theorem with a few examples.

(a) Applying Theorem 2.6 to $F = xy + xz^2 + y^2z - z^6$, we have $m = 5, \mu = \min\{5, 6\} = 5$ and $\delta(5) = 1 \neq 0$, so F represents an $\mathbf{A}_4$ singularity. The variable change $Y = y + z^2$ gives $F = xY + (Y - z^2)^2 z - z^6 = xY + Y^2 z - 2Yz^3 + z^5 - z^6$ so that $H(z) = z^5 - z^6$ has order 5. After the sequence of variable changes suggested, the z^5 term survives while the terms involving x, Y eventually factor.

(b) For the singularity given by

$$F = xy + xz^4 + y^2 z^6 + y^3 z^2 + y^4 z^{25} + x^2 z$$

we have $m = \mu = 14$ and $\delta(14) = 0$, so we make the variable change $Y = y + z^4$ suggested by the theorem. Then we have

$$F = xY + (Y - z^4)^2 z^6 + (Y - z^4)^3 z^2 + (Y - z^4)^4 z^{25} + x^2 z$$

When multiplying this out, the z^{14} term drops out (because $\delta(14) = 0$), but that the new incarnation of F has linear Y-terms, namely

$$f = xY + Y(-2z^{10} + 3z^{10}) + \cdots + x^2 z = xY + Yz^{10} + \cdots + x^2 z$$

where the dots represent higher power of Y terms. Continuing with $X = x + z^{10}$ gives

$$f = XY + \cdots + X^2 z - 2Xz^{11} + z^{21}$$

and it becomes clear that we are dealing with an $\mathbf{A}_{20}$.

(c) Proposition 2.4 follows readily from Theorem 2.6 as (with x, y reversed) we have $\mu = mn$ and $\delta(mn) = u^n v \neq 0$, yielding an $\mathbf{A}_{mn-1}$ singularity.

3. Two multiple curves intersect at a point

In this section we give a solution to Problem 1.1 when $Z = Z_1 \cup Z_2$ is a union of two multiple curves of embedding dimension two with respective smooth supports C_1, C_2 meeting transversely at p under the condition that Z_1 and Z_2 do not share the same Zariski tangent space at p. In other words, we consider the following two cases:

(1) No Tangency: C_1 is not tangent to Z_2 and C_2 is not tangent to Z_1.
(2) Mixed Tangency: C_1 is tangent to Z_2 but C_2 is not tangent to Z_1.

For each of these we find canonical forms for the local ideals (Propositions 3.3 and 3.5) and determine the local Picard groups at the corresponding fixed singularity on the very general surface containing the curve (Propositions 3.4, 3.6, 3.7 and 3.8). The following local algebra lemma will facilitate computing the intersection of ideals $I_Z = I_{Z_1} \cap I_{Z_2}$.

Lemma 3.1. *Let R be a regular (local) ring. For $a, b, c, d \in R$, assume that a, c, d form a regular sequence and that $d \in (a, b)$. Then $(a, b) \cap (c, d) = (ac, bc, d)$.*

Proof. Write $d = as + br$ with $s, r \in R$. Since d is a non-zero divisor mod (a), the same is true of r, so that a, r and a, b also form regular sequences in R. Since $(a, d) = (a, br)$, the ideals (a, b) and (a, r) are linked by the complete intersection (a, d). It follows that $(a, b) \cap (c, d)$ is linked to (a, r) by the complete intersection $(ac, d) = (a, d) \cap (c, d)$. The inclusion of ideals $(ac, d) \subset (a, r)$ lifts to a map of the corresponding Koszul complexes

$$\begin{array}{ccccccc} 0 & \to & R & \xrightarrow{(-d,ac)} & R^2 & \xrightarrow{(ac,d)} & (ac,d) \\ & & \downarrow \alpha & & \downarrow \beta & & \downarrow \\ 0 & \to & R & \xrightarrow{(-r,a)} & R^2 & \xrightarrow{(a,r)} & (a,r) \end{array}$$

where $\beta(A, B) = (Ac + Bs, Bb)$ and $\alpha(C) = Cbc$. By the mapping cone construction for liaison [19, Prop. 2.6], the ideal $(a, b) \cap (c, d)$ is the image of $R^3 \to R$ given by the direct sum of $\alpha^\vee$ and $(-d, ac)^\vee$, so the ideal is $(bc, -d, ac) = (ac, bc, d)$. $\square$

Example 3.2. Lemma 3.1 fails if a, c, d do not form a regular sequence in R, for example $R = k[x, y, z], a = c = x, b = d = y$ when $(x, y) \cap (x, y) \neq (x^2, yx, y) = (x^2, y)$.

Proposition 3.3. *Let $Z = Z_1 \cup Z_2$ be the union of two multiplicity structures on smooth curves C_1, C_2 meeting at p with respective multiplicities $t \leq n$. Assume Z_i is contained in a local smooth surface $S_i, i = 1, 2$, C_1 is not tangent to S_2 and C_2 is not tangent to S_1. Then there are local coordinates x, y, z at p for which $I_{Z_1} = (x, z^t)$, $I_{Z_2} = (y, z^n)$ and*

$$I_Z = (xy, yz^t, z^n).$$

Proof. Locally we may assume that S_1 is given by equation $x = 0$ and S_2 is given by equation $y = 0$. Letting $z = 0$ be the equation of a smooth surface containing both C_1 and C_2 near p, the lack of tangency conditions imply that x, y, z is a regular system of parameters at p and we obtain $I_{C_1} = (x, z)$ and $I_{C_2} = (y, z)$. Given that $Z_i \subset S_i$ with the multiplicities given, it's clear that $I_{Z_1} = (x, z^t)$ and $I_{Z_2} = (y, z^n)$. Taking $a = x, b = z^t, c = y, d = z^n$, we have $d \in (a, b)$ because $t \leq n$, so the intersection ideal is (xy, yz^t, z^n) by Lemma 3.1. $\square$

Proposition 3.4. *For Z as in Proposition 3.3 above, the general surface S containing Z has an $\mathbf{A}_{n-1}$ singularity at p and C_1 (resp. C_2) maps to 1 (resp. -1) under the isomorphism $\operatorname{Cl} \widehat{\mathcal{O}}_{S,p} \cong \mathbb{Z}/n\mathbb{Z}$.*

Proof. The general element of I_Z has the form $xy + byz^t + cz^n$ with units $b, c \in \mathcal{O}_{\mathbb{P}^3,p}$. In the language of Theorem 2.6, $\mu = n$ and $\delta(n) = c \neq 0$, so the singularity is of type $\mathbf{A}_{n-1}$. The first change of variables $X = x + bz^t$ is the only one necessary, giving us immediately (up to units) the form $XY - z^n$; furthermore, the ideal defining C_1 is is $(x, z) = (X - cz^n, z) = (X, z)$ and $I_{C_2} = (y, z) = (Y, z)$, so these curves correspond to the canonical generators ± 1 for the group $\operatorname{Cl} \widehat{\mathcal{O}}_{S,p} \cong \mathbb{Z}/n\mathbb{Z}$ by Examples 2.1 (a) and (b). □

The mixed tangency case is more complicated.

Proposition 3.5. *Let C_1, C_2 be smooth curves meeting transversely at p, $C_i \subset S_i$ local smooth surfaces, and $Z_1 = tC_1 \subset S_1, Z_2 = nC_2 \subset S_2$ multiplicity structures. Assume that C_1 meets S_2 transversely and C_2 is tangent to S_1 of order $q > 1$. Then there are local coordinates x, y, z at p for which*

$$I_{Z_1} = (x - z^q, z^t), \quad I_{Z_2} = (y, x^n)$$

and the intersection ideal $I_{Z_1 \cup Z_2} = I_{Z_1} \cap I_{Z_2}$ takes the form:

(a) If $t \leq q$, then $I_Z = (xy, yz^t, x^n)$.
(b) If $q < t < qn$, then $I_Z = (y(x - z^q), yz^t, x^n)$.
(c) If $t \geq qn$, then $I_Z = (y(x - z^q), yz^t, x^n z^{t-qn})$.

Proof. Let $x = 0$ (resp. $y = 0$) be a local equation for S_1 (resp. S_2). Since C_1 meets S_2 transversely, we can extend x, y to a regular sequence x, y, z with $I_{C_1} = (x, z)$. Locally C_2 meets S_1 tangentially to order $q > 1$, so we may write $I_{C_2} = (x + \alpha, y)$ with $\alpha \in (x, y, z)^q$. Now $I_{C_2 \cap S_1} = (x, y, \alpha)$ defines a scheme of length q, so $\alpha = uz^q$ modulo (x, y) for some unit u; writing $\alpha = uz^q + xf + yg$ we have

$$I_{C_2} = (x + \alpha, y) = (x + uz^q + xf + yg, y) = (x(1 + f) + uz^q, y)$$

where $(1 + f)$ is a unit. Replacing x with $\dfrac{x(1+f)}{u} + z^q$ we have

$$I_{Z_1} = (x - z^q, z^t) \qquad I_{Z_2} = (y, x^n)$$

and it remains to find the intersection $I_Z = I_{Z_1} \cap I_{Z_2}$.

If $t \leq q$ (including the case $q = \infty \Rightarrow \alpha = 0 \Rightarrow C_2 \subset S_1$), then $I_{Z_1} = (x, z^t)$. Lemma 3.1 applies with $a = x, b = z^t, c = y$ and $d = x^n$, showing that $I_Z = (xy, yz^t, x^n)$.

If $q < t < qn$, then $z^{qn} = z^t \cdot z^{qn-t} \in I_{Z_1}$ and also $(x - z^q)|(x^n - z^{qn}) \Rightarrow x^n - z^{qn} \in I_{Z_1}$ so $x^n \in I_{Z_1}$. Application of Lemma 3.1 with $a = x - z^q, b = z^t, c = y, d = x^n$ produces the ideal $I_Z = (y(x - z^q), yz^t, x^n)$.

If $qn \leq t$, then we have the telescoping sum

$$z^t + z^{t-q}(x - z^q) + xz^{t-2q}(x - z^q) + \cdots + x^{n-1}z^{t-qn}(x - z^q) = x^n z^{t-qn} \in I_{Z_1}$$

and so we can again apply Lemma 3.1 with $a = x - z^q, b = z^t, c = y, d = x^n z^{t-qn}$ to obtain $I_Z = (y(x - z^q), yz^t, x^n z^{t-qn})$. □

Proposition 3.6. *For $Z = Z_1 \cup Z_2$ as in Proposition 3.5 (a) with $t \leq q$, the general surface S containing Z has a singularity of type $\mathbf{A}_{tn-1}$ at p and C_1 (resp. C_2) maps to 1 (resp. $tn - t$) under the isomorphism $\operatorname{Cl} \widehat{\mathcal{O}}_{S,p} \cong \mathbb{Z}/tn\mathbb{Z}$.*

Proof. In view of Prop. 3.5 (a), the general surface S containing $Z_1 \cup Z_2$ has local equation $xy - uyz^t - vx^n$ with u, v units in $\mathcal{O}_{\mathbb{P}^3,p}$ and x, y, z a regular sequence of parameters. Since C_1 is given by the ideal (x, z) and C_2 is given by (x, y), the result follows from Proposition 2.4. □

Proposition 3.7. *For $Z = Z_1 \cup Z_2$ as in Proposition 3.5 (b) with $q < t < qn$, the general surface S containing Z has a singularity of type $\mathbf{A}_{qn}$ at p and C_1 (resp. C_2) maps to 1 (resp. $qn-q$) under the isomorphism $\mathrm{Cl}\,\widehat{\mathcal{O}}_{S,p} \cong \mathbb{Z}/qn\mathbb{Z}$.*

Proof. The general surface containing Z has local equation $xy - yz^q - uyz^t - vx^n$ for units $u, v \in \mathcal{O}_{\mathbb{P}^3,p}$ and since $w = 1 + uz^{t-q}$ is a unit we may write the equation as $xy - wyz^q - vx^n$ and we can apply Proposition 2.4 to see the $\mathbf{A}_{qn-1}$ singularity and that C_1 with ideal (x,z) corresponds to $1 \in \mathbb{Z}/qn\mathbb{Z} \cong \mathrm{Cl}\,\widehat{\mathcal{O}}_{S,p}$ while C_2 with ideal (x,y) corresponds to $qn-q$. □

Proposition 3.8. *For $Z = Z_1 \cup Z_2$ as in Proposition 3.5 (c) with $qn \le t$, the very general surface S containing Z has a singularity of type A_{t-1} and C_1 (resp. C_2) maps to 1 (resp. $t-q$) under the isomorphism $\mathrm{Cl}\,\widehat{\mathcal{O}}_{S,p} \cong \mathbb{Z}/t\mathbb{Z}$.*

Proof. By Proposition 3.5 (c), the general surface S containing Z has local equation

$$xy - yz^q + uyz^t + vx^n z^{t-qn}$$

for units u, v in $\mathcal{O}_{\mathbb{P}^3,p}$. Using Theorem 2.6 with the roles of x and y reversed, we see that $\mu = qn + (t - qn) = t$; moreover, $\delta(\mu) = \pm v \neq 0$, so S has a type $\mathbf{A}_{t-1}$ singularity.

In order to determine the classes of C_1 and C_2 in $\mathrm{Cl}\,\widehat{\mathcal{O}}_{S,p} \cong \mathbb{Z}/t\mathbb{Z}$, however, it is expedient to calculate a coordinate change. Therefore set $X = x - z^q + uz^t = x - \beta z^q$ with β a unit in the local ring. With this change of coordinates the defining equation for S becomes

$$Xy + v(X + \beta z^q)^n z^{t-qn}.$$

Write this last expression as

$$X(\underbrace{y + X^{n-1}z^{t-qn} + n\beta X^{n-2}z^{t-q(n-1)} + \cdots + \beta^{n-1}nz^{t-q}}_{Y}) + \beta^n z^t.$$

The ideal of C_1 is $(x,z) = (X,z)$, corresponding to the generator 1 by Example 2.1(a). The ideal of C_2 is $(x,y) = (X + \beta z^q, y)$; evaluating the expression for Y at $X = -\beta z^q$ shows that this ideal is $(X + \beta z, Y - \beta^{n-1}z^{t-q})$, which corresponds to the element $t - q$ by Example 2.1(c). □

4. Multiple structures on a smooth curve

Let Z be a locally Cohen-Macaulay multiplicity structure supported on a smooth connected curve $C \subset \mathbb{P}^3$. The ideal sheaves $\mathcal{I}_Z + \mathcal{I}_C^i$ define one-dimensional subschemes of Z and after removing the embedded points we arrive at the Cohen-Macaulay filtration

$$C = Z_1 \subset Z_2 \subset \cdots \subset Z_m = Z \tag{9}$$

from work of Banica and Forster [1] (see also [16, §2]). If Z has generic embedding dimension two, then the quotient sheaves $\mathcal{I}_{Z_j}/\mathcal{I}_{Z_{j+1}} = L_j$ are line bundles on C and the multiplicity of Z is m (note that this use of the symbol m is different from that in Theorem 2.6 and thereabouts). In this section we solve Problem 1.1 for such a multiplicity structure Z at a point p of embedding dimension three assuming that Z_{m-1} or Z_{m-2} has local embedding dimension two at p. This class of curves contains all multiplicity structures on C with multiplicity $m \le 4$. First we describe the local ideal of Z at p.

Proposition 4.1. *Let $Z \subset \mathbb{P}^3_{\mathbb{C}}$ be the subscheme with ideal $\mathcal{I}_Z = (x^2, xy, xz^q - y^{m-1}, y^m)$ for $m \ge 3$. Then the very general surface S containing Z has an $\mathbf{A}_{(m-1)q-1}$ singularity at $p = (0,0,0,1)$ and $\mathrm{Picloc}\,p \cong \mathbb{Z}/(m-1)\mathbb{Z}$ is generated by C.*

Proof. For appropriate units $a, b, c \in \mathcal{O}_{\mathbb{P}^3,p}$ the general surface S containing Z has equation

$$xy + ax^2 + by^{m-1} + cy^m - bxz^q = x\underbrace{(y + ax)}_{y_1} + \underbrace{(b + cy)}_{u}y^{m-1} - bxz^q. \tag{10}$$

With $y_1 = y + ax$ and unit $u = b + cy$ as above, we may write

$$xy_1 + u(y_1 - ax)^{m-1} - bxz^q = x_1y_1 + c(y_1 + x_1)^{m-1} - dx_1z^q \tag{11}$$

for new units c, d after setting $x_1 = -ax$ and multiplying by $-1/a$. If $m = 3$, the first two terms are a homogeneous quadratic form: for general a, b, c, this factors into two linear terms and making the corresponding change of variable brings the equation to the form $XY + (AX + BY)z^q$ for units A, B.

For $m \geq 4$, expand the $(m-1)$st power in equation (11) as

$$c(y_1 + x_1)^{m-1} = y_1\underbrace{[c{y_1}^{m-2}]}_{g_1} + x_1\underbrace{[c((m-1){y_1}^{m-2} + \cdots + {x_1}^{m-2})]}_{h_1}$$

and set $x_2 = x_1 + g_1, y_2 = y_1 + h_1$, when equation (11) becomes

$$x_2y_2 - g_1h_1 - dxz^q \tag{12}$$

with $g_1h_1 \in (x, y)^{2m-4} \subset (x, y)^m$. Applying Lemma 2.3, we make another change of variables from x_2, y_2 to X, Y for which $x_2y_2 - g_1h_1 = XY$. Looking at the last term, one observes that $dx = d(x_2 - cy^{m-2}) = d(x_2 - c(y_2 - h_1)^{m-2})$ with $h_1 \in (x, y)^{m-2}$; extracting the multiples of x_2 in the resulting power series this may be written $A_1x_2 + B_1y_2^{m-2}$ with A_1, B_1 units. Switching to the variables X, Y and noting that $x_2 \equiv X \bmod (x, y)^{m-1}$ and $y_2 \equiv Y \bmod (x, y)^{m-1}$ by the proof of Lemma 2.3 ($g_1h_1 \in (x, y)^m$), this may be written $AX + BY^{m-2}$ with A, B units. Thus S has local equation

$$XY - (AX + BY^{m-2})z^q$$

with units A, B. Setting $Y_2 = Y - Az^q$, we obtain $XY_2 - B(Y_2 + Az^q)^{m-2}z^q$. Multiplying out the $(m-2)$nd power, the second term may be written $Y_2L + BA^{m-2}z^{(m-1)q}$ with

$$L = B(Y_2^{m-3} + A(m-2)Y_2^{m-4}z^q + \cdots + A^{m-3}(m-2)z^{(m-3)q})z^q.$$

Setting $X_2 = X - L$ gives the form

$$X_2Y_2 - A^{m-2}Bz^{(m-1)q},$$

showing that S has an $\mathbf{A}_{(m-1)q-1}$ singularity.

We now follow the ideal (x, y) through its coordinate changes:

$$(x, y) = (x_1, y_1) = (x_2, y_2) = (X, Y) = (X, Y_2 + Az^q) = (X_2 + L, Y_2 - Az^q).$$

Working modulo $Y_2 - Az^q$, replacing Y_2 with Az^q reduces L to

$$BA^{m-3}[1 + (m-2) + \binom{m-2}{2} \cdots + (m-2)]z^{(m-2)q} = (2^{m-2} - 1)BA^{m-3}z^{(m-2)q},$$

so the final form for our ideal is $(X_2 + (2^{m-2} - 1)BA^{m-3}z^{(m-2)q}, Y_2 - Az^q)$, which has class q in $\mathrm{Cl}\,\widehat{\mathcal{O}}_{S,p} \cong \mathbb{Z}/(m-1)\mathbb{Z}$ by Example 2.1 (c). $\square$

Proposition 4.2. *Let $C \subset \mathbb{P}^3$ be a smooth curve and Let Z be a locally Cohen-Macaulay multiplicity m structure on a C of generic embedding dimension two with filtration (9). Let $p \in Z$ be a point of embedding dimension three at which Z_{m-1} has embedding dimension two. Then*

(a) *There are local coordinates x, y, z for which Z has local ideal*

$$I_Z = (x^2, xy, xz^q - y^{m-1}).$$

(b) *The general surface S containing Z has an $\mathbf{A}_{(m-1)q-1}$ singularity at p and and the class of C maps to $q \in \operatorname{Cl} \widehat{\mathcal{O}}_{S,p} \cong \mathbb{Z}/q(m-1)\mathbb{Z}$.*

Proof. Consider the exact sequence $0 \to I_Z \to I_{Z_{m-1}} \xrightarrow{\pi} L_{m-1} \to 0$ near p. Since Z_{m-1} has embedding dimension two at p, it locally lies on a smooth surface with equation $x = 0$ and $I_C = (x, y)$ for suitable y, whence $I_{Z_{m-1}} = (x, y^{m-1})$. Now I_Z appears locally as the kernel of a surjection $\pi : (x, y^{m-1}) \to \mathcal{O}/(x, y)$. If $\pi(y^{m-1}) = u$ is a unit and $\pi(x) = \bar{h}$ with $h \in \mathcal{O}$, then $x - u^{-1}hy^{m-1} \in \operatorname{Ker}(\pi) = I_Z$, but then $x - u^{-1}hy^{m-1} \notin \mathfrak{m}_p^2$ implies Z has embedding dimension two at p, contrary to assumption. Therefore we may assume $\pi(y^{m-1}) = \bar{h}$ for $h \in \mathfrak{m}_p$ and $\pi(x) = 1$ in which case $I_Z = \operatorname{Ker}(\pi) = (x^2, xy, hx - y^{m-1})$. If $h \in \mathcal{O}/(x, y)$ vanishes to order $q > 0$ at p, we may write $h = uz^q \bmod (x, y)$ where u is a unit and z is a local parameter for C at p. Absorbing u into x gives part (a).

The local ideal of Z is $(x^2, xy, xz^q - y^{m-1}, y^m)$, the generator y^m being redundant. By Proposition 4.1, then, the associated class group is $\mathbb{Z}/(m-1)\mathbb{Z} \subset \mathbb{Z}/q(m-1)\mathbb{Z}$. □

Proposition 4.3. *Let $C \subset \mathbb{P}^3$ be a smooth curve and Let Z be a locally Cohen-Macaulay multiplicity m structure on a C of generic embedding dimension two with filtration (9). Fix a point $p \in Z$ of embedding dimension three. If Z_{m-2} has embedding dimension two at p and Z_{m-1} does not, then there are local coordinates x, y, z for which Z has local ideal*

(a) *$(x^2, xy^2, xyz^q - y^{m-1}, xy - uz^w(z^q x - y^{m-2}))$, u a unit and $w \geq 0$, or*
(b) *$(x^2, xy^2, fxy - (z^q x - y^{m-2}))$ where $f = 0$ or $f = uz^w$ for some $w > 0$ and unit u.*

Proof. Use Prop. 4.2 (a) to write $I = I_{Z_{m-1}} = (x^2, xy, xz^q - y^{m-2})$. At the level of sheaves, the map π factors through $\mathcal{F} = \mathcal{I}_{Z_{m-1}} \otimes \mathcal{O}_C/\{\text{torsion}\}$, a vector bundle on C which has rank two because Z_{m-1} is a generic local complete intersection. Working in the free $\mathcal{O}/(x, y)$-module $F = (I \otimes \mathcal{O}/(x, y))/\{\text{torsion}\}$ near p,

$$z^q(x^2) = x(xz^q - y^{m-1}) + y^{m-2}(xy) = 0$$

shows that x^2 is torsion, hence zero. Therefore $F \cong (\mathcal{O}/(x, y))^2$ is freely generated by xy and $xz^q - y^{m-2}$.

The kernel of the map $I \to F$ is

$$(x^2) + (x, y)I = (x^2, xy^2, xyz^q - y^{m-1}).$$

and we obtain $I_Z = \operatorname{Ker} \pi$ by adding the Koszul relation for the surjection of free modules $F \to \mathcal{O}/(x, y)$. Surjectivity implies that $\pi(xy)$ or $\pi(z^q x - y^{m-2})$ is a unit in $\mathcal{O}/(x, y)$. If $\pi(z^q x - y^{m-2}) = 1$ and $\pi(xy) = f \in \mathcal{O}/(x, y)$, the Koszul relation is $xy - f(z^q x - y^{m-2})$; here $f = uz^w$ for some unit u, since $f = 0$ leads to the ideal (x^2, xy, y^{m-1}) which does not have generic embedding dimension two; this gives the ideal in part (a). Otherwise take $\pi(xy) = 1$ and $\pi(z^q x - y^{m-1}) = f$ where $f = 0$ or $f = uz^w$ for some $w > 0$ and unit u, when the Koszul relation is $fxy - (z^q x - y^{m-2})$, giving ideal in part (b). □

Remark 4.4. Propositions 4.2 and 4.3 give a local description of ideals of certain multiplicity structures Z on a smooth curve C. Using the ideals in the Propositions to define multiple curves in $\mathbb{A}^3$, one can obtain global examples by taking closures in $\mathbb{P}^3$. When $C \subset \mathbb{P}^3$ is a line and $m \leq 4$, all such global structures have been classified [16, 17].

Proposition 4.5. *Let Z be a multiplicity-m structure on a smooth curve C with local ideal at p as in Prop. 4.3(a). Then at p the general surface S containing Z has an $\mathbf{A}_{(m-2)(q+w)+w-1}$ singularity and C has class $q + w \in \mathbb{Z}/((m-2)(q+w)+w)\mathbb{Z} \cong \operatorname{Cl} \widehat{\mathcal{O}}_{S,p}$.*

Proof. The very general surface S of sufficiently high degree containing Z has equation

$$xy + ax^2 + bxy^2 + cy^m + dxyz^q - dy^{m-1} - uxz^{q+w} + uy^{m-2}z^w$$

for units a, b, c, d, u in the local ring at the origin p by assumption.

We first apply the preparation steps in the first part of subsection 2.3 to bring the equation into a form recognizable by Theorem 2.6, beginning with changing variables to $Y = y + ax$. Expanding and gathering xY-terms brings the equation to the form

$$\text{unit}\cdot xY + u_1x^3 + u_2Y^m + u_3x^m + u_4x^2z^q + u_5Y^{m-1} + u_6x^{m-1} - uxz^{q+w} + uY^{m-2}z^w + u_7x^{m-2}z^w$$

where $u_1 = a^2b, u_2 = c, u_3 = c(-a)^m, u_4 = -ad, u_5 = -d, u_6 = -d(-a)^m$ and $u_7 = u(-a)^{m-2}$ are units. Applying Theorem 2.6 we have $r_1 = q + w$, $\mu = (m-2)(q+w) + w$, and since $\delta((m-2)(q+w)+w) \neq 0$ for general choice of units, the singularity type is $\mathbf{A}_{(m-2)(q+w)+w-1}$.

To determine the class of C in the completed local ring, we will look at the resolution of the singularity and determine which exceptional curve meets the strict transform of C. On the patch $Z = 1$ on the first blowup, the singularities must lie on the exceptional locus $z = 0$. This gives the equation (recycling the symbols x and y) $xy + ax^2 = 0$; partials similarly give $x = 0$ and $y + 2ax = 0$, so the blown-up surface is singular only at the origin on this patch. On $X = 1$ the exceptional locus has equations $x = 0, y = 0$, which is smooth, and similarly on the other patch. This situation persists until we get to the $(q+w)^{\text{th}}$ blow-up, which on the patch $Z = 1$ has equation

$$xy + ax^2 - ux + bxy^2z^{q+w} + dxyz^q + dy^{m-1}z^{(q+w)(m-3)} + uy^{m-2}z^{(q+w)(m-4)+w}.$$

This surface is smooth at the origin and singular at $(0, u, 0)$. Changing variables to $y' = y + u$ produces an equation of the form

$$xy' + \text{ (terms of order at least 2 in } x \text{ and } y' \text{ times powers of } z) + \text{ unit } \cdot z^{(q+w)(m-4)+w}.$$

As in subsection 2.3, this becomes $XY - Z^{(q+w)(m-4)+w}$ where the variable changes to obtain X and Y do not affect z and then Z is a unit times z.

To determine the class of C, note that its strict transform passes through the origin all the way to the $(q+w)^{\text{th}}$ blowup, at which point it still passes through the origin but misses the singular point. This gives C the class $q+w$ in the complete local Picard group $\mathbb{Z}/((q+w)(m-2)+w)\mathbb{Z}$. As C generates the class group of the original singular point, the order of this group depends on the greatest common divisor of $q+w$ and $(q+w)(m-2)+w$. $\square$

Proposition 4.6. *For $p \in Z$ as in Prop. 4.3(b) with $C = \operatorname{Supp} Z$, let S be the general surface containing Z. Then locally the equation of S at p has the form*

$$F = ax^2 + bxy^2 + e(fxy - (z^qx - y^{m-2}))$$

for local parameters x, y, z general units $a, b, e \in \mathcal{O}_{\mathbb{P}^3,p}$ and $f = uz^w$ for some $w > 0$ (interpret $w = \infty$ as $f = 0$) and C has order $m-2$ in $\operatorname{Cl}\widehat{\mathcal{O}}_{S,p}$. Furthermore

1. *If $m = 4$, then S has an $\mathbf{A}_{2q-1}$ singularity at p and $C \mapsto q \in \mathbb{Z}/2q\mathbb{Z} \cong \operatorname{Cl}\widehat{\mathcal{O}}_{S,p}$.*
2. *If $m > 4, q = 1$, then S has an $\mathbf{A}_{m-3}$ singularity at p and $C \mapsto 1 \in \mathbb{Z}/(m-2)\mathbb{Z}$.*
3. *If $m = 5, q = 2$, then S has an $\mathbf{E}_6$ singularity at p and $C \mapsto 1 \in \mathbb{Z}/3\mathbb{Z} \cong \operatorname{Cl}\widehat{\mathcal{O}}_{S,p}$.*
4. *For $m = 5$ and $q \geq 3$ or $m \geq 6$ and $q \geq 2$, the singularity of S at p is not a rational double point.*

Proof. The local equation for S follows immediately from Prop. 4.3 (b). To see that C has order $m-2$, first observe that $(m-2)C$ is Cartier on S at p simply because

$$(x, F) = (x, ax^2 + bxy^2 + fxy - (z^qx - y^{m-2})) = (x, y^{m-2}).$$

This shows that the order of C divides $m-2$ and it remains to show the order cannot be less. For this, recall that by construction dC has local ideal (x, y^d) for all $d \leq m-2$, so we must show that (x, y^d) is not Cartier on S at p for $d < m-2$. By Nakayama's lemma, this is equivalent to showing that the $\mathcal{O}/(x,y,z)$-vector space

$$\frac{(x, y^d)}{(x,y,z)(x,y^d) + (F)}$$

has dimension > 1, but this is clear because $F \in (x,y,z)(x,y^d)$ for $d < m-2$, so the dimension is 2.

First assume $m = 4$. Take $e = -1$ so that the local equation for S at p is

$$F = ax^2 - y^2 - dy^3 + cy^4 + bxy^2 + dxyz^q - fxy + xz^q.$$

By Lemma 2.2 a has a square root $\sqrt{a}$ in the complete local ring; set $x_1 = \sqrt{a}x + y$ and $y_1 = \sqrt{a}x - y$ so that the equation takes the form

$$x_1 y_1 + G + \frac{(x_1 + y_1)}{2\sqrt{a}} z^q$$

with $G \in (x_1, y_1)^2 \mathfrak{m}$. By Lemma 2.3 there is a coordinate change X, Y for which

$$F = XY + (AX + BY)z^q$$

where A, B are units for general choices of a, b, c, d. Making the elementary transformation $X_1 = X + Bz^q$ and $Y_1 = Y + Az^q$ brings the equation to the form $F = X_1 Y_1 - ABz^{2q}$ displaying the $\mathbf{A}_{2q-1}$ singularity. Tracing the class of the supporting curve we have

$$(x, y) = (x_1, y_1) = (X, Y) = (X_1 - Bz^q, Y_1 - Az^q)$$

which has class $q \in \mathbb{Z}/(2q)\mathbb{Z} \cong \operatorname{Cl} \widehat{\mathcal{O}}_{S,p}$ by Example 2.1 (c).

Now assume $m > 4$ and $q = 1$; again take $e = -1$ so that the local equation for S at p is

$$xz - y^{m-2} - fxy + ax^2 + bxy^2 + cy^m + d(xyz - y^{m-1})$$

for units $a, b, c, d \in \mathcal{O}_{\mathbb{P}^3, p}$. For $Z = z + fy + ax + by^2 + dyz^q$ and unit $u = 1 + dy - cy^2$ this takes the form $xZ - uy^{m-2}$, defining an $\mathbf{A}_{m-3}$ singularity. The class of the curve with ideal (x, y) is $1 \in \mathbb{Z}/(m-2)\mathbb{Z}$ by Example 2.1 (a).

The case $m = 5$ and $q = 2$ has a different flavor: Write the equation for S as

$$x^2 + 2axy^2 + by^5 + 2cxyz^2 - cy^4 + \underbrace{2uxyz^w}_{\text{or } 0} - 2vxz^2 + 2vy^3;$$

let $x_1 = x + y^2 + cyz^2 + uyz^w - vz^2$, and note that the expression takes on the form

$$x_1^2 + \alpha y^3 + \beta y^2 z^2 + \gamma z^4,$$

where α, β, γ are units. We may assume $\alpha = 1$; rewrite this expression as

$$x_1^2 + \left(y + \frac{\beta}{3} z^2\right)^3 + \gamma' z^4$$

where γ' is another unit. Taking $y_1 = y + \frac{\beta}{3} z^2$ shows that S has an $\mathbf{E}_6$ singularity at the origin with $\operatorname{Cl} \widehat{\mathcal{O}}_{S,p} \cong \mathbb{Z}/3\mathbb{Z}$. (The ideal for the curve C has become $(x_1 - \gamma z^4, y)$.)

For the case $m = 5, q \geq 3$ in part (4), a calculation entirely analogous to the previous one gives the form $X^2 + Y^3 + aYZ^4 + bZ^6$, a, b units. Lemma 2.2 shows that b has a square root, so the ideal of C can be written $(X + i\sqrt{b}Z^3, Y)$. The first blow-up of this surface on the patch (recycling variables as usual) has equation $X^2 + Y^3 z + aYz^3 + bz^4$, which is not the equation for a rational double point, since it is congruent to a square mod $\mathfrak{m}^4$ (see the classification given in

[14, §24]). Therefore the original singularity is not a rational double point, since the resolution of a rational double point only involves other rational double points.

Finally, for $m \geq 6, q \geq 2$ in part (4), after the first algebraic step of completing the square, we see that the equation for S is congruent to a square mod $\mathfrak{m}^4$, so again the singularity is not a rational double point. □

5. Global Picard groups of normal surfaces

In this section we give a formula for the Picard group of very general high degree surfaces containing a fixed curve Z with at most finitely many points of embedding dimension three. The solution to Problem 1.1 is required to apply the formula and we illustrate this with the examples worked out in the previous two sections.

Theorem 5.1. *Let $Z \subset \mathbb{P}^3$ be a closed one-dimensional subscheme with curve components $Z_1, \dots Z_r$ having respective supports C_i and suppose that the set T of points where Z has embedding dimension three is finite. If S is a very general surface of degree $d \gg 0$ containing Z with plane section H, then*

(1) S is normal and $\operatorname{Cl} S$ is freely generated by H and the C_i.
(2) The Picard group of S is

$$\operatorname{Pic} S = \bigcap_{p \in T} \operatorname{Ker}(\operatorname{Cl} S \to \operatorname{Cl} \mathcal{O}_{S,p}) \cap \langle Z_1, Z_2, \dots, Z_r, H \rangle \subset \operatorname{Cl} S. \tag{13}$$

Remark 5.2. The condition $d \gg 0$ can be expressed effectively, namely that $\mathcal{I}_Z(d-1)$ is generated by global sections and either (1) Z is reduced of embedding dimension $le2$ at each point or (2) $h^0(\mathcal{I}_Z(d-2)) \neq 0$ [2, Sections 1 and 2].

Proof. Part (1) is [2, Thm. 1.1]. It follows from sequence (1) in the introduction that

$$\operatorname{Pic} S = \bigcap_{p \in \operatorname{Sing} S} \operatorname{Ker}(\operatorname{Cl} S \to \operatorname{Cl} \mathcal{O}_{S,p}). \tag{14}$$

Along with the fixed singularities T, which forcibly lie on every surface S containing Z, there are *moving singularities* p, which vary with the surface and lie on exactly one component Z_i of multiplicity $m_i > 1$ [2, Prop. 2.2]: these are $\mathbf{A}_{m_i-1}$ singularities and the corresponding map $\operatorname{Cl} S \to \operatorname{Cl} \widehat{\mathcal{O}}_{S,p} \cong \mathbb{Z}/m_i\mathbb{Z}$ sends C_i to 1 and the remaining C_i to 0, therefore the corresponding kernel is $\langle C_1, C_2, \dots, m_i C_i = Z_i, C_{i+1}, \dots, C_r, H \rangle$. Intersecting these subgroups for $1 \leq i \leq r$ yields $\langle Z_1, Z_2, \dots, Z_r, H \rangle$, which gives equation (13) provided there is at least one component Z_i of multiplicity $m_i > 1$. If there are no components of multiplicity $m_i > 1$, then Z is reduced and there are no moving singularities: here formula (13) still works because $Z_i = C_i$ for each $1 \leq i \leq r$ and hence $\langle Z_1, \dots, Z_r, H \rangle = \langle C_1, \dots, C_r, H \rangle = \operatorname{Cl} S$. □

We first note some easy special cases.

Corollary 5.3. *Let Z and S be as in Theorem 5.1. Then*

(a) If Z is reduced of embedding dimension at most two, then $\operatorname{Pic} S = \operatorname{Cl} S$.
(b) If Z is reduced, then $\operatorname{Pic} S = \bigcap_{p \in F} \operatorname{Ker}(\operatorname{Cl} S \to \operatorname{Cl} \mathcal{O}_{S,p})$.
(c) If Z has embedding dimension ≤ 2, then $\operatorname{Pic} S = \langle Z_1, \dots, Z_r, H \rangle$.

Proof. (a) Here T is empty and $Z_i = C_i$ for each i, so part (2) of the theorem says that $\operatorname{Pic} S$ is generated by H and the C_i, which is exactly $\operatorname{Cl} S$ by part (1). (b) Here $Z_i = C_i$ again, so $\langle Z_i, H \rangle = \operatorname{Cl} S$. (c) Here again T is empty. □

Remarks 5.4. We make a few observations about Corollary 5.3.

(1) [2, Cor. 1.3] highlights some other special cases of the Theorem.
(2) In part (a), S is in fact globally smooth by [18, Thm. 1.2]; see also [2, Cor. 2.3].
(3) We see that the Picard groups aren't very interesting for nicely behaved curves, which explains why we have focused on non-reduced base locus Z. For example, if Z has at worst nodes, then by Corollary 5.3 (a) the Picard group of S is freely generated by H and the components of Z, extending Lopez' theorem [15, II, Thm. 3.1].

Example 5.5. To see what can happen to smooth curves intersecting at a point, consider the simplest case when $Z = \bigcup_{i=1}^{r} L_i$ is the union of r lines passing through p.

(a) If $r = 2$, then S is smooth at p and $\operatorname{Pic} S$ is freely generated by the two lines and H by Corollary 5.3 (a).
(b) If $3 \leq r \leq 5$ and the lines are not coplanar, then p is a fixed singularity, but a mild one. Even when the lines are in general position with respect to containing p, S has an $\mathbf{A}_1$ singularity at p and the map $\operatorname{Cl} S \to \operatorname{Cl} \mathcal{O}_{S,p}$ takes each line to $1 \in \mathbb{Z}/2\mathbb{Z}$. Therefore the Picard group is
$$\operatorname{Pic} S = \{\sum a_i L_i + bH : 2 | \sum a_i\}$$
in this case. We had worked out the case $r = 4$ in [2, Ex. 1.4].
(c) If $r > 5$ and the lines are in general position, then by [3, Cor. 5.2] p is a non-rational singularity and the local class group $\mathcal{O}_{S,p}$ contains an Abelian variety. Moreover the images of the lines are involved in no relations in $\mathcal{O}_{S,p}$, so that $\operatorname{Pic} S = \langle H \rangle$.
(d) There are many ways that the lines can lie in special position. We have not explored all of them, but we did work out the case of r planar lines $L_1, \dots, L_r$ through p union a line L_0 not in the plane, this configuration resembles a pinwheel [3, Ex. 5.3 (b)]. Here the point p is an $\mathbf{A}_{r-1}$-singularity on S and the map $\operatorname{Cl} S \to \widehat{\operatorname{Cl}} \mathcal{O}_{S,p} \cong \mathbb{Z}/r\mathbb{Z}$ sends L_0 to 1 and the other lines L_i to -1. Therefore $\operatorname{Pic} S = \langle rL_0, L_0 + L_1, L_0 + L_2, \dots L_0 + L_r, H \rangle$.

Now we consider examples in which Z is non-reduced, but the set T of embedding dimension three points is non-empty.

Example 5.6. Consider the very general high-degree surface S containing a locally Cohen-Macaulay m-structure Z of generic embedding dimension two supported on a line L.

(a) If Z has embedding dimension two at each point (note that this always holds if $m = 2$), then $\operatorname{Pic} S = \langle H, Z \rangle$ by Cor. 5.3 (c).
(b) If the underlying $(m-1)$-structure has embedding dimension two but Z itself does not, then S has an $\mathbf{A}_{(m-1)q-1}$-singularity at p for some $q > 0$ and the restriction map $\operatorname{Cl} S \to \widehat{\operatorname{Cl}} \mathcal{O}_{S,p} \cong \mathbb{Z}/q(m-1)\mathbb{Z}$ sends L to $q \in \mathbb{Z}/q(m-1)\mathbb{Z}$ by Prop. 4.2. Applying Theorem 5.1 (c) we have
$$\operatorname{Pic} S = \langle mL, H \rangle \cap \langle (m-1)L, H \rangle = \langle m(m-1)L, H \rangle.$$
For example, the very general surface S containing a typical triple line Z supported on L has Picard group $\operatorname{Pic} S = \langle 6L, H \rangle$.
(c) The story is more complicated if the underlying $(m-2)$-structure has embedding dimension two and the underlying $(m-1)$-structure does not because there are two possibilities for the local ideal of Z at p by Prop. 4.3. For the form of the local ideal given in Proposition 4.6 (b), L has order $m-2$ in $\operatorname{Cl} \mathcal{O}_{S,p}$ and Theorem 5.1 gives
$$\operatorname{Pic} S = \langle mL, H \rangle \cap \langle (m-2)L, \mathcal{O}_S(1) \rangle = \langle LCM(m, m-2)L, H \rangle.$$
The actual singularity may be an $\mathbf{A}_n$, an $\mathbf{E}_6$ or even irrational.

Example 5.7. In section 3 consider the very general high degree surface S containing a union of two multiple lines Z_1, Z_2 supported on L_1, L_2.

(a) If $Z_1 \cap Z_2 = \emptyset$, then $\operatorname{Pic} S = \langle Z_1, Z_2, H \rangle$ by Corollary 5.3 (c).
(b) Now suppose $I_{Z_1} = (x, z^m)$ and $I_{Z_2} = (y, z^n)$ with $m \leq n$ so that $Z_1 \cap Z_2$ is a length n subscheme supported at $p = (0, 0, 0, 1)$. By Props. 3.3 and 3.4, p is an $\mathbf{A}_{n-1}$ singularity of S and the restriction map $\operatorname{Cl} S \to \operatorname{Cl} \widehat{\mathcal{O}}_{S,p} \cong \mathbb{Z}/n\mathbb{Z}$ takes L_1, L_2 to $1, -1$. Taking the kernel of this map we find that $\operatorname{Pic} S = \langle nL_1, L_1 + L_2, H \rangle$.
(c) Now replace Z_1 with the multiple line with having (x^m, z). The support of $Z_1 \cup Z_2$ is the same as the last example, but now L_2 is contained in the plane $S_1 : \{z = 0\}$ containing Z_1, so L_2 has order of tangency $q = \infty$ to S_1. According to Proposition 3.6, S has an $\mathbf{A}_{mn-1}$ singularity at p and the restriction $\operatorname{Cl} S \to \operatorname{Cl} \widehat{\mathcal{O}}_{S,p} \cong \mathbb{Z}/mn\mathbb{Z}$ takes L_1 to 1 and L_2 to $mn - m$. Therefore $\operatorname{Pic} S = \langle mnL_1, L_2 - (mn - m)L_1, H \rangle$.

Example 5.8. These results can be used in combination, so we close with an example illustrating several behaviors at once. Start with three non-planar lines L_1, L_2, L_3 meeting at p_1. Let Z_4 be a 4-structure on a line L_4 intersecting L_1 at $p_2 \neq p_1$, and assume that Z_4 is contained in a smooth quadric surface Q which is tangent to L_1. Let Z_5 be a 3-structure on a line L_5 which intersects L_2 in a reduced point $p_3 \neq p_1$, and suppose that Z_5 has at least one point $p_4 \neq p_3$ of embedding dimension three. Finally, let Z_6 be a double line supported on L_6 which intersects Z_5 at a point $p_5 \neq p_4, p_3$ and assume that L_6 intersects a local surface S_5 defining Z_5 in a double point. Finally let $Z = L_1 \cup L_2 \cup L_3 \cup Z_4 \cup Z_5 \cup Z_6$ and consider the very general surface S of high degree containing Z.

By Theorem 5.1 (a), $\operatorname{Cl} S$ is freely generated by H and $L_1, L_2, \dots, L_6$ and to find $\operatorname{Pic} S$ we must compute the kernels of the maps $\operatorname{Cl} S \to \operatorname{Cl} \mathcal{O}_{S,p_i}$ for $1 \leq i \leq 5$:

(1) By Ex. 5.5 (b) the kernel at p_1 is $\langle 2L_1, L_2 - L_1, L_3 - L_1, L_4, L_5, L_6, H \rangle$ and S has an $\mathbf{A}_1$ singularity at p_1.
(2) By Prop. 3.8 with $m = 4, n = 1, q = 2$, the natural restriction map is given by $L_4 \mapsto 1, L_1 \mapsto 2 \in \mathbb{Z}/4\mathbb{Z}$, so the kernel at p_2 is $\langle L_1 - 2L_4, L_2, L_3, 4L_4, L_5, L_6, H \rangle$ and S has an $\mathbf{A}_3$ singularity at p_2.
(3) By Prop. 4.2, the kernel at p_3 is $\langle L_1, L_2 + L_3, 3L_3, L_4, L_5, L_6, H \rangle$.
(4) By Ex. 5.6 (b) the kernel at p_4 is $\langle L_1, L_2, L_3, L_4, 6L_5, L_6, H \rangle$.
(5) By Prop. 3.8 with $n = q = 2, m = 3$, the kernel at p_5 is $\langle L_1, L_2, L_3, L_4, L_6 - 2L_5, 4L_5, H \rangle$.

Using Hermite Normal Form and Mathematica, we compute the intersection of

$$\langle L_1, L_2, L_3, 4L_4, 3L_5, 2L_6, H \rangle$$

and the above kernels to be $\operatorname{Pic} S = \langle 2L_1, 6L_2, L_3 + L_2, 4L_4, 12L_5, 2L_6, H \rangle$.

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Journal *of* Singularities
Volume 4 (2012), 171-179

received 6 June 2012
in revised form 25 October 2012
DOI: 10.5427/jsing.2012.4j

PURITY OF BOUNDARIES OF OPEN COMPLEX VARIETIES

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Abstract. We study the boundary of an open smooth complex algebraic variety U. We ask when the cohomology of the geometric boundary $Z = X \setminus U$ in a smooth compactification X is pure with respect to the mixed Hodge structure. Knowing the dimension of singularity locus of some singular compactification, we give a bound for k above which the cohomology $H^k(Z)$ is pure. The main ingredient of the proof is purity of the intersection cohomology sheaf.

1. Introduction

Let U be a smooth complex algebraic variety which is not compact. We study cohomological properties of U which are invariant with respect to modifications of the interior of U. In other words, we investigate cohomological properties of the boundary.

The boundary itself can have at least two meanings. First of all, from the topological point of view, we may treat an open smooth variety as the interior of a compact manifold with boundary. In this case, the boundary would mean an odd-dimensional real manifold. We call it the link at infinity. On the other hand, from the geometric point of view, we may compactify our variety in the category of algebraic varieties. The boundary is then a subvariety of the compactification. In addition, we may require that the compactification is smooth. The condition that the boundary is a normal crossing divisor is irrelevant to us, although it is hidden in the construction of the mixed Hodge structure. The link at infinity can be identified with the link of the boundary in the compactification. Despite the differences, we show that the topological and geometric boundaries have a great deal in common where the mixed Hodge structure is concerned.

To some extent, we try to avoid specific methods of Hodge theory, having in mind possible applications (or rather open questions) to real algebraic geometry, as well as some questions about torsion for cohomology of complex varieties. The sections §2-§4 are valid in that generality. Nevertheless, the results of §5 cannot be generalized and they hold only for rational cohomology of algebraic varieties. The strong functoriality of the weight filtration implies that the lower weight subspaces of the topological and geometric boundaries coincide; see Proposition 3. In similar situations, this phenomenon was already described in [10, Prop. 7.1] and [2, Prop. 5.1].

The proof of the main result of §6 uses even stronger techniques. The purity of the intersection cohomology sheaf [3] imposes some conditions on the link of the geometric boundary. We prove Theorem 11, which can be shortened to the following statement:

Theorem 1. *Suppose U is a complex smooth algebraic variety. Assume that U admits a singular compactification Y. Suppose that the singularity of the pair $(Y, Y \setminus U)$ is of dimension s. Then, for every smooth compactification X, the boundary $X \setminus U$ has pure cohomology $H^k(X \setminus U)$, for $k \geq \dim(U) + s$.*

2000 *Mathematics Subject Classification.* 32S35, 55N33, 14E15, 32S20.

Key words and phrases. Mixed Hodge theory of singular varieties, intersection cohomology, resolution of singularities.

Theorem 1, in the case where U admits a one-point compactification, already appeared in [4, Th. 2.1.11]. A vast generalization was given in [13]. The present version gives a better bound for purity, although the situation considered here is less general.

One can treat Theorem 1 as a contractibility criterion. For a subvariety $Z \subset X$: if $H^k(Z)$ is not pure for $k \geq \dim(U) + s$, then the pair (X, Z) cannot be contracted to a pair (Y, W) with singularities of dimension less than or equal to s. Although Theorem 1 resembles the Grauert criterion, it is of a different nature. In the Grauert criterion, the intersection form on Z depends on the embedding $Z \subset X$, whereas, here, the mixed Hodge structure of Z does not. Needless to say, our criterion is not sufficient for the existence of a contraction.

In contrast to our previous paper [13], we try to present the subject in a form as elementary as possible. We have avoided using mixed the Hodge modules of [11], taking for granted that the cohomology with coefficients in a complex of sheaves of geometric origin has a natural mixed Hodge structure. By "geometric origin", we mean "obtained by the standard sheaf-theoretic operations". We assume that the varieties are defined over the real numbers or over the complex numbers, and we use classical topology. In our arguments, we will apply resolution of singularities, although parts of the results depend only on the formal properties of mixed Hodge modules or Weil sheaves.

2. Topological boundary: the link at infinity

Let us begin with a description of some invariants of open manifolds which can be defined using just topology and basic properties of resolutions of singularities. Working with $\mathbb{Z}/2$-cohomology, we can apply our construction also for real algebraic manifolds. For complex manifolds, we can use any coefficients, not necessarily $\mathbb{Q}$.

The first invariant we propose to consider is the cohomology of the link at infinity:

$$H^*(L_\infty U) := \varinjlim_{K \subset U} H^*(U \setminus K)\,,$$

where K runs through compact sets contained in U. The group $H^*(L_\infty U)$ is exactly the cohomology of the link L_Z, which is the link of the boundary set $Z = X \setminus U$, where X is a compactification of U. (For various approaches to the link of a subvariety, see [5].) This cohomology group is of finite dimension. It can be expressed in terms of sheaf operations on X:

$$H^*(L_\infty U) = H^*(L_Z) = H^*(Z; i^* Rj_* \mathbb{Q}_U)\,,$$

where $j : U \hookrightarrow X$ and $i : X \setminus U \hookrightarrow X$ are the inclusions.

3. Geometric boundary: image of boundary cycles

Another invariant which we consider is the image of the boundary cocycles

$$IB^*(U) = \operatorname{im}(H^*(Z) \to H^{*+1}_c(U)) = \ker(H^{*+1}_c(U) \to H^{*+1}(X))\,,$$

where X is a **smooth** compactification of U and $Z = X \setminus U$. The maps come from the long exact sequence of the pair (X, Z).

To show the independence of X, we start with a purely topological lemma.

Lemma 2. *Suppose we have a map of real smooth oriented closed manifolds*

$$f : X_1 \to X_2,$$

which is an isomorphism of some open subsets

$$f_{|U_1} : U_1 = f^{-1}(U_2) \stackrel{\simeq}{\to} U_2\,.$$

Denote by

$$IB_i^* = \ker(H_c^{*+1}(U_i) \to H^{*+1}(X_i))$$

the kernels of the natural maps for $i = 1, 2$. Then f induces an isomorphism

$$f^* : IB_2^* \to IB_1^* .$$

Proof. The map $f^* : H^*(X_2) \to H^*(X_1)$ is injective, since it is a map of degree one of compact manifolds. The map f induces the transformation

$$\begin{array}{ccccc} IB_2^k & \hookrightarrow & H_c^{k+1}(U_2) & \to & H^{k+1}(X_2) \\ \downarrow & & \simeq \downarrow & & {\scriptstyle mono} \downarrow \\ IB_1^k & \hookrightarrow & H_c^{k+1}(U_1) & \to & H^{k+1}(X_1). \end{array}$$

It follows that $IB_2^k \to IB_1^k$ is an isomorphism. □

To prove the independence of $IB^*(U)$ on the compactification, simply note that any two smooth compactifications are dominated by a third one.

4. Basic exact sequences

We will need three exact sequences to relate the described invariants. These exact sequences may be constructed topologically, but it is important to know that they come from distinguished triangles in the derived category of sheaves. It will follow that, for complex varieties, the maps of the described exact sequences preserve the mixed Hodge structure.

We start with the sequence relating the cohomology of U and the cohomology of its link at infinity. Let X be any compactification and $Z = X \setminus U$. It is possible to find a neighbourhood N of Z which retracts to Z and such that the boundary ∂N is homeomorphic to the link of Z. Considering the pair $(X \setminus N, \partial N)$, we arrive at the long exact sequence

$$\cdots \to H^k(U) \to H^k(L_Z) \xrightarrow{\delta} H_c^{k+1}(U) \to H^{k+1}(U) \to \cdots . \tag{1}$$

This exact sequence may, in fact, be obtained from the fundamental distinguished triangle (in the category of mixed Hodge modules on X)

$$\begin{tikzcd} i_*i^!G \arrow[rr] & & G \arrow[dl] \\ & Rj_*j^*G \arrow[ul, "{[+1]}"] & \end{tikzcd} \tag{2}$$

where $G = j_!\mathbb{Q}_U$. By duality, we obtain the triangle

$$\begin{tikzcd} i_!i^*Rj_*\mathbb{Q}_U \arrow[dr, "{[+1]}"'] & & Rj_*\mathbb{Q}_U \arrow[ll] \\ & j_!\mathbb{Q}_U \arrow[ur] & \end{tikzcd}$$

since $j_!j^!Rj_*\mathbb{Q}_U \simeq j_!\mathbb{Q}_U$. Applying cohomology, we obtain the sequence (1).

We also need an exact sequence relating $H^k(Z)$ and $H^k(L_Z)$. Topologically, we have a deformation retraction from the closure $\overline{N} \to Z$. The exact sequence for the manifold with boundary $(\overline{N}, \partial N)$,

$$\cdots \to H^k(\overline{N}) \to H^k(\partial N) \to H^{k+1}(\overline{N}, \partial N) \to H^{k+1}(\overline{N}) \to \cdots ,$$

becomes

$$\cdots \to H^k(Z) \to H^k(L_Z) \to H^{k+1}(X, U) \to H^{k+1}(Z) \to \cdots . \tag{3}$$

The sheaf theoretic definition is given below. Let us restrict the triangle (2) with $G = \mathbb{Q}_X$ to Z. We have $i^* i_* i^! \mathbb{Q}_X = i^! \mathbb{Q}_X$, and we obtain the triangle

$$\begin{array}{ccc} i^!\mathbb{Q}_X & \longrightarrow & \mathbb{Q}_Z \\ {\scriptstyle [+1]}\nwarrow & & \swarrow \\ & i^* Rj_* \mathbb{Q}_U . & \end{array}$$

The associated sequence of cohomology is just (3). It plays a fundamental role in our further consideration.

Of course, the third exact sequence that we use is the sequence of the pair (X, Z)

$$\cdots \to H^k(X) \to H^k(Z) \to H_c^{k+1}(U) \to H^{k+1}(X) \to \cdots . \tag{4}$$

To relate the groups $H^*(L_\infty(U)) = H^*(L_Z)$ and $IB^*(U) = \operatorname{im}(H^k(Z) \to H_c^{k+1}(U))$, we apply the map of exact sequences from (4) to (1) induced by the inclusion

$$(X \setminus N, \partial N) \subset (X, N) \overset{htp}{\simeq} (X, Z) .$$

We obtain the commutative diagram

$$\begin{array}{ccc} H^k(Z) & \to & H_c^{k+1}(U) \\ \downarrow & & \| \\ H^k(L_Z) & \to & H_c^{k+1}(U) \end{array} .$$

We see that

$$IB^k(U) = \operatorname{im}(H^k(Z) \to H_c^{k+1}(U)) \subset \operatorname{im}(H^k(L_\infty(U)) \to H_c^{k+1}(U)) .$$

In general, this inclusion is proper.

5. Mixed Hodge structure

From now on, we consider only complex algebraic varieties and rational cohomology.

The invariants $H^k(L_\infty U)$ and $IB^k(U)$ are equipped with mixed Hodge structures. The first one is given by the sheaf-theoretic description:

$$H^*(L_\infty U) = H^*(Z; i^* Rj_* \mathbb{Q}_U) .$$

The second one, $IB^*(U)$, has a structure induced from $H^*(Z)$. In the situation of Lemma 2, the map $IB_2^k \to IB_1^k$ preserves the quotient mixed Hodge structures and since it is an isomorphism of vector spaces it must be also an isomorphism of all weight subspaces. In fact, by the definition of the mixed Hodge structure, we have

$$IB^k(U) = W_k H_c^{k+1}(U) .$$

For us, the most interesting part is the weight subspace W_{k-1}. Using basic properties of the mixed Hodge structure, we will give three descriptions of that weight space.

Proposition 3. *Let X be a smooth compactification of U and $Z = X \setminus U$. Then the following groups are isomorphic:*

(1) $W_{k-1} H^k(L_\infty U)$,
(2) $W_{k-1} H_c^{k+1}(U)$,
(3) $W_{k-1} H^k(Z)$.

Note that, in the statement of the theorem, we do not assume that Z is a smooth divisor with a normal crossings. As a corollary to Proposition 3, we have

Corollary 4. *Let X be a smooth compactification of U and $Z = X \setminus U$. The cohomology $H^k(Z)$ is pure, of weight k, if and only if $H^k(L_Z)$ is of weight $\geq k$.*

Also we note (compare [10, Prop. 7.1]):

Corollary 5. *The impure part of cohomology of the boundary set $W_{k-1}H^k(Z)$ does not depend on the smooth compactification.*

Remark 6. Note that the group $W_{k-1}H^k(Z)$ is a topological invariant of Z, since, by [12], it is the kernel of the canonical map to intersection cohomology $H^k(Z) \to IH^k(Z)$. Also, by the construction of the mixed Hodge structure, we have

$$W_{k-1}H^k(Z) = ker(g^* : H^k(Z) \to H^k(\widetilde{Z}))\,,$$

where $g : \widetilde{Z} \to Z$ is any dominating proper map from a smooth variety, possibly of bigger dimension.

The entire cohomology of the boundary of a smooth compactification is not an invariant of U. Of course, when we blow up something at the boundary, the cohomology is modified; nevertheless, the lower parts of the weight filtration remain unchanged.

Remark 7. With the help of the Decomposition Theorem of [3], we have better insight into what happens with the cohomology of the boundary. Let f be a map of pairs $(X_1, Z_1) \to (X_2, Z_2)$ which is an isomorphism outside Z_1. The push-forward of the constant sheaf on X_1 decomposes:

$$Rf_*\mathbb{Q}_{X_1} \simeq \mathbb{Q}_{X_2} \oplus \bigoplus_\alpha IC(V_\alpha; L_\alpha)\,.$$

The supports of the intersection sheaves $IC(V_\alpha; L_\alpha)$ are contained in Z_2; therefore

$$H^*(Z_1) = H^*(Z_2; (Rf_*\mathbb{Q}_{X_1})_{|Z_2}) \simeq H^*(Z_2) \oplus \bigoplus_\alpha IH^*(V_\alpha; L_\alpha)\,.$$

Again, we see that the difference between $H^*(Z_1)$ and $H^*(Z_2)$ is pure, since $\bigoplus_\alpha IH^*(V_\alpha; L_\alpha)$ is a summand of $H^*(X_1)$.

Remark 8. Using another powerful tool, namely the Weak Factorization Theorem [1], we can trace how the cohomology of the boundary may change. Each time, when we blow up a smooth center S contained in the boundary, the pure summand $\operatorname{coker}\big(H^*(S) \to H^*(\mathbb{P}N_{S/X})\big)$ contributes to the cohomology of the blown-up boundary. Here, $H^*(\mathbb{P}N_{S/X})$ is the projectivization of the normal bundle of S in X.

The proof of Proposition 3 is divided into Lemmas 9 and 10.

Lemma 9. *We have*

$$W_{k-1}H^k(Z) \simeq W_{k-1}H^{k+1}_c(U)\,.$$

Proof. We recall that $H^k(X)$ is of weight k and $H^{k+1}(X)$ is of weight $k+1$. Therefore, the long exact sequence

$$\cdots \to H^k(X) \to H^k(Z) \xrightarrow{\delta} H^{k+1}_c(U) \to H^{k+1}(X) \to \cdots$$

induces an isomorphism of graded pieces for $\ell < k$

$$Gr^W_\ell H^k(Z) \simeq Gr^W_\ell H^{k+1}_c(U)\,.$$

It follows that the boundary map $W_{k-1}H^k(Z) \to W_{k-1}H^{k+1}_c(U)$ is an isomorphism. □

Lemma 10. *We have*

$$W_{k-1}H^k(L_Z) \simeq W_{k-1}H^{k+1}_c(U)\,.$$

Proof. We consider the long exact sequence (1). Since U is smooth, $W_{k-1}H^k(U)=0$. Therefore, for $\ell<k$
$$Gr^W_\ell H^k(L_Z)\simeq Gr^W_\ell H^{k+1}_c(U)\,.$$
Again, the boundary map $W_{k-1}H^k(L_Z)\to W_{k-1}H^{k+1}_c(U)$ is an isomorphism. □

6. Singular versus smooth compactifications

Let $W\subset Y$ be a pair of varieties. Assume that $Y\setminus W$ is smooth. By the "singularity of the pair", we mean the set of points at which W in Y analytically does not look like a submanifold (of any dimension) in a manifold. The singularity set consists of points at which W or Y is singular. Below, we give the exact statement of our main result.

Theorem 11. *Let U be a smooth variety. Suppose that U admits a compactification Y and let $W=Y\setminus U$ be the boundary set. Denote by s the dimension of the singularity of the pair (Y,W). Let X be a smooth compactification of U and $Z=X\setminus U$. For $k\geq\dim(U)+\dim(W)$, we have:*
i) the cohomology of the link $H^k(L_Z)$ is of weight $\geq k+1$,
ii) the restriction map $H^k(Z)\to H^k(L_Z)$ vanishes.

For $k\geq\dim(U)+s$ we have:
iii) the cohomology of the boundary $H^k(Z)$ is pure, of weight k, that is
$$W_{k-1}H^k(Z)=0$$
iv) the cohomology of the link $H^k(L_Z)$ is of weight $\geq k$.

Note that, by Proposition 3, claim iv) does not depend on the choice of the smooth compactification X.

Let $n=\dim(U)$. By Poincaré duality, we have
$$H^k(Z)^*=H^{2n-k}(X,U)(n)\,,$$
$$H^k(L_Z)^*=H^{2n-1-k}(L_Z)(n)\,,$$
where (n) denotes the Tate twist shifting the weights by $2n$. The dual version of Theorem 11 is the following:

Theorem 12. *With the assumptions of Theorem 11:*

For $k\leq\dim(U)-\dim(W)$, we have
i') the cohomology of the link $H^{k-1}(L_Z)$ is of weight $\leq k-1$,
ii') the boundary map $H^{k-1}(L_Z)\to H^k(X,U)$ vanishes.

For $k\leq\dim(U)-s$, we have
iii') the cohomology $H^k(X,U)$ is pure, of weight k, that is
$$W_kH^k(X,U)=H^k(X,U)$$
iv') the cohomology of the link $H^{k-1}(L_Z)$ is of weight $\leq k$.

To distinguish two copies of U in X and in Y, we will use the letter V for the copy of U in Y. The identification map $U\to V$ is denoted by f:
$$\begin{array}{ccccc} Z=X\setminus U & \subset & X=\overline{U} & \supset & U\\ & & & \simeq & \downarrow f\\ W=Y\setminus V & \subset & Y=\overline{V} & \supset & V\end{array}$$

Remark 13. In our setup, we can apply completion and resolution of singularities. Therefore, X can be replaced by a dominating smooth variety for which the map f extends to the boundary.

Some information about the weights of the cohomology of the link and the boundary can be deduced when we have a proper map $f : U \to V$ and a compactification of V. A statement which generalizes i) and ii) in terms of the *defect of semismallness* (introduced in [4]) is formulated in [13]. The direct generalization of iii) and iv) would involve precise information about the singularities of the perverse cohomology sheaves ${}^p\mathcal{H}^k Rf_*\mathbb{Q}_U$.

Theorem 11 can be localized around a topological component of $X \setminus U$. Precisely, consider the set of ends, i.e., $U_\infty = \pi_0(X \setminus U)$. This set does not depend on the choice of X, provided that X is normal. A map of algebraic varieties which is proper induces a map of their ends. To deduce purity of the cohomology of a part of the boundary of U, it is enough to have information about a singular completion of the corresponding end.

7. Proofs

Before the proof of Theorem 11, let us recall the key property of the link of a subvariety.

Theorem 14 ([5])**.** *Let Y be a variety and let W be a compact subvariety. Let us assume that $Y \setminus W$ is smooth. Then, $H^k(L_W)$ is of weight $\leq k$, for degrees $k < \dim(Y) - \dim(W)$.*

Theorem 14 immediately follows from the purity of the intersection sheaf [6, 3], since the stalk cohomology $\mathcal{H}^k(IC_Y)$ is isomorphic to $\mathcal{H}^k(Rj_*\mathbb{Q}_V)$ for $k < \dim(Y) - \dim(W)$ and $H^*(L_W) = H^*(W; (Rj_*\mathbb{Q}_V)_{|W})$.

Remark 15. In [2, §6], the Decomposition Theorem of [3] was used to give estimates for the dimension of intersection cohomology of the link by means of resolution. But it seems that the purity of the intersection sheaf was not used directly.

Proof of (12.i'-ii').
By Remark 13, we assume that the map f extends to X. The extended map (denoted by the same letter) induces a map of sheaves $i'^*Rj'_*\mathbb{Q}_V \to Rf_*i^*Rj_*\mathbb{Q}_U$. Therefore, the mixed Hodge structures of the isomorphic groups $H^*(L_W)$ and $H^*(L_Z)$ coincide. By Theorem 14 and the assumption on the dimension of W, the cohomology $H^{k-1}(L_W)$ is of weight $\leq k-1$. Claim 12.ii' follows from the long exact sequence (3): the boundary map

$$H^{k-1}(L_Z) \to H^k(X, U)$$

vanishes because the first term is of weight $\leq k-1$ and the second term is of weight $\geq k$.

Proof of (11.i-ii) follows by duality.

Proof of (11.iii-iv) If $W = Sing(Y)$, then $s = \dim(W)$, and statement i) is even stronger than iii). Also by Proposition 3 we have $W_{k-1}H^k(Z) = W_{k-1}H^k(L_Z) \subset W_kH^k(L_Z) = 0$, therefore iv) follows.

Suppose now that $Sing(Y) \subsetneq W$. We may assume that f extends to a map $X \to Y$ and also we may assume that the map f is a resolution of singularities of the pair (Y, W). Let $\widetilde{W} \subset X$ be the proper transform of W. Denote by $E \subset X$ the exceptional set of f and let $F = E \cap \widetilde{W}$. Consider the Mayer-Vietoris exact sequence for $Z = E \cup \widetilde{W}$:

$$\cdots \to H^{k-1}(E) \oplus H^{k-1}(\widetilde{W}) \xrightarrow{\alpha} H^{k-1}(F) \xrightarrow{\delta} H^k(Z) \to H^k(E) \oplus H^k(\widetilde{W}) \to \cdots.$$

By (11.i), applied to the map $(X, E) \to (Y, f(E))$, the cohomology of the link $H^k(L_E)$ is of weight $\geq k+1$ for $k \geq \dim(X) + s$. Hence, by Proposition 3, the cohomology $H^k(E)$ is pure for

$k \geq \dim(X) + s$. Of course, $H^k(\widetilde{W})$ is pure since we assume that $\widetilde{W}$ is smooth. To prove the purity of $H^k(Z)$, it remains to show that the map δ of the Mayer-Vietoris sequence is trivial.

By (11.ii), applied to $F \subset \widetilde{W}$, the map

$$H^{k-1}(F) \to H^{k-1}(L_F)$$

vanishes for $k - 1 \geq \dim(W) + s$. By the exact sequence (3) of that pair, the restriction map

$$H^{k-1}(\widetilde{W}, \widetilde{W} \setminus F) \to H^{k-1}(F)$$

is surjective. The above map factors through $H^{k-1}(\widetilde{W})$; therefore, the map

$$H^{k-1}(\widetilde{W}) \to H^{k-1}(F)$$

is surjective. It follows that the restriction map α is surjective and the boundary map δ is trivial for $k \geq \dim(X) + s \geq \dim(W) + s + 1$. This completes the proof.

Proof of (12.iii'-iv') follows by duality. □

Remark 16. If the singularity set is empty, then $s = -\infty$ by convention. Claims (11.i-ii) hold in all degrees for trivial reasons.

The special case when W is a point (an isolated singularity resolution) was studied from the very beginning of the theory. In that case, both maps $H^{n-1}(L_Z) \to H^n(X, U)$ and $H^n(X) \to H^n(L_Z)$ are trivial. The map $H^n(X, U) \to H^n(Z)$ is an isomorphism. After the identification $H^n(X, U) = H^n(Z)^*$, we obtain a nondegenerate intersection form, which was studied for example in [7].

8. Questions about real algebraic varieties

Hodge theory for real algebraic varieties and $\mathbb{Z}/2$ coefficients is not available. The approach of [8, 9] does not lead to a strongly functorial weight filtration. Nevertheless, one defines impure cohomology of a singular compact variety X: it is the kernel of $H^*(X) \to H^*(\tilde{X})$, where $\tilde{X}$ is any resolution. We say that the cohomology of a real variety is pure if the kernel $H^*(X) \to H^*(\tilde{X})$ is trivial. The definition does not depend on $\tilde{X}$. Now one can ask the question about a generalization of Theorem 11:

Question 17. *With the assumption of Theorem 3 for real algebraic varieties: What properties of (Y, W) would guarantee purity of $H^*(Z)$ in some range of degrees?*

The dimension of the singularity set is far too weak of an invariant. It is well-known that any real algebraic set can be contracted to a point.

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Journal *of* Singularities
Volume 4 (2012), 180-187

received 11 June 2012
in revised form 14 November 2012
DOI: 10.5427/jsing.2012.4k

ON A NEWTON FILTRATION FOR FUNCTIONS ON A CURVE SINGULARITY

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Abstract. In a previous paper, there was defined a multi-index filtration on the ring of functions on a hypersurface singularity corresponding to its Newton diagram generalizing (for a curve singularity) the divisorial one. Its Poincaré series was computed for plane curve singularities non-degenerate with respect to their Newton diagrams. Here we use another technique to compute the Poincaré series for plane curve singularities without the assumption that they are non-degenerate with respect to their Newton diagrams. We show that the Poincaré series only depends on the Newton diagram and not on the defining equation.

Introduction

In [2, 3], there were defined two multi-index filtrations on the ring $\mathcal{O}_{\mathbb{C}^n,0}$ of germs of holomorphic functions in n variables associated to a Newton diagram Γ in $\mathbb{R}^n$ and to a germ of an analytic function $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ with this Newton diagram. We assumed that the function f was non-degenerate with respect to its Newton diagram Γ. These filtrations are essentially filtrations on the ring $\mathcal{O}_{V,0} = \mathcal{O}_{\mathbb{C}^n,0}/(f)$ of germs of functions on the hypersurface singularity $V = \{f = 0\}$. They correspond to the quasihomogeneous valuations on the ring $\mathcal{O}_{\mathbb{C}^n,0}$ defined by the facets of the diagram Γ. These facets correspond to some components of the exceptional divisor of a toric resolution of the germ f constructed from the diagram Γ. Such a component defines the corresponding divisorial valuation on the ring $\mathcal{O}_{\mathbb{C}^n,0}$. For $n \geqslant 3$ (and for a Γ-non-degenerate f) these valuations induce divisorial valuations on the ring $\mathcal{O}_{V,0}$ and define the corresponding multi-index filtration on it. The filtration defined in [2] was expected to be a certain "simplification" of the divisorial one. This appeared not to be the case. For example, a general formula for the Poincaré series of this filtration is not known even for the number of variables $n = 2$. For Newton diagrams of special type, A. Lemahieu identified this filtration with a so called embedded filtration on $\mathcal{O}_{V,0}$ [7]. In [7], a formula for the Poincaré series of the embedded filtration for a hypersurface singularity was given. H. Hamm studied the embedded filtration and the corresponding Poincaré series for complete intersection singularities [6].

In [3], there was given an "algebraic" definition of the divisorial valuation corresponding to a Newton diagram (for $n \geqslant 3$) somewhat similar to the definition in [2]. Roughly speaking, the difference consists in using the ring $\mathcal{O}_{\mathbb{C}^n,0}[x_1^{-1}, \ldots, x_n^{-1}]$ instead of $\mathcal{O}_{\mathbb{C}^n,0}$. For $n = 2$, this definition does not give, in general, a valuation, but an order function (see the definition below). For a Γ-non-degenerate $f \in \mathcal{O}_{\mathbb{C}^2,0}$, this order function was described as a "generalized divisorial valuation" defined by the divisorial valuations corresponding to all the points of intersection of the resolution (normalization) $\widetilde{V}$ of the curve V with the corresponding component of the exceptional divisor. This permitted to apply the technique elaborated in [1] and to compute the corresponding Poincaré series. (This technique has no analogue which could be applied to

Partially supported by the DFG (Eb 102/7–1), the Russian government grant 11.G34.31.0005, RFBR–10-01-00678, NSh–4850.2012.1 and Simons–IUM fellowship. Keywords: filtrations, curve singularities, Newton diagrams, Poincaré series. AMS 2010 Math. Subject Classification: 32S05, 14M25, 16W70.

degenerate f, or to the case $n > 2$, or to the filtration defined in [2].) In this case the Poincaré series depends only on the Newton diagram Γ and does not depend on the function f with $\Gamma_f = \Gamma$.

The definitions in [2] and [3] make also sense for functions f degenerate with respect to their Newton diagrams. Here we compute the Poincaré series of the filtration introduced in [3] for $n = 2$ directly from the definition without the assumption that f is non-degenerate with respect to the Newton diagram. We show that the answer is the same as in [3, Corollary 1] for non-degenerate f. Thus, for $n = 2$, the Poincaré series of this filtration depends only on the Newton diagram Γ. One can speculate that the same holds for $n \geqslant 3$ and for the Poincaré series of the filtration defined in [2].

We hope that some elements of the technique used here can be applied to the case $n \geqslant 3$ and/or to the filtration defined in [2] as well.

One motivation to study (multi-variable) Poincaré series of filtrations comes from the fact that they are sometimes related or even coincide with appropriate monodromy zeta functions or with Alexander polynomials (see e.g. [1]). We show that the obtained formula for the Poincaré series has a relation to the (multi-variable) Alexander polynomial of a collection of functions.

1. Filtrations associated to Newton diagrams

Let $(V, 0)$ be a germ of a complex analytic variety and let $\mathcal{O}_{V,0}$ be the ring of germs of holomorphic functions on $(V, 0)$. A map $v : \mathcal{O}_{V,0} \to \mathbb{Z}_{\geqslant 0} \cup \{+\infty\}$ is an *order function* on $\mathcal{O}_{V,0}$ if $v(\lambda g) = v(g)$ for a non-zero $\lambda \in \mathbb{C}$ and $v(g_1 + g_2) \geqslant \min\{v(g_1), v(g_2)\}$. (If, moreover, $v(g_1 g_2) = v(g_1) + v(g_2)$, the map v is a *valuation* on $\mathcal{O}_{V,0}$.) A collection $\{v_1, v_2, \ldots, v_r\}$ of order functions on $\mathcal{O}_{V,0}$ defines a multi-index filtration on $\mathcal{O}_{V,0}$:

$$J(\underline{v}) := \{g \in \mathcal{O}_{V,0} : \underline{v}(g) \geqslant \underline{v}\} \tag{1}$$

for $\underline{v} = (v_1, \ldots, v_r) \in \mathbb{Z}^r_{\geqslant 0}$, $\underline{v}(g) = (v_1(g), \ldots, v_r(g))$, $\underline{v}' = (v'_1, \ldots, v'_r) \geqslant \underline{v}'' = (v''_1, \ldots, v''_r)$ iff $v'_i \geqslant v''_i$ for all $i = 1, \ldots, r$. (It is convenient to assume that the equation (1) defines the subspaces $J(\underline{v}) \subset \mathcal{O}_{V,0}$ for all $\underline{v} \in \mathbb{Z}^r$.) The *Poincaré series* $P_{\{v_i\}}(\underline{t})$ ($\underline{t} = (t_1, \ldots, t_r)$) of the filtration (1) can be defined as

$$P_{\{v_i\}}(\underline{t}) := \frac{\left(\sum_{\underline{v} \in \mathbb{Z}^r} \dim(J(\underline{v})/J(\underline{v}+\underline{1}))\underline{t}^{\underline{v}}\right) \prod_{i=1}^r (t_i - 1)}{(t_1 t_2 \cdots t_r - 1)}, \tag{2}$$

where $\underline{1} = (1, \ldots, 1) \in \mathbb{Z}^r$, $\underline{t}^{\underline{v}} = t_1^{v_1} \cdots t_r^{v_r}$ (see e.g. [1]; it is defined when the dimensions of all the factor spaces $J(\underline{v})/J(\underline{v}+\underline{1})$ are finite). In [1] it was explained that the Poincaré series (2) is equal to the integral with respect to the Euler characteristic

$$P_{\{v_i\}}(\underline{t}) = \int_{\mathbb{P}\mathcal{O}_{V,0}} \underline{t}^{\underline{v}(g)} d\chi \tag{3}$$

over the projectivization $\mathbb{P}\mathcal{O}_{V,0}$ of the space $\mathcal{O}_{V,0}$. (In the integral, $t_i^{+\infty}$ has to be assumed to be equal to zero.)

Let $f \in \mathcal{O}_{\mathbb{C}^n,0}$ be a function germ with the Newton diagram $\Gamma = \Gamma_f \subset \mathbb{R}^n$, $V := \{f = 0\}$. Let γ_i, $i = 1, \ldots, r$, be (all) the facets of the diagram Γ and let $\ell_i(\bar{k}) = c_i$ be the reduced equation of the hyperplane containing the facet γ_i. One has $\ell_i(\bar{k}) = \sum_{j=1}^n \ell_{ij} k_j$ ($\bar{k} = (k_1, \ldots, k_n)$), where ℓ_{ij} are positive integers, $\gcd(\ell_{i1}, \ldots, \ell_{in}) = 1$.

For $g \in \mathcal{O}_{\mathbb{C}^n,0}[x_1^{-1}, \ldots, x_n^{-1}]$, $g = \sum\limits_{\bar{k} \in \mathbb{Z}^n} a_{\bar{k}} \bar{x}^{\bar{k}}$ ($\bar{x} = (x_1, \ldots, x_n)$), let $\operatorname{supp} g := \{\bar{k} \in \mathbb{Z}^n : a_{\bar{k}} \neq 0\}$ and let

$$u_i(g) := \min_{\bar{k} : a_{\bar{k}} \neq 0} \ell_i(\bar{k}).$$

One can see that u_i is a valuation on $\mathcal{O}_{\mathbb{C}^n,0} \subset \mathcal{O}_{\mathbb{C}^n,0}[x_1^{-1},\dots,x_n^{-1}]$. For a Newton diagram Λ in $\mathbb{R}^n$, let

$$u_i(\Lambda) := \min_{\bar{k}\in\Lambda} \ell_i(\bar{k}).$$

(It is also equal to $u_i(g)$ for any germ g with the Newton diagram Λ.) Let $g_{\gamma_i}(\bar{x}) := \sum\limits_{\bar{k}:\ell_i(\bar{k})=u_i(g)} a_{\bar{k}}\bar{x}^{\bar{k}}$.

The following two collections of order functions on $\mathcal{O}_{\mathbb{C}^n,0}$ corresponding to the pair (Γ, f) were defined in [2] and [3] respectively:

$$v_i'(g) := \sup_{h\in\mathcal{O}_{\mathbb{C}^n,0}} u_i(g+hf)\,, \tag{4}$$

$$v_i''(g) := \sup_{h\in\mathcal{O}_{\mathbb{C}^n,0}[x_1^{-1},\dots,x_n^{-1}]} u_i(g+hf)\,. \tag{5}$$

(v_i' and v_i'' are, in general, not valuations, at least when $n=2$ or when f is degenerate with respect to its Newton diagram Γ.) They can be considered as order functions on the ring $\mathcal{O}_{V,0} = \mathcal{O}_{\mathbb{C}^n,0}/(f)$ as well. (These order functions and moreover the corresponding Poincaré series are, in general, different.)

Assume that the function f is non-degenerate with respect to its Newton diagram Γ and let $p : (X,D) \to (\mathbb{C}^n, 0)$ be a toric resolution of f corresponding to the Newton diagram Γ. The facets $\gamma_1, \dots, \gamma_r$ of Γ correspond to some components (say, $E_1, \dots, E_r$) of the exceptional divisor D. Let $\widetilde{V}$ be the strict transform of the hypersurface singularity V (it is a smooth complex manifold) and let $\mathcal{E}_i := \widetilde{V}\cap E_i$, $i=1,\dots,r$.

For $n \geqslant 3$, the set $\mathcal{E}_i$ is an irreducible component of the exceptional divisor $\mathcal{D} = D\cap\widetilde{V}$ of the resolution $p_{|\widetilde{V}} : (\widetilde{V},\mathcal{D}) \to (V,0)$. The divisorial valuation $v_{\mathcal{E}_i}$ on $\mathcal{O}_{V,0}$ defined by this component coincides with v_i'': see [3]. For $n=2$, the set $\mathcal{E}_i$ is, in general, reducible (if the integer length s_i of the facet (edge) γ_i is greater than 1). Let $\mathcal{E}_i = \bigcup\limits_{j=1}^{s_i} \mathcal{E}_i^{(j)}$ be the decomposition into irreducible components ($\mathcal{E}_i^{(j)}$ are points on the curve $\widetilde{V}$). One can show that in this case $v_i''(g) = \min\limits_j v_{\mathcal{E}_i^{(j)}}(g)$, where $v_{\mathcal{E}_i^{(j)}}$ are the corresponding divisorial valuations on $\mathcal{O}_{V,0}$. This order function v_i'' can be regarded as a generalized divisorial valuation.

2. The Poincaré series

Let Γ be a Newton diagram in $\mathbb{R}^2$ with the facets (edges) $\gamma_1, \dots, \gamma_r$ and let f be a function germ $(\mathbb{C}^2,0)\to(\mathbb{C},0)$ with the Newton diagram Γ. One can see that $f = x^a y^b \prod\limits_{i=1}^{r} f_i$, where f_i is such that $f_{\gamma_i} = \lambda_i \bar{x}^{\bar{k}_i}(f_i)_{\gamma_i}$ for certain $\lambda_i\in\mathbb{C}^*$ and $\bar{k}_i \in \mathbb{Z}^2_{\geqslant 0}$. The Newton diagram Γ_i of the germ f_i consists of one segment congruent (by a shift; in particular, parallel) to the facet γ_i with the vertices on the coordinate lines in $\mathbb{R}^2$.

Let $\underline{M}_i := \underline{u}(\Gamma_i)$, i.e. $\underline{M}_i = (M_{i1},\dots,M_{ir})$, where $M_{ij} := u_j(\Gamma_i)$. (One can see that $\underline{M}_i = s_i\underline{m}_i$ in the notations of [3].)

Theorem 1. *One has*

$$P_{\{v_i''\}}(\underline{t}) = \frac{\prod\limits_{i=1}^{r}(1-\underline{t}^{\underline{M}_i})}{(1-\underline{t}^{\underline{u}(x)})(1-\underline{t}^{\underline{u}(y)})}\,. \tag{6}$$

Corollary 1. *For the number of variables $n=2$, the Poincaré series $P_{\{v_i''\}}(\underline{t})$ depends only on the Newton diagram Γ and does not depend on f with $\Gamma_f = \Gamma$.*

For the proof of Theorem 1 we need some auxiliary statements. We first introduce some notations.

For a Newton diagram Λ in $\mathbb{R}^2$, let Σ_Λ be the corresponding Newton polygon: $\Sigma_\Lambda := \bigcup_{\bar{q}\in\Lambda} \left(\bar{q} + \mathbb{R}^2_{\geqslant 0}\right)$. Let $\mathcal{O}^\Lambda$ be the set of functions $g \in \mathcal{O}_{\mathbb{C}^2,0}$ with the Newton diagram $\Gamma_g = \Lambda$. For $\underline{v} \in \mathbb{Z}^r_{\geqslant 0}$, let $\mathcal{O}^\Lambda_{\underline{v}} := \{g \in \mathcal{O}^\Lambda : \underline{v}''(g) = \underline{v}\}$. The set $\mathcal{O}_{\mathbb{C}^2,0} \setminus \{0\}$ is the disjoint union of the sets $\mathcal{O}^\Lambda$ over all diagrams Λ. According to (3) one has

$$P_{\{v''_i\}}(\underline{t}) = \sum_{\Lambda} \int_{\mathbb{P}\mathcal{O}^\Lambda} \underline{t}^{\underline{v}''(g)} d\chi\,. \tag{7}$$

For $g \in \mathcal{O}^\Lambda$, one has $v''_i(g) \geqslant u_i(\Lambda)$. We shall first show that the integrals in (7) can be restricted only to functions $g \in \mathbb{P}\mathcal{O}^\Lambda$ with $\underline{v}''(g) = \underline{u}(\Lambda)$.

Proposition 1. *For a Newton diagram Λ in $\mathbb{R}^2$, let $\underline{v} \in \mathbb{Z}^r_{\geqslant 0}$ be such that $\underline{v} > \underline{u}(\Lambda)$, i.e. $v_i \geqslant u_i(\Lambda)$ for all $i = 1, \ldots, r$ and $v_i > u_i(\Lambda)$ for some i. Then the set $\mathbb{P}\mathcal{O}^\Lambda_{\underline{v}}$ has Euler characteristic equal to zero.*

Remark. The direct analogue of this proposition does not hold for the filtration defined by the order functions $\{v'_i\}$. As an example one can take $f(x,y) = y^5 + xy^2 + x^2y + x^5$ whose Newton diagram Γ has the set of vertices $\{(0,5),(1,2),(2,1),(5,0)\}$. One has $\ell_1(\bar{k}) = 3k_x + k_y$, $\ell_2(\bar{k}) = k_x + k_y$, $\ell_3(\bar{k}) = k_x + 3k_y$. Let Λ be the Newton diagram with the set of vertices $\{(0,5),(1,2)\}$. One has $\underline{u}(\Lambda) = (5,3,7)$. Let us take $\underline{v} = (7,3,7)$. One can see that for the order functions $\{v'_i\}$ the set $\mathcal{O}^\Lambda_{\underline{v}}$ consists of the germs $g(x,y) = \sum a_{ij}x^iy^j$ from $\mathcal{O}^\Lambda$ with $a_{05} = a_{12} \neq 0$ and $a_{06} = a_{13} = 0$. This gives $\chi(\mathbb{P}\mathcal{O}^\Lambda_{\underline{v}}) = 1$. For the order functions $\{v''_i\}$ the set $\mathcal{O}^\Lambda_{\underline{v}}$ consists of the germs with $a_{05} = a_{12} \neq 0$, $a_{06} = a_{13} = 0$ and $a_{07} - a_{14} + a_{21} \neq 0$. This gives $\chi(\mathbb{P}\mathcal{O}^\Lambda_{\underline{v}}) = 0$ in accordance with Proposition 1.

For the proof of Proposition 1 we need two lemmas.

Let $\mathcal{O}^\Lambda_{\underline{v}}$ be non-empty. For $g \in \mathcal{O}^\Lambda$ with $\underline{v}''(g) = \underline{v}$ and for $i = 1, \ldots, r$, one can find $h(= h_i) \in \mathcal{O}_{\mathbb{C}^2,0}[x^{-1}, y^{-1}]$ such that the Newton diagram of $g + hf$ lies in the (closed) half-plane $H_i := \{\bar{k} : \ell_i(\bar{k}) \geqslant v_i\}$, but there are no h for which the Newton diagram of $g + hf$ lies in the open half-plane $\{\bar{k} : \ell_i(\bar{k}) > v_i\}$. Let Λ^* be the union of the compact edges of the (infinite) polygon $\Sigma^*_\Lambda := \left(\bigcap_{i=1}^r H_i\right) \cap \Sigma_\Lambda$, where Σ_Λ is the Newton polygon corresponding to Λ. (Λ^* is not, in general, a Newton diagram since it may have non-integral vertices. Nevertheless we shall use the name "diagram" for it.)

Lemma 1. *In the situation described above, there exists an index i ($1 \leqslant i \leqslant r$) such that Λ^* has an edge δ_i parallel to γ_i and (strictly) longer than γ_i.*

Proof. We shall prove that there exists an edge of the diagram Λ^* which is (strictly) longer than the edge of Λ parallel to it. This implies that this edge is parallel to a certain edge γ_i of the diagram Γ and is longer than it. Since we assumed $\mathcal{O}^\Lambda_{\underline{v}}$ being non-empty, all edges of Λ^* are parallel to edges of Λ. Let $a_0 < a_1 < \ldots < a_\sigma$ be the k_x-coordinates of all the vertices of Λ. Let $b_0 \leqslant b_1 \leqslant \ldots \leqslant b_\sigma$ be the k_x-coordinates of the corresponding vertices of Λ^*, i.e. $[b_{i-1}, b_i]$ is the projection of the segment of Λ^* parallel to the segment of Λ projected to $[a_{i-1}, a_i]$ (b_{i-1} and b_i may coincide). One can see that $a_0 = b_0$ and $a_\sigma \leqslant b_\sigma$. Then either $b_i = a_i$ for all $i = 0, 1, \ldots, \sigma$ or $[a_{i-1}, a_i] \subsetneq [b_{i-1}, b_i]$ for some $i \in \{0, 1, \ldots, \sigma\}$. But the first case cannot happen since $\Lambda^* \neq \Lambda$. $\square$

We shall also use the following generalized version of the division with remainder for Laurent polynomials.

Lemma 2. *Let $p(z)$ and $q(z)$ be Laurent polynomials in z. Assume that* $\operatorname{supp} q$ *has length s, i.e. $q(z) = \sum\limits_{i=0}^{s} b_i z^{d+i}$ with $b_0 \neq 0$, $b_s \neq 0$, and let d' be an integer. Then the polynomial $p(z)$ has a unique representation of the form $p(z) = q(z)a(z) + r(z)$ with $r(z) = \sum\limits_{i=0}^{s-1} c_i z^{d'+i}$.*

Proof of Proposition 1. Let i be as in Lemma 1. Let the integer length of γ_i be equal to s_i. Then the segment δ_i contains at least s_i integer points. Let $Q_1, \ldots, Q_{s_i}$ be s_i consecutive integer points on the segment δ_i. Let $g \in \mathcal{O}^\Lambda$ be such that $\underline{v}''(g) = \underline{\upsilon}$ $(> \underline{u}(\Lambda))$ and let $\widetilde{g} = g + hf$ be a Laurent polynomial such that $\operatorname{supp} \widetilde{g} \subset H_i = \{\ell_i(\bar{k}) \geqslant \upsilon_i\}$. Lemma 2 implies that $\widetilde{g}_{\gamma_i}(x,y) = f_{\gamma_i}(x,y)p_i(x,y) + r_i(x,y)$, where $\operatorname{supp} r_i \subset \{Q_1, \ldots, Q_{s_i}\}$ and $r_i \neq 0$ (otherwise $v_i(g) > \upsilon_i = u_i(\Lambda^*)$). (Let us recall that $\operatorname{supp} f_{\gamma_i}$ consists of $s_i + 1$ consecutive points on the line containing γ_i.) Moreover the polynomial r_i depends only on g and does not depend on the choice of $\widetilde{g}$ (i.e. on the choice of h).

One can see that all functions g' of the form $g + (\lambda - 1)r_i$ with $\lambda \neq 0$ lie in $\mathcal{O}^\Lambda$ and satisfy the condition $v_i''(g') = v_i''(g)$. Thus the set $\mathbb{P}\mathcal{O}^\Lambda_{\underline{\upsilon}}$ is fibred by $\mathbb{C}^*$-families and therefore its Euler characteristic is equal to zero. □

Proposition 1 implies that

$$P_{\{v_i''\}}(\underline{t}) = \sum_{\Lambda} \int_{\mathbb{P}\mathcal{O}^\Lambda_{\underline{u}(\Lambda)}} \underline{t}^{\underline{v}''(g)} d\chi \,. \tag{8}$$

Proposition 2. *Suppose that a Newton diagram Λ contains an edge δ not congruent to any edge of Γ, i.e. either not parallel to all the edges γ_i, or parallel to one of them, but of another length. Then $\chi(\mathbb{P}\mathcal{O}^\Lambda_{\underline{u}(\Lambda)}) = 0$.*

Proof. Assume first that the edge δ is either not parallel to all the edges γ_i, or it is parallel to γ_i, but is shorter than γ_i. Let $\bar{q} = (q_x, q_y)$ and $\bar{q}' = (q_x', q_y')$, $q_x > q_x'$, be the vertices of the edge δ and let Λ' be the set of points $\bar{k}$ in Λ with $k_x \geqslant q_x$. A function germ $g \in \mathcal{O}^\Lambda_{\underline{u}(\Lambda)}$ can be represented as $g_1 + g_2$, where $\operatorname{supp} g_1 \subset \Lambda'$, $\operatorname{supp} g_2 \subset \Sigma_\Lambda \setminus \Lambda'$. (Note that $g_1 \neq 0$ and $g_2 \neq 0$.) One can see that all the functions of the form $g_1 + \lambda g_2$ with $\lambda \neq 0$ lie in $\mathcal{O}^\Lambda_{\underline{u}(\Lambda)}$. Thus the set $\mathbb{P}\mathcal{O}^\Lambda_{\underline{u}(\Lambda)}$ is fibred by $\mathbb{C}^*$-families and therefore its Euler characteristic is equal to zero.

Now assume that the edge δ of the diagram Λ is parallel to γ_i and is longer than it. Let $\bar{q} = (q_x, q_y)$ and $\bar{q}' = (q_x', q_y')$, $q_x > q_x'$, (respectively $\bar{q}_0 = (q_{0x}, q_{0y})$ and $\bar{q}_0' = (q_{0x}', q_{0y}')$, $q_{0x} > q_{0x}'$) be the vertices of the edge δ (respectively of the edge γ_i) and let Λ' be defined as above: $\Lambda' = \{\bar{k} \in \Lambda : k_x \geqslant q_x\}$. Let $g(\bar{x}) = \sum a_{\bar{k}} \bar{x}^{\bar{k}} \in \mathcal{O}^\Lambda_{\underline{u}(\Lambda)}$, $f(\bar{x}) = \sum c_{\bar{k}} \bar{x}^{\bar{k}}$ $(\bar{x} = (x, y))$ and let

$$g_1(\bar{x}) = g_{\Lambda'}(\bar{x}) - a_{\bar{q}} \bar{x}^{\bar{q}} + (a_{\bar{q}}/c_{\bar{q}_0}) f_{\gamma_i}(\bar{x}) \cdot \bar{x}^{\bar{q} - \bar{q}_0},$$

where $g_{\Lambda'}(\bar{x}) = \sum\limits_{\bar{k} \in \Lambda'} a_{\bar{k}} \bar{x}^{\bar{k}}$, $g_2 = g - g_1$. One has $\operatorname{supp} g_1 \subset \Lambda' \cup (\bar{q}, \bar{q}')$, $\operatorname{supp} g_2 \subset \Sigma_\Lambda \setminus \Lambda'$, $f_{\gamma_i} \not| (g_2)_{\gamma_i}$ (in $\mathcal{O}_{\mathbb{C}^2,0}[x^{-1}, y^{-1}]$), where $(\bar{q}, \bar{q}')$ denotes the open line segment connecting the two points. All the functions of the form $g_1 + \lambda g_2$ with $\lambda \neq 0$ lie in $\mathcal{O}^\Lambda_{\underline{u}(\Lambda)}$. Thus the set $\mathbb{P}\mathcal{O}^\Lambda_{\underline{u}(\Lambda)}$ is again fibred by $\mathbb{C}^*$-families and therefore its Euler characteristic is equal to zero. □

Proposition 3. *Let the Newton diagram Λ consist (only) of segments congruent to γ_i for $i \in I \subset \{1, \ldots, r\}$. Then $\chi(\mathbb{P}\mathcal{O}^\Lambda_{\underline{u}(\Lambda)}) = (-1)^{\#I}$.*

Proof. For $I = \emptyset$, the statement is obvious. Let $I \neq \emptyset$. Let $\mathbb{P}\mathcal{O}_i^\Lambda$ be the set of functions $g \in \mathbb{P}\mathcal{O}^\Lambda_{\underline{u}(\Lambda)}$ with $f_{\gamma_i} | g_{\gamma_i}$ (in $\mathcal{O}_{\mathbb{C}^2,0}[x^{-1}, y^{-1}]$). One has

$$\mathbb{P}\mathcal{O}^\Lambda_{\underline{u}(\Lambda)} = \mathbb{P}\mathcal{O}^\Lambda \setminus \bigcup_{i\in I} \mathbb{P}\mathcal{O}_i^\Lambda .$$

Therefore

$$\chi(\mathbb{P}\mathcal{O}^\Lambda_{\underline{u}(\Lambda)}) = \chi(\mathbb{P}\mathcal{O}^\Lambda) + \sum_{I'\subset I, I'\neq\emptyset} (-1)^{\#I'} \chi\left(\bigcap_{i\in I'} \mathbb{P}\mathcal{O}_i^\Lambda\right) . \tag{9}$$

Let $\bar{q}_i$, $i = 0, 1, \ldots, \#I$, be the vertices of the diagram Λ. The set $\mathbb{P}\mathcal{O}^\Lambda$ consists of functions $g(\bar{x}) = \sum a_{\bar{k}} \bar{x}^{\bar{k}}$ with $a_{\bar{q}_i} \neq 0$ for $i = 0, 1, \ldots, \#I$ and $a_{\bar{k}} = 0$ for $\bar{k} \notin \Sigma^\Lambda$. Its Euler characteristic is equal to zero. Assume that $I' \subsetneq I$, $I' \neq \emptyset$. Let $[\bar{q}, \bar{q}']$ be an edge of Λ congruent to γ_i, $i \in I \setminus I'$ ($\bar{q} = (q_x, q_y)$, $\bar{q}' = (q'_x, q'_y)$, $q_x > q'_y$). Let $\Lambda' = \{\bar{k} \in \Lambda : k_x \geqslant q_x\}$. For $g \in \bigcap_{i\in I'} \mathbb{P}\mathcal{O}_i^\Lambda$, let $g_1(\bar{x}) = g_{\Lambda'}(\bar{x})$, $g_2 = g - g_1$. All the functions of the form $g_1 + \lambda g_2$ with $\lambda \neq 0$ belong to $\bigcap_{i\in I'} \mathbb{P}\mathcal{O}_i^\Lambda$. Thus $\bigcap_{i\in I'} \mathbb{P}\mathcal{O}_i^\Lambda$ is fibred by $\mathbb{C}^*$-families and therefore $\chi(\bigcap_{i\in I'} \mathbb{P}\mathcal{O}_i^\Lambda) = 0$.

Let $f_I(\bar{x}) := \prod_{i\in I} f_i(\bar{x})$. The intersection $\bigcap_{i\in I} \mathbb{P}\mathcal{O}_i^\Lambda$ (the Euler characteristic of which corresponds to $I' = I$ in (9)) consists of the functions $g \in \mathbb{P}\mathcal{O}_{\mathbb{C}^2,0}$ such that $g_\Lambda(\bar{x}) = \lambda \bar{x}^{\bar{a}} (f_I)_{\Gamma_{f_I}}$ where Γ_{f_I} is the Newton diagram of f_I, $\lambda \neq 0$ and $\bar{x}^{\bar{a}}$ is a certain monomial. Therefore

$$\chi\left(\bigcap_{i\in I} \mathbb{P}\mathcal{O}_i^\Lambda\right) = 1.$$

□

Proof of Theorem 1. Propositions 1 and 2 imply that

$$P_{\{v_i''\}}(\underline{t}) = \sum_\Lambda \int_{\mathbb{P}\mathcal{O}^\Lambda_{\underline{u}(\Lambda)}} \underline{t}^{\underline{v}''(g)} d\chi \tag{10}$$

where the sum runs over all diagrams Λ consisting only of edges congruent to some of the edges γ_i of the diagram Γ. Let the edges of Λ be congruent to the edges γ_i with $i \in I = I(\Lambda)$. Proposition 3 implies that the summand in (10) corresponding to such a diagram Λ is equal to $(-1)^{\#I} \underline{t}^{\underline{u}(\Lambda)}$. All the diagrams of this sort are obtained from the diagrams $\Gamma_I = \Gamma_{f_I}$ by shifts by non-negative integral vectors $\bar{k}$, i.e. $\Lambda = \bar{k} + \Gamma_I$. One has $\underline{u}(\Lambda) = \underline{\ell}(\bar{k}) + \sum_{i\in I} \underline{M}_i$. Therefore

$$\begin{aligned} P_{\{v_i''\}}(\underline{t}) &= \sum_{\bar{k}\in\mathbb{Z}^2_{\geqslant 0}} \sum_{I\subset\{1,\ldots,r\}} (-1)^{\#I} \underline{t}^{\underline{\ell}(\bar{k}) + \sum_{i\in I} \underline{M}_i} \\ &= \left(\sum_{\bar{k}\in\mathbb{Z}^2_{\geqslant 0}} \underline{t}^{\underline{\ell}(\bar{k})}\right) \cdot \left(\sum_{I\subset\{1,\ldots,r\}} (-1)^{\#I} \underline{t}^{\sum_{i\in I} \underline{M}_i}\right) \\ &= \frac{\prod_{i=1}^r (1 - \underline{t}^{\underline{M}_i})}{(1 - \underline{t}^{\underline{u}(x)})(1 - \underline{t}^{\underline{u}(y)})}. \end{aligned}$$

□

3. Relation with an Alexander polynomial

One can see that the equation (6) gives the Poincaré series $P_{\{v''_i\}}(\underline{t})$ as a finite product/ratio of "cyclotomic" binomials of the form $(1-\underline{t}^{\underline{M}})$ with $\underline{M} \in \mathbb{Z}^r_{>0}$. This looks similar to the usual A'Campo type expressions for monodromy zeta functions or for Alexander polynomials of algebraic links [4]. Here we shall describe a relation between the Poincaré series (6) and a certain Alexander polynomial.

A notion of the multi-variable Alexander polynomial for a finite collection of germs of functions on $(\mathbb{C}^n, 0)$ was defined in [8]: see Proposition 2.6.2 therein. (In [8] it is called the (multi-variable) zeta function.) In a somewhat more precise form this definition can be found in [5]. (The definition in [5] gives the one for a collection of functions if one considers the corresponding principal ideals.)

As above, let Γ be a Newton diagram in $\mathbb{R}^2$ with the edges $\gamma_1, \ldots, \gamma_r$ of integer lengths $s_1, \ldots, s_r$ and let $p : (X, D) \to (\mathbb{C}^2, 0)$ be a toric modification of $(\mathbb{C}^2, 0)$ corresponding to the diagram Γ. For $i = 1, \ldots, r$, let $\widetilde{C}_i$ be a germ of a smooth curve on X transversal to the component E_i of the exceptional divisor D. Let $C_i = p(\widetilde{C}_i)$ be the image of $\widetilde{C}_i$ in $(\mathbb{C}^2, 0)$ (a *curvette* corresponding to the component E_i) and let $L_i = C_i \cap S^3_\varepsilon$ be the corresponding knot in the 3 sphere $S^3_\varepsilon = S^3_\varepsilon(0)$ for $\varepsilon > 0$ small enough. The curve C_i can be defined by an equation $g_i = 0$ where g_i is a function germ $(\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ with the Newton diagram consisting of one segment parallel to γ_i, with (integer) length 1 and with the vertices on the coordinate lines.

Let $\Delta_{\underline{g}}(\underline{t})$ and $\Delta_{\underline{g}^{\underline{s}}}(\underline{t})$ be the Alexander polynomials of the collections of functions $\underline{g} = (g_1, \ldots, g_r)$ and $\underline{g}^{\underline{s}} = (g_1^{s_1}, \ldots, g_r^{s_r})$ respectively. The polynomial $\Delta_{\underline{g}}(\underline{t})$ is the classical Alexander polynomial $\Delta^L(\underline{t})$ of the link $L = \bigcup L_i$ (see e.g. [4]). A one-variable analogue of $\Delta_{\underline{g}^{\underline{s}}}(\underline{t})$ is considered in [4, I.5] as the Alexander polynomial of the multilink $L(\underline{s}) := \bigcup s_i L_i$. One has

$$\Delta_{\underline{g}}(\underline{t}) = \frac{\prod_{i=1}^r (1-\underline{t}^{\underline{m}_i})}{(1-\underline{t}^{\underline{u}(x)})(1-\underline{t}^{\underline{u}(y)})}\,,$$

$$\Delta_{\underline{g}^{\underline{s}}}(\underline{t}) = \frac{\prod_{i=1}^r (1-\underline{t}^{\underline{s}\,\underline{m}_i})}{(1-\underline{t}^{\underline{u}(x)})(1-\underline{t}^{\underline{u}(y)})}\,,$$

where $\underline{s}\,\underline{m}_i = (s_1 m_{1i}, s_2 m_{2i}, \ldots, s_r m_{ri})$. The main result of [1] says that $\Delta_{\underline{g}}(\underline{t}) = \Delta^L(\underline{t})$ coincides with the Poincaré series of the filtration corresponding to the Newton diagram of the function $\prod_{i=1}^r g_i$.

Let the *reduced Poincaré series* of the filtration defined by $\{v''_i\}$ be

$$\widetilde{P}_{\{v''_i\}}(\underline{t}) := P_{\{v''_i\}}(\underline{t}) / P_{\{u_i\}}(\underline{t}), \quad \text{where} \quad P_{\{u_i\}}(\underline{t}) = \frac{1}{(1-\underline{t}^{\underline{u}(x)})(1-\underline{t}^{\underline{u}(y)})}$$

is the Poincaré series of the filtration defined by the quasihomogeneous valuations $\{u_i\}$ on $\mathcal{O}_{\mathbb{C}^2,0}$. One has

$$\widetilde{P}_{\{v''_i\}}(\underline{t}) = \prod_{i=1}^r (1-\underline{t}^{s_i \underline{m}_i}). \tag{11}$$

Let

$$\widetilde{\Delta}_{\underline{g}^{\underline{s}}}(\underline{t}) := \Delta_{\underline{g}^{\underline{s}}}(\underline{t}) / \Delta^x_{\underline{g}^{\underline{s}}}(\underline{t}) \cdot \Delta^y_{\underline{g}^{\underline{s}}}(\underline{t})$$

where $\Delta^x_{\underline{g}^{\underline{s}}}(\underline{t})$ and $\Delta^y_{\underline{g}^{\underline{s}}}(\underline{t})$ are the Alexander polynomials of the sets of functions $\underline{g}^{\underline{s}} = \{g_1^{s_1}, \ldots, g_r^{s_r}\}$ restricted to the coordinate axes $\mathbb{C}_x$ and $\mathbb{C}_y$ respectively. One can regard $\widetilde{\Delta}_{\underline{g}^{\underline{s}}}(\underline{t})$ as the Alexander polynomial of the set of functions $\underline{g}^{\underline{s}}$ restricted to the complex torus $(\mathbb{C}^*)^2 \subset \mathbb{C}^2$. One has

$$\widetilde{\Delta}_{\underline{g}^{\underline{s}}}(\underline{t}) = \prod_{i=1}^{r}(1 - \underline{t}^{\underline{s}\,\underline{m}_i}). \tag{12}$$

One can see that a relation between (11) and (12) can be described in the following way. Consider products of r ordered cyclotomic binomials in r variables. Such a product

$$\prod_{i=1}^{r}(1 - \underline{t}^{\underline{N}_i}), \quad \underline{N}_i = (N_{i1}, \ldots, N_{ir}),$$

can be described by the corresponding $r \times r$-matrix $N := (N_{ij})$. The transposition of the matrix induces an involution on the set of products of this sort. One can see that this involution maps the product (11) for the Poincaré series to the product (12) for the Alexander polynomial.

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Journal *of* Singularities
Volume 4 (2012), 188-195

received 29 July 2012
DOI: 10.5427/jsing.2012.4l

ON RELATIVE INVARIANTS AND DETERMINACY OF PLANE CURVES

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Introduction

In current work on flat singularity theory, I have been led to consider the local invariants of a curve C relative to the divisor D defined by its tangent cone. It turned out in the calculation of these invariants that the fact that D consisted of the tangent cone was not used, only the fact that it was a collection of lines through the origin. This led me to ask whether any restriction at all on D was required. The first object of this note is to show that it is not. The second object is to obtain general estimates for the degree of determinacy of C relative to D.

We begin by recalling some standard notations and ideas of singularity theory. Then we apply them to plane curves, which may be given either by a parametrisation f or by an equation ϕ. Throughout we study only germs (at the origin) of curves, so omit 'germ' from our terminology. We work throughout in the complex analytic framework. All curves will be assumed to be reduced.

Next we will recall the definitions of the invariants $\delta(C)$, $\mu(C)$ and $\tau(C)$ and the calculations of the codimensions $d_e(f, \mathcal{L}) = 2\delta(C)$, $d_e(f, \mathcal{A}) = \tau(C) - \delta(C)$, $d_e(\phi, \mathcal{R}) = \mu(C)$, and $d_e(\phi, \mathcal{K}) = \tau(C)$. We then introduce the relative versions of the above concepts. Our first main result is the calculation of the codimensions in the relative case: the results are

$$d_e(f, \mathcal{L}_D) = 2\delta(C) + C.D, \quad d_e(f, \mathcal{A}_D) = \tau(C \cup D) - \delta(C) - C.D - \tau(D),$$

$$d_e(\phi, \mathcal{R}_D) = \mu(C) + C.D + \mu(D) - 1 - \tau(D), \quad d_e(\phi, \mathcal{K}_D) = \tau(C \cup D) - C.D - \tau(D).$$

The degree of determinacy for C up to right or left equivalence is bounded by the Milnor number $\mu(C)$. In the last section we obtain corresponding bounds for C relative to D; such bounds are also needed for the work on flat singularity theory.

1. Singularities of plane curves

We now recall the methods and notations of singularity theory, following Mather [5] (see e.g. [8]). Write $\mathcal{O}_x$ for the ring of germs of functions on N at x and $\mathfrak{m}_x$ for its maximal ideal. Denote the tangent bundle $\pi_N : TN \to N$ and write θ_N for the set of germs at x of sections of π_N (i.e. vector fields on N); we think of θ_N as the tangent space at the identity to the group $Diff(N, x)$ of germs of diffeomorphisms. Introduce corresponding notations for (P, y).

For $g : (N, x) \to (P, y)$ a map-germ, we consider the diagram

$$\begin{array}{ccc} TN & \xrightarrow{Tg} & TP \\ \pi_N \downarrow & & \pi_P \downarrow \\ N & \xrightarrow{g} & P \end{array},$$

and write θ_g for the set of germs of maps $\xi : N \to TP$ with $\pi_P \circ \xi = g$. Then composition with Tg induces a map $tg : \theta_N \to \theta_g$ which we think of as the tangent map to the action of $\mathcal{R} = Diff(N)$ on $Map(N, P)$ by composition; composition with g induces a map $\omega g : \theta_P \to \theta_g$ tangent to the action of $\mathcal{L} = Diff(P)$. Set $\mathcal{A} := \mathcal{R} \times \mathcal{L}$.

The respective images of tg and ωg are denoted $T\mathcal{R}^e g$ and $T\mathcal{L}^e g$, with sum $T\mathcal{A}^e g$. We write $T\mathcal{R}g := tg(\mathfrak{m}_x.\theta_N)$, $T\mathcal{L}g := \omega g(\mathfrak{m}_y.\theta_P)$ and $T\mathcal{A}g := T\mathcal{R}g + T\mathcal{L}g$. In the classification of map-germs up to $\mathcal{A}$-equivalence, $T\mathcal{A}g$ serves as the tangent space to the $\mathcal{A}$-equivalence class of g, but for unfolding theory we no longer fix the source and target points, so use the extended tangent space $T\mathcal{A}^e g$; similarly for $\mathcal{R}$ and $\mathcal{L}$. We also set $T\mathcal{C}g := g^*\mathfrak{m}_y.\theta_g$, $T\mathcal{K}g := T\mathcal{R}g + T\mathcal{C}g$ and $T\mathcal{K}^e g := T\mathcal{R}^e g + T\mathcal{C}g$.

Following the notation introduced in [8], for any equivalence relation $\mathcal{B}$ on map-germs, I write $d_e(g, \mathcal{B}) := \dim_{\mathbb{C}}(\theta_g/T\mathcal{B}^e g)$. This is the dimension of the miniversal unfolding space for g under $\mathcal{B}$-equivalence.

A plane curve C may be given as $\phi^{-1}(0)$ for an equation $\phi : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ and (if r denotes the number of branches) as the image of a parametrisation $f : \bigcup_{i=1}^r(\mathbb{C}, 0) \to (\mathbb{C}^2, 0)$. From now on, we write $\mathcal{O}_{x,y}$ for the local ring at the origin in $\mathbb{C}^2$, $\mathfrak{m}_{x,y}$ for its maximal ideal and $\theta_{x,y}$ for the set of germs of vector fields, which is a free $\mathcal{O}_{x,y}-$module with basis $\{\partial_x, \partial_y\}$ (where we write ∂_x for $\partial/\partial x$); the corresponding items for $\mathbb{C}$ are denoted $\mathfrak{m}_t \lhd \mathcal{O}_t$ and θ_t. We denote the source variables of f by t_i ($1 \le i \le r$), with local rings $\mathcal{O}_{t_i}$ and the constituent maps f_i, and set $\mathcal{O}_T := \bigoplus_i \mathcal{O}_{t_i}$, $\mathfrak{m}_T := \oplus_i \mathfrak{m}_{t_i}$ and $\theta_T := \oplus_i \theta_{t_i}$.

The module θ_f is free over $\mathcal{O}_T$ on ∂_x, ∂_y. The map $tf : \theta_T \to \theta_f$ is the sum of the maps $tf_i : \theta_{t_i} \to \theta_{f_i}$ induced by df_i/dt, and the map $\omega f : \theta_{x,y} \to \theta_T$ agrees on each co-ordinate with the ring homomorphism $f^* : \mathcal{O}_{x,y} \to \mathcal{O}_T$. The local ring of C is defined to be $\mathcal{O}_C := f^*\mathcal{O}_{x,y}$; its integral closure in its quotient ring coincides with $\mathcal{O}_T$; as the kernel of $f^* : \mathcal{O}_{x,y} \to \mathcal{O}_T$ is the ideal $\langle\phi\rangle$, we can also identify $\mathcal{O}_C$ with $\mathcal{O}_{x,y}/\langle\phi\rangle$. The module θ_ϕ is free over $\mathcal{O}_{x,y}$ on a single generator, and we identify it with this ring; $t\phi(\theta_{x,y})$ is the (Jacobian) ideal $\langle\phi_x, \phi_y\rangle$ (where we write ϕ_x for $\partial_x\phi$), and $\phi^*\mathfrak{m}_t.\theta_\phi$ is the ideal $\langle\phi\rangle$.

We say that two curves C and C' are equivalent if there is a local diffeomorphism of $\mathbb{C}^2$ taking C to C': this holds if and only if ϕ, ϕ' are $\mathcal{K}$-equivalent if and only if f, f' are $\mathcal{A}$-equivalent. For an equation ϕ, we also have $\mathcal{R}$-equivalence, and for a parametrisation f have $\mathcal{L}$-equivalence.

The basic invariants of a reduced plane curve C are the number r of branches, the 'double point number' defined as $\delta(C) := \dim(\mathcal{O}_T/\mathcal{O}_C)$, and the Milnor and Tjurina numbers defined respectively by

$$\mu(C) := d_e(\mathcal{R}, \phi), \qquad \tau(C) := d_e(\mathcal{K}, \phi).$$

The following identities are well-known: $\mu(C) = 2\delta(C) - r + 1$ [6], $\delta(C \cup C') = \delta(C) + \delta(C') + C.C'$ and (hence) $\mu(C \cup C') = \mu(C) + \mu(C') + 2C.C' - 1$.

We also have calculations of the codimensions $d_e(\mathcal{L}, f) = 2\delta(C)$ (trivial), and $d_e(\mathcal{A}, f) = \tau(C) - \delta(C)$ [4, Theorem 2.59].

2. Relative singularity theory

We define two curves C, C' to be equivalent relative to a curve D if there is a diffeomorphism of $\mathbb{C}^2$ which preserves D and takes C to C'. The diffeomorphisms which preserve D form a group $Diff_D(\mathbb{C}^2)$, whose tangent space is the module θ_D of 'logarithmic' vector fields tangent to D. The definitions of right- and of left-equivalence of curves C relative to D are obtained by replacing $Diff(\mathbb{C}^2)$ by $Diff_D(\mathbb{C}^2)$ throughout. Each of these fits into the general framework of 'geometric groups' introduced by Damon [3], and we have a general unfolding theory. The tangent spaces for the relative notions of equivalence are obtained from those in the absolute case by replacing $\theta_{\mathbb{C}^2}$ by θ_D throughout: thus $T\mathcal{L}^e_D f := \omega f(\theta_D)$, $T\mathcal{A}^e_D f := T\mathcal{R}^e f + T\mathcal{L}^e_D f$, $T\mathcal{R}^e_D\phi := t\phi(\theta_D)$, $T\mathcal{K}^e_D\phi := T\mathcal{R}^e_D\phi + T\mathcal{C}\phi$ and, for each $\mathcal{B}$, $d_e(g, \mathcal{B}_D) := \dim_{\mathbb{C}}(\theta_g/T\mathcal{B}^e_D g)$.

The case of relative singularity theory when D is a straight line L has been investigated by Arnol'd [1] under the name of 'boundary singularities'.

We seek formulae expressing $d_e(f,\mathcal{L}_D)$, $d_e(f,\mathcal{A}_D)$, $d_e(\phi,\mathcal{R}_D)$ and $d_e(\phi,\mathcal{K}_D)$ in terms of the invariants of C and D. We will also require the (local) intersection number $C.D$, and $\tau(C\cup D)$.

Since any plane curve is a free divisor, the $\mathcal{O}_{x,y}-$module θ_D is free of rank 2. It will be convenient to choose generators for θ_D: write them as

$$\xi_1 = a_1\partial_x + b_1\partial_y, \qquad \xi_2 = a_2\partial_x + b_2\partial_y. \tag{1}$$

By a result of Saito [7] we may take the equation of D as $\psi := a_1b_2 - a_2b_1$.

Proposition 2.1. *We have* $d_e(f,\mathcal{L}_D) = 2\delta(C) + C.D$.

Proof. By definition, $T\mathcal{L}^e_D f$ is the $\mathcal{O}_T-$submodule of $\mathcal{O}^2_T$ generated over $\mathcal{O}_C$ by $(a_1\circ f, b_1\circ f)$ and $(a_2\circ f, b_2\circ f)$. Hence $\mathcal{O}^2_C/T\mathcal{L}^e_D f$ has the same composition factors as $\mathcal{O}_C/((a_1b_2-a_2b_1)\circ f).\mathcal{O}_C = \mathcal{O}_C/(\psi\circ f).\mathcal{O}_C$. Thus $d_e(f,\mathcal{L}_D) = \dim(\mathcal{O}^2_T/\mathcal{O}^2_C) + \dim(\mathcal{O}_C/(\psi\circ f).\mathcal{O}_C)$: the first term is equal to $2\delta(C)$ and the second to $\dim(\mathcal{O}_{x,y}/\langle\phi,\psi\rangle)$ and hence to $C.D$. □

Lemma 2.2. *The map* $\overline{\omega g} : \theta_{\mathbb{C}^2} \to \operatorname{Coker}(tg)$ *induced by* ωg *has kernel* θ_D.

Proof. The kernel in question is the set of vector fields $\xi \in \theta_{\mathbb{C}^2}$ such that $\omega g(\xi) = tg(\eta)$ for some $\eta \in \theta_T$. In particular, at each point of D we must have ξ tangent to D. But this is just the condition defining θ_D. □

Now let f, g parametrise C, D respectively, and write h for the pair (f,g), so that h parametrises $C\cup D$. We have

Proposition 2.3. *We have* $d_e(f,\mathcal{A}_D) = \tau(C\cup D) - \delta(C) - C.D - \tau(D)$.

Proof. Since th is the direct sum of tf and tg, we can regard the following as a short exact sequence of chain complexes:

$$\begin{array}{ccccccccc} 0 & \to & 0 & \to & \theta_{\mathbb{C}^2} & = & \theta_{\mathbb{C}^2} & \to & 0 \\ & & \downarrow & & \downarrow \overline{\omega h} & & \downarrow \overline{\omega g} & & \\ 0 & \to & \operatorname{Coker}(tf) & \to & \operatorname{Coker}(tf)\oplus\operatorname{Coker}(tg) & \to & \operatorname{Coker}(tg) & \to & 0 \end{array}.$$

Now the Cokernels of $\overline{\omega h}$ and $\overline{\omega g}$ are $\theta_h/T\mathcal{A}^e h$ and $\theta_g/T\mathcal{A}^e g$ and, by Lemma 2.2, the kernel of $\overline{\omega g}$ is θ_D. Hence the exact homology sequence of the diagram is

$$\theta_D \to \operatorname{Coker}(tf) \to \theta_h/T\mathcal{A}^e h \to \theta_g/T\mathcal{A}^e g \to 0.$$

Thus we have an exact sequence

$$0 \to \theta_f/T\mathcal{A}^e_D(f) \to \theta_h/T\mathcal{A}^e(h) \to \theta_g/T\mathcal{A}^e(g) \to 0,$$

so $\dim(\theta_f/T\mathcal{A}^e_D(f)) = \dim(\theta_h/T\mathcal{A}^e(h)) - \dim(\theta_g/T\mathcal{A}^e(g))$. Here the terms on the right hand side are $\tau(C\cup D) - \delta(C\cup D)$ and $\tau(D) - \delta(D)$, and the result follows on substituting for $\delta(C\cup D)$. □

We now consider C as defined by ϕ and introduce the relative Tjurina number

$$\tau_D C := \tau_D\phi := d_e(\phi,\mathcal{K}_D) = \dim(\mathcal{O}_{x,y}/\langle\theta_D\phi,\phi\rangle).$$

Suppose we have three curve-germs C, D_1 and D_2, no two with a common component, with respective equations ϕ, ψ_1 and ψ_2.

Lemma 2.4. *We have exact sequences*

$$0 \to \frac{\mathcal{O}_{x,y}}{\langle\theta_{D_1\cup D_2}\phi,\phi\rangle} \xrightarrow{Y} \frac{\mathcal{O}_{x,y}}{\langle\theta_{D_2}(\phi\psi_1),\phi\psi_1\rangle} \xrightarrow{Q} \frac{\mathcal{O}_{x,y}}{\langle\psi_1,\phi\theta_{D_2}(\psi_1)\rangle} \to 0, \tag{2}$$

$$0 \to \frac{\mathcal{O}_{x,y}}{\langle\psi_1,\theta_{D_2}(\psi_1)\rangle} \xrightarrow{X} \frac{\mathcal{O}_{x,y}}{\langle\psi_1,\phi\theta_{D_2}(\psi_1)\rangle} \xrightarrow{P} \frac{\mathcal{O}_{x,y}}{\langle\psi_1,\phi\rangle} \to 0, \tag{3}$$

where Y, X are induced by multiplication by ψ_1, ϕ respectively and Q, P are the projections.

Proof. (i) For $a \in \mathcal{O}_{x,y}$, the condition that $a\psi_1 \in \langle \theta_{D_2}(\phi\psi_1), \phi\psi_1 \rangle$ means that for some $\xi \in \theta_{D_2}$ and $b \in \mathcal{O}_{x,y}$ we have $a\psi_1 = \xi(\phi\psi_1) + b\phi\psi_1$. Since $\xi(\phi\psi_1) = \phi\xi(\psi_1) + \psi_1\xi(\phi)$, it follows that $\xi(\psi_1)$ is divisible by ψ_1, in other words, that $\xi \in \theta_{D_1}$, and hence that $\xi \in \theta_{D_1 \cup D_2}$.

Conversely, if $a = \xi(\phi) + b\phi$ with $\xi \in \theta_{D_1 \cup D_2}$, we have $a\psi_1 = \xi(\phi\psi_1) - \phi\xi(\psi_1) + b\phi\psi_1$, and since $\xi(\psi_1)$ is divisible by ψ_1, this belongs to $\langle \theta_{D_2}(\phi\psi_1), \phi\psi_1 \rangle$.

We have shown that the map Y is well defined and injective. Its cokernel is the quotient of $\mathcal{O}_{x,y}$ by $\langle \psi_1, \theta_{D_2}(\phi\psi_1) \rangle$. Using again the identity $\xi(\phi\psi_1) = \phi\xi(\psi_1) + \psi_1\xi(\phi)$ and absorbing the second term of this sum, we see that this module is the same as $\langle \psi_1, \phi\theta_{D_2}(\psi_1) \rangle$.

(ii) For $a \in \mathcal{O}_{x,y}$, if $a\phi \in \langle \psi_1, \phi\theta_{D_2}(\psi_1) \rangle$, we can write $a\phi = b\psi_1 + \phi\xi(\psi_1)$ for some $b \in \mathcal{O}_{x,y}$ and some $\xi \in \theta_{D_2}$. It follows that $b\psi_1$, and hence b is divisible by ϕ, say $b = c\phi$. Thus $a = c\psi_1 + \xi(\psi_1) \in \langle \psi_1, \theta_{D_2}(\psi_1) \rangle$. The converse is again easy, so the first map exists and is injective; the cokernel is as given. □

Proposition 2.5. *We have*

(i) $\tau_{D_2}(D_1 \cup C) = \tau_{D_1 \cup D_2}(C) + \tau_{D_2}(D_1) + D_1.C$, *and*
(ii) $d_e(\phi, \mathcal{K}_D) = \tau_D(C) = \tau(C \cup D) - C.D - \tau(D)$.

Proof. The dimensions of the first two terms in (2) of Lemma 2.4 are $\tau_{D_1 \cup D_2}(C)$ and $\tau_{D_2}(D_1 \cup C)$ respectively: thus $\dim(\mathcal{O}_{x,y}/\langle \psi_1, \phi\theta_{D_2}(\psi_1) \rangle) = \tau_{D_2}(D_1 \cup C) - \tau_{D_1 \cup D_2}(C)$. Since the first term in (3) has dimension $\tau_{D_2}(D_1)$ and the third has dimension $D_1.C$, it follows that this expression is equal to $\tau_{D_2}(D_1) + D_1.C$. This proves (i), and (ii) follows on setting $D_2 = \emptyset$ (and $D_1 = D$). □

In fact the apparent extra generality of (i) is spurious: (i) follows from (ii) on substituting for each of the terms $\tau_D C$.

The following turns out to be the most difficult of our four cases; indeed the result is not what I had originally guessed.

It will be convenient to write, for a a function, $[a]$ for the curve defined by $a = 0$ and $[a].[b]$ for the local intersection number of $a = 0$ and $b = 0$, which is equal to $\dim(\mathcal{O}_{x,y}/\langle a, b \rangle)$. We will manipulate such intersection numbers using identities of the forms (i) $[a].[b + ca] = [a].[b]$ and (ii) $[a].[bc] = [a].[b] + [a].[c]$.

Proposition 2.6. *We have $d_e(\phi, \mathcal{R}_D) = C.D + \mu(C) + \mu(D) - 1 - \tau(D)$.*

Proof. Let ξ_1 and ξ_2, as in (1), generate θ_D. Since $\theta_D\phi$ is an ideal with 2 generators, its codimension is equal to the intersection number of the curves they define, hence to $[\xi_1(\phi)].[\xi_2(\phi)]$. We begin by writing

$$[\xi_1(\phi)].[\xi_2(\phi)] = [\xi_1(\phi)].[b_1\xi_2(\phi)] - [\xi_1(\phi)].[b_1].$$

Now manipulate using (i) to reduce the first term to $[\xi_1(\phi)].[(b_1a_2 - b_2a_1)\phi_x] = [\xi_1(\phi)].[\psi\phi_x]$ and the second to $-[a_1\phi_x].[b_1]$. Next use (ii) to obtain

$$[\xi_1(\phi)].[\psi] + [\xi_1(\phi)].[\phi_x] - [a_1].[b_1] - [\phi_x].[b_1] =$$

$$[\xi_1(\phi)].[\psi] + [b_1\phi_y].[\phi_x] - [a_1].[b_1] - [\phi_x].[b_1] = [\xi_1(\phi)].[\psi] + [\phi_y].[\phi_x] - [a_1].[b_1].$$

The second term here is equal to $\mu(C)$.

We pause to establish the

Claim 2.1. *We have $[x].D + [\xi_1(\phi)].D = C.D + [a_1].D$.*

First note that, since ξ_1 is tangent to D, we can set

$$a_1(\alpha_i(t_i), \beta_i(t_i)) = \lambda_i \alpha_i'(t_i) \text{ and } b_1(\alpha_i(t_i), \beta_i(t_i)) = \lambda_i \beta_i'(t_i)$$

for some $\lambda_i(t_i)$, so that

$$\xi_1(\phi)(\alpha_i(t_i), \beta_i(t_i)) = (\lambda_i \alpha_i'(t_i)\phi_x + \lambda_i \beta_i' \phi_y)(\alpha_i(t_i), \beta_i(t_i)) = \lambda_i d\phi(\alpha_i(t_i), \beta_i(t_i))/dt_i.$$

We calculate intersection numbers with $[\psi] = D$ by choosing a parametrisation $(\alpha_i(t_i), \beta_i(t_i))$ for each branch of D, so that $[\psi].[\chi] = \sum_i \mathrm{ord}_{t_i}(\chi(\alpha_i(t_i), \beta_i(t_i)))$. Taking in turn χ equal to x, ϕ, a_1 and $\xi_1(\phi)$, we obtain

$[x].D = \sum_i \mathrm{ord}_{t_i}(\alpha_i(t_i))$,

$C.D = \sum_i \mathrm{ord}_{t_i} \phi(\alpha_i(t_i), \beta_i(t_i))$,

$[a_1].D = \sum_i \mathrm{ord}_{t_i}(\lambda_i \alpha_i'(t_i)) = \sum_i (\mathrm{ord}_{t_i}(\lambda_i) + \mathrm{ord}_{t_i}(\alpha_i(t_i)) - 1)$,

and

$$[\xi_1(\phi)].D = \sum_i \mathrm{ord}_{t_i}(\xi_1(\phi)(\alpha_i(t_i), \beta_i(t_i))) = \\ \sum_i \mathrm{ord}_{t_i}(\lambda_i d\phi(\alpha_i(t_i), \beta_i(t_i))/dt_i) = \sum_i (\mathrm{ord}_{t_i} \lambda_i + \mathrm{ord}_{t_i} \phi(\alpha_i(t_i), \beta_i(t_i)) - 1).$$

The claim follows from these four equations.

We also have

$$[a_1].D = [a_1].[b_1 a_2 - b_2 a_1] = [a_1].[b_1 a_2] = [a_1].[b_1] + [a_1].[a_2].$$

Combining this with our Claim, we obtain

$$d_e(\phi, \mathcal{R}_D) = [\xi_1(\phi)].[\psi] + [\phi_y].[\phi_x] - [a_1].[b_1] =$$

$$C.D + [a_1].D - [x].D + \mu(C) - [a_1].[b_1] = C.D - [x].D + \mu(C) + [a_1].[a_2].$$

Since ξ_1, ξ_2 generate θ_D, the coefficients a_1, a_2 generate the ideal $I := \{\alpha \in \mathcal{O}_{x,y} \,|\, \alpha\psi_x \in \langle \psi_y, \psi \rangle\}$. This ideal contains ψ_x and ψ which have no common factor, hence neither do a_1 and a_2, so $[a_1].[a_2] = \dim(\mathcal{O}_{x,y}/\langle a_1, a_2 \rangle) = \dim(\mathcal{O}_{x,y}/I)$. We have an exact sequence

$$0 \to \mathcal{O}_{x,y}/I \xrightarrow{\psi_x} \mathcal{O}_{x,y}/\langle \psi_y, \psi \rangle \to \mathcal{O}_{x,y}/\langle \psi_x, \psi_y, \psi \rangle \to 0.$$

The third term has dimension $\tau(D)$; the second has dimension $[\psi_y].[\psi]$, and we have $[\psi].[\psi_y] = [\psi_x].[\psi_y] + [x].[\psi_y]$: we can prove this by the same method as the Claim or appeal to [9, Lemma 6.5.7]. Thus,

$$[a_1].[a_2] = \dim(\mathcal{O}_{x,y}/I) = [\psi_x].[\psi_y] + [x].[\psi_y] - \tau(D) = \mu(D) - \tau(D) + [x].D - 1.$$

The Proposition follows by substituting this in the above formula. $\square$

Corollary 2.7. $d_e(\phi, \mathcal{R}_D) - d_e(\phi, \mathcal{K}_D) = \mu(C \cup D) - 1 - \tau(C \cup D)$.

This follows from Propositions 2.6 and 2.5, and compares with the equation

$$d_e(\phi, \mathcal{R}) - d_e(\phi, \mathcal{K}) = \mu(C) - \tau(C).$$

Corollary 2.8. *We have* $d_e(\phi, \mathcal{K}_D) = d_e(f, \mathcal{A}_D) + \delta(C)$.

This follows from Propositions 2.3 and 2.5, and compares with $d_e(\phi,\mathcal{K}) = d_e(f,\mathcal{A}) + \delta(C)$.

We refer to [4, Theorem 2.59] for a discussion of semi-universal deformations of plane curve-germs, and in particular for the result that the δ constant stratum, which in the case that the central curve is parametrised consists precisely of those curve-germs that can be simultaneously parametrised, has codimension δ in the deformation space, and has a smooth normalisation. Moreover, this normalisation can be identified with the semi-universal deformation of the parametrised curve.

For singularity theory relative to D, we have a deformation of C (given by equations) for which the tangent space to the unfolding space U_e maps isomorphically to $\theta_\phi/T\mathcal{K}^e_D\phi$, and a deformation given by parametrised curves for which the tangent space to the unfolding space U_p maps isomorphically to $\theta_f/T\mathcal{A}^e_D f$. We can construct a map $U_p \to U_e$; its image will certainly lie in the $\delta-$constant part of U_e which, as U_e is certainly versal in the usual sense, is of codimension δ with a smooth normalisation to which our map lifts. We expect this lift to be a (local) isomorphism; the expectation is supported by Corollary 2.8.

3. Determinacy

The theory of determinacy was developed mainly by Mather [5]. We say that f is $m-\mathcal{B}$-determined if any g whose Taylor expansions up to degree m agree with those of f (or equivalently, with $g - f \in \mathfrak{m}_N^{m+1}.\theta_f$) is $\mathcal{B}$-equivalent to f, and f is finitely $\mathcal{B}$-determined if it is $m-\mathcal{B}$-determined for some m. Mather characterised determinacy for $\mathcal{A}$-equivalence, and gave estimates for the degree of determinacy; better estimates can be found in [2]. We recall a key result of that paper, in simplified form.

Theorem 3.1. [2, Theorem 1.9] *Suppose $\mathcal{G}$ a subgroup of $\mathcal{K}$ such that*

(i) *for each s, $J^s\mathcal{G}$ is a closed algebraic subgroup of $J^s\mathcal{K}$,*
(ii) *$J^1\mathcal{G}$ is unipotent, e.g., trivial.*

Then, f is $r-\mathcal{G}$-determined if and only if $\mathfrak{m}^{r+1}.\theta(f) \subseteq T\mathcal{G}f$.

The formulation of this theorem refers only to germs at a single point. However if we consider germs at a finite set (say, with a common target), all the arguments involved go through without other than notational change. We will use this extension without further comment.

For $\mathcal{R}$-equivalence, write J for the Jacobian ideal $\langle\partial_x\phi, \partial_y\phi\rangle$ and recall that $\mu = \dim(\mathcal{O}_{x,y}/J)$. Thus, not all the inclusions

$$\mathfrak{m}^{\mu+1} + J \subseteq \ldots \subseteq \mathfrak{m}^{i+1} + J \subseteq \mathfrak{m}^i + J \ldots \subseteq \mathfrak{m} + J \subseteq \mathcal{O}$$

can be proper, so for some $i \leq \mu$, $\mathfrak{m}^{i+1} + J = \mathfrak{m}^i + J$, so by Nakayama's lemma, $J \supseteq \mathfrak{m}^i \supseteq \mathfrak{m}^\mu$.

We can take $\mathcal{G}$ as the subgroup $\mathcal{R}_1$ of $\mathcal{R}$ of diffeomorphisms with trivial 1-jet. Since $T\mathcal{R}_1\phi = t\phi(\mathfrak{m}^2.\theta_{x,y}) = \mathfrak{m}^2.J \supseteq \mathfrak{m}^{\mu+2}$, ϕ is $(\mu+1)-\mathcal{R}$-determined. Experiment soon shows that this well-known estimate is usually very poor, though it is best possible for singularities of type A_k.

For $\mathcal{L}$-equivalence, a similar result holds, but can be improved. Let C have branches B_i $(1 \leq i \leq r)$; write $B_i^* := C \setminus B_i$, and set $K_C := \max_i(\mu(B_i) + B_i.B_i^*)$: thus if $r > 1$ we have $K_C < \mu(C)$. Write $m(C)$ for the multiplicity of C: thus, for C not of type A_*, we have $m(C) \geq 3$.

Proposition 3.2. *We have*

(i) *$f^*\mathcal{O}_{x,y} \supseteq \mathfrak{m}_T^{K_C}$, and*
(ii) *if $m(C) \geq 3$, $f^*\mathfrak{m}^2_{x,y} \supseteq \mathfrak{m}_T^{K_C}$.*

Proof. (i) We have $f_i^*\mathcal{O}_{x,y} \supseteq \mathfrak{m}_i^{\mu(B_i)}$ (see e.g. [9, 4.3.3, 6.3.2], thus for each $k \geq \mu(B_i)$ there exists $\alpha_k \in \mathcal{O}_{x,y}$ with $f_i^*\alpha_k$ of order k. If β_i is a defining equation for B_i^*, $f_i^*(\beta_i)$ has order

$B_i.B_i^*$. Hence $f^*(\alpha_k\beta_i)$ vanishes on B_j for $j \neq i$ and has order $k + B_i.B_i^*$ on B_i. As this yields all orders $\geq K_C$, the result follows.

(ii) It will suffice to show that for any $g \in (\mathfrak{m}_{x,y} \setminus \mathfrak{m}_{x,y}^2)$, we have $f^*g \notin \mathfrak{m}_T^{K_C}$. Set $L := g^{-1}(0)$; this is a smooth curve-germ at $(0,0)$. The order of f_i^*g is the intersection number $L.B_i$. We need to show that, for some i, $L.B_i < K_C$.

Suppose the multiplicity sequence for (infinitely near points of) B_i has r instances of $m(B_i)$ followed by an integer $m' < m(B_i)$ (see e.g. [9, 3.5.1] for this sequence). Since the following point in the sequence is proximate to a point other than O_0, it cannot belong to L, so $L.B_i \leq rm(B_i) + m'$; while (see e.g. [9, 6.5.9]) $\mu(B_i) \geq rm(B_i)(m(B_i) - 1) + m'(m' - 1)$. We now distinguish cases.

If $m(B_i) \geq 3$, $L.B_i \leq rm(B_i) + m' \leq \frac{1}{2}\mu(B_i) + 1 < \mu(B_i) \leq K_C$ (here the '+1' is only needed if m' is 1 or 2).

If $m(B_i) = 2$ and $r \geq 2$, $L.B_i \leq rm(B_i) + 1 < \mu(B_i) + B_i.B_i^* \leq K_C$.

If $m(B_i) = 1$ and $r \geq 2$, the mutual orders of contact of L and the B_i are equal to the intersection numbers. Choose i with $L.B_i$ minimum. As the least two of $L.B_i$, $L.B_j$, $B_i.B_j$ are equal, $L.B_i \leq B_i.B_j \leq K_C$, with equality only if $r = 2$. □

Corollary 3.3. *If $m(C) \geq 3$, C is $(K_C - 1) - \mathcal{L}$-determined.*

We apply Theorem 3.1, taking $\mathcal{G}$ to be the group $\mathcal{L}_1$ of left equivalences with trivial 1-jet. Since $T\mathcal{L}_1 f = \omega f(\mathfrak{m}_{x,y}^2.\theta_{x,y})$, it follows from (ii) that if $m(C) \geq 3$, $T\mathcal{L}_1 f \supseteq \mathfrak{m}_T^{K_C}.\theta_T$; the result follows.

If $m(C) = 2$ then either C has type A_{2k-1} for some $k \geq 1$, we have $K_C = k$ and the degree of determinacy is k; or C has type A_{2k}, $K_C = 2k$ and the degree of determinacy is $2k + 1$.

We turn to relative determinacy. We would like to apply Theorem 3.1 taking $\mathcal{G}$ to be the group $Diff_D(\mathbb{C}^2)$ with tangent space θ_D acting on the right on equations and on the left on parametrisations. However this is not always jet unipotent. We thus take $\mathcal{G}$ as the group of diffeomorphisms preserving D and with identity 1-jet, so $T\mathcal{G} = \theta_D \cap \mathfrak{m}_{x,y}^2.\theta_{x,y}$. Set $e_D := \dim(\theta_D/(\theta_D \cap \mathfrak{m}_{x,y}^2.\theta_{x,y}))$.

I conjecture that if D is not weighted homogeneous, then $\theta_D \subset \mathfrak{m}_{x,y}^2.\theta_{x,y}$ (so $e_D = 0$). If D is weighted homogeneous but not of type A_*, then $\theta_D \subset \mathbb{C}.\{x\partial_x + y\partial_y\} + \mathfrak{m}_{x,y}^2.\theta_{x,y}$ (so $e_D = 1$). If D has type A_*, then $e_D = 2$ and if $D = \{y = 0\}$, then $e_D = 4$.

Proposition 3.4. *If C has equation ϕ, the degree of $\mathcal{R}_D$-determinacy of ϕ is at most*

$$\mu(C) + C.D + \mu(D) - 1 - \tau(D) + e_D.$$

Proof. As in the absolute case it follows, using Nakayama's lemma, that $\theta_D\phi \supseteq \mathfrak{m}^k$, where $k = \dim(\mathcal{O}_{x,y}/\theta_D\phi) = d_e(\phi, \mathcal{R}_D)$; by Proposition 2.6, we have $d_e(\phi, \mathcal{R}_D) = \mu(C) + C.D + \mu(D) - 1 - \tau(D)$.

Since $T\mathcal{G}$ is an $\mathcal{O}_{x,y}-$module, the same argument gives $T\mathcal{G}\phi \supseteq \mathfrak{m}^{k+e_D}$. The result now follows from Theorem 3.1. □

Proposition 3.5. *If C has parametrisation f, then f is $(K_{C\cup D} - 1) - \mathcal{L}_D$-determined.*

Proof. Let D have parametrisation g, then $C \cup D$ has parametrisation $h = (f, g)$. We claim that if h is $k - \mathcal{L}$-determined, then f is $k - \mathcal{L}_D$-determined.

For let $j^k f_1 = j^k f$, and set $h_1 := (f_1, g)$. Then h and h_1 have the same $k-$jet, so are $\mathcal{L}$-equivalent. Thus there is a diffeomorphism A of $\mathbb{C}^2$ with $A \circ f = f_1$ and $A \circ g = g$. Hence A is an $\mathcal{L}_D$-equivalence of f and f_1.

By Corollary 3.3, if $m(C \cup D) \geq 3$, we can take k to be $K_{C\cup D} - 1$. The result follows. □

If we set $K_{C,D} := \max_i(\mu(B_i) + B_i.B_i^* + B_i.D)$, then $K_{C\cup D} = \max(K_{C,D}, K_{D,C})$. It seems likely that a direct approach might allow a sharpening of $K_{C\cup D}$ to $K_{C,D}$ above, but this is not useful for our application.

In particular, if $C \cup D$ is reduced, C is finitely determined relative to D in each sense.

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Journal *of* Singularities
Volume 4 (2012), 196-205

received 29 June 2012
in revised form 7 December 2012
DOI: 10.5427/jsing.2012.4m

ON THE COBORDISM GROUPS OF COORIENTED, CODIMENSION ONE MORIN MAPS

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Abstract. We compute cobordism groups of fold maps, cusp maps, and more general Morin maps of oriented n-dimensional manifolds to R^{n+1}. The results for fold maps in dimensions $n \leq 10$ are complete. In general we express the results through the stable homotopy groups of spheres and that of the infinite projective space.

Introduction

We consider cobordism groups of maps of n-dimensional manifolds into R^{n+1} such that they have at most:

(a) $\Sigma^{1,0}$ type singularities, i.e., fold maps (in Part 1),
(b) $\Sigma^{1,1}$ type singularities, i.e., cusp maps (in Part 2),
(c) Σ^{1_r}-type singularities (in Part 3).

We express these groups completely through the stable homotopy groups of spheres and those of the infinite projective space in case a) and modulo some small prime components for the cases b) and c).

The main tools of the computation are:

(1) the classifying spaces of the cobordisms of maps with a given set of allowed singularities, see [RSz], [Sz3],
(2) a fibration connecting these classifying spaces, the so-called key bundle in [Sz3], [T],
(3) identification of the boundary map in the homotopy exact sequence of the above mentioned fibration with a well-studied map in homotopy theory, namely the so called Kahn–Priddy map.

Part 1. Cobordism of fold maps and the Kahn–Priddy map

1.1. Formulation of the result

Let us denote by $\mathrm{Cob}\,\Sigma^{1,0}(n)$ the cobordism group of cooriented, codimension 1 fold maps of closed, smooth, n–dimensional manifolds in R^{n+1} (see [Sz3]).

(A fold map may have only $\Sigma^{1,0}$–type (or A_1–) singular points, see [AGV].)

Theorem A.

(a) *$\mathrm{Cob}\,\Sigma^{1,0}(n)$ is a finite Abelian group.*
(b) *Its odd torsion part is isomorphic to that of the n^{th} stable homotopy group of spheres, i.e., for any odd prime p, $\mathrm{Cob}\,\Sigma^{1,0}(n)_p \approx \pi^s(n)_p$, where the lower index p denotes the p–primary part. The isomorphism is induced by the natural forgetting map $\pi^s(n) \to \mathrm{Cob}\,\Sigma^{1,0}(n)$.*
(c) *Its 2–primary part is isomorphic to the kernel of the Kahn–Priddy homomorphism [KP]:*

$$\lambda_*\colon\ \pi^s_{n-1}(RP^\infty) \longrightarrow \pi^s(n-1).$$

Remark. The group $\pi^s_{n-1}(RP^\infty)$ is a 2–primary group and λ_* is onto the 2–primary part of $\pi^s(n-1)$, see [KP].

Corollary. $\operatorname{Cob}\Sigma^{1,0}(n) \approx \pi^s(n)_{\{\text{odd torsion part}\}} \oplus \operatorname{Ker}\big(\lambda^*\colon\ \pi^s_{n-1}(RP^\infty) \longrightarrow \pi^s(n-1)\big)$.

1.2. The Kahn–Priddy map ([KP], [K])

Let us consider the composition of the following maps:

a) $RP^{q-1} \hookrightarrow O(q)$. A line $L \subset R^q$, $[L] \in RP^{q-1}$ is mapped into the reflection in its orthogonal hyperplane.

b) $O(q) \hookrightarrow \Omega^q S^q$ maps $A \in O(q)$ to the map

$$\begin{array}{ccccl} S^q = R^2 \cup \infty & \longrightarrow & S^q = R^q \cup \infty & \text{defined as} \\ x & \longrightarrow & A(x) & \text{for} \ \ x \in R^q \ \ \text{and} \\ \infty & \longrightarrow & \infty. & \end{array}$$

Take the adjoint of the composition map $RP^{q-1} \longrightarrow \Omega^q S^q$. It is a map

$$\lambda\colon\ \Sigma^q RP^{q-1} \longrightarrow \Omega^q S^q.$$

If $n < q$, then the homotopy groups $\pi_{q+n}(\Sigma^q\, RP^{q-1})$ and $\pi_{q+n+1}(S^{q+1})$ are stable, and

$$(*) \qquad \pi^s_n(RP^{q-1}) \approx \pi^s_n(RP^\infty).$$

The Kahn–Priddy homomorphism $\lambda_*\colon\ \pi^s_n(RP^\infty) \longrightarrow \pi^s(n)$ is the homomorphism induced by λ in the stable homotopy groups (precomposed with the isomorphism $(*)$).

Theorem 1 (Kahn–Priddy [KP])**.** *λ_* is onto the 2–primary component of $\pi^s(n)$.*

1.3. Koschorke's interpretation of λ_*

Ulrich Koschorke gave a very geometric description of the Kahn–Priddy homomorphism through the so-called "figure 8 construction". Given an immersion of an $(n-1)$–dimensional (unoriented) manifold N^{n-1} into R^n the figure 8 construction associates with it an immersion of an oriented n–dimensional manifold M^n into R^{n+1} as follows:

Let us consider the composition $N^{n-1} \looparrowright R^n \hookrightarrow R^{n+1}$.

This has normal bundle of the form $\varepsilon^1 \oplus \zeta^1$, where ε^1 is the trivial line bundle (the $(n+1)^{\text{th}}$ coordinate direction in R^{n+1}) and ζ^1 is the normal line bundle of N^{n-1} in R^n.

Let us put a figure 8 in each fiber of $\varepsilon^1 \oplus \zeta^1$ symmetrically with respect to the reflection in the fiber ζ^1. Choosing these figures 8 smoothly their union gives the image of an immersion of an oriented n–dimensional manifold M^n into R^{n+1}. (Clearly M^n is the total space of the circle bundle $S(\varepsilon^1 \oplus \zeta^1)$ over N^{n-1}.)

This construction gives a map $8_*\colon\ \pi^s_n(RP^\infty) \longrightarrow \pi^s_n(n)$. Indeed, the cobordism group of immersion of unoriented $(n-1)$–dimensional manifolds in R^n is isomorphic to $\pi^s_n(RP^\infty)$, and that of oriented n–dimensional manifolds in R^{n+1} is $\pi^s(n)$.

Since the figure 8 construction respects the cobordism relation (i.e. it associates to cobordant immersions such ones) we obtain a map of the cobordism groups.

Theorem 2 (Koschorke, [K, Theorem 2.1])**.** *The maps λ_* and 8_* coincide.*

This theorem of Koschorke will be the main tool in the computation of the cobordism groups of fold maps.

1.4. Generalities on the cobordisms of singular maps

In [Sz3] we considered cobordism groups of singular maps with a given set τ of allowed local forms. (Such a map was called a τ–map.) The cobordism group of (cooriented) τ–maps of n–dimensional manifolds in Euclidean space was denoted by $\mathrm{Cob}_\tau(n)$. A classifying space X_τ has been constructed for τ–maps with the property that its homotopy groups are isomorphic to the groups $\mathrm{Cob}_\tau(n)$.

An ancestor of the spaces X_τ was the classifying space for the cobordism groups of immersions. Namely given a vector bundle ξ^k we denote by $\mathrm{Imm}^\xi(n)$ the cobordism group of immersions of n–manifolds in R^{n+k} such that the normal bundle is induced from ξ. There is a classifying space $Y(\xi)$ such that

$$\pi_{n+k}(Y(\xi)) \approx \mathrm{Imm}^\xi(n).$$

Namely $Y(\xi) = \Gamma(T\xi)$, where $T\xi$ denotes the Thom space of the bundle ξ, and $\Gamma = \Omega^\infty S^\infty$. (This follows by a slight modification from [W].)

Next we recall the so-called "key bundle", that is the main tool in handling cobordism groups of singular maps.

Let τ be a list of allowed local forms, and let η be a maximal element in it. (The set of local forms has a natural partial ordering, η is greater than η' if an isolated η–germ (at the origin) has an η'–point arbitrarily close to the origin.)

Let τ' be $\tau \setminus \{\eta\}$ (i.e. we omit the maximal element η).

Note that the stratum of η points is immersed. We have established in [Sz3] that there is a universal bundle – denoted by $\widetilde{\xi}_\eta$ – for the normal bundles of η–strata from which these normal bundles always can be induced (with the smallest possible structure group).

In particular to the cobordism class $[f]$ of a τ–map $f\colon M^n \longrightarrow R^{n+k}$ we can associate the element in $\mathrm{Imm}^{\widetilde{\xi}_\eta}(m)$ represented by the restriction of f to its η–stratum. (Here m is the dimension of the η–stratum.) Hence a homomorphism $\mathrm{Cob}_\tau(n) \longrightarrow \mathrm{Imm}^{\widetilde{\xi}_\eta}(m)$ arises. Both these groups are homotopy groups (of X_τ and $\Gamma T\widetilde{\xi}_\eta$ respectively). It turns out that this map is induced by a map of the classifying spaces $X_\tau \longrightarrow \Gamma T\widetilde{\xi}_\eta$. Moreover the latter is a Serre fibration with (homotopy) fiber $X_{\tau'}$. (This was shown in [Sz3] using some nontrivial homotopy theory. Terpai in [T] gave an elementary proof for it. This fibration is called the "key bundle".)

1.5. Computation of the groups $\mathrm{Cob}\,\Sigma^{1,0}(n)$

In the case of fold maps $\tau = \{\Sigma^0, \Sigma^{1,0}\}$ where Σ^0 denotes the germ of maximal rank and $\Sigma^{1,0}$ denotes that of a Whitney umbrella $(R^2, 0) \longrightarrow (R^3, 0)$ (multiplied by the germ of identity $(R^{n-2}, 0) \longrightarrow (R^{n-2}, 0)$).

Hence here $\eta = \Sigma^{1,0}$ and $\tau' = \Sigma^0$. Note that a τ'–map is nothing else but an immersion (cooriented and of codimension 1). Hence $X_{\tau'} = \Gamma S^1$. Now the key bundle looks as follows:

$$(**) \qquad X\Sigma^{1,0} \xrightarrow{\Gamma S^1} \Gamma T\widetilde{\xi}_{\Sigma^{1,0}}\,.$$

It is not hard to see (see also [RSz]) that the bundle $\widetilde{\xi}_{\Sigma^{1,0}}$ is $2\varepsilon^1 \oplus \gamma^1$, and so $T\widetilde{\xi}_{\Sigma^{1,0}} = S^2RP^\infty$.

Now the bundle $(**)$ gives the following exact sequence of homotopy groups:

$$\pi_{n+1}(\Gamma S^1) \longrightarrow \pi_{n+1}(X\Sigma^{1,0}) \longrightarrow \pi_{n+1}(\Gamma S^2RP^\infty) \xrightarrow{\partial} \pi_n(\Gamma S^1), \quad \text{i.e.,}$$

$$\pi^s(n) \longrightarrow \mathrm{Cob}\,\Sigma^{1,0}(n) \longrightarrow \pi^s_{n-1}(RP^\infty) \xrightarrow{\partial} \pi^s(n-1) \longrightarrow$$

Lemma 3. *The boundary map ∂ coincides with the map 8_* and hence with the Kahn–Priddy homomorphism λ_*.*

Proof of Theorem A. is immediate from Theorem 1, Theorem 2 and Lemma 3. □

Proof of Lemma 3. The boundary map $\partial\colon \pi^s_{n+1}(S^2RP^\infty) \approx \pi_{n+1}(X\Sigma^{1,0}, \Gamma S^1) \longrightarrow \pi_n(\Gamma S^1)$ can be interpreted geometrically as follows:

The source group $\pi_{n+1}(X\Sigma^{1,0}, \Gamma S^1)$ is isomorphic to the cobordism group of fold maps

$$f\colon (M^n, \partial M^n) \longrightarrow (D^{n+1}, S^n) \text{ such that } f^{-1}(S^n) = \partial M^n,$$

and the map $\partial f = f\Big|_{\partial M^n}$ is an immersion of ∂M^n into S^n. (Here M^n is an oriented compact smooth n–dimensional manifold with boundary ∂M^n.)

Let $[f]$ denote the (relative) cobordism class of f, and let $[\partial f]$ be that of the immersion $\partial f\colon \partial M^n \longrightarrow S^n$. Then $\partial[f] = [\partial f]$.

Now let V denote the set of singular points of f. This is a submanifold of M^n of codimension 2. The restriction of f to V is an immersion, its image we denote by $\widetilde{V}$ $(= f(V))$. Let $\widetilde{T}$ be the (immersed) tubular neighbourhood of $\widetilde{V}$. More precisely there exist a D^3–bundle $T' \longrightarrow V$ over V, a submersion F of T' into D^n, $(F(T') = \widetilde{T})$ and F extends the immersion $f \mid V\colon V \longrightarrow D^n$. The bundle $T' \longrightarrow V$ has the form $2\varepsilon^1 \oplus \zeta^1$, where ζ^1 is a line bundle.

Let T be the tubular neighbourhood of V in M such that $f(T) \subset \widetilde{T}$. The map $f\Big|_T\colon T \longrightarrow \widetilde{T}$ can be decomposed into a map $\widehat{f}\colon T \longrightarrow T'$ and the submersion $F\colon T' \longrightarrow \widetilde{T}$, where $\widehat{f}$ maps each fiber D^2 of the bundle $T \longrightarrow V$ into a fiber D^3 of $T' \longrightarrow V$ as a Whitney umbrella and $\widehat{f}^{-1}(\partial T') = \partial T$. On the boundary of each fiber D^3 we obtain a "curved figure 8" as image of $\widehat{f}$. The manifold with boundary $M \setminus T$ will be denoted by W. Note that its boundary is $\partial W = \partial_1 W \sqcup \partial_2 W$, where $\partial_1 W = \partial M$, and $\partial_2 W = \partial T$.

The image of $\partial_2 W$ at f is the union of the (F–images of the) above mentioned curved figures 8. This is a codimension 2 framed immersed submanifold in D^{n+1}, we will denote it by $\widetilde{V}$. (The first framing is the F–image of the inside normal vector of $\partial T'$ in T'. The second framing is the normal vector of the curved figure 8 in $S^2 = \partial D^3$.)

It remained to show the following two claims.

Claim a). *$\widetilde{V}$ with the given 2–framing is framed cobordant to the immersion $\partial f\colon \partial M \looparrowright S^n$ (compared with the framed embedding $S^n \subset D^{n+1}$).*

Claim b). *$\widetilde{V}$ is obtained from the immersion $f\big|V\colon V^{n-2} \looparrowright D^{n+1}$ by the figure 8 construction.*

We have to make some remarks in order to clarify the above statements a) and b).

To a): The framed immersion $\partial f\colon \partial M \looparrowright S^n$ and its composition with $i : S^n \subset D^{n+1} \subset R^{n+1}$ (with the added second framing, the inside normal vectors of S^n in D^{n+1}) represent the same element in $\pi^s(n-1)$. Indeed, the composition with i corresponds to applying the suspension homomorphism in homotopy groups of spheres. But the cobordism group of framed *immersions* is isomorphic to the corresponding *stable* homotopy group of spheres, so the suspension homomorphism gives the identity map of these groups.

To b): The figure 8 construction was defined for a codimension one immersed submanifold in a Euclidean space. Here we apply it to the codimension 3 immersed submanifold $\widetilde{V}^{n-2}$ in D^{n+1}. But $\widetilde{V}^{n-2}$ has two linearly independent normal vector fields in D^{n+1} as was described above. Identify D^{n+1} with R^{n+1} and apply the so-called multicompression theorem by Rourke–Sanderson [RS], thus one can make the two normal vectors parallel to the last two coordinate axes in R^{n+1}, we can project the immersion to R^{n-1} and then we have a codimension 1–immersion, so claim b) makes sense. (This needs some more clarification since the multicompression theorem deals with embeddings. See below.)

Proof of Claim a). It can be supposed that the center c of D^{n+1} does not belong to $\partial\widetilde{T} \cup f(M)$. Let us omit from D^{n+1} a small ball centered around c and still disjoint from $\partial\widetilde{T} \cup f(M)$. In the remaining manifold $S^n \times I$ the direction of I will be called vertical. Take the product with an R^q for big enough q, so that the immersions $f\big|_W \colon W^n \looparrowright S^n \times I$ and $\partial T \looparrowright S^n \times I$ become embeddings after small perturbations.

Now W is embedded in $S^n \times I \times R^q$, it is framed with $q+1$ normal vectors (q are parallel to the coordinate axes of R^q).

One can suppose that the two boundary components of W are embedded as follows

1) $\partial_1 W = \partial M$ is embedded into $S^n \times \{0\} \times R^q$.

2) $\partial_2 W = \partial T$ is embedded into the interior part $\operatorname{int}(S^n \times I) \times R^q$.

Both have $(q+2)$-framings.

Now applying the multicompression theorem we make by an isotopy the first framing vector (the one coming from the normal vector of $\partial\widetilde{T}$ in $\widetilde{T}$) vertical, i.e. parallel to the direction of I in $S^n \times I \times R^q$, while the q last framing vectors (coming from R^q) we keep parallel to themselves. The other boundary component $\partial_1 W \subset S^n \times \{0\} \times R^q$ is kept fixed.

We arrive at such an embedding of W in $S^n \times I \times R^q$ for which the outward normal vector along $\partial_2 W$ in W is vertical (i.e. parallel to the direction of I). Now by a vertical shift we can deform $\partial_2 W$ into $S^n \times \{1\} \times R^q$.

This deformation can be extended to W. Projecting into $S^n \times I$ this new position of W in $S^n \times I \times R^q$ we obtain an immersion cobordism between the immersions of the two boundary components.

On the first component $(\partial_1 W)$ we obtain $\partial[f]$. On the second component we obtain the same framed cobordism class as was that of $\widetilde{V}$ in $\partial\widetilde{T} \subset D^{n+1}$ (the union of curved figures 8). Claim a) is proved. $\square$

Proof of Claim b). Deform the immersed manifold $\widetilde{V}^{n-1}$ (formed by the union of curved figures 8) as follows. Contract each curved figure 8 by an isotopy along the corresponding sphere S^2 into a small neighbourhood of its double point obtaining an almost flat (very small) figure 8. As we have noticed the normal bundle of $f(V)$ in D^{n+1} has the form $2\varepsilon^1 \oplus \zeta^1 = \varepsilon^1_1 \oplus \varepsilon^1_2 \oplus \zeta^1$. The first trivial normal line bundle ε^1_1 can be identified with the direction of the double line of the umbrella, the second one ε^1_2 can be the symmetry axes of the figures 8 (both of the curved and the flattened ones). ζ^1 is the direction orthogonal to the symmetry axes.

Now considering the maps of V and $\widetilde{V}$ in R^{n+1} (instead of D^{n+1}) applying again the multicompression theorem we make the two trivial normal directions ε^1_2 and ε^1_1 parallel to the last two coordinate axes and then project V to R^{n-1}. In this way we obtain a new immersion $g \colon V^{n-2} \looparrowright R^{n-1}$ and ε^1_2 (the symmetry axes of the figures 8) will be parallel to the normal vector of R^{n-1} in R^n. Now the (flattened) figures 8 (of $\widetilde{V}$) are placed exactly as by the original figure 8 construction applied to g.

It remained to note that the described deformations do not change the cobordism class of a framed immersion. Claim b) is proved. $\square$

Thus Theorem A is also proved. $\square$

Remark. The stable homotopy groups of RP^∞ were computed by Liulevicius [Liu] in dimensions not greater than 9.

Below in the first line we show his result, in the second one the stable homotopy groups of spheres. These two lines by Theorem A give the groups $\operatorname{Cob}\Sigma^{1,0}(n)$ for $n \leq 10$ given in the third line. (Here for example $(Z_2)^3$ stands for $Z_2 \oplus Z_2 \oplus Z_2$.)

n	1	2	3	4	5	6	7	8	9	10
$\pi_n^s(RP^\infty)$	Z_2	Z_2	Z_8	Z_2	0	Z_2	$Z_{16}\oplus Z_2$	$(Z_2)^3$	$(Z_2)^4$	?
$\pi^s(n)$	Z_2	Z_2	Z_{24}	0	0	Z_2	Z_{240}	$(Z_2)^2$	$(Z_2)^3$	Z_6
$\mathrm{Cob}\,\Sigma^{1,0}(n)$	0	0	Z_3	0	Z_2	0	Z_{15}	Z_2	Z_2	Z_6

Remark. It follows from part b) of Theorem A that all the odd torsion part of $\mathrm{Cob}\,\Sigma^{1,0}(n)$ can be represented by immersions. In particular, in dimensions $n = 3$ and $n = 7$ all the elements can be represented by immersions of the sphere S^3 and S^7, respectively, since the J-homomorphism is onto in these dimensions.

Part 2. Cusp maps

2.1. Formulation of the results

Here we consider cusp maps, i.e. maps having at most cusp singularities. (In the previous terms these are τ–maps for $\tau = \{\Sigma^0, \Sigma^{1,0}, \Sigma^{1,1}\}$.) The cobordism group of cusp maps of oriented n–dimensional manifolds in R^{n+1} will be denoted by $\mathrm{Cob}\,\Sigma^{1,1}(n)$. We shall compute these groups modulo their 2–primary and 3–primary parts. Let $C_{\{2,3\}}$ be the minimal class of groups containing all 2–primary and 3–primary groups.

Theorem B. $$\mathrm{Cob}\,\Sigma^{1,1}(n) \underset{\mathcal{C}_{\{2,3\}}}{\approx} \pi^s(n)\oplus\pi^s(n-4)$$

where $\underset{\mathcal{C}_{\{2,3\}}}{\approx}$ *means isomorphism modulo the class* $\mathcal{C}_{\{2,3\}}$*, and* $\pi^s(m)$ *denotes the* m^{th} *stable homotopy group of spheres.*

2.2. Preliminaries on Morin maps

Morin maps are those of types $\Sigma^{1,0}, \Sigma^{1,1,0}, \ldots, \Sigma^{1_r,0}, \ldots,\ r = 1, 2, \ldots$ (See [AGV].)

For $\eta = \Sigma^{1_r,0}$ the universal normal bundle $\widetilde{\xi}_\eta$ will be denoted by $\widetilde{\xi}_r$. It was established in [RSz] and [R] that the structure group of $\widetilde{\xi}_r$ is Z_2 and the bundle $\widetilde{\xi}_r$ is associated to a representation $\lambda_2\colon Z_2 \longrightarrow O(2r+1)$ with the property that $\lambda_2(Z_2) \subset SO(2r+1)$ precisely when r is even. It follows that $\widetilde{\xi}_r$ is the direct sum $i\cdot\gamma^1 \oplus j\cdot\varepsilon^1$, where $i+j = 2r+1$, and $i \equiv r \bmod 2$. Here ε^1, γ^1 are the trivial and the universal line bundles respectively. Hence the Thom space $T\widetilde{\xi}_r$ is $S^j(RP^\infty/RP^{i-1})$. It is easy to see that for any odd p the reduced mod p cohomology $\overline{H}^*(T\widetilde{\xi}_r; Z_p)$ vanishes if r is odd, and the natural inclusion $S^{2r+1} \subset T\widetilde{\xi}_r$ (as a "fiber") induces isomorphism of the cohomology groups with Z_p–coefficients for r even. Consequently by Serre's generalization of the Whitehead theorem [S2] – the inclusion $\Gamma S^{2r+1} \subset \Gamma T\widetilde{\xi}_r$ (recall $\Gamma = \Omega^\infty S^\infty$) induces isomorphism of the odd torsion parts of the homotopy groups for r even, while for r odd $\pi_*(\Gamma T\widetilde{\xi}_r)$ are finite 2–primary groups.

2.3. Computation of the cusp cobordism groups

In the case of cusps $r = 2$ and we have that the inclusion $\Gamma S^5 \subset \Gamma T\widetilde{\xi}_2$ is a mod $\mathcal{C}_2$ homotopy equivalence ($\mathcal{C}_2$ is the class of 2–primary groups).

Let us consider the following pull-back diagram defining the space $X^{\mathrm{fr}}\Sigma^{1,1}$

$$\begin{tikzcd} X^{\mathrm{fr}}\Sigma^{1,1} \arrow[r] \arrow[d, "X\Sigma^{1,0}"] & X\Sigma^{1,1} \arrow[d, "X\Sigma^{1,0}"] \\ \Gamma S^5 \arrow[r] & \Gamma T\widetilde{\xi}_2 \end{tikzcd}$$

(Note that $X^{\mathrm{fr}}\Sigma^{1,1}$ is the classifying space of those cusp maps for which the normal bundle of the $\Sigma^{1,1}$–stratum in the target is trivialized. Equivalently these are the cusp maps for which the kernel of the differential is trivialized over the cusp-stratum.)

The horizontal maps of the diagram induce isomorphisms of the odd-torsion parts of the homotopy groups. Now we show that the homotopy exact sequence of the left-hand side fibration "almost has a splitting".

Definition. Let $p\colon E \xrightarrow{F} B$ be a fibration and let t be a natural number. We say that this fibration has an algebraic t–splitting if for each i there is homomorphism $S_i\colon \pi_i(B) \longrightarrow \pi_i(E)$ such that the composition of S_i with the map p_* induced by p is a multiplication by t. We say that the fibration p has a geometric t–splitting if it has an algebraic one such that all S_i are induced by a map $s\colon B \longrightarrow E$ (the same map s for each i).

Lemma 4. *The fibration $X^{\mathrm{fr}}\Sigma^{1,1} \longrightarrow \Gamma S^5$ has a 6–splitting.*

Remark. We shall prove this only in algebraic sense, since we will need only that. For the existence of geometric splitting we give only a hint.

Proof of Theorem B. is immediate from Lemma 4. Indeed,

$$\operatorname{Cob}\Sigma^{1,1}(n) \approx \pi_{n+1}(X\Sigma^{1,1}) \underset{\mathcal{C}_2}{\approx} \pi_{n+1}(X^{\mathrm{fr}}\Sigma^{1,1}).$$

Now the homotopy exact sequence of the fibration $p^{\mathrm{fr}}\colon X^{\mathrm{fr}}\Sigma^{1,1} \xrightarrow{X\Sigma^{1,0}} \Gamma S^5$ has a 6–splitting, hence modulo the class $\mathcal{C}_{\{2,3\}}$ we have

$$\pi_{n+1}(X^{\mathrm{fr}}\Sigma^{1,1}) \underset{\mathcal{C}_{\{2,3\}}}{\approx} \pi^s_{n+1}(S^5) \oplus \pi_{n+1}(X\Sigma^{1,0}) \underset{\mathcal{C}_2}{\approx} \pi^s(n-4) \oplus \pi^s(n).$$

(In the last mod $\mathcal{C}_2$ isomorphism we used Theorem A.)

Theorem B is proved except Lemma 4. □

Proof of Lemma 4. will follow from the following two claims.

Claim 1. *If there is a map of an oriented 4–dimensional manifold into R^5 with t cusp points (algebraically counting them), then the fibration p^{fr} has a t–splitting (algebraically).*

Claim 2. *There is a cusp-map $f\colon M^4 \longrightarrow R^5$ with t–cusp points (counting algebraically).*

Proof of Claim 1. Let $f\colon M^4 \longrightarrow R^5$ be a cusp-map with t cusp-points. Let x be an element in $\pi_m(\Gamma S^5) \approx \pi^s(m-5)$. It can be represented by a framed, immersed $(m-5)$–dimensional manifold A^{m-5} in R^m, let us denote its immersion by α.

Take the product $A^{m-5} \times M^4$ and its map into the direct product $A^{m-5} \times D^5$ by $\mathrm{id}_A \times f$. Now the target $A^{m-5} \times D^5$ can be mapped by a submersion F into R^m onto the immersed tubular neighbourhood of $\alpha(A)$ using the framing to map the D^5–fibers). The composition $A \times M^4 \longrightarrow R^m$ is clearly a cusp map and its cusp-singularity stratum represents the element $t \cdot x$ in $\pi^s(m-5)$.

Claim 1 is proved (at least its algebraic version. The geometric one follows from the fact that we use the same element $[f\colon M^4 \longrightarrow D^5]$ for any element $x \in \pi^s(i)$ and for any dimension i to construct the element $S_i(x)$. The classifying space ΓS^5 can be obtained as the limit of target spaces of codimension 5 framed immersions.) □

Proof of Claim 2. is a compilation of the following two theorems.

Theorem ([Sz1], [Sz2], [L])**.** *Given a generic immersion $g\colon M \looparrowright Q \times R^1$ and a natural number r, let us denote by $\Delta_{r+1}(g)$ the manifold of (at least) $(r+1)$–tuple points in M^n. Let $f\colon M \longrightarrow Q$ be the composition of g with the projection $Q \times R^1 \longrightarrow Q$. Let us denote by $\Sigma^{1_r}(f)$ the closure* *of the set of Σ^{1_r} singular points of f. Then the manifolds $\Delta_{r+1}(g)$ and $\Sigma^{1_r}(f)$ are cobordant. If M and Q are oriented and* $\dim Q - \dim M$ *is odd, then these manifolds are oriented and they are oriented-cobordant.*

Theorem 5 (Eccles–Mitchell [EM])**.** *There is an oriented closed 4–dimensional manifold M^4 and an immersion $g\colon M^4 \looparrowright R^6$ with (algebraically) 2 triple points.*

□

Theorem B is proved. □

Part 3. Higher Morin maps

Most of the previous arguments can be applied in the computation of cobordism groups of Morin maps having at most Σ^{1_r} singular points for any r (the codimensions of the considered maps are still equal to one, and the maps are cooriented). The only problem is that we need a generalization of the Theorem of Eccles–Mitchell.

Below we shall give a weak form of such a generalization. This will allow us to compute the groups $\operatorname{Cob}\Sigma^{1_r}(n)$ modulo the p–primary part for $p \le r+1$.

Notation: Let $\mathcal{C}\{p \le 2r+1\}$ denote the minimal class of groups containing all p–primary groups for any prime $p \le 2r+1$. The main result of this Part 3 is the following.

Theorem C. *Let us denote by $\operatorname{Cob}\Sigma^{1_i}(n)$ the cobordism group of Σ^{1_i}–map of oriented n–manifolds in R^{n+1} (i.e. τ–maps for $\tau = \{\Sigma^0, \Sigma^{1,0}, \dots, \Sigma^{1,1,\dots,1},\ i$ digits $1\}$). Then for any r*

$$\operatorname{Cob}\Sigma^{1_{2r+1}}(n) \underset{\mathcal{C}_2}{\approx} \operatorname{Cob}\Sigma^{1_{2r}}(n) \underset{\mathcal{C}\{p\le 2r+1\}}{\approx} \bigoplus_{i=0}^{r} \pi^s(n-4i).$$

The proof is very similar to that given in Parts 1 and 2. It goes by induction on r. First we give a weak analogue of the Theorem of Eccles and Mitchell.

Lemma 1. a) *For any natural number k there is a positive integer $t(k)$ such that for any immersion of an oriented, closed, smooth $4k$–dimensional manifold in R^{4k+2} the algebraic number of $(2k+1)$–tuple points is divisible by $t(k)$, and there is a case when this number is precisely $t(k)$.*

b) *The number $t(k)$ coincides with the order of the cokernel of the stable Hurewicz homomorphism*

$$\pi^s_{4k+2}(CP^\infty) \longrightarrow H_{4k+2}(CP^\infty).$$

Before proving Lemma 1 it will be useful to recall a result on the cokernel of the stable Hurewicz map.

Theorem (Arlettaz [A])**.** *Let X be a $(b-1)$–connected space and let ϱ_j be the exponent of the stable homotopy group of spheres $\pi^s(j)$. Let $h_m\colon \pi^s_m(X) \longrightarrow H_m(X)$ be the stable Hurewicz homomorphism. Then $(\varrho_1 \dots \varrho_{m-b-1})(\operatorname{coker} h_m) = 0$.*

Next we recall a theorem of Serre on the prime divisors of the numbers ϱ_j.

Theorem (Serre [S1])**.** $\pi^s(i) \otimes Z_p = 0$ *if $i < 2p-3$ and $\pi^s(2p-3) \otimes Z_p = Z_p$.*

Hence ϱ_j is not divisible by a prime p if $p > \frac{j+3}{2}$, in other words, ϱ_j may have a prime p as a divisor only if $p \le \frac{j+3}{2}$.

Applying Arlettaz' theorem to $X = CP^\infty$, $b = 2$, $m = 4r+2$ we obtain that $t(r)$ has no prime divisor greater than $2r+1$.

Proof of Theorem C. should be clear now, since it is completely analogous to that of Theorem B.

First we consider the "key bundle"

$$X\Sigma^{1_{2r+1}} \longrightarrow \Gamma T\widetilde{\xi}_{2r+1} \text{ with fiber } X\Sigma^{1_{2r}}.$$

Remember that $H^*(\Gamma T\widetilde{\xi}_{2r+1}; Z_p) = 0$ for any odd p, so – by the mod $\mathcal{C}$ Whitehead theorem [S2] we obtain the first (mod $\mathcal{C}_2$) isomorphism in the Theorem

$$\operatorname{Cob}\Sigma^{1_{2r+1}}(n) \underset{\mathcal{C}_2}{\approx} \operatorname{Cob}\Sigma^{1_{2r}}(n).$$

In order to prove the second (mod $\mathcal{C}\{p \le 2r+1\}$) isomorphism recall that modulo the class $\mathcal{C}_2$ the key bundle

$$X\Sigma^{1_{2r}} \longrightarrow \Gamma T\widetilde{\xi}_{2r} \quad \text{with fibre } X\Sigma^{1_{2r-1}}$$

can be replaced by the bundle

$$X^{\mathrm{fr}}\Sigma^{1_{2r}} \longrightarrow \Gamma S^{4r+1} \text{ with the same fibre.}$$

The later bundle has a(n algebraic) $t(r)$–splitting.

By Lemma 1 and the theorems of Arlettaz and Serre $t(r)$ has no prime divisor greater than $2r+1$, hence by induction on r we obtain the second isomorphism in Theorem C. □

Proof of Lemma 1. By Herbert's theorem the algebraic number of the $(2k+1)$–tuple points of an immersion $f\colon M^{4k} \looparrowright R^{4k+2}$ is $\langle \bar{p}_1{}^k, [M^{4k}]\rangle$. The immersion $f$represents an element $[f]$ of the corresponding cobordism group of immersions of oriented $4k$–manifolds in R^{4k+2}. This cobordism group is isomorphic to the group $\pi^s_{4k+2}(MSO(2))$, the element of the later group corresponding to $[f]$ will be denoted by $[\alpha_f]$. Here α_f is the Pontrjagin–Thom map $S^{q+4k+2} \longrightarrow S^q MSO(2)$, for q big enough.

Let us consider the following composition of maps

$$\pi^s_{4k+2}(MSO(2)) \xrightarrow[①]{} H_{4k+2+q}\big(S^q MSO(2)\big) \xrightarrow[②]{\approx} H_{4k}(BSO(2)) \xrightarrow[③]{\approx} Z.$$

Here ① is the stable Hurewicz homomorphism, ② is the Thom isomorphism in the homologies

$$x \longrightarrow S^q U_2 \cap x$$

where U_2 is the Thom class of $MSO(2)$ and S^qU_2 its q^{th} suspension, and also the Thom class of $S^q MSO(2)$.

③ is the evaluation on the class ${p_1}^k$

$$y \longrightarrow \langle y, {p_1}^k\rangle.$$

Since the maps ② and ③ are isomorphisms, the cokernel of this composition is the same as the cokernel of ①, i.e. of the stable Hurewicz homomorphism.

On the other hand, we show that the image of this composition map is $t(k)Z$, and that will prove part b) of Lemma 1. (Part a) follows then as well, since the rational stable Hurewicz homomorphism

$$\pi^s_m(X) \otimes Q \longrightarrow H_m(X; Q) \text{ is an isomorphism.)}$$

Claim. *The composition of the maps ①, ②, ③ has image $t(k)\cdot Z$.*

Proof It is enough to show that the image of $[\alpha_f] \in \pi^s_{4k+2}(MSO(2))$ is $\langle \bar{p}_1{}^k, [M^{4k}]\rangle$.

$[\alpha_f]$ goes by the map ① to $(\alpha_f)_*[S^{q+4k+2}]$, that is mapped by ② to $(\alpha_f)_*[S^{q+4k+2}] \cap S^qU_2$.

Let ν be the normal bundle of f, let us denote by $T\nu$ its Thom space, let $pr\colon S^{q+4k+2} \longrightarrow S^qT\nu$ be the Pontrjagin–Thom map, $\beta_f\colon S^qT\nu \longrightarrow S^q\, MSO(2)$ the fiberwise map of Thom spaces that on the base spaces is the map $\nu_f\colon M \longrightarrow BSO(2)$ inducing the normal bundle ν.

Now $(\alpha_f)_*[S^{q+4k+2}] = (\beta_f)_* \circ pr_*[S^{q+4k+2}] = (\beta_f)_*[S^qT\nu]$. Here $[S^qT\nu]$ is the fundamental homology class of $S^qT\nu$. Therefore

$$\begin{aligned}\left\langle {p_1}^k, (\alpha_f)_*[S^{q+4k+2}] \cap S^qU_2 \right\rangle &= \left\langle {p_1}^k, (\beta_f)_*[S^qT\nu] \cap S^qU_2 \right\rangle \\ &= \left\langle {p_1}^k, (\nu_f)_*[M] \right\rangle = \left\langle {\nu_f}^* {p_1}^k, [M] \right\rangle = \left\langle {\overline{p}_1}^k, [M] \right\rangle.\end{aligned}$$

□

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ISSN #1949-2006

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