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ON DIVISORIAL FILTRATIONS ASSOCIATED WITH NEWTON DIAGRAMS

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Abstract. We consider divisorial filtrations on the rings of functions on hypersurface singularities associated with Newton diagrams and their analogues for plane curve singularities. We compute the multi-variable Poincaré series for the latter ones.

Introduction

A multi-index filtration on the ring $\mathcal{O}_{V,0} = \mathcal{O}_{\mathbb{C}^n,0}/(f)$ of functions on a hypersurface singularity $(V,0) = \{f = 0\}$ defined by the Newton diagram $\Gamma = \Gamma_f$ of the germ f was considered in [4]. The initial idea was to look for a filtration corresponding to a Newton diagram for which the Poincaré series could be computed and compared with the corresponding monodromy zeta function. This was inspired by the coincidence of Poincaré series and monodromy zeta functions in some cases (e.g. in [1]) and relations between them in some other cases (e.g. in [3]). A somewhat natural filtration on the ring $\mathcal{O}_{V,0}$ corresponding to the Newton diagram $\Gamma = \Gamma_f$ is the divisorial filtration defined by the divisors in a toric resolution of f corresponding to the facets of the diagram. However, at that moment the divisorial valuation was regarded as being complicated to treat. The filtration defined in [4] was regarded as a certain "simplification" of the divisorial one. This seems not to be the case. It is rather complicated to compute the Poincaré series of that filtration and moreover the assertion of Theorem 1 of [4] for $s > 2$ appeared to be wrong. Another filtration corresponding to a Newton diagram was considered in [5].

Here we discuss an analogue of the divisorial valuation corresponding to a Newton diagram, describe its generalization for plane curve singularities, and compute the Poincaré series of the latter one.

For a germ $(V,0)$ of a complex analytic variety, let $\pi : (\mathcal{X}, \mathcal{D}) \to (V,0)$ be a resolution of $(V,0)$ with the exceptional divisor $\mathcal{D} = \pi^{-1}(0)$ being a normal crossing divisor on $\mathcal{X}$. For an irreducible component $\mathcal{E}$ of $\mathcal{D}$ and for $g \in \mathcal{O}_{V,0}$, let $v_{\mathcal{E}}(g)$ be the order of the zero of the lifting $\widetilde{g} = g \circ \pi$ of the germ g to the space $\mathcal{X}$ of the resolution along $\mathcal{E}$. The function $v_{\mathcal{E}} : \mathcal{O}_{V,0} \to \mathbb{Z}_{\geq 0} \cup \{+\infty\}$ is called a *divisorial valuation* on $\mathcal{O}_{V,0}$. One can consider the multi-index filtration defined by a collection $\mathcal{E}_1, \ldots, \mathcal{E}_r$ of components of the exceptional divisor:

$$J(\underline{v}) = \{g \in \mathcal{O}_{V,0} : \underline{v}(g) \geq \underline{v}\}\,, \tag{1}$$

where $\underline{v} = (v_1, \ldots, v_r) \in \mathbb{Z}^r_{\geq 0}$, $\underline{v}(g) = (v_1(g), \ldots, v_r(g))$, $v_i(g) = v_{\mathcal{E}_i}(g)$, $\underline{v}' = (v'_1, \ldots, v'_r) \geq \underline{v}$ if and only if $v'_i \geq v_i$ for $i = 1, \ldots, r$. This filtration is called a *divisorial* one. The notion of the Poincaré series of a multi-index filtration was introduced in [2] (see also [1]). In [1] it was explained that the Poincaré series of a filtration defined by a formula like (1) is equal to the

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integral with respect to the Euler characteristic

$$P_{\{v_i\}}(\underline{t}) = \int_{\mathbb{P}\mathcal{O}_{V,0}} \underline{t}^{\underline{v}(g)} d\chi \tag{2}$$

over the projectivization $\mathbb{P}\mathcal{O}_{V,0}$ of $\mathcal{O}_{V,0}$ ($\underline{t} = (t_1, \ldots, t_r)$, $\underline{t}^{\underline{v}} = t_1^{v_1} \cdots t_r^{v_r}$). In this integral, t_i^{∞} has to be assumed to be equal to zero. Also in [1] it was shown that the Poincaré series of the divisorial filtration corresponding to all the components of the exceptional divisor of a resolution (uniformization) of a plane curve singularity $(C, 0) = \{f = 0\} \subset (\mathbb{C}^2, 0)$ (that is to all the components of the curve $(C, 0)$) coincides with the Alexander polynomial (in several variables) of the corresponding link $C \cap S^3_\varepsilon \subset S^3_\varepsilon$, where S^3_ε is the sphere of small radius ε centred at the origin in $\mathbb{C}^2$. (The Alexander polynomial becomes the monodromy zeta function of the left hand side f of the equation of the curve $(C, 0)$ after identification of all the variables.)

For the definition of a multi-index filtration by the formula (1), it is not necessary to assume that all the $v_i : \mathcal{O}_{V,0} \to \mathbb{Z}_{\geq 0} \cup \{+\infty\}$ are valuations (i.e. that they satisfy the condition $v_i(g_1 g_2) = v_i(g_1) + v_i(g_2)$). It is sufficient to require that all of them are so called *order functions*. This means that they satisfy the condition $v_i(g_1 + g_2) \geq \min\{v_i(g_1), v_i(g_2)\}$, but, in general not the condition $v_i(g_1 g_2) = v_i(g_1) + v_i(g_2)$. We shall use order functions to define the filtrations below.

1. Divisorial filtration corresponding to a Newton diagram

Let $\mathbb{C}^n$ be the complex space with the coordinates $x_1, \ldots, x_n$ and let $f \in \mathcal{O}_{\mathbb{C}^n,0}$ be a function germ non-degenerate with respect to its Newton diagram $\Gamma = \Gamma_f$. Let $p : (X, D) \to (\mathbb{C}^n, 0)$ be a toric resolution of f corresponding to the Newton diagram Γ. The facets of Γ correspond to some components (say, $E_1, \ldots, E_r$) of the exceptional divisor D. Let $(V, 0) = \{f = 0\}$, let $\widetilde{V}$ be the strict transform of the hypersurface singularity V, and let $\mathcal{E}_i := \widetilde{V} \cap E_i$.

For $n \geq 3$ the $\mathcal{E}_i$ are the irreducible components of the exceptional divisor $\mathcal{D} = D \cap \widetilde{V}$ of the resolution $p_{|\widetilde{V}} : (\widetilde{V}, \mathcal{D}) \to (V, 0)$. Thus one can consider the divisorial valuations v_i defined by these components and the corresponding (multi-index) filtration on $\mathcal{O}_{V,0}$. For $n = 2$ the intersections $\mathcal{E}_i$ are not, in general, irreducible (if the corresponding facets of Γ have integer points in their interiors). Therefore for $n = 2$ the corresponding definition has to be modified.

Let us first reformulate the definition of the divisorial valuations (for $n \geq 3$) in terms of the Newton diagram Γ. Let $\gamma_1, \ldots, \gamma_r$ be the facets of the diagram Γ and let $\ell_i(\bar{k}) = c_i$ be the reduced equation of the facet γ_i, $i = 1, \ldots, r$. This means that $\ell_i(\bar{k}) = a_{i1}k_1 + \ldots + a_{in}k_n$ ($\bar{k} = (k_1, \ldots, k_n)$), where $a_{i1}, \ldots, a_{in}$ are positive integers with greatest common divisor equal to 1.

For $g \in \mathcal{O}_{\mathbb{C}^n,0}[x_1^{-1}, \ldots, x_n^{-1}]$, $g(\bar{x}) = \sum_{\bar{k}} c_{\bar{k}} \bar{x}^{\bar{k}}$ ($\bar{x} = (x_1, \ldots, x_n)$), and for $i = 1, \ldots, r$, let $u_i(g) := \min_{\bar{k}: c_{\bar{k}} \neq 0} \ell_i(\bar{k})$, and let $g_{\gamma_i}(\bar{x}) = \sum_{\bar{k}: \ell_i(\bar{k}) = u_i(g)} c_{\bar{k}} \bar{x}^{\bar{k}}$. For $g \in \mathcal{O}_{\mathbb{C}^n,0}/(f)$ (or rather for $g \in \mathcal{O}_{\mathbb{C}^n,0}$) let us define $\widehat{v}_i(g)$ by

$$\widehat{v}_i(g) = \sup_{h \in \mathcal{O}_{\mathbb{C}^n,0}[x_1^{-1}, \ldots, x_n^{-1}]} u_i(g + hf)\,. \tag{3}$$

One can see that, for $n = 2$, $\widehat{v}_i : \mathcal{O}_{\mathbb{C}^n,0}/(f) \to \mathbb{Z}_{\geq 0} \cup \{+\infty\}$ is not, in general, a valuation, but only an order function.

Example. Let $f(x, y) = y^3 + y^2 x - x^5$ and let γ_1 be the facet of Γ_f defined by the equation $2k_y + k_x = 5$. Let $g_1(x, y) = y + x^2$, $g_2(x, y) = y - x^2$. One has $\widehat{v}_1(g_i) = u_1(g_i) = 2$ for $i = 1, 2$, but $\widehat{v}_1(g_1 g_2) = u_1(g_1 g_2 - x^{-1} f) = u_1(-y^3 x^{-1}) = 5$.

Remark. One can see that this definition resembles the definition used in [4] where similar order functions were defined by equation (3) with $\mathcal{O}_{\mathbb{C}^n,0}[x_1^{-1},\ldots,x_n^{-1}]$ substituted by $\mathcal{O}_{\mathbb{C}^n,0}$.

Proposition 1. *For $n \geq 3$, $i = 1, \ldots, r$, and $g \in \mathcal{O}_{\mathbb{C}^n,0}$ one has*

$$\widehat{v}_i(g) = v_i(g)\,.$$

Proof. The claim follows from the following statements:

1) $v_i(g) \geq \widehat{v}_i(g)$;
2) if $f_{\gamma_i} \not| g_{\gamma_i}$, then $v_i(g) = u_i(g)$;
3) if $f_{\gamma_i} | g_{\gamma_i}$, then there exists $h \in \mathcal{O}_{\mathbb{C}^n,0}[x_1^{-1},\ldots,x_n^{-1}]$ such that $u_i(g+hf) > u_i(g)$.

Indeed, by iterated applications of 2) and 3) one obtains that either $\widehat{v}_i(g) = \infty$ or there exists $h \in \mathcal{O}_{\mathbb{C}^n,0}[x_1^{-1},\ldots,x_n^{-1}]$ such that $v_i(g) = u_i(g+hf)$. Therefore $\widehat{v}_i(g) = \sup\limits_{h\in\mathcal{O}_{\mathbb{C}^n,0}[x_1^{-1},\ldots,x_n^{-1}]} u_i(g+hf) \geq v_i(g)$ and 1) implies the assertion.

Statement 1) follows from the facts that: $u_i(g)$ is the order of vanishing of the lifting $g \circ \pi$ of g along E_i; $v_i(g)$ is the order of vanishing of $g \circ \pi_{|\widetilde{V}}$ along $\mathcal{E}_i \subset E_i$ and therefore $v_i(g) \geq u_i(g)$; $v_i(g) = v_i(g + hf)$ for any $h \in \mathcal{O}_{\mathbb{C}^n,0}[x_1^{-1},\ldots,x_n^{-1}]$.

If $f_{\gamma_i} \not| g_{\gamma_i}$, then the intersection $\widetilde{\{g = 0\}} \cap E_i$ of the strict transform $\widetilde{\{g = 0\}}$ with the component E_i does not contain $\mathcal{E}_i$. Therefore the order of vanishing of $g \circ \pi_{|\widetilde{V}}$ along $\mathcal{E}_i$ coincides with the order of vanishing of $g \circ \pi$ along E_i, equal to $u_i(g)$. This gives 2).

If $g_{\gamma_i} = hf_{\gamma_i}$ ($h \in \mathcal{O}_{\mathbb{C}^n,0}[x_1^{-1},\ldots,x_n^{-1}]$), then $(g - hf)_{\gamma_i}$ contains with non-zero coefficients only monomials $\bar{x}^{\bar{k}}$ with $\ell_i(\bar{k}) > u_i(\bar{k})$. This gives 3). □

As it was mentioned above, for $n = 2$ the intersections $\mathcal{E}_i = E_i \cap \widetilde{V}$ may be reducible: i.e. consist of several points. In this case there is no divisorial valuation associated to $\mathcal{E}_i$. Let us modify (generalize) the definition of a divisorial valuation in the following way. Let $\mathcal{E} = \bigcup\limits_{j=1}^{s} \mathcal{E}^{(j)}$ be the union of some of the irreducible components of the exceptional divisor $\mathcal{D}$ of the resolution $\pi : (\widetilde{V}, \mathcal{D}) \to (V, 0)$ and for $g \in \mathcal{O}_{\mathbb{C}^n,0}$ let

$$v_{\mathcal{E}}(g) := \min_{j=1,\ldots,s} v_{\mathcal{E}^{(j)}}(g)\,.$$

The function $v_{\mathcal{E}} : \mathcal{O}_{V,0} \to \mathbb{Z}_{\geq 0} \cup \{+\infty\}$ is not, in general, a valuation (for $s > 1$), but an order function. The number $v_{\mathcal{E}}(g)$ can also be defined as the minimum over all arcs γ on $\widetilde{V}$ at points of $\mathcal{E}$ of the order of g along γ.

One can easily see that this definition gives order functions v_i on $\mathcal{O}_{V,0}$ corresponding to the facets of the Newton diagram $\Gamma = \Gamma_f$ for $n = 2$ as well so that Proposition 1 also holds in this case.

2. Plane curve singularities

Here we consider analogues of the order functions v_i corresponding to the facets of the Newton diagram Γ (for $n = 2$) for plane curve singularities not associated with Newton diagrams (say, for those whose components may have more than one Puiseux pair). We compute the Poincaré series of the corresponding filtration and give its specialization for the filtration defined by a Newton diagram. It seems to be less involved to carry out computations in this way than to produce them directly by considering Newton diagrams.

Let $(C, 0) \subset (\mathbb{C}^2, 0)$ be a plane curve singularity with an embedded resolution $\pi : (X, D) \to (\mathbb{C}^2, 0)$ such that

1) C is the union of irreducible components $C = \bigcup_{i,j} C_{ij}$, where $i = 1, \ldots, r$, $j = 1, \ldots, s_i$ ($s_i > 0$);
2) for each i the strict transforms $\widetilde{C}_{i1}, \ldots \widetilde{C}_{is_i}$ of the components $C_{i1}, \ldots C_{is_i}$ intersect one and the same component E_i of the exceptional divisor D;
3) for $i_1 \neq i_2$ the strict transforms $\widetilde{C}_{i_1j_1}$ and $\widetilde{C}_{i_2j_2}$ intersect different components of D (one can say that $E_1, \ldots, E_r$ are part of the set $\{E_\sigma : \sigma \in \Sigma\}$ of irreducible components of D).

For an irreducible component E_σ of the exceptional divisor D, $\sigma \in \Sigma$, let $w_\sigma : \mathcal{O}_{\mathbb{C}^2,0} \setminus \{0\} \to \mathbb{Z}_{\geq 0}$ be the corresponding divisorial valuation.

For $i = 1, \ldots, r$, $j = 1, \ldots, s_i$, let $\varphi_{ij} : (\mathbb{C}, 0) \to (\mathbb{C}^2, 0)$ be a uniformization of the component C_{ij}. For $g \in \mathcal{O}_{\mathbb{C}^2,0}$ let $v_{ij}(g)$ be the order of vanishing of $g \circ \varphi_{ij}$ at the origin. The function $v_{ij}(g) : \mathcal{O}_{\mathbb{C}^2,0} \to \mathbb{Z}_{\geq 0} \cup \{+\infty\}$ is a valuation on $\mathcal{O}_{\mathbb{C}^2,0}$. Let

$$v_i(g) := \min_{j=1,\ldots,s_j} v_{ij}(g)\,.$$

The function $v_i : \mathcal{O}_{\mathbb{C}^2,0} \to \mathbb{Z}_{\geq 0} \cup \{+\infty\}$ is, in general, not a valuation, but an order function (if $s_i > 1$).

The order functions $v_1, \ldots, v_r$ define in the usual way an r-index filtration on $\mathcal{O}_{\mathbb{C}^2,0}$:

$$J(\underline{v}) = \{g \in \mathcal{O}_{\mathbb{C}^2,0} : \underline{v}(g) \geq \underline{v}\}\,, \tag{4}$$

where, as usual, $\underline{v} = (v_1, \ldots, v_r) \in \mathbb{Z}^r_{\geq 0}$, $\underline{v}(g) = (v_1(g), \ldots, v_r(g))$. We shall call it the *generalized divisorial filtration.*

Let $\{E_\sigma : \sigma \in \Sigma\}$ be the set of all irreducible components of the exceptional divisor D ($\Sigma \supset \{1, \ldots, r\}$). Each component E_σ is isomorphic to the complex projective line $\mathbb{CP}^1$. For $\sigma \in \Sigma$, let $\overset{\bullet}{E}_\sigma$ be the "smooth part" of the component E_σ in the exceptional divisor D, that is E_σ itself minus the intersection points with all the other components of D, and let $\overset{\circ}{E}_\sigma$ be the "smooth part" of the component E_σ in the total transform of the curve C, that is E_σ itself minus the intersection points with other components of D and also with the strict transform of the curve C. (One has $\overset{\circ}{E}_\sigma = \overset{\bullet}{E}_\sigma$ for $\sigma \notin \{1, \ldots, r\}$; for $\sigma = i \in \{1, \ldots, r\}$, $\overset{\circ}{E}_\sigma$ is $\overset{\bullet}{E}_\sigma$ minus s_i points.)

For $\sigma \in \Sigma$, let $\widetilde{L}_\sigma$ be a smooth arc on the space X of the resolution transversal to E_σ at a smooth point (i.e. at a point of $\overset{\bullet}{E}_\sigma$). Let the (irreducible) curve $L_\sigma = \pi(\widetilde{L}_\sigma)$ be given by an equation $g_\sigma = 0$ ($g_\sigma \in \mathcal{O}_{\mathbb{C}^2,0}$). The curve L_σ (or sometimes the function g_σ) is called *a curvette* at E_σ. Let $m_{\sigma\delta}$ ($\sigma, \delta \in \Sigma$) be the order of vanishing of g_σ along the component E_δ, that is $m_{\sigma\delta} = w_\delta(g_\sigma)$. One can show that $m_{\sigma\delta} = m_{\delta\sigma}$ and the matrix $(m_{\sigma\delta})$ is minus the inverse matrix of the intersection matrix $(E_\sigma \circ E_\delta)$ of the components E_σ on the manifold X. For $\sigma \in \Sigma$, let $\underline{m}_{\,\sigma} := (m_{\sigma 1}, \ldots, m_{\sigma r}) \in \mathbb{Z}^r_{\geq 0}$.

Theorem 1. *The Poincaré series of the generalized divisorial filtration (4) is equal to*

$$P_{\{v_i\}}(\underline{t}) = \prod_{\sigma\in\Sigma} (1 - \underline{t}^{\underline{m}_\sigma})^{-\chi(\overset{\bullet}{E}_\sigma)} \cdot \prod_{i=1}^{r} (1 - \underline{t}^{s_i \underline{m}_i})\,. \tag{5}$$

Example. Let $s_i = 1$ for $i = 1, \ldots, r$. In this case $\chi(\overset{\circ}{E}_\sigma) = \chi(\overset{\bullet}{E}_\sigma)$ for $\sigma \notin \{1, \ldots, r\}$ and $\chi(\overset{\circ}{E}_i) = \chi(\overset{\bullet}{E}_i) - 1$. Therefore one has

$$P_{\{v_i\}}(\underline{t}) = \prod_{\sigma\in\Sigma} (1 - \underline{t}^{\underline{m}_\sigma})^{-\chi(\overset{\circ}{E}_\sigma)}\,.$$

This is just the formula from [1].

Let $\pi : (X, D) \to (\mathbb{C}^2, 0)$ be a toric resolution corresponding to the Newton diagram $\Gamma = \Gamma_f$ of a (Γ-non-degenerate) germ $f \in \mathcal{O}_{\mathbb{C}^2,0}$. The dual graph of the resolution π is a chain. The extreme vertices of this graph correspond to the components of the exceptional divisor intersecting the strict transforms of the coordinate lines in $\mathbb{C}^2$. (Therefore $\{x = 0\}$ and $\{y = 0\}$ are curvettes corresponding to these components.) For these two components one has $\chi(\overset{\bullet}{E}_\sigma) = 1$, for all others $\chi(\overset{\bullet}{E}_\sigma) = 0$. Therefore one has

Corollary 1. *The Poincaré series of the filtration associated with the Newton diagram Γ and defined by the order function $\widehat{v}_i$ corresponding to the facets of Γ is equal to*

$$P_{\{\widehat{v}_i\}}(\underline{t}) = \frac{\prod_{i=1}^{r}(1 - \underline{t}^{s_i \underline{m}_i})}{(1 - \underline{t}^{\underline{v}(x)})(1 - \underline{t}^{\underline{v}(y)})}\,. \tag{6}$$

Remark. A function germ f which is non-degenerate with respect to the Newton diagram $\Gamma = \Gamma_f$ can be represented in the form $f = x^a y^b \prod_{i=1}^{r} f_i$ where $\{f_i = 0\}$ is the union of the components of $\{f = 0\}$ whose strict transforms intersect the component E_i of the exceptional divisor of a toric resolution. One can see that the number s_i of irreducible factors in a decomposition of f_i is equal to the integer length of the facet γ_i (i.e. to the number of integer points in its interior plus one) and the Newton diagram Γ_i of f_i is just the facet γ_i of Γ translated to the origin inside the positive octant as far as possible. Moreover, the jth component of $s_i \underline{m}_i$ is equal to $\min_{\bar{k} \in \Gamma_i} \ell_j(\bar{k})$.

Proof of Theorem 1. Let $\mathcal{J}^N_{\mathbb{C}^2,0} = \mathcal{O}_{\mathbb{C}^2,0}/\mathfrak{m}^{N+1}$ be the space of N-jets of functions on $(\mathbb{C}^2, 0)$ ($\mathfrak{m}$ is the maximal ideal of $\mathcal{O}_{\mathbb{C}^2,0}$). One can see that for a function $g \in \mathcal{O}_{\mathbb{C}^2,0}$ with $w_\sigma(g) \leq N$ for all $\sigma \in \Sigma$, the values $w_\sigma(g)$ and also $v_i(g)$ are defined by the N-jet $j^N g$ of g. (This follows from the fact that, for $h \in \mathfrak{m}^{N+1}$, all $w_\sigma(h)$ and $v_i(h)$ are greater than N.) Let $\widehat{\mathcal{J}}^N \subset \mathcal{J}^N_{\mathbb{C}^2,0}$ be the set of N-jets g with $w_\sigma(g) \leq N$ for all $\sigma \in \Sigma$. The equation (2) implies that

$$P_{\{v_i\}}(\underline{t}) \equiv \int_{\mathbb{P}\widehat{\mathcal{J}}^N} \underline{t}^{\underline{v}(g)} d\chi$$

modulo terms of degree $> N$. Recall that here t_i^∞ should be assumed to be equal to 0.

Without loss of generality, we can suppose that, for any function $g \in \mathcal{O}_{\mathbb{C}^2,0}$ with $w_\sigma(g) \leq N$ for all $\sigma \in \Sigma$, the strict transform $\widetilde{\{g = 0\}}$ of the zero level curve of g intersects the exceptional divisor D only at smooth points, i.e. at points of $\overset{\bullet}{D} = \bigcup_\sigma \overset{\bullet}{E}_\sigma$. Such a resolution can be obtained, if necessary, by additional blow-ups of intersection points of the components of D. Each such blow-up produces an additional component E_σ with $\chi(\overset{\bullet}{E}_\sigma) = 0$ and therefore it does not effect the right hand side of the equation (5).

Let

$$Y = \coprod_{\{k_\sigma\}} \left(\prod_\sigma S^{k_\sigma} \overset{\bullet}{E}_\sigma \right) = \prod_\sigma \left(\coprod_{k=0}^{\infty} S^k \overset{\bullet}{E}_\sigma \right)$$

be the configuration space of all effective divisors on $\overset{\bullet}{D} = \bigcup \overset{\bullet}{E}_\sigma$ and let $\underline{w} : Y \to \mathbb{Z}^r_{\geq 0}$ be the function which maps the component $\prod_\sigma S^{k_\sigma} \overset{\bullet}{E}_\sigma$ of Y to $\sum_\sigma k_\sigma \underline{m}_\sigma$. For a function $g \in \mathcal{O}_{\mathbb{C}^2,0}$ with $w_\sigma(g) \leq N$ for all $\sigma \in \Sigma$, let $I(g) \in Y$ be the intersection of the strict transform $\widetilde{\{g = 0\}}$ of $\{g = 0\}$ with D, i.e. the collection of the intersection points with multiplicities. One can see that $I(g)$ only depends on the N-jet of g, $(w_1(g), \ldots, w_r(g)) = \underline{w}(I(g))$ and also $(v_1(g), \ldots, v_r(g)) =$

$\underline{w}(I(g))$ if (and only if) for each $i = 1, \dots, r$, the effective divisor $I(g)$ does not contain all the points $p_{i1}, \dots, p_{is_i}$. (If $I(g)$ contains all the points $p_{i1}, \dots, p_{is_i}$, then $v_i(g)$ is not determined by $I(g)$.)

For a component E_σ of D let $g_{\sigma q} = g_{\sigma q}(x, y)$ be an analytic family of functions such that $\{g_{\sigma q} = 0\}$ is a curvette corresponding to the component E_σ and its strict transform passes through the point $q \in \overset{\bullet}{E}_\sigma$. (One can take two functions $g_{\sigma,q'}$ and $g_{\sigma,q''}$ with the described properties for two different points q' and q'' from $\overset{\bullet}{E}_\sigma$ and define $g_{\sigma q}$ as $\lambda g_{\sigma,q'} + \mu g_{\sigma,q''}$ with appropriate λ and μ.)

If $A = B \coprod C$, then

$$\coprod_{k=0}^{\infty} S^k A = \left(\coprod_{k=0}^{\infty} S^k B \right) \times \left(\coprod_{k=0}^{\infty} S^k C \right).$$

This permits to rewrite Y as $Y' \times Y''$, where

$$Y' = \prod_{\sigma} \left(\coprod_{k=0}^{\infty} S^k \overset{\circ}{E}_\sigma \right), \quad Y'' = \prod_{i} \left(\coprod_{k=0}^{\infty} S^k P_i \right),$$

where P_i is the set $\{p_{i1}, \dots, p_{is_i}\}$ consisting of s_i points.

For $y \in Y$, $y = \sum_{\sigma,j} \ell'_{\sigma j} q_{\sigma j} + \sum_{i=1}^{r} \sum_{j=1}^{s_i} \ell''_{ij} p_{ij}$, where $q_{\sigma j}$ are points of $\overset{\circ}{E}_\sigma$, let

$$g_y := \prod_{\sigma,j} g_{\sigma q_{\sigma j}}^{\ell'_{\sigma j}} \cdot \prod_{i=1}^{r} \prod_{j=1}^{s_i} f_{ij}^{\ell''_{ij}},$$

where $g_{\sigma q_{\sigma j}}$ is the curvette corresponding to E_σ through the point $q_{\sigma j}$. One can see that $I(g_y) = y$.

For an element $g \in \widehat{\mathcal{J}}^N$ with $I(g) = y$, one has $I(g) = I(g_y)$, i.e. the strict transforms of $\{g = 0\}$ and $\{g_y = 0\}$ intersect the exceptional divisor D at the same points with the same multiplicities. This means that the ratio $g_y \circ \pi / g \circ \pi$ of the liftings of g and g_y is regular (has no zeros and poles) on D and therefore it is constant (say, equal to a) on it. If $g \neq g_y$, let $h_\lambda := g_y + \lambda(ag - g_y)$ for $\lambda \in \mathbb{C}^*$. One can see that $w_\sigma(h_\lambda)$ and $v_i(h_\lambda)$ do not depend on λ. In this way we decompose the space of elements of $\mathbb{P}\widehat{\mathcal{J}}^N$ different from all g_y into $\mathbb{C}^*$-families with constant values of $\underline{v}$. Since the Euler characteristic of $\mathbb{C}^*$ is equal to zero, this means that the integral (with respect to the Euler characteristic) of $\underline{t}^{\underline{v}}$ over the complement of $\{g_y\}$ is equal to zero and therefore (up to terms of degree $> N$)

$$P_{\{v_i\}}(\underline{t}) \equiv \int_{Y} \underline{t}^{\,\underline{v}(g_y)} d\chi.$$

For $y \in Y$, $v_i(g_y)$ is finite if and only if y does not contain all the points $p_{i,1}, \dots, p_{i,s_i}$. If, for each i, y does not contain all the points $p_{i,1}, \dots, p_{i,s_i}$, one has $\underline{v}(g_y) = \underline{w}(y)$. Therefore

$$\int_{Y} \underline{t}^{\,\underline{v}(g_y)} d\chi = \int_{Y'} \underline{t}^{\,\underline{w}(y')} d\chi \cdot \int_{Y''_0} \underline{t}^{\,\underline{w}(y'')} d\chi\,, \tag{7}$$

where $Y''_0 \subset Y''$ is the set of elements $\sum\limits_{i=1}^{r} \sum\limits_{j=1}^{s_j} \ell_{ij} p_{ij}$ such that for each i at least one of the coefficients ℓ_{ij} is equal to zero.

One has

$$\int_{Y'} \underline{t}^{\,\underline{w}(y')} d\chi = \prod_{\sigma \in \Sigma} \left(\sum_{k=0}^{\infty} \chi(S^k \overset{\circ}{E}_\sigma) \underline{t}^{\,k \underline{m}_\sigma} \right).$$

Due to the equation

$$\sum_{k=0}^{\infty} \chi(S^k Z) t^k = (1-t)^{-\chi(Z)}$$

one has

$$\int\limits_{Y'} \underline{t}^{\,\underline{w}(y')} d\chi = \prod (1-\underline{t}^{\,\underline{m}_\sigma})^{-\chi(\overset{\circ}{E}_\sigma)}\,. \tag{8}$$

(This is just the computation from [1].)

For the second factor in (7) one has

$$\begin{aligned} \int\limits_{Y_0''} \underline{t}^{\,\underline{w}(y'')} d\chi &= \prod_{i=1}^{r} \left(\sum_{(\ell_{i1},\ldots,\ell_{is_i}) \in \mathbb{Z}_{\geq 0}^{s_i} \setminus \mathbb{Z}_{>0}^{s_i}} \underline{t}^{\,(\sum \ell_{ij}) \underline{m}_i} \right) \\ &= \prod_{i=1}^{r} \left(\sum_{(\ell_{i1},\ldots,\ell_{is_i}) \in \mathbb{Z}_{\geq 0}^{s_i}} \underline{t}^{\,(\sum \ell_{ij}) \underline{m}_i} - \sum_{(\ell_{i1},\ldots,\ell_{is_i}) \in \mathbb{Z}_{>0}^{s_i}} \underline{t}^{\,(\sum \ell_{ij}) \underline{m}_i} \right) \\ &= \prod_{i=1}^{r} \left[(1-\underline{t}^{\,\underline{m}_i})^{-s_i} - \underline{t}^{\,s_i \underline{m}_i} (1-\underline{t}^{\,\underline{m}_i})^{-s_i} \right] = \prod_{i=1}^{r} (1-\underline{t}^{\,\underline{m}_i})^{-s_i} (1-\underline{t}^{\,s_i \underline{m}_i})\,. \end{aligned} \tag{9}$$

Since $\chi(\overset{\bullet}{E}_\sigma) = \chi(\overset{\circ}{E}_\sigma) + s_i$, the equations (7), (8), and (9) imply (5). □

Remark. Here, in contrast to [1], we make computations of integrals with respect to the Euler characteristic not over $\mathbb{P}\mathcal{O}_{\mathbb{C}^2,0}$, but over a subspace of $\mathbb{P}\mathcal{J}^N_{\mathbb{C}^2,0}$ since the set of functions $\{g_y | y \in Y\}$ is not measurable in $\mathbb{P}\mathcal{O}_{\mathbb{C}^2,0}$ (i.e. its Euler characteristic is not defined).

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FOLIATIONS ON $\mathbb{P}^2$ ADMITTING A PRIMITIVE MODEL

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Abstract. Given a foliation $\mathcal{F}$ on $\mathbb{P}^2_{\mathbb{C}}$, by fixing a line $L \subset \mathbb{P}^2_{\mathbb{C}}$, the polar pencil of $\mathcal{F}$ with axis L is the set of all polar curves of $\mathcal{F}$ with respect to points $l \in L$. In this work we study foliations $\mathcal{F}$ which admit a polar pencil whose generic element is reducible. To such an $\mathcal{F}$ is associated a primitive model, which is a foliation $\widetilde{\mathcal{F}}$ whose polar pencil, besides having irreducible generic element, is such that its fibers are contained in those of the polar pencil of $\mathcal{F}$. This work focuses on relating geometric properties of a foliation $\mathcal{F}$ with those of its primitive model $\widetilde{\mathcal{F}}$.

1. Introduction

This work deals with reducibility properties of the pencil of algebraic curves

$$\mathcal{P}: \{\alpha P(x,y) + \beta Q(x,y) = 0;\ (\alpha : \beta) \in \mathbb{P}^1\},$$

where $(x,y) \in \mathbb{C}^2$ and $P(x,y)$ and $Q(x,y)$ are polynomials in $\mathbb{C}[x,y]$. More specifically, we want to give conditions that identify when the generic element of this pencil is reducible. One situation is obvious: if the generators P and Q have a common irreducible factor, then this will be a factor for all elements in this pencil. Thus we can suppose P and Q relatively prime. In this case, Stein's factorization Theorem (see [3]) asserts that the generic element of $\mathcal{P}$ is reducible if and only if there are polynomials $\widetilde{P}(x,y)$ and $\widetilde{Q}(x,y)$ and a rational function $r : \mathbb{P}^1 \to \mathbb{P}^1$ of degree greater than one such that

$$\frac{P(x,y)}{Q(x,y)} = r\left(\frac{\widetilde{P}(x,y)}{\widetilde{Q}(x,y)}\right).$$

To this situation we associate two foliations on the projective plane $\mathbb{P}^2$: a foliation $\mathcal{F}$ induced in affine coordinates $(x,y) \in \mathbb{C}^2$ by the polynomial vector field

$$\mathbf{v} \;=\; P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y},$$

and a second foliation $\widetilde{\mathcal{F}}$ induced in the same affine coordinates by the vector field

$$\widetilde{\mathbf{v}} \;=\; \widetilde{P}(x,y)\frac{\partial}{\partial x} + \widetilde{Q}(x,y)\frac{\partial}{\partial y}.$$

We call $\mathcal{F}$ a *non-primitive* foliation and, if the generic element of the pencil

$$\widetilde{\mathcal{P}}: \{\alpha \widetilde{P}(x,y) + \beta \widetilde{Q}(x,y) = 0;\ (\alpha : \beta) \in \mathbb{P}^1\}$$

is irreducible, we say that $\widetilde{\mathcal{F}}$ is a *primitive* foliation, which is a *primitive model* for $\mathcal{F}$. Our idea is to study this configuration by relating geometric properties of $\mathcal{F}$ and $\widetilde{\mathcal{F}}$. As a byproduct, we

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will obtain information about problem of the reducibility of the generic element of the pencil $\mathcal{P}$ itself.

After presenting basic facts about foliations on $\mathbb{P}^2$ in section 2, we develop in section 3 the concept of primitive and non-primitive foliations. We prove that a non-primitive foliation and its primitive model have the same singularities in the affine plane $\mathbb{C}^2$ and, in Proposition 2, we establish a relation between their Milnor numbers. A consequence of this fact is that a foliation having only non-degenerate singularities is primitive. This, in its turn, implies that the generic foliation in the space of foliations of degree d on $\mathbb{P}^2$ is non-primitive.

We finally dedicate section 4 to the study of the singularities of $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ that lie over the line at infinity L_∞. Proposition 3 asserts that a non-primitive foliation always has singularities in L_∞. We also consider the case where L_∞ is invariant by $\mathcal{F}$ and the sum of its Milnor numbers over L_∞ is minimal, equal to the degree of the foliation plus one. By Proposition 5, this occurs if and only if all singularities of $\mathcal{F}$ in L_∞ are either non-degenerate or saddle nodes having L_∞ as a weak separatrix. Proposition 7 says that, when both a non-primitive foliation $\mathcal{F}$ and its primitive model $\widetilde{\mathcal{F}}$ leave L_∞ invariant, then the sum of the Milnor numbers at L_∞ is minimal for $\mathcal{F}$ if and only if it is minimal for $\widetilde{\mathcal{F}}$. This apparently contrasts to what happens to Milnor numbers of singularities on the affine plane $\mathbb{C}^2$: the passage from the primitive model $\widetilde{\mathcal{F}}$ to the non-primitive $\mathcal{F}$ "degenerates" these singularities, in the sense that their Milnor numbers increase, as shown in Proposition 2.

2. Preliminaries

A foliation $\mathcal{F}$ of degree $d \geq 0$ in $\mathbb{P}^2 = \mathbb{P}^2_{\mathbb{C}}$ is induced in homogeneous coordinates $(X : Y : Z) \in \mathbb{P}^2$ by a 1-form

$$\omega \;=\; A(X,Y,Z)dX + B(X,Y,Z)dY + C(X,Y,Z)dZ, \tag{1}$$

where A, B and C are homogeneous polynomials of degree $d+1$ satisfying the Euler condition

$$XA(X,Y,Z) + YB(X,Y,Z) + ZC(X,Y,Z) = 0. \tag{2}$$

This means that we have a foliation of dimension two on $\mathbb{C}^3$ which contains in its leaves the lines through the origin, so that the foliation goes down to a foliation of dimension one on $\mathbb{P}^2$. The singular set of $\mathcal{F}$, denoted by $\mathrm{Sing}(\mathcal{F})$, is the set of common zeroes of A, B and C. We suppose, throughout this text, that $\mathrm{Sing}(\mathcal{F})$ has codimension two, which amounts to requiring that A, B and C have no common factor. In the affine plane $Z = 1$ with affine coordinates $x = X/Z$ and $y = Y/Z$ the foliation $\mathcal{F}$ is induced by the 1-form

$$\omega \;=\; A(x,y,1)dx + B(x,y,1)dy.$$

The foliation $\mathcal{F}$ is also given by the integral curves of the dual vector field of ω:

$$\mathbf{v} \;=\; P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}.$$

Here $P(x,y) = -B(x,y,1)$ and $Q(x,y) = A(x,y,1)$. We have two situations: if the line at infinity $L_\infty : \{Z = 0\}$ is invariant by $\mathcal{F}$ then Z divides A and B. Furthermore, for $k > 1$, Z^k is not a common factor for A and B, since otherwise Z would be a common factor of A, B and C by the Euler condition. This implies that $\max\{\deg P, \deg Q\} = d$. On the other hand, if the line at infinity is not invariant by $\mathcal{F}$, then Z is not a factor of both A and B, thus $P(x,y) = -B(x,y,1)$ as well as $Q(x,y) = A(x,y,1)$ have degree $d+1$. The Euler condition written in affine coordinates reads

$$xA(x,y,1) + yB(x,y,1) + C(x,y,1) = xQ(x,y) - yP(x,y) + C(x,y,1) = 0.$$

The terms of degree $d+2$ in the above relation give the equation

$$xQ_{d+1}(x,y) - yP_{d+1}(x,y) = 0,$$

where P_{d+1} and Q_{d+1} stand for the homogeneous part of degree $d+1$ of P and Q, respectively. Thus, there is a homogenous polynomial $G(x,y)$ of degree d such that $P_{d+1}(x,y) = xG(x,y)$ and $Q_{d+1}(x,y) = yG(x,y)$. We conclude that, when L_∞ is not invariant, $\mathcal{F}$ is induced by a vector field of the type

$$\mathbf{v} \;=\; (xG(x,y) + \hat{P}(x,y))\frac{\partial}{\partial x} + (yG(x,y) + \hat{Q}(x,y))\frac{\partial}{\partial y}, \tag{3}$$

where $\hat{P}$ and $\hat{Q}$ comprise the terms of degree d and lower of P and Q.

Reciprocally, let $\mathcal{F}$ be a foliation induced in affine coordinates (x,y) by a polynomial vector field of the form

$$\mathbf{v} \;=\; (xG(x,y) + \hat{P}(x,y))\frac{\partial}{\partial x} + (yG(x,y) + \hat{Q}(x,y))\frac{\partial}{\partial y}\,,$$

where G, when non-zero, is a homogeneous polynomial of degree d, while $\hat{P}$ and $\hat{Q}$ are either polynomials of degree d, when $G = 0$, or of degree d or lower, when $G \neq 0$. Then $\mathcal{F}$ is a foliation of degree d and L_∞ is $\mathcal{F}$-invariant if and only if $G = 0$.

Let now $\mathcal{F}$ be a germ of foliation at $p = (0,0) \in \mathbb{C}^2$, which is induced in local coordinates (x,y) by a vector field

$$\mathbf{v} \;=\; P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y},$$

where $P, Q \in \mathcal{O}_p$ are relatively prime germs of analytic functions. The *Milnor number* of $\mathcal{F}$ at p is defined as

$$\mu_p(\mathcal{F}) \;=\; \dim_{\mathbb{C}} \frac{\mathcal{O}_p}{(P,Q)},$$

where $(P,Q) \subset \mathcal{O}_p$ refers to the ideal generated by P and Q. Evidently, $\mu_p(\mathcal{F})$ is a non-negative integer, which is non-zero if and only p is a singularity for $\mathcal{F}$ (see [1] for more details).

Suppose now that the germ of foliation $\mathcal{F}$ has a smooth separatrix S, that is, a germ of holomorphic invariant curve passing through $p = (0,0)$. If we take local coordinates such that $S = \{y = 0\}$ then $\mathcal{F}$ will be induced by a vector field of the form

$$\mathbf{v} \;=\; P(x,y)\frac{\partial}{\partial x} + y\overline{Q}(x,y)\frac{\partial}{\partial y},$$

which, restricted to S, is the vector field $\mathbf{v}_{|S} \;=\; P(x,0)\partial/\partial x$. We define the *relative Milnor number* of $\mathcal{F}$ with respect to S as the order of $\mathbf{v}_{|S}$ at $x = 0$, that is

$$\mu_p(\mathcal{F}, S) \;=\; \dim_{\mathbb{C}} \frac{\mathcal{O}_p}{(P,y)} \;=\; \text{order}_{x=0}\mathbf{v}_{|S} \;=\; \text{order}_{x=0}P(x,0).$$

It comes straight from the definition that $\mu_p(\mathcal{F}, S) \leq \mu_p(\mathcal{F})$. We also remark that, when p is a regular point for $\mathcal{F}$, both numbers are zero.

Now, if S is a germ of a smooth analytic curve at p, non-invariant by $\mathcal{F}$, we take local coordinates (x,y) such that $p = (0,0)$ and $S : \{y = 0\}$, so that $Q(x,0) \neq 0$. The *order of tangency* between $\mathcal{F}$ and S at p is the following number:

$$\tau_p(\mathcal{F}, S) \;=\; \text{order}_{x=0}Q(x,0).$$

The invariants $\mu_p(\mathcal{F})$, $\mu_p(\mathcal{F}, S)$ and $\tau_p(\mathcal{F}, S)$ are independent of the local coordinates and of the local expression of a vector field representing $\mathcal{F}$.

Next we state some global results about these invariants which will be used in the sequel. Let $\mathcal{F}$ be a foliation of degree d on $\mathbb{P}^2$. First of all, given a line $L \subset \mathbb{P}^2$ non-invariant by $\mathcal{F}$, then

$$\sum_{p \in \mathbb{P}^2} \tau_p(\mathcal{F}, L) \;=\; d.$$

In fact, we can take a system of affine coordinates $(x, y) \in \mathbb{C}^2$ for which that L has equation $y = 0$ and such that $\mathcal{F}$ and L are not tangent at $q = L \cap L_\infty$, that is $\tau_q(\mathcal{F}, L) = 0$. Here L_∞ denotes the line at infinity. We can also suppose that L_∞ is not $\mathcal{F}$-invariant, so that $\mathcal{F}$ is induced by a polynomial vector field as in (3). Simple calculations show that the fact that $\tau_q(\mathcal{F}, L) = 0$ is equivalent to the degree of $\hat{Q}$ in (3) being d. Furthermore, since L is not invariant by $\mathcal{F}$, the variable y does not divide $\hat{Q}$, so that $\hat{Q}(x, 0)$ actually has degree d. The result follows by noticing that at each point $p = (x_0, 0) \in L$, the order of tangency $\tau_p(\mathcal{F}, L)$ is the multiplicity of x_0 as a root of $\hat{Q}(x, 0)$.

Now, if $L \subset \mathbb{P}^2$ is an $\mathcal{F}$-invariant line it holds

$$\sum_{p \in L} \mu_p(\mathcal{F}, L) \;=\; d + 1. \tag{4}$$

To see this, it suffices to take an affine coordinate system $(x, y) \in \mathbb{C}^2$ such that L_∞ is not invariant by $\mathcal{F}$, L has equation $y = 0$ and $q = L \cap L_\infty$ is a regular point for $\mathcal{F}$, so that $\mu_q(\mathcal{F}, L) = 0$. Thus, supposing that $\mathcal{F}$ is induced by a vector field as in (3), for a point $p = (x_0, 0) \in L$, we have that $\mu_p(\mathcal{F}, L)$ is the order of x_0 as a root of $P(x, 0) = xG(x, 0) + \hat{P}(x, 0)$. The result follows from the fact that, since $q \notin \operatorname{Sing}(\mathcal{F})$, this polynomial has degree $d + 1$.

Finally, the sum of Milnor numbers of $\mathcal{F}$ on $\mathbb{P}^2$ gives a Bézout type theorem for $\mathcal{F}$, which reads

$$\sum_{p \in \mathbb{P}^2} \mu_p(\mathcal{F}) \;=\; d^2 + d + 1, \tag{5}$$

where d is the degree of $\mathcal{F}$. To see this we suppose that $\mathcal{F}$ is induced in affine coordinates $(x, y) \in \mathbb{C}^2$ by the polynomial vector field $\mathbf{v} \;=\; P(x, y)\partial/\partial x + Q(x, y)\partial/\partial y$. By an appropriate choice of the line at infinity L_∞ we may suppose that it does not contain any of the singularities of $\mathcal{F}$. This also implies that L_∞ is not invariant by $\mathcal{F}$, so that P and Q have degree $d+1$. Bézout's Theorem for the projective curves defined by P and Q give that the sum of their intersection numbers is $(d + 1)^2 = d^2 + 2d + 1$. The sum corresponding to points contained in the affine plane $\mathbb{C}^2$ equals the sum of the Milnor numbers of singularities of $\mathcal{F}$. The result is achieved by noticing that the two curves have d points of intersection over L_∞, with multiplicities counted.

3. Primitive models of foliations

Let $\mathcal{F}$ be a foliation on $\mathbb{P}^2$. Given a point $l \in \mathbb{P}^2$, the *polar curve* of $\mathcal{F}$ with *center* at $l \in \mathbb{P}^2$ is the closure of the set of points $p \in \mathbb{P}^2 \setminus \operatorname{Sing}(\mathcal{F})$ such that $T_p^{\mathbb{P}}\mathcal{F}$ passes through l:

$$P_l^{\mathcal{F}} \;=\; \overline{\{p \in \mathbb{P}^2 \setminus \operatorname{Sing}(\mathcal{F});\; l \in T_p^{\mathbb{P}}\mathcal{F}\}}.$$

Here $T_p^{\mathbb{P}}\mathcal{F}$ is the line through p with direction $T_p\mathcal{F}$. When $\mathcal{F}$ is induced in affine coordinates $(X : Y : Z) \in \mathbb{P}^2$ by a polynomial 1-form

$$\omega \;=\; A(X, Y, Z)dX + B(X, Y, Z)dY + C(X, Y, Z)dZ$$

as in (1), the polar curve with center $l = (\alpha : \beta : \gamma)$ has equation

$$\alpha A(X, Y, Z) + \beta B(X, Y, Z) + \gamma C(X, Y, Z) = 0.$$

It follows that if $\mathcal{F}$ has degree $d \geq 1$ then $P_l^{\mathcal{F}}$ is a curve of degree $d + 1$. Furthermore $P_l^{\mathcal{F}}$ contains all singularities of $\mathcal{F}$ as well as the point l. This object was studied in [2] and [4].

As the point $l \in \mathbb{P}^2$ moves, the curves $P_l^{\mathcal{F}}$ form a linear system of dimension two, the *polar net* of $\mathcal{F}$. If we fix a line $L \subset \mathbb{P}^2$ and take all polar curves of $\mathcal{F}$ whose centers lie in L we have the *polar pencil* of $\mathcal{F}$ with *axis* L. It is the set of curves

$$\alpha A(X,Y,Z) + \beta B(X,Y,Z) + \gamma C(X,Y,Z) = 0 \quad , \ (\alpha : \beta : \gamma) \in L \, ,$$

and will be denoted by $\mathcal{P}(\mathcal{F}, L)$.

Proposition 1. *Let $L \subset \mathbb{P}^2$ be an $\mathcal{F}$-invariant line. Then L is a fixed component of $\mathcal{P}(\mathcal{F}, L)$ with multiplicity one. Reciprocally, the only fixed component admitted in $\mathcal{P}(\mathcal{F}, L)$ is the line L, in which case it is $\mathcal{F}$-invariant and of multiplicity one. In particular, if L is not invariant by $\mathcal{F}$ then $\mathcal{P}(\mathcal{F}, L)$ has no fixed components.*

Proof. Suppose first that L is $\mathcal{F}$-invariant and fix $l \in L$. Then, the $\mathcal{F}$-invariance of L gives that $l \in T_p^{\mathbb{P}}\mathcal{F}$ for every $p \in L \setminus \mathrm{Sing}(\mathcal{F})$. Thus, $L \subset P_l^{\mathcal{F}}$. Since $l \in L$ is arbitrary, we have $L \subset \mathcal{P}(\mathcal{F}, L)$. In what concerns its multiplicity, putting $L : \{Z = 0\}$ in the above system of homogeneous coordinates, we have

$$\mathcal{P}(\mathcal{F}, L) = \{\alpha A(X,Y,Z) + \beta B(X,Y,Z) = 0; \ (\alpha : \beta) \in \mathbb{P}^1\}.$$

Thus, if L were a fixed element of the pencil with multiplicity $k > 1$, then Z^k would be a divisor of both A and B, and the Euler condition (2) would imply that Z^{k-1} would be a divisor of C and we would find a component of codimension one in $\mathrm{Sing}(\mathcal{F})$, which is not allowed. For the converse, we first remark that if $\mathcal{P}(\mathcal{F}, L)$ has a line L' in its base, then $L' = L$. Actually, if $p \in L' \setminus \mathrm{Sing}(\mathcal{F})$ then $l \in T_p^{\mathbb{P}}\mathcal{F}$ for every $l \in L$. But, if $L' \neq L$ and if $p \notin L$, then $T_p^{\mathbb{P}}\mathcal{F}$ intersects L in only one point. Thus, the only possibility left is that $L' = L$. Then for a fixed $l \in L$ and for every $p \in L \setminus \mathrm{Sing}(\mathcal{F})$ we have $l \in T_p^{\mathbb{P}}\mathcal{F}$. This means that $T_p^{\mathbb{P}}\mathcal{F} = L$ for every $p \in L \setminus \mathrm{Sing}(\mathcal{F})$, which gives the $\mathcal{F}$-invariance of L. By the first part of the proof, L has multiplicity one. Finally, an irreducible fixed component of $\mathcal{P}(\mathcal{F}, L)$ of degree greater than one with equation $F(X,Y,Z) = 0$ would mean that F is a factor of both A and B and thus, by the Euler condition, it would be a factor of C, giving rise to a codimension one component in $\mathrm{Sing}(\mathcal{F})$, which is impossible. ☐

Let $\mathcal{F}$ be a foliation in $\mathbb{P}^2$ as before. Its *modified polar pencil* with axis at the line $L \subset \mathbb{P}^2$, denoted by $\mathcal{P}^*(\mathcal{F}, L)$, is the pencil obtained from $\mathcal{P}(\mathcal{F}, L)$ in the following way:

$$\mathcal{P}^*(\mathcal{F}, L) \ = \ \begin{cases} \mathcal{P}(\mathcal{F}, L) - L & \text{if } L \text{ is } \mathcal{F}\text{-invariant} \\ \mathcal{P}(\mathcal{F}, L) & \text{if } L \text{ is not } \mathcal{F}\text{-invariant} \end{cases}$$

Evidently $\mathcal{P}^*(\mathcal{F}, L)$ is free of fixed components.

We now choose an affine system of coordinates $(x, y) \in \mathbb{C}^2$ such that L is the line at infinity by making $L : \{Z = 0\}$, $x = X/Z$ and $y = Y/Z$, where $\mathcal{F}$ is induced by the vector field

$$\mathbf{v} \ = \ P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}.$$

In the coordinates (x, y), both $\mathcal{P}(\mathcal{F}, L)$ and $\mathcal{P}^*(\mathcal{F}, L)$ are given by

$$\{\alpha P(x,y) + \beta Q(x,y) = 0; \ (\alpha : \beta) \in \mathbb{P}^1\}.$$

By means of Bertini's Theorem concerning linear systems whose generic element is reducible, it is proved in [4] that the generic element of the polar net of a foliation on $\mathbb{P}^2$ is irreducible. However, it comes out that the polar net of a foliation might contain a pencil whose generic element is reducible. Evidently, if L is a line invariant by $\mathcal{F}$, then L belongs to all elements of the polar pencil having L as an axis, that is L is a fixed element of the polar pencil. By removing

L from the pencil, we can again ask if its generic element is reducible. Taking affine coordinates $(x,y)\in\mathbb{C}^2$ such that $L=L_\infty$ is the line at infinity then the polar pencil becomes

$$\{\alpha P(x,y)+\beta Q(x,y)=0;\ (\alpha:\beta)\in\mathbb{P}^1\}.$$

We remark that now there are no elements of codimension one in the pencil, since the fact that $\mathrm{Sing}(\mathcal{F})$ has codimension 2 implies that P and Q have no common factor. We can then apply Stein's factorization Theorem (see [3]): the generic element of the pencil $\{\alpha P(x,y)+\beta Q(x,y)=0\ ,\ (\alpha:\beta)\in\mathbb{P}^1\}$ is reducible if and only if there are polynomials $\widetilde{P}(x,y)$ and $\widetilde{Q}(x,y)$ and a rational function $r:\mathbb{P}^1\to\mathbb{P}^1$ of degree greater than one such that

$$\frac{P(x,y)}{Q(x,y)}=r\left(\frac{\widetilde{P}(x,y)}{\widetilde{Q}(x,y)}\right).$$

This means that the pencil induced by P and Q "factors" through the one induced by $\widetilde{P}$ and $\widetilde{Q}$. We can ask once again if the generic element of the pencil $\{\alpha\widetilde{P}(x,y)+\beta\widetilde{Q}(x,y)=0;\ (\alpha:\beta)\in\mathbb{P}^1\}$ is reducible. If true, we can repeat the process above, until we reach a situation where $\tilde{d}$ is minimal and the generic element of $\{\alpha\widetilde{P}(x,y)+\beta\widetilde{Q}(x,y)=0;\ (\alpha:\beta)\in\mathbb{P}^1\}$ is irreducible.

We say that a foliation $\mathcal{F}$ on $\mathbb{P}^2$ is *primitive* if for every line $L\subset\mathbb{P}^2$ the modified polar pencil of $\mathcal{F}$ with axis L has irreducible generic element. If for some line $L\subset\mathbb{P}^2$ the modified polar pencil of $\mathcal{F}$ with respect to L has reducible generic element, we say that $\mathcal{F}$ is *non-primitive* (with respect to L). In this case, taking affine coordinates $(x,y)\in\mathbb{C}^2$ with respect to which L is the line at infinity, and a polynomial vector field

$$\mathbf{v}\ =\ P(x,y)\frac{\partial}{\partial x}+Q(x,y)\frac{\partial}{\partial y} \tag{6}$$

that induces $\mathcal{F}$, we find polynomials $\widetilde{P}(x,y)$ and $\widetilde{Q}(x,y)$ and a rational function $r:\mathbb{P}^1\to\mathbb{P}^1$ of degree $m=\deg(r)\geq 2$ such that $P/Q=r(\widetilde{P}/\widetilde{Q})$ and so that the pencil $\mathcal{P}(\widetilde{P},\widetilde{Q})$ has irreducible generic element. Notice that, putting $t=z/w$, we write $r(t)=r(z/w)=S(z,w)/T(z,w)$, where S and T are homogeneous polynomials of degree m, so that

$$\begin{cases}P(x,y)=S(\widetilde{P}(x,y),\widetilde{Q}(x,y))\\ Q(x,y)=T(\widetilde{P}(x,y),\widetilde{Q}(x,y)).\end{cases} \tag{7}$$

We now define a foliation $\widetilde{\mathcal{F}}$ on $\mathbb{P}^2$ induced, in the same system of affine coordinates (x,y), by the vector field

$$\widetilde{\mathbf{v}}\ =\ \widetilde{P}(x,y)\frac{\partial}{\partial x}+\widetilde{Q}(x,y)\frac{\partial}{\partial y}.$$

Since $\mathcal{P}(\widetilde{P},\widetilde{Q})$ has irreducible generic element, $\widetilde{P}$ and $\widetilde{Q}$ are relatively prime, so $\mathrm{Sing}(\widetilde{\mathcal{F}})$ has codimension two. $\widetilde{\mathcal{F}}$ is said to be a *primitive model* for $\mathcal{F}$. The number $m=\deg(r)$ will be called *degree of ramification* of $\mathcal{F}$. We remark that the property of being a non-primitive foliation and that of being the primitive model of a foliation involves fixing an affine plane with coordinates $(x,y)\in\mathbb{C}^2$ and a line at infinity $L_\infty\subset\mathbb{P}^2$. The degree of the vector field (6) inducing $\mathcal{F}$ is called *the affine degree* of $\mathcal{F}$, and is denoted by $\deg_a(\mathcal{F})$. If $\mathcal{F}$ is a non-primitive foliation admitting a primitive model $\widetilde{\mathcal{F}}$, we evidently have

$$\deg_a(\mathcal{F})\ =\ m\deg_a(\widetilde{\mathcal{F}}),$$

where m is the degree of ramification.

Fix an affine plane in $\mathbb{P}^2$ with coordinates $(x,y)\in\mathbb{C}^2$. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be foliations on $\mathbb{P}^2$ induced, respectively, by polynomial vector fields

$$\mathbf{v}_1 \;=\; P_1(x,y)\frac{\partial}{\partial x}+Q_1(x,y)\frac{\partial}{\partial y} \quad \text{and} \quad \mathbf{v}_2 \;=\; P_2(x,y)\frac{\partial}{\partial x}+Q_2(x,y)\frac{\partial}{\partial y}.$$

Definition 1. The foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ are said to be *linearly equivalent* if there exist $a,b,c,d\in\mathbb{C}$ such that $ad-bc\neq 0$ and

$$\begin{cases} P_1(x,y) \;=\; aP_2(x,y)+bQ_2(x,y) \\ Q_1(x,y) \;=\; cP_2(x,y)+dQ_2(x,y). \end{cases}$$

The notion of linear equivalence defines equivalence classes in the space of foliations on $\mathbb{P}^2$. From the expression (3) it is easy to see that, in such an equivalence class, all foliations have the same degree d and leave L_∞ invariant, with the possible exception of one, which has degree $d-1$ and for which L_∞ is not invariant. Nevertheless, the affine degree is the same for all foliation in a class of linear equivalence. Therefore, a foliation of degree d for which L_∞ is not invariant is always linear equivalent to a foliation of degree $d+1$ which leaves L_∞ invariant. Evidently, two primitive models for the same foliation are linearly equivalent. On the other hand, two non-primitive foliations which are linearly equivalent have the same class of primitive models.

In the next two examples we introduce two classes of foliation which will appear in Theorem 1 below.

Example 1. We say that a foliation $\mathcal{F}$ on $\mathbb{P}^2$ is *homogeneous* with center at $l\in\mathbb{P}^2$ if $\mathcal{F}$ is induced in affine coordinates $(x,y)\in\mathbb{C}^2$ for which $l=(0,0)$ by a polynomial vector field

$$\mathbf{v} \;=\; P(x,y)\frac{\partial}{\partial x}+Q(x,y)\frac{\partial}{\partial y}$$

such that $P(x,y)$ and $Q(x,y)$ are homogeneous polynomials of the same degree. One outstanding property of a foliation $\mathcal{F}$ which is homogeneous with center at $l\in\mathbb{P}^2$ is that its polar curve with center at l is $\mathcal{F}$-invariant and consists of $d+1$ lines passing through l, with multiplicities counted. If $\mathcal{F}$ is a homogeneous foliation centered at $l=(0,0)$ as above, then the line at infinity is invariant by $\mathcal{F}$ and $d=\deg(\mathcal{F})=\deg_a(\mathcal{F})$. The only singularity in $\mathbb{C}^2$ is $l=(0,0)$, which has Milnor number $\mu_l(\mathcal{F})=d^2$. Observe that this, along with expression (5), implies that

$$\sum_{p\in L_\infty}\mu_p(\mathcal{F})=d+1.$$

All the singularities of $\mathcal{F}$ on the line at infinity L_∞ are at the intersection of L_∞ and one of the invariant lines L which form the polar curve with center l. If L has multiplicity one as a component of $P_l^{\mathcal{F}}$, then $p=L\cap L_\infty$ is a non-degenerate singularity, meaning that the linear part of any vector field which induces $\mathcal{F}$ near p has two non-zero eigenvalues. On the other hand, if this multiplicity is $k>1$, then $p=L\cap L_\infty$ is a saddle-node whose weak separatrix is contained in L_∞. We finally observe that any curve in the polar pencil of $\mathcal{F}$ with axis at L_∞ consists of $d+1$ lines passing through $(0,0)\in\mathbb{C}^2$.

Example 2. Let $\mathcal{F}$ be a foliation on $\mathbb{P}^2$. We say that $\mathcal{F}$ is a foliation *in one variable* if in some affine coordinate system $(x,y)\in\mathbb{C}^2$ it is induced by a polynomial vector field of the kind

$$P(x)\frac{\partial}{\partial x}+Q(x)\frac{\partial}{\partial y},$$

where P and Q are polynomials depending only on the variable x. Since $\mathrm{Sing}(\mathcal{F})$ has codimension two, P and Q are without common factors, which results that $\mathcal{F}$ has no singularities in the affine plane $\mathbb{C}^2$. It is easy to see that the line at infinity is $\mathcal{F}$-invariant, for its non-invariance would imply, from expression (3), that the higher order terms of P and Q would depend on both x and y. Thus, $d = \deg(\mathcal{F}) = \deg_a(\mathcal{F})$. We also remark that, if x_0 is a root of $Q(x)$, then the line $x = x_0$ is $\mathcal{F}$-invariant. These invariant lines all meet L_∞ at a singularity p. If $\deg(P) < \deg(Q)$, then this is the only singularity of $\mathcal{F}$. If $\deg(P) \geq \deg(Q)$ there is still another singularity on L_∞. For a foliation in one variable as above, any element of the polar pencil with axis at L_∞ consists of $d+1$ vertical lines, with multiplicities counted.

Theorem 1. *Let $\mathcal{F}$ be a non-primitive foliation on $\mathbb{P}^2$ which admits a primitive model of affine degree one. Then either $\mathcal{F}$ is a homogeneous foliation or it is a foliation in one variable.*

Proof. Let $\widetilde{\mathcal{F}}$ be a primitive model for $\mathcal{F}$, induced in affine coordinates $(x, y) \in \mathbb{C}^2$ by the polynomial vector field $\widetilde{P}(x, y)\partial/\partial x + \widetilde{Q}(x, y)\partial/\partial y$.

1st case: Either $\widetilde{P}$ or $\widetilde{Q}$ is a constant. Then, by means of a linear equivalence, we may suppose that $\widetilde{\mathcal{F}}$ is induced by a vector field of the form $(ax + by)\partial/\partial x + \partial/\partial y$, where $a \neq 0$ or $b \neq 0$. If $a = 0$, evidently $\widetilde{\mathcal{F}}$ is a foliation in one variable. If $a \neq 0$, by applying the affine change of coordinates $(u, v) = (ax + by, y)$, we arrive to the same conclusion.

2nd case: Both $\widetilde{P}$ and $\widetilde{Q}$ have degree one. Let us put $\widetilde{P} = ax + by + e$ and $\widetilde{Q} = cx + dy + f$. We first consider the situation where $\widetilde{P}$ and $\widetilde{Q}$ have no common root in the affine plane $\mathbb{C}^2$. This means that $ax + by$ is a multiple of $cx + dy$ by a non-zero constant. Thus, by linear equivalence, we can suppose that $\widetilde{P} = ax + by$ and $\widetilde{Q} = 1$ and we come to the first case, where $\mathcal{F}$ is a foliation in one variable. We then suppose that $\widetilde{P}$ and $\widetilde{Q}$ have a common root in $\mathbb{C}^2$. By an affine change of coordinates, we can suppose that this root is $(0, 0)$, which makes $\widetilde{P} = ax + by$ and $\widetilde{Q} = cx + dy$. If $r(t)$ is the rational map such that $P/Q = r(\widetilde{P}/\widetilde{Q})$, writing $t = z/w$, we have $r(z/w) = F(z, w)/G(z, w)$, where F and G are homogeneous polynomials of degree equal to the degree of r. We finally conclude that

$$P(x, y) = F(ax + by, cx + dy) \qquad \text{and} \qquad Q(x, y) = G(ax + by, cx + dy)$$

which says that $\mathcal{F}$ is a homogeneous foliation. □

If $\mathcal{F}$ is a non-primitive foliation with primitive model $\widetilde{\mathcal{F}}$ then, in the affine plane $\mathbb{C}^2$, the singular points for $\mathcal{F}$ and for $\widetilde{\mathcal{F}}$ are the same. In fact, with the notation of (7), we know that $P(x, y) = S(\widetilde{P}(x, y), \widetilde{Q}(x, y))$ and $Q(x, y) = T(\widetilde{P}(x, y), \widetilde{Q}(x, y))$. Evidently, the common zeroes of $\widetilde{P}$ and $\widetilde{Q}$ are zeroes of both P and Q, which gives $\mathrm{Sing}(\widetilde{\mathcal{F}})_{|\mathbb{C}^2} \subset \mathrm{Sing}(\mathcal{F})_{|\mathbb{C}^2}$. Reciprocally, the existence of a point (x_0, y_0) in $\mathbb{C}^2$ which is singular for $\mathcal{F}$ but not for $\widetilde{\mathcal{F}}$ would imply the existence of a common factor for $S(z, w)$ and $T(z, w)$. Thus we actually have $\mathrm{Sing}(\widetilde{\mathcal{F}})_{|\mathbb{C}^2} = \mathrm{Sing}(\mathcal{F})_{|\mathbb{C}^2}$.

Proposition 2. *Let $\mathcal{F}$ be a non-primitive foliation having $\widetilde{\mathcal{F}}$ as primitive model and m as the degree of ramification. If $p \in \mathbb{C}^2$ then*

$$\mu_p(\mathcal{F}) = m^2 \mu_p(\widetilde{\mathcal{F}}).$$

Proof. We keep the notation of (7). We consider the following maps from $\mathbb{C}^2$ to $\mathbb{C}^2$:

$$\begin{cases} \Phi(x, y) = (P(x, y), Q(x, y)), \\ \widetilde{\Phi}(x, y) = (\widetilde{P}(x, y), \widetilde{Q}(x, y)), \\ H(z, w) = (S(u, v), T(u, v)). \end{cases}$$

We have $\Phi = H \circ \widetilde{\Phi}$. We first remark that the Milnor number of the vector field $P\partial/\partial x + Q\partial/\partial y$ at a singularity p is the number of pre-images of $\Phi = (P, Q)$ lying near p of any point q sufficiently near $(0, 0) \in \mathbb{C}^2$. The result follows by noticing that, since S and T are homogeneous of degree

m and without common factors, the Milnor number of $S\partial/\partial u + T\partial/\partial v$ at $(0,0)$ is m^2 (see [1], section 2). □

Corollary 1. *If $\mathcal{F}$ is a foliation having three non-aligned singularities each of them having the property that its Milnor number is not divisible by some m^2, where $m \in \mathbb{Z}$ and $m \geq 2$. Then $\mathcal{F}$ is a primitive foliation. In particular, if $\mathcal{F}$ has three non-aligned non-degenerate singularities, then $\mathcal{F}$ is primitive.*

Corollary 2. *Let $\mathcal{F}$ be a foliation having only non-degenerate singularities. Then $\mathcal{F}$ is primitive.*

Proof. Since all singularities of $\mathcal{F}$ have Milnor number 1, the above corollary implies that all singularities of $\mathcal{F}$ would lie in L_∞ if $\mathcal{F}$ were non-primitive. Summing up their Milnor numbers we have $\sum_{p\in L_\infty} \mu_p(\mathcal{F}) = d^2+d+1$, where d is the degree of $\mathcal{F}$. If L_∞ were $\mathcal{F}$-invariant, we would have $\sum_{p\in L_\infty} \mu_p(\mathcal{F}, L_\infty) = d+1$, which leads to a contradiction since $\mu_p(\mathcal{F}) = \mu_p(\mathcal{F}, L_\infty) = 1$ for a non-degenerate singularity. If L_∞ were non-invariant, then $\sum_{p\in L_\infty} \tau_p(\mathcal{F}, L_\infty) = d$, which is a contradiction since, when $p \in \mathrm{Sing}(\mathcal{F})$ is non-degenerate, it holds $\tau_p(\mathcal{F}, L_\infty) = \mu_p(\mathcal{F}) = 1$. □

Corollary 3. *Let $\mathcal{F}ol(d)$ be the space of foliations of degree d in $\mathbb{P}^2$. Then the set of primitive foliations contain a non-empty Zariski open set.*

4. The study of the singularities on L_∞

We have seen in the previous section that a non-primitive foliation $\mathcal{F}$ and its primitive model $\widetilde{\mathcal{F}}$ have the same singularities in the affine plane $\mathbb{C}^2$, and its Milnor numbers are related by Proposition 2. The objective of this section is to explore the consequences of this fact to the singularities of $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ that lie over L_∞.

Let us consider a non-primitive foliation $\mathcal{F}$ of degree d_0 having a primitive model $\widetilde{\mathcal{F}}$ of degree $\tilde{d}_0$. We denote the affine degrees of $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ respectively by d and $\tilde{d}$. By summing up Milnor numbers we get

$$\begin{aligned}
\sum_{\mathbb{P}^2} \mu_p(\mathcal{F}) &= \sum_{\mathbb{C}^2} \mu_p(\mathcal{F}) + \sum_{L_\infty} \mu_p(\mathcal{F}) \\
&= m^2 \sum_{\mathbb{C}^2} \mu_p(\widetilde{\mathcal{F}}) + \sum_{L_\infty} \mu_p(\mathcal{F}) \\
&= m^2 \left(\sum_{\mathbb{P}^2} \mu_p(\widetilde{\mathcal{F}}) - \sum_{L_\infty} \mu_p(\widetilde{\mathcal{F}}) \right) + \sum_{L_\infty} \mu_p(\mathcal{F})
\end{aligned}$$

thus, using (5), we obtain

$$\begin{aligned}
\sum_{L_\infty} \mu_p(\mathcal{F}) - m^2 \sum_{L_\infty} \mu_p(\widetilde{\mathcal{F}}) &= \sum_{\mathbb{P}^2} \mu_p(\mathcal{F}) - m^2 \sum_{\mathbb{P}^2} \mu_p(\widetilde{\mathcal{F}}) \\
&= (d_0^2 + d_0 + 1) - m^2 (\tilde{d}_0^{\,2} + \tilde{d}_0 + 1). \qquad (8)
\end{aligned}$$

The values of d_0 and $\tilde{d}_0$ in terms of the affine degrees d and $\tilde{d}$ depend only on the fact of L_∞ being $\mathcal{F}$-invariant or not. We consider three cases:

1st case: L_∞ is $\widetilde{\mathcal{F}}$-invariant but not $\mathcal{F}$-invariant.

We have $d_0 = d-1$ and $\tilde{d}_0 = \tilde{d}$ and, putting this in equation (8),

$$\begin{aligned}\sum_{L_\infty} \mu_p(\mathcal{F}) - m^2 \sum_{L_\infty} \mu_p(\widetilde{\mathcal{F}}) &= (d^2-d+1) - m^2(\tilde{d}^2+\tilde{d}+1) \\ &= ((m\tilde{d})^2 - m\tilde{d}+1) - m^2(\tilde{d}^2+\tilde{d}+1) \\ &= -m^2\tilde{d} - m\tilde{d} - m^2 + 1. \end{aligned} \tag{9}$$

This allows us to conclude the following:

Proposition 3. *Let $\mathcal{F}$ be a non-primitive foliation. Then $\mathcal{F}$ has some singularity in L_∞.*

Proof. If L_∞ is $\mathcal{F}$ invariant then formula (4) implies that it must contain some singularity. Suppose now that L_∞ is not invariant by $\mathcal{F}$. By linear equivalence, we can suppose that $\widetilde{\mathcal{F}}$ leaves L_∞ invariant. If $\mathrm{Sing}(\mathcal{F}) \cap L_\infty = \emptyset$ then $\sum_{L_\infty} \mu_p(\mathcal{F}) = 0$. The above formula gives

$$-m^2 \sum_{L_\infty} \mu_p(\widetilde{\mathcal{F}}) = -m^2\tilde{d} - m\tilde{d} - m^2 + 1.$$

Thus, m would be a divisor of the right side of the equation, which is absurd. $\square$

Suppose now that L_∞ is $\widetilde{\mathcal{F}}$-invariant and that $\sum_{L_\infty} \mu_p(\widetilde{\mathcal{F}}) = \tilde{d}_0 + 1 = \tilde{d}+1$. In this case, expression (9) reads

$$\sum_{L_\infty} \mu_p(\mathcal{F}) - m^2(\tilde{d}+1) = -m^2\tilde{d} - m\tilde{d} - m^2 + 1,$$

which implies

$$\sum_{L_\infty} \mu_p(\mathcal{F}) = -m\tilde{d} + 1.$$

This is a contradiction, since the right side is negative. We get the following conclusion:

Proposition 4. *Let $\mathcal{F}$ be a non-primitive foliation having a primitive model $\widetilde{\mathcal{F}}$ leaving L_∞ invariant. Suppose that $\sum_{L_\infty} \mu_p(\widetilde{\mathcal{F}}) = \tilde{d}+1$, where $\tilde{d} = \deg(\widetilde{\mathcal{F}})$. Then L_∞ is $\mathcal{F}$-invariant. In particular, if all singularities of $\widetilde{\mathcal{F}}$ in L_∞ are non-degenerate, then L_∞ is $\mathcal{F}$-invariant.*

In the situation of the Proposition 4, relation (4) reads $\sum_{L_\infty} \mu_p(\widetilde{\mathcal{F}}, L_\infty) = \tilde{d}+1$. Thus, the hypothesis $\sum_{L_\infty} \mu_p(\widetilde{\mathcal{F}}) = \tilde{d}+1$ is a condition of minimality on the Milnor numbers of $\widetilde{\mathcal{F}}$ over L_∞, as explained in the next result:

Proposition 5. *Let $\mathcal{F}$ be a germ of foliation having a singularity at $p \in \mathbb{C}^2$ and let L be a germ of smooth separatrix at p. Then $\mu_p(\mathcal{F}, L) \le \mu_p(\mathcal{F})$. Furthermore, equality occurs if and only if one of the two alternatives holds:*

(i) p is a non-degenerate singularity of $\mathcal{F}$;

(ii) p is a saddle-node having L as its weak separatrix.

Proof. Suppose that $\mathcal{F}$ is induced at p by a local vector field $P\partial/\partial x + Q\partial/\partial y$, where $P, Q \in \mathcal{O}_p$. Let us denote $\mu_p(P,Q) := \mu_p(\mathcal{F})$. For a vector field $P\partial/\partial x + Q_1Q_2\partial/\partial y$, where $Q_1, Q_2 \in \mathcal{O}_p$, we have $\mu_p(P, Q_1Q_2) = \mu_p(P,Q_1) + \mu_p(P,Q_2)$ (see [1]). Let us suppose that the separatrix L has equation $y = 0$, so that $\mathcal{F}$ is induced by a vector field of the form $P\partial/\partial x + yQ_1\partial/\partial y$ for some $Q_1 \in \mathcal{O}_p$. Thus

$$\mu_p(\mathcal{F}) = \mu_p(P, yQ_1) = \mu_p(P,y) + \mu_p(P,Q_1) = \mu_p(\mathcal{F}, L) + \mu_p(P, Q_1),$$

where we used that $\mu_p(\mathcal{F}, L) = \mu_p(P, y)$. The result follows by noticing that $\mu_p(P,Q_1) \ge 0$. Now, equality holds if and only if $\mu_p(P,Q_1) = 0$. This means that the vector field $P\partial/\partial x + Q_1\partial/\partial y$

is non-singular at p. Since $P(p) = 0$ we must have $Q_1(p) \neq 0$. This gives at least one non-zero eigenvalue for $P\partial/\partial x + Q\partial/\partial y$, which implies (i) or (ii). Reciprocally, if p is a non-degenerate singularity, then $\mu_p(\mathcal{F}) = \mu_p(\mathcal{F}, L) = 1$. In the case of a saddle-node having $y = 0$ as weak separatrix, after an analytic change of coordinates, we may suppose that we have the normal form of the saddle node: $x^{k+1}\partial/\partial x + y(1 + \lambda x^k)\partial/\partial y$, where $\lambda \in \mathbb{C}$ and $k \geq 0$. Its easy to see that $\mu_p(\mathcal{F}) = \mu_p(\mathcal{F}, L) = k + 1$. □

2nd case: L_∞ is $\mathcal{F}$-invariant but not $\widetilde{\mathcal{F}}$-invariant. We have $d_0 = d$ and $\tilde{d}_0 = \tilde{d} - 1$. Equation (8) gives

$$\begin{aligned}
\sum_{L_\infty} \mu_p(\mathcal{F}) - m^2 \sum_{L_\infty} \mu_p(\widetilde{\mathcal{F}}) &= (d^2 + d + 1) - m^2((\tilde{d} - 1)^2 + \tilde{d}) \\
&= ((m\tilde{d})^2 + m\tilde{d} + 1) - m^2(\tilde{d}^2 - \tilde{d} + 1) \\
&= m^2(\tilde{d} - 1) + m\tilde{d} + 1.
\end{aligned}$$

Let us suppose that the sum of Milnor numbers of $\mathcal{F}$ at L_∞ is minimal, that is $\sum_{L_\infty} \mu_p(\mathcal{F}) = d_0 + 1 = d + 1$. This gives

$$-m^2 \sum_{L_\infty} \mu_p(\widetilde{\mathcal{F}}) \;=\; m^2(\tilde{d} - 1).$$

This implies that $\tilde{d} = 1$ and $\sum_{L_\infty} \mu_p(\widetilde{\mathcal{F}}) = 0$, that is, $\widetilde{\mathcal{F}}$ is the radial foliation. Thus, $\mathcal{F}$ is a homogeneous foliation. As commented on Example 1, for a homogeneous foliation $\mathcal{F}$ of degree d_0, it holds $\sum_{L_\infty} \mu_p(\mathcal{F}) = d_0 + 1$. We can thus state the following result:

Proposition 6. *Let $\mathcal{F}$ be a non-primitive foliation of degree d_0 which leaves L_∞ invariant, having a primitive model $\widetilde{\mathcal{F}}$ for which L_∞ is non-invariant. It holds $\sum_{L_\infty} \mu_p(\mathcal{F}) = d_0 + 1$ if and only if $\mathcal{F}$ is a homogeneous foliation and, in this case, $\widetilde{\mathcal{F}}$ is the radial foliation.*

3rd case: L_∞ is both $\mathcal{F}$-invariant and $\widetilde{\mathcal{F}}$-invariant. We have $d_0 = d$ and $\tilde{d}_0 = \tilde{d}$, thus

$$\begin{aligned}
\sum_{L_\infty} \mu_p(\mathcal{F}) - m^2 \sum_{L_\infty} \mu_p(\widetilde{\mathcal{F}}) &= (d^2 + d + 1) - m^2(\tilde{d}^2 + \tilde{d} + 1) \\
&= ((m\tilde{d})^2 + m\tilde{d} + 1) - m^2(\tilde{d}^2 + \tilde{d} + 1) \\
&= -m^2\tilde{d} + m\tilde{d} - m^2 + 1.
\end{aligned}$$

Suppose now that $\sum_{L_\infty} \mu_p(\widetilde{\mathcal{F}}) = \tilde{d}_0 + 1 = \tilde{d} + 1$. This is equivalent to

$$\sum_{L_\infty} \mu_p(\mathcal{F}) - m^2(\tilde{d} + 1) \;=\; -m^2\tilde{d} + m\tilde{d} - m^2 + 1,$$

which in its turn is equivalent to

$$\sum_{L_\infty} \mu_p(\mathcal{F}) \;=\; m\tilde{d} + 1 \;=\; d + 1 \;=\; d_0 + 1.$$

We reach the following conclusion:

Proposition 7. *Let $\mathcal{F}$ be a non-primitive foliation of degree d_0 having a primitive model $\widetilde{\mathcal{F}}$ of degree $\tilde{d}_0$. Suppose that both foliations leave L_∞ invariant. Then $\sum_{L_\infty} \mu_p(\widetilde{\mathcal{F}}) = \tilde{d}_0 + 1$ if and only if $\sum_{L_\infty} \mu_p(\mathcal{F}) = d_0 + 1$.*

This results shows an interesting behavior concerning non-primitive foliations and their primitive models. If $\mathcal{F}$ is a non-primitive foliation having $\widetilde{\mathcal{F}}$ as primitive model, both of them having the line at infinity invariant, then the passage from $\widetilde{\mathcal{F}}$ to $\mathcal{F}$ degenerates all singularities in the affine plane $\mathbb{C}^2$, in the sense that $\mu_p(\mathcal{F}) = m^2\mu_p(\widetilde{\mathcal{F}})$ for every $p \in Sing(\mathcal{F})_{|\mathbb{C}^2} = Sing(\widetilde{\mathcal{F}})_{|\mathbb{C}^2}$, where m is the degree of ramification. On the other hand, this process does not degenerate the singularities of $\widetilde{\mathcal{F}}$ lying in L_∞, in the sense that, considering Proposition 5, if all singularities of $\widetilde{\mathcal{F}}$ in L_∞ are either non-degenerate or saddle-nodes with weak separatrix over L_∞, then the same property holds for the singularities of $\mathcal{F}$ in L_∞.

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THE ALUFFI ALGEBRA

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Abstract. We deal with the *quasi-symmetric algebra* introduced by Paolo Aluffi, here named the (embedded) Aluffi algebra. This algebra is a sort of "intermediate" algebra between the symmetric algebra and the Rees algebra of an ideal, which serves the purpose of introducing the characteristic cycle of a hypersurface in intersection theory. The results described in the present paper have an algebraic flavor and naturally connect with various themes of commutative algebra, such as standard bases à la Hironaka, Artin–Rees like questions, Valabrega–Valla ideals, ideals of linear type, relation type and analytic spread.

We give estimates for the dimension of the Aluffi algebra and show that, pretty generally, the latter is equidimensional whenever the base ring is a hypersurface ring. There is a converse to this under certain conditions that essentially subsume the setup in Aluffi's theory, thus suggesting that this algebra will not handle cases other than the singular locus of a hypersurface. The torsion and the structure of the minimal primes of the algebra are clarified.

In the case of a projective hypersurface the results are more precise and one is naturally led to look at families of projective plane singular curves to understand how the property of being of linear type deforms/specializes for the singular locus of a member. It is fairly elementary to show that the singular locus of an irreducible curve of degree at most 3 is of linear type. This is roundly false in degree larger than 4 and the picture looks pretty wild as we point out by means of some families. Degree 4 is the intriguing case. Here we are able to show that the singular locus of the generic member of a family of rational quartics, fixing the singularity type, is of linear type. We conjecture that *every* irreducible quartic has singular locus of linear type.

Introduction

This work is inspired on a paper of P. Aluffi ([2]) that shows, in the case of a hypersurface, how to define a so-called characteristic cycle in parallel to the well-known conormal cycle in intersection theory. To accomplish it, Aluffi introduces an intermediate algebra between a symmetric algebra of an ideal and the corresponding Rees algebra (blowup).

Aluffi has dubbed his construction a *quasi-symmetric* algebra. Since there are many homomorphic images of the symmetric algebra that could equally benefit from this terminology, we have decided to call it an *embedded Aluffi algebra*. This has the advantage of indicating that the algebra itself has a more complex behavior for more general schemes than for hypersurfaces and, as such, it will often tilt to the other end of the spectrum, namely, become a honest blowup algebra.

The definition of the algebra is based on taking ideals $J \subset I \subset R$ in an arbitrary ring, by setting

$$\mathcal{A}_{R\twoheadrightarrow R/J}(I/J) := \mathcal{S}_{R/J}(I/J) \otimes_{\mathcal{S}_R(I)} \mathcal{R}_R(I).$$

The R-embedded Aluffi algebra is functorial in the following sense: let $R \to R'$ be a ring homomorphism, let $J' \subset I' \subset R'$ denote the respective images of $J \subset I$ under this map. If this

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map induces an isomorphism $R/J \simeq R'/J'$ then it induces a ring surjection

$$\mathcal{A}_{R\twoheadrightarrow R/J}(I/J) \twoheadrightarrow \mathcal{A}_{R'\twoheadrightarrow R'/J'}(I'/J').$$

Thus, it makes sense to take the inverse limit of such ring surjections. Letting A denote the common target and $\mathfrak{a}$ the common ideal in the target, Aluffi takes

$$\mathcal{A}(\mathfrak{a}) :\, = \lim_{R\twoheadrightarrow A} \mathcal{A}_{R\twoheadrightarrow A}(\mathfrak{a}).$$

A point made in his work is that $\mathcal{A}(\mathfrak{a})$ is actually independent of the choice of the source R (i.e., of the presentation $R/J \simeq A$) provided R is constrained to be regular. Thus, if R is indeed regular then $\mathcal{A}(\mathfrak{a}) \simeq \mathcal{A}_{R\twoheadrightarrow A}(\mathfrak{a})$ by the structural map.

Motivated by the case where R is regular, we will study a single member $\mathcal{A}_{R\twoheadrightarrow A}(\mathfrak{a})$ of this inverse system and, accordingly, omit "R-embedded" if this causes no confusion.

The overall goal of this work is to study the nature of the algebra *ab initio* and then apply it to a concrete case. Now, Aluffi focused on the case of a hypersurface – or, so to say, a Cartier divisor. Though not explicitly, his work suggests that the application to intersection theory as he had in mind may not turn out to be suitable for more general varieties. One of our results explains this insufficiency by showing that if the Aluffi algebra is equidimensional – actually, it suffices to know that two of the minimal primes have the same dimension – then the variety has codimension one. Moreover, if the Aluffi algebra is actually pure-dimensional (i.e., no embedded primes and equidimensional) then the variety has to be a Cartier divisor.

Grosso modo the material presented here encloses two sorts of results. First, one studies the properties of the Aluffi algebra in a quite general ring-theoretic setup, bringing in some of the typical objects and invariants of commutative algebra. This will take up the first two sections. The third section deals with the special case of a projectively embedded hypersurface and its singular locus ("gradient ideal"), which is the main background in Aluffi's work for this sort of embedding.

One has a better view of how more structured is the algebra in the case of a homogeneous equation than in that of its affine companion. In characteristic zero the intervenience of the Euler formula becomes crucial in order to obtain the specifics of the algebra.

A major case, as already pointed out by Aluffi, is the case where the ideal I is of *linear type* – meaning that $\mathcal{S}_R(I) = \mathcal{R}_R(I)$. As it follows immediately from the definition, this assumption implies that $\mathcal{A}_{R\twoheadrightarrow R/J}(I/J) = \mathcal{S}_{R/J}(I/J)$. We show that the converse holds in the case where J is a principal ideal generated by a regular element.

It is our belief that the algebra is relevant on its own and may play a role in situations other than having J a principal ideal. Therefore, we deal as much as possible with its structure in the general case – i.e., when the ideal J has arbitrary codimension. We find that it is closely related to known themes of commutative algebra, such as standard bases (à la Hironaka), Artin–Rees number and relation type of an ideal.

One aspect of this to identify the torsion of the Aluffi algebra as the so-called Valabrega–Valla module. This module – actually an ideal in the Aluffi algebra – has been mainly considered in [16, 5.1] in connection to the situation in which J is a reduction of the ideal I. However, in this case the structural surjection $\mathcal{A}_{R\twoheadrightarrow A}(I/J) \twoheadrightarrow \mathcal{R}_{R/J}(I/J)$ to the relative blowup is actually an equality in all high degrees, hence the two algebras are finite R/J-modules, a case one can dismiss as of no interest for the present theory, as we are mainly interested in the case where I has a regular element modulo J – or at least when $\mathrm{ht}\,(J) < \mathrm{ht}\,(I)$ (strict inequality).

An equally meaningful topic is the nature of the associated primes of the Aluffi algebra. This could throw some light on the summands of the so-called *shadow* of the characteristic cycle, a notion introduced by Aluffi in [loc.cit.] (we thank R. Bedregal for calling our attention to this matter). We get pretty close to describing its minimal primes. Since often the algebra is just the symmetric algebra of an ideal, getting hold of its associated primes undergoes the same hardship one faces for the latter. Actually, as we will show, the basic intuition one has about the minimal primes of the symmetric algebra will work for the Aluffi algebra as well.

Our motivation for the last section comes from Aluffi's quest of the nature of the algebra in the case that J is generated by the equation of a reduced hypersurface and I defines its singular locus. Our main interest is to understand its impact in the case of a projective hypersurface and even more modestly, on the nature of the singularities of plane such curves in low degrees.

Thus, let $J = (f) \subset R$ be a principal ideal, where $R = k[X_1, \ldots, X_n]$. We will focus on the Jacobian ideal $I = I_f = (f, \partial f/\partial X_1, \ldots, \partial f/\partial X_n)$. We are particularly motivated by the problem as to when I is an ideal of linear type. Now, in general f will not be Eulerian, hence the local number of generators of I_f maybe an early obstruction – examples of this sort are easily available.

At the other extreme, if f is Eulerian – e.g., if f homogeneous in the standard grading of the polynomial ring and and its degree is not a multiple of the characteristic – then it seems like a good bet to expect that I often be of linear type over R. Of course, $I/(f)$ over $R/(f)$ will never be of linear type – not even generated by analytically independent elements for that matter – as the defining equations of the dual variety to $V(f)$ is a permanent obstruction.

In order to stress the partial derivatives of the homogeneous polynomial f we will call I_f the *gradient* ideal of f. We will assume throughout that $\operatorname{char}(k) = 0$ or at least that the latter does not divide the degree of f. In this case, by the Euler formula, $f \in I_f$. One may name the Aluffi embedded algebra in this case the *gradient Aluffi algebra* of f.

We show that, for a regular element $a \in R$, the Aluffi algebra of a pair $(a) \subset I \subset R$ is equidimensional. However, its structure is still intricate even when a is the equation of an irreducible projective hypersurface and I, its gradient ideal. A sufficiently tidy case is that of the singular locus of f being set-theoretically a nonempty set of points. Algebraically, this translates into the gradient ideal I_f being a (strict) almost complete intersection, a situation which is fairly manageable.

We will by and large consider the property of being of linear type for I_f along certain families of plane curves. Part of the difficulty of the theory is that, perhaps unexpectedly, the notion of being of linear type is neither kept by specialization nor by generization.

We discuss the property of being of linear type for the gradient ideal, giving evidence that its behavior may be rather erratic. The main question is to understand how the nature of the singularities reflect on the algebra and on its minimal primes, with an eye to the cycle components of Aluffi's characteristic shadow.

Here we are able to show that the singular ideal of the general member of a family of irreducible rational quartics, fixing the singularity type, is of linear type. The proof is of some substance as it uses a classification of these curves in terms of quadratic Cremona maps. We conjecture that any irreducible quartic has gradient ideal of linear type. In the case of rational quartics, a classification of the possible families allows for a computational verification of this conjecture. However, we have found no theoretical argument that works for all rational quartics and not just for the general member of each of these families.

1. The embedded Aluffi algebra

1.1. **Preliminaries.** Let A be a ring and $\mathfrak{a}$ an ideal of A. Let R be a ring surjecting onto A and let I denote the inverse image of $\mathfrak{a}$ in R. Note that the symmetric algebra $\mathcal{S}_R(I)$ maps surjectively both to the Rees algebra $\mathcal{R}_R(I)$ and (by functoriality of the symmetric algebra) to $\mathcal{S}_A(\mathfrak{a})$.

Definition 1.1. The *R-embedded Aluffi algebra* of $\mathfrak{a}$ is defined by

$$\mathcal{A}_{R\twoheadrightarrow A}(\mathfrak{a})\colon=\mathcal{S}_A(\mathfrak{a})\otimes_{\mathcal{S}_R(I)}\mathcal{R}_R(I).$$

We develop a few general preliminaries about the R-embedded Aluffi algebra. The first is a useful presentation that has already been observed in [2, Theorem 2.9] in the context of schemes.

Lemma 1.2. *In the above setup, write $I/J:=\mathfrak{a}\subset R/J:=A$, where $I\subset R$ is the inverse image of $\mathfrak{a}$ in R. There are natural A-algebra isomorphisms*

$$\mathcal{A}_{R\twoheadrightarrow A}(\mathfrak{a})\simeq\frac{\mathcal{R}_R(I)}{(J,\widetilde{J})\,\mathcal{R}_R(I)}\simeq\bigoplus_{t\geq 0}I^t/JI^{t-1}.$$

where J is in degree 0 and $\widetilde{J}$ is in degree 1. In particular, there is a surjective A-algebra homomorphism $\mathcal{A}_{R\twoheadrightarrow A}(\mathfrak{a})\twoheadrightarrow\mathcal{R}_A(\mathfrak{a})$.

Proof. By the universal property of the symmetric algebra, one sees that

$$\mathcal{S}_{R/J}(I/J)\simeq\frac{\mathcal{S}_R(I)}{(J,\tilde{J})\,\mathcal{S}_R(I)}.$$

Tensoring with $\mathcal{R}_R(I)$ gives the first isomorphism. The second one is now immediate from the definition of $\tilde{J}$. □

From the definition and Lemma 1.2, the Aluffi algebra is squeezed as

$$\mathcal{S}_{R/J}(I/J)\twoheadrightarrow\mathcal{A}_{R\twoheadrightarrow R/J}(I/J)\twoheadrightarrow\mathcal{R}_{R/J}(I/J)\tag{1}$$

and is moreover a residue ring of the ambient Rees algebra $\mathcal{R}_R(I)$.

If no confusion arises, for a fixed ambient R we will simply refer to this algebra as the Aluffi algebra of $\mathfrak{a}=I/J$. Unless stated otherwise, we will assume that $J\subsetneq I\subsetneq R$.

Note that if the ideal I is of linear type – i.e., the natural surjection $\mathcal{S}_R(I)\twoheadrightarrow\mathcal{R}_R(I)$ is injective – then trivially $\mathcal{S}_A(I/J)\simeq\mathcal{A}_{R\twoheadrightarrow A}(I/J)$. The following example shows that in general there is no converse to this statement even when R is a hypersurface domain.

Example 1.3. Let $R=k[x,y,z]=k[X,Y,Z]/(XY-Z^2)$, with $J=(x,z)$ (the ideal of a ruling in the affine cone) and $I=(x,y,z)$. Then $R/J\simeq k[Y]$ and $I/J\simeq(Y)$. Therefore, I/J is of linear type, hence $\mathcal{S}_{R/J}(I/J)\simeq\mathcal{A}_{R\twoheadrightarrow R/J}(I/J)\simeq\mathcal{R}_{R/J}(I/J)$.

This is a particular instance in the following large class: take $(R,\mathfrak{m})$ to be a non-regular local ring – or a non-degenerate standard graded algebra over a field and its irrelevant ideal – with $J\subset I=\mathfrak{m}$ such that R/J is regular. Then I/J is generated by a regular sequence on R/J, hence is of linear type, while $\mathfrak{m}$ is never of linear type.

It would be interesting to find such examples with $(R,\mathfrak{m})$ a regular local ring and $J\subset\mathfrak{m}I$.

In the special case where J is a hypersurface, no such examples exist as we now indicate.

Proposition 1.4. *Let R be a Noetherian ring and let I denote an ideal. If $a\in I$ is a regular element then I is of linear type if and only if the natural surjection*

$$\mathcal{S}_{R/(a)}(I/(a))\twoheadrightarrow\mathcal{A}_{R\twoheadrightarrow R/(a)}(I/(a))$$

is an isomorphism.

Proof. The trivial implication has already been mentioned above. For the reverse direction, consider an R-algebra presentation $S:=R[\mathbf{T}]\twoheadrightarrow\mathcal{R}_R(I)$ based on a set of generators $\mathbf{b}=\{b_1,\ldots,b_n\}$ of I. Write $\mathcal{J}=\bigoplus_{i\geq 1}\mathcal{J}_i$ for the kernel of this map, where $\mathcal{J}_i$ stands for the homogenous part of degree i of $\mathcal{J}$ in the standard grading of S. Note that $\mathcal{J}_1S\subset\mathcal{J}$ defines likewise the symmetric algebra of I on S, so we need to show that for any $r\geq 0$, $\mathcal{J}_r\subset\mathcal{J}_1S$.

We induct on r, the result being trivial if $r = 1$. Thus, let $r \geq 2$. By Lemma 1.2 one has

$$\mathcal{J} \subset (\mathcal{J}_1, \tilde{a}, a)S.$$

Let $F = F(\mathbf{T}) \in \mathcal{J}_r$. Then $F = L + \tilde{a}G + aH$ where $L \in \mathcal{J}_1 S_{r-1}$, $G \in S_{r-1}$ and $H \in S_r$. Note that, if $a = \sum_{j=1}^{n} c_j b_j$ then $\tilde{a} = \sum_{j=1}^{n} c_j T_j$, hence $\tilde{a}(\mathbf{b}) = a$, i.e., evaluating $\tilde{a}$ on the generators of I gives back a. Therefore

$$\begin{aligned} 0 &= F(\mathbf{b}) = L(\mathbf{b}) + \tilde{a}(\mathbf{b})G(\mathbf{b}) + aH(\mathbf{b}) \\ &= a \cdot (G + H)(\mathbf{b}), \end{aligned}$$

since $L \in \mathcal{J}$. As a is a regular element, $G + H \in \mathcal{J}$, hence, by homogeneity, $G \in \mathcal{J}_{r-1}$ and $H \in \mathcal{J}_r$.

By the inductive hypothesis, $G \in \mathcal{J}_1 S_{r-2}$, hence $\tilde{a}G \in \mathcal{J}_1 S_{r-1}$. Therefore, $F \in (\mathcal{J}_1 S_{r-1})S + a\,\mathcal{J}_r S$, thus showing the equality of ideals $\mathcal{J}_r S = (\mathcal{J}_1 S_{r-1})S + a\,\mathcal{J}_r S$. By the graded version of Nakayama's lemma, this implies that $\mathcal{J}_r S = (\mathcal{J}_1 S_{r-1})S$, as was to be shown. $\square$

There is the following consequence, for which we claim no priority.

Corollary 1.5. *Let R be a Noetherian ring and let $\{a_1, \ldots, a_m\} \subset R$ be a regular sequence. If $I \subset R$ is an ideal containing $\{a_1, \ldots, a_m\}$ such that $I/(a_1, \ldots, a_m)$ is of linear type on $R/(a_1, \ldots, a_m)$ then I is of linear type on R.*

Proof. Induct on m. For $m = 1$, it readily follows from (1) and Proposition 1.4.

Next assume that $m \geq 2$ and write $J = (a_1, \ldots, a_m)$. Set $\bar{R} := R/(a_1, \ldots, a_{m-1})$ and, likewise,

$$\bar{J} := J/(a_1, \ldots, a_{m-1}) = (\overline{a_m}) \subset \bar{I} := I/(a_1, \ldots, a_{m-1}).$$

Clearly, $\bar{I}/(\overline{a_m}) \simeq I/J$ in $\bar{R}/(\overline{a_m}) \simeq R/J$. Therefore, the assumption that I/J is of linear type on R/J implies that $\bar{I}/(\overline{a_m})$ is of linear type on $\bar{R}/(\overline{a_m})$, where $\overline{a_m}$ is a regular element in $\bar{R}$. By the first part, $\bar{I}$ is of linear type on $\bar{R}$. Then, by the inductive hypothesis, I is of linear type on R. $\square$

1.2. **Dimension.** A few routine statements follow from the preliminaries of the previous subsection.

Proposition 1.6. *Let $J \subsetneq I \subsetneq R$ be ideals of the Noetherian ring R.*

(a) *If J has a regular element then* $\dim \mathcal{A}_{R \twoheadrightarrow R/J}(I/J) \leq \dim R$.

(b) *If I/J has a regular element then*

$$\min\{\dim R + 1\,,\, \dim \mathcal{S}_{R/J}(I/J)\,\} \geq \dim \mathcal{A}_{R \twoheadrightarrow R/J}(I/J) \geq \dim R/J + 1.$$

Proof. (a) Since $\mathcal{R}_R(I)$ is R-torsionfree, one has $\operatorname{ht} J\mathcal{R}_R(I) \geq 1$. Therefore

$$\dim \mathcal{A}_{R \twoheadrightarrow R/J}(I/J) \leq \dim \mathcal{R}_R(I)/J\mathcal{R}_R(I) \leq \dim R + 1 - 1 = \dim R.$$

(b) This follows immediately from (1) by the well-known dimension formula for the Rees algebra of an ideal containing a regular element. $\square$

Remark 1.7. In (a) this is all one can assert in such generality because if, e.g., a power of the ideal I is contained in J, then $\dim \mathcal{A}_{R \twoheadrightarrow R/J}(I/J) = \dim R/J$.

Perhaps less routine is the following result.

Theorem 1.8. *Let R be a catenary, equidimensional and equicodimensional Noetherian ring and let $I \subsetneq R$ be an ideal containing a regular element a. Then $\mathcal{A}_{R \twoheadrightarrow R/J}(I/(a))$ is equidimensional and* $\dim \mathcal{A}_{R \twoheadrightarrow R/J}(I/(a)) = \dim R$.

Proof. Under the assumptions on R, $I/(a)$ and a, one can apply Proposition 1.6, (a), and the right hand inequality of (b) to conclude that $\dim \mathcal{A}_{R\twoheadrightarrow R/J}(I/(a)) = \dim R - 1 + 1 = \dim R$.

To prove the equidimensionality part, we will show that $\mathcal{A}_{R\twoheadrightarrow R/(a)}(I/(a))$ is equidimensional locally at every prime ideal $\mathcal{P} \subset \mathcal{R}_R(I)$ in its support. Localizing first at $\mathcal{P}\cap R$ in the base ring one can assume that $(R, \mathfrak{m})$ is local, with $\mathcal{P}\cap R = \mathfrak{m}$ and $I \subset \mathfrak{m}$. Now, $\mathcal{M} = (\mathfrak{m}, Iu) \subset \mathcal{R}_R(I) \subset R[Iu]$ is not a minimal prime of $\mathcal{A}_{R\twoheadrightarrow R/(a)}(I/(a))$. This is because the Aluffi algebra is graded, with grading induced from $\mathcal{R}_R(I)$, hence $\mathcal{M}$ would actually be its unique associated prime, which is impossible as $\dim R \geq 1$. Thus, for the purpose of showing equidimensionality, we may assume that $\mathcal{P}$ is a homogeneous ideal properly contained in $\mathcal{M}$.

Let $I = (b_1, \ldots, b_n)$. Note that in the present situation, one has by Lemma 1.2:

$$\mathcal{A}_{R\twoheadrightarrow R/(a)}(I/(a)) \simeq \mathcal{R}_R(I)/(a, \tilde{a}).$$

Write $\mathcal{R}_R(I) = R[b_1u, \ldots, b_nu] \subset R[u]$, so that $\tilde{a} = \sum_{j=1}^{n} c_jb_ju$, for suitable $c_j \in R$.

Suppose first that $(Iu) \not\subset \mathcal{P}$. Say, $b_1u \notin \mathcal{P}$. Localizing at $\mathcal{P}$ yields

$$\begin{aligned}
\mathcal{A}_{R\twoheadrightarrow R/(a)}(I/(a))_{\mathcal{P}} \quad &\simeq \quad R[Iu]_{\mathcal{P}}/(a, \tilde{a})_{\mathcal{P}} \simeq R\left[\frac{I}{b_1}, b_1u, (b_1u)^{-1}\right]_{\mathcal{P}'} \Bigg/ \left(a, c_1 + \sum_{j=2}^{m} c_j\frac{b_j}{b_1}\right)_{\mathcal{P}'} \\
&= \quad R\left[\frac{I}{b_1}, b_1u, (b_1u)^{-1}\right]_{\mathcal{P}'} \Bigg/ \left(a, \frac{a}{b_1}\right)_{\mathcal{P}'} \\
&= \quad R\left[\frac{I}{b_1}, b_1u, (b_1u)^{-1}\right]_{\mathcal{P}'} \Bigg/ \left(\frac{a}{b_1}\right)_{\mathcal{P}'},
\end{aligned}$$

where $\mathcal{P}'$ denotes the corresponding image of $\mathcal{P}$. The rightmost ring above is a factor ring of a catenary, equidimensional and equicodimensional ring by a principal ideal generated by a regular element, hence it is equidimensional and so is $\mathcal{A}_{R\twoheadrightarrow R/(a)}(I/(a))_{\mathcal{P}}$ too.

Suppose now that $(Iu) \subset \mathcal{P}$. Then $\mathfrak{m} \not\subset \mathcal{P}$ since $\mathcal{P} \subsetneq \mathcal{M}$, hence $p := \mathcal{P} \cap R \subsetneq \mathfrak{m}$. Note that p is a prime containing a.

If $I \not\subset p$ then $\mathcal{A}_{R\twoheadrightarrow R/(a)}(I/(a))_{\mathcal{P}}$ is a localization of the ring

$$R_p[I_pu]/(a, \tilde{a}) = R_p[u]/(a, au) = R_p[u]/(a)$$

and we conclude as above. If $I \subset p$ then $\mathcal{A}_{R\twoheadrightarrow R/(a)}(I/(a))_{\mathcal{P}}$ is a localization of the Aluffi algebra $\mathcal{A}_{R_p\twoheadrightarrow R_p/(a)}(I/(a))$ and we conclude by induction on $\dim R$.

□

A geometric version of Theorem 1.8 case is stated in [2, Corollary 2.18].

We will have more to say about the equidimensionality of the Aluffi algebra in subsequent sections.

1.3. **Local or graded case.** In this part we assume that $(R, \mathfrak{m})$ is a Noetherian local ring and its maximal ideal or a standard graded algebra over a field and its maximal irrelevant ideal. Throughout $R/\mathfrak{m}$ is an infinite field.

Let $J \subset I \subset \mathfrak{m}$. We confront ourselves with two quite opposite situations, namely, when $J \subset \mathfrak{m}I$ and when J contains minimal generators of I. Note that if J is a reduction of I then $J \subset \mathfrak{m}I$ would entail $I^t = JI^{t-1} \subset \mathfrak{m}I^t$, for $t >> 0$, hence $I^t = \{0\}$, i.e., I would be nilpotent.

Drawing upon a terminology of geometry, let us agree to say that the pair $J \subset I$ of ideals is *non-degenerate* if $J \subset \mathfrak{m}I$. If on the other extreme, $J \subset I$ is generated by a subset of minimal generators of I, we may call the pair $J \subset I$ *totally degenerate*. The latter case can usually be disposed of by a standard argument (see Proposition 2.2).

We recall that the *analytic spread* of I, denoted $\ell(I)$, is the dimension of the $R/\mathfrak{m}$-standard algebra $\mathcal{R}_R(I)/\mathfrak{m}\mathcal{R}_R(I)$. It can be shown that $\ell(I)$ coincides with the number of minimal generators of a least possible reduction of I, but we shall have no occasion to use this result. The behavior of $\ell(I)$ in face of other numerical invariants related to I is as follows:

$$\operatorname{ht} I \leq \ell(I) \leq \min\{\mu(I), \dim R\},$$

where $\mu(I)$ denotes minimal number of generators. We will say that I has *maximal analytic spread* if $\ell(I) = \dim R$. Note that this forces $\mu(I) \geq \dim R$.

Proposition 1.9. *Let $(R, \mathfrak{m})$ be as above with $R/\mathfrak{m}$ infinite. Suppose that $J \subset I$ is a non-degenerate pair and that J has a regular element. Then*

(i) $\ell(I) \leq \dim \mathcal{A}_{R \twoheadrightarrow R/J}(I/J) \leq \dim R$; *in particular, if I has maximal analytic spread then* $\dim \mathcal{A}_{R \twoheadrightarrow R/J}(I/J) = \dim R$.

(ii) *If I has maximal analytic spread and, moreover, $\mu(I) = \dim R$, then $\mathfrak{m}\mathcal{R}_R(I)$ is a minimal prime of $\mathcal{A}_{R \twoheadrightarrow R/J}(I/J)$ of maximal dimension.*

(iii) *If $J \subset I^2$ then* $\dim \mathcal{A}_{R \twoheadrightarrow R/J}(I/J) = \dim R$.

Proof. (i) $J \subset \mathfrak{m}I$ implies $JI^{t-1} \subset \mathfrak{m}I^t$ for every $t \geq 0$. This yields a surjective homomorphism $\mathcal{A}_{R \twoheadrightarrow R/J}(I/J) \twoheadrightarrow \mathcal{R}_R(I)/\mathfrak{m}\mathcal{R}_R(I)$, from which follows the leftmost inequality.

The other inequality stems from Proposition 1.6, (a).

(ii) The assumption $\ell(I) = \mu(I) = \dim R$ implies that I is generated by analytically independent elements and the latter entails that $\mathcal{R}_R(I)/\mathfrak{m}\mathcal{R}_R(I)$ is a polynomial ring over $R/\mathfrak{m}$. In particular, $\mathfrak{m}\mathcal{R}_R(I)$ is a prime ideal of $\mathcal{R}_R(I)$. Since $JI^{t-1} \subset \mathfrak{m}I^t \subset \mathfrak{m}$ then $(J, \tilde{J}) \subset \mathfrak{m}\mathcal{R}_R(I)$ as ideals of $\mathcal{R}_R(I)$. Therefore $\mathcal{A}_{R \twoheadrightarrow R/J}(I/J)/\mathfrak{m}\,\mathcal{A}_{R \twoheadrightarrow R/J}(I/J) \simeq \mathcal{R}_R(I)/\mathfrak{m}\mathcal{R}_R(I)$, hence $\mathfrak{m}\mathcal{A}_{R \twoheadrightarrow R/J}(I/J)$ is a prime ideal with

$$\dim \mathcal{A}_{R \twoheadrightarrow R/J}(I/J)/\mathfrak{m}\,\mathcal{A}_{R \twoheadrightarrow R/J}(I/J) = \dim R = \dim \mathcal{A}_{R \twoheadrightarrow R/J}(I/J)$$

by the first part.

(iii) Write $\operatorname{gr}_I(R)$ for the associated graded ring of I. Since $J \subset I^2$, one has $JI^{t-1} \subset I^{t+1}$ for every $t \geq 0$. This yields a surjective homomorphism $\mathcal{A}_{R \twoheadrightarrow R/J}(I/J) \twoheadrightarrow \operatorname{gr}_I(R)$, showing that $\dim \mathcal{A}_{R \twoheadrightarrow R/J}(I/J) \geq \dim R$. The reverse inequality follows from Proposition 1.6, (a). $\square$

We wrap up with a comment on the last result. Namely, we actually have

$$\dim \mathcal{A}_{R \twoheadrightarrow R/J}(I/J) \geq \max\{\ell(I), \dim R/J + 1\}$$

provided I has a regular element modulo J. The interesting case is when $\ell(I) \geq \dim R/J + 1$. If, say, R is catenary and equidimensional, it would entail

$$\dim \mathcal{A}_{R \twoheadrightarrow R/J}(I/J) \geq \frac{\dim R + 1}{2}.$$

2. Structural properties

In this section one looks more closely at the internal structure of the Aluffi algebra and relate some of the elements of this structure to well-known notions in ideal theory.

2.1. **Torsion and minimal primes.** By Lemma 1.2 one has

$$\mathcal{A}_{R\twoheadrightarrow R/J}(I/J) \simeq \mathcal{R}_R(I)/(J,\tilde{J})\mathcal{R}_R(I) = \bigoplus_{t\geq 0} I^t/JI^{t-1}.$$

Since $\mathcal{R}_{R/J}(I/J) = \bigoplus_{t\geq 0}(I^t, J)/J \simeq \bigoplus_{t\geq 0} I^t/J\cap I^t$, the kernel of the natural surjection $\mathcal{A}_{R\twoheadrightarrow R/J}(I/J) \twoheadrightarrow \mathcal{R}_{R/J}(I/J)$ is the homogeneous ideal

$$\mathbb{W}_{J\subset I} := \bigoplus_{t\geq 2} \frac{J\cap I^t}{JI^{t-1}}, \tag{2}$$

dubbed as the *module of Valabrega–Valla* (see [13], also [16, 5.1])

We retrieve a result of Valla ([15, Theorem 2.8]):

Corollary 2.1. *Let $J \subset I \subsetneq R$ be ideals of the local ring R. If I/J is of linear type over R/J (e.g., if I is generated by a regular sequence modulo J) then $J\cap I^t = JI^{t-1}$ for every positive integer t.*

Proof. This follows immediately from the structural "squeezing" (1). □

Note that the assumption in [6, Proposition 3.10] to the effect that I be of linear type over R does not intervene in the above statement.

Here is a useful explicit situation, where we write $I = (J, \mathfrak{a})$, with no particular care for minimal generation.

Proposition 2.2. *Let $I = (J, \mathfrak{a})$. If $J \cap \mathfrak{a}^t \subset J\mathfrak{a}^{t-1}$, for every $t \geq 0$ then $\mathcal{A}_{R\twoheadrightarrow R/J}(I/J) \twoheadrightarrow \mathcal{R}_{R/J}(I/J)$ is an isomorphism.*

Proof. One has:

$$J\cap I^t = J\cap (J,\mathfrak{a})^t = J\cap (J(J,\mathfrak{a})^{t-1}, \mathfrak{a}^t) = J(J,\mathfrak{a})^{t-1} + J\cap \mathfrak{a}^t \subset JI^{t-1} + J\mathfrak{a}^{t-1} \subset JI^{t-1}.$$

□

Remark 2.3. In the notation of the previous proposition, one of the main results of [7] is that if $\mathfrak{a}$ is generated by a d-sequence modulo J then the assumption of the proposition is fulfilled. Therefore, under the hypothesis of [*loc.cit.*], the surjection $\mathcal{A}_{R\twoheadrightarrow R/J}(I/J) \twoheadrightarrow \mathcal{R}_{R/J}(I/J)$ is an isomorphism. This result, however, is a special case of Corollary2.1 if one uses that an ideal generated by a d-sequence is of linear type. Of course, the proof of this fact requires some non-trivial work on itself and is previous to the later results, such as [6].

When $J = (a)$ is a principal ideal, one has a result somewhat subsumed in the spirit of [15].

Proposition 2.4. *Let $\mathfrak{a}$ be an ideal in the ring R and let $a \in R$ be an element such that $\mathfrak{a}^t : a = \mathfrak{a}^t$ for every integer $t \geq 0$. Then the inclusion $(a) \subset (a,\mathfrak{a})$ induces an isomorphism $\mathcal{A}_{R\twoheadrightarrow R/(a)}((a,\mathfrak{a})/(a)) \simeq \mathcal{R}_{R/(a)}((a,\mathfrak{a})/(a))$.*

Proof. The assumption means that $(a)\cap \mathfrak{a}^t = a\mathfrak{a}^t$ for every $t \geq 0$, hence $(a)\cap (a,\mathfrak{a})^t = a(a,\mathfrak{a})^{t-1} + (a)\cap \mathfrak{a}^t = a(a,\mathfrak{a})^{t-1}$ for $t > 0$. □

The Valabrega–Valla module gives the torsion in as many cases as the ones in which the Rees algebra is the symmetric algebra modulo torsion.

Proposition 2.5. *Let $J \subset I \subsetneq R$ be ideals of the Noetherian ring R. If I/J has a regular element then the R/J-torsion of the embedded Aluffi algebra of I/J is the kernel of the natural surjection $\mathcal{A}_{R\twoheadrightarrow R/J}(I/J) \twoheadrightarrow \mathcal{R}_{R/J}(I/J)$.*

Proof. Consider the general elementary observation: given a ring A and A-modules

$$N \twoheadrightarrow M \twoheadrightarrow K$$

such that the A-torsion of N is the kernel of the composite $N \twoheadrightarrow K$ then the A-torsion of M is the kernel of $M \twoheadrightarrow K$. We apply this to the situation in (1), by recalling that if $\mathfrak{a} \subset A$ has a regular element in the ring A then the A-torsion of the symmetric algebra $\mathcal{S}_A(\mathfrak{a})$ is the kernel of the natural surjection $\mathcal{S}_A(\mathfrak{a}) \twoheadrightarrow \mathcal{R}_A(\mathfrak{a})$. $\square$

Recall that, given a ring S, an ideal $\mathfrak{b} \subset S$ and an S-module E, one denotes by $\mathrm{H}^0_{\mathfrak{b}}(E)$ the zeroth local cohomology of E with respect to $\mathfrak{b}$. One has

$$\mathrm{H}^0_{\mathfrak{b}}(E) \simeq E : \mathfrak{b}^{\infty} := \{\, \epsilon \in E \,|\, \exists n \geq 0,\ \mathfrak{b}^n \epsilon = \{0\}\,\}.$$

Corollary 2.6. *Let $J \subset I \subsetneq R$ be ideals of the Noetherian ring R. If I/J has a regular element then $\mathbb{W}_{J\subset I} = \mathrm{H}^0_{I/J}(\mathcal{A})$, where $\mathcal{A}$ denotes the Aluffi algebra and $\mathbb{W}_{J\subset I}$ is the Valabrega–Valla module as introduced above.*

Proof. By Proposition 2.5, $\mathbb{W}_{J\subset I}$ is the R/J-torsion of $\mathcal{A}$. On the other hand, localizing at primes of the base R/J not containing I/J makes the surjection $\mathcal{S}_{R/J}(I/J) \twoheadrightarrow \mathcal{R}_{R/J}(I/J)$ an isomorphism, hence also the surjection $\mathcal{A} \twoheadrightarrow \mathcal{R}_{R/J}(I/J)$. Therefore, $\mathcal{A}$ is torsionfree locally at those primes. Since I/J has regular elements, the result follows easily (see, e.g., [11, Lemma 5.2]). $\square$

Remark 2.7. The last result says, in particular, that there exists an integer $k \geq 0$ such that $I^k\,(J \cap I^t) \subset JI^{t-1}$ for every $t \geq 1$. Later on we will relate such an exponent to the so-called Artin–Rees number.

By the same principle, one can get a hold of the minimal primes of the Aluffi algebra. Quite generally, to any ideal $\mathfrak{a} \subset R$ we associate its extended–contracted ideal

$$\tilde{\mathfrak{a}} := \mathfrak{a}R[u] \cap R[Iu] = \sum_{t\geq 0} (\mathfrak{a} \cap I^t)u^t$$

in the Rees algebra $\mathcal{R}_R(I) \simeq R[Iu] \subset R[u]$ (u a variable over R).

Proposition 2.8. *Let $J \subset I \subsetneq R$ be ideals of the Noetherian ring R. Any minimal prime $\wp$ of $\mathcal{A}_{R\twoheadrightarrow R/J}(I/J)$ on $\mathcal{R}_R(I)$ is either of the form $\wp = \tilde{p}$ for some minimal prime of R/J on R, or else has the form $(q, \wp_+)$ where $q := \wp \cap R$ contains a minimal prime of R/I on R and $\wp_+ = \wp \cap (Iu)$.*

Proof. By Corollary 2.6 – rather by its proof – a power of I annihilates the kernel of $\mathcal{A}_{R\twoheadrightarrow R/J}(I/J) \twoheadrightarrow \mathcal{R}_{R/J}(I/J)$ lifted to $\mathcal{R}_R(I)$ – call it $\mathcal{K}$. If $\wp \subset \mathcal{R}_R(I)$ is a minimal prime of $\mathcal{A}_{R\twoheadrightarrow R/J}(I/J)$ it follows that $\wp$ contains either $\mathcal{K}$ or I. In the first case, it contains a minimal prime of $\mathcal{R}_R(I)/\mathcal{K} \simeq \mathcal{R}_{R/J}(I/J)$ hence must be a minimal prime of the latter on $\mathcal{R}_R(I)$. But, it is well known that the above extending-contracting operation induces a bijection between the minimal primes of R/J on R and the minimal primes of $\mathcal{R}_{R/J}(I/J)$ on $\mathcal{R}_R(I)$.

In the case $I \subset \wp$, since $\wp$ is homogeneous in the natural $\mathbb{N}$-grading of $\mathcal{R}_R(I)$, then it is clear that $\wp = (q, \wp_+)$, where $\wp_+ = \wp \cap \mathcal{R}_R(I)_+$; obviously, q contains a minimal prime of R/I on R. $\square$

Remark 2.9. Note that if $\wp \subset \mathcal{R}_R(I)$ is a minimal prime of $\mathcal{A}_{R\twoheadrightarrow R/J}(I/J)$ containing I then $\wp_+$ behaves erratically: it can actually be zero in certain cases (see, e.g., Proposition 1.9, (ii)). On the other hand, its contraction $q \subset R$ may turn out to be an embedded associated prime of R/I and not a minimal one (see Section 3).

Corollary 2.10. *Let $J \subset I \subsetneq R$ be ideals of a Noetherian ring R such that I/J has a regular element. If $\mathcal{A}_{R\twoheadrightarrow R/J}(I/J)$ is equidimensional then*

$$\dim \mathcal{A}_{R\twoheadrightarrow R/J}(I/J) = \dim R/J + 1.$$

Proof. This readily follows from Proposition 2.8 and the known value of the dimension of $\mathcal{R}_{R/J}(I/J)$ under the present assumption on I/J. $\square$

Equidimensionality of $\mathcal{A}_{R\twoheadrightarrow R/J}(I/J)$ if J has codimension ≥ 2 may be quite rare. The next result shows that, at least in the local or graded case, pure-dimensionality is really infrequent under this assumption.

Proposition 2.11. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and its maximal ideal $\mathfrak{m}$ or a standard graded algebra over a field and its maximal irrelevant ideal $\mathfrak{m}$. Assume that $R/\mathfrak{m}$ infinite. Let $J \subset I \subset \mathfrak{m}$, with J having a regular element and I having a regular element modulo J. Suppose that:*

(i) *$J \subset I \subset \mathfrak{m}$ is a non-degenerate pair and $\ell(I) = \mu(I) = \dim R$; or else*
(ii) *$J \subset I^2$.*

If $\mathcal{A}_{R\twoheadrightarrow R/J}(I/J)$ is equidimensional then J has height one. If $\mathcal{A}_{R\twoheadrightarrow R/J}(I/J)$ is pure-dimensional then J is an ideal of pure height one (hence, principal if R is regular).

Proof. By Proposition 2.8, $\mathcal{R}_{R/J}(I/J)$ is equidimensional. In particular, one has

$$\dim \mathcal{A}_{R\twoheadrightarrow R/J}(I/J) = \dim \mathcal{R}_{R/J}(I/J) = \dim R/J + 1 \leq \dim R - \operatorname{ht} J + 1.$$

On the other hand, by Proposition 1.9, (ii) or (iii), $\dim \mathcal{A}_{R\twoheadrightarrow R/J}(I/J) = \dim R$. Therefore, $\operatorname{ht} J \leq 1$ (hence $\operatorname{ht} J = 1$ since J has a regular element by hypothesis).

Now, assume that $\mathcal{A}_{R\twoheadrightarrow R/J}(I/J)$ is pure-dimensional. Let $p \in \operatorname{Ass}_R(R/J)$ have height ≥ 2. By Proposition 2.8 and the the pure-dimensionality of $\mathcal{A}_{R\twoheadrightarrow R/J}(I/J)$, the prime $\tilde{p}$ satisfies

$$\begin{aligned}\dim \mathcal{A}_{R\twoheadrightarrow R/J}(I/J) &= \dim \mathcal{A}_{R\twoheadrightarrow R/J}(I/J)/\tilde{p} = \dim \mathcal{R}_R(I)/\tilde{p} = \dim \mathcal{R}_{R/p}((p, I)/p) \\ &= \dim R/p + 1 \leq \dim R - \operatorname{ht} p + 1 \leq \dim R - 2 + 1 = \dim R - 1.\end{aligned}$$

Again Proposition 1.9, (ii) or (iii) gives a contradiction. $\square$

More difficult is to get hold of non-trivial embedded primes of $\mathcal{A}_{R\twoheadrightarrow A}(I/J)$. In the case where J is the ideal of a homogeneous hypersurface in projective space there are often embedded primes containing the irrelevant ideal.

We wrap up with the following

Question 2.12. Suppose as above that $J = (a) \subset \mathfrak{m}I$, $I/(a)$ has a regular element and I has maximal analytic spread. To what extent can we assert that, conversely, $\mathcal{A}_{R\twoheadrightarrow A}(I/J)$ is pure-dimensional?

This seems to be the case in a variety of situations such as the one considered in Section 3.

2.2. **Relation to the Artin–Rees number.** A close associate to $\mathbb{W}_{J\subset I}$ is the well-known ideal

$$\ker\left(\operatorname{gr}_I(R) \twoheadrightarrow \operatorname{gr}_{I/J}(R/J)\right) = \bigoplus_{t\geq 0} (I^{t+1} + J \cap I^t)/I^{t+1},$$

generated by the I-initial forms of elements of J. Recall that the I-initial form of an element $a \in R$ is the residue class a^* of a in $I^{\nu(a)}/I^{\nu(a)+1}$ where $\nu(a)$ is the Ith order of a (i.e., $a \in I^{\nu(a)} \setminus I^{\nu(a)+1}$, setting $\nu(a) = \infty$ and $a^* = 0$ if $a \in \cap_{t\geq 0} I^t$).

A set of elements of J is called an *I-standard base* of J if their initial forms in $\operatorname{gr}_I(R)$ generate the above ideal. If R is Noetherian local then an I-standard base of J is a generating set of J (see [5, Lemma 6]). We will shorten $\nu(a_i)$ to ν_i if a_i is sufficiently clear from the context.

The following basic result will be used throughout.

Theorem 2.13. ([13]) *Let $J = (a_1, \dots, a_m)$ be an ideal of ring R. Then $\{a_1, \dots, a_m\}$ is an I-standard base of J if and only if*

$$J \cap I^t = \sum_{i=1}^{m} a_i I^{t-\nu_i}$$

for every positive integer t.

This result implies immediately:

Proposition 2.14. *Let R be a Noetherian local ring and $J \subset I$ be ideals of R. Let $\{a_1, \dots, a_m\}$ be an I-standard base of J such that $1 \leq \nu_1 \leq \nu_2 \leq \cdots \leq \nu_m$.*

(a) *The surjection $\mathcal{A}_{R \twoheadrightarrow R/J}(I/J) \twoheadrightarrow \mathcal{R}_{R/J}(I/J)$ is an isomorphism if and only if $\nu_m = 1$ (i.e., $\nu_1 = \cdots = \nu_m = 1$)*

(b) *More generally*

$$\mathfrak{R} \cap I^{\nu_m - 1}\mathcal{R}_R(I) \subset \mathfrak{A} \subset \mathfrak{R} \cap I^{\nu_1 - 1}\mathcal{R}_R(I),$$

where $\mathfrak{R} = \bigoplus_{t \geq 0} J \cap I^t$ and $\mathfrak{A} = \bigoplus_{t \geq 0} JI^{t-1}$.

(c) *If $\nu_1 > 1$ and J is not contained in any minimal prime of R, then $\dim \mathcal{A}_{R \twoheadrightarrow R/J}(I/J) = \dim R$.*

Proof. (a) One direction is obvious from Theorem 2.13. For the reverse implication, assume that $J \cap I^t = JI^{t-1}$ for every positive integer t. By the above remark, one may assume that $a_1, \dots, a_m$ is a minimal set of generators of J. If for some i, $a_i \in I^2$ then by hypothesis $a_i \in JI$, which clearly contradicts the minimality of $a_1, \dots, a_m$.

(b) By definition, we want to show the two inclusions

$$J \cap I^{t+\nu_m-1} \subset JI^{t-1} \subset J \cap I^{t+\nu_1-1}$$

as subideals of $J \cap I^t$, for every $t \geq 1$.

This is however a straightforward consequence of Theorem 2.13 as one has thereof

$$J \cap I^{\nu_m+t-1} = \sum_{i=1}^{m} a_i I^{\nu_m+t-1-\nu_i} \subset \sum_{i=1}^{m} a_i I^{t-1} = JI^{t-1},$$

and similarly

$$J \cap I^{\nu_1+t-1} = \sum_{i=1}^{m} a_i I^{\nu_1+t-1-\nu_i} \supset \sum_{i=1}^{m} a_i I^{t-1} = JI^{t-1}.$$

(c) Quite generally, for a positive integer r one has

$$\begin{aligned} \dim \mathcal{R}_R(I)/\mathfrak{R} \cap I^r\mathcal{R}_R(I) &\geq \max\{\dim \mathcal{R}_R(I)/\mathfrak{R}\ ,\ \dim \mathcal{R}_R(I)/I^r\mathcal{R}_R(I)\} \\ &= \max\{\dim \mathcal{R}_{R/J}(I/J),\ \dim \mathrm{gr}_I(R)\} = \dim R. \end{aligned}$$

On the other hand, since J is assumed to be of positive height, by Proposition 1.6 (a) one has $\dim \mathcal{A}_{R \twoheadrightarrow R/J}(I/J) \leq \dim R$. The result follows now at once. □

We close with yet another condition for the surjection $\mathcal{A}_{R \twoheadrightarrow R/J}(I/J) \twoheadrightarrow \mathcal{R}_{R/J}(I/J)$ to be an isomorphism.

For this recall that, pretty generally, given ideals $J, I \subset R$ the *Artin–Rees number* of J relative to I is the integer

$$\min\{k \geq 0 \mid J \cap I^t = (J \cap I^k)I^{t-k}\ \forall\ t \geq k\}.$$

We observe that if $J \subset I \subsetneq R$, where R is Noetherian and J has regular elements then the Artin–Rees number of J relative to I is ≥ 1.

Proposition 2.15. *Let $J \subset I \subsetneq R$ be ideals of a Noetherian ring R and let $k \geq 1$ be an upper bound for the Artin–Rees number of J relative to I, i.e., $J \cap I^t = (J \cap I^k)I^{t-k}\ \forall\ t \geq k$.*

Then I^{k-1} annihilates the kernel of the surjection$\mathcal{A}_{R \twoheadrightarrow R/J}(I/J) \twoheadrightarrow \mathcal{R}_{R/J}(I/J)$. Moreover, the latter is an isomorphism if and only if the Artin–Rees number of J relative to I is 1.

Proof. One has $(J \cap I^t)I^{k-1} = (J \cap I^k)I^{t-k}I^{k-1} = (J \cap I^k)I^{t-1} \subset JI^{t-1}$ for $t \geq k$. On the other hand, for $t \leq k-1$, one has $I^{k-1} \subset I^{t-1}$, hence $(J \cap I^t)I^{k-1} \subset (J \cap I^t)I^{t-1} \subset JI^{t-1}$.

The second assertion is clear. $\square$

More generally:

Lemma 2.16. *Let $J \subset I \subset R$ be ideals of a ring R. Assume that ℓ is an upper bound for the Artin–Rees number of J relative to I such that $J \cap I^t = JI^{t-1}$ for every $t \leq \ell$. Then $\mathcal{A}_{R \twoheadrightarrow R/J}(I/J) \twoheadrightarrow \mathcal{R}_{R/J}(I/J)$ is an isomorphism.*

Proof. By assumption $J \cap I^t = I^{t-\ell}(J \cap I^\ell)$ for $t \geq \ell$. Now use the assumed equality $J \cap I^\ell = JI^{\ell-1}$ to get $J \cap I^t = JI^{t-1}$ for $t \geq \ell$. $\square$

Given a ring A and an ideal $\mathfrak{a} = (a_1, \ldots, a_n) \subset A$, one lets $R[T_1, \ldots, T_n] \to \mathcal{R}_R(\mathfrak{a}) = R[\mathfrak{a}T]$ be the graded map sending T_i to $a_i T$. The *relation type* of $\mathfrak{a}$ is the largest degree of any minimal system of homogeneous generators of the kernel $\mathcal{J}$. Since the isomorphism $R[T_1, \ldots, T_m]/\mathcal{J} \simeq \mathcal{R}_R(\mathfrak{a})$ is graded, an application of the Schanuel lemma to the graded pieces shows that the notion is independent of the set of generators of $\mathfrak{a}$.

Corollary 2.17. *Let R be a Notherian ring and let $J \subset I$ be ideals in R such that*

(i) *I/J has relation type at most ℓ as an ideal of R/J.*
(ii) *$J \cap I^t = JI^{t-1}$ for every $t \leq \ell$.*

Then $\mathcal{A}_{R \twoheadrightarrow R/J}(I/J) \twoheadrightarrow \mathcal{R}_{R/J}(I/J)$ is an isomorphism.

Proof. By Lemma 2.16, it suffices to show that ℓ is an upper bound for the Artin–Rees number of J relative to I. Thus, Let I be generated by elements $a_1, \ldots, a_n$ and let let $a \in J \cap I^t$, with $t \geq \ell$. Then there exists a homogeneous $F \in R[\mathbf{T}] = R[T_1, \ldots, T_n]$, of degree t, such that $F(a_1, \ldots, a_n) = a$. Since $a \in J$, reducing modulo J shows that F is a polynomial relation on $a_1, \ldots, a_n$ modulo J. Then, by assumption there are polynomials $G_i, H_i \in R[\mathbf{T}]$ of degrees $\ell, t-\ell$, respectively, such that $G_i(a_1, \ldots, a_n) \equiv 0 \pmod{JR[\mathbf{T}]}$ for $i = 1, \ldots, n$ and $F \equiv \sum_i G_i H_i \pmod{JR[\mathbf{T}]}$. Write $F = \sum G_i H_i + L$ for some homogeneous polynomial $L \in JR[\mathbf{T}]$ of degree t. Since

$$L(a_1, \ldots, a_n) \in JI^t \subset I^{t-\ell}(J \cap I^\ell),$$

then $G_i(a_1, \ldots, a_n) \subset J \cap I^\ell$ and $H_i(a_1, \ldots, a_n) \in I^{t-\ell}$ for $t \geq \ell$. This shows that the element $a = F(a_1, \ldots, a_n) \in I^{t-\ell}(J \cap I^\ell)$, that is, $J \cap I^t = (J \cap I^\ell)I^{t-\ell}$ for $t \geq \ell$. $\square$

Using the notion of standard base we can add a tiny bit on the problem of describing the associated primes of the Aluffi algebra.

Proposition 2.18. *Let $J \subset I \subsetneq R$ be ideals of a Noetherian ring R such that I/J has a regular element. If $p \in \operatorname{Ass}_R(R/J)$ then $\tilde{p} \in \operatorname{Ass}_{\mathcal{R}_R(I)}(\mathcal{A}_{R \twoheadrightarrow R/J}(I/J))$.*

Proof. Let $p \in \operatorname{Ass}_R(R/J)$. Since I/J contains a regular element, one has $I \not\subset p$. Let $a_1, \ldots, a_m$ be an I-standard base of p, so that $p = (a_1, \ldots, a_m)$ and

$$\tilde{p} = \bigoplus_{t \geq 0} p \cap I^t = \bigoplus_{t \geq 0} \left(\sum_{i=1}^m a_i I^{t-\nu_i} \right),$$

where $\nu_i = \nu_I(a_i)$. Write $\nu = \max_i \nu_i$ for $i = 1, \ldots, m$ and take $b \in I^{\nu-1} \setminus p$ (note that if $\nu = 0$, which is a possibility, one means any $b \notin p$).

Write $\mathfrak{R} = \bigoplus_{t\geq 0} J \cap I^t$ and $\mathfrak{A} = \bigoplus_{t\geq 0} JI^{t-1}$.

Say, $p = J : a$. We claim that $\tilde{p} = \mathfrak{A} : ab$ which will prove that $\tilde{p}$ is an associated prime of $\mathcal{A}_{R\twoheadrightarrow R/J}(I/J))$ on $\mathcal{R}_R(I)$.

For this, let $c_t \in J \cap I^t \subset p \cap I^t = \sum_{i=1}^m a_i I^{t-\nu_i}$, with $t \geq 0$. Then

$$bc_t \in \sum_{i=1}^m a_i\, bI^{t-\nu_i} \subset \sum_{i=1}^m a_i I^{t-\nu_i+\nu-1} \subset \sum_{i=1}^m a_i I^{t-1} = p\, I^{t-1} = (J : a)I^{t-1} \subset JI^{t-1} : a,$$

hence $b\tilde{p} \subset \mathfrak{A} : a$, hence $\tilde{p} \subset \mathfrak{A} : ab$.

For the inverse inclusion, since $\mathfrak{A} \subset \mathfrak{R}$, it follows that $\mathfrak{A} : ab \subset \mathfrak{R} : ab = (\mathfrak{R} : a) : b$. Note that $\widetilde{J : a} = (J : a)R[t] \cap \mathcal{R}_R(I) = (JR[t] : a) \cap \mathcal{R}_R(I) = (JR[t] \cap \mathcal{R}_R(I)) : a = \mathfrak{R} : a$.

Therefore $\mathfrak{A} : ba \subset \tilde{p} : b = \tilde{p}$. Thus, $\tilde{p} = \mathfrak{A} : ab$ as was to be shown. □

2.3. **Selected examples.** Let us agree to call a pair of ideals $J \subset I \subset R$ an *$\mathcal{A}$-torsionfree pair* if the map $\mathcal{A}_{R\twoheadrightarrow A}(I/J) \twoheadrightarrow \mathcal{R}_{R/J}(I/J)$ is injective.

The examples we have in mind in this part are of the two sorts mentioned previously, namely, of totally degenerate or non-degenerate pairs. The first kind will be based on Proposition 2.2. For these, we let $R = k[\mathbf{X}]$ be an $\mathbb{N}$-graded polynomial ring over a field k, $J \subset R$ is a homogeneous ideal and $I \subset R$ is the Jacobian ideal of J, by which we always mean the ideal $(J, \mathcal{I}_r(\Theta))$ where $r = \mathrm{ht}\,(J)$ and Θ stands for the Jacobian matrix of a set of generators of J. One knows that this maybe a slack ideal, but it is well defined modulo J.

First we state a general format that implies an $\mathcal{A}$-torsionfree pair $J \subset I$.

Example 2.19. Let $J \subset R = k[\mathbf{x}]$ be an ideal generated by forms of the same degree $d \geq 1$. If $I = (J, \mathfrak{m}^r)$, where $\mathfrak{m} = (\mathbf{x})$ and $r \geq d$, then the pair $J \subset I$ is $\mathcal{A}$-torsionfree.

To see this, one uses Proposition 2.2. Namely, it suffices to show that $J \cap \mathfrak{m}^{rt} \subset J\mathfrak{m}^{r(t-1)}$ for every $t \geq 1$. Let $a_1, \ldots, a_m$ be generators of J of degree d and let F be a form in the a_i's such that $F \in \mathfrak{m}^{rt}$. Then $F = \sum_{i=1}^m G_i a_i$ where $G_i = \sum a_{\underline{\alpha}} \mathbf{x}^\alpha \in R_{rt-d+\delta}$ for $\delta \geq 0$, since $R_{rt-d+\delta} = R_{r-d+\delta}\, R_{rt-r}$, so we can rewrite G_i as

$$G_i = \sum_{\substack{|\alpha|=r-d+\delta \\ |\beta|=rt-r}} a_{\alpha,\beta}\, \mathbf{x}^{\alpha+\beta}, \quad \text{hence} \quad F = \sum_{|\alpha|=r-d+\delta} \mathbf{x}^\alpha \left(\sum_{\substack{i=1 \\ |\beta|=rt-r}}^{s} (\mathbf{x}^\beta) a_i \right)$$

Therefore $F \in J\mathfrak{m}^{rt-r} = J\mathfrak{m}^{r(t-1)}$, as required.

Instances of this situation seem to be any of the following ideals J with respect to the respective Jacobian ideal I.

(a) The defining ideal of the rational normal curve;
(b) The defining ideal of the Segre embedding of $\mathbb{P}^r \times \mathbb{P}^s$, with $r > 1$ or $s > 1$;
(c) The defining ideal of the 2–Veronese embedding of a projective space;
(d) The ideal generated by the 4×4 Pfaffians.

It is well-known that J is the ideal of 2–minors of the generic Hankel, square, symmetric matrix, respectively, and, lastly, the ideal generated by the maximal Pfaffians of a 5×5 skew-symmetric.

Let I denote the Jacobian ideal of J on R. Set $\mathfrak{m} = (\mathbf{x})$ and write $\mathrm{ht}\, J = r \geq 2$. We would need to prove that $I_r(\Theta) = \mathfrak{m}^r$, where Θ is the Jacobian matrix of the natural generators of J.

A calculation of a good deal of cases gives evidence to this equality – actually, it may be the expression of a more general fact disguised under an inductive procedure.

Note that the case of the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ is exceptional, essentially because it is a hypersurface. Here the ideal of 1-minors is $\mathbf{x}$ which is of linear type, but clearly the defining ideal of the relative blowup contains the equation of the dual to the (self-dual) quadric surface.

Example 2.20. Consider the monomial $x_1 \cdots x_n \in R = k[x_1, \ldots, x_n]$ $(n \geq 3)$ and let $J \subset R$ be the ideal generated by its partial derivatives $a_i =: x_1 x_2 \cdots \widehat{x_i} \cdots x_n$, for $i = 1, ..., n$. If I is the Jacobian ideal of J the pair $J \subset I$ is $\mathcal{A}$-torsionfree.

Proof. Its well-known and easy that J is a codimension 2 perfect ideal with Hilbert–Burch matrix

$$\varphi = \begin{pmatrix} x_1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & x_2 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & x_{n-2} & \ldots & 0 \\ 0 & x_{n-1} & 0 & \ldots & 0 \\ -x_n & -x_n & -x_n & \ldots & -x_n \end{pmatrix}.$$

Setting $\Delta_{i,j} := \frac{\partial a_j}{\partial x_i} = x_1 x_2 ... \widehat{x_i} ... \widehat{x_j} ... x_n$, and inspecting the Hessian matrix Θ of $x_1 \cdots x_n$ – a symmetric matrix – one finds three basic types of 2×2 minors, namely

- Principal minors:
$$\det \begin{pmatrix} 0 & \Delta_{i,j} \\ \Delta_{i,j} & 0 \end{pmatrix} = \Delta_{i,j}^2,$$
one for each pair $i < j$;
- vanishing minors:
$$\det \begin{pmatrix} \Delta_{i,j} & \Delta_{i,j'} \\ \Delta_{i',j} & \Delta_{i',j'} \end{pmatrix} = 0,$$
one for each choice of row indices $1 \leq i, i' \leq n$ and column indices $1 \leq j, j' \leq n$;
- semidiagonal minors of typical form
$$\Lambda_j := \det \begin{pmatrix} \Delta_{i,j} & 0 \\ * & \Delta_{i',i} \end{pmatrix}.$$

Since clearly, $\Lambda_j \in J$, we get that the Jacobian ideal I of J is generated by J and the squares of the second partial derivatives of $x_1 \cdots x_n$, i.e., $I = (J, \Delta_{i,j}^2)$ for $1 \leq i < j \leq n$.

As a side curiosity we note that $I = (J, \mathcal{I}_2(\Theta)) = (J, \mathcal{I}_{n-2}(\varphi)^2)$, hence $\sqrt{I} = \mathcal{I}_{n-2}(\varphi)$, so in particular I/J has codimension one. This example will therefore yield a case of a height one ideal in R/J which is $\mathcal{A}$-torsionfree, but clearly not of linear type because its number of generators on R/J is too large.

Setting $\Delta = (\Delta_{i,j}^2 \mid 1 \leq i < j \leq n)$, the usual algorithmic procedure to find generators of the intersection of monomial ideals yields for any $t \geq 2$

$$J \cap \Delta^t = ((x_i, x_j)\Delta_{i,j}^{2t}, (\mathfrak{F}))$$

where $\mathfrak{F}$ is the set of all monomials in Δ^t excluding the monomials $\Delta_{i,j}^{2t}$ for $1 \leq i < j \leq n$. Another calculation shows that $(x_i, x_j)\Delta_{i,j}^{2t} \in J\Delta^{t-1}$, for $1 \leq i < j \leq n$, and that $\mathfrak{F} \subset J^2 \Delta^{t-2}$. This proves that $J \cap \Delta^t \subset JI^{t-1}$. $\square$

Question 2.21. (k algebraically closed) Let $J \subset R = k[x_1, \ldots, x_n]$ denote the homogeneous defining ideal of an arrangement of linear subspaces of dimension $n-3$ of $\mathbb{A}^n$. If I is the Jacobian ideal of J, when is $J \subset I$ an $\mathcal{A}$-torsionfree pair?

Plausibly, a similar question can be posed about the Jacobian ideal of a hyperplane arrangement.

Example 2.22. Let $J \subset R = k[x_1, \ldots, x_n]$ denote the ideal of the coordinate points in projective space $\mathbb{P}^{n-1}$, i.e., $J = (x_i x_j \mid 1 \leq i < j \leq n)$. If I is the Jacobian ideal of J the pair $J \subset I$ is $\mathcal{A}$-torsionfree.

Proof. Since J contains all square-free monomials of degree 2, it is rather transparent that the Jacobian ideal I of J is generated by J and pure powers of the variables. Moreover, a closer inspection shows that, more precisely,

$$I = (J,\ x_1^{n-1},\ \ldots,\ x_n^{n-1}).$$

Setting $\Delta := (x_1^{n-1},\ \ldots,\ x_n^{n-1})$, a procedure based on finding generators of the intersection of monomial ideals yields for any $t \geq 2$

$$J \cap \Delta^t = (x_i^{t(n-1)}(x_1, \ldots, \widehat{x_i}, \ldots, x_n), (\mathfrak{F})),$$

where $(\mathfrak{F})$ is the set of all monomials in Δ^t excluding the monomials $x_i^{t(n-1)}$ for $i = 1, \cdots, n$. A calculation shows that $x_i^{t(n-1)}(x_1, \ldots, \widehat{x_i}, \ldots, x_n) \in J\Delta^{n-1}$ for $i = 1, \cdots, n$ and that $(\mathfrak{F}) \subset J^2\Delta^{t-2}$. This proves that $J \cap \Delta^t \subset JI^{t-1}$.

Example 2.23. Let $J \subset R = k[x, y, z]$ denote the homogeneous defining ideal of the four points $(1:0:0), (0:1:0), (0:0:1)$ and $(1:1:1)$ in the projective plane $\mathbb{P}^2_k$ and let I denote its Jacobian ideal. An easy calculation gives

$$J = (x^2 - xz, y^2 - yz) \quad \text{and} \quad I = (x^2 - xz, y^2 - yz, x(2y - z), y(2x - z), (2x - z)(2y - z)).$$

In terms of the internal grading of the algebra, using the description in Proposition 2.5, the torsion is generated by the appropriate residues of $\{xz^2(x - z), yz^2(y - z)\} \subset J \cap I^2$. For further details see [10, 1.1].

The last two examples motivate the following

Question 2.24. (k algebraically closed) Let $J \subset R = k[x_1, \ldots, x_n]$ denote a radical homogeneous ideal of codimension $n - 1$ (i.e., the ideal of a reduced set of points). If I is the Jacobian ideal of J, when is $J \subset I$ an $\mathcal{A}$-torsionfree pair?

3. The gradient Aluffi algebra of a projective hypersurface

In this section we will deal with the case where J is generated by the equation of a reduced hypersurface.

Thus, let $J = (f) \subset R$ be a principal ideal, where $R = k[X_1, \ldots, X_n]$. We will focus on the Jacobian ideal $I_f = (f, \partial f/\partial X_1, \ldots, \partial f/\partial X_n)$. We are particularly motivated by the problem as to when I_f is an ideal of linear type. In general, if f is not Eulerian, the local number of generators of I_f maybe an early obstruction to this property. We will consider the case where f is homogeneous in the standard grading of the polynomial ring and its degree is not a multiple of the characteristic – hence, $f \in I_f$. In this context the ideal I_f will often be of linear type.

We call I_f the *gradient* ideal of f and the corresponding algebra $\mathcal{A}_{R\twoheadrightarrow R/(f)}(I_f/(f))$, the *gradient Aluffi algebra* of f. By Proposition 1.6 (d), it is equidimensional of dimension $\dim R = n$.

3.1. **Preliminaries on the gradient ideal.** If f defines a smooth hypersurface then I_f is irrelevant, i.e., is generated by a regular sequence, hence is of linear type. We regard this case as uninteresting and assume that f has singularities. This entails $\operatorname{ht}(I_f/(f)) \leq n-2$. If moreover f is reduced then $\operatorname{ht}(I_f/(f)) \geq 1$. For $n=3$ we therefore find $\operatorname{ht}(I_f/(f)) = 1$. Ideals of height 1 in non-regular rings of dimension 2 are a tall order and typically involve a non-trivial primary decomposition.

Thus, even over projective plane curves the structure of the gradient Aluffi algebra seems to be fairly intricate. Note that for $n=3$, the ideal I_f is an almost complete intersection. Since we regard the linear type case as a limit situation we would like to understand this case first.

Now, for an almost complete intersection this property is fairly under control. For convenience we file the following general result, which collects in a more detailed version several known facts about an almost complete intersection (see [12, Proposition 3.7], also [6, Proposition 8.4, Proposition 10.4, Remark 10.5]).

Lemma 3.1. *Let R denote a Cohen–Macaulay local ring and let $I \subset R$ denote a proper ideal of height $h \geq 0$. Assume that*

- *I is a strict almost complete intersection (i.e., minimally generated by $h+1$ elements)*
- *R/I is equidimensional (i.e., $\dim R/I = \dim R/P$ for every minimal prime P of R/I)*
- *I satisfies the so-called sliding depth inequality* $\operatorname{depth} R/I \geq \dim R/I - 1$.

Let $R^m \xrightarrow{\varphi} R^{h+1} \longrightarrow I \longrightarrow 0$ stand for a minimal free presentation of I as an R-module. The following conditions are equivalent:

(1) $\operatorname{ht} I_1(\varphi) \geq \operatorname{ht} I + 1$
(2) *I_P is a complete intersection for every minimal prime P of R/I*
(3) *I is of linear type.*

Proof. (1) $\Rightarrow$ (2) Localizing at such a prime leaves some element of $I_1(\varphi)$ invertible, so up to an elementary transformation on φ_P the local presentation has the form

$$R_P^{m-1} \oplus R_P \xrightarrow{\varphi_P} R_P^h \oplus R_P \longrightarrow I_P \longrightarrow 0,$$

with

$$\varphi_P = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & \psi \end{array}\right)$$

Therefore, we get a free presentation $R_P^{m-1} \xrightarrow{\psi} R_P^h \longrightarrow I_P \longrightarrow 0$, thus showing that I_P is generated by (at most) h elements.

(2) $\Rightarrow$ (3) By [6, Proposition 10.4] the symmetric algebra of I is a Cohen–Macaulay ring. Therefore, by [6, Proposition 8.4] it suffices to show that $\operatorname{ht} I_Q \leq \operatorname{ht} Q$ for every prime $Q \subset R$ containing I. Let P be a minimal prime of R/I contained in Q. If $P = Q$ the hypothesis guarantees the inequality. Otherwise $\operatorname{ht}(Q) \geq h+1$. But $\operatorname{ht}(I_Q) = \operatorname{ht} I_P = h$ because R/I is equidimensional, hence we are through.

(3) $\Rightarrow$ (1) By [6, Lemma 8.2 and Proposition 8.4] one has $\operatorname{ht} I_t(\varphi) \geq \operatorname{rank}(\varphi) - \mathrm{t} + 2$ for every $1 \leq t \leq \operatorname{rank}(\varphi) = \mathrm{h}$. In particular, $\operatorname{ht} I_1(\varphi) \geq h - 1 + 2 = \operatorname{ht} I + 1$.

$\square$

Corollary 3.2. *Let $f \in R = k[x_1, \ldots, x_n]$ stand for a reduced homogeneous polynomial. Assume that the singular locus of $V(f) \subset \mathbb{P}^{n-1}$ consists of a nonempty set of points. The following are equivalent:*

(1) *The coordinates of the vector fields of $\mathbb{P}^{n-1}$ vanishing on f generate an irrelevant ideal.*
(2) *Locally at each singular point of $V(f)$ the gradient ideal is a complete intersection.*

(3) *The gradient ideal of f is of linear type.*

Proof. A vector field $v = \sum_{i=1}^n a_i\partial/\partial x_i$ vanishes on f if and only if $\sum_{i=1}^n a_i\partial f/\partial x_i = 0$. Therefore the coordinates of all such vector fields generate the ideal of 1-minors of a syzygy matrix of the gradient ideal. The result then follows from Lemma 3.1 once its hypotheses are verified in this setup, as we next proceed to see.

Since f is assumed to be reduced, whose singular locus is a nonempty set of isolated singularities, its gradient ideal is a (homogeneous) ideal of codimension $n-1$, hence can only have minimal primes of codimension $n-1$ in R. Therefore it is equidimensional.

Finally, the depth condition is trivially verified for the numbers in question. $\square$

So much for the linear type property. Clearly, this property implies that the partial derivatives are algebraically independent over k. The latter property in turn reads geometrically to the effect that the polar map associated to the hypersurface $V(f) \subset \mathbb{P}^{n-1}$ is dominant. In characteristic zero this is tantamount to saying that the Hessian of f does not vanish (cf. [4] for a detailed account on this).

The following result collects parts of the main backstage for the Aluffi gradient algebra.

Theorem 3.3. *Let k denote an infinite field, let $f \in R = k[\mathbf{x}] = k[x_1, \ldots, x_n]$ be a reduced homogeneous polynomial whose degree is not a multiple of the characteristic of k and let $I_f \subset R$ denote the corresponding gradient ideal. Assume that*

(i) *The singular locus of $V(f) \subset \mathbb{P}^{n-1}$ consists of a nonempty set of points (equivalently, $\dim R/I_f = 1$)*
(ii) *The partial derivatives of f are algebraically independent over k.*

Then:

(a) *The minimal primes of the gradient Aluffi algebra on $\mathcal{R}_R(I_f)$ are*
  - *The minimal prime ideals of $\mathcal{R}_{R/(f)}(I_f/(f))$, all of the form $\sum_{t\geq 0}(p) \cap I^t$ for a prime factor p of f*
  - *The extended ideal $(\mathbf{x})\mathcal{R}_R(I_f)$*
  - *Prime ideals whose lifting to $R[\mathbf{T}] = R[T_1, \ldots, T_n]$ from a presentation $R[\mathbf{T}]/\mathcal{A} \simeq \mathcal{R}_R(I)$ have the form $(P, \mathfrak{f})$, where $P \subset R$ is a minimal prime of R/I_f and $\mathfrak{f}$ is an irreducible homogeneous polynomial in $k[\mathbf{T}]$.*

(b) *The gradient Aluffi algebra has non-trivial torsion.*
(c) *I_f is an ideal of linear type (respectively, weakly of linear type) if and only if the natural surjection*
$$\mathcal{S}_{R/(f)}(I_f/(f)) \twoheadrightarrow \mathcal{A}_{R\twoheadrightarrow R/(f)}(I_f/(f))$$
is an isomorphism (respectively, an isomorphism in all high degrees).
(d) *The symmetric algebra $\mathcal{S}_{R/(f)}(I_f/(f))$ is Cohen–Macaulay; in particular, if I_f is of linear type then the gradient Aluffi algebra is Cohen–Macaulay.*

Proof. (a) We apply Proposition 2.8, from which the first set of minimal primes is clear.

To see that $(\mathbf{x})\mathcal{R}_R(I_f)$ is a minimal prime thereof one proceeds as follows. There is a presentation of the gradient Aluffi algebra

$$\mathcal{A}_{R\twoheadrightarrow R/(f)}(I_f/(f)) \simeq R[\mathbf{T}]/(\mathcal{J}_f, f, \sum_{i=1}^n x_iT_i), \tag{3}$$

where $\mathcal{J}_f$ denotes the defining ideal of the Rees algebra $\mathcal{R}_R(I_f)$ on $R[\mathbf{T}]$.

Since the partials are homogeneous of the same degree algebraic independence over k is tantamount to analytic independence (i.e., the relations of the generators of I_f have coefficients in the ideal $(\mathbf{x})$). Therefore, the result follows from Proposition 1.9, (ii).

Let $\mathcal{P}$ be a minimal prime of $\mathcal{A}_{R\twoheadrightarrow R/(f)}(I_f/(f))$ whose contraction P to R contains I_f and is properly contained in $(\mathbf{x})$. By Proposition 2.8, $\mathcal{P} = (P, \mathcal{P}_+)$. By assumption (i) it follows that P is a minimal prime of R/I_f, hence has height $n-1$. By Theorem 1.8, $\mathcal{A}_{R\twoheadrightarrow R/(f)}(I_f/(f))$ is equidimensional. Therefore the lifting of $\mathcal{P}$ to $R[\mathbf{T}]$ has height n. Since the lifting of any minimal generator of $(\mathcal{P}_+)$ is irreducible in $k[\mathbf{T}]$ it follows that the lifting of $\mathcal{P}$ to $R[\mathbf{T}]$ has the required form.

(b) The defining equation of the dual curve to f belongs to the presentation ideal of $\mathcal{R}_{R/(f)}(I_f/(f))$ on $R[\mathbf{T}]$ and not to $(\mathbf{x})R[\mathbf{T}]$, hence by (a) it does not belong to the defining ideal of $\mathcal{A}_{R\twoheadrightarrow R/(f)}(I_f/(f))$ on $R[\mathbf{T}]$.

(c) This is a straightforward application of Proposition 1.4. The argument for the weak version of the property of being of linear type is exactly the same as in [loc.cit.].

(d) We apply the criterion of [6, Theorem 10.1]. Namely, we have to verify the following conditions:

(A) $\mu(I_f/(f)_{P/f}) \leq \mathrm{ht}\,(P/(f)) + 1 = \mathrm{ht}\,P$, for every prime ideal $P \supset I_f$ of R.

(B) $\mathrm{depth}\,(H_i)_{P/(f)} \geq \mathrm{ht}\,(P/(f)) - \mu(I_f/(f)_{P/(f)}) + i = \mathrm{ht}\,P - \mu(I_f/(f)_{P/(f)}) + i - 1$, for every prime ideal $P \supset I_f$ of R and every i such that $0 \leq i \leq \mu(I_f/(f)_{P/(f)}) - \mathrm{ht}\,(I_f/(f)_{P/(f)})$, and where H_i denotes the ith Koszul homology module of the partial derivatives on $R/(f)$.

Note that the primes containing I_f are $\mathfrak{m} = (x_1, \ldots, x_n)$ and its minimal primes, the latter all of height $n-1$.

(A) Since I_f itself is generated by n elements, it suffices to check the minimal primes. Thus, let $P \subset R$ be such a prime. Say, without lost of generality, that $x_n \notin P$. Because of the Euler relation, $\partial f/\partial x_n \in I_f$ locally at P and module (f). Therefore, locally at P and module (f), I_f is generated by $n-1 = \mathrm{ht}\,(P)$ elements.

(B) If P is a minimal prime of I_f we saw in (A) that $\mu(I_f/(f)_{P/(f)}) = n-1$. Since $\mathrm{ht}\,P = n-1$, the condition is trivially verified as $i = 0, 1$.

Thus, let $P = \mathfrak{m}$. Again, an easy inspection of the numbers tell us that only the case where $i = 2$ needs an argument and, in this case, one has to prove that $\mathrm{depth}\,(H_2)_{\mathfrak{m}/(f)} \geq 1$. Localize R at $\mathfrak{m}$ and update the notation, so $R := R_{\mathfrak{m}} \supset I_f := I_{f\,\mathfrak{m}} \supset (f) = (f)_{\mathfrak{m}}$ and $H_2 := (H_2)_{P/(f)}$.

Now, f is a nonzero element in I_f and I_f has grade $n-1$ in R. Thus, there is a regular sequence in I_f of length $n-1$ starting with f. Write $L \subset I_f$ for the ideal generated by this regular sequence.

The following isomorphism is well know (see, e.g., [3, Theorem 1.6.16]):

$$H_2 \simeq \mathrm{Hom}_{R/(f)}\left(\frac{R/(f)}{I_f/(f)}, \frac{R/(f)}{L/(f)}\right) \simeq \mathrm{Hom}_{R/(f)}\left(R/I_f, R/L\right).$$

Therefore $\mathrm{Ass}\,_{R/(f)}(H_2) = \mathrm{Supp}_{R/(f)}(R/I_f) \cap \mathrm{Ass}\,_{R/(f)}(R/L) \subset \mathrm{Ass}\,_{R/(f)}(R/L)$. But L is generated by a regular sequence of length $n-2$ modulo f by construction, while $\dim R/(f) = n-1$. It follows that $\mathfrak{m}/(f) \notin \mathrm{Ass}\,_{R/(f)}(R/(J,f))$, hence $\mathfrak{m}/(f) \notin \mathrm{Ass}\,_{R/(f)}(H_2)$. $\square$

Remark 3.4. For $n \leq 4$, if the partial derivatives are k-linearly independent then the result of Gordan–Hesse–Noether implies that they are algebraically independent over k (see [4, Proposition 2.7] for a proof of the case $n = 3$ based on an observation of Zak). Thus, the assumption in this range is just linear independence.

As to (c), it's valid with no restriction when f is reduced since the defining equations of the dual variety to the hypersurface $V(f)$ belong to the presentation ideal $\mathcal{A}_f$ and, moreover, the ideal generated by these contains properly the defining ideal of the polar map of $V(f)$ (see [4, Remark 2.4]).

Example 3.5. Here is a simple illustration. Let $f = x^2y^2 + x^2z^2 + y^2z^2$, the equation of a plane quartic with 3 ordinary nodes. The minimal primes of the corresponding gradient Aluffi algebra, lifted to $k[x, y, z, T, U, V]$, are $(x, y, z)k[x, y, z, T, U, V]$, (x, y, V), (x, z, U), (y, z, T) and its lifted torsion. Since I_f is of linear type (see next section), these are of course the minimal primes of the symmetric algebra $\mathcal{S}_{R/(f)}(I_f/(f))$.

3.2. **Gradient ideals of linear type along a family.** In general, the gradient ideal I_f will not be of linear type. This subsection will prepare the ground to showing that if f is the equation of an irreducible plane rational *quartic* then I_f is an ideal of linear type.

We can immediately show simple cases of rational plane quintics whose corresponding gradient ideals are not of linear type. Moreover, though the corresponding gradient Aluffi algebras are equidimensional, they tend to behave quite erratically from the viewpoint of the Cohen–Macaulay locus and of the associated primes. It is apparent that this behavior reflects the nature of the singularities, but it is in general quite misterious.

Example 3.6. Let $f = y^4z+x^5+x^3y^2$. Then $I_f = (x^2(5x^2+3y^2), y(2x^3+4y^2z), y^4)$. Canceling the common factor among the last two generators, gives rise to the obvious Koszul relation. From this it immediately follows that the radical of the ideal generated by the coordinates of the syzygies of I_f has x, y among its minimal generators. The rest follows by inspection, as it is not difficult to verify that no syzygy coordinate has as term a pure z-power. By Corollary 3.2, I_f is not of linear type.

Of course everything in this example is easily obtained by machine computation. The three algebras $\mathcal{S}_{R/(f)}(I_f/(f)) \twoheadrightarrow \mathcal{A}_{R\twoheadrightarrow R/(f)}(I_f/(f)) \twoheadrightarrow \mathcal{R}_{R/(f)}(I_f/(f))$ are all distinct, but of the same dimension. The leftmost is Cohen–Macaulay, while the Aluffi algebra has no embedded primes though it is not Cohen–Macaulay.

Now let $f = zy^2(x^2+y^2)+x^5+y^5+x^3y^2$. Here the symmetric algebra is Cohen–Macaulay, while the Aluffi algebra has embedded primes.

In this part we study families of singular plane curves and a certain "partial" gradient ideal for the linear type property and the corresponding Aluffi algebra. We start by making clear what we mean by a family for our purposes. Note that the considerations that follow work *ipsis litteris* for hypersurfaces.

Let $k[\mathbf{u}] = k[u_1, \ldots, u_m]$ stand for a polynomial ring over the field k and let $F \in S := k[\mathbf{u}][x, y, z]$ denote a polynomial which is a form on x, y, z. We give S the structure of standard graded ring over $k[\mathbf{u}]$. The basic assumption is that the content of F with respect to the $\mathbf{u}$-coefficients is 1. Then F is a non-zero-divisor on $k[\mathbf{u}]/I$ for every ideal $I \subset k[\mathbf{u}]$, hence $\mathrm{Tor}^1_{k[\mathbf{u}]}(k[\mathbf{u}]/I, k[\mathbf{u}][x, y, z]/(F)) = \{0\}$ for any such ideal. This gives that the inclusion $k[\mathbf{u}] \subset k[\mathbf{u}][x, y, z]/(F)$ is flat, hence defines a family of curves in $\mathbb{P}^2$ over the parameters $\mathbf{u}$.

Thus, we speak of a *family of plane curves* over the *parameters* $\mathbf{u}$ when referring to this setup. We will of course adhere to the terminology of calling *general member* of the family the equation of the plane curve obtained by substituting general values in k for $\mathbf{u}$. Moreover, our interest lies on the case where the general member of the family is a reduced singular plane curve. In this case we speak of a *family of plane singular curves*.

In the sequel we will assume moreover that $m \leq \binom{d+2}{2} - 1$, where d is the (homogeneous) degree of F in x, y, z and that F has the form

$$F = \varphi_0(x,y,z) + \sum_{j=1}^{m} u_j \varphi_j(x,y,z), \tag{4}$$

where $\{\varphi_j(x,y,z) \,|\, 0 \leq j \leq m\}$ is a set of monomials of degree d in x, y, z, and $\varphi_0(x,y,z) \neq 0$.

Note that the form of F depends on the singular points of the general member. Thus, it makes sense to speak about a *normal form* or *canonical form* of F depending on this singular locus. Our convention is that such a normal form is to be obtained through projective transformations applied to the x, y, z-coordinates allowing coefficients from $k[\mathbf{u}]$. Besides, in order to account for *degeneration of singularities* of the general member we need correspondingly to consider certain *degeneration ideals* in the parameter ring $k[\mathbf{u}]$.

Write

$$F \equiv \varphi_0(x,y,z) + \psi(x,y,z,u_1,\ldots,u_m),$$

as in (4), where $\varphi_0(x,y,z)$ involves the singularity type in terms of the projectivized tangent cones on suitable affine pieces.

Example 3.7. Let us write a normal form for the family of irreducible singular quartic plane curves such that the singular locus of the general member consists of one simple node - note that at this point it is not totally clear that there exists at all such a family in the sense we established, since we must first obtain some $F \in S$ that works. By projectivities one can assume that the node is $P = (0:0:1)$ and the tangent cone at $z \neq 0$ has equation xy. Since the general member ought to vanish at P then we may omit the terms in z^4, z^3x and z^3y. Thus, an intermediate step towards a normal form is

$$F = xyz^2 + u_1x^3z + u_2x^2yz + u_3xy^2z + u_4y^3z + u_5x^4 + u_6x^3y + u_7x^2y^2 + u_8xy^3 + u_9y^4.$$

We can see that the specialization of F by k-values factors properly if both u_1 and u_5 have vanishing k-values; similarly, if both u_4 and u_9 have vanishing k-values. Thus, for writing a normal form we may incorporate x^4 and y^4 as terms of $\varphi_0(x,y,z)$. Finally, the projectivity $x = x, y = y, z = z - \frac{1}{2}(u_2x + u_3y)$ (characteristic $\neq 2$) allows to eliminate the terms in x^2yz and xy^2z. Up to renaming parameters, this yields the following normal form:

$$F = xyz^2 + x^4 + y^4 + u_1x^3z + u_2y^3z + u_3x^3y + u_4x^2y^2 + u_5xy^3$$

3.3. **Degeneration for the linear type condition.** A piece of difficulty regarding the question as to how the property of being of linear type moves on a family is that this property is neither kept by specialization nor by generization. This difficulty permeates the theory by often conflicting with the usual degeneration conditions considered in the realm of families of hypersurfaces.

The normal form has degenerations to other normal forms whose general member has more involved singularities or even acquires new singular points. The following example may illuminate this phenomenon.

Example 3.8. Consider the family of irreducible rational plane quartics with exactly three nodes. In [8, Lemma 11.3] a normal form is given of a family whose general member is an irreducible quartic with three double points, namely

$$F = \lambda x^2y^2 + \mu x^2z^2 + \nu y^2z^2 + 2xyz(u_1x + u_2y + u_3z), \ \lambda\nu\mu \neq 0.$$

To get a normal form whose general member is an irreducible quartic with three nodes, substituting $x = (\nu/\lambda\mu)^{1/4}x,\ y = (\mu/\lambda\nu)^{1/4}y,\ z = (\lambda/\mu\nu)^{1/4}z$ and renaming, one obtains the normal form

$$F = x^2y^2 + x^2z^2 + y^2z^2 + 2xyz(u_1x + u_2y + u_3z).$$

Note that for k-values $u_1 = \pm 1$, one of the nodes degenerates into a cusp and, similarly, for $u_2 = \pm 1$ or $u_3 = \pm 1$. Thus, the general member requires that the k-values of the triple (u_1, u_2, u_3) do not lie on the hypersurface $V((u_1^2 - 1)(u_2^2 - 1)(u_3^2 - 1))$ in order that it have exactly three nodes.

Requiring that the general member acquire no new singular points besides the three nodes imposes yet another obstruction. Of course, in the present low degree 4, because of genus reason there will be new singular points only if the general member properly factors. As we will see this obstruction is precisely given by the hypersurface whose equation is the discriminant $2u_1u_2u_3 + u_1^2 + u_2^2 + u_3^2 - 1$ of a suitable conic (see Section 3.4).

The following is a basic result for this part. It would mostly suffice for it to assume that the $\mathbf{u}$-coefficients of the terms of F be algebraically independent over k. We observe that a similar result holds for families of hypersurfaces whose general member is reduced and irreducible and, moreover, the singular locus is a nonempty set of points.

Theorem 3.9. *Let F denote a family of singular plane curves of degree $d \geq 2$, on parameters $\mathbf{a} = u_1, ..., u_m$, whose general member is reduced and irreducible. Write $S = k[\mathbf{u}][x, y, z]$. Let $I_F \subset S$ denote the ideal generated by the x, y, z-partial derivatives of F and let $\mathcal{I} \in S$ stand for the ideal of 1–minors of the syzygy matrix of I_F. Then:*

(a) *I_F has codimension 2*
(b) *$\mathcal{I}$ has codimension at most 3*
(c) *If k is algebraically closed of characteristic zero, the following are equivalent:*
  (i) *$\mathcal{I}$ has codimension 3.*
  (ii) *The contraction of the ideal $\mathcal{I} : (x, y, z)S^{\infty}$ to $k[\mathbf{u}]$ has codimension ≥ 1.*
  (iii) *The plane projective curve $F(\alpha) \in k[x, y, z]$ obtained by evaluating $\mathbf{u} \mapsto \alpha$ off a set of codimension ≥ 1 in $\mathbb{A}^m_k$ has gradient ideal of linear type.*
  (iv) *There is some $\alpha \in \mathbb{A}^m_k$ for which the evaluated ideal $\mathcal{I}(\alpha) \in k[x, y, z]$ has codimension 3.*

Proof. (a) Clearly, $\operatorname{codim}(I_F) \leq 3$. Since the general member is singular and reduced its gradient ideal has codimension 2. This forces $\operatorname{codim}(I_F) = 2$.

(b) We go more algebraic: the ring $S = k[\mathbf{u}][x, y, z]$ is standard graded with $S_0 = k[\mathbf{u}]$. Since F is a homogeneous polynomial in this grading, its partial derivatives are homogeneous of same degree. We claim that any syzygy of the partial derivatives has coefficients in $(x, y, z)S$. Indeed, since the partial derivatives are homogeneous of the same degree $d \geq 1$ in x, y, z, any syzygy is homogeneous of non-negative degree in x, y, z. Now, if a syzygy would happen to be of degree 0, i.e., with all its coordinates in the zero degree part $k[\mathbf{u}]$, this would force, by reading the relation in degree 0, a polynomial relation among the coefficients of degree 0 of the partials, hence a polynomial relation of $\mathbf{u}$, which is nonsense since these are indeterminates over k.

Incidentally, note this argument breaks down for syzygies of a higher order as the first syzygies may have different degrees in S.

(c) (i) $\Rightarrow$ (ii) Write $\mathfrak{m} = (x, y, z)S$. Since $\operatorname{codim}(\mathcal{I}) = 3$ and $\mathfrak{m}$ is a minimal prime therein by the proof of (b), the saturation $\mathcal{I} : \mathfrak{m}^{\infty}$ picks up the primary components of $\mathcal{I}$ not containing $\mathfrak{m}$. This shows that $(\mathcal{I} : \mathfrak{m}^{\infty}) \cap k[\mathbf{u}] \neq \{0\}$.

(ii) $\Rightarrow$ (iii) Let $g = g(\mathbf{u}) \in (\mathcal{I} : \mathfrak{m}^\infty) \cap k[\mathbf{u}]$ be any nonzero element. By hypothesis, g conducts a power of $(x, y, z)S$ inside $\mathcal{I}$. Giving $\mathbf{u}$ k-values α off $V((\mathcal{I} : \mathfrak{m}^\infty) \cap k[\mathbf{u}])$ yields a power of the maximal ideal $(x, y, z) \subset k[x, y, z]$ inside the image $\mathcal{I}(\alpha)$ of $\mathcal{I}$ by this evaluation. Let $f = F(\alpha) \in k[x, y, z]$ denote the member of the family thus obtained. Then $\mathcal{I}(\alpha) \subset I_1(\varphi)$, where φ denotes the syzygy matrix of the partial derivatives of f. This shows that $I_1(\varphi)$ is (x, y, z)-primary. Therefore, the result follows from Corollary 3.2.

(iii) $\Rightarrow$ (iv) The hypothesis is that the gradient ideal of the general member of the family is of linear type. Again by Corollary 3.2 this implies that the ideal of 1-minors of such a plane curve has codimension 3. On the other hand, for general value α of $\mathbf{u}$, one has $\mathcal{I}(\alpha) = I_1(\varphi_\alpha)$, where φ_α stands for the syzygy matrix of $F(\alpha)$.

(iv) $\Rightarrow$ (i) By definition, $\mathcal{I}(\alpha) = (\mathcal{I}, \mathbf{u} - \alpha)/(\mathbf{u} - \alpha)$ upon identifying $k[\mathbf{u}, x, y, z]/(\mathbf{u} - \alpha) = k[x, y, z]$ under the surjection $k[\mathbf{u}][x, y, z] \twoheadrightarrow k[x, y, z]$ such that $\mathbf{u} \mapsto \alpha$. Since $\mathbf{u} - \alpha$ is a regular sequence in a polynomial ring we easily get

$$\begin{aligned} &\mathrm{ht}\,((\mathcal{I}, \mathbf{u} - \alpha)/(\mathbf{u} - \alpha)) = \mathrm{ht}\,(\mathcal{I}, \mathbf{u} - \alpha) - \mathrm{ht}\,(\mathbf{u} - \alpha) \leq \\ &\mathrm{ht}\,(\mathcal{I}) + \mathrm{ht}\,(\mathbf{u} - \alpha) - \mathrm{ht}\,(\mathbf{u} - \alpha) = \mathrm{ht}\,(\mathcal{I}), \end{aligned}$$

which implies the result. $\square$

The following example illustrates the various obstructions.

Example 3.10. The one-parameter family $F = y^4z + x^5 + ux^3y^2$ (see Example 3.6) is such that $\mathcal{I}$ has codimension 2, hence the gradient ideal of the general member $F(\alpha)$ of the family is not of linear type. Clearly, it follows that $\mathcal{I}(\alpha)$ has height ≤ 2 for any $\alpha \in k$. A computation with *Macaulay* gives moreover that the associated primes of $S/\mathcal{I}$ are $(x, y) \subset (x, y, z)$. Perhaps surprisingly, the special member $F(\mathbf{0})$ is easily seen to have gradient ideal of linear type, i.e., the ideal of 1-minors of the syzygy matrix of the special member $F(\mathbf{0})$, obtained by evaluating F at $\mathbf{0}$, has codimension 3. This simple example shows that the property in question does not deform to the generic member.

Under the equivalent assumptions of item (c) of Theorem3.9, one can give the approximate structure of the contracted ideal in item (ii).

Proposition 3.11. *Let the assumptions be those of Theorem 3.9 and assume that $\mathcal{I}$ has codimension* 3. *If $\mathcal{I}$ has a minimal prime of codimension* 3 *other than $\mathfrak{m} = (x, y, z)S$ then $(\mathcal{I} : \mathfrak{m}^\infty) \cap k[\mathbf{u}]$ has codimension* 1. *If, moreover, $\mathcal{I}$ is pure-dimensional then this contraction is a principal ideal.*

Proof. Let $\mathfrak{p}$ be a minimal prime of codimension 3 of $\mathcal{I}$ other than $\mathfrak{m}$. Since $I_F \subset \mathcal{I}$ (because of the Koszul relations) then $\mathfrak{p}$ contains a minimal prime of I_F. But the latter are of two sorts: either the extensions of the minimal primes (in $k[x, y, z]$) of the singular points of the general curve of the family, or else minimal primes of codimension 3. In the first case, $\mathfrak{p}$ contains two independent 1-forms in $k[x, y, z]$ which, up to a projective change of coordinates, can be assumed to be x, y. Clearly, these forms are part of a minimal set of generators of $\mathfrak{p}$ and $(x, y, \mathcal{I}) \subset \mathfrak{p}$. But $\mathcal{I}$ is homogeneous in the variables $k[x, y, z]$, generated in positive such degrees (see the argument in the proof of (b)), hence there exist suitable polynomials $g_j(\mathbf{u}) \in k[\mathbf{u}]$ and integers $k_j \geq 1$, $1 \leq j \leq s$, such that

$$(x, y, \mathcal{I}) = (x, y,\, z^{k_j} g_j(\mathbf{u}), 1 \leq j \leq s).$$

Since $z \notin \mathfrak{p}$, necessarily some $g_j(\mathbf{u}) \in \mathfrak{p}$. But then $\mathfrak{p} = (x, y, p(\mathbf{u}))$, for some prime factor of $g_j(\mathbf{u})$.

We now deal with the case where $\mathfrak{p}$ contains a minimal prime of I_F of codimension 3, hence coincides with it. Let $\mathfrak{p}_i, 1 \leq i \leq r$ denote the minimal primes of codimension 3 of $\mathcal{I}$. Each of these, by the previous argument, has a minimal generator $p_i(\mathbf{u}) \in k[\mathbf{u}]$. Then

$$\left(\prod_i \mathfrak{p}_i\right) \cap k[\mathbf{u}] = (\prod p_i(\mathbf{u})),$$

a principal ideal. On the other hand,

$$\sqrt{\mathcal{I} : \mathfrak{m}^\infty} \subset \bigcap \sqrt{\mathcal{P}_i : \mathfrak{m}^\infty} \subset \bigcap (\sqrt{\mathcal{P}_i} : \mathfrak{m}^\infty) = \bigcap \sqrt{\mathcal{P}_i} = \bigcap \mathfrak{p}_i = \sqrt{\prod_i \mathfrak{p}_i},$$

where $\mathcal{P}_i$ denotes the $\mathfrak{p}_i$th primary component of $\mathcal{I}$. This proves that the contraction $(\mathcal{I} : \mathfrak{m}^\infty) \cap k[\mathbf{u}]$ has codimension ≤ 1, hence is exactly 1 by Theorem 3.9.

The additional assertion at the end of the statement is now clear. □

3.4. **Rational quartics.** We review some preliminaries about rational quartics, the basic reference being [17].

An irreducible rational quartic having only double points can be obtained as a rational transform from a non-degenerate conic by means of one of the three basic plane quadratic Cremona maps:

(1) $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ with defining coordinates $(yz : zx : xy)$
 The base locus of this Cremona map consists of the points $(1 : 0 : 0)$, $(0 : 1 : 0)$ and $(0 : 0 : 1)$, each with multiplicity one (in the classical terminology, three proper points – see [1]).
(2) $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ with defining coordinates $(xz : yz : y^2)$
 The base locus of this Cremona map consists of the points $(0 : 0 : 1)$ and $(1 : 0 : 0)$, with multiplicity 1 and 2, respectively (in the classical terminology, one proper point and another proper point with a point in its first neighborhood).
(3) $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ with defining coordinates $(y^2 - xz : yz : z^2)$
 The base locus of this Cremona map consists of the point $(1 : 0 : 0)$ with multiplicity 3, a so-called triple structure on a point (in the classical terminology, one proper point with a point in its first neighborhood and a point in its second neighborhood).

Theorem 3.12. (k algebraically closed of characteristic zero) *Let $F = F(\mathbf{u}, x, y, z) \in k[\mathbf{u}, x, y, z]$ be a family of rational plane curves of degree 4 with a fixed set of singular points in the sense previously defined. Then the general member $f = F(\alpha) \in k[x, y, z]$ in this family has gradient ideal of linear type.*

Proof. We will actually show a bit more, namely, that any irreducible rational quartic falls within a family whose general member has the required property for its gradient ideal. In this vein, we can and will assume that the members of any family are singular. This is because the gradient ideal of any smooth plane curve is generated by a regular sequence, hence trivially of linear type.

Now, any irreducible rational quartic $f \in k[x, y, z]$ has at least one double singular point and at most a triple point. Let us first consider the situation where f has a double point – hence has at most 3 such points and no triple point.

In this case, as explained above, f comes from a conic by means of a Cremona map.

Let $Q = u_1x^2 + u_2y^2 + u_3z^2 + 2u_4yz + 2u_5zx + 2u_6xy$ be the equation of the conic as above, assumed non-singular, i.e., the corresponding symmetric matrix has nonzero determinant $\Delta = u_1u_2u_3 + 2u_4u_5u_6 - u_1u_4^2 - u_2u_5^2 - u_3u_6^2$ (the *discriminant* of Q).

Applying the above Cremona maps, we obtain, respectively:

(1) A quartic with exactly three double points at $P_1 = (1:0:0)$, $P_2 = (0:1:0)$ and $P_3 = (0:0:1)$, where P_1 (respectively, P_2, P_3) is a node except when the principal minor $u_1u_2 - u_6^2$ vanishes (respectively, except when the principal minors $u_1u_3 - u_5^2$, $u_2u_3 - u_4^2$ vanish).

Here we may harmlessly assume that $u_1 = u_2 = u_3 = 1$ provided they are all nonzero. In this block belong the following families.

(a) Three nodes:

$$\tilde{f} = y^2z^2 + x^2z^2 + x^2y^2 + 2xyz(u_4x + u_5y + u_6z), \qquad u_4, u_5, u_6 \neq \pm 1, \quad \Delta \neq 0$$

where $\Delta = 2u_4u_5u_6 - u_4^2 - u_5^2 - u_6^2 + 1$.

(b) Two nodes and one cusp:

$$\tilde{f} = y^2z^2 + x^2z^2 + x^2y^2 + 2xyz^2 + 2xyz(u_4x + u_5y), \qquad u_4, u_5 \neq \pm 1, \quad \Delta \neq 0$$

where $\Delta = 2u_4u_5 - u_4^2 - u_5^2 = -(u_4 - u_5)^2$.

(c) One node and two cusps:

$$\tilde{f} = y^2z^2 + x^2z^2 + x^2y^2 + 2xy^2z + 2xyz^2 + 2u_4x^2yz, \quad u_4 \neq \pm 1$$

(d) Three cusps :

$$\tilde{f} = y^2z^2 + x^2z^2 + x^2y^2 - 2xyz(x + y + z)$$

(2) A quartic with a double point at $P_1 = (0:0:1)$ (which is a node or a cusp according as to whether $u_2 \neq 0$ or $u_2 = 0$) and a double point at $P_2 = (1:0:0)$ (which is either a tacnode or a ramphoid cusp according as to whether the principal minor $u_1u_3 - u_5^2$ is nonzero or vanishes).

Here we may assume that $u_1 = u_3 = 1$ and $u_6 = 0$.

It comprises the following families.

(e) One tacnode and one cusp:

$$\tilde{f} = x^2z^2 + y^4 + 2y^3z + 2u_5xy^2z, \qquad u_5 \neq \pm 1$$

(f) One tacnode and one node :

$$\tilde{f} = z^2(x^2 + y^2) + y^4 + 2y^2z(u_4y + 2u_5x), \qquad u_5 \neq \pm 1, \quad \Delta = u_4^2 + u_5^2 - 1 \neq 0$$

(g) One ramphoid cusp and one node:

$$\tilde{f} = x^2z^2 + y^4 + 2zy^3 + 2xy^2z + u_2z^2y^2, \qquad u_2 \neq 0$$

(h) One ramphoid cusp and one cusp:

$$\tilde{f} = x^2z^2 + y^4 + 2zy^3 + 2xy^2z$$

(3) A quartic with an oscnode at $P_1 = (1:0:0)$ if $u_2 \neq 0$; else, a singularity of type A_6.

Here we may assume that $u_1 = 1$ and $u_5 = u_6 = 0$. Namely, we get the following forms.

(i) One oscnode :

$$\tilde{f} = (y^2 - xz)^2 + y^2z^2 + u_3z^4, \qquad u_3 \neq 0$$

(j) One singularity of type A_6:

$$\tilde{f} = (y^2 - xz)^2 + 2yz^3$$

An irreducible rational quartic having only double points – ordinary or not – falls within the following families up to coordinate change, according to the nature of its singularities. We have written $\tilde{f}$ instead of F to help us think of the general member instead of the family itself. To keep track of the parameters in each case we have maintained the original indices, however anaesthetical they may look.

Note the two kinds of degeneration: first, whether a member of the family factors is controlled by the vanishing of the corresponding value of Δ; second, whether the non-general member goes across stratified subfamilies is controlled by the vanishing of another ideal in $k[\mathbf{u}]$ - we will call the latter ideal the *strata degeneration locus*.

We now consider the case where the quartic has a triple point, say, at $P = (0 : 0 : 1)$.

Generally, for a plane irreducible curve of degree d with a singular point of multiplicity $d-1$ (hence, a rational curve), it is frequently easier to look at the linear type condition. In the case of a quartic, up to a projective change of coordinates, the equation of the curve has the form $\varphi(x,y)\,z + \psi(x,y) = 0$, where φ can moreover be brought up to one of the forms $x(y^2 - x^2)$, xy^2 and y^3, and ψ may be further normalized in such a way that the resulting family has as few parameters as possible.

(4) After these reductions, any irreducible plane quartic having $(0 : 0 : 1)$ as a triple point falls within three basic families, according to the nature of the triple point:

(k) An ordinary triple point:

$$\tilde{f} = x(y^2 - x^2)z + y^4 + x^2y(u_1y + u_2x)$$

(l) A triple point with double tangent:

$$\tilde{f} = xy^2z + x^4 + y^4 + u_1x^3y$$

(m) A higher cusp :

$$\tilde{f} = y^3z + x^4 + u_1x^2y^2.$$

The proof proceeds by dealing with each of the above types.

Block (1)

(a) By Theorem 3.9, it suffices to show that $\mathcal{I}(\mathbf{0})$ has codimension 3. From the symmetrical parametric structure of $\tilde{f}$, the inclusion $\mathcal{I}(\mathbf{0}) \subset I_{\tilde{f}(\mathbf{0})}$ is an equality. On the other hand, by direct verification, the latter ideal admits the following syzygies:

$$\begin{pmatrix} xy^2 - xz^2 \\ -y^3 - yz^2 \\ y^2z + z^3 \end{pmatrix}, \quad \begin{pmatrix} -x^3 - xz^2 \\ x^2y - yz^2 \\ x^2z + z^3 \end{pmatrix}$$

Bringing in besides the generators of the gradient ideal, one obtains after a calculation $x^3, y^3, z^3 \in \mathcal{I}(\mathbf{0})$. Thus, $\mathcal{I}(\mathbf{0})$ has codimension 3.

We first note that (b)–(d) are obvious successive strata of the family (a).

(b) In this first stratum the evaluated ideal $\mathcal{I}(\mathbf{0})$ has codimension 2, so one may try another evaluation. Note that one cannot blindly apply the result of (a) to claim that, here too, there is some general value α for which the assertion holds, since α could lie outside the open set obtained in (a). (Of course, it goes without saying that an explicit such open set can be computed by way of Theorem 3.9, (ii) and we even have a conjecture about its form.) Instead, we resort to a painful hand verification, namely, the following is a syzygy of $\mathcal{I}$:

$$\begin{pmatrix} x^2(u_4^2-1)+xy(u_4u_5-1)+2xz(u_4-u_5)+yz(u_4-u_5) \\ y^2(u_5^2-1)+xy(u_4u_5-1)+xz(u_5-u_4)+2yz(u_5-u_4) \\ 3z^2(u_4-u_5)+xy(u_4-u_5)+xz(2u_4^2-u_5u_4-1)+yz(-2u_5^2+u_5u_4+1) \end{pmatrix}$$

Looking at the first summand of each coordinate, one sees that for any "value" $\alpha=(u_4,u_5)\in\mathbb{A}_k^2$ with $u_4\neq u_5$ and $u_4\neq\pm1$, $u_5\neq\pm1$, pure powers of x,y,z remain and the ideal generated by the coordinates will have codimension 3, hence also the corresponding $\mathcal{I}(\alpha)$.

(c) In this stratum by the same token, we look at the following two syzygies:

$$\begin{pmatrix} -xy+xz \\ 3y^2+2xyu_4+xz+3yz \\ -3z2-2xzu_4-2xy-3yz \end{pmatrix}, \quad \begin{pmatrix} x^2(u_4+1)+3/2xy+3/2xz+yz \\ 3/2y^2-xy(u_4+1)+1/2yz \\ 3/2z^2+2/2yz+xz(u_4+1) \end{pmatrix}$$

An identical analysis as above, looking at the pure powers, allow to choose any "value" $u_4\neq-1$.

(d) Here too it suffices to look at the following syzygies

$$\begin{pmatrix} x^2+xy+2/3xz-2/3yz \\ -xy-y^2+2/3xz-2/3yz \\ -1/3xz+1/3yz \end{pmatrix}, \quad \begin{pmatrix} xy+xz-2/3yz \\ -y^2+1/3yz \\ 1/3yz-z^2 \end{pmatrix}$$

Once more, pure powers of the variables are easily located.

Block (2)

(e) As in the proof of (a), here too the ideal $\mathcal{I}(\mathbf{0})$ contains the ideal of 1-minors of the syzygy matrix of $I_{\tilde{f}(\mathbf{0})}$. One can check that the latter ideal admits the following syzygies:

$$\begin{pmatrix} 2x^2-3y^2 \\ xz \\ -2xz \end{pmatrix}, \quad \begin{pmatrix} 2xy+3xz \\ yz \\ -2yz-3z2 \end{pmatrix}.$$

Bringing in the generator $\partial\tilde{f}(\mathbf{0})/\partial y=4y^3+6y^2z$, one readily sees that $\mathcal{I}(\mathbf{0})$ has codimension 3.

(f) This follows the same pattern as (e). The relevant syzygies to look at are

$$\begin{pmatrix} x^2+y^2 \\ 0 \\ -xz \end{pmatrix}, \quad \begin{pmatrix} 2xy^2+xz^2 \\ yz^2 \\ -2y^2z-z3 \end{pmatrix}$$

and the calculation to get suitable powers of x,y,z inside $\mathcal{I}(\mathbf{0})$ is pretty straightforward.

(g) If we indulge ourselves allowing a computation with *Macaulay*, we get $u_2^2\in(\mathcal{I}\colon\mathfrak{m}^\infty)$. By Theorem 3.9, *every* member in this family has gradient ideal of linear type. Alternatively, one can look for $\mathcal{I}$ evaluated, say, at $u_2\mapsto1$. One column turns out to be

$$\begin{pmatrix} 5x^2-y^2+xz-yz \\ y^2+xy+xz+yz \\ -z^2-2y^2-xz-2yz \end{pmatrix}$$

Thus, the ideal generated by the three coordinates above already has codimension 3.

(h) Again, a computation with *Macaulay* gives the following syzygies:

$$\begin{pmatrix} 2y^2-3xz \\ -yz \\ 3z^2 \end{pmatrix}, \quad \begin{pmatrix} x^2-27/50xz \\ 1/5xy+1/5y^2+3/25xz-9/50yz \\ -2/5y^2-xz-6/25yz+27/50z^2 \end{pmatrix}.$$

Thus, we locate pure powers as terms of the coordinates. Alternatively, note the coordinates $3z^2$ and $x^2-27/50xz = x(x-27/50z)$ are invertible locally at (x,y) and (y,z), respectively. Since the latter are the two singular minimal primes of the quartic, this shows that the gradient ideal is locally a complete intersection at these primes, hence is of linear type.

Block (3)

(i) The discussion of this case is analogous to the one of (g). A computation with *Macaulay* yields get $u_3 \in (\mathcal{I}\colon \mathfrak{m}^\infty)$. Therefore, *every* member in this family has gradient ideal of linear type. As previously enacted, the ideal $\mathcal{I}(1)$ has codimension 3 through a convenient analysis of its terms.

(j) The following vectors are directly seen to be syzygies of the gradient ideal:

$$\begin{pmatrix} 6y^2+xz \\ 3yz \\ -z^2 \end{pmatrix}, \quad \begin{pmatrix} 7x^2+18yz \\ 3xy \\ 6y^2-7xz \end{pmatrix}.$$

A straightforward calculation gives the right codimension.

Block (4) (Triple point)

(k) The discussion of this case is much like the one of (a) in that the parametric structure of $\tilde{f}$ allows to see that the syzygies of $I_{\tilde{f}(\mathbf{0})}$ are contained in the syzygies of $I_{\tilde{f}}$ evaluated at $\mathbf{0}$. It then suffices to check that $I_{\tilde{f}(\mathbf{0})}$ is of linear type. We do this by arguing that it is locally a complete intersection at its unique singular prime (x,y). For this, it suffices to consider the syzygy

$$\begin{pmatrix} x^2-2/3y^2+1/6xz \\ 1/6yz \\ -(3x+1/2z)z \end{pmatrix}$$

where the last coordinate is invertible locally at (x,y).

(l) As in (k), the syzygies of $I_{\tilde{f}(\mathbf{0})}$ are contained in the syzygies of $I_{\tilde{f}}$ evaluated at $\mathbf{0}$. Here it is elementary to guess the syzygy

$$\begin{pmatrix} 0 \\ xy \\ -4y^2-2xz \end{pmatrix}$$

Using further the generators of the gradient ideal of $\tilde{f}(\mathbf{0})$, it is readily seen powers of the variables among the entries.

(m) This case is like the previous one, only more elementary. We argue that $\mathcal{I}(\mathbf{0})$ has codimension 3 as before by looking at the obvious syzygy of $I_{\tilde{f}(\mathbf{0})}$

$$\begin{pmatrix} 0 \\ y \\ -3z \end{pmatrix}$$

which clearly tells us that the ideal is locally a complete intersection at (x,y). $\square$

Remark 3.13. We have drawn quite a bit on computation to verify all cases of the theorem. Using Theorem 3.9, to have a computation-free argument it would suffice to show that the contraction $(\mathcal{I}\colon \mathfrak{m}^\infty) \cap k[\mathbf{u}]$ coincides set-theoretically with the product $\Delta\mathfrak{a}$ of the discriminant and the strata degeneration locus $\mathfrak{a} \subset k[\mathbf{u}]$. We conducted a computational verification of this fact for all four blocks of families. Thus, morally, the conjecture for rational quartics is settled. However, we have found no immediate theoretical reason pointing at least to an ideal inclusion $(\mathcal{I}\colon \mathfrak{m}^\infty) \cap k[\mathbf{u}] \subset \Delta$. Note that, according to Proposition 3.11, one expects that the contraction also have codimension one, whereas the strata degeneration locus is given by a principal ideal.

It seems natural to conjecture that any *irreducible* quartic has gradient ideal of linear type.

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Combinatorial computation of the motivic Poincaré series

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Abstract

We give an explicit algorithm computing the motivic generalization of the Poincaré series of a plane curve singularity introduced by A. Campillo, F. Delgado and S. Gusein-Zade. It is done in terms of the embedded resolution. The result is a rational function depending of the parameter q, at $q = 1$ it coincides with the Alexander polynomial of the corresponding link. For irreducible curves we relate this invariant to the Heegaard-Floer knot homology constructed by P. Ozsváth and Z. Szabó. Many explicit examples are considered.

Contents

1 Introduction

In the series of articles (e.g. [3],[4]) A. Campillo, F. Delgado and S. Gusein-Zade proved that the Alexander polynomial of the link of the plane curve singularity is related to the generating function arising in the purely algebraic setup.

Let $C = \cup_{i=1}^{r} C_i$ be a germ of a plane curve,

$$\gamma_i : (\mathbb{C}, 0) \to (C_i, 0)$$

are the uniformizations of its components. If $f \in \mathcal{O} = \mathcal{O}_{\mathbb{C}^2,0}$ is a germ of a function on $(\mathbb{C}^2, 0)$, we define

$$v_i(f) = Ord_0 f(\gamma_i(t)),$$

and the Poincaré series of the curve C is defined ([4]) as the integral with respect to the Euler characteristic

$$P^C(t_1, \ldots, t_r) = \int_{\mathbb{P}\mathcal{O}} t_1^{v_1} \cdot \ldots \cdot t_r{}^{v_r} d\chi, \tag{1}$$

where $\mathbb{P}\mathcal{O}$ denotes the projectivization of $\mathcal{O}$ as a vector space. For example, if C is irreducible, we can define the decreasing filtration

$$\mathcal{O} \supset J_1 \supset J_2 \supset \ldots, \qquad J_n = \{f \in \mathcal{O} | v_1(f) \geq n\}, \tag{2}$$

and

$$P^C(t) = \sum_{n=0}^{\infty} t^n \dim J_n/J_{n+1}. \tag{3}$$

Let $\Delta^C(t_1, \ldots, t_n)$ denote the Alexander polynomial of the intersection of C with a small sphere centered at the origin. The theorem of Campillo, Delgado and Gusein-Zade says that if $r = 1$, then

$$(1 - t)P^C(t) = \Delta^C(t), \tag{4}$$

and if $r > 1$, then

$$P^C(t_1, \ldots, t_r) = \Delta^C(t_1, \ldots, t_r).$$

In [5] there was proposed the following natural generalization of the Poincaré series. One can naturally define the motivic measure on the space of functions, and consider the following motivic integral, generalizing (1):

$$P_g^C(t_1, \ldots, t_r) = \int_{\mathbb{P}\mathcal{O}} t_1^{v_1} \cdot \ldots \cdot t_r{}^{v_r} d\mu. \tag{5}$$

If $r = 1$, we can rewrite (5) as the generalization of (3):

$$P_g^C(t) = \sum_{n=0}^{\infty} t^n \frac{q^{\mathrm{codim} J_n} - q^{\mathrm{codim} J_{n+1}}}{1-q}, \tag{6}$$

therefore in this case one can deduce $P_g(t)$ from $P(t)$. If r is greater than 1, the situation becomes more complicated. Nevertheless, the explicit algorithm for the computation of the motivic Poincaré series is presented in Theorem 3.

Definition: The reduced motivic Poincaré series is the power series

$$\overline{P}_g(t_1, \ldots, t_r) = (1 - qt_1) \cdot \ldots \cdot (1 - qt_r) \cdot P_g(t_1, \ldots, t_r). \tag{7}$$

We prove that the reduced motivic Poincaré series satisfies the following properties.

1. **Polynomiality.** $\overline{P}_g(t_1, \ldots, t_r; q)$ is a polynomial in variables $t_1, \ldots, t_r$ and q. We give a bound for its degree on $t_1, \ldots, t_r$.

2. **Reduction to the Alexander polynomial.** If $n = 1$, then

   $$\overline{P}_g(t; q = 1) = \Delta(t),$$

   where Δ denote the Alexander polynomial of the link of the corresponding plane curve singularity. If $n > 1$, then

   $$\overline{P}_g(t_1, \ldots, t_r; q = 1) = \Delta(t_1, \ldots, t_r) \cdot \prod_{i=1}^{r} (1 - t_i).$$

3. **Forgetting components.** Let C be a curve with r components, and C_1 be an irreducible curve. Then

   $$\overline{P}_g^{C \cup C_1}(t_1, \ldots, t_r, t_{r+1} = 1) = (1 - q)\overline{P}_g^C(t_1, \ldots, t_r). \tag{8}$$

   If C has only one component, then

   $$\overline{P}_g^C(t = 1) = 1.$$

   This property is clear from the equation (5), but seems to be curious and, for example, does not hold for the Alexander polynomial (we cannot reconstruct the Alexander polynomial of a sublink from the Alexander polynomial of a link by setting the corresponding variable to 1).

4. **Symmetry.** Let μ_α be the Milnor number ([2]) of C_α, let $(C_\alpha \circ C_\beta)$ be the intersection index of C_α and C_β, let $\mu(C)$ be the Milnor number of C. Let

   $$l_\alpha = \mu_\alpha + \sum_{\beta \neq \alpha} (C_\alpha \circ C_\beta), \quad \delta(C) = (\mu(C) + r - 1)/2.$$

Remark that $\sum_{\alpha=1}^{r} l_\alpha = 2\delta(C)$.

It is known that the Alexander polynomial is symmetric in a sense that

$$\Delta(t_1^{-1},\ldots,t_r^{-1}) = \prod_{\alpha=1}^{r} t_\alpha^{1-l_\alpha}\cdot\Delta(t_1,\ldots,t_r),\quad r>1$$

and

$$\Delta(t^{-1}) = t^{-\mu}\Delta(t),\quad r=1.$$

In Theorem 4 we prove a generalization of this identities that holds for any r, namely,

$$\overline{P}_g(\frac{1}{qt_1},\ldots,\frac{1}{qt_r}) = q^{-\delta(C)}\prod_{\alpha} t_\alpha^{-l_\alpha}\cdot\overline{P}_g(t_1,\ldots,t_r).$$

5. **Relation to the knot homology.** For irreducible curves we prove that $\overline{P}_g(t)$ can be related by the simple procedure to the Poincaré polynomial of the Heegaard-Floer knot homology constructed by P. Ozsváth and Z. Szabó. This homology theory is a "categorification" of the Alexander polynomial, tightly related with the symplectic topology and Seiberg-Witten theory. Since the origins of our and their construction are quite far, the relation between them seems to be interesting. No conceptual proof for this fact is known, and we just use that both answers are determined by the Alexander polynomial in the same way.

The paper is organized in the following way. In the section 2 we recall the definition of the Poincaré series of a plane curve singularity. Then we recall the definition of the motivic measure on the space of functions and give, following [5], two definitions of the motivic Poincaré series as a motivic integral and in terms of the multi-index filtration associated with the curve. We give the simple method of deduction of the motivic Poincaré series from the ordinary Poincaré series for irreducible curves. In Theorem 2 we recall the formula from [5] expressing the motivic Poincaré series in terms of the embedded resolution of a curve. This formula is proved by Campillo, Delgado and Gusein-Zade using thorough analysis of the geometry of the functional spaces defined by the embedded resolution of a curve.

In the section 3 we apply Theorem 2 to a nonsingular curve and explain step-by-step the calculation of all sums involved. It turns out to be a curious exercise, and this simplest example is a toy model for the consequent combinatorial work.

The section 4 contains several steps of the simplification of Theorem 2. In the result (Lemma 6) the motivic Poincaré series is expressed in terms of some quantities $c_K(n)$. In Lemma 5 the generating function for these quantities is explicitly written in the closed form. This allows to compute the motivic Poincaré series.

Applying Lemma 6 directly, we get a lot of similar summands which cancel after all substitutions, but this cancellation is not clear from lemmas 5 and 6. For example, it is not even clear, that the answer is a polynomial.

Therefore in the rest of section 4 we discuss the analogues of the identity

$$\sum_{n=0}^{\infty} t^n q^{\frac{n^2+3n}{2}}(q^{-n}-tq) = 1$$

arising in the nonsingular case. The result of this investigation is Theorem 3, where we formulate an explicit algorithm of calculation of the motivic Poincaré series. This algorithm does not involve infinite sums, and can be implemented as a short `Mathematica` program.

The algorithm is presented in the same manner as in Lemma 6: the motivic Poincaré series is expressed in terms of some quantities $d_P(n)$, which fit into the explicitly defined generating function $H_P(u)$. This function is generally more complicated than the one from Lemma 5, but in some examples (Lemma 9) it has more or less compact form.

Section 5 contains a bunch of explicit answers for the curves with resolutions containing up to 3 divisors.

In the section 6 we prove the symmetry property for the motivic Poincaré series (Theorem 4). It generalizes the known symmetry property for the Alexander polynomial of a link. From the viewpoint of the algebraic geometry, it is related to the Gorenstein property of the coordinate ring of a curve ([6]), thus it seems to be related to the Kapranov's functional equation ([11],[10]) for the motivic zeta function of a curve.

We prove the symmetry property by proving the analogous statements for all steps of our algorithm: the function $H_P(u)$ is symmetric, what implies some relations for its coefficients $d_P(n)$ and, therefore, for the Poincaré series.

The main result of the section 7 is Theorem 6 describing the surprising relation between the motivic Poincaré series of an irreducible plane curve singularity and another deformation of the Alexander polynomial, namely, the Poincaré polynomial for the Heegaard-Floer knot homology ([18],[19]). The proof is based on the fact that in both cases the Poincaré polynomial (and series) is defined by the Alexander polynomial. We also give some corollaries from this fact which look more geometric. A filtered complex of $\mathbb{Z}[U]$-modules analogous to the Ozsváth-Szabó complex $CFL^{-}(K)$ is constructed. This gives an algebraic model for the minus- and hat-versions of the Heegaard-Floer complexes for algebraic knots.

We also compare the motivic Poincaré series with the Heegaard-Floer homologies of two-component links, corresponding to the singularities of type A_{2n-1}.

The motivic Poincaré series has been independently studied by J. Moyano-Fernandez and W. Zuniga-Galindo in [14]. Their approach is based on the study of the multi-dimensional semigroup of the singularity instead of its resolution. In particular, they gave alternative proofs of the Theorems 3 and 4 of this article.

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2 Poincaré series and its generalization

2.1 Poincaré series

Let $C = \cup_{i=1}^r C_i$ be a reduced plane curve singularity at the origin in $\mathbb{C}^2$, and C_i are its irreducible components. Let $\gamma_i : (\mathbb{C}, 0) \to (C_i, 0)$ be the uniformizations of these components.

We define r integer-valued functions on the space $\mathcal{O} = \mathcal{O}_{\mathbb{C}^2,0}$ by the formula

$$v_i(f) = \mathrm{Ord}_0(f(\gamma_i(t)))$$

and $\mathbb{Z}^r$-indexed filtration

$$J_{\underline{v}} = \{f \in \mathcal{O} | v_i(f) \geq v_i\}.$$

Note that $J_{\underline{v}}$ are also defined for negative values of $\underline{v}$. This filtration is decreasing in a sense that if $\underline{v}_1 \prec \underline{v}_2$, then $J_{\underline{v}_1} \supset J_{\underline{v_2}}$. Consider the Laurent series

$$L_C(t_1, \ldots, t_r) = \sum_{\underline{v}} t_1^{v_1} \ldots t_r^{v_r} \cdot \dim J_{\underline{v}} / J_{\underline{v}+\underline{1}}.$$

Definition:([6], [3]) The Poincaré series of the curve C is defined by the formula

$$P_C(t_1, \ldots, t_r) = \frac{L_C(t_1, \ldots, t_r) \cdot \prod_{i=1}^r (t_i - 1)}{t_1 \cdot \ldots \cdot t_r - 1}.$$

For example, if $r = 1$, we have

$$P_C(t) = \sum_{v=0}^{\infty} t^v \cdot \dim J_v / J_{v+1}.$$

One can prove, that P_C is always a power series. More geometric meaning of this definition is given by the following interpretation of the Poincaré series as an integral with respect to the Euler characteristic.

Proposition.([4]) Let $\mathbb{P}\mathcal{O}$ denote the projectivization of the functional space $\mathcal{O}$ as a vector space. Then the following equation holds:

$$P_C(t_1, \ldots, t_r) = \int_{\mathbb{P}\mathcal{O}} t_1^{v_1} \cdot \ldots \cdot t_r^{v_r} d\chi. \tag{9}$$

On the other hand, consider the link of C – the intersection of C with a small three-dimensional sphere centred at the origin. We denote its multi-variable Alexander polynomial by $\Delta_C(t_1, \ldots, t_r)$. Campillo, Delgado and Gusein-Zade proved the following

Theorem 1 *([4]) If $r = 1$ then*

$$P_C(t)(1 - t) = \Delta_C(t), \tag{10}$$

and if $r > 1$ then

$$P_C(t_1, \ldots, t_r) = \Delta_C(t_1, \ldots, t_r). \tag{11}$$

2.2 Motivic measure

Let $\mathcal{O} = \mathcal{O}_{\mathbb{C}^2,0}$ be the space of formal germs of analytic functions at the origin on the plane. It is the set of formal power series $f(x, y)$ (without degree 0 term). Let $\mathcal{O}_n$ be the space of n-jets of such arcs, let $\pi_n : \mathcal{O} \to \mathcal{O}_n$ be the natural projection.

Let $K_0(Var_{\mathbb{C}})$ be the Grothendieck ring of complex quasiprojective varieties. It is generated by the isomorphism classes of complex quasiprojective varieties modulo the relations $[X] = [Y] + [X \setminus Y]$, where Y is a Zariski locally closed subset of X. Multiplication is given by the formula $[X] \cdot [Y] = [X \times Y]$. Let $\mathbb{L} = [\mathbb{C}] \in K_0(Var_{\mathbb{C}})$ be the class of the affine line in this ring.

The Euler characteristic provides a ring homomorphism

$$\chi : K_0(Var_{\mathbb{C}}) \to \mathbb{Z}.$$

Consider the ring $K_0(Var_{\mathbb{C}})[\mathbb{L}^{-1}]$ with the following filtration: F_k is generated by the elements of the type $[X] \cdot [\mathbb{L}^{-n}]$ with $n - \dim X \geq k$. Let $\mathcal{M}$ be the completion of the ring $K_0(Var_{\mathbb{C}})[\mathbb{L}^{-1}]$ corresponding to this filtration.

On an algebra of subsets of $\mathcal{O}$ Campillo, Delgado and Gusein-Zade ([5]), following the ideas of Kontsevich, Denef and Loeser ([7]) constructed a measure μ with values in the ring $\mathcal{M}$.

Definition:([5]) A subset $A \subset \mathcal{O}$ is said to be cylindric if there exist n and a constructible set $A_n \subset \mathcal{O}_n$ such that $A = \pi_n^{-1}(A_n)$. For the cylindric set A define its *motivic measure* by the formula

$$\mu(A) = [A_n] \cdot \mathbb{L}^{-\frac{(n+1)(n+2)}{2}}.$$

Remark that $\dim \mathcal{O}_n = \frac{(n+1)(n+2)}{2}$, hence the definition of the motivic measure is in fact independent on n. In a full analogy with [7], this measure can be extended to an countable-additive $\mathcal{M}$-valued measure on a suitable algebra of subsets of $\mathcal{O}$.

Definition: A function $f : \mathcal{O} \to G$ with values in an abelian group G is called simple, if its image is countable or finite, and for every $g \in G$ the set $f^{-1}(g)$ is measurable. Using this measure, one can define in the natural way the motivic integral for simple functions on $\mathcal{O}$ as

$$\int_{\mathcal{O}} f d\mu = \sum_{g \in G} g \cdot \mu(f^{-1}(g)),$$

if the right hand side sum converges in $G \otimes \mathcal{M}$.

Remark. Note that for cylindric sets the Euler characteristic can be defined by the formula $\chi(A) = \chi(A_n)$. This gives a $\mathbb{Z}-$valued measure on the algebra of cylindric sets. However, it cannot be extended to the algebra of measurable sets. This measure provides a notion of an integral with respect to the Euler characteristic for functions on $\mathcal{O}$ with cylindric level sets. It is clear that for such functions

$$\chi(\int_{\mathcal{O}} f d\mu) = \int_{\mathcal{O}} f d\chi.$$

Using the same construction, one can define the motivic measure on the projectivization $\mathbb{P}\mathcal{O}$ of the functional space.

As a direct generalisation of the equation (9) Campillo, Delgado and Gusein-Zade proposed the following

Definition: Motivic Poincaré series is the motivic integral

$$P_g^C(t_1, \ldots, t_r) = \int_{\mathbb{P}\mathcal{O}} t_1^{v_1} \cdot \ldots \cdot t_r^{v_r} d\mu \tag{12}$$

As above, this definition can be reformulated in terms of the multi-index filtration on the space of functions. Let $q = \mathbb{L}^{-1}$ be a formal variable. Let $h(\underline{v}) = \mathrm{codim} J_{\underline{v}}$, and

$$L_g(t_1, \ldots, t_r, q) = \sum_{\underline{v} \in \mathbb{Z}^r} \frac{q^{h(\underline{v})} - q^{h(\underline{v}+\underline{1})}}{1-q} \cdot t_1^{v_1} \ldots t_r^{v_r}.$$

Then the following equation holds ([5]):

$$P_g^C(t_1, \ldots, t_r; q) = \frac{L_g^C(t_1, \ldots, t_r) \cdot \prod_{i=1}^r (t_i - 1)}{t_1 \cdot \ldots \cdot t_r - 1}. \tag{13}$$

An example of the calculation of the motivic Poincaré series for the singularities of type A_{2n-1} directly from the equation (13) is presented in the section 7.4 below.

2.3 Irreducible case

If $r = 1$, the equation (13) has a very clear form, since in this case

$$P_g^C(t) = L_g^C(t).$$

Remark that

$$\mathrm{codim} J_v = \dim \mathcal{O}/J_1 + \dim J_1/J_2 + \ldots + \dim J_{v-1}/J_v, \tag{14}$$

so the series $P_g^C(t)$ can be reconstructed from the series $P_C(t)$.

The functional $v(f) = \mathrm{Ord}_0 f(\gamma(t))$ is a valuation on the ring $\mathcal{O}$.The set of values of v is an integer semigroup $S = \{\sigma_1, \sigma_2, \sigma_3, \ldots\}$. For example, for the singularity $x^p = y^q$ (its link is the torus (p, q) knot) we have $x(t) = t^q, y(t) = t^p$, so the corresponding semigroup is generated by p and q. The coefficient at t^v in $P_C(t)$ vanishes, if $J_v = J_{v+1}$ (or, equivalently, v does not belong to the semigroup S) , and equals to 1 otherwise. Therefore we have

$$P_C(t) = 1 + t^{\sigma_1} + t^{\sigma_2} + t^{\sigma_3} + \ldots.$$

Now the equation (14) implies the following formula for the motivic Poincaré series:

$$P_g^C(t; q) = 1 + qt^{\sigma_1} + q^2 t^{\sigma_2} + q^3 t^{\sigma_3} + \ldots. \tag{15}$$

Example. Consider the cusp $x^2 = y^3$. Its semigroup is generated by 2 and 3, the Poincaré series is equal to

$$P(t) = 1 + t^2 + t^3 + t^4 + \ldots,$$

the motivic Poincaré series is equal to

$$P_g(t) = 1 + qt^2 + q^2t^3 + q^3t^4 + \dots.$$

Note that

$$P(t)(1-t) = 1 - t + t^2,$$

what equals to the Alexander polynomial of the trefoil knot.

2.4 Formula of Campillo, Delgado and Gusein-Zade

In [5] Campillo, Delgado and Gusein-Zade gave a formula for the generalized Poincaré series in terms of the resolution.

Let $\pi : (X, D) \to (\mathbb{C}^2, 0)$ be an embedded resolution where $D = \cup_{i=1}^{s} E_i$ is the exceptional divisor. Let $E_i^{\bullet}$ be E_i without intersection points of E_i with other components of D, E_i° be $E_i^{\bullet}$ without intersection points of E_i with the components of the strict transform of our curve. Let $A = (E_i \circ E_j)$ be the intersection matrix and $M = -A^{-1}$.

Let $I_0 = \{(i,j) : i < j, E_i \cap E_j = pt\}$, $K_0 = \{1, \dots, r\}$. For $\sigma \in I_0$, $\sigma = (i,j)$ let $i(\sigma) = i$, $j(\sigma) = j$. For $I \subset I_0$, $K \subset K_0$ let

$$\mathcal{N}_{I,K} := \{\underline{\mathbf{n}} = (n_i, n'_\sigma, n''_\sigma, \tilde{n}'_k, \tilde{n}''_k) : n_i \geq 0, i = 1 \dots, s$$

$$n'_\sigma, n''_\sigma, \sigma \in I; \tilde{n}'_k > 0, \tilde{n}''_k > 0, k \in K\}.$$

For $\underline{\mathbf{n}} \in \mathcal{N}_{I,K}, i = 1, \dots, s$, let

$$\hat{n}_i = n_i + \sum_{\sigma \in I : i(\sigma) = i} n'_\sigma + \sum_{\sigma \in I : j(\sigma) = i} n''_\sigma + \sum_{k \in K : i(k) = i} \tilde{n}'_k. \tag{16}$$

Let

$$F(\underline{\mathbf{n}}) = \frac{1}{2}(\sum_{i,j=1}^{s} m_{ij}\hat{n}_i\hat{n}_j + \sum_{i=1}^{s} \hat{n}_i(\sum_{j=1}^{s} m_{ij}\chi(E_j^{\bullet}) + 1)) + \sum_{k \in K} \tilde{n}''_k, \tag{17}$$

$$\overline{F}(\underline{\hat{n}}) = \frac{1}{2}(\sum_{i,j=1}^{s} m_{ij}\hat{n}_i\hat{n}_j + \sum_{i=1}^{s} \hat{n}_i(\sum_{j=1}^{s} m_{ij}\chi(E_j^{\bullet}) + 1)),$$

and

$$\underline{w}(\underline{\mathbf{n}}) = \sum_{i=1}^{s} \hat{n}_i \underline{m}_i, v_k(\underline{\mathbf{n}}) := w_{i(k)}(\underline{\mathbf{n}}) + \tilde{n}''_k.$$

Theorem 2 *([5])*

$$P_g(t_1, \dots, t_r, q) = \sum_{I \subset I_0, K \subset K_0} \sum_{\underline{\mathbf{n}} \in \mathcal{N}_{I,K}} q^{F(\underline{\mathbf{n}}) - \sum_{i=1}^{s} n_i - |I| - |K|} \cdot (1-q)^{|I|+|K|} \times$$

$$\times \prod_{i=1}^{s} \left(\sum_{j=0}^{\min\{n_i, 1-\chi(E_i^{\circ})\}} (-1)^j \binom{1-\chi(E_i^{\circ})}{j} q^j \right) \cdot t^{\underline{v}(\underline{\mathbf{n}})}.$$

We briefly recall the sketch of the proof from [5]. Consider a function $f \in \mathcal{O}$ and its pullback $\pi^* f$ on the space of resolution X. Now let $I(f)$ be the set of intersection points in D such that there are components of the strict transform of X passing through them, $K(f)$ is the analogous set of intersection points of strict transform of C with D. Now $n_i(f)$ is the intersection index of the strict transform of f with the smooth part of E_i, n'_σ and n''_σ are intersection indices of the component of the strict transform of f passing through σ with $E_{i(\sigma)}$ and $E_{j(\sigma)}$ respectively, $\tilde{n}'_k$ and $\tilde{n}''_k$ are intersection indices of the component passing through the point k with $E_{i(k)}$ and corresponding component of C respectively.

Given these sets and multiplicities, the value of the function $t_1^{v_1(f)} \cdot \ldots \cdot t_r^{v_r(f)}$ is equal to $t^{\underline{v}(\mathbf{n})}$. Every summand in Theorem 2 is equal to this value multiplied by the motivic measure of the set of functions providing such set of data.

3 Example: nonsingular curve

Let us check that for the nonsingular curve the complicated expression from Theorem 2 coincides with the expected one.

We have one divisor and one component of the strict transform of the curve. We have $I_0 = \emptyset$, $K_0 = \{1\}$. Also we have $\chi(E^\circ) = 1, \chi(E^\bullet) = 2$, hence $1 - \chi(E^\circ) = 0$. To sum over $K \subset K_0$, consider two cases:

1) $K = \emptyset$. In this case $F(n) = \frac{1}{2}(n^2 + 3n)$, and we have a sum

$$\sum_{n=0}^{\infty} t^n q^{\frac{n^2+3n}{2}} \cdot q^{-n}$$

2) $K = \{1\}$. In this case $F(n) = \frac{1}{2}(\hat{n}^2 + 3\hat{n}) + n''$, and we have a sum

$$\sum_{\hat{n}=1}^{\infty} q^{\frac{\hat{n}^2+3\hat{n}}{2}} t^{\hat{n}} \sum_{n=0}^{\hat{n}-1} q^{-n-1}(1-q) \sum_{n''=1}^{\infty} q^{n''} t^{n''} = \sum_{\hat{n}=1}^{\infty} q^{\frac{\hat{n}^2+3\hat{n}}{2}} t^{\hat{n}} (q^{-\hat{n}} - 1) \cdot \frac{qt}{1-qt}.$$

Summing these two expressions, we get

$$1 + \sum_{n=1}^{\infty} t^n q^{\frac{n^2+3n}{2}} (q^{-n} + \frac{qt}{1-qt}(q^{-n} - 1)) = 1 + \frac{1}{1-qt} \sum_{n=1}^{\infty} t^n q^{\frac{n^2+3n}{2}} (q^{-n} - qt) =$$

$$1 + \frac{1}{1-qt} (\sum_{n=1}^{\infty} t^n q^{\frac{n(n+1)}{2}} - \sum_{n=1}^{\infty} t^{n+1} q^{\frac{(n+1)(n+2)}{2}}).$$

In the last sum all coefficients at t^n for $n \geq 2$ cancel, therefore

$$P_g(t; q) = 1 + \frac{tq}{1-qt} = \frac{1}{1-qt}.$$

4 Combinatorics

4.1 Preliminary simplification

Let

$$P_{k,n}(q) = \sum_{j=0}^{n} (-1)^j q^j \binom{k}{j}$$

(k can be negative, but n should be non-negative and integer).

Lemma 1 *Let $S^n X$ denote the nth symmetric power of a space X. Then*

$$[S^n(\mathbb{CP}^1 - k\{pt\})] = q^{-n} P_{k-1,n}(q).$$

Proof. If Y denote the union of k points on $\mathbb{C}^1$, then we have

$$S^m(\mathbb{CP}^{\nVdash}) = \sqcup_{i=0}^{m} S^i(Y) \times S^{m-i}(\mathbb{CP}^1 \setminus Y),$$

what is equivalent to the following multiplicativity property:

$$\sum_{n=0}^{\infty} t^n [S^n(\mathbb{CP}^1)] = \sum_{n=0}^{\infty} t^n [S^n(Y)] \cdot \sum_{n=0}^{\infty} t^n [S^n(\mathbb{CP}^1 \setminus Y)].$$

Since

$$\sum_{n=0}^{\infty} t^n [S^n(\mathbb{CP}^1)] = \sum_{n=0}^{\infty} t^n [\mathbb{CP}^n] = \frac{1}{(1-t)(1-\mathbb{L}t)},$$

we get

$$\sum_{n=0}^{\infty} t^n [S^n(\mathbb{CP}^1 - k\{pt\})] = \frac{(1-t)^{k-1}}{(1-\mathbb{L}t)} =$$

$$\sum_{a,b} (-1)^a \binom{k-1}{a} t^a \mathbb{L}^b t^b = \sum_{n=0}^{\infty} t^n \sum_{a=0}^{n} (-1)^a \binom{k-1}{a} \mathbb{L}^{n-a} =$$

$$\sum_{n=0}^{\infty} t^n q^{-n} P_{k-1,n}(q).$$

$\square$

Let us fix some notations.

Definition: Let

$$f_i(I,K) = \sum_{\sigma \in I : i(\sigma) = i} 1 + \sum_{\sigma \in I : j(\sigma) = i} 1 + \sum_{k \in K : i(k) = i} 1,$$

$$f_i(I) = \sum_{\sigma \in I : i(\sigma) = i} 1 + \sum_{\sigma \in I : j(\sigma) = i} 1.$$

Note that $\sum_{i=1}^{s} f_i(I,K) = 2|I| + |K|, \sum_{i=1}^{s} f_i(I) = 2|I|$.

To any divisor E_i we associate the factor

$$\phi_i(I,K,\hat{n}) = P_{1-\chi(E_i^\circ)-f_i(I,K),\hat{n}_i-f_i(I,K)},$$

and let

$$G(I,K,\hat{n}) = q^{|I|}(1-q)^{|I|+|K|}\prod_i \phi_i(I,K,\hat{n}).$$

Now we can start the simplification of the algorithm proposed in Theorem 2. The next two lemmas will allow us to reduce the summation over all quadruples $(n_i, n'_\sigma, n''_\sigma, \widetilde{n}'_k)$ to the summation by a single variable $\hat{n}_i$ defined by (16).

Lemma 2 *Let us fix $\hat{n}_i$. Then*

$$\sum_{n_i,n'_\sigma,n''_\sigma,\widetilde{n}'_k} q^{-n_i-f_i(I,K)}P_{1-\chi(E_i^\circ),n_i}(q) = q^{-\hat{n}_i}\phi_i(I,K,\hat{n}). \tag{18}$$

Proof. By Lemma 1 we have

$$\sum_{n_i,n'_\sigma,n''_\sigma,\widetilde{n}'_k} q^{-n_i-f_i(I,K)}P_{1-\chi(E_i^\circ),n_i}(q) = \sum_{n_i,n'_\sigma,n''_\sigma,\widetilde{n}'_k} q^{-f_i(I,K)}[S^{n_i}(E_i^\circ)].$$

Consider a n_i-tuple of points on E_i°, intersection points $\sigma \in I$ such that $i(\sigma) = i$ with multiplicities $n'_\sigma - 1$, intersection points $\sigma \in I$ such that $j(\sigma) = i$ with multiplicities $n''_\sigma - 1$, intersection points $k \in K$ such that $i(k) = i$ with multiplicities $\widetilde{n}'_k - 1$. We get the unordered $\hat{n}_i - f_i$-tuple of points on $E_i^\circ \cup f_i(I,K)$. Thus the sum (18) equals to

$$q^{-f_i(I,K)}[S^{\hat{n}_i-f_i(I,K)}(E_i^\circ \cup f_i(I,K))] = q^{-\hat{n}_i}P_{1-\chi(E_i^\circ)-f_i(I,K),\hat{n}_i-f_i(I,K)}(q).$$

□

Lemma 3

$$P_g(t_1,\ldots,t_r,q) = \sum_{I\subset I_0,K\subset K_0}\sum_{\hat{n}_i\geq f_i(I,K)} \underline{t}^{M\underline{\hat{n}}}q^{\overline{F}(\underline{\hat{n}})}\prod_{i=1}^{s} q^{-\hat{n}_i}\phi_i(I,K,\hat{n})\times \tag{19}$$

$$q^{|I|}(1-q)^{|I|+|K|}\prod_{k\in K}\frac{qt_k}{1-qt_k}.$$

Proof. First, remark that for every k

$$\sum_{\tilde{n}''_k>0} q^{\tilde{n}''_k}t_k^{\tilde{n}''_k} = \frac{t_kq}{1-t_kq},$$

so from now on we can forget about summation over $\tilde{n}''_k$.

We have

$$q^{-\sum_{i=1}^{s}n_i-|I|-|K|} = q^{|I|}\prod_{i=1}^{s} q^{-n_i-f_i(I,K)},$$

therefore we can reformulate the statement of Theorem 2 in the form

$$P_g(t_1,\dots,t_r,q)=\sum_{I\subset I_0,K\subset K_0} q^{|I|}(1-q)^{-|I|}\sum_{\hat{n}_i\geq f_i(I,K)} \underline{t}^{M\underline{\hat{n}}}q^{\overline{F}(\underline{\hat{n}})}\times$$

$$\prod_{i=1}^{s}\left[\sum_{n_i,n'_\sigma,n''_\sigma,\widetilde{n}'_k} q^{-n_i-f_i(I,K)}P_{1-\chi(E_i^\circ),n_i}(q)\right].$$

Now the equation (19) follows from the Lemma 2. $\square$

Definition: By the reduced motivic Poincaré series from now on we mean

$$\overline{P}_g(t_1,\dots,t_r)=P_g(t_1,\dots,t_r)\cdot\prod_{j=1}^{r}(1-t_jq).$$

Lemma 4

$$\sum u^{\hat{n}}G(K,I,\hat{n})=q^{|I|}(1-q)^{|I|+|K|}\prod_i \frac{u_i^{f_i(K,I)}}{1-u_i}(1-u_iq)^{1-\chi(E_i^\circ)-f_i(I,K)} \qquad (20)$$

The proof of this lemma can be found in the Appendix.

Definition: Let

$$c_K(n)=\sum_I\sum_{K_1\subset K}(-1)^{|K|-|K_1|}G(K_1,I,n),$$

$$A_K(u)=\sum_n u^n c_K(n).$$

The next lemma provides a closed formula for the function $A_K(u)$, which can be considered as a generating function for the quantities $c_K(n)$.

Lemma 5

$$A_K(u)=(-1)^{|K|}\prod_i(1-u_iq)^{|\overline{K}\cap E_i|-1}(1-u_i)^{|K\cap E_i|-1}\prod_\sigma(1-qu_{i(\sigma)}-qu_{j(\sigma)}+qu_{i(\sigma)}u_{j(\sigma)}).$$

The proof of this lemma can be found in the Appendix. The next lemma expresses the reduced motivic Poincaré series in terms of the quantities $c_K(n)$.

Lemma 6

$$\overline{P}_g(t_1,\dots,t_r,q)=\sum_n t^{Mn}q^{F(n)-\sum n_i}\sum_K t_Kq^{|K|}c_K(n). \qquad (21)$$

Proof. From the equation (19) we get

$$P_g(t_1,\ldots,t_r,q)=\sum_{I\subset I_0,K\subset K_0}\sum_{\hat{n}_i\geq f_i(I,K)}\underline{t}^{M\underline{\hat{n}}}q^{\overline{F}(\underline{\hat{n}})}\prod_{i=1}^{s}q^{-\hat{n}_i}\phi_i(I,K,\hat{n})\times$$

$$q^{|I|}(1-q)^{|I|+|K|}\prod_{k\in K}\frac{qt_k}{1-qt_k}=$$

$$\sum_{I\subset I_0,K\subset K_0}\sum_{\hat{n}_i\geq f_i(I,K)}\underline{t}^{M\underline{\hat{n}}}q^{\overline{F}(\underline{\hat{n}})}\prod_{i=1}^{s}q^{-\hat{n}_i}\phi_i(I,K,\hat{n})\times q^{|I|}(1-q)^{|I|+|K|}\prod_{k\in K}\frac{qt_k}{1-qt_k}=$$

$$\frac{1}{\prod_{i=1}^{n}(1-qt_i)}\sum_{\hat{n}}t^{Mn}q^{F(n)-\sum n_i}\sum_{K}t_Kq^{|K|}\sum_{I\subset I_0}\sum_{K_1\subset K}(-1)^{|K|-|K_1|}G(K_1,I,\hat{n})=$$

$$\frac{1}{\prod_{i=1}^{n}(1-qt_i)}\sum_{\hat{n}}t^{Mn}q^{F(n)-\sum n_i}\sum_{K}t_Kq^{|K|}c_K(\hat{n}).$$

□

Lemma 6 together with Lemma 5 gives the explicit description of $\overline{P}_g(t)$: it is expressed in terms of some quantities $c_K(n)$, which fit together into the generating function $A_K(u)$. Lemma 5 provides a closed formula for this generating function.

Nevertheless, as the model example with a nonsingular curve shows, lots of summands in the sum (21) have the same power in t, and for n large enough we have a huge number of cancellations.

4.2 Cancellations

We say that a subset $K\subset K_0$ is *proper everywhere*, if for all i $K\cap E_i$ is a proper subset of $K_0\cap E_i$. We denote the set of proper everywhere subsets by $\mathcal{P}$. For any $K\subset K_0$ let $E(K)$ be the set of divisors such that for $i\in E(K)$ the set $K\cap E_i$ is empty. Sometimes we will write $i\in P$, if $i\notin E(P)$.

Using these notations, every subset $K\subset K_0$ can be presented (uniquely) in the following way: we fix a proper everywhere subset $P(K)$ and a set of divisors $E\subset E(P(K))$ where all intersection points with K_0 belong to K.

For a set E of divisors let $\Delta(E)$ be the number of pairs of intersecting divisors from E. Let $\mu_i(E)=1$, if $i\in E$ and $\mu_i(E)=0$ otherwise.

Lemma 7 *For a proper everywhere set P let*

$$\widetilde{H}_P(u_1,\ldots,u_s)=\sum_{E\subset E(P)}(-1)^{|K_0\cap E|}\prod u_i^{-\sum a_{ij}\mu_j}\cdot q^{\Delta(E)}\prod_{i\in E}(q-u_i)^{k_i-1}\prod_{i\notin(P\cup E)}(1-qu_i)^{k_i-1}$$
$$\times\prod_{\sigma}(1-q^{1-\mu_{i(\sigma)}(E)}u_{i(\sigma)}-q^{1-\mu_{j(\sigma)}(E)}u_{j(\sigma)}+q^{1-\mu_{i(\sigma)}(E)-\mu_{j(\sigma)}(E)}u_{i(\sigma)}u_{j(\sigma)}). \tag{22}$$

Then the polynomial $\widetilde{H}_P$ is divisible by $\prod_{i\in E(P)}(1-u_i)$.

The proof of this lemma can be found in the Appendix.

The next lemma explains the relation of the function $\widetilde{H}_P(u_1,\ldots,u_s)$ (which is a modification of the function $A_K(u)$) to the coefficients $c_K(n)$ defined above. It is the main technical instrument in the study of the cancellations.

Lemma 8

$$\sum_n u^n \sum_{E\subset E(P)} q^{-\sum_{i\in E} n_i-\Delta(E)-\sum_{i\in E} a_{ii}-|E|} q^{|K_0\cap E|}\times c_{P\cup E}(n_i+\sum a_{ij}\mu_j(E))=$$

$$(-1)^{|P|}\prod_{i\in P}[(1-qu_i)^{k_i-p_i-1}(1-u_i)^{p_i-1}]\cdot\frac{1}{\prod_{i\in E(P)}(1-u_i)}\widetilde{H}_P(u_1,\ldots,u_s).$$

The proof of this lemma can be found in the Appendix.

Definition: For a proper everywhere set P define the quantities $d_P(n)$ by the equation

$$H_P(u)=\sum_n d_P(n)u^n d_P(n)=\frac{\prod_{i\in P}[(1-qu_i)^{k_i-p_i-1}(1-u_i)^{p_i-1}]}{\prod_{i\in E(P)}(1-u_i)}\widetilde{H}_P(u_1,\ldots,u_s). \quad (23)$$

Remark that by Lemma 7 the function $H_P(u)$ is polynomial in u, so we have only finite number of non-zero coefficients $d_P(n)$.

Combining the statements of Lemma 6 and Lemma 8, we get the following result.

Theorem 3 *Then*

$$\overline{P}_g(t_1,\ldots,t_r)=\sum_{P\in\mathcal{P}}(-1)^{|P|}q^{|P|}t_P\times\sum_n d_P(n)t^{Mn}q^{F(n)-\sum n_i}.$$

Proof. From Lemma 6 we have

$$\overline{P}(t)=\sum_{n_1}t^{Mn_1}q^{F(n_1)-\sum n_i}\sum_{K\subset K_0}t_K q^{|K|}c_K(n_1)=$$

$$\sum_{P\in\mathcal{P}}q^{|P|}t_P\sum_{n_1}t^{Mn_1}q^{F(n_1)-\sum n_i}\sum_{E\subset E(P)}t_E q^{|K_0\cap E|}c_{P\cup E}(n_1).$$

Let us collect the coefficient at t^{Mn}. We have

$$Mn_1+\sum\mu_j(E)=Mn,\quad n_1=n+\sum a_{ij}\mu_j(E).$$

and

$$(\overline{F}(n)-\sum n_i)-(\overline{F}(n_1)-\sum n_{1i})=\frac{1}{2}[-2\sum m_{ij}n_i a_{js}\mu_j(E)$$

$$-\sum m_{ij}a_{is}\mu_s(E)a_{jl}\mu_l(E)-\sum m_{ij}\chi(E_i^\bullet)a_{js}\mu_s(E)+\sum a_{ij}\mu_j(E)].$$

Remark that

$$\sum_{i\neq j}a_{ij}=2-\chi(E_j^\bullet),$$

hence

$$(\overline{F}(n) - \sum n_i) - (\overline{F}(n_1) - \sum n_{1i}) = \sum_{i \in E} n_i + \Delta(E) + \sum_{i \in E} a_{ii} + |E|.$$

Thus

$$\overline{P}(t) = \sum_{P \in \mathcal{P}} q^{|P|} t_P \sum_n t^{Mn} q^{F(n) - \sum n_i} \sum_{E \subset E(P)} q^{-\sum_{i \in E} n_i - \Delta(E) - \sum_{i \in K} a_{ii} - |E|}$$

$$\times q^{|K_0 \cap E|} c_{P \cup E}(n + \sum a_{ij} \mu_j(E)).$$

Now we apply Lemma 8.

□

Corollary 1 *The power series $\overline{P}_g(t_1, \ldots, t_r)$ is a polynomial.*

4.3 The algorithm

If every line E_i is intersected by the one component of the strict transform, any proper everywhere set should be empty. Therefore we get the following statement as a corollary of Theorem 3.

Lemma 9 *Suppose that each divisor E_i is intersected by exactly one component of the strict transform of the curve. Then the reduced motivic Poincaré series can be computed using the following algorithm.*

1. Consider the polynomial

$$A(u_1, \ldots, u_r) = \prod_\sigma (1 - q u_{i(\sigma)} - q u_{j(\sigma)} + q u_{i(\sigma)} u_{j(\sigma)}).$$

2. Consider the Laurent polynomial

$$\widetilde{H}(u_1, \ldots, u_t) = \sum_{K \subset K_0} (-1)^{|K|} q^{\Delta(K)} \prod u_i^{-\sum a_{ij} \mu_j} \cdot A(u_1 q^{-\mu_1(K)}, \ldots, u_r q^{-\mu_r(K)}).$$

3. This polynomial is divisible by $\prod(1 - u_i)$. Let

$$H(u_1, \ldots, u_r) = \frac{\widetilde{H}(u_1, \ldots, u_r)}{\prod_{i=1}^r (1 - u_i)}.$$

4. Expand this polynomial:

$$H(u_1, \ldots, u_r) = \sum d_{\underline{n}} u^{\underline{n}},$$

and now

$$\overline{P}_g(t_1, \ldots, t_r) = \sum d_{\underline{n}} t^{M\underline{n}} q^{F(\underline{n}) - \sum n_i}.$$

5 Examples

5.1 One divisor

We consider the singularity
$$x^{k_0} - y^{k_0} = 0,$$
which is geometrically a union of k_0 pairwise transversal lines. Its minimal resolution has one divisor and k_0 components of the strict transform intersecting it. In particular, for $k_0 = 1$ we get a non-singular case considered above. For $0 < k < k_0$ let the numbers $c_k(n)$ be defined by the equation
$$A_k(u) = \sum_{n=0}^{\infty} u^n c_k(n) = (1 - uq)^{k_0-k-1}(1 - u)^{k-1},$$
and for $k = 0$ let the numbers $c_0(n)$ be defined by the equation
$$A_0(u) = \sum_{n=0}^{\infty} u^n c_0(n) = \frac{(1 - uq)^{k_0-1} - u(u - q)^{k_0-1}}{1 - u}.$$
The polynomials $A_k(u)$ have degree $k_0 - 2$ for $k > 0$, $A_0(u)$ has degree $k_0 - 1$, so we have a finite number of non-zero $c_k(n)$.

From the Theorem 3 we conclude that
$$\overline{P}_g(t_1, \ldots, t_{k_0}) = \sum_{K \subset_{\neq} K_0} (-1)^{|K|} q^{|K|} t_K \sum_{n=0}^{\infty} c_{|K|}(n)(t_1 \ldots t_{k_0})^n q^{\frac{n(n+1)}{2}}.$$

For example, if $k_0 = 2$,
$$A_1(u) = 1, A_0(u) = \frac{1 - uq - u(u - q)}{1 - u} = 1 + u,$$
so
$$\overline{P}_g(t_1, t_2) = 1 - qt_1 - qt_2 + qt_1t_2.$$

If $k_0 = 3$,
$$A_1(u) = 1 - qu, A_2(u) = 1 - u, A_0(u) = 1 + (1 - 2q - q^2)u + u^2,$$
so
$$\overline{P}_g(t_1, t_2, t_3) = 1 - q(t_1 + t_2 + t_3) + q^2(t_1t_2 + t_1t_3 + t_2t_3) + q(1 - 2q - q^2)t_1t_2t_3+$$
$$q^3t_1t_2t_3(t_1 + t_2 + t_3) - q^3t_1t_2t_3(t_1t_2 + t_1t_3 + t_2t_3) + q^3t_1^2t_2^2t_3^2.$$
This answer can be rewritten as
$$\overline{P}_g(t_1, t_2, t_2) = (1 - qt_1)(1 - qt_2)(1 - qt_3) - q^3t_1t_2t_3(1 - t_1)(1 - t_2)(1 - t_3) + q(1 - q)^2t_1t_2t_3.$$

5.2 Two divisors

Suppose that the second divisor is intersected by two components of the strict transform, and the first one by one component. This corresponds to the singularity

$$x \cdot (y - x^2) \cdot (y + x^2) = 0.$$

The matrix M is equal to

$$M = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix},$$

$$\chi(E_1^{\bullet}) = \chi(E_2^{\bullet}) = 1,$$

so

$$F(n_1, n_2) = \frac{1}{2}(n_1^2 + 2n_1n_2 + 2n_2^2 + 2n_1 + 3n_2).$$

If $P = \emptyset$, we get

$$\widetilde{H}_{\emptyset}(u_1, u_2) = (1 - qu_1 - qu_2 + qu_1u_2)(1 - qu_2) - (1 - u_1 - qu_2 + u_1u_2)(1 - qu_2)u_1^2u_2^{-1}$$

$$+(1 - qu_1 - u_2 + u_1u_2)(q - u_2)u_1^{-1}u_2 - q(1 - u_1 - u_2 + q^{-1}u_1u_2)(1 - qu_2)u_1 =$$

$$\frac{1}{u_1u_2}(1 - u_1)(1 - u_2)(-u_1^3 + u_1u_2 + u_1^2u_2 - qu_1^2u_2 - q^2u_1^2u_2 + qu_1^3u_2$$

$$+qu_2^2 + u_1u_2^2 - qu_1u_2^2 - q^2u_1u_2^2 + u_1^2u_2^2 - u_2^3),$$

if P is one point on the second divisor, we get

$$\widetilde{H}_{pt}(u_1, u_2) = (1 - qu_1 - qu_2 + qu_1u_2) - (1 - u_1 - qu_2 + u_2)u_1^2u_2^{-1} =$$

$$-\frac{1}{u_2}(1 - u_1)(u_1^2 - u_2 - u_1u_2 + qu_1u_2 - u_1^2u_2 + qu_2^2).$$

Finally we get the following answer (t_0 corresponds to the first divisor):

$$\overline{P}_g(t_0, t_1, t_2) = 1 - qt_0 - qt_1 + q^2t_0t_1 - qt_2 + q^2t_0t_2 + q^2t_1t_2 + qt_0t_1t_2 - q^2t_0t_1t_2 - q^3t_0t_1t_2$$

$$-q^2t_0t_1^2t_2 + q^3t_0t_1^2t_2 - q^2t_0t_1t_2^2 + q^3t_0t_1t_2^2 + q^2t_0t_1^2t_2^2 - q^3t_0t_1^2t_2^2 - q^4t_0t_1^2t_2^2 + q^4t_0^2t_1^2t_2^2$$

$$+q^4t_0t_1^3t_2^2 - q^4t_0^2t_1^3t_2^2 + q^4t_0t_1^2t_2^3 - q^4t_0^2t_1^2t_2^3 - q^4t_0t_1^3t_2^3 + q^4t_0^2t_1^3t_2^3.$$

This answer can be rewritten as

$$\overline{P}_g(t_0, t_1, t_2) = (1 - qt_0)(1 - qt_1)(1 - qt_2) - q^4t_0t_1^2t_2^2(1 - t_0)(1 - t_1)(1 - t_2)$$

$$+(1 - q)qt_0t_1t_2(1 - qt_1 - qt_2 + qt_1t_2).$$

If $q = 1$, we get the known Alexander polynomial:

$$\overline{P}_g(t_0, t_1, t_2; q = 1) = (1 - t_0)(1 - t_1)(1 - t_2)(1 - t_0t_1^2t_2^2).$$

If $t_2 = 1$, we get the known answer for A_1 singularity:

$$\overline{P}_g(t_0, t_1, 1) = (1 - q)(1 - qt_0 - qt_1 + qt_0t_1).$$

If $t_0 = 1$, we get the answer for A_3 singularity:

$$\overline{P}_g(1,t_1,t_2) = (1-q)(1-qt_1-qt_2+qt_1t_2+q^2t_1t_2-q^2t_1^2t_2-q^2t_1t_2^2+q^2t_1^2t_2^2),$$

so

$$\overline{P}_g^{A_3}(t_1,t_2) = (1-qt_1)(1-qt_2)+qt_1t_2(1-qt_1-qt_2+qt_1t_2) =$$
$$(1-qt_1)(1-qt_2)+q^2t_1t_2(1-t_1)(1-t_2)+(1-q)qt_1t_2.$$

This answer agrees with the general answer for the singularities of type A_{2n-1} in the section 7.5.

5.3 Three divisors

For simplicity we assume that each divisor is intersected by one component of the strict transform. This corresponds to the singularity

$$x \cdot y \cdot (x^2 - y^3) = 0.$$

Matrix M is equal to

$$M = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 6 \end{pmatrix},$$

$$\chi(E_1^\bullet) = \chi(E_2^\bullet) = 1, \chi(E_3^\bullet) = 0,$$

so

$$F(n_1,n_2,n_2) = \frac{1}{2}(n_1^2+2n_2^2+6n_3^2+2n_1n_2+4n_1n_3+6n_2n_3+n_1+2n_2+4n_3).$$

Now

$$A(u_1,u_2,u_3) = (1-qu_1-qu_3+qu_1u_3)(1-qu_2-qu_3+qu_2u_3),$$

so

$$E(u_1,u_2,u_3) = \frac{1}{u_1u_2u_3^2}({u_2}^3u_3u_1-{u_1}^3{u_3}^2q+{u_1}^4u_3u_2-{u_1}^2{u_2}^2{u_3}^2-{u_2}^2{u_3}^2u_1+$$
$${u_1}^4{u_2}^3u_3-{u_3}^3{u_1}^2q-{u_1}^3u_2{u_3}^2+{u_1}^3{u_2}^3u_3+{u_1}^2{u_2}^3u_3-{u_3}^3qu_2-$$
$${u_1}^3{u_2}^2{u_3}^2-{u_3}^3u_1q-{u_2}^2{u_3}^2q-{u_1}^2u_2{u_3}^2-{u_3}^2u_1u_2+{u_2}^2{u_1}^4u_3-{u_1}^3{u_2}^3qu_3+$$
$${u_2}^2{u_3}^2{u_1}^2q-{u_1}^4u_3{u_2}^2q-{u_1}^4{u_3}^2u_2q-{u_2}^3{u_3}^2u_1q-{u_2}^3u_3{u_1}^2q+{u_3}^3u_1q^2u_2+$$
$${u_2}^2{u_3}^2u_1q^2+{u_1}^3{u_2}^2u_3q^2+{u_1}^3{u_3}^2u_2q^2-{u_1}^4{u_2}^3+{u_1}^2{u_3}^3+{u_3}^3u_1+{u_3}^2{u_1}^2u_2q+$$
$${u_1}^3{u_3}^3+{u_3}^3{u_2}^2+{u_3}^3u_2+{u_3}^3-{u_3}^4),$$

and

$$\overline{P}_g(t_1,t_2,t_3) = 1-t_3q+{t_1}^2{t_2}^3{t_3}^7q^7+{t_1}^2{t_2}^2{t_3}^5q^5+t_1t_2{t_3}^3q^3+t_1{t_2}^2{t_3}^4q^4-{t_1}^2{t_2}^4{t_3}^7q^7+$$
$$t_2t_3q^2-t_1t_2{t_3}^3q^2+t_1t_2q^2-t_1{t_2}^2{t_3}^4q^3-{t_1}^2{t_2}^2{t_3}^5q^4-t_1{t_2}^2{t_3}^2q^2-{t_1}^2{t_2}^3{t_3}^5q^5-$$

$$t_1{}^3t_2{}^3t_3{}^7q^7 - t_1{}^3t_2{}^4t_3{}^6q^7 + t_1{}^2t_2{}^3t_3{}^5q^4 + t_1{}^2t_2{}^2t_3{}^4q^3 + t_1{}^2t_2{}^2t_3{}^3q^4 - t_1{}^2t_2{}^2t_3{}^3q^3 +$$
$$t_1{}^2t_2{}^3t_3{}^4q^5 + t_1{}^2t_2{}^4t_3{}^6q^7 + t_1t_2{}^2t_3{}^2q^3 - t_1{}^2t_2{}^3t_3{}^6q^7 - t_1{}^2t_2{}^2t_3{}^4q^5 - t_1t_2{}^2t_3{}^3q^4 -$$
$$t_1t_2t_3q^3 + t_1t_2{}^2t_3{}^3q^2 - t_2q + t_1t_3q^2 - t_1t_2t_3{}^2q^2 + t_1{}^3t_2{}^4t_3{}^7q^7 +$$
$$t_1t_2t_3{}^2q - t_1q - t_1{}^2t_2{}^3t_3{}^4q^4 + t_1{}^3t_2{}^3t_3{}^6q^7.$$

It can be rewritten as

$$\overline{P}_g(t_1, t_2, t_3) = (1 - t_1q)(1 - t_2q)(1 - t_3q) - t_1{}^2t_2{}^3t_3{}^6q^7(1 - t_1)(1 - t_2)(1 - t_3) -$$
$$t_1t_2t_3^2q(q - 1)(1 - t_2q)(1 - t_3q) - t_1^2t_2^2t_3^4q^4(q - 1)(1 - t_2)(1 - t_3) -$$
$$t_1t_2^2t_3^3q^2(q - 1)(1 - t_1q) + t_1t_2^2t_3^4q^3(q - 1)(1 - t_1).$$

In this presentation the symmetry of $\overline{P}_g$ is clear, since every line in the right hand side is invariant under the change $t_i \leftrightarrow q^{-1}t_i^{-1}$.

If we set $q = 1$, we get

$$\overline{P}_g(t_1, t_2, t_3, q = 1) = (1 - t_1^2t_2^3t_3^6)(1 - t_1)(1 - t_2)(1 - t_3).$$

If we consider only singularity of type A_2, we set $t_1 = t_2 = 1, t_3 = t$, and

$$\overline{P}_g(1, 1, t) = (1 - q)^2(1 - tq + t^2q),$$

so

$$P_g(1, 1, t) = \frac{1 - tq + t^2q}{1 - tq} = 1 + \sum_{k=2}^{\infty} t^k q^{k-1}.$$

This answer coincides with the one obtained in the section 2.3.

6 Symmetry

In this section we prove the symmetry property for the reduced motivic Poincaré series (Theorem 4). The strategy of the proof passes along the lines of the computation described in Lemma 6: namely, we prove the symmetry property for the generating function $A_K(u)$ in Lemma 10, deduce from it a certain relations on its coefficients $c_K(n)$ in Lemma 11. Since we can express the motivic Poincaré series in terms of $c_K(n)$, we can finish the proof by fitting this relations to the statement of Theorem 4.

Lemma 10

$$A_K(\frac{1}{qu_1}, \ldots, \frac{1}{qu_s}) = q^{1-|K|} \prod_{i=1}^{s} u_i^{\chi(E_i^\circ)} \cdot A_{\overline{K}}(u_1, \ldots, u_s).$$

Proof.

$$A_K(\frac{1}{qu}) = (-1)^{|K|}\prod_i(1-\frac{1}{u_i})^{|\overline{K}\cap E_i|-1}(1-\frac{1}{u_iq})^{|K\cap E_i|-1}\prod_\sigma(1-\frac{1}{u_{i(\sigma)}}-\frac{1}{u_{j(\sigma)}}+\frac{1}{qu_{i(\sigma)}u_{j(\sigma)}}) =$$

$$A_{\overline{K}}(u)\prod_i u_i^{1-|\overline{K}\cap E_i|}u_i^{1-|K\cap E_i|}q^{1-|K\cap E_i|}\prod_\sigma(qu_{i(\sigma)}u_{j(\sigma)})^{-1} =$$

$$A_{\overline{K}}(u)q^{s-|K|-|I_0|}\prod u_i^{2-|K_0\cap E_i|+\chi(E_i^\bullet)-2}.$$

It rests to note that $|I_0| = s-1$ and $\chi(E_i^\circ) = \chi(E_i^\bullet) - |K_0\cap E_i|$. $\square$

Lemma 11

$$c_K(n_1,\ldots,n_s) = q^{1-|K|+n}c_{\overline{K}}(-\chi(E_1^\circ)-n_1,\ldots,-\chi(E_s^\circ)-n_s),$$

where $n = \sum_{i=1}^s n_i$.

Proof.

$$A_K(\frac{1}{qu_1},\ldots,\frac{1}{qu_s}) = \sum_{\underline{n}} c_K(n_1,\ldots,n_s)\underline{u}^{-\underline{n}}q^{-n} = q^{1-|K|}\prod u_i^{\chi(E_i^\circ)}\sum_{\underline{z}} c_{\overline{K}}(z_1,\ldots,z_s)\underline{u}^{\underline{z}}.$$

We have

$$z_i + \chi(E_i^\circ) = -n_i,\ \ z_i = -\chi(E_i^\circ) - n_i.$$

$\square$

Theorem 4 *Let μ_α be the Milnor number of C_α, and $(C_\alpha\circ C_\beta)$ is the intersection index of $C_\alpha\circ C_\beta$, $\mu(C)$ is the Milnor number of C. Let $l_\alpha = \mu_\alpha + \sum_{\beta\neq\alpha}(C_\alpha\circ C_\beta)$ and $\delta(C) = (\mu(C)+r-1)/2$. Then*

$$\overline{P}_g(\frac{1}{qt_1},\ldots,\frac{1}{qt_r}) = q^{-\delta(C)}\prod_\alpha t_\alpha^{-l_\alpha}\cdot\overline{P}_g(t_1,\ldots,t_r).$$

The theorem follows from Lemma 11 describing the symmetry of the coefficients $c_K(n)$ and Lemma 6 describing $\overline{P}_g(t_1,\ldots,t_r)$ in terms of $c_K(n)$. The detailed proof is rather technical and can be found in the Appendix.

Corollary 2 *The degree of the polynomial $\overline{P}_g(t_1,\ldots,t_r)$ with respect to the variable t_i is equal to l_i. The greatest monomial in it equals to $q^{\delta(C)}\prod_{i=1}^r t_i^{l_i}$.*

Alternative proof of the symmetry property for the motivic Poincaré series can be found in [14], where it is deduced from the theorem of Campillo, Delgado and Kiyek on the symmetry of the multi-variable Poincaré series of a plane curve singularity.

7 Relation to the Heegaard-Floer knot homology

7.1 Heegaard-Floer homology

In the series of articles (e.g. [18],[19],[20],[22], see also [23]) P. Ozsváth and Z. Szabó constructed new powerful knot invariants, Heegaard-Floer knot (and link) homology. To each link $L = \cup_{i=1}^{r} K_i$ they assign the collection of homology groups $\widehat{HFL}_d(L, \underline{h})$, where d is an integer and $\underline{h}$ belongs to some r-dimensional lattice. Their original description was based on the constructions from the symplectic topology, later ([12],[13]) there were elaborated combinatorial models for them. All of these homologies are invariants of the link L, and they have the following properties ([19], [13]).

First, they give a "categorification" of the Alexander polynomial of L: if $r = 1$, then

$$\sum_h \chi(\widehat{HFL}_*(L,h))t^h = \Delta^s(t),$$

where $\Delta^s(t) = t^{-\deg\Delta/2}\Delta(t)$ is a symmetrized Alexander polynomial of L. If $r > 1$, then

$$\sum_{\underline{h}} \chi(\widehat{HFL}_*(L,h))\underline{t}^h = \prod_{i=1}^{r}(t_i^{1/2} - t_i^{-1/2}) \cdot \Delta^s(t_1, \ldots, t_r).$$

Second, they have the symmetry extending the symmetry of the Alexander polynomial:

$$\widehat{HFL}_d(L,h) \cong \widehat{HFL}_{d-2H}(L,-h),$$

where $H = \sum_{i=1}^{r} h_i$.

These properties are similar to the ones of the polynomials $\overline{P}_g(t)$, and one could be interested in comparison of these objects. It turns out, that for knots (of course, $\overline{P}_g(t)$ is defined only for the algebraic ones) this comparison can be done.

In [22] for the relatively large class of knots, containing all algebraic knots, the following statement was proved.

Theorem 5 *([22]) Let the symmetrized Alexander polynomial have the form*

$$\Delta^s(t) = (-1)^k + \sum_{i=1}^{k}(-1)^{k-i}(t^{n_i} + t^{-n_i})$$

for some integers $0 < n_1 < n_2 < \ldots < n_k$. Let $n_{-j} = -n_j, n_0 = 0$. For $-k \le i \le k$ let us introduce the numbers δ_i by the formula

$$\delta_i = \begin{cases} 0, & \text{if } i=k \\ \delta_{i+1} - 2(n_{i+1} - n_i) + 1, & \text{if } k\text{-}i \text{ is odd} \\ \delta_{i+1} - 1, & \text{if } k\text{-}i > 0 \text{ is even.} \end{cases}$$

Then $\widehat{HFL}(K,j) = 0$, if j does not coincide with any n_i, and $\widehat{HFL}(K,n_i) = \mathbb{Z}$ belongs to the homological grading δ_i.

In what follows we will need more detailed algebraic structure of the Heegaard-Floer homology which can be described in the following way ([19]).

Consider the ring
$$R = \mathbb{Z}[U_1, \dots, U_r].$$
For every r-component link L there exists a $\mathbb{Z}^r$-filtered chain complex $CFL^-(S^3, L)$ of R-modules, whose filtered homotopy type is an invariant of the link L. Filtrations naturally correspond to the components of the link L. The operators U_i lowers the homological grading by 2 and the filtration level by $\underline{1}$. The homologies of the associated graded object are denoted as $HFL^-(S^3, L)$. If one sets $U_1 = U_2 = \ldots = U_r = 0$, he gets a new $\mathbb{Z}^r$-filtered chain complex of $\mathbb{Z}$-modules, which will be denoted as $\widehat{CFL}(L)$. The homology of the associated graded object are denoted as $\widehat{HFL}(L)$, and they are the homology discussed above.

The filtration on the second complex is compatible with the forgetting of components (proposition 7.1 in [19]). Namely, let M be the two-dimensional graded vector space with one generator in grading 0 and one in grading -1.

Proposition. Let L be an oriented, r-component link in S^3 and distinguish the first component K_1. Consider the complex $\widehat{CFL}(L)$ viewed as a $\mathbb{Z}^{n-1}$-filtered chain complex where the filtration corresponding to the first component is omitted. The filtered homotopy type of this complex is identified with $\widehat{CFL}(L - K_1) \otimes M$.

If we forget all components of L, we get either the complex
$$\hat{CF}(S^3) \otimes M^{r-1},$$
where $\hat{CF}(S^3)$ has one-dimensional homology in grading 0 or
$$CF^-(S^3) = \mathbb{Z}[U],$$
where all U_i acts by the multiplication by U.

This proposition is a direct analogue to the equation (8).

The three-manifolds with simplest Heegaard-Floer homology are the rational homology spheres Y, for which the rank of the Heegaard-Floer homology is equal to the order of the first (singular) homology, i.e.
$$\text{rk } \widehat{HF(Y)} = |H_1(Y; Z)|.$$
These manifolds are called L-spaces, for example, lens spaces are L-spaces. In the case that some positive surgery on K gives an L-space, we call K an L-space knot. It was proved by M. Hedden in [9] that all algebraic knots (i.e. links of irreducible plane curve singularities) belong to the class of L-space knots.

It was proved in [22], that for the L-space knot K and any filtration level n
$$\text{rk } H^*(CFL^-(K, n)/U_1(CFL^-(K, n))) = 1. \tag{24}$$
This is a key geometric ingredient in the proof of Theorem 5.

7.2 Matching the answers

Consider the Poincaré polynomial for the Heegaard-Floer homologies:

$$HFL(t,u)=\sum u^d t^s \dim \widehat{HFL}_{d,s}(K).$$

It categorifies the Alexander polynomial in the sense that

$$HFL(t,-1)=t^{-\deg\Delta/2}\Delta(t).$$

Remark that the coefficients in $\overline{P}_g(t,q)$ are always equal to 0 or to ±1. It can be proved from the equation (15).

Theorem 6 *Take $\overline{P}_g(t,q)$ and let us make a following change in it: $t^\alpha q^\beta$ is transformed to $t^\alpha u^{-2\beta}$, and $-t^\alpha q^\beta$ is transformed to $t^\alpha u^{1-2\beta}$. We get a polynomial $\widetilde{\Delta}_g(t,u)$. Then*

$$\widetilde{\Delta}_g(t^{-1},u)=t^{-\deg\Delta/2}HFL(t,u). \tag{25}$$

Example. For $(3,5)$ torus knot we have

$$P_g(t,q)=1+qt^3+q^2t^5+q^3t^6+\frac{q^4t^8}{1-qt},$$

$$\overline{P}_g(t,q)=1-qt+qt^3-q^2t^4+q^2t^5-q^4t^7+q^4t^8,$$

$$\widetilde{\Delta_g}(t,q)=1+u^{-1}t+u^{-2}t^3+u^{-3}t^4+u^{-4}t^5+u^{-7}t^7+u^{-8}t^8,$$

and

$$HFL(t,u)=t^4+u^{-1}t^3+u^{-2}t+u^{-3}t^0+u^{-4}t^{-1}+u^{-7}t^{-3}+u^{-8}t^{-4}.$$

Proof. To prove (25) we match Theorem 5 with the equation (15).

In the notation of Theorem 5 the non-symmetrized Alexander polynomial equals to

$$\Delta=\sum_{i=k}^{-k}(-1)^{k-i}t^{n_k-n_i}=\sum_{i=0}^{2k}(-1)^i t^{n_k-n_{k-i}},$$

$$P(t)=\frac{\Delta}{1-t}=\sum_{i=0}^{k-1}\sum_{j=n_k-n_{k-2i}}^{n_k-n_{k-2i-1}-1}t^j+\frac{t^{2n_k}}{1-t}.$$

Note that for $i>0$

$$\delta_{k-2i}=\delta_{k-2i+1}-1=\delta_{k-2(i-1)}-2(n_{k-2i+2}-n_{k-2i+1}),$$

so

$$P_g(t,q)=\sum_{i=0}^{k-1}\sum_{j=n_k-n_{k-2i}}^{n_k-n_{k-2i-1}-1}q^{(j-n_k+n_{k-2i})-\delta_{k-2i}/2}t^j+\frac{t^{2n_k}q^{n_k}}{1-qt},$$

$$\overline{P}_g(t,q)=\sum_{i=0}^{k-1}(q^{-\delta_{k-2i}/2}t^{n_k-n_{k-2i}}-q^{-\delta_{k-2i-1}/2}t^{n_k-n_{k-2i-1}})+t^{2n_k}q^{n_k}.$$

Now

$$\widetilde{\Delta_g}(t,u)=\sum_{i=0}^{k-1}(u^{\delta_{k-2i}}t^{n_k-n_{k-2i}}+u^{\delta_{k-2i-1}}t^{n_k-n_{k-2i-1}})+t^{2n_k}u^{-2n_k},$$

$$t^{n_k}\widetilde{\Delta_g}(t^{-1},u)=\sum_{i=0}^{k-1}(u^{\delta_{k-2i}}t^{n_{k-2i}}+q^{\delta_{k-2i-1}}t^{n_{k-2i-1}})+t^{2n_k}u^{-2n_k}=\sum_{i=-k}^{k}u^{\delta_i}t^{n_i}=HFL(t,u).$$

□

7.3 Comparing filtered complexes

In this section we try to describe the relation between the knot filtration on the Heegaard-Floer complexes and the filtration on the space of functions defined by a curve.

To be more close to the algebraic setup, we reverse all signs for filtrations and for the homological (Maslov) grading as well (so we get cohomology groups). The Alexander grading is also changed to get the non-symmetrized Alexander polynomial. In another words, the Poincaré polynomial of the resulting cohomology coincides with $\widetilde{\Delta_g}(t,u^{-1})$. The operator U will now increase the homological grading by 2.

Consider a $\mathbb{Z}_{\ge 0}$-indexed filtration J_n by vector subspaces (with finite codimensions) on a infinite-dimensional complex vector space J_0. It induces a filtration by projective subspaces $\mathbb{P}J_n$ on $\mathbb{P}J_0=\mathbb{CP}^\infty$:

$$\mathbb{P}J_0\overset{j_1}{\hookleftarrow}\mathbb{P}J_1\overset{j_2}{\hookleftarrow}\mathbb{P}J_2\overset{j_3}{\hookleftarrow}\ldots,$$

so we have a sequence of corresponding Gysin maps in cohomology:

$$H^*(\mathbb{P}J_0)\overset{(j_1)_*}{\hookleftarrow}H^{*-2\cdot\mathrm{codim}_{J_1}}\mathbb{P}J_1\overset{(j_2)_*}{\hookleftarrow}H^{*-2\cdot\mathrm{codim}_{J_2}}\mathbb{P}J_2\overset{(j_3)_*}{\hookleftarrow}\ldots.$$

We get a $\mathbb{Z}_{\ge 0}$-indexed filtration

$$F_k=(j_k)_*(H^*(\mathbb{P}J_k))$$

in $H^*(\mathbb{CP}^\infty)=\mathbb{Z}[U]$, which is compatible with the multiplication by U. If we also know (as for the filtration defined by the orders on the curve), that $\dim J_k/J_{k+1}\le 1$, we conclude that U increase the filtration level at least by 1.

The motivic Poincaré series in this setup can be written as

$$P_g(t,q)=\sum_{k,n}t^kq^{n/2}\dim H^n(F_k/F_{k+1}).$$

The situation is similar to the Heegaard-Floer complexes, but U may increase the filtration level more that by 1. To avoid this problem, we should modify the complex.

Example. Consider the following filtered complex T: it has generators U^ka_0, U^ka_1 and U^ka_2. The homological degree of U^la_j equals to $2l+j$ and its filtration level equals to $l+j$. The differential is defined as

$$d(a_1)=a_2+Ua_0.$$

One can check that

$$\sum_{k,n} t^k u^n \dim H^n(T_k/T_{k+1}) = 1 + u^2t^2 + u^4t^3 + u^6t^4 + \dots$$

(so this complex corresponds to minus-version of the Heegaard-Floer homology of the trefoil knot) and $\mathrm{rk} H^*(T_k/UT_k) = 1$ for all k. Remark that if $\widehat{T}^k = T_k/UT_{k-1}$, then

$$\sum_{k,n} t^k u^n \dim H^n(\widehat{T}_k/\widehat{T}_{k+1}) = 1 + ut + u^2t^2,$$

what is the Poincaré polynomial for the hat-version of the Heegaard-Floer homology of the trefoil.

Let us turn to the general case. Consider the complex

$$\mathcal{C}_0 = F_0[U_1] + (F_0[1])[U_1] \tag{26}$$

with the filtration

$$\mathcal{C}_n = \bigoplus_{k+l=n} U_1^l F_k \oplus \bigoplus_{k+l=n-1} U_1^l F_k[1]$$

and the natural action of the operator U_1 of homological degree 2. The differential is given by the equation

$$d(x) = U_1 \cdot x + Ux.$$

One can check that this differential preserves the filtration $\mathcal{C}_n$ and commutes with U_1.

Lemma 12

$$H^*(\mathcal{C}_n/\mathcal{C}_{n+1}) = F_n/F_{n+1}, \mathrm{rk} \quad H^*(\mathcal{C}_n/U_1(\mathcal{C}_n)) = 1.$$

Proof. We have

$$\mathcal{C}_n/\mathcal{C}_{n+1} = \bigoplus_{k+l=n} U_1^l(F_k/F_{k+1}) \oplus \bigoplus_{k+l=n-1} U_1^l(F_k/F_{k+1})[1].$$

Since the U_1-increasing component of the differential

$$d_1(U_1^l x[1]) = U_1^{l+1} x$$

gives the isomorphism

$$d_1 : U_1^l(F_k/F_{k+1}) \to U_1^{l+1}(F_k/F_{k+1}),$$

we have

$$H^*(\mathcal{C}_n/\mathcal{C}_{n+1}) = F_n/F_{n+1}.$$

Also we have

$$\mathcal{C}_n/U_1(\mathcal{C}_n) = F_0 \oplus F_0[1] \bigoplus_{k+l=n, l>0} U_1^l(F_k/F_{k+1}) \oplus \bigoplus_{k+l=n-1, l>0} U_1^l(F_k/F_{k+1})[1],$$

and up to the isomorphisms d_1 we have the complex $F_0 \oplus F_0[1]$ with the differential

$$d_2(x[1]) = Ux,$$

so

$$rk\ \ H^*(\mathcal{C}_n/U_1(\mathcal{C}_n)) = 1.$$

$\square$

The properties of the complex $\mathcal{C}_0$ are similar to the ones of the complex $CFL^-(K)$. More precisely, the calculations of [22] (lemma 3.1 and lemma 3.2) imply the following

Proposition. Suppose that a cochain complex $\mathcal{C}$ has a filtration $\mathcal{C}_k,\ \ k \geq 0$ and an injective operator U of homological degree 2 acting on it such that

1)$U(\mathcal{C}_k) \subset C_{k+1}$ and $U^{-1}(\mathcal{C}_k) \subset \mathcal{C}_{k-1}$ (this means that U increase the level of filtration exactly by 1)

2)$H^*(\mathcal{C}_k/U(\mathcal{C}_k))$ has rank 1 for all k.

Then

3) For all k the rank of $H^*(\mathcal{C}_k/\mathcal{C}_{k+1})$ is at most 1.

Let $\{0, \sigma_1, \sigma_2, \ldots\}$ is the set of k such that this rank is 1. Then

4) $H^*(\mathcal{C}_{\sigma_k}/\mathcal{C}_{\sigma_k+1})$ belongs to degree $2k$.

Let

$$Q(t,q) = \sum_{k=0}^{\infty} q^k t^{\sigma_k},\ \ \overline{Q}(t,q) = Q(t,q)(1-qt).$$

Let us make a following change in $\overline{Q}$: $t^\alpha q^\beta$ is transformed to $t^\alpha u^{2\beta}$, and $-t^\alpha q^\beta$ is transformed to $t^\alpha u^{2\beta-1}$.

5) The result is equal to

$$\sum_{k,n} t^k u^n \dim H^n(\mathcal{C}_k/(\mathcal{C}_{k+1} + U\mathcal{C}_{k-1})).$$

The second condition is analogous to the equation (24) for the Heegaard-Floer homology of the L-space knots.

The last result can be reformulated as follows. Consider the complex $\widehat{\mathcal{C}_k} = \mathcal{C}_k/U\mathcal{C}_{k-1}$, then the last homology is the homology of the associated graded object $\widehat{\mathcal{C}}_k/\widehat{\mathcal{C}}_{k-1}$. The multiplication by $1-qt$ corresponds to the exact sequence

$$0 \to \mathcal{C}_{k-1}/\mathcal{C}_k \xrightarrow{U} \mathcal{C}_k/\mathcal{C}_{k+1} \to \widehat{\mathcal{C}}_k/\widehat{\mathcal{C}}_{k+1} \to 0.$$

As a corollary we get that the series $Q(t,1)$ determines completely all discussed cohomology. Since for the filtered complexes $\mathcal{C}$ and CFL^- we have $Q(t,1) = \Delta(t)/(1-t)$ for both, we have the equality of the cohomology of the associated graded objects and the more clear proof of the Theorem 6. As an another corollary, we get the equation

$$H^*(CFL^-(S^3)/CFL_s^-(S^3,K)) \cong H^*(\mathbb{P}(\mathcal{O}/J_s)), \tag{27}$$

which looks more geometric than the Theorem 6.

Remarks.

1. It would be interesting to construct the analogous $\mathbb{Z}^n$-filtered complex of $\mathbb{Z}[U_1,\ldots,U_n]$ for multi-component links which would carry the information about the Poincaré series of the corresponding multi-index filtration.

2. It would be also interesting to compare these results with the ones of [15], [16] and [17] computing the Seiberg-Witten and Heegaard-Floer invariants of links of surface singularities.

7.4 Example: A_{2n-1} singularities

Since the algorithm of computation of the (reduced) motivic Poincaré series is quite complicated, it is useful to have a series of answers where the motivic Poincaré series and the Heegaard-Floer link homology can be computed.

Proposition. Consider the singularity of type A_{2n-1} given by the equation

$$y^2 = x^{2n}.$$

From the topological viewpoint this corresponds to the 2-component link, whose components are unknotted, all intersections are positive and the linking number of the components equals to n. Then

$$P_g(t_1,t_2) = 1 + qt_1t_2 + \ldots + q^{n-1}t_1^{n-1}t_2^{n-1} + \frac{q^n(1-q)t_1^nt_2^n}{(1-t_1q)(1-t_2q)}.$$

Proof. For the proof we use the equation (13). Parametrisations of the components are

$$(x(t_1),y(t_1)) = (t_1,t_1^n), \quad \text{and} \quad (x(t_2),y(t_2)) = (t_2,-t_2^n),$$

so

$$x^ay^b|_{C_1} = t_1^{a+bn}, \quad x^ay^b|_{C_2} = (-1)^bt_2^{a+bn}.$$

If $a<n$, then every function with order a on C_1 has a form $x^a+\ldots$, so its order on C_2 is also equal to a.

For every $a,b \geq n$ consider the function $x^{a-n}(x^n+y)+x^{b-n}(x^n-y)$. Its restrictions on C_1 and C_2 are respectively equal to $2t_1^a$ and $2t_2^b$, therefore

$$\dim J_{a,b}/J_{a+1,b} = \dim J_{a,b}/J_{a,b+1} = 1.$$

The codimensions $h(v_1,v_2)$ are equal to v_1+v_2-n, if $v_1,v_2\geq n$, to v_2, if $v_1<n, v_2\geq n$, to v_1, if $v_2<n, v_1\geq n$, and to $\max(v_1,v_2)$, if $0\leq v_1,v_2<n$. We have

$$L_g^{A_{2n-1}}(t_1,t_2,q) = \sum_{0\leq \max(v_1,v_2);\min(v_1,v_2)<n} t_1^{v_1}t_2^{v_2}q^{\max(v_1,v_2)} + (1+q)\sum_{v_1,v_2=n}^{\infty} t_1^{v_1}t_2^{v_2}q^{v_1+v_2-n},$$

hence

$$L_g^{A_{2n-1}}(t_1-1)(t_2-1) = -1+(1-q)t_1t_2+\ldots+(q^{n-2}-q^{n-1})t_1^{n-1}t_2^{n-1}+q^{n-1}(1-q+q^2)t_1^nt_2^n$$
$$+\frac{q^{n+1}t_1^{n+1}t_2^n(q-1)}{1-qt_1}+\frac{q^{n+1}t_1^nt_2^{n+1}(q-1)}{1-qt_2}+\frac{q^nt_1^{n+1}t_2^{n+1}(1+q)(1-q)^2}{(1-qt_1)(1-qt_2)},$$

and

$$P_g^{A_{2n-1}} = \frac{L_g^{A_{2n-1}}(t_1-1)(t_2-1)}{t_1t_2-1} = 1+qt_1t_2+\ldots+q^{n-1}t_1^{n-1}t_2^{n-1}+\frac{q^n(1-q)t_1^nt_2^n}{(1-qt_1)(1-qt_2)}.$$

□

Corollary 3

$$\overline{P}_g^{A_{2n-1}}(t_1,t_2) = [1+(q+q^2)t_1t_2+\ldots+(q^{n-1}+q^n)t_1^{n-1}t_2^{n-1}+q^nt_1^nt_2^n] \tag{28}$$
$$-(t_1+t_2)[q+q^2t_1t_2+\ldots+q^nt_1^{n-1}t_2^{n-1}].$$

In [19] Ozsváth and Szabó computed the Heegaard-Floer homology of the corresponding links. In their notation the answer has the following form (everywhere we write the Poincaré polynomials of the corresponding complexes). Let

$$Y^l_{(d)}(t_1,t_2,u) = u^d(t_1^l+t_1^{l-1}t_2+\ldots+t_2^l)+u^{d-1}(t_1^{l-1}+\ldots+t_2^{l-1}),$$
$$B_{(d)}(t_1,t_2,u) = u^d+(t_1+t_2)u^{d+1}+u^{d+2}t_1t_2.$$

Then

$$HFL_{A_{2n-1}}(t_1,t_2,u) = Y^0_{(0)}t_1^{n/2}t_2^{n/2}+Y^1_{(-1)}t_1^{n/2-1}t_2^{n/2-1}+\sum_{i=2}^{n}B_{(-2i)}t_1^{n/2-i}t_2^{n/2-i}.$$

Since $Y^0_{(0)}=1$ and $Y^1_{(-1)}=u^{-1}(t_1+t_2)+u^{-2}$ one can simplify this as

$$HFL_{A_{2n-1}}(t_1,t_2,u) = t_1^{n/2}t_2^{n/2}+(u^{-1}(t_1+t_2)+u^{-2})t_1^{n/2-1}t_2^{n/2-1}$$
$$+\sum_{i=2}^{n}(u^{-2i}+(t_1+t_2)u^{-2i+1}+u^{-2i+2}t_1t_2)t_1^{n/2-i}t_2^{n/2-i},$$

so

$$t_1^{n/2}t_2^{n/2}HFL_{A_{2n-1}}(t_1^{-1},t_2^{-1},u) = 1+(u^{-1}(t_1+t_2)+u^{-2}t_1t_2)$$
$$+\sum_{i=2}^{n}(u^{-2i}t_1^it_2^i+(t_1+t_2)u^{-2i+1}t_1^{i-1}t_2^{i-1}+u^{-2i+2}t_1^{i-1}t_2^{i-1}) =$$
$$[1+2u^{-2}t_1t_2+\ldots+2u^{-2n+2}t_1^{n-1}t_2^{n-1}+u^{-2n}t_1^nt_2^n]$$
$$-(t_1+t_2)[u^{-1}+u^{-3}t_1t_2+\ldots+u^{-2n+1}t_1^{n-1}t_2^{n-1}].$$

The last expression is similar to (28) in analogy with the Theorem 6.

8 Appendix

Proof of Lemma 4.
We have

$$\sum u^{\hat{n}_i}\phi_i(I,K,\hat{n}) = \sum_j \sum_{\hat{n}=j+f_i(K,I)}^{\infty} u^{\hat{n}_i}(-1)^j \binom{1-\chi(E_i^\circ)-f_i(I,K)}{j} q^j =$$

$$\frac{u^{f_i(K,I)}}{1-u}\sum_j (-1)^j \binom{1-\chi(E_i^\circ)-f_i(I,K)}{j}(uq)^j = \frac{u^{f_i(K,I)}}{1-u}(1-uq)^{1-\chi(E_i^\circ)-f_i(I,K)},$$

and

$$\sum u^{\hat{n}} G(K,I,\hat{n}) = q^{|I|}(1-q)^{|I|+|K|}\prod_i \frac{u_i^{f_i(K,I)}}{1-u_i}(1-u_i q)^{1-\chi(E_i^\circ)-f_i(I,K)}.$$

□

Proof of Lemma 5

$$A_K(u) = \sum_I q^{|I|}(1-q)^{|I|}\sum_{K_1}(-1)^{|K|-|K_1|}(1-q)^{|K_1|}\sum_n u^n \prod_i \phi_i(I,K_1,n).$$

We have

$$\sum_n u^n \prod_i \phi_i(I,K_1,n) = \prod_i \frac{u_i^{f_i(K,I)}(1-u_i q)^{1-\chi(E_i^\circ)-f_i(I,K)}}{1-u_i}.$$

Now

$$\sum_{K_{1i}\subset(K\cap E_i)} (-1)^{|K\cap E_i|-|K_{i1}|}(1-q)^{|K_{1i}|}\frac{1}{1-u_i}u_i^{f_i(K_1,I)}(1-u_i q)^{1-\chi(E_i^\circ)-f_i(I,K_1)} =$$

$$\frac{1}{1-u_i}u_i^{f_i(K,I)}(1-u_i q)^{1-\chi(E_i^\circ)-f_i(K,I)}\times$$

$$\sum_{K_{1i}}(-1)^{|K\cap E_i|-|K_{1i}|}(1-q)^{|K_{1i}|}u_i^{|K_{1i}|-|K\cap E_i|}(1-u_i q)^{|K\cap E_i|-|K_{1i}|} =$$

$$\frac{1}{1-u_i}u_i^{f_i(K,I)}(1-u_i q)^{1-\chi(E_i^\circ)-f_i(K,I)}(1-q-\frac{1-u_i q}{u_i})^{|K\cap E_i|} =$$

$$\frac{1}{1-u_i}(-1)^{|K\cap E_i|}u_i^{f_i(K,I)-|K\cap E_i|}(1-u_i q)^{1-\chi(E_i^\circ)-f_i(K,I)}(1-u_i)^{|K\cap E_i|}.$$

Remark that $f_i(K,I)-|K\cap E_i| = f_i(I)$ and

$$\chi(E_i^\circ)+f_i(K,I) = \chi(E_i^\bullet)-|K_0\cap E_i|+|K\cap E_i|+f_i(I),$$

hence the last expression can be rewritten in a form

$$(-1)^{|K\cap E_i|}u_i^{f_i(I)}(1-u_i q)^{1-\chi(E_i^\bullet)+|\overline{K}\cap E_i|-f_i(I)}(1-u_i)^{|K\cap E_i|-1}.$$

Also

$$\sum_I q^{|I|}(1-q)^{|I|}\prod_i u_i^{f_i(I)}(1-u_iq)^{-f_i(I)}=\prod_\sigma(1+q(1-q)u_{i(\sigma)}u_{j(\sigma)}(1-u_{i(\sigma)}q)^{-1}(1-u_{j(\sigma)}q)^{-1})=$$

$$\prod_i(1-u_iq)^{\chi(E_i^\bullet)-2}\prod_\sigma(1-qu_{i(\sigma)}-qu_{j(\sigma)}+qu_{i(\sigma)}u_{j(\sigma)}).$$

Therefore

$$A_K(u)=(-1)^{|K|}\prod_i(1-u_iq)^{1-\chi(E_i^\bullet)+|\overline{K}\cap E_i|}(1-u_i)^{|K\cap E_i|-1}\times$$

$$\times\prod_i(1-u_iq)^{\chi(E_i^\bullet)-2}\prod_\sigma(1-qu_{i(\sigma)}-qu_{j(\sigma)}+qu_{i(\sigma)}u_{j(\sigma)})=$$

$$(-1)^{|K|}\prod_i(1-u_iq)^{|\overline{K}\cap E_i|-1}(1-u_i)^{|K\cap E_i|-1}\prod_\sigma(1-qu_{i(\sigma)}-qu_{j(\sigma)}+qu_{i(\sigma)}u_{j(\sigma)}).$$

□

Proof of Lemma 7

We have to prove that $\widetilde{H}_P=0$ at $u_\beta=1$ for $\beta\in E(P)$. Suppose that E_β is intersected by $E_{\alpha_1},\ldots,E_{\alpha_k}$. For every set E of divisors not containing E_β let us compare the summands corresponding to E and to $E\cup E_\beta$.

For E at $u_\beta=1$ we have

$$\prod_{i\neq\beta}u_i^{-\sum a_{ij}\mu_j}(-1)^{|K_0\cap E|}q^{\Delta(E)}\prod_{i\in E}(q-u_i)^{k_i-1}(1-q)^{k_\beta-1}\prod_{i\notin(P\cup E)}(1-qu_i)^{k_i-1}$$

$$\times\prod_{\sigma\notin E_\beta}(1-q^{1-\mu_{i(\sigma)}(E)}u_{i(\sigma)}-q^{1-\mu_{j(\sigma)}(E)}u_{j(\sigma)}+q^{1-\mu_{i(\sigma)}(E)-\mu_{j(\sigma)}(E)}u_{i(\sigma)}u_{j(\sigma)})\cdot(1-q)^k.$$

For $E\cup E_1$ at $u_\beta=1$ we have

$$\prod_{j=1}^k u_{\alpha_j}\prod_{i\neq\beta}u_i^{-\sum a_{ij}\mu_j}(-1)^{k_\beta+|K_0\cap E|}q^{\Delta(E\cup E_1)}(q-1)^{k_\beta-1}\prod_{i\in E}(q-u_i)^{k_i-1}\prod_{i\notin(E\cup P)}(1-qu_i)^{k_i-1}$$

$$\times\prod_{\sigma\notin E_\beta}(1-q^{1-\mu_{i(\sigma)}(E)}u_{i(\sigma)}-q^{1-\mu_{j(\sigma)}(E)}u_{j(\sigma)}+q^{1-\mu_{i(\sigma)}(E)-\mu_{j(\sigma)}(E)}u_{i(\sigma)}u_{j(\sigma)})\cdot\prod_{j=1}^k(1-q)q^{-\mu_{\alpha_j}(E)}u_{\alpha_j}.$$

It rests to note that $\Delta(E\cup E_\beta)-\Delta(E)=\sum_{j=1}^k\mu_{\alpha_j}(E)$.

□

Proof of Lemma 8.

$$\sum_n u^n\sum_{E\subset E(P)}q^{-\sum_{i\in E}n_i-\Delta(E)-\sum_{i\in E}a_{ii}-|E|}q^{|K_0\cap E|}\times c_{P\cup E}(n_i+\sum a_{ij}\mu_j(E))=$$

$$\sum_{E\subset E(P)}\prod u_i^{-\sum a_{ij}\mu_j(E)}\cdot q^{\sum a_{ij}\mu_i(E)\mu_j(E)}\cdot q^{-\Delta(E)-\sum_{i\in I}a_{ii}+|K_0\cap E|-|E|}$$

$$\times \sum_{n_1} \prod_i (u_i q^{-\mu_i(E)})^{n_{1i}} \cdot c_{P\cup E}(n_1) =$$

$$\sum_{E\subset E(P)} \prod u_i^{-\sum a_{ij}\mu_j(E)} \cdot A_{P\cup E}(u_i q^{-\mu_i(E)}) q^{\Delta(E)+|K_0\cap E|-|E|} =$$

$$(-1)^{|P|} \sum_{E\subset E(P)} \prod u_i^{-\sum a_{ij}\mu_j(E)} \cdot (-1)^{|K_0\cap E|} q^{\Delta(E)+|K_0\cap E|-|E|} \prod_{i\in E} [(1-u_i)^{-1}(1-u_i q^{-1})^{k_i-1}]$$

$$\times \prod_{i\in P} [(1-qu_i)^{k_i-p_i-1}(1-u_i)^{p_i-1}] \prod_{i\notin(P\cup E)} [(1-qu_i)^{k_i-1}(1-u_i)^{-1}]$$

$$\times \prod_\sigma (1-q^{1-\mu_{i(\sigma)}(E)} u_{i(\sigma)} - q^{1-\mu_{j(\sigma)}(E)} u_{j(\sigma)} + q^{1-\mu_{i(\sigma)}(E)-\mu_{j(\sigma)}(E)} u_{i(\sigma)} u_{j(\sigma)}) =$$

$$(-1)^{|P|} \prod_{i\in P} [(1-qu_i)^{k_i-p_i-1}(1-u_i)^{p_i-1}] \cdot \frac{1}{\prod_{i\in E(P)}(1-u_i)}$$

$$\times \sum_{E\subset E(P)} (-1)^{|K_0\cap E|} \cdot \prod u_i^{-\sum a_{ij}\mu_j(E)} \cdot q^{\Delta(E)} \prod_{i\in E} (q-u_i)^{k_i-1} \prod_{i\notin E} (1-qu_i)^{k_i-1}$$

$$\times \prod_\sigma (1-q^{1-\mu_{i(\sigma)}(E)} u_{i(\sigma)} - q^{1-\mu_{j(\sigma)}(E)} u_{j(\sigma)} + q^{1-\mu_{i(\sigma)}(E)-\mu_{j(\sigma)}(E)} u_{i(\sigma)} u_{j(\sigma)}).$$

□

Proof of Theorem 4.
Let $k_i = |K_0 \cap E_i|$. From Lemma 6 we get

$$\overline{P}_g(\frac{1}{qt_1}, \ldots, \frac{1}{qt_r}) = (t_1 \cdot \ldots \cdot t_r)^{-1} \sum_{\underline{n}} \underline{t}^{-M\underline{n}} q^{-\sum m_{ij}k_i n_j} q^{F(n)-\sum n_i} \sum_K t_{\overline{K}} c_K(n) =$$

$$\underline{t}^{-\underline{1}-M\chi(\underline{E}^\circ)} \sum_{\underline{n}} \underline{t}^{M(\chi(\underline{E}^\circ)-\underline{n})} q^{-\sum m_{ij}k_i n_j} q^{F(n)-\sum n_i}$$

$$\times \sum_K q^{1-|K|+n} \cdot t_{\overline{K}} \cdot c_{\overline{K}}(-\chi(E_i^\circ) - n_i). \tag{29}$$

Let

$$\xi_i = -\chi(E_i^\circ), \quad \underline{n}_1 = \underline{\xi} - \underline{n}.$$

Then

$$F(n) - \sum n_i = \frac{1}{2}[\sum m_{ij}n_i n_j + \sum m_{ij} n_i \chi(E_j^\bullet) - \sum n_i],$$

so

$$2[F(n_1) - \sum n_{1i} - F(n) + \sum n_i] =$$

$$\sum m_{ij}(\xi_i - n_i)(\xi_j - n_j) + \sum m_{ij}(\xi_i - n_i)\chi(E_j^\bullet) - \sum(\xi_i - n_i)$$

$$-\sum m_{ij}n_i n_j - \sum m_{ij} n_i \chi(E_j^\bullet) + \sum n_i =$$

$$-2\sum m_{ij}(\xi_i + \chi(E_i^\bullet))n_j + 2\sum n_j + 2(F(\xi) - \sum \xi_i) =$$

$$-2\sum m_{ij}k_in_j + 2\sum n_j + 2(F(\xi) - \sum \xi_i).$$

Thus (29) is equal to

$$t^{-1-M\xi}q^{-F(\xi)+\sum\xi_i}q^{1-|K_0|}\sum t^{Mn_1}q^{F(n_1)-\sum n_{1i}}\sum_K t_{\overline{K}}q^{|\overline{K}|}c_{\overline{K}}(n_1).$$

It rests to compute the powers of t_α and of q.

Remark that $\sum \xi_i = |K_0| - 2$, so $\sum \xi_i + 1 - |K_0| = -1$.

Also

$$2F(\xi) = \sum m_{ij}k_ik_j - 2\sum m_{ij}k_i\chi(E_j^\bullet) + \sum m_{ij}\chi(E_i^\bullet)\chi(E_j^\bullet) +$$
$$\sum m_{ij}k_i\chi(E_j^\bullet) - \sum m_{ij}\chi(E_i^\bullet)\chi(E_j^\bullet) + \sum \xi_i =$$
$$\sum m_{ij}k_ik_j - \sum m_{ij}k_i\chi(E_j^\bullet) + |K_0| - 2.$$

The formula of A'Campo ([1]) says that

$$1-\mu = \sum m\chi(S_m) = \sum \chi(E_i^\circ)m_{ij}k_j = \sum m_{ij}(\chi(E_i^\bullet) - k_i)k_j,$$

so

$$2F(\xi) = \mu - 1 + |K_0| - 2 = 2\delta - 2.$$

Thus $-F(\xi) - 1 = -\delta$.

Also for every α one has

$$1 - \mu_\alpha = \sum_{j\neq i(\alpha)} m_{i(\alpha)j}\chi(E_j^\bullet) + m_{i(\alpha),i(\alpha)}(\chi(E_{i(\alpha)}^\bullet) - 1),$$

and for $\beta \neq \alpha$

$$C_\alpha \circ C_\beta = m_{i(\alpha),i(\beta)},$$

so

$$\sum_{\beta\neq\alpha} C_\alpha \circ C_\beta = \sum_{j\neq i(\alpha)} m_{i(\alpha),j}k_j + m_{i(\alpha),i(\alpha)}(k_{i(\alpha)} - 1)$$

and

$$1 - \mu_\alpha - C_\alpha \circ C_\beta = \sum_j m_{i(\alpha),j}\chi(E_j^\circ).$$

□

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Linear components of the tangent cone in the Nash modification of a complex surface singularity

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Abstract

We prove that each linear component of the tangent cone of a complex surface singularity corresponds to at least one singular point in the normalized Nash modification, whenever the minimal resolution factors through the blow-up of the origin of the germ. We give an example of a surface whose tangent cone has no linear component and the normalized Nash modification is singular.

1 Introduction

Let (X, x) be a germ of equidimensional complex analytic singularity. The Nash modification of (X, x) is a modification that consists in replacing each singular point of a representative of the germ by all the possible limits of directions of tangent spaces. More precisely, let $X \subset \mathbb{C}^n$ be a representative of $(X, 0)$, and suppose it has pure dimension d. Call $\mathrm{Sing}(X)$ the singular locus of X. We define the map:

$$\begin{array}{rrcl} \lambda : & X \setminus \mathrm{Sing}(X) & \rightarrow & \mathbf{G}(d,n) \\ & y & \mapsto & T_yX \end{array}$$

where $\mathbf{G}(d, n)$ is the Grassmannian of d-dimensional vector space in $\mathbb{C}^n$ and T_yX is the direction of tangent space to X at y. The closure of the graph of λ in $X \times \mathbf{G}(d, n)$ is a complex analytic space of dimension d. Call it $\tilde{X}$ and endow it with the restriction ν of the projection on the first factor. The datum $\nu : \tilde{X} \to X$ is the Nash modification of X.

If y is a singular point of X, the fibre $\nu^{-1}(y)$ consists of the vector spaces T obtained as limits of directions of tangent spaces $T_{y_n}X$ of a sequence of non singular points of X converging to y. We will simply call it a limit of tangent spaces to X at y. The limits of tangent spaces to analytic varieties have been studied in the 70's and 80's by D.T Lê, B. Teissier, J.P.G. Henry, M. Merle, T. Gaffney and others, see [8, 6, 5, 2].

We will focus in this work on the 2-dimension case. Let (S, s) be a complex surface singularity. We will denote by $C_{S,s}$ its tangent cone. We recall that this is the algebraic variety defined by the ideal generated by all the initial forms at s of the holomorphic functions of the ideal defining the surface S. The tangent cone is strongly related to the limits of tangent spaces to the surface. This relation is established and explained in [7].

In particular, any two-dimensional plane, tangent to the tangent cone is a limit of tangent spaces to the surface at s. So whenever $C_{S,s}$ has a linear two-dimensional plane as an irreducible component, this plane will correspond to a limit of tangent spaces to S at s. These linear components will be called the planar components of the tangent cone.

The Nash modification of a surface singularity is not necessarily normal, not even when the original surface is normal. Call $n : \bar{S} \to \tilde{S}$ the normalization of the surface obtained by Nash modification. The composition map $\nu \circ n : \bar{S} \to S$ will be called the normalized Nash modification of the surface singularity (S, s).

The normalization map being finite, each point in the exceptional fiber of the Nash modification $\tilde{S}$ has finitely many pre-images in the normalized Nash modification. In particular, since each planar component of the tangent cone corresponds to a point in the exceptional fiber of ν, then each of these components corresponds to finitely many points in the normalized Nash modified surface $\bar{S}$.

In terms of this last correspondence, our main result states that whenever (S, s) is a surface singularity for which the minimal resolution factors through the blow-up of the point s, then any planar component of the tangent cone corresponds to at least a singular point in the normalized Nash modification of (S, s).

Then we give an example showing that the converse is false. Namely we exhibit a minimal singularity whose tangent cone has no planar component and the normalized Nash modification has two singular points.

This example shows some limit of the analogy between the pairs (Nash modification, polar curves) and (point blow-up, hyperplane sections) for normal surfaces. Indeed, the singular points of the normalized blow-up of the origin of a normal germ of a surface are fixed points of the family of polar curves. Meanwhile a singular point of the normalized Nash modification of a normal germ of surface need not be a fixed point of the family of hyperplane sections ; see [10, 11].

2 The result

Let (S, s) be a germ of complex surface singularity and let $f_1, \dots, f_r$ be holomorphic functions on it. For each $\alpha \in \mathbb{P}^{r-1}$ we define the curve $\mathcal{C}_\alpha$ to be the zero set on (S, s) of a linear combination $\Sigma_{i=1}^r a_i f_i$, where $(a_1 : \dots : a_r)$ are homogeneous co-ordinates of α. The family of curves $(\mathcal{C}_\alpha)_\alpha$ is the linear system of curves generated by $f_1, \dots, f_r$ and parametrised by $\mathbb{P}^{r-1}$.

Consider now a modification $\mu : X \to (S, s)$.

Definition 2.1. *A point $\eta \in \mu^{-1}(s)$ is called a fixed point, or a base point, of the family of curves $(\mathcal{C}_\alpha)_\alpha$ if there exists an open set $U \subset \mathbb{P}^{r-1}$, such that for any $\alpha \in U$ the strict transform of the curve $\mathcal{C}_\alpha$ by the modification μ contains the point η.*

In [11, Thms 3.2 and 4.2], we proved the following:

Proposition 2.2. *Let (S, s) be a reduced equidimensional germ of complex surface singularity. The normalized Nash modification of (S, s) factors through the blow-up of the point s in S if and only if the tangent cone $C_{S,s}$ does not have any planar component.*

Moreover, the planar components of the tangent cone correspond exactly to the fixed points of the linear system of hyperplane sections on S at s in the Nash modification.

The aim of this short note is to show how the planar components of the tangent cone contribute in the singularities of the surface obtained by Nash modification.

Theorem 2.3. *Let (S, s) be a germ of reduced and equidimensional complex surface singularity. Suppose that the minimal resolution of (S, s) factors through the blow-up of the point s. Then to any planar component of the tangent cone $C_{S,s}$ corresponds at least one singular point in the surface obtained by the normalized Nash modification.*

Proof:

Let P be a planar component of the tangent cone $C_{S,s}$. It is a limit of tangent planes to the surface at s. So it corresponds to a point η in the exceptional fibre of the Nash Modification. Call $\eta_1, \ldots, \eta_d$ the inverse image of η by the normalization map.

Let us consider the following commutative diagram:

$$\begin{tikzcd}
\bar{X} \arrow[r, "\tau"] \arrow[d, "\pi"'] & X \arrow[d, "\rho"] \\
\bar{S} \arrow[d, "n"'] & S' \arrow[d, "e"] \\
\tilde{S} \arrow[r, "\nu"] & S
\end{tikzcd}$$

where $n : \bar{S} \to \tilde{S}$ is the normalization of the Nash modified surface, the map $\pi : \bar{X} \to \bar{S}$ is the minimal resolution of the singularities of $\bar{S}$, $e : S' \to S$ is the blow-up of the point s and $\rho : X \to S'$ is the minimal resolution of the singularities of S'.

Since the composition $\nu \circ n \circ \pi$ is a resolution of S it factors through the minimal resolution of S which coincides by hypothesis with the minimal resolution of S'. Let us call $\tau : \bar{X} \to X$ the factorisation map.

By proposition 2.2, η is a fixed point of the linear system of hyperplane sections of S at s in the surface $\tilde{S}$. Since the normalization map is finite, at least one of the points $\eta_i \in \bar{S}$ is still a fixed point of the hyperplane sections. Suppose η_1 is. And suppose it is not a singular point of the surface $\bar{S}$. Then, the minimal resolution π induces an isomorphism over a neighborhood of η_1. So the inverse image $\pi^{-1}(\eta_1)$ is again a fixed point of the hyperplane sections of S at s in the resolution $\bar{X}$ and hence in the minimal resolution X and in the blow-up of the origin S'.

The universal property of the blowing up asserts that the linear system of hyperplane sections does not have any fixed point in the blown-up surface. So this contradicts the assumption of η_1 being a non singular point of $\bar{S}$.

So at least one of the $\eta_i \in \bar{S}$ is a singular point. □

Remark 2.4. *There exists a class of normal two-dimensional singularities, called rational surface singularities having the property that any resolution factors through the blow-up of the singularity. For more details on rational surface singularities see for example [1] and [9]. So our hypothesis in theorem 2.3 is satisfied by a non trivial class of surface singularities. However it would be interesting to see if the conclusion of theorem 2.3 is valid for a general complex reduced purely two-dimensional singularity.*

Example 2.5. *The A_2 singularity given by the equation $x^2+y^2+z^3$ is a rational double point singularity. Its tangent cone is a union of two planes. The Nash modification of this surface was studied by G. Gonzalez-Sprinberg in [4]. There it is shown that the Nash modification of this surface has exactly two singular points corresponding precisely to the planes of the tangent cone at the origin.*

3 The converse is false

Consider a normal two-dimensional singularity $(S,0)$ having the diagram of figure 1 as dual graph of its minimal resolution $\pi : X \to S$.

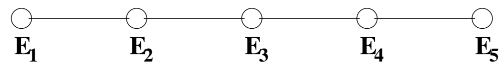

Figure 1: Dual graph of the minimal resolution

Where all the irreducible components E_i, $1 \leq i \leq 5$, of the exceptional divisor are rational smooth curves intersecting transversally. The self intersections are given as follows: $E_1^2 = E_5^2 = -3$ and $E_2^2 = E_3^2 = E_4^2 = -2$. This singularity is rational, and actually even minimal, i.e. the fundamental cycle (or equivalently in this case, the maximal ideal cycle) in the minimal resolution is $Z = \Sigma_{i=1}^5 E_i$. A surface with minimal singularity has also the property that its tangent cone is reduced.

It is well known that the surface S' obtained by the contraction of the irreducible components of the exceptional divisor satisfying the property $Z \cdot E_i = 0$ (known as Tyurina components) is isomorphic to the surface obtained by the blow-up of the origin in S. The image under this contraction of the irreducible components of the exceptional fibre such that $Z \cdot E_i < 0$ (the non-Tyurina components) is precisely the projective tangent cone of the surface $(S,0)$.

So in this example, the projective tangent cone has two (reduced) and irreducible components obtained by the images of E_1 and E_5 in the contraction of E_2, E_3 and E_4. Moreover the degree of each of these components is given by the (positive) number $-Z \cdot E_i$. So the projective curve associated to the tangent cone is the union of two irreducible and reduced curves of degree two, intersecting in one point. This intersection point is the only singular point of the surface S'; it is an A_3 singularity. In particular, the tangent cone $C_{S,0}$ has no planar component.

A normalized modification of a normal surface singularity factors through the Nash modification if and only if the family of (local and absolute) polar curves of the surface at the singular point have no fixed point in the normalized modification ([12, Thm. III.1.2]). M. Spivakovsky studied extensively Nash modification on normal surfaces, generalizing a previous work by G. González-Sprinberg in [3]. In particular he gave a precise characterization of the fixed points of the polar curves in the minimal resolution of a minimal surface singularity, and determined the components of the exceptional fibre that intersect a general polar curve; see [12, Thm.III 5.4.]. Let us state this result:

Consider $\pi : X \to S$ the minimal resolution of a minimal surface singularity. Call

$E = \bigcup_i E_i$ the decomposition of the exceptional fiber into its irreducible components and Γ the dual graph associated to it. Let V_i be the vertex of Γ corresponding to E_i. The cycle $Z = \Sigma_i E_i$ is in the case the so called fundamental cycle. It satisfies $Z \cdot E_i \leq 0$ for all i. Let V_{i_0} be a vertex such that $Z \cdot E_{i_0} = 0$. The Tyurina component of Γ containing V_{i_0} is the connected component of $\Gamma \setminus \{V_i,\ Z \cdot E_i < 0\}$ that contains V_{i_0}. We define the integer s_i associated to V_i as follows:

if $Z \cdot E_i < 0$ then $s_i = 1$

if $Z \cdot E_i = 0$ call Δ the Tyurina component of Γ containing V_i. In this case the value s_i is the minimal number of edges between V_i and $\Gamma \setminus \Delta$ plus 1.

Let V_i and V_j be two adjacent vertices in Γ. The edge joining both vertices is called a central edge if $s_i = s_j$.

A vertex V_i is called a central vertex if there exist at least two vertices V_j and V_k adjacent to V_i such that $V_j = V_k = V_i - 1$.

The criterion established by M. Spivakovsky in [12, Thm.III 5.4.] says the following:

The fixed points of the polar curves in the minimal resolution are precisely the points of X corresponding to the central edges of the graph Γ. Away from these points, the general polar curve intersects a component E_i if and only if V_i is a central vertex or $Z \cdot E_i \leq -2$.

Applying this criterion to our example, we obtain that:

- The dual graph in figure 1 has no central edge, and hence the minimal resolution has no fixed point of the polar curves. So the minimal resolution $\pi : X \to S$ factors through the Nash modification, and hence it is also the minimal resolution of the normalization $\bar{S}$ of the Nash modification $\tilde{S}$ of S.

- The general polar curve intersects exactly the irreducible components E_1, E_3 and E_5. So the surface obtained by the contraction of the irreducible curves E_2 and E_4 is the normalized Nash modified surface $\bar{S}$.

The surface $\bar{S}$ has then two A_1 singularities.

This examples shows that it is possible the have a rational surface singularity whose tangent cone has no planar component and whose normalized Nash modification has singularities.

Remark 3.1. *In [10], we proved that a singular point of the normalized blow-up of the origin of a germ of a normal surface singularity is always a base point of the linear system of polar curves. We used to think that the similar behavior of the hyperplane sections in the blow-up of the origin with the polar curves in the Nash modification of normal surfaces would suggest that all singular points of the normalized Nash modification of a normal surface are fixed points of the family of hyperplane sections. However this example shows that it is not always the case.*

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A COMPLETE CHARACTERIZATION OF $\mathcal{A}_0$-SUFFICIENCY OF PLANE-TO-PLANE JETS OF RANK 1

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Abstract. Sufficient conditions for $\mathcal{A}_0$-sufficiency of plane-to-plane r-jets are known. These conditions are stated in the form of two Łojasiewicz inequalities which have to be satisfied. The first of these inequalities is known to be necessary for $\mathcal{A}_0$-sufficiency, and in this article we prove that the second inequality is also necessary for $\mathcal{A}_0$-sufficiency of all jets of rank 1. We also prove that a simpler Łojasiewicz inequality is equivalent to the second inequality for rank 1 jets.

1. Introduction

Let $\mathcal{E}_{[r]}(n,p)$ be the set of C^r map germs $(\mathbb{R}^n,0) \to (\mathbb{R}^p,0)$. Two map germs f and g in $\mathcal{E}_{[r]}(n,p)$ are $\mathcal{A}_s$-equivalent if there exist germs of C^s diffeomorphisms $h : (\mathbb{R}^n,0) \to (\mathbb{R}^n,0)$ and $k : (\mathbb{R}^p,0) \to (\mathbb{R}^p,0)$ such that $g = k \circ f \circ h^{-1}$. If $f,g \in \mathcal{E}_{[r]}(2,2)$ are $\mathcal{A}_s$-equivalent, then we write $f \sim_{\mathcal{A}_s} g$. If f and g are $\mathcal{A}_0$-equivalent, then we say that they are topologically equivalent, and if f and g are not $\mathcal{A}_0$-equivalent, then they are topologically different. A jet $\omega \in J^r(n,p)$ is $\mathcal{A}_0$-sufficient in $\mathcal{E}_{[r]}(n,p)$ if every $f \in \mathcal{E}_{[r]}(n,p)$ with $j^r f(0) = \omega$ is $\mathcal{A}_0$-equivalent to ω. There exists no general theorem giving necessary and sufficient conditions for $\mathcal{A}_0$-sufficency of r-jets in $\mathcal{E}_{[r]}(n,p)$ for arbitrary n and p. Known results include a characterization of $\mathcal{A}_0$-sufficient jets with 0 as an isolated singular point (see [1]), and a study of $\mathcal{A}_0$-sufficiency in $\mathcal{E}_{[r]}(2,2)$ of jets from $\mathbb{R}^2$ to $\mathbb{R}^2$ (see [2]). The result in [2] gives a complete characterization of $\mathcal{A}_0$-sufficent plane-to-plane jets for a restricted class of jets, and it is the aim of this article to extend the result of [2] to a complete characterization of $\mathcal{A}_0$-sufficient plane-to-plane jets of rank 1.

We identify r-jets in $J^r(2,2)$ with polynomial maps $\mathbb{R}^2 \to \mathbb{R}^2$ of degree $\leq r$ with zero constant term. Let $\omega \in J^r(2,2)$. Let $J\omega(p)$ denote the Jacobian determinant of ω at p and let $\Sigma(\omega) = J\omega^{-1}(0)$ denote the singular set of ω. $\Sigma(\omega)$ is an algebraic set. Let $B(x,\rho)$ denote the open ball in $\mathbb{R}^2$ with center x and radius ρ. If ω is a nonzero singular jet, then there is a real number $\rho_0 > 0$ and a natural number N such that $(\Sigma(\omega) \setminus \{0\}) \cap B(0,\rho)$ has exactly N topological components whenever $0 < \rho < \rho_0$. These components are called *branches* of ω.

Let $C_1, C_2, \ldots, C_N$ denote the branches of ω. Since $\Sigma(\omega)$ is an algebraic set, the Curve Selection Lemma implies that each of these branches has a well defined tangent direction at the origin. We think of these directions as points on S^1. If all these points are distinct, then we say that ω has *different tangent directions* at 0. Note that a line through the origin represents two different tangent directions corresponding to antipodal points on S^1.

Identify $J^1(2,2)$ with $\mathbb{R}^4$ by identifying $(ax+by, cx+dy)$ with (a,b,c,d) and let $\Sigma = \{(a,b,c,d) | ad - bc = 0\} \subset J^1(2,2)$. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be a C^r map with $r \geq 2$. The germ of F at a singular point p is a *fold singularity* if two conditions are satisfied. The first condition is that $j^1F \pitchfork \Sigma$ at p. If the first condition is satisfied, then $\Sigma(F)$ is a C^{r-1} manifold in a neighbourhood of p. The second condition for fold singularities is that $T_p\Sigma(F) + \ker D(JF)(p) = \mathbb{R}^2$. Whether or not the germ of F at a point p in the source of F is a fold singularity is determined by the non-constant part of the 2-jet extension of F at p, i.e. the 2-jet extension at 0 of the map $q \mapsto F(q+p) - F(p)$ which will be denoted by $J^2F(p)$.

An element of $J^2(2,2)$ is then thought of as a polynomial map as above. We may use the coefficients of these polynomials as coordinates of $J^2(2,2)$, and hence identify $J^2(2,2)$ with $\mathbb{R}^4 \times \mathbb{R}^6$ by identifying the polynomial map given by

$$(x,y) \mapsto (ax + by + ex^2 + 2fxy + gy^2, cx + dy + hx^2 + 2ixy + jy^2)$$

with $(L,H) = ((a,b,c,d),(e,f,g,h,i,j))$. It is shown in [2] that in these coordinates, the set of singular 2-jets which are not folds is given by

$$\Gamma = \left\{ (a,\dots,j) \,|\, ad - bc = 0, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} aj - bi - cg + df \\ -ai + bh + cf - de \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}.$$

For every C^2 map $F : \mathbb{R}^2 \to \mathbb{R}^2$ we may define a map $(L_F, H_F) : \mathbb{R}^2 \to J^2(2,2) = \mathbb{R}^4 \times \mathbb{R}^6$ induced by J^2F via the identifications above.

Let $d(\cdot, \Sigma)$ denote the distance function from a point in $\mathbb{R}^4$ to Σ with respect to the norm on $J^1(2,2)$ induced by the Euclidean norm on $\mathbb{R}^4$. For all $f \in \mathcal{E}_{[r]}(2,2)$, define

$$d_f(p) = d(j^1f(p), \Sigma).$$

For all $\epsilon, \rho > 0$ and $f \in \mathcal{E}_{[r]}(2,2)$, define

$$H_{\epsilon,\rho}(f) = \left\{ p \,|\, d(j^1f(p), \Sigma) \le \epsilon \|p\|^{r-1}, 0 < \|p\| < \rho \right\}.$$

$H_{\epsilon,\rho}(\omega)$ is a semialgebraic set with $\Sigma(\omega) \cap B(0,\rho) \setminus \{0\} \subset H_{\epsilon,\rho}(\omega)$.

Proposition 1.1 (Proposition 2.1 of [2]). *Let $r \ge 2$ and let $\omega \in J^r(2,2)$ be a singular, nonzero jet such that 0 is not isolated in $\Sigma(\omega)$. Let Γ and $C_1,\dots,C_N$ and $H_{\epsilon,\rho}(\omega)$ be as explained above. Consider the following condition:*

(I) *There is a neighbourhood U of 0 and a constant $C > 0$ such that if $p \in U$ and $(L,H) \in \Gamma$, then*

$$\|L_\omega(p) - L\| + \|H_\omega(p) - H\| \, \|p\| \ge C \|p\|^{r-1}.$$

Assume that condition (I) *is satisfied. Then there exist $\epsilon_0 > 0$ and $\rho_0 > 0$ such that the following is satisfied: For each ρ such that $0 < \rho < \rho_0$, and for each ϵ such that $0 < \epsilon < \epsilon_0$, $H_{\epsilon,\rho}(\omega)$ has exactly N connected components and we can label these components by $H^1_{\epsilon,\rho},\dots,H^N_{\epsilon,\rho}$, such that for $i = 1,\dots,N$, $C_i \subset H^i_{\epsilon,\rho}$.*

Theorem 1.2 (Theorem 2.3 of [2]). *If $\omega \in J^r(2,2)$ has an isolated singularity at the origin, then ω is $\mathcal{A}_0$-sufficient in $\mathcal{E}_{[r]}(2,2)$ if and only if inequality (I) of Proposition 1.1 holds.*

In this article, whenever ω is an r-jet which satisfies (I) and we speak about $H_{\epsilon,\rho}(\omega)$, it is understood that $\epsilon < \epsilon_0$ and $\rho < \rho_0$ where ϵ_0 and ρ_0 have the properties stated in Proposition 1.1.

Theorem 1.3 (Main Theorem of [2]). *Let $r > 2$ and let $\omega \in J^r(2,2)$ be a jet as described in Proposition 1.1 . Let Γ, $C_1,\dots,C_N$ and $H_{\epsilon,\rho}(\omega)$ be as defined above and assume that condition* (I) *from Proposition 1.1 is satisfied. Let ρ_0 and ϵ_0 be as in the conclusion of 1.1. Consider the following condition :*

(II) *There exist $\rho > 0$ with $\rho < \rho_0$ and $\epsilon > 0$ with $\epsilon < \epsilon_0$ and a constant C such that if $H^i_{\epsilon,\rho}(\omega)$*

and $H^j_{\epsilon,\rho}(\omega)$ *are distinct components of* $H_{\epsilon,\rho}(\omega)$ *and* $p \in H^i_{\epsilon,\rho}(\omega) \cup \{0\}$ *and* $q \in H^j_{\epsilon,\rho}(\omega) \cup \{0\}$ *then*

$$\|\omega(p) - \omega(q)\| \geq C(\|p\|^{r-1} + \|q\|^{r-1}) \|p - q\| .$$

Assume also that the condition (II) *above is satisfied, then* ω *is* $\mathcal{A}_0$*-sufficient in* $\mathcal{E}_{[r]}(2,2)$ *.*

Moreover, the condition (I) *of Proposition 1.1 is a necessary condition for* $\mathcal{A}_0$*-sufficiency in* $\mathcal{E}_{[r]}(2,2)$ *for all jets in* $J^r(2,2)$ *with* $r > 2$*, and if we consider singular, nonzero jets* ω *where* 0 *is not isolated in* $\Sigma(\omega)$*, and where* ω *has different tangent directions at 0, then condition* (II) *above is also a necessary condition for* $\mathcal{A}_0$*-sufficiency in* $\mathcal{E}_{[r]}(2,2)$*.*

Proposition 1.4. *If* $\omega \in J^r(2,2)$ *satisfies (I), then every* C^r *realization of* ω *has only regular points and fold singularities outside the origin. If* ω *does not satisfy (I), then there is a* C^r *realization of* ω *with a sequence of simple cusp points converging to the origin. Furthermore, simple cusps are topologically different from folds and regular points.*

Proof. The first assertion follows from the defining property of Γ and Lemma 4.1 of [2]. The second assertion is the content of Lemma 6.3 of [2]. The last assertion is the content of Lemma 6.6 of [2]. □

Proposition 1.5. *If* $\omega \in J^r(2,2)$ *satisfies (I) and (II), then the restriction of every* C^r *realization of* ω *to its singular set is injective. If* ω *has different tangent directions and satisfies (I) but does not satisfy (II), then there is a* C^r *realization of* ω *having a sequence of singular double points converging to the origin.*

Proof. The first part of the Proposition follows from Lemma 4.12 of [2] and the last part follows from Lemma 6.4 of [2]. □

Definition 1.6. A map germ $z = (z_1, z_2) : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ of rank 1 is in *standard form* if $z_1(x, y) = x$.

Theorem 1.3 can be quite difficult to apply in practice. In the case of rank 1 jets in standard form, the following theorem gives the neat conditions that characterize $\mathcal{A}_0$-sufficient jets.

Theorem 1.7. *Let* $r > 2$ *and let* $\omega(x,y) = (x, f(x,y)) \in J^r(2,2)$ *and let* $C_1, \ldots, C_N$ *be as above. Then* ω *is* $\mathcal{A}_0$*-sufficient in* $\mathcal{E}_{[r]}(2,2)$ *if and only if the conditions (i) and (ii) below are satisfied:*

(i) *There are a neighbourhood* U *of 0 and a constant* $C > 0$ *such that if* $p \in U$*, then*

$$|f_y(p)| + |f_{yy}(p)| \, \|p\| \geq C \, \|p\|^{r-1} .$$

(ii) *There are a neighbourhood* U *of 0 and a constant* $C > 0$ *such that if* C_i *and* C_j *are different components of* $\Sigma(\omega) \setminus \{0\}$ *and* $p = (x, y) \in C_i \cup \{0\} \cap U$ *and* $q = (x, v) \in C_j \cup \{0\} \cap U$*, then*

$$|f(p) - f(q)| > C(\|p\|^{r-1} + \|q\|^{r-1})|y - v|.$$

There is also an analogue of Theorem 1.2 for rank 1 jets in standard form.

Theorem 1.8. *If* $\omega \in J^r(2,2)$ *is in standard form and has an isolated singularity at the origin, then* ω *is* $\mathcal{A}_0$*-sufficient in* $\mathcal{E}_{[r]}(2,2)$ *if and only if (i) of Theorem 1.7 holds.*

Proof. This follows immediately from Theorem 1.2 and Lemma 2.2 of Section 2.2 which says that for jets in standard form, (i) and (I) are equivalent. □

From now on we consider only singular jets where 0 is not an isolated singularity. The main step in the proof of Theorem 1.7 is to prove the following proposition:

Proposition 1.9. *For jets of rank 1 in standard form, (II) $\Leftrightarrow$ (ii).*

The virtue of Theorem 1.7 is that both the set Γ and the sets $H_{\epsilon,\rho}$ are left out of the theorem. Also, when verifying (ii) one only needs to consider pairs of points with the same x-components. Finally, the validity of Theorem 1.7 is not restricted to the case of jets with different tangent directions at 0.

Theorem 1.7 holds for rank 1 jets given in a special form. For rank 1 jets in general, the following theorem holds.

Theorem 1.10. *Let $r > 2$ and let $\omega \in J^r(2,2)$ be a jet of rank 1. Then ω is $\mathcal{A}_0$-sufficient in $\mathcal{E}_{[r]}(2,2)$ if and only if (I) of Theorem 1.3 and (II') below hold:*

(II') *There is a neighbourhood U of 0 and a constant $C > 0$ such that if $i \neq j$ and $p \in C_i \cap U$ and $q \in C_j \cap U$, then*

$$\|\omega(p) - \omega(q)\| > C(\|p\|^{r-1} + \|q\|^{r-1}) \|p - q\| .$$

The article is organized as follows: In Section 2 we prove that Theorem 1.7 implies Theorem 1.10. Section 3 contains a thorough study of the hornshaped neighbourhoods $H_{\epsilon,\rho}$. This enables us to prove that inequality (II') implies inequality (II) for rank 1 jets. This is the topic of Section 4. In Section 4 we also give the proof of Proposition 1.9. This proposition is the key to the construction of a certain Whitney field in Section 5. This Whitney field is the main technical tool in the proof of the necessity of (ii) for all rank 1 jets in standard form, and will conclude the demonstration of Theorem 1.7 and Theorem 1.10.

In the rest of the article, $\mathcal{A}_0$-sufficiency of an r-jet is understood to mean $\mathcal{A}_0$-sufficiency in $\mathcal{E}_{[r]}(2,2)$. Sometimes only the term 'sufficiency' will be used.

Notation 1 ($\lesssim$, $\gtrsim$, $\sim$)**.** Let F and G be two nonnegative real-valued functions defined on some subset of some Euclidean space E. We will use the notation $F \gtrsim G$ if there is a constant $a > 0$ such that $F \geq aG$. The notation $F \lesssim G$ means that there is a constant $b > 0$ such that $F \leq bG$. If $F \lesssim G$ and $F \gtrsim G$, then we write $F \sim G$. For two sequences (p_n) and (q_n) in E and positive real valued functions F and G, $F(p_n) \gtrsim G(q_n)$ means that there is a positive constant a and a natural number N such that $F(p_n) \geq aG(q_n)$ when $n > N$. Similarly, $F(p_n) \lesssim G(q_n)$ means that there is a positive constant b and a natural number N such that $F(p_n) \leq bG(q_n)$ when $n > N$. Of course, $F(p_n) \sim G(q_n)$ means that $F(p_n) \gtrsim G(q_n)$ and $F(p_n) \lesssim G(q_n)$.

Notation 2 (O, o)**.** If F and G are real-valued functions defined in a neighbourhood of 0 in some Euclidean space, then $F(x) = o(G(x))$ means that $F(x)/G(x) \to 0$ as $x \to 0$. If (p_n) and (q_n) are sequences converging to 0, then $F(p_n) = o(G(q_n))$ means that $F(p_n)/G(q_n) \to 0$ as $n \to \infty$. For fractional power series β and γ, $O(\beta)$ denotes the order of β and $\beta = o(\gamma)$ means that $O(\beta) > O(\gamma)$.

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2. Coordinate changes

2.1. **Suitable coordinates.** To establish the connection between Theorem 1.7 and Theorem 1.10, we have to investigate how our Łojasiewicz inequalities behave under coordinate changes. Let $\omega \in J^r(2,2)$ and let $\omega' = k \circ \omega \circ h^{-1}$ where h and k are germs of C^r diffeomorphisms $(\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$.

Lemma 2.1. *ω is $\mathcal{A}_0$-sufficient if and only if $j^r\omega'$ is $\mathcal{A}_0$-sufficient.*

Proof. Assume that ω is sufficient and let $\tilde{\omega}$ be a C^r realization of $j^r\omega'$. Then $j^r(k^{-1}\circ\tilde{\omega}\circ h) = j^r(k^{-1}\circ j^r\omega'\circ h) = \omega$. Thus $\tilde{\omega} \sim_{\mathcal{A}_0} j^r\omega'$, and hence, $j^r\omega'$ is sufficient. Conversely, suppose $j^r\omega'$ is sufficient and let $\bar{\omega}$ be a C^r realization of ω. Then clearly $j^r(k\circ\bar{\omega}\circ h^{-1}) = j^r(k\circ\omega\circ h^{-1}) = j^r\omega'$. Thus $\bar{\omega} \sim_{\mathcal{A}_0} k\circ\bar{\omega}\circ h^{-1} \sim_{\mathcal{A}_0} j^r\omega' \sim_{\mathcal{A}_0} \omega' \sim_{\mathcal{A}_0} \omega$, which shows that ω is sufficient. □

Lemma 2.2. *ω satisfies (I) $\Leftrightarrow$ $j^r\omega'$ satisfies (I).*

Proof. Assume that ω satisfies (I) and that $j^r\omega'$ does not satisfy (I). By Proposition 1.4, $j^r\omega'$ has a C^r realization $\tilde{\omega}$ with a sequence of singular points converging to 0, all topologically different from folds. Then $\omega = j^r(k^{-1}\circ\tilde{\omega}\circ h)$ has a realization which has a sequence of singular points converging to 0, all of which are topologically different from folds. This contradicts the assumption that ω satisfies (I).

Let $\omega_2 = j^r\omega'$. Then $\omega = j^r(k^{-1}\circ\omega'\circ h) = j^r(k^{-1}\circ\omega_2\circ h)$, and hence, the other implication follows from the first implication. □

Lemma 2.3. *Let z and z' in $\mathcal{E}_{[r]}(2,2)$ be such that $z' = k\circ z\circ h^{-1}$ for some germs at the origin of origin-preserving C^r diffeomorphisms h and k . For each $\epsilon,\rho > 0$, there are $\epsilon',\rho' > 0$ such that $h(H_{\epsilon',\rho'}(z)) \subset H_{\epsilon,\rho}(z')$.*

Proof. It is enough to show that $\|p\| \sim \|h(p)\|$ and $d_z(p) \sim d_{z'}(h(p))$. An application of Taylor's formula gives $\|p\| \sim \|h(p)\|$. We also have

$$\begin{aligned} d_z(p) &= \inf\{\|Dz(p)v\| \mid \|v\| = 1\} \qquad \text{(by (3.11) in [2])}\\ &\sim \inf\{\|D(k\circ z)(p)v\| \mid \|v\| = 1\}\\ &\sim \inf\{\|Dz'(h(p))v\| \mid \|v\| = 1\}\\ &= d_{z'}(h(p)), \end{aligned}$$

and the lemma follows. □

Lemma 2.4. *Suppose z and z' in $\mathcal{E}_{[r]}(2,2)$ are such that $j^rz(0) = j^rz'(0)$. Let $\epsilon,\rho > 0$. Then there are $\epsilon',\rho' > 0$ such that*

$$H_{\epsilon',\rho'}(z') \subset H_{\epsilon,\rho}(z).$$

Proof. Assume that z and z' satisfy the premises of the lemma. Let $\tilde{z} = z - z'$. Then $j^r\tilde{z}(0) = 0$, and hence, $\|D\tilde{z}(p)\| = o(\|p\|^{r-1})$. Using this, we see that

$$\begin{aligned} d_z(p) = \inf\{\|Dz(p)v\| \mid \|v\| = 1\} &\le \inf\{\|Dz'(p)v\| + \|D\tilde{z}(p)v\| \mid \|v\| = 1\}\\ &\le \inf\{\|Dz'(p)v\| \mid \|v\| = 1\} + \sup\{\|D\tilde{z}(p)v\| \mid \|v\| = 1\} = d_{z'}(p) + o(\|p\|^{r-1}). \end{aligned}$$

The lemma follows. □

Lemma 2.5. *For every sequence (p_n) of points converging to 0 such that $d(j^1\omega(p_n),\Sigma) = o(\|p_n\|^{r-1})$, there is a subsequence $(p_{n(k)})$ of (p_n) and a C^r realization ω_p of ω such that $p_{n(k)} \in \Sigma(\omega_p)$ for every k.*

Proof. Let (p_n) be as in the lemma. Choose $p_{n(k)}$ such that $\|p_{n(k+1)}\| < \frac{1}{2}\|p_{n(k)}\|$. For every k, let M_k be a matrix such that $\|M_k\| = d(j^1\omega(p_{n(k)}),\Sigma)$ and $D\omega(p_{n(k)}) + M_k$ is singular. Let Q be the r-th order Taylor field defined on $K = \{0\}\cup(\cup_k\{p_{n(k)}\})$ with values in $\mathbb{R}^2$ given by $Q^1(p) = M_k$ for $p = p_{n(k)}$ and $Q = 0$ otherwise. It is clear that Q is a Whitney field. Let h be a C^r extension of Q. Then $j^rh(0) = 0$. Let $\omega_p = \omega + h$. It is clear that ω_p satisfies the conditions in the lemma. □

Lemma 2.6. *(I) and (II) hold for ω $\Leftrightarrow$ (I) and (II) hold for $j^r\omega'$.*

Proof. Assume that (I) holds and (II) fails for ω. By Lemma 2.2, (I) holds for $j^r\omega'$ as well. We proceed to show that (II) fails for $j^r\omega'$. Since (II) fails for ω, there are sequences (p_n) and (q_n) of points converging to 0 such that

$$d(j^1\omega(p_n),\Sigma) = o(\|p_n\|^{r-1}) \text{ and } d(j^1\omega(q_n),\Sigma) = o(\|q_n\|^{r-1})$$

and

$$\|\omega(p_n) - \omega(q_n)\| = o(\|p_n\|^{r-1} + \|q_n\|^{r-1})\,\|p_n - q_n\|$$

and p_n and q_n belong to different components of $H_{\tilde\epsilon,\tilde\rho}(\omega)$. Since h and k are germs of diffeomorphisms, an application of Taylor's formula shows that $\|h(p)\| \sim \|k(p)\| \sim \|p\|$ for all p close to 0. Furthermore, since h and k are diffeomorphisms, the definition of differentiability gives $\|h(p)-h(q)\| \sim \|k(p)-k(q)\| \sim \|p-q\|$ for p,q close to 0. Furthermore, $j^r\omega' = \omega' + \tilde\omega$ where $j^r\tilde\omega(0) = 0$ and hence, $\|\tilde\omega(p)\| = o(\|p\|^r)$ and $\|D\tilde\omega(p)\| = o(\|p\|^{r-1})$. Using this and the Mean Value Theorem, we get

$$\begin{aligned}
&\|j^r\omega'(h(p_n)) - j^r\omega'(h(q_n))\| \\
&\le \|k\circ\omega(p_n) - k\circ\omega(q_n)\| + \|\tilde\omega(h(p_n)) - \tilde\omega(h(q_n))\| \\
&\lesssim \|\omega(p_n)-\omega(q_n)\| + \sup_{t\in[0,1]} \|D\tilde\omega(th(p_n)+(1-t)h(q_n))\|\,\|h(p_n)-h(q_n)\| \\
&= o(\|p_n\|^{r-1} + \|q_n\|^{r-1})\,\|p_n-q_n\| + o(\|h(p_n)\|^{r-1} + \|h(q_n)\|^{r-1})\,\|h(p_n)-h(q_n)\| \\
&= o(\|h(p_n)\|^{r-1} + \|h(q_n)\|^{r-1})\,\|h(p_n)-h(q_n)\|\,.
\end{aligned}$$

By Lemma 2.5 there are subsequences $(p_{n(k)})$ and $(q_{n(k)})$ of (p_n) and (q_n) and C^r realizations ω_p and ω_q of ω such that for each k, $p_{n(k)} \in \Sigma(\omega_p)$ and $q_{n(k)} \in \Sigma(\omega_q)$. Hence, for each of the sequences $(h(p_{n(k)}))$ and $(h(q_{n(k)}))$, there are C^r realizations of $j^r(\omega')$ having singular points along the sequence. It follows that, given small positive ϵ and ρ, then eventually the sequences $(h(p_{n(k)}))$ and $(h(q_{n(k)}))$ are in $H_{\epsilon,\rho}(j^r\omega')$.

We need to show that for small ϵ,ρ, eventually the sequences $(h(p_{n(k)}))$ and $(h(q_{n(k)}))$ lie in different components of $H_{\epsilon,\rho}(j^r\omega')$. To this end, use Lemma 2.3 to pick ϵ',ρ' so small that $h^{-1}(H_{\epsilon',\rho'}(\omega')) \subset H_{\tilde\epsilon,\tilde\rho}(\omega)$ where $\tilde\epsilon$ and $\tilde\rho$ are as above, i.e. such that (p_n) and (q_n) lie in different components of $H_{\tilde\epsilon,\tilde\rho}(\omega)$. Then use Lemma 2.4 to pick ϵ,ρ such that $H_{\epsilon,\rho}(j^r\omega') \subset H_{\epsilon',\rho'}(\omega')$. Assume that there are subsequences $(h(p_{n(k(l))}))$ and $(h(q_{n(k(l))}))$ which lie in the same component of $H_{\epsilon,\rho}(j^r\omega')$. Since h^{-1} is a homeomorphism, the component of $H_{\epsilon,\rho}(j^r\omega')$ containing $(h(p_n))$ and $(h(q_n))$ is mapped by h^{-1} into one component of $H_{\tilde\epsilon,\tilde\rho}(\omega)$. This contradicts the assumption that (p_n) and (q_n) lie in different components of $H_{\tilde\epsilon,\tilde\rho}(\omega)$. Hence, (II) fails for $j^r\omega'$.

To finish the proof, observe that $\omega = j^r(k^{-1}\circ j^r\omega'\circ h)$, and hence the other implication follows from the first. □

2.2. **Łojasiewicz inequality (I) for rank 1 jets.** When ω is in standard form, we have a particularly convenient version of inequality (I).

Lemma 2.7. *Let $\omega(x,y) = (x, f(x,y))$ be an r-jet in standard form. Then (I) holds for ω if and only if (i) of Theorem 1.7 holds for ω.*

Proof. To prove that (I) implies (i), notice that

$$(L,H) = (1,0,f_x,0,0,0,0,f_{xx},f_{xy},0)(p) \in \Gamma$$

for all p, and hence, if (I) holds, then

$$|f_y(p)| + |f_{yy}(p)|\,\|p\| = \|L_\omega(p) - L\| + \|H_\omega(p) - H\|\,\|p\| \ge C\,\|p\|^{r-1}\,.$$

Conversely, if (I) fails, then there are a sequence (p_n) in $\mathbb{R}^2$ converging to 0 and a sequence $(L_n, H_n) \in \Gamma$ such that

$$\|L_\omega(p_n) - L_n\| + \|H_\omega(p_n) - H_n\| \, \|p_n\| = o(\|p_n\|^{r-1}).$$

Let $(L_n, H_n) = (a_n, \ldots, d_n, e_n, \ldots, j_n) \in \mathbb{R}^{10}$. We get that $a_n = 1 - o(\|p_n\|^{r-1})$ and $b_n = o(\|p_n\|^{r-1})$. Also, since L_n is singular, $d_n = c_n b_n / a_n = o(\|p_n\|^{r-1})$, which implies $|f_y(p_n)| = o(\|p_n\|^{r-1})$. We have $H_\omega(p_n) = (0, 0, 0, f_{xx}, f_{xy}, f_{yy})(p_n)$. Thus we also have $e_n, f_n, g_n = o(\|p_n\|^{r-2})$. Furthermore, from the definition of Γ, we get that

$$a_n(a_n j_n - b_n i_n - c_n g_n + d_n f_n) + b_n(-a_n i_n + b_n h_n + c_n f_n - d_n e_n) = 0.$$

It follows that $j_n = o(\|p_n\|^{r-2})$ and hence, $|f_{yy}(p_n)| = o(\|p_n\|^{r-2})$. This shows that (i) fails. □

Lemma 2.8. *Let a be a real number and let Φ be the diffeomorphism $\Phi(x, y) = (\Phi_1(x, y), \Phi_2(x, y)) = (x, ax + y)$. Let $\omega \in J^r(2, 2)$ be in standard form. Then $\omega_\Phi = \omega \circ \Phi^{-1}$ is an r-jet in standard form and ω satisfies (i) and (ii) if and only if ω_Φ satisfies (i) and (ii).*

Proof. The first assertion is clear from the form of Φ. For the second assertion, assume that ω satisfies (i) but not (ii). Lemma 2.2 and Lemma 2.7 imply that $\omega \circ \Phi^{-1}$ satisfies (i). Since ω does not satisfy (ii), there are distinct components C_i and C_j of $\Sigma(\omega) \setminus \{0\}$ and sequences $p_n = (x_n, y_n) \in C_i$ and $q_n = (x_n, v_n) \in C_j$, both converging to 0 and such that

$$|f(p_n) - f(q_n)| = o(\|p_n\|^{r-1} + \|q_n\|^{r-1})|y_n - v_n|.$$

From the definition of Φ, it is clear that $\omega_\Phi(x, y) = (x, f_\Phi(x, y))$ is in standard form. Furthermore, $\Phi(C_i)$ and $\Phi(C_j)$ are different components of $\Sigma(\omega_\Phi)$ and

$$\begin{aligned} |f_\Phi(\Phi(p_n)) - f_\Phi(\Phi(q_n))| &= |f(p_n) - f(q_n)| = o(\|p_n\|^{r-1} + \|q_n\|^{r-1})|y_n - v_n| \\ &= o(\|\Phi(p_n)\|^{r-1} + \|\Phi(q_n)\|^{r-1})|\Phi_2(p_n) - \Phi_2(q_n)|, \end{aligned}$$

and hence (ii) fails for ω_Φ.

Observe that $\Phi^{-1}(x, y) = (x, -ax + y)$, and hence the other implication follows directly from the argument above. □

Lemma 2.9. *Let ω be an r-jet which satisfies (I), and let ω' be a C^r map germ with $\omega \sim_{\mathcal{A}_r} \omega'$. Then (II') holds for ω if and only if (II') holds for $j^r\omega'$.*

Proposition 2.10. *If ω is an r-jet in standard form satisfying (i), then (ii) and (II') are equivalent for ω.*

The proofs of Lemma 2.9 and Proposition 2.10 will be postponed until Section 4.

Proof that Theorem 1.7 ⇒ Theorem 1.10. Assume that Theorem 1.7 is true. Assume now that (I) and (II') hold for an r-jet $\omega \in J^r(2, 2)$ of rank 1. By Lemma 2.2, Lemma 2.7, Lemma 2.9 and Proposition 2.10, we may choose C^r coordinates transforming ω to the standard form $\bar{\omega}(x, y) = (x, f(x, y))$ such that (i) and (ii) hold for $j^r\bar{\omega}$. By Theorem 1.7, $j^r\bar{\omega}$ is $\mathcal{A}_0$-sufficient. Lemma 2.1 implies that ω is $\mathcal{A}_0$-sufficient.

Conversely, if (I) fails for ω, then, by Lemma 2.2 and Lemma 2.7, (i) fails for $j^r\bar{\omega}$ and hence, $j^r\bar{\omega}$ is not sufficient by Theorem 1.7. By Lemma 2.1, ω is not sufficient. If (I) holds and (II') fails for ω, then (II') fails for $j^r\bar{\omega}$ by Lemma 2.9. By Proposition 2.10, (ii) fails for $j^r\bar{\omega}$. Theorem 1.7 shows that $j^r\bar{\omega}$ is not $\mathcal{A}_0$-sufficient, and hence, by Lemma 2.1 again, ω is not $\mathcal{A}_0$-sufficient. □

3. Hornshaped neighbourhoods

3.1. **Consequences of inequality (i).** Let $\omega(x,y) = (x, f(x,y))$ be an r-jet of rank 1 in standard form for which (I), or equivalently (i) holds. By Lemma 2.8, we may choose coordinates such that no branch of $\Sigma(\omega)$ is tangent to the x-axis. Let

$$\tilde{H}_{\epsilon,\rho} = \{p : |f_y(p)| \leq \epsilon \|p\|^{r-1}, 0 < \|p\| \leq \rho\}.$$

Recall from (3.3) in [2] that

$$d_\omega(p) = d(j^1\omega(p), \Sigma) \sim \frac{|J\omega(p)|}{\|D\omega(p)\|} \sim |f_y(p)|.$$

It follows that for every $\epsilon > 0$ there are $\epsilon_1, \epsilon_2, \epsilon_3 > 0$ such that

$$H_{\epsilon_1,\rho}(\omega) \subseteq \tilde{H}_{\epsilon_2,\rho}(\omega) \subseteq H_{\epsilon,\rho}(\omega) \subseteq \tilde{H}_{\epsilon_3,\rho}(\omega).$$

Lemma 3.1. *Proposition 1.1 holds when we replace $H_{\epsilon,\rho}$ by $\tilde{H}_{\epsilon,\rho}$.*

Proof. Let

$$S = \{(x,y) | \nabla f_y(x,y) \cdot (y, -x) = 0\}.$$

The proof of Proposition 1.1 in [2] applies to $\tilde{H}_{\epsilon,\rho}(\omega)$ once we have shown that

$$\left|(f_y|S)(p)\right| \gtrsim \|p\|^{r-1}. \tag{3.1}$$

This corresponds to Lemma 3.1 in [2]. Let

$$D = \{p \in S : |f_y(p)| \leq |f_y(q)| \text{ for all } q \in S \text{ with } \|p\| = \|q\| \neq 0\}.$$

An application of the Tarski-Seidenberg Theorem shows that D is semialgbraic. Assume that (3.1) does not hold. Then $0 \in \overline{D}$ and the Curve Selection Lemma implies that we can find an analytic curve $\gamma = (\gamma_1, \gamma_2) : [0, \delta) \to \mathbb{R}^2$ with $\gamma(0) = 0$, $\gamma(0, \delta) \subset D$ and $|f_y(\gamma(t))| = o(\|\gamma(t)\|^{r-1})$. Assume that $\|\gamma(t)\| \sim t^s$ and $|f_y(\gamma(t))| \sim t^d$. Then $\frac{d}{s} > r - 1$. Also, $\|\gamma'(t)\| \sim t^{s-1}$ and

$$\left| \nabla f_y(\gamma(t)) \cdot \frac{\gamma'(t)}{\|\gamma'(t)\|} \right| \sim t^{d-s}.$$

Let $v(t) = (\gamma_2(t), -\gamma_1(t)) / \|\gamma(t)\|$ and $w(t) = \gamma'(t) / \|\gamma'(t)\|$. Then $v(t) \cdot w(t) \to 0$ as $t \to 0^+$. Let $e_2(t) = \frac{\partial}{\partial y} \circ \gamma(t)$. Then $e_2(t) = a(t)v(t) + b(t)w(t)$ where $|a(t)| < 2$ and $|b(t)| < 2$. Using that $\gamma(t) \in S$, it follows that

$$|f_{yy}(\gamma(t))| = |\nabla f_y(\gamma(t)) \cdot e_2(t)| \lesssim t^{d-s} = o(\|\gamma(t)\|^{r-2}),$$

and hence (i) fails along γ, contrary to our assumptions. Therefore (3.1) must hold and the rest of the proof goes as the proof of Proposition 1.1 in [2]. □

In the rest of the article, when we consider jets in standard form, we will only talk about $\tilde{H}_{\epsilon,\rho}$ and by abuse of notation, it will be denoted by $H_{\epsilon,\rho}$. Lemma 3.1 gives very specific geometric information about $H_{\epsilon,\rho}$. The situation for $\epsilon < \epsilon_0$ and $\rho < \rho_0$ is illustrated in Figure 1.

For the proof of Theorem 1.7 we need information about $H_{\epsilon,\rho}$ of more quantitative character. This section and the next contain the results we need.

Lemma 3.2. *There is a $\delta > 0$ and a neighbourhood U of 0 such that*

$$\{(x,y) \in \mathbb{R}^2 \mid |x| \leq \delta |y|^{r-1}\} \cap \Sigma(\omega) \cap U \setminus \{0\} = \emptyset.$$

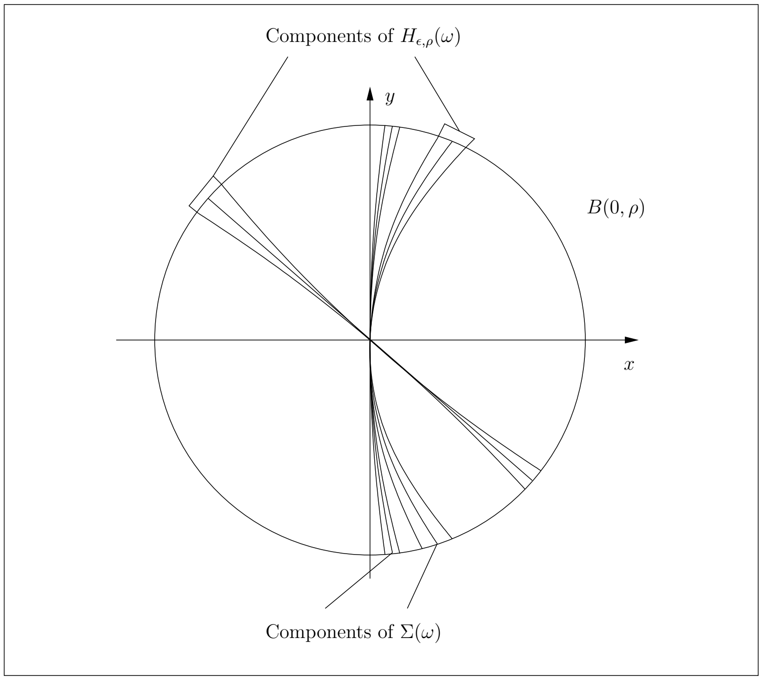

FIGURE 1. The figure shows 6 different components of $H_{\epsilon,\rho}$. The branches of $\Sigma(\omega)$ are contained in different components of $H_{\epsilon,\rho}$.

Proof. Assume that the lemma is false. Then there is a branch of $\Sigma(\omega)$ parametrized by an analytic curve $\alpha(t) = (\alpha_1(t), \alpha_2(t))$ with $\alpha(0) = 0$ and such that $\alpha_1(t) = o(|\alpha_2(t)|^{r-1})$. Let $m = O(\alpha_1(t))$, $n = O(\alpha_2(t))$. Then $m > n(r-1)$. We compute

$$0 = \frac{d}{dt} f_y(\alpha(t)) = \nabla f_y(\alpha(t)) \cdot \alpha'(t) = f_{yx}(\alpha(t))\alpha_1'(t) + f_{yy}(\alpha(t))\alpha_2'(t). \tag{3.2}$$

By (i),

$$|f_{yy}(\alpha(t))\alpha_2'(t)| \gtrsim \|\alpha(t)\|^{r-2} t^{n-1} \sim t^{n(r-2)+n-1} = t^{n(r-1)-1}.$$

By continuity of f_{yx} at 0, we have that $O(f_{yx}(\alpha(t))\alpha_1'(t)) \geq m-1 > n(r-1)-1$. It follows that (3.2) cannot hold, and this contradiction proves the lemma. $\square$

Lemma 3.3. *If ϵ and t are small enough, then $(0,t) \notin H_{\epsilon,\rho}$.*

Proof. It is enough to check that the order in t of $f_y(0,t)$ is not greater than $r-1$. Assume that $O(f_y(0,t)) > r-1$. We have

$$\frac{d}{dt} f_y(0,t) = f_{yy}(0,t),$$

and our assumption implies that $O(f_{yy}(0,t)) > r-2$. This contradicts (i). $\square$

3.2. **Newton-Puiseux roots of $J\omega$.** The real polynomial $J\omega = f_y$ has a Newton-Puiseux factorisation of the form

$$f_y(x,y) = u(x,y) \cdot x^E \cdot \prod_{i=1}^{p} [y - \beta_i(x)]$$

where $u \in \mathbb{C}\{x, y\}$ is a unit, $E \geq 0$ and each β_i is a formal fractional power series in x with complex coefficients. We may assume that $O(\beta_i) > 0$ for all i. Furthermore, all of the fractions occuring as exponents in these formal fractional power series have a common denominator N. This means that for each i, the formal fractional power series obtained by substituting t^N for x is an ordinary formal power series in t. This factorization is a purely algebraic rewriting of the original polynomial, but since the product is a holomorphic function, each of the power series $\beta_i(t^N)$ are in fact convergent power series, and hence, they are holomorphic functions of t for small t. We call the β_i *convergent* fractional power series.

Lemma 3.3 implies that $E = 0$, and we also assume that $O(\beta_i) \leq 1$ for each i. This can always be obtained by composition of ω with a diffeomorphism of the type in Lemma 2.8.

Lemma 3.4. *For each branch C of ω contained in the first quadrant of $\mathbb{R}^2$ there is a uniquely determined index i with $1 \leq i \leq p$ such that $t \mapsto (t^N, \beta_i(t^N)), t > 0$, is a parametrization of C.*

Proof. Let C be a branch of ω contained in the first quadrant of $\mathbb{R}^2$. The Curve Selection Lemma gives an analytic parametrization $\gamma(t)$ of C for $t > 0$. By a change of parameter if necessary, we may assume that $\gamma(t) = (t^{M\cdot N}, \tilde{\gamma}(t^N))$. Now,

$$f_y(\gamma(t)) = u(\gamma(t)) \cdot \prod_{i=1}^{p} [\tilde{\gamma}(t^N) - \beta_i(t^{M\cdot N})] \equiv 0.$$

This is an equality between analytic functions, and hence, for some i, $\tilde{\gamma}(t^N) \equiv \beta_i(t^{M\cdot N})$. It only remains to show that $\beta_i = \beta_j \Rightarrow i = j$. If there are $i \neq j$ such that $\beta_i = \beta_j$, then $f_{yy}(t^N, \beta_i(t^N)) = f_y(t^N, \beta_i(t^N)) = 0$, and this contradicts (i). $\square$

For real $x > 0$, we may think of the β_i as complex valued functions of x. By Lemma 3.4, each branch of ω in the first quadrant is a part of the graph of one of these functions $\beta_i(x)$. Any such fractional power series β_i can have only real coefficients, for we may write $\beta_i(x) = \text{Re}\, \beta_i(x) + \mathfrak{I}\text{Im}\, \beta_i(x)$ where $\mathfrak{I}$ is the imaginary unit and both terms on the right side are convergent fractional power series of x. If $\text{Im}\, \beta_i \neq 0$, then $\text{Im}\, \beta_i(x) \neq 0$ for small x, and this cannot be the case. We may assume that $\beta_1, \beta_2, \ldots, \beta_s$ correspond to the components of $\Sigma(\omega) \setminus \{0\}$ in the first quadrant and that $\beta_1(x) < \beta_2(x) < \ldots < \beta_s(x)$ for small x. The corresponding components will be denoted by $C_1, C_2, \ldots, C_s$.

In our factorisation of f_y, we have in effect solved the equation $f_y(x, y) = 0$ in terms of x. We might equally well have solved the same equation in terms of y and obtained another factorisation

$$f_y(x, y) = u'(x, y) \cdot y^F \cdot \prod_{i=1}^{q} [x - \beta_i^*(y)]$$

where $u' \in \mathbb{C}\{x, y\}$ is a unit, $F \geq 0$ and each β_i^* is a convergent fractional power series in y with $O(\beta_i^*) \geq 0$. As before, we may assume that $y \mapsto (\beta_i^*(y), y), y > 0$ is a parametrization of C_i for $i = 1, \ldots, s$. For $(x, y) \in C_i$, $(x, \beta_i(x)) = (\beta_i^*(y), y)$, and hence, both $\beta_i \circ \beta_i^*$ and $\beta_i^* \circ \beta_i$ are the identity maps. In our case, $F = 0$ and $O(\beta_i^*) \geq 1$ for $i = 1, \ldots s$ because $O(\beta_i) \leq 1$ for $i = 1, \ldots, s$.

We will call the β_i the x-roots of f_y and the β_i^* the y-roots of f_y.

Notice that if $\gamma \neq 0$ is a convergent real fractional power series in x for which the exponents in the powers of x in the terms of γ have a common denominator N and the term of lowest order has positive coefficient, then $\gamma(t^N) = g(t)$ for some real analytic function $g(t) = t^m h(t)$ where h is real analytic and $h(0) > 0$. Then $s = t(h(t))^{\frac{1}{m}}$ is a real analytic change of parameter near $t = 0$, and $t = k(s)$ for some real analytic function k. We have $(t^N, g(t)) = (k(s)^N, s^m)$. Thus, if we set $\gamma^*(y) = k(y^{\frac{1}{m}})^N$, then we get a fractional power series γ^* such that $\gamma^* \circ \gamma = \gamma \circ \gamma^*$ is the identity map.

Lemma 3.5. *Let β be a convergent fractional power series with real coefficients. Let c be the coefficient of the lowest-order term of β. Assume that $c > 0$. Then $O(\beta) \cdot O(\beta^*) = 1$ and the coefficient of the lowest-order term of β^* is $c^{-\frac{1}{O(\beta)}}$.*

Proof. Let d be the coefficient of the lowest-order term of β^*. Since β and β^* are both convergent fractional power series, $y = \beta \circ \beta^*(y) = cd^{O(\beta)} y^{O(\beta) \cdot O(\beta^*)}$+terms of higher order. The conclusion follows immediately from this. □

Lemma 3.6. *Let β_i be one of the x-roots of f_y, and let β_j^* be a y-root of f_y. Let $a \in \mathbb{Q}_+$ and let $t \in \mathbb{R}$ and let $\gamma_s(x) = \beta_i(x) + sx^a + \alpha(x)$ and let $\sigma_t^*(y) = \beta_j^*(y) + ty^a + \alpha(y)$, where α is a convergent fractional power series with $O(\alpha) > a$. Then there are finite sets $S(i,a) \subset \mathbb{R}$ and $T(j,a) \subset \mathbb{R}$, independent of α such that $0 \in S(i,a) \cap T(j,a)$ and $O_x(f_y(x, \gamma_s(x))$ and $O_y(f_y(\sigma_t^*(y), y)$ are constant numbers A and B, respectively, for all $s \notin S(i,a)$ and $t \notin T(j,a)$. If $s \in S(i,a)$, then $O_x(f_y(x, \gamma_s(x)) > A$, and if $t \in T(j,a)$, then $O_y(f_y(\sigma_t^*(y), y)) > B$.*

Proof. We prove only the part of the lemma concerning the x-roots, since the other part is completely analogous. From the factorisation above we get

$$f_y(x, \gamma_s(x)) = u(x, \gamma_s(x)) \cdot (sx^a + \alpha(x)) \cdot \prod_{j \neq i} [\gamma_s(x) - \beta_j(x)].$$

The coefficient of the term of lowest order in this fractional power series is a nonzero polynomial in s. Let $S(i,a)$ be the set of real zeros of this polynomial. It is clear by definition that $s = 0$ has to be a root of this polynomial. □

Definition 3.7. Let β_i be an x-root of f_y and let β_j^* be a y-root of f_y.

We say that a fractional power series γ is an *a-perturbation of* β_i if $\gamma(x) = \beta_i(x) + sx^a + \alpha(x)$ and α is a convergent fractional power series with $O(\alpha) > a$. We say that γ is a *generic* a-perturbation of β_i if $s \notin S(i,a)$ and either $a \neq O(\beta_i)$ or $O(\gamma) = O(\beta_i)$.

We say that a fractional power series σ^* is an *a-perturbation of* β_j^* if $\sigma^*(y) = \beta_j^*(y) + ty^a + \alpha(y)$ and α is a convergent fractional power series with $O(\alpha) > a$. We say that σ^* is a *generic a*-perturbation of β_j^* if $t \notin T(j,a)$ and either $a \neq O(\beta_j^*)$ or $O(\sigma^*) = O(\beta_j^*)$.

Lemma 3.8. *Let $a = O(\beta_j)$ and let γ be a generic a-perturbation of β_j. Then γ^* is a generic $\frac{1}{a}$-perturbation of β_j^*.*

Proof. Assume $\beta_j(x) = cx^a + \beta(x)$ where $O(\beta) > a$. Let $\gamma_s(x) = \beta_j(x) + sx^a + \alpha(x)$. Then $\gamma(x) = \gamma_{\tilde{s}}(x)$ for some $\tilde{s} \notin S(j,a)$. Since γ is a generic a-perturbation of β_j, $\tilde{s} \neq -c$. Therefore $\gamma(x) = (c + \tilde{s})x^a + \beta(x) + \alpha(x)$ is of order a. It follows that

$$\beta_j^*(y) = \frac{1}{c^{1/a}} y^{1/a} + \bar{\beta}(y)$$

and

$$\gamma^*(y) = \frac{1}{(c + \tilde{s})^{1/a}} y^{1/a} + \bar{\alpha}(y).$$

Since $S(j,a)$ is finite and $\tilde{s} \notin S(j,a)$, $\gamma_s(x)$ is generic for s in some small interval I containing $\tilde{s}$ and such that $-c \notin I$. Therefore, $O_x(f_y(x, \gamma_s(x)))$ is constant for $s \in I$, and hence,

$$O_y(f_y(\gamma_s^*(y), y)) = \frac{1}{a} O_x(f_y(x, \gamma_s(x)))$$

is constant for $s \in I$. Since $T(j, \frac{1}{a})$ is finite , this means that $1/(c + \tilde{s})^{1/a} \notin T(j, \frac{1}{a})$. It follows that $\gamma^*(y)$ is a generic $\frac{1}{a}$-perturbation of β_j^*. □

3.3. **Width of $H_{\epsilon,\rho}(\omega)$.** To obtain the necessary estimates of the next section, it is of great importance to know more about how large $H_{\epsilon,\rho}(\omega)$ is and, in some sense, how well separated the components of $H_{\epsilon,\rho}(\omega)$ are.

For every $j = 1, \ldots, s-1$, the map $y \mapsto |f_y(x, y)|$ has a local maximum $\gamma_j(x) \in (\beta_j(x), \beta_{j+1}(x))$. The $\gamma_j(x)$ have to lie in the open intervals because, by (i), $|f_{yy}(p)| > 0$ for all $p \in \Sigma(\omega) \setminus \{0\}$. The functions γ_j have to be Newton-Puiseux roots of f_{yy}, and are therefore convergent fractional power series in x with real coefficients.

For a convergent real fractional power series β, we denote by $G(\beta)$ the set $\{(x, \beta(x))|x > 0\}$ and by $G^*(\beta^*)$ the set $\{(\beta^*(y), y)|y > 0\}$.

Lemma 3.9. *If $i > j$, $a = O(\beta_i - \beta_j)$ and $O(\beta_i) = O(\beta_j)$, then for every generic a-perturbation β of β_i and β_j, there are $\epsilon > 0, \rho > 0$ such that $G(\beta) \cap H_{\epsilon,\rho}(\omega) = \emptyset$.*

Proof. There is a root γ of f_{yy} with $\beta_j(x) < \gamma(x) < \beta_i(x)$. Since $O(\beta_i) = O(\beta_j)$, γ has to be an a-perturbation of β_i and β_j. Łojasiewicz inequality (i) implies that $|f_y(x, \gamma(x))| \gtrsim \|(x, \gamma(x))\|^{r-1}$. Since β is a generic a-perturbation, it follows that $O(f_y(x, \beta(x))) \leq O(f_y(x, \gamma(x)))$. We also have $O(\beta) = O(\beta_i) = O(\beta_j) = O(\gamma)$, and hence, $\|(x, \gamma(x))\| \sim \|(x, \beta(x))\|$. Altogether this shows that $|f_y(x, \beta(x))| \gtrsim \|(x, \beta(x))\|^{r-1}$, and the conclusion follows. $\square$

Lemma 3.10. *Let $b = O(\beta_i^*)$, and let β^* be a generic b-perturbation of β_i^*. Then, for small enough $\epsilon, \rho > 0$, $G^*(\beta^*) \cap H_{\epsilon,\rho}(\omega) = \emptyset$.*

Proof. The fractional power series $\gamma^*(y) = 0$ is a b-perturbation of β_i^*, and from Lemma 3.3 we know that $|f_y(\gamma^*(y), y)| \gtrsim \|(\gamma^*(y), y)\|^{r-1}$. Since β^* is a generic b-perturbation of β_i^*, $O_y(f_y(\beta^*(y), y)) \leq O_y(f_y(\gamma^*(y), y))$, and since we also have $\|(\beta^*(y), y)\| \sim \|(\gamma^*(y), y)\| \sim y$, the lemma follows. $\square$

Lemma 3.11. *Let $a = O(\beta_i)$, and let β be a generic a-perturbation of β_i. Then there are $\epsilon > 0, \rho > 0$ such that $G(\beta) \cap H_{\epsilon,\rho}(\omega) = \emptyset$.*

Proof. Using Lemma 3.8, we see that β^* is a generic $\frac{1}{a}$-perturbation of β_i^*, and by Lemma 3.10, for small $\epsilon > 0, \rho > 0$ we have $G^*(\beta^*) \cap H_{\epsilon,\rho}(\omega) = G(\beta) \cap H_{\epsilon,\rho}(\omega) = \emptyset$. $\square$

Lemma 3.12. *Let ϵ_n, ρ_n be sequences of real numbers such that $\epsilon_n \to 0$ and $\rho_n \to 0$ and let $p_n = (x_n, y_n)$ and $q_n = (u_n, v_n)$ be in $H_{\epsilon_n,\rho_n}(\omega)$. If $u_n < 0 < x_n$, then*

$$\|\omega(p_n) - \omega(q_n)\| \gtrsim \|p_n\|^{r-1} + \|q_n\|^{r-1}.$$

Proof. We claim that $x_n \gtrsim \|p_n\|^{r-1}$ and $|u_n| \gtrsim \|q_n\|^{r-1}$. Any branch of $\Sigma(\omega)$ may be parametrized by some convergent fractional power series $\beta(x)$ which by Lemma 3.2 must satisfy $O(\beta) \geq \frac{1}{r-1}$. By Lemma 3.11 there is a generic $O(\beta_s)$-perturbation $\tilde{\beta}$ of β_s such that $\tilde{\beta}(x) > \beta_s(x)$. By Lemma 3.2, $O(\tilde{\beta}) = O(\beta_s) \geq \frac{1}{r-1}$ and this shows that $y_n < \tilde{\beta}(x_n) < \delta x_n^{\frac{1}{r-1}}$ for some $\delta > 0$. Consider $\omega_\Phi = \omega \circ \Phi$ where $\Phi(x, y) = (x, -y)$. From Lemma 2.8 we know that (i) holds for ω_Φ, and it is clear that $H_{\epsilon,\rho}(\omega) = H_{\epsilon,\rho}(\omega_\Phi)$. It is also obvious that the branches of $\Sigma(\omega_\Phi)$ in the first quadrant correspond to the branches of $\Sigma(\omega)$ in the fourth quadrant. A similar analysis of ω_Φ as the above analysis of ω will show that $-\delta x_n^{\frac{1}{r-1}} < y_n$. This shows that $x_n \gtrsim \|p_n\|^{r-1}$. Let $\Psi(x, y) = (-x, y)$. A similar analysis of $\omega \circ \Psi$ shows that $|u_n| \gtrsim \|q_n\|^{r-1}$. Altogether we get

$$\|\omega(p_n) - \omega(q_n)\| \geq |x_n - u_n| \gtrsim \|p_n\|^{r-1} + \|q_n\|^{r-1}.$$

$\square$

3.4. **Preliminary estimates.** The proof of Theorem 1.7 depends on a number of estimates. The actual proofs of those estimates are a bit lengthy and quite delicate, so we include them here in a separate section.

3.4.1. *The first quadrant.* For $i = 1, \dots, s$, let $H^i_{\epsilon,\rho}(\omega)$ be the component of $H_{\epsilon,\rho}(\omega)$ containing $G(\beta_i) \cap H_{\epsilon,\rho}(\omega)$. Let $\epsilon_n, \tilde{\epsilon}_n, \rho_n$ and $\tilde{\rho}_n$ be sequences of positive real numbers converging to 0. Let $1 \le j < i \le s$ and let $p_n = (x_n, y_n) \in H^i_{\epsilon_n,\rho_n}(\omega)$ and $q_n = (u_n, v_n) \in H^j_{\epsilon_n,\rho_n}(\omega)$ be two sequences. We assume that (II) fails along these sequences, that is,

$$\|\omega(p_n) - \omega(q_n)\| = o(\|p_n\|^{r-1} + \|q_n\|^{r-1}) \|p_n - q_n\| . \tag{3.3}$$

Let $\tilde{p}_n = (x_n, \tilde{y}_n) \in H^i_{\tilde{\epsilon}_n,\tilde{\rho}_n}(\omega)$ and $\tilde{q}_n = (u_n, \tilde{v}_n) \in H^j_{\tilde{\epsilon}_n,\tilde{\rho}_n}(\omega)$. We want to see that

$$\|\omega(\tilde{p}_n) - \omega(\tilde{q}_n)\| = o(\|\tilde{p}_n\|^{r-1} + \|\tilde{q}_n\|^{r-1}) \|\tilde{p}_n - \tilde{q}_n\| .$$

To this end we need to show that

(1) $\|\tilde{p}_n\| = \|p_n\| + o(\|p_n\|)$
(2) $\|\tilde{q}_n\| = \|q_n\| + o(\|q_n\|)$
(3) $\|p_n - q_n\| = \|\tilde{p}_n - \tilde{q}_n\| + o(\|p_n - q_n\|)$
(4) $\|p_n - \tilde{p}_n\| = o(\|p_n - q_n\|)$
(5) $\|q_n - \tilde{q}_n\| = o(\|p_n - q_n\|)$.

We have assumed that $\beta_i(x) > \beta_j(x)$. Let $\delta > 0$ be a small number. We claim that there are generic $O(\beta_i - \beta_j)$-perturbations $\underline{\beta}_i$ and $\overline{\beta}_i$ of β_i and generic a-perturbations $\underline{\beta}_j$ and $\overline{\beta}_j$ of β_j where $a = O(\beta_i - \beta_j)$ if $O(\beta_i) = O(\beta_j)$ and $a = O(\beta_j)$ if $O(\beta_j) > O(\beta_i)$, such that for small x,

$$\underline{\beta}_j(x) < \beta_j(x) < \overline{\beta}_j(x) < \underline{\beta}_i(x) < \beta_i(x) < \overline{\beta}_i(x), \tag{3.4}$$

$$\overline{\beta}_j(x) - \underline{\beta}_j(x) < \delta(\underline{\beta}_i(x) - \overline{\beta}_j(x)) \tag{3.5}$$

and

$$\overline{\beta}_i(x) - \underline{\beta}_i(x) < \delta(\underline{\beta}_i(x) - \overline{\beta}_j(x)). \tag{3.6}$$

To justify the claim, assume first that $O(\beta_i) = O(\beta_j)$ and let $\gamma_t(x) = t\beta_i(x) + (1-t)\beta_j(x)$. Let

$$\begin{aligned}
\overline{\beta}_i(x) &= \gamma_{1+\epsilon}(x) \\
\underline{\beta}_i(x) &= \gamma_{1-\epsilon}(x) \\
\overline{\beta}_j(x) &= \gamma_{\epsilon}(x) \\
\underline{\beta}_j(x) &= \gamma_{-\epsilon}(x).
\end{aligned}$$

All these fractional power series are generic $O(\beta_i - \beta_j)$ perturbations of β_i and β_j for all but finitely many choices of ϵ. We compute

$$\overline{\beta}_i - \underline{\beta}_i = \overline{\beta}_j - \underline{\beta}_j = \frac{2\epsilon}{1-2\epsilon}(\underline{\beta}_i - \overline{\beta}_j).$$

The claim follows in this case if we choose $\epsilon < \min\{\frac{1}{4}, \frac{\delta}{4}\}$. If $O(\beta_i) < O(\beta_j)$, then we choose $\overline{\beta}_i$ and $\underline{\beta}_i$ as before, but we choose

$$\begin{aligned}
\overline{\beta}_j(x) &= (1+\epsilon)\beta_j(x) \\
\underline{\beta}_j(x) &= (1-\epsilon)\beta_j(x).
\end{aligned}$$

Again, for all but finitely many ϵ, these fractional power series are generic $O(\beta_j)$-perturbations of β_j and we compute

$$\begin{aligned}\overline{\beta}_i - \underline{\beta}_i &= 2\epsilon(\beta_i - \beta_j)\\ \overline{\beta}_j - \underline{\beta}_j &= 2\epsilon\beta_j\\ \underline{\beta}_i - \overline{\beta}_j &= (1-\epsilon)\beta_i - \beta_j.\end{aligned}$$

Since $O(\beta_i) < O(\beta_j)$, $\beta_j(x) < \frac{1}{2}\beta_i(x)$ for small x. So for small x,

$$\begin{aligned}\overline{\beta}_i(x) - \underline{\beta}_i(x) &< \frac{2\epsilon}{\frac{1}{2}-\epsilon}(\underline{\beta}_i(x) - \overline{\beta}_j(x))\\ \overline{\beta}_j(x) - \underline{\beta}_j(x) &< \frac{2\epsilon}{\frac{1}{2}-\epsilon}(\underline{\beta}_i(x) - \overline{\beta}_j(x))\end{aligned}$$

and the claim follows from choosing $\epsilon < \min\{\frac{1}{4}, \frac{\delta}{8}\}$.

Lemma 3.13. *There are $\epsilon > 0$ and $\rho > 0$ such that $H^i_{\epsilon,\rho} \cup H^j_{\epsilon,\rho} \subset \{(x,y) \mid \underline{\beta}_i(x) < y < \overline{\beta}_i(x)$ or $\underline{\beta}_j(x) < y < \overline{\beta}_j(x)\}$.*

Proof. It is enough to check that

$$(G(\underline{\beta}_j) \cup G(\overline{\beta}_j) \cup G(\underline{\beta}_i) \cup G(\overline{\beta}_i)) \cap (H^i_{\epsilon,\rho} \cup H^j_{\epsilon,\rho}) = \emptyset.$$

This follows directly from Lemma 3.9 and Lemma 3.11. □

Estimates (1) and (2) above can be shown by the same argument. To show (1), let $\delta > 0$ be arbitrary and notice that by Lemma 3.13 and (3.5), there is an N such that $|\,\|p_n\| - \|\tilde{p}_n\|\,| \leq |\,\|p_n - \tilde{p}_n\|\,| = |y_n - \tilde{y}_n| < |\overline{\beta}_i(x_n) - \underline{\beta}_i(x_n)| \leq \delta(\underline{\beta}_i(x_n) - \overline{\beta}_j(x_n)) < \delta\underline{\beta}_i(x_n) < \delta y_n$ for all $n > N$. Estimate (1) follows since $\|p_n\| \sim y_n$. To justify (3), (4) and (5) we introduce a pair of new sequences which help clarify the geometry of the situation. Let ϵ and ρ be given by Lemma 3.13. Let n be so large that ϵ_n and $\tilde{\epsilon}_n$ are less than ϵ and ρ_n and $\tilde{\rho}_n$ are less than ρ. Let $\bar{p}_n = (u_n, \underline{\beta}_i(u_n))$ and $\bar{q}_n = (x_n, \overline{\beta}_j(x_n))$. One possible configuration of these sequences is illustrated in Figure 2.

We have

$$\begin{aligned}\|\tilde{p}_n - \tilde{q}_n\| &\geq \|p_n - q_n\| - \|p_n - \tilde{p}_n\| - \|q_n - \tilde{q}_n\|\\ &\geq \|p_n - q_n\| - \delta\,\|p_n - \bar{q}_n\| - \delta\,\|q_n - \bar{p}_n\|\,.\end{aligned}$$

We consider the cases $x_n > u_n$ and $x_n \leq u_n$ separately. If $x_n > u_n$, then both $\|p_n - \bar{q}_n\|$ and $\|\bar{p}_n - q_n\|$ are less than or equal to $\|p_n - q_n\|$. In this case, $\|\tilde{p}_n - \tilde{q}_n\| \geq (1-2\delta)\,\|p_n - q_n\|$.

Next is the case $x_n \leq u_n$. If there is a $K > 0$ such that

$$\frac{q_n - \bar{q}_n}{\|q_n - \bar{q}_n\|} \cdot (1,0) > K,$$

then $\|q_n - \bar{q}_n\| < |x_n - u_n|/K = o(\|p_n - q_n\|)$. The last inequality follows from (3.3). If

$$\frac{q_n - \bar{q}_n}{\|q_n - \bar{q}_n\|} \cdot (1,0) \to 0 \text{ as } n \to \infty,$$

then we may assume that either $v_n < \overline{\beta}_j(x_n)$ for all n or that $v_n > \overline{\beta}_j(x_n)$ for all n by passing to a subsequence. If $v_n \leq \overline{\beta}_j(x_n)$, then $\|p_n - \bar{q}_n\| \leq \|p_n - q_n\|$. Now, assume that $v_n > \overline{\beta}_j(x_n)$. In this case, $O(\overline{\beta_j}) < 1$. To see this, let θ_n be the angle between $q_n - \bar{q}_n$ and $(1,0)$. If $O(\overline{\beta_j}) = 1$, then $|\tan\theta_n| \leq 2\overline{\beta_j}'(x_n) < 2M$ for a bound M on $\overline{\beta_j}'$. It follows that $\cos\theta_n$ is bounded away from 0, and that $(q_n - \bar{q}_n)\cdot(1,0)/\,\|q_n - \bar{q}_n\|$ does not converge to 0, contrary to our current assumptions. Therefore $O(\overline{\beta_j}) < 1$. Lemma 3.2 also implies that $\beta^*_k(y) \gtrsim y^{r-1}$ for $k = i, j$. This implies

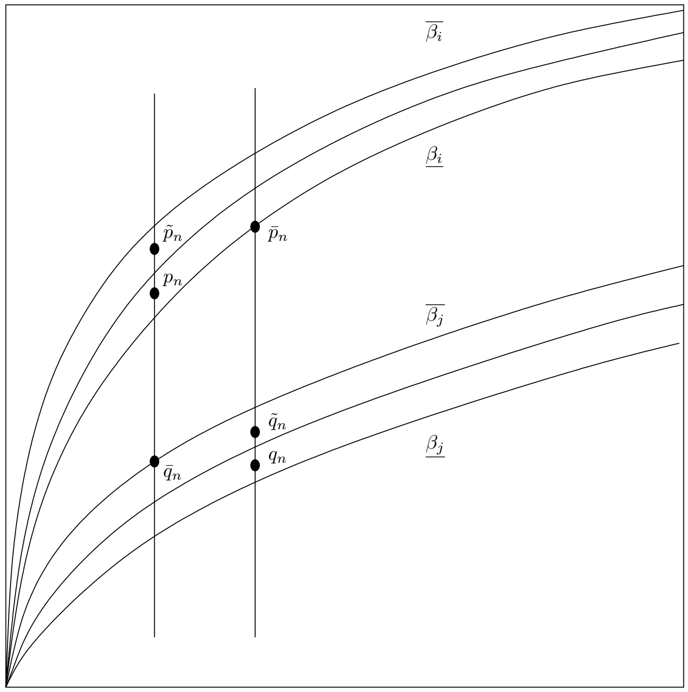

FIGURE 2. Example of a possible configuration of points when $x_n < u_n$.

that $\beta_k(x) \lesssim x^{1/(r-1)}$ and $\beta_k'(x) \lesssim x^{-(r-2)/(r-1)}$ for $k = i, j$. Since $O(\overline{\beta}_j) = O(\beta_j)$, similar inequalities must hold for $\overline{\beta}_j^*$ and $\overline{\beta}_j$ as well. We also claim that $\|q_n\| / \|p_n\|$ is bounded. Assume this is not the case. Then, by passing to a subsequence, we may assume that $\|p_n\| = o(\|q_n\|)$ for large n. Then $y_n = o(v_n)$ for large n, but by Lemma 3.2 again, this implies

$$|x_n - u_n| > |\underline{\beta_i}^*(y_n) - \overline{\beta_j}^*(v_n)| > |\overline{\beta_j}^*(y_n) - \overline{\beta_j}^*(v_n)| \gtrsim v_n^{r-1} \sim \|q_n\|^{r-1}$$

which is false, because, since (II) fails,

$$|x_n - u_n| = o(\|p_n\|^{r-1} + \|q_n\|^{r-1}) \|p_n - q_n\| .$$

This proves the claim. Using these observations, we see that

$$\begin{aligned}
\|q_n - \bar{q}_n\| &\leq |x_n - u_n| (\overline{\beta}_j'(x_n) + 1) \\
&\lesssim |x_n - u_n| \frac{1}{x_n^{\frac{r-2}{r-1}}} && \text{(since } O(\overline{\beta}_j') = O(\beta_j') \geq -\tfrac{r-2}{r-1}\text{)} \\
&\lesssim |x_n - u_n| \frac{1}{\|p_n\|^{r-2}} && \text{(since } \|p_n\| \sim y_n \sim \beta_i(x_n) \lesssim x_n^{\frac{1}{r-1}}\text{)} \\
&= \frac{o(\|p_n\|^{r-1} + \|q_n\|^{r-1}) \|p_n - q_n\|}{\|p_n\|^{r-2}} && \text{(by (3.3))} \\
&= o(\|p_n - q_n\|). && \text{(since } \|q_n\| / \|p_n\| \text{ is bounded)}
\end{aligned}$$

We conclude that

$$\|p_n - \bar{q}_n\| \leq \|p_n - q_n\| + \|q_n - \bar{q}_n\| = \|p_n - q_n\| + o(\|p_n - q_n\|).$$

Completely analogous arguments show that $\|q_n - \bar{p}_n\| \leq \|p_n - q_n\| + o(\|p_n - q_n\|)$. Altogether we have

$$\|\tilde{p}_n - \tilde{q}_n\| \geq \|p_n - q_n\| - \delta \|p_n - \bar{q}_n\| - \delta \|q_n - \bar{p}_n\| \geq (1 - 3\delta) \|p_n - q_n\|.$$

To finish the justification of (3), let δ_k be a sequence of positive real numbers converging to 0. By the above, for each k there is a natural number $N(k)$ such that $\|\tilde{p}_n - \tilde{q}_n\| \geq (1 - 3\delta_k) \|p_n - q_n\|$ when $n > N(k)$. Since $\delta_k \to 0$, (3) follows. To justify (4), notice that there is a natural number $M(k)$ such that $\|p_n - \tilde{p}_n\| < \delta_k \|p_n - \bar{q}_n\| \leq \delta_k(\|p_n - q_n\| + o(\|p_n - q_n\|))$ when $n > M(k)$. This clearly implies (4), and (5) follows by similar arguments.

3.4.2. *The other quadrants.* Let $\Phi(x, y) = (x, -y)$ and let $\omega_\Phi = \omega \circ \Phi$. By Lemma 2.2 and Lemma 2.7, (i) holds for ω_Φ. Hence, we may parametrize the components of $\Sigma(\omega_\Phi)$ in the first quadrant by Newton-Puiseux roots $\beta_{\Phi,i}, i = 1, \ldots, s_\Phi$ and the analysis of Section 3.4.1 holds for ω_Φ as well.

The fractional power series $-\beta_{\Phi,i}, i = 1, \ldots, s_\Phi$ parametrize the components of $\Sigma(\omega)$ contained in the fourth quadrant, and also, $H_{\epsilon,\rho}(\omega_\Phi) = \Phi(H_{\epsilon,\rho}(\omega))$. Hence, if we instead of β_i and β_j consider $-\beta_{\Phi,i}$ and $-\beta_{\Phi,j}$ in the discussion of Section 3.4.1, we get the same estimates (1)-(5). If we instead of β_i and β_j consider $-\beta_{\Phi,i}$ and β_j, we also obtain (1)-(5) after a minor modification of the justification of (3)-(5). In the latter case the corresponding branches of $\Sigma(\omega)$ have different tangent directions.

To study $H_{\epsilon,\rho}(\omega)$ in the second and third quadrant, let $\Psi(x, y) = (-x, y)$, and study the r-jet $\omega_\Psi = \omega \circ \Psi$. The components of $H_{\epsilon,\rho}(\omega_\Psi)$ contained in the first and fourth quadrant can be studied in the manner explained above, and since $H_{\epsilon,\rho}(\omega_\Psi) = \Psi(H_{\epsilon,\rho}(\omega))$, this gives the estimates (1)-(5) when we consider parametrizations of components of $\Sigma(\omega)$ in the second and/or third quadrant instead of β_i and β_j.

Since, by Lemma 3.12, (II) only fails along pairs of sequences on the same side of the y-axis, this establishes our estimates in all possible cases.

4. Relations between the Łojasiewicz inequalities

Le ω be an r-jet of rank 1 such that (I) holds. Let $\{C_i\}$ be the components of $\Sigma(\omega) \setminus \{0\}$. Recall the second Lojasiewicz inequality of Theorem 1.10:

There is a constant $C > 0$ and a neighbourhood U of 0 such that if $p \in C_i \cap U$ and $q \in C_j \cap U$ for some $i \neq j$, then

$$\|\omega(p) - \omega(q)\| \geq C(\|p\|^{r-1} + \|q\|^{r-1}) \|p - q\| \tag{II'}$$

Proposition 4.1. *If ω is of rank 1 and in standard form, then (II) holds for ω iff (II') holds.*

Proof. (II) $\Rightarrow$ (II') is obvious, (II') being a weakening of (II). We assume $\omega(x, y) = (x, f(x, y))$ and proceed to show that (II') $\Rightarrow$ (II). If (II) fails, then there are $i \neq j$ and sequences ϵ_n and ρ_n of positive real numbers converging to 0 and sequences $p_n = (x_n, y_n) \in H^i_{\epsilon_n,\rho_n}$ and $q_n = (u_n, v_n) \in H^j_{\epsilon_n,\rho_n}$. Then we have $f_y(p_n) = o(\|p_n\|^{r-1})$, $f_y(q_n) = o(\|q_n\|^{r-1})$ and

$$\|\omega(p_n) - \omega(q_n)\| = o(\|p_n\|^{r-1} + \|q_n\|^{r-1}) \|p_n - q_n\|. \tag{4.1}$$

Let $\tilde{p}_n$ and $\tilde{q}_n$ be the points on C_i and C_j having the same x-component as p_n and q_n respectively. These points exist by Lemma 3.2. By the estimates (3)-(5) of the previous section we have

$$\|p_n - q_n\| \sim \|\tilde{p}_n - \tilde{q}_n\| \tag{4.2}$$

and

$$\|p_n - \tilde{p}_n\| = o(\|p_n - q_n\|) \tag{4.3}$$

and

$$\|q_n - \tilde{q}_n\| = o(\|p_n - q_n\|). \tag{4.4}$$

As remarked in Section 3.4.2, (1)-(5) hold regardless of whether (p_n) and (q_n) are in the same quadrant or not. By (1) and (2), $\|p_n\| \sim \|\tilde{p}_n\|$ and $\|q_n\| \sim \|\tilde{q}_n\|$. Let $\epsilon < \epsilon_0$ where ϵ_0 is given by Proposition 1.1. Assume that n is so large that $\epsilon_n < \epsilon$. Then $p_n, \tilde{p}_n \in H^i_\epsilon$ and since H^i_ϵ is semialgebraic and connected, the line segment between p_n and $\tilde{p}_n$ must be contained in H^i_ϵ. If b_n is a sequence such that for every n, b_n lies on the line segment between p_n and $\tilde{p}_n$ or on the line segment between q_n and $\tilde{q}_n$, then $\|b_n\| \sim \|p_n\|$ or $\|b_n\| \sim \|q_n\|$, and since (I), and therefore (i) holds, we must have $|f_{yy}(x_n, y)| > 0$ on the open line segment between p_n and $\tilde{p}_n$. It follows that $|f_y(b_n)| < |f_y(p_n)| = o(\|p_n\|^{r-1}) = o(\|b_n\|^{r-1})$. In a similar fashion we obtain similar inequalities for points on the line segment between q_n and $\tilde{q}_n$. Now, using the Mean Value Theorem, we can find c_n on the line segment between p_n and $\tilde{p}_n$ and d_n on the line segment between q_n and $\tilde{q}_n$ such that

$$\begin{aligned}
\|\omega(\tilde{p}_n) - \omega(\tilde{q}_n)\| &\le \|\omega(\tilde{p}_n) - \omega(p_n)\| + \|\omega(p_n) - \omega(q_n)\| \\
&\quad + \|\omega(q_n) - \omega(\tilde{q}_n)\| \\
&= |f_y(c_n)|\,\|\tilde{p}_n - p_n\| + o(\|p_n\|^{r-1} + \|q_n\|^{r-1})\,\|p_n - q_n\| \\
&\quad + |f_y(d_n)|\,\|q_n - \tilde{q}_n\| \\
&= o(\|p_n\|^{r-1}) o(\|p_n - q_n\|) + o(\|p_n\|^{r-1} + \|q_n\|^{r-1})\,\|p_n - q_n\| \\
&\quad + o(\|q_n\|^{r-1}) o(\|p_n - q_n\|) \\
&= o(\|\tilde{p}_n\|^{r-1} + \|\tilde{q}_n\|^{r-1})\,\|\tilde{p}_n - \tilde{q}_n\|.
\end{aligned}$$

This shows that (II') fails. ☐

Lemma 4.2 (=Lemma 2.9). *Let ω be an r-jet which satisfies (I), and let ω' be a C^r map germ with $\omega \sim_{\mathcal{A}_r} \omega'$. Then (II') holds for ω if and only if (II') holds for $j^r\omega'$.*

Proof. Let ω be an r-jet, h and k C^r-diffeomorphisms of neighbourhoods of 0 and $\omega' = k \circ \omega \circ h^{-1}$. We may assume that ω is in standard form. Assume that (I) holds for ω and that (II'), and hence (II), fails for ω along sequences in H_i and H_j which are different components of $H_{\epsilon,\rho}(\omega)$. Let C_i and C_j be the branches of $\Sigma(\omega)$ corresponding to H_i and H_j respectively. From the proof of Lemma 2.6 we know that in a small neighbourhood of 0, $h(C_i)$ and $h(C_j)$ are in different components of $H_{\epsilon,\rho}(j^r\omega')$. Let C'_i and C'_j denote the components of $\Sigma(j^r\omega')$ contained in the same components of $H_{\epsilon,\rho}(j^r\omega')$ as $h(C_i)$ and $h(C_j)$ respectively. Since $h^{-1}(C'_i)$ and $h^{-1}(C'_j)$ belong to the singular set of $k^{-1} \circ j^r\omega' \circ h$, which is a C^r realization of ω, $h^{-1}(C'_i)$ and $h^{-1}(C'_j)$ belong to $H^i_{\epsilon,\rho}(\omega)$ and $H^j_{\epsilon,\rho}(\omega)$ for every small ϵ. It now follows from the proof of Proposition 4.1 that (II) fails for ω along sequences in $h^{-1}(C'_i)$ and $h^{-1}(C'_j)$. Then it follows from the proof of Lemma 2.6 again that (II) fails for $j^r\omega'$ along sequences in C'_i and C'_j. This shows that (II') fails for $j^r\omega'$ and finishes the proof of the lemma. ☐

Proposition 4.3 (=Proposition 2.10)**.** *If ω is in standard form and satisfies (i), then (II') and (ii) are equivalent.*

Proof. (II')$\Rightarrow$ (ii) is obvious, (ii) being a weakening of (II'). Assume that ω is in standard form and satisfies (i), but not (II'). Since (i) is satisfied, Lemma 3.12 implies that (II') fails along sequences on the same side of the y-axis. Assume they are in the 1st or 4th quadrant. Note that Lemma 3.4 also holds for singular branches in the 4th quadrant, and by arguments similar to the arguments in Section 3.4.2, we may parametrize the branches of $\Sigma(\omega)$ in the 4th quadrant by convergent fractional power series. Let now β_i, $i = 1, \dots, S$ be parametrizations of the S branches of $\Sigma(\omega)$ in these quadrants. Then there are $i \neq j$ and sequences $p_n = (x_n, y_n)$ and $q_n = (u_n, v_n)$ both converging to 0 such that $p_n \in G(\beta_i)$, $q_n \in G(\beta_j)$ and

$$\|\omega(p_n) - \omega(q_n)\| = o(\|p_n\|^{r-1} + \|q_n\|^{r-1}) \, \|p_n - q_n\| \, .$$

We may assume that $x_n > u_n > 0$ Let $\tilde{v}_n = \beta_j(x_n)$. Then $\tilde{q}_n = (x_n, \tilde{v}_n) \in G(\beta_j)$. Let $\beta(t) = (\beta^1(t), \beta^2(t))$ be the parametrization of $G(\beta_j)$ by arclength with $\beta(0) = 0$ and $\beta(t) \in G(\beta_j)$ for $t > 0$. Assume that $(u_n, v_n) = \beta(t_{u_n})$ and $(x_n, \tilde{v}_n) = \beta(t_{x_n})$. Then there are parameter values c_n and d_n between t_{u_n} and t_{x_n} such that

$$\begin{aligned}
\|\omega(u_n, v_n) - \omega(x_n, \tilde{v}_n)\| &= \|\omega(\beta(t_{u_n})) - \omega(\beta(t_{x_n}))\| \\
&= \left\| \begin{pmatrix} \beta^1(t_{u_n}) - \beta^1(t_{x_n}) \\ Df(\beta(c_n)) \cdot \beta'(c_n)(t_{u_n} - t_{x_n}) \end{pmatrix} \right\| \\
&= \left\| \begin{pmatrix} \frac{d}{dt}\beta^1(d_n) \\ f_x(\beta(c_n))\frac{d}{dt}\beta^1(c_n) \end{pmatrix} \right\| |t_{u_n} - t_{x_n}| \quad (\text{since } f_y(\beta(t)) \equiv 0) \\
&\lesssim \max\left\{ \left|\frac{d}{dt}\beta^1(c_n)\right|, \left|\frac{d}{dt}\beta^1(d_n)\right| \right\} |t_{u_n} - t_{x_n}|.
\end{aligned}$$

If $O(\beta_j) = 1$, then $t \sim \|\beta(t)\| \sim |\beta^1(t)|$, and in that case,

$$\|\omega(u_n, v_n) - \omega(x_n, \tilde{v}_n)\| \lesssim |x_n - u_n|.$$

If $O(\beta_j) < 1$, then

$$\left|\frac{d}{dt}\beta^1(c_n)\right| \sim \left|\frac{\frac{d}{dt}\beta^1(c_n)}{\frac{d}{dt}\beta^2(c_n)}\right|,$$

since β is parametrised by arclength. Since we have assumed that $x_n > u_n$, we have $t_{x_n} > t_{u_n}$. Then

$$\left|\frac{\frac{d}{dt}\beta^1(c_n)}{\frac{d}{dt}\beta^2(c_n)}\right| < |\beta_j'(x_n)|^{-1}.$$

Now, since $O(\beta_j) < 1$, there is a small $\epsilon > 0$ such that $|\beta_j(x)|$ is a concave function on $[0, \epsilon)$. This implies that for large enough n,

$$|\beta_j'(x_n)| < \left|\frac{v_n - \tilde{v}_n}{x_n - u_n}\right| < \frac{|\beta_j(x_n)|}{|x_n|}.$$

But since β_j is a fractional power series in x, $|\beta_j(x_n)| \sim |x_n||\beta_j'(x_n)|$. Thus

$$|\beta_j'(x_n)| \sim \left|\frac{v_n - \tilde{v}_n}{x_n - u_n}\right|,$$

and hence,

$$\left|\frac{d}{dt}\beta^1(c_n)(t_{u_n} - t_{x_n})\right| \lesssim \left|\frac{x_n - u_n}{v_n - \tilde{v}_n}\right| |v_n - \tilde{v}_n| = |x_n - u_n|.$$

The same holds if we replace c_n with d_n. In any case,

$$\|\omega(u_n, v_n) - \omega(x_n, \tilde{v}_n)\| \lesssim |x_n - u_n| = o(\|p_n\|^{r-1} + \|q_n\|^{r-1}) \|p_n - q_n\| .$$

Using this we get

$$\begin{aligned}\|\omega(p_n) - \omega(\tilde{q}_n)\| &\leq \|\omega(p_n) - \omega(q_n)\| + \|\omega(q_n) - \omega(\tilde{q}_n)\| \\ &= o(\|p_n\|^{r-1} + \|q_n\|^{r-1}) \|p_n - q_n\| \\ &= o(\|p_n\|^{r-1} + \|\tilde{q}_n\|^{r-1}) \|p_n - \tilde{q}_n\| ,\end{aligned}$$

which means that (ii) fails to hold. The last equality needs some justification. Notice that $u_n < x_n$ implies that $\|q_n\| < \|\tilde{q}_n\|$. We also have to show that $\|p_n - q_n\| \lesssim \|p_n - \tilde{q}_n\|$. We claim that $u_n = x_n + o(x_n)$. If not, then $|x_n - u_n| = x_n - u_n \sim x_n$. By Lemma 3.2, $x_n \gtrsim \|p_n\|^{r-1}$. This implies that $|x_n - u_n| \gtrsim \|p_n\|^{r-1}$ which contradicts the failure of (II'). Therefore, we may assume that $u_n = x_n + o(x_n)$. This gives $|\beta_j(u_n)| \sim |\beta_j(x_n)|$ and hence, $\|q_n\| \sim \|\tilde{q}_n\|$. Assume that $\|q_n\| = o(\|p_n\|)$. In this case, $\|\tilde{q}_n\| = o(\|p_n\|)$ and it follows that $\|p_n\| \sim \|p_n - q_n\| \sim \|p_n - \tilde{q}_n\|$. Assume now that $\|p_n\| \lesssim \|q_n\|$. We have

$$\|p_n - q_n\| \leq \|p_n - \tilde{q}_n\| + |\beta_j(x_n) - \beta_j(u_n)| + |x_n - u_n|.$$

Using that $|\beta_j(x_n) - \beta_j(u_n)| \leq (|\beta_j'(x_n)| + |\beta_j'(u_n)|)|x_n - u_n|$, we get

$$\|p_n - q_n\| \leq \|p_n - \tilde{q}_n\| + (|\beta_j'(x_n)| + |\beta_j'(u_n)| + 1)|x_n - u_n|.$$

As in the justification of (3) in Section 3.4.1, Lemma 3.2 implies that $|\beta_j'(x_n)| \lesssim x_n^{-\frac{r-2}{r-1}} \lesssim 1/\|\tilde{q}_n\|^{r-2} \sim 1/\|q_n\|^{r-2}$ and similarly, $|\beta_j'(u_n)| \lesssim 1/\|q_n\|^{r-2}$. Now we have

$$\begin{aligned}\|p_n - q_n\| &\leq \|p_n - \tilde{q}_n\| + (1 + \frac{2}{\|q_n\|^{r-2}})|x_n - u_n| \\ &= \|p_n - \tilde{q}_n\| + (1 + \frac{2}{\|q_n\|^{r-2}})o(\|p_n\|^{r-1} + \|q_n\|^{r-1}) \|p_n - q_n\| \\ &= \|p_n - \tilde{q}_n\| + o(\|p_n - q_n\|).\end{aligned}$$

The last equality follows from the assumption that $\|p_n\| \lesssim \|q_n\|$. This completes the proof of Proposition 4.3. $\square$

Proof of Proposition 1.9. This is a direct consequence of Proposition 4.1 and Proposition 4.3. $\square$

5. Construction of Whitney field and proof of Theorem 1.7

This section deals with the construction of a Whitney field which leads to the proof of the only if part of Theorem 1.7. Let $\omega(x, y) = (x, f(x, y))$ be an r-jet of rank 1 in standard form having no branches of its singular set tangent to the x-axis. Assume that (i) holds and (ii) fails for ω. We only consider the case when (ii) fails along sequences in the first quadrant. Then there are sequences $p_n = (x_n, y_n) \in C_i$ and $q_n = (x_n, v_n) \in C_j$ such that $\|p_n\| \to 0$, $\|q_n\| \to 0$ and

$$|f(p_n) - f(q_n)| = o(\|p_n\|^{r-1} + \|q_n\|^{r-1})|y_n - v_n|.$$

In this case, $\|p_n\| \sim y_n$ and $\|q_n\| \sim v_n$. We assume that $y_n > v_n$ and that $\|p_n - q_n\| = o(\|p_n\| + \|q_n\|)$ and thus, $\|p_n\| \sim \|q_n\|$.

Lemma 5.1. *There are sequences of real positive numbers $\tilde{\epsilon}_n$ and $\tilde{\rho}_n$ converging to 0 and sequences $\tilde{p}_n = (x_n, \tilde{y}_n) \in H^i_{\tilde{\epsilon}_n, \tilde{\rho}_n}$ and $\tilde{q}_n = (x_n, \tilde{v}_n) \in H^j_{\tilde{\epsilon}_n, \tilde{\rho}_n}$ such that*

$$f_y(x_n, \tilde{y}_n) = f_y(x_n, \tilde{v}_n) = \frac{f(x_n, \tilde{y}_n) - f(x_n, \tilde{v}_n)}{\tilde{y}_n - \tilde{v}_n} = o(\|\tilde{p}_n\|^{r-1} + \|\tilde{q}_n\|^{r-1}).$$

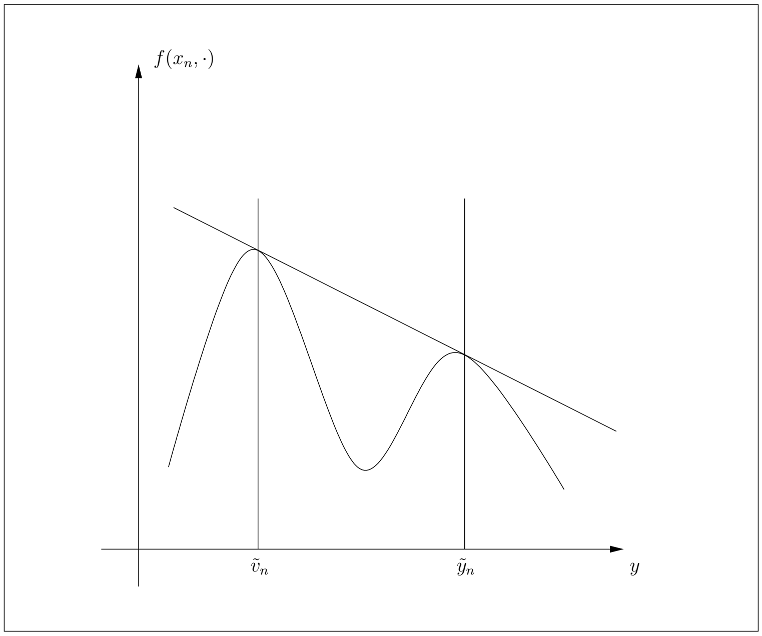

FIGURE 3. Illustration of the geometric idea behind Lemma 5.1.

The points $\tilde{y}_n$ and $\tilde{v}_n$ are chosen such that we obtain the geometric situation illustrated in Figure 3.

Proof of Lemma 5.1. If there are subsequences (p_{n_k}) and (q_{n_k}) of (p_n) and (q_n) respectively such that $f(x_{n_k}, y_{n_k}) = f(x_{n_k}, v_{n_k})$, then, since $f_y(x_n, y_n) = f_y(x_n, v_n) = 0$, we may take $\tilde{y}_n = y_n$ and $\tilde{v}_n = v_n$. If there are no such subsequences, let $p_n(t) = (x_n, y_n + t)$ and $q_n(s) = (x_n, v_n + s)$. Recall that we have assumed that (p_n) and (q_n) are in the first quadrant. We have also assumed that $y_n > v_n$, and hence, $y_n + t > v_n + s$ and $\|p_n(t)\| > \|q_n(s)\|$ for small s and t. In particular, $\|p_n(t)\| > \|q_n(s)\|$ when $p_n(t) \in H^i_{\epsilon,\rho}$ and $q_n(s) \in H^j_{\epsilon,\rho}$. Since (i) holds, there is a constant $C > 0$ such that

$$|f_y(p)| + |f_{yy}(p)|\,\|p\| \geq C\,\|p\|^{r-1}$$

for all p in a neighbourhood $B(0, \rho)$ of 0. Let $\epsilon < C$ and as always, assume that $\epsilon < \epsilon_0$ where ϵ_0 is chosen such that the conclusion of Lemma 3.1 holds. Since $f_{yy}(p) \neq 0$ for all $p \in H_{\epsilon,\rho}$, the restriction of the function $u \mapsto f_y(p + (0, u))$ to any component of the set $\{u | p + (0, u) \in H_{\epsilon,\rho}\}$ is injective. Assume that ρ is large enough to ensure that

$$\sup\{\|p_n(t)\| \mid p_n(t) \in H^i_{\epsilon,\rho}\} < \rho.$$

Since $\|p_n(t)\| > \|q_n(s)\|$, $\{f_y(q_n(s)) \mid q_n(s) \in H^j_{\epsilon,\rho}\} \subset \{f_y(p_n(t)) \mid p_n(t) \in H^i_{\epsilon,\rho}\}$. In fact, both these sets are intervals. Using that $f_{yy}(p) \neq 0$ for all $p \in H_{\epsilon,\rho}$ together with the definition of the $H_{\epsilon,\rho}$ and the assumption on ρ, we see that there are real numbers s_1, s_2, t_1, t_2 such that

$$\{f_y(q_n(s)) \mid q_n(s) \in H^j_{\epsilon,\rho}\} = [-\epsilon\,\|q_n(s_1)\|^{r-1}, \epsilon\,\|q_n(s_2)\|^{r-1}]$$

and

$$\{f_y(p_n(t)) \mid p_n(t) \in H^i_{\epsilon,\rho}\} = [-\epsilon\,\|p_n(t_1)\|^{r-1}, \epsilon\,\|p_n(t_2)\|^{r-1}].$$

It follows that when $q_n(s) \in H^j_{\epsilon,\rho}$, the equation $f_y(q_n(s)) = f_y(p_n(t))$ has a unique solution $t = h(s)$ with $p_n(t) \in H^i_{\epsilon,\rho}$.

Let $F : \mathbb{R}^2 \to \mathbb{R}$ be given by $F(s,t) = f_y(p_n(t)) - f_y(q_n(s))$. We have

$$\frac{\partial F}{\partial t}(s,t) = f_{yy}(p_n(t)) \neq 0$$

when $p_n(t) \in H_{\epsilon,\rho}$. The function $h(s)$ above satisfies $F(s,h(s)) = 0$, and by the Implicit Function Theorem, h is a smooth function.

Define the function G by

$$G(s) = \frac{f(p_n(h(s))) - f(q_n(s))}{p_n(h(s)) - q_n(s)} - f_y(q_n(s)).$$

Clearly G is continuous near $s = 0$. Let ϵ_n be defined by

$$|f(p_n) - f(q_n)| = \epsilon_n(\|p_n\|^{r-1} + \|q_n\|^{r-1})|y_n - v_n|. \tag{5.1}$$

Note that $\epsilon_n > 0$ and that $\epsilon_n \to 0$ as $n \to \infty$. For constants K and indices n such that $|K|\epsilon_n < \epsilon$, let $S(K,n) \in \{s \mid q_n(s) \in H^j_{\epsilon,\rho}\}$ be defined by

$$f_y(q_n(S(K,n))) = K\epsilon_n \|q_n(S(K,n))\|^{r-1}.$$

This definition is unambiguous because $f_{yy} \neq 0$ in $H_{\epsilon,\rho}$. There are eight cases to consider, one for each possible value of

$$\operatorname{Sign}(\omega) = \left(\frac{f(p_n) - f(q_n)}{|f(p_n) - f(q_n)|}, \frac{f_{yy}(p_n)}{|f_{yy}(p_n)|}, \frac{f_{yy}(q_n)}{|f_{yy}(q_n)|}\right) \in \{-1,1\}^3.$$

The denominators in the definition of $\operatorname{Sign}(\omega)$ cause no problems, because $f_{yy}(p) \neq 0$ when $f_y(p) = 0$ as a consequence of (i). Suppose $\operatorname{Sign}(\omega) = (-1,-1,-1)$. This situation is illustrated in Figure 3. Since $h(0) = 0$, $G(0) < 0$. Let $S = S(K,n)$ for some fixed $K < 0$ and assume that $G(S) < 0$. Notice that necessarily, $S > 0$. There is a sequence ρ_n converging to 0 such that $p_n(h(S)) \in H^i_{|K|\epsilon_n,\rho_n}$ and $q_n(S) \in H^j_{|K|\epsilon_n,\rho_n}$. We get

$$f(p_n(h(S))) - f(q_n(S)) < f_y(q_n(S))(y_n + h(S) - v_n - S) < 0.$$

By our assumption that $\|p_n - q_n\| = o(\|p_n\| + \|q_n\|)$, we have $\|p_n\| = \|q_n\| + o(\|q_n\|)$. By the estimates (1)-(3) of Section 3, $|y_n + h(S) - v_n - S| = |y_n - v_n| + o(|y_n - v_n|)$, $\|p_n(h(S))\| = \|p_n\| + o(\|p_n\|)$ and $\|q_n(S)\| = \|q_n\| + o(\|q_n\|)$. This gives

$$|f(p_n(h(S))) - f(q_n(S))| > \frac{|K|}{4}\epsilon_n(\|p_n\|^{r-1} + \|q_n\|^{r-1})|y_n - v_n|.$$

Let $0 < \delta < \frac{1}{4}$. By estimates (4) and (5) of Section 3, $\|p_n(h(S)) - p_n\| = o(|y_n - v_n|)$ and $\|q_n(S) - q_n\| = o(|y_n - v_n|)$. Furthermore, since $f_{yy} \neq 0$ in $H_{\epsilon,\rho}$, the maximum of $\left|f_y|\overline{p_n p_n(h(S))}\right|$ is $|f_y(p_n(h(S)))|$ and the maximum of $\left|f_y|\overline{q_n q_n(S)}\right|$ is $|f_y(q_n(S))|$. Using this and our assumption that $\|p_n\| = \|q_n\| + o(\|q_n\|)$, we get

$$\begin{aligned}
|f(p_n) - f(q_n)| &\geq |f(p_n(h(S))) - f(q_n(S))| - |f(p_n(h(S))) - f(p_n)| - |f(q_n(S)) - f(q_n)| \\
&\geq |f(p_n(h(S))) - f(q_n(S))| - |K|\epsilon_n \|q_n(S)\|^{r-1} (\|p_n(h(S)) - p_n\| + \|q_n(S) - q_n\|) \\
&\geq |f(p_n(h(S))) - f(q_n(S))| - |K|\epsilon_n\delta(\|q_n\|^{r-1} + \|p_n\|^{r-1})|y_n - v_n| \\
&\geq \frac{|K|}{4}\epsilon_n(1 - 4\delta)(\|p_n\|^{r-1} + \|q_n\|^{r-1})|y_n - v_n|.
\end{aligned}$$

When n is large, we may take $|K| > 4/(1-4\delta)$ and this contradicts (5.1), and hence, $G(S(K,n)) > 0$. By the Intermediate Value Theorem, there is a sequence (s_n) with $0 < s_n < S(K,n)$ such that $G(s_n) \equiv 0$. The proof is finished in this case by choosing $\tilde{\epsilon}_n = |K|\epsilon_n$, $\tilde{y}_n = y_n + h(s_n)$ and $\tilde{v}_n = v_n + s_n$.

All the other seven cases are checked by essentially the same argument. It is just a matter of keeping track of the signs and the directions of the inequalities, so the details are left out. $\square$

Let $\tilde{p}_n = (x_n, \tilde{y}_n)$ and $\tilde{q}_n = (x_n, \tilde{v}_n)$ be the sequences given by Lemma 5.1. We may assume that for all n, $\|\tilde{p}_{n+1}\| < \frac{1}{2}\|\tilde{q}_n\|$. Remember that we have also assumed that $\|\tilde{p}_n\| \geq \|\tilde{q}_n\|$ and $\|\tilde{p}_n - \tilde{q}_n\| = o(\|\tilde{p}_n\|)$. Let $K = \{0\} \cup \bigcup_n \{\tilde{p}_n, \tilde{q}_n\}$. We define an r-th order Taylor field Q on K with values in $\mathbb{R}$ by

$$Q^m(p) = \begin{cases} f(\tilde{q}_n) - f(\tilde{p}_n), & p = \tilde{p}_n,\ m = (0,0); \\ -\frac{f(\tilde{p}_n) - f(\tilde{q}_n)}{\tilde{y}_n - \tilde{v}_n}, & p = \tilde{p}_n, \tilde{q}_n,\ m = (0,1); \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 5.2. *Q is a Whitney field.*

Proof. Let $X = (x, y)$. We have to show that for all $p, q \in K$, $m \in \mathbb{N}^2$,

$$\begin{aligned}(R_qQ)^m(p) &= Q^m(p) - \frac{\partial^{|m|}}{\partial X^m}\left(\sum_\alpha (\frac{1}{\alpha!}Q^\alpha(q)(X-q)^\alpha)\right)\Bigg|_{X=p} \\ &= o(\|p-q\|^{r-|m|}).\end{aligned}$$

There are a number of cases to consider, each of which is straightforward. In any of the cases $(p,q) = (\tilde{p}_n, \tilde{q}_n)$ or $(p,q) = (\tilde{q}_n, \tilde{p}_n)$, the definition of Q gives us that $(R_qQ)^m(p) = 0 = o(\|p-q\|^{r-|m|})$ for $m = (0,0)$ and $m = (0,1)$. In the remaining combinations, $\|p-q\| > \frac{1}{2}\max\{\|p\|, \|q\|\}$ and $(R_qQ)^m(p) = o((\max\{\|p\|, \|q\|\})^{r-|m|})$ for $m = (0,0)$ and $m = (0,1)$. Since $(R_qQ)^m(p) \equiv 0$ when $m = (1,0)$ or $|m| > 1$, it follows that Q is a Whitney field. $\square$

Proof of Theorem 1.7. Assume first that (i) and (ii) hold for ω of rank 1 in standard form $\omega(x,y) = (x, f(x,y))$. By Lemma 2.7 and Proposition 1.9, (I) and (II) holds for ω as well. Then we may use Theorem 1.3 to conclude that ω is $\mathcal{A}_0$-sufficient.

Now, suppose that (i) fails for ω. By Lemma 2.7, (I) also fails for ω, and by Theorem 1.3, ω is not $\mathcal{A}_0$-sufficient.

Finally, suppose that (i) holds and (ii) fails for ω. Then there are distinct components C_i and C_j of $\Sigma(\omega)$ and sequences $p_n = (x_n, y_n) \in C_i$ and $q_n = (x_n, v_n) \in C_j$ such that

$$|f(p_n) - f(q_n)| = o(\|p_n\|^{r-1} + \|q_n\|^{r-1})|y_n - v_n|.$$

By passing to a subsequence, we may also assume that $\|p_n\| \geq \|q_n\|$ and $\|p_{n+1}\| < \frac{1}{2}\|q_n\|$ for all n. If there are subsequences (p_{n_k}) and (q_{n_k}) of (p_n) and (q_n), respectively, with

$$\|p_{n_k} - q_{n_k}\| \sim \max\{\|p_{n_k}\|, \|q_{n_k}\|\},$$

then it is easy to see that the Taylor field

$$Q_1^m(p) = \begin{cases} f(q_{n_k}) - f(p_{n_k}), & p = p_{n_k},\ m = (0,0); \\ 0, & \text{otherwise} \end{cases}$$

is a Whitney field. By Whitney's Extension Theorem ([3]), we may extend Q_1 by a C^r map h_1 defined in a neighbourhood of 0. By construction of Q_1, $j^r h_1(0) = 0$, and hence, $\omega + h_1$ is a C^r realization of ω. However, p_{n_k} and q_{n_k} are singular points of $\omega + h_1$ for every n, and $(\omega + h_1)(p_{n_k}) = (\omega + h_1)(q_{n_k})$. This gives sequences of singular double points of $\omega + h_1$ converging to 0, and it is shown in [2] that a sufficient jet cannot have any such representative. Thus, ω is not $\mathcal{A}_0$-sufficient.

If there are no subsequences as above, then we may assume that $\|p_n - q_n\| = o(\|p_n\| + \|q_n\|)$. By Lemma 2.1, Lemma 2.6, Lemma 2.8 and Proposition 1.9, we may assume that p_n and q_n are in the first quadrant and that $\|p_n\| \sim y_n$ and $\|q_n\| \sim v_n$. In this situation, we can find the sequences $(\tilde{p}_n)$ and $(\tilde{q}_n)$ of Lemma 5.1 and construct the Whitney field Q of Lemma 5.2. By Whitney's Extension Theorem again, we may extend Q by a C^r map h defined in a neighbourhood of 0. By construction of Q, $j^r h(0) = 0$, and hence, $\omega + h$ is a C^r realization of ω. Again, $\tilde{p}_n$ and $\tilde{q}_n$ are singular points of $\omega + h$ for every n, and $(\omega + h)(\tilde{p}_n) = (\omega + h)(\tilde{q}_n)$. This gives sequences of singular double points of $\omega + h$ converging to 0, and hence, ω is not $\mathcal{A}_0$-sufficient. The proof is finished. □

6. Examples

Example 6.1 (Example 1 of [2] revised)**.** Let $r > 3$ and let $\omega(x, y) = (x, f(x, y)) = (x, xy + y^r)$. Since ω is given in standard form, we can apply Theorem 1.7 to prove that ω is $\mathcal{A}_0$-sufficient in $\mathcal{E}_{[r]}(2, 2)$. We have $f_y(x, y) = x + ry^{r-1}$ and $f_{yy}(x, y) = r(r-1)y^{r-2}$.

Assume that (i) does not hold for ω. Then there is a sequence $p_n = (x_n, y_n)$ converging to 0 such that

$$|f_y(p_n)| + |f_{yy}(p_n)|\,\|p_n\| = o(\|p_n\|^{r-1}).$$

Thus, $|f_y(p_n)| = |x_n + ry_n^{r-1}| = o(\|(x_n, y_n)\|^{r-1})$ and this implies that $|x_n| \sim |y_n|^{r-1}$ and $\|p_n\| \sim |y_n|$. But then $f_{yy}(p_n) \geq \|p_n\|^{r-2}$, which contradicts that (i) fails. This proves that ω satisfies (i).

If r is even, then ω has one branch on each side of the y-axis, and (ii) is trivially satisfied. Assume that r is odd. Then $\Sigma(\omega) = \{(x, y) | x = -ry^{r-1}\}$. Let $p = (-ry^{r-1}, y)$ and $q = (-ry^{r-1}, -y)$. Then $\|p - q\| \sim \|p\| = \|q\| \sim |y|$ and we get

$$|f(p) - f(q)| = |2ry^r + 2y^r| \gtrsim (\|p\|^{r-1} + \|q\|^{r-1})|y|.$$

This shows that (ii) holds, and by Theorem 1.7, ω is sufficient as claimed.

Example 6.2. Let $a > b > c > 0$ and let $\omega(x, y) = (x, f(x, y))$ in $J^7(2, 2)$ be such that $f_y(x, y) = (x - ay^2)(x - by^2)(x - cy^2)$. Let

$$F(x, y) = x - y - \frac{1}{3}(a + b + c)(x^3 - y^3) + \frac{1}{5}(ab + ac + bc)(x^5 - y^5) - \frac{1}{7}abc(x^7 - y^7).$$

We claim that ω is $\mathcal{A}_0$-sufficient in $\mathcal{E}_{[7]}(2, 2)$ if $0 \notin \{F(a^{-\frac{1}{2}}, b^{-\frac{1}{2}}), F(a^{-\frac{1}{2}}, c^{-\frac{1}{2}}), F(b^{-\frac{1}{2}}, c^{-\frac{1}{2}})\}$. This means that we need to verify (i) and (ii) of Theorem 1.7 for ω with $r = 7$.

Assume that (i) fails. Then there is a sequence $p_n = (x_n, y_n)$ converging to 0 such that

$$|f_y(p_n)| + |f_{yy}(p_n)|\,\|p_n\| = o(\|p_n\|^6).$$

From the expression for f_y we conclude that (i) can only fail along the sequence if $x_n = dy_n^2 + o(y_n^2)$ for some $d \in \{a, b, c\}$. We also have

$$f_{yy}(x, y) = -2(a + b + c)x^2y + 4(ab + ac + bc)xy^3 - 6abcy^5.$$

Suppose $x_n = ay_n^2 + o(y_n^2)$. Then $\|p_n\| \sim |y_n|$ and

$$f_{yy}(p_n) = -[2a^2(a + b + c) - 4a(ab + ac + bc) + 6abc]y_n^5 + o(y_n^5).$$

But $f_{yy}(p_n) = o(y_n^5)$ since (i) fails, and hence,

$$2a^2(a + b + c) - 4a(ab + ac + bc) + 6abc = 0.$$

Since $a \neq 0$, this implies the equation

$$(a - b)(a - c) = 0 \tag{6.1}$$

which cannot hold since $a > b > c$. The same argument applies when $x_n = by_n^2 + o(y_n^2)$ and when $x_n = cy_n^2 + o(y_n^2)$ and gives equations

$$(b-c)(b-a) = 0$$

and

$$(c-a)(c-b) = 0.$$

None of these two equations can have a solution with $a > b > c$. Altogether this shows that (i) holds for ω when $r = 7$.

To verify (ii), notice that for $s, t > 0$,

$$\left| f(x, \sqrt{\frac{x}{s}}) - f(x, \sqrt{\frac{x}{t}}) \right| = x^{\frac{7}{2}} |F(s^{-\frac{1}{2}}, t^{-\frac{1}{2}})|.$$

This proves that (ii) holds with $r = 7$, since we have assumed that

$$0 \notin \{F(a^{-\frac{1}{2}}, b^{-\frac{1}{2}}), F(a^{-\frac{1}{2}}, c^{-\frac{1}{2}}), F(b^{-\frac{1}{2}}, c^{-\frac{1}{2}})\}.$$

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APPARENT CONTOURS OF STABLE MAPS INTO THE SPHERE

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Abstract. For a stable map $\varphi\colon M \to S^2$ of a closed and connected surface into the sphere, let $c(\varphi)$ and $n(\varphi)$ denote the numbers of cusps and nodes respectively. In this paper, for each integer $i \geq 1$, in the given homotopy class with i fold curve components, we will determine the minimal number $c+n$.

1. Introduction

Let M be a closed and connected surface and N a connected surface. Let $\varphi\colon M \to N$ be a C^∞ map. Define the set of singular points of φ as

$$S(\varphi) = \{p \in M \mid \text{rank } d\varphi_p < 2\}.$$

We call $\varphi(S(\varphi))$ the *apparent contour* (or *contour* for short) of φ and denote it by $\gamma(\varphi)$.

A C^∞ map $\varphi\colon M \to N$ is said to be *stable* if it satisfies the following two properties.

(1) The map germ at each $p \in M$ is C^∞ right-left equivalent to one of the map germs at $0 \in \mathbb{R}^2$ below;
  - $(a,x) \mapsto (a,x)$: p is a regular point,
  - $(a,x) \mapsto (a,x^2)$: p is a fold point,
  - $(a,x) \mapsto (a,x^3+ax)$: p is a cusp point.

  Hence, $S(\varphi)$ is a finite disjoint union of circles.

(2) For each $q \in \gamma(\varphi)$, the map germ $(\varphi|_{S(\varphi)}, \varphi^{-1}(q) \cap S(\varphi))$ is right-left equivalent to one of the three multi-germs as depicted in Figure 1.

According to a classical result of Whitney [8], stable maps form an open everywhere dense set in the space of all C^∞ maps $M \to N$. Thus, for a C^∞ map $M \to N$, there is a stable map $M \to N$ homotopic to the C^∞ map.

In this paper, we consider stable maps with singular points. When φ is stable, $S(\varphi)$ is called the *fold curve* of φ, and the numbers of cusps, fold curve components and nodes on $\gamma(\varphi)$ are denoted by $c(\varphi)$, $i(\varphi)$ and $n(\varphi)$ respectively.

An oriented closed surface of genus g is denoted by Σ_g. The 2-dimensional sphere and the plane are denoted by S^2 and $\mathbb{R}^2$ respectively.

Let $\varphi_0\colon M \to S^2$ be a C^∞ map and $\varphi\colon M \to S^2$ be a stable map which is homotopic to φ_0 and whose contour consists of i components. Then, call $\gamma(\varphi)$ an *i-minimal contour* of φ_0 if the number $c+n$ for $\gamma(\varphi)$ is the smallest among the contours of stable maps which are homotopic to φ_0 and whose contours consist of i components. A 1-minimal contour, which is called a *minimal contour* in [4], of a C^∞ map $M \to \mathbb{R}^2$ was studied by Pignoni [4]. A 1-minimal contour of a C^∞ map $M \to S^2$ was studied by Demoto [1], Kamenosono and the second author [2]. They obtained the following result:

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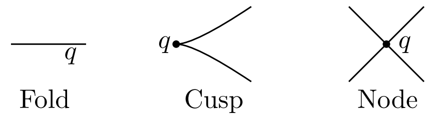

FIGURE 1. The multi-germs of $\varphi|_{S(\varphi)}$

Theorem 1.1 ([1], [2])**.** Let $d \geq 0$ and $f\colon \Sigma_g \to S^2$ be a degree d stable map whose contour consists of one component. The contour $\gamma(f)$ is 1-minimal if and only if the pair (c,n) for $\gamma(f)$ is one of the items below:

$$(c,n) = \begin{cases} (2d,0) & \text{if } g=0, \\ (2(d-1),4) \text{ or } (2d+2,0) & \text{if } g=1 \text{ and for each } d\geq 1, \\ (2,4) \text{ or } (6,0) & \text{if } (d,g)=(1,2), \\ (2(d-g),2g+2) & \text{if } d \geq g > 1, \\ (2,d+g+1) & \text{if } d \leq g \text{ and } g \not\equiv d \pmod 2,\ (d,g)\neq(1,2), \\ (0,d+g+2) & \text{if } d\leq g \text{ and } g\equiv d \pmod 2,\ (d,g)\neq(1,1). \end{cases}$$

On the other hand, the second author [9] introduced and studied a (c,i,n)-minimal contour of a C^∞ map $\Sigma_g \to S^2$: The apparent contour of a stable map $\varphi\colon M \to S^2$ is a (c,i,n)-minimal contour of a C^∞ map $\varphi_0\colon M \to S^2$ if the triple $(c(\varphi), i(\varphi), n(\varphi))$ is the smallest with respect to the lexicographic order among the stable maps homotopic to φ_0. Furthermore, he introduced some lemmas concerning apparent contours of stable maps $M \to S^2$ whose contours consist of some components.

In this paper, we will study an i-minimal contour of a C^∞ map $\Sigma_g \to S^2$ for each $i \geq 2$. Note that, for each number $i \geq 1$, there is a C^∞ map $\Sigma_g \to S^2$ whose contour consists of i components.

Recall that by virtue of Hopf's theorem (see [3] for example), two C^∞ maps $\Sigma_g \to S^2$ are homotopic if and only if their degrees coincide. Thus, the homotopy class of stable maps $\Sigma_g \to S^2$ of degree d is represented by the pair (d,g).

The main theorem of this paper is the following.

Theorem 1.2. Let $f\colon \Sigma_g \to S^2$ be a degree d stable map whose contour consists of i components. Then, the contour $\gamma(f)$ is i-minimal if and only if the pair (c,n) for $\gamma(f)$ is one of the items below:

$g=0$**:**

$$(c,n) = \begin{cases} \text{(0-i)} \ \ (2(|d|-i+1),0) & \text{if } 1\leq i \leq |d|+1, \\ \text{(0-ii)} \ \ (2,0) & \text{if } i \geq |d|+2,\ i\equiv d \pmod 2, \\ \text{(0-iii)} \ \ (0,0) & \text{if } i\geq |d|+2,\ i\not\equiv d \pmod 2, \end{cases}$$

$g=1$**:**

$$(c,n) = \begin{cases} \text{(1-i)} \ \ (2(|d|-i),4) \text{ or } (2(|d|-i)+4,0) & \text{if } 1\leq i\leq |d|, \\ \text{(1-ii)} \ \ (2,2) & \text{if } (d,i)=(0,1), \\ \text{(1-iii)} \ \ (2,0) & \text{if } i\geq |d|+1,\ i\not\equiv d \pmod 2 \text{ except } (d,i)=(0,1), \\ \text{(1-iv)} \ \ (0,0) & \text{if } i \geq |d|+1,\ i\equiv d \pmod 2, \end{cases}$$

$g = 2$**:**

$$(c,n) = \begin{cases} \text{(2-i)} \ \ (2(|d|-i-1), 6) & \text{if } 1 \le i \le |d|-1, \\ \text{(2-ii)} \ \ (2,4) \text{ or } (6,0) & \text{if } i = |d|, \\ \text{(2-iii)} \ \ (0,4) & \text{if } i = |d|+1, \\ \text{(2-iv)} \ \ (2,2) & \text{if } (d,i) = (0,2), \\ \text{(2-v)} \ \ (2,0) & \text{if } i \ge |d|+2,\ i \equiv d \pmod 2 \text{ except } (d,i) = (0,2), \\ \text{(2-vi)} \ \ (0,0) & \text{if } i \ge |d|+2,\ i \not\equiv d \pmod 2, \end{cases}$$

$g \ge 3$**:**

$$(c,n) = \begin{cases} \text{(g-i)} \ \ (2(|d|-g-i+1), 2+2g) & \text{if } 1 \le i \le |d|-g+1, \\ \text{(g-ii)} \ \ (2, |d|+g-i+2) & \text{if } |d|-g+2 \le i < |d|+g-1 \text{ and } d+g \equiv i \pmod 2, \\ \text{(g-iii)} \ \ (0, |d|+g-i+3) & \text{if } |d|-g+2 \le i \le |d|+g-1 \text{ and } d+g \not\equiv i \pmod 2, \\ \text{(g-iv)} \ \ (2,2) & \text{if } (d,i) = (0,g), \\ \text{(g-v)} \ \ (2,0) & \text{if } i \ge |d|+g,\ i \equiv d+g \pmod 2 \text{ except } (d,i) = (0,g), \\ \text{(g-vi)} \ \ (0,0) & \text{if } i \ge |d|+g,\ i \not\equiv d+g \pmod 2. \end{cases}$$

Theorem 1.2 yields the following corollaries.

Corollary 1.3. Let $f\colon \Sigma_g \to S^2$ be a degree d stable map whose contour consists of i components. Then, the contour $\gamma(f)$ is i-minimal if and only if the number $c+n$ for $\gamma(f)$ is one of the items below:

$g = 0$**:**

$$c+n = \begin{cases} 2(|d|-i+1) & \text{if } 1 \le i \le |d|+1, \\ 2 & \text{if } i \ge |d|+2,\ i \equiv d \pmod 2, \\ 0 & \text{if } i \ge |d|+2,\ i \not\equiv d \pmod 2. \end{cases}$$

$g \ge 1$**:**

$$c+n = \begin{cases} 2(|d|-i+2) & \text{if } 1 \le i \le |d|-g+1, \\ |d|+g-i+4 & \text{if } |d|-g+2 \le i < |d|+g-1 \text{ and } d+g \equiv i \pmod 2, \\ |d|+g-i+3 & \text{if } |d|-g+2 \le i \le |d|+g-1 \text{ and } d+g \not\equiv i \pmod 2, \\ 4 & \text{if } (d,i) = (0,g), \\ 2 & \text{if } i \ge |d|+g,\ i \equiv d+g \pmod 2 \text{ except } (d,i) = (0,g), \\ 0 & \text{if } i \ge |d|+g,\ i \not\equiv d+g \pmod 2, \end{cases}$$

Corollary 1.4. (1) For each i, any i-minimal contour of a C^∞ between S^2 has no node.
(2) For each i, the number of nodes on any i-minimal contour of a C^∞ map $\Sigma_g \to S^2$ is an even number.

We remark that the number of cusps on each stable map $\Sigma_g \to S^2$ is an even number, see [6] for details.

Note that for each d and i, there is a degree d stable map $\Sigma_g \to S^2$ whose contour consists of i components and whose contour has odd number of nodes.

This paper is organized as follows: In §2, we introduce some notions concerning the apparent contour of a stable map between surfaces. In §3, some stable maps $\Sigma_g \to S^2$ are described. In §4, Theorem 1.2 is proved. In §5, we consider the case of a stable map which has no cusps. In §6, some problems are posed.

Throughout this paper, all surfaces are connected and of class C^∞, and all maps are of class C^∞. The symbols $d, g \geq 0, i \geq 1$ denote integers unless stated otherwise.

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2. Preliminaries

In the following, we describe some notions concerning the apparent contour of a stable map $M \to S^2$ of a closed surface which is not necessary orientable.

Let M be a closed surface and $\varphi\colon M \to S^2$ a stable map with singular points. Let $S(\varphi) = S_1 \cup \cdots \cup S_\ell$ be the decomposition of $S(\varphi)$ into the connected components and set $\gamma_i = \varphi(S_i)$ $(i = 1, \ldots, \ell)$. Then, $\gamma(\varphi) = \gamma_1 \cup \cdots \cup \gamma_\ell$. Denote by $n_1(\varphi)$ the total number of self-intersection points of γ_i $(i = 1, \ldots, \ell)$ and $n_2(\varphi)$ the total of the number of points $\gamma_i \cap \gamma_j$ for all i and j with $i \neq j$. Note that $n_2(\varphi)$ is an even number and that $n(\varphi) = n_1(\varphi) + n_2(\varphi)$. Let $m(\varphi)$ be the smallest number of elements in the set $\varphi^{-1}(y)$, where $y \in S^2$ runs over all regular values of φ. Fix a regular value ∞ such that $\varphi^{-1}(\infty)$ consists of $m(\varphi)$ points. For each γ_i, denote by U_i the component of $S^2 \setminus \gamma_i$ which contains ∞. Note that $\partial U_i \subset \gamma_i$.

Orient γ_i so that at each fold point image, the surface is "folded to the left". More precisely, for a point $y \in \gamma_i$ which is not a cusp or a node of γ_i, choose a normal vector v of γ_i at y such that $\varphi^{-1}(y')$ contains more elements than $\varphi^{-1}(y)$, where y' is a regular value of φ close to y in the direction of v. Let τ be a tangent vector of γ_i at y with respect to the above orientation of γ_i. Then, orient S^2 by the ordered pair (τ, v). It is easy to see that this gives a well-defined orientation of S^2.

Definition 2.1. A point $y \in \partial U_i \setminus \{\text{cusps, nodes}\}$ is said to be *positive* if the normal orientation v at y points toward U_i. Otherwise, it is said to be *negative*.

A component γ_i is said to be *positive* if all points of $\partial U_i \setminus \{\text{cusps, nodes}\}$ are positive; otherwise, γ_i is said to be *negative*. The numbers of positive and negative components are denoted by i^+ and i^- respectively. Note that there is at least one negative component unless $S(\varphi) = \emptyset$.

Definition 2.2. A point $y \in \partial U_i \setminus \{\text{cusps, nodes}\}$ is called an *admissible starting point* if

(1) y is a positive point of a positive component γ_i or
(2) y is a negative point of a negative component γ_i.

Note that for each i, there always exists an admissible starting point in γ_i.

Definition 2.3. Let $y \in \gamma_i$ be an admissible starting point. Suppose that $Q \in \gamma_i$ is a node, and let $\alpha\colon [0, 1] \to \gamma_i$ be a parameterization consistent with the orientation which is singular only when the image is a cusp such that $\alpha^{-1}(y) = \{0, 1\}$. Then, there are two numbers $t_1 < t_2$ satisfying $\alpha(t_1) = \alpha(t_2) = Q$.

We say that Q is *positive* if the orientation of S^2 at Q defined by the ordered pair $(\alpha'(t_1), \alpha'(t_2))$ coincides with that of S^2 at Q; *negative*, otherwise. See Figure 2 for details.

The numbers of positive and negative nodes on γ_i are denoted by N_i^+ and N_i^- respectively. The definition of a positive (or negative) node of γ_i depends on the choice of an admissible starting point y. However, it is known that the algebraic number $N_i^+ - N_i^-$ does not depend on the choice of y, see [7] for details. Thus, the algebraic number $N^+ - N^- = \sum_{i=1}^k (N_i^+ - N_i^-)$ is well defined. Note that nodes arising from $\gamma_i \cap \gamma_j$ $(i \neq j)$ play no role in the computation.

Then, the following formula was obtained in [2].

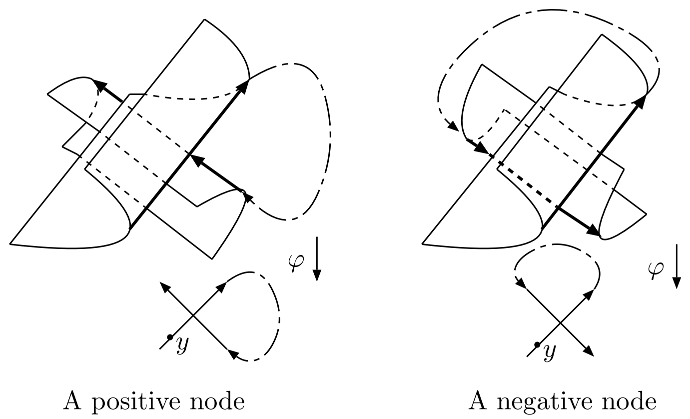

FIGURE 2. A positive node and a negative node.

Proposition 2.4 ([2]). *For a stable map $\varphi\colon M \to S^2$ of a closed surface of genus g, we have*

$$g = \varepsilon(M)\left[(N^+ - N^-) + \frac{c(\varphi)}{2} + (1 + i^+ - i^-) - m(\varphi)\right] \tag{2.1}$$

where $\varepsilon(M)$ is equal to 1 if M is orientable and 2 if M is not orientable.

The second author has obtained an extension of the formula (2.1) to a stable map $M \to \Sigma_h$ ($h \geq 1$) whose contour consists of one component that will be published in the forthcoming paper [10].

In the following, we assume $\gamma_i \cap \gamma_j = \emptyset$ for all $i \neq j$. Denote by $U_\infty \subset S^2 \setminus \gamma(\varphi)$ the component which contains ∞. Denote by γ_1 the component of $\gamma(\varphi)$ which contains ∂U_∞. Note that γ_1 is a negative component of φ. Then, the following lemmas and corollary were obtained in [9].

Lemma 2.5. If γ_1 has a node, then it has a negative node.

Lemma 2.6. If a positive component γ_i has a node, then it has a positive node.

Corollary 2.7. If the number of negative components of $\gamma(\varphi)$ is equal to one and $\gamma(\varphi)$ has a node, then it has a negative node.

3. STABLE MAPS $\Sigma_g \to S^2$

In this section, we introduce some stable maps $\Sigma_g \to S^2$ which we employ the following sections. In the following, the symbol $f_{a,b,c}$ denote the degree a stable map of Σ_b into S^2 having c connected components of singular set.

For each $g \geq 0$, define a degree zero stable map $f_{0,g,g+1}\colon \Sigma_g \to S^2$ by $f_{0,g,g+1} = \iota \circ p_g$, where $p_g\colon \Sigma_g \to \mathbb{R}^2$ is defined by Figure 3 and ι is the inclusion $\iota\colon \mathbb{R}^2 \hookrightarrow \mathbb{R}^2 \cup \{\infty\} = S^2$. Then, the triple (c, n, i) for $\gamma(f_{0,g,g+1})$ is equal to $(0, 0, g+1)$.

The following lemma can be easily proven as illustrated in Figure 4.

Lemma 3.1. Let $f\colon \Sigma_g \to S^2$ be a degree d stable map. Then, there is a degree d stable map $\widetilde{f}\colon \Sigma_g \to S^2$ whose triple (c, n, i) is equal to $(c(f), n(f), i(f) + 2)$ such that $\gamma(\widetilde{f}) = \gamma(f) \coprod S^1 \coprod S^1$.

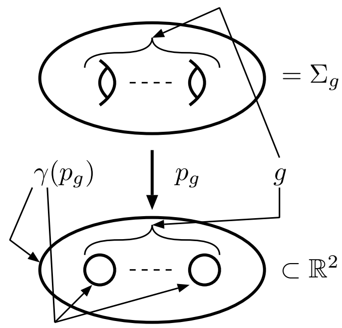

FIGURE 3. The contour $\gamma(p_g)$

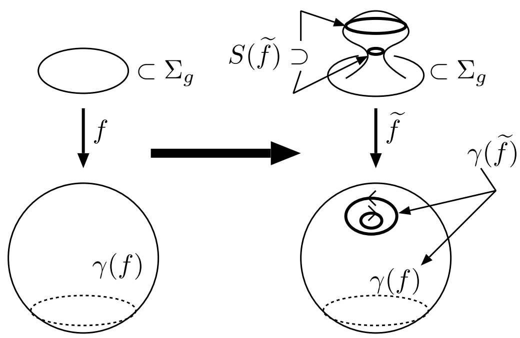

FIGURE 4. Proof of Lemma 3.1.

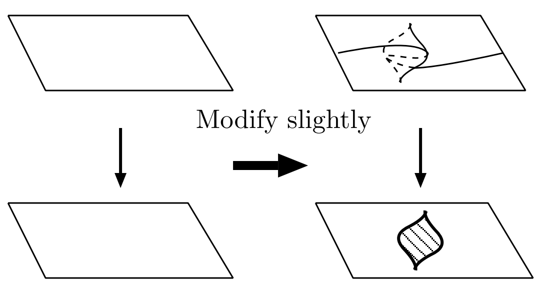

FIGURE 5. Making a pleat

By applying Lemma 3.1 inductively to $f_{0,g,g+1}$, we obtain the degree zero stable map $f_{0,g,i}\colon \Sigma_g \to S^2$ whose triple (c, n, i) is equal to $(0, 0, i)$ for each pair (g, i) which satisfies $i \geq g+1$ and $i \equiv g + 1 \pmod 2$.

By making a pleat to $f_{0,g,i}$ (see Figure 5 for details), we obtain a degree zero stable map $f_{0,g,i+1}\colon \Sigma_g \to S^2$ whose triple (c, n, i) is equal to $(2, 0, i + 1)$.

For each odd number g, by attaching $(g - 1)$ handles vertically (see Figure 6 for details) to a degree zero stable map $T^2 \to S^2$ whose contour is in Figure 7(a) with $\ell_1 = 0$, we obtain a degree zero stable map $f_{0,g,g}\colon \Sigma_g \to S^2$ whose contour is in Figure 7(a) with $\ell_1 = (g - 1)$. Similarly, for each even number $g \geq 2$, by attaching $(g - 2)$ handles vertically to a degree zero stable

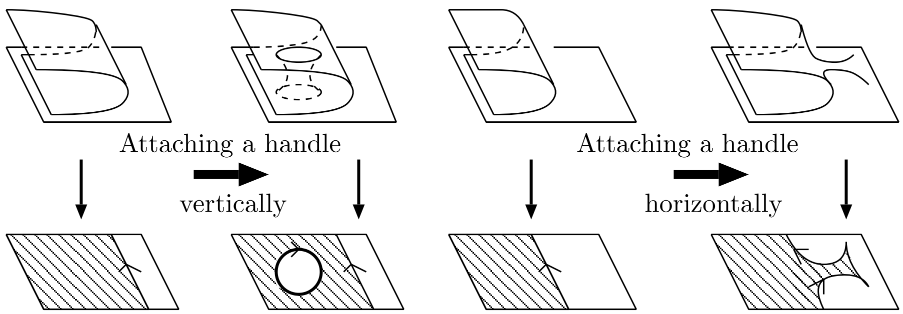

FIGURE 6. Attaching a handle

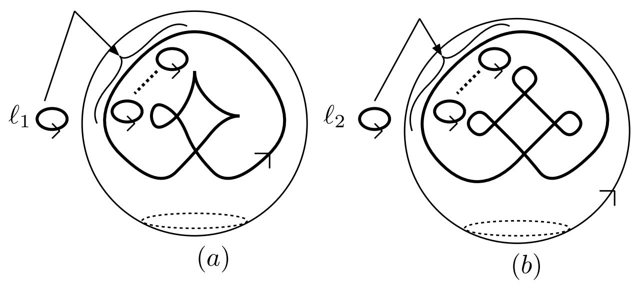

FIGURE 7. The contours $\gamma(f_{0,g,g})$ (g is odd), and $\gamma(f_{0,g,g-1})$ (g is even)

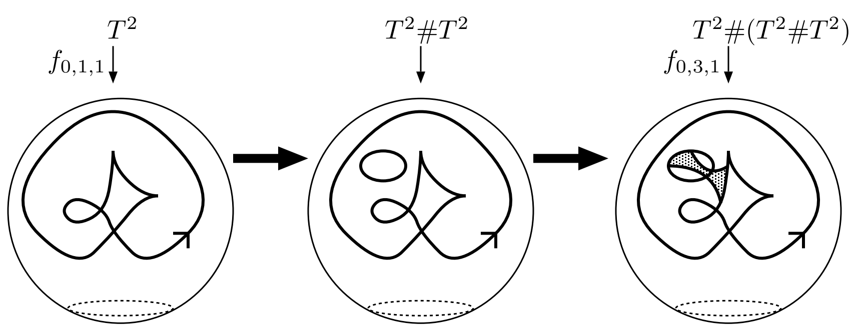

FIGURE 8. Attaching a pair of handles to $f_{0,1,1}$

map $\Sigma_2 \to S^2$ whose contour is in Figure 7(b) with $\ell_2 = 0$, we obtain a degree zero stable map $f_{0,g,g-1}\colon \Sigma_g \to S^2$ whose contour is in Figure 7(b) with $\ell_2 = (g-2)$. Remark that the degree zero stable maps $f_{0,1,1}$ and $f_{0,2,1}$ were obtained in [2].

For each $g \geq 1$, by attaching a pair of handles, attaching a handle vertically first and attaching a handle horizontally, see Figure 6 for details, second, see Figure 8 for example, or by attaching a handle vertically inductively to the degree zero stable map $\Sigma_g \to S^2$ whose contour is 1-minimal, the degree zero stable map is in Theorem 1.1, we obtain a degree zero stable map $f_{0,g,i}\colon \Sigma_g \to S^2$

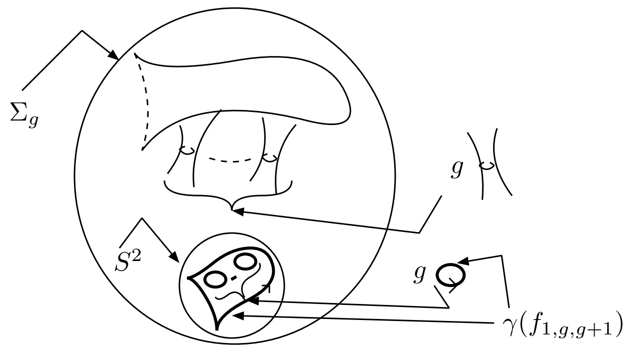

FIGURE 9. The stable map $f_{1,g,g+1}$

whose contour consists of i components and whose pair (c, n) is equal to

$$(c,n) = \begin{cases} (2, g-i+2) & \text{if } 1 \leq i \leq g \text{ and } i \equiv g \pmod 2, \\ (0, g-i+3) & \text{if } 1 \leq i \leq g \text{ and } i \not\equiv g \pmod 2. \end{cases}$$

Thus, we obtain the following maps.

Proposition 3.2. For each $i \geq 1$ and $g \geq 0$, there is a degree zero stable map $f_{0,g,i}\colon \Sigma_g \to S^2$ whose contour consists of i components and whose pair (c, n) is one of the items below:

$$(c,n) = \begin{cases} \text{(a)} \ \ (2, g-i+2) & \text{if } 1 \leq i \leq g \text{ and } i \equiv g \pmod 2, \\ \text{(b)} \ \ (0, g-i+3) & \text{if } 1 \leq i \leq g \text{ and } i \not\equiv g \pmod 2, \\ \text{(c)} \ \ (2, 0) & \text{if } i \geq g+1 \text{ and } i \equiv g \pmod 2, \\ \text{(d)} \ \ (0, 0) & \text{if } i \geq g+1 \text{ and } i \not\equiv g \pmod 2. \end{cases}$$

For a sufficiently large sphere whose center is the origin of $\mathbb{R}^3$, make a pleat. Then, by attaching g handles to the sphere, we obtain a Σ_g as in Figure 9. Then, define the map $f_{1,g,g+1}\colon \Sigma_g \to S^2$ by $\pi|_{\Sigma_g}$, where $\pi\colon \mathbb{R}^3 \setminus \{0\} \to S^2$ defined by $\pi(x) = x/|x|$. Thus, we obtain the following Lemma.

Proposition 3.3. The map $f_{1,g,g+1}\colon \Sigma_g \to S^2$ is a degree one stable map whose triple (c, n, i) is equal to $(2, 0, g+1)$.

4. Proof of Theorem 1.2

In this section, we prove Theorem 1.2. Note that for a C^∞ map $\Sigma_g \to S^2$ of degree d, by changing the orientation of Σ_g, we obtain a C^∞ map $\Sigma_g \to S^2$ of degree $-d$. In the following, we assume $d \geq 0$.

Proof of Theorem 1.2. The contour $\gamma(f_{0,g,i})$, the degree zero stable map $f_{0,g,i}$ in Proposition 3.2(d), is trivially i-minimal.

The following lemma can be easily proven as illustrated in Figure 10 where $(\Sigma_g)_-$ denotes the closure of the set of regular points whose neighborhoods are orientation reversed by the map.

Lemma 4.1. Let $f\colon \Sigma_g \to S^2$ be a degree d stable map having a singular point. Then, there is a degree $d+1$ stable map $f'\colon \Sigma_g \to S^2$ such that $\gamma(f') = \gamma(f) \coprod S^1$. The triple (c, n, i) for $\gamma(f')$ is equal to $(c(f), n(f), i(f)+1)$.

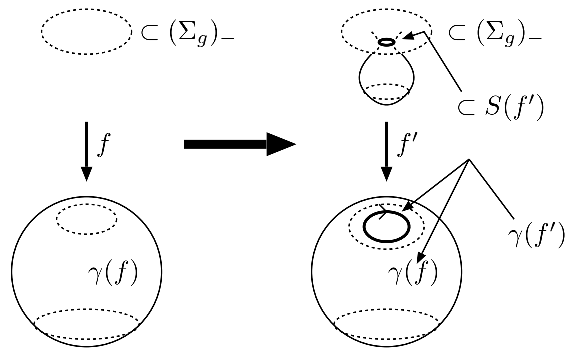

FIGURE 10. Proof of Lemma 4.1

Thus, the contour of the map $\Sigma_g \to S^2$ which is obtained by applying Lemma 4.1 inductively to the degree zero stable map $f_{0,g,i}$ in Proposition 3.2(d) is trivially i-minimal. The cases (0-iii), (1-iv), (2-vi) and (g-vi) of Theorem 1.2 are proved.

We introduce the following lemma.

Lemma 4.2. Let $f\colon \Sigma_g \to S^2$ be a degree d stable map whose contour consists of i components. If the number $d+g+i$ is even, then $\gamma(f)$ has at least two cusps.

Proof. To prove this Lemma, apply a result of Quine [5]: for a stable map $f\colon M \to N$ between oriented surfaces, we have

$$\chi(M) - 2\chi(M_-) + \sum_{q_k:\mathrm{cusp}} \mathrm{sign}(q_k) = (\deg f)(\chi(N)),$$

where M_- denotes the closure of the set of regular points whose neighborhoods are orientation reversed by f, and $\mathrm{sign}(q_k) = \pm 1$ the sign of a cusp q_k, see [5] for definition.

Apply our situation to the Quine's formula:

$$\sum_{q_k:\mathrm{cusp}} \mathrm{sign}(q_k) = 2(d+g-1+\chi((\Sigma_g)_-)). \tag{4.1}$$

Note that $\chi((\Sigma_g)_-) \equiv i \pmod 2$. Then, it follows immediately. □

Lemma 4.2 shows that the following:

Proposition 4.3. (1) The contour of the degree zero stable map $f_{0,g,i}$ in Proposition 3.2(c) is i-minimal.

(2) The contour of the degree one stable map $f_{1,g,g+1}$ in Proposition 3.3, is $(g+1)$-minimal for each $g \geq 1$.

Thus, the contours of the maps $\Sigma_g \to S^2$ which are obtained by applying Lemma 4.1 inductively to $f_{0,g,i}$ in Proposition 3.2(c) and $f_{1,g,g+1}$ in Proposition 3.3 are i-minimal. The cases (0-ii), (1-iii), (2-v) and (g-v) of Theorem 1.2 are proved.

We prove the remaining cases of Theorem 1.2.

4.1. **The case of** $g = 0$**.** Let us consider the case (0-i) of Theorem 1.2. For a fixed $d \geq 0$ and each $i \leq d+1$, the formula (4.1) shows that the contour of a degree d stable map between S^2 whose contour consists of i components has at least $2(d-i+1)$ cusps. This shows that the contour of a degree $d+1$ stable map between S^2 which obtained by applying Lemma 4.1 to a degree d stable map between S^2 whose contour is 1-minimal is 2-minimal. By applying this inductively, the case (0-i) of Theorem 1.2 is proved.

4.2. **The case of** $g = 1$**.** Note that the case (1-ii) is contained in Thorem 1.1. Let us consider the case (1-i) of Theorem 1.2. The formula (2.1) for a degree d stable map $\Sigma_g \to S^2$ whose contour consists of i components induces the following equality:

$$m(f) + g + 2i^- = (N^+ - N^-) + \frac{c}{2} + (1+i)$$

Thus, by $i^- \geq 1$ and $m(f) \geq d$, we obtain the following inequality for the stable map

$$d + g + 1 \leq (N^+ - N^-) + \frac{c}{2} + i. \tag{4.2}$$

Note that the formula (2.1) for a degree $d+1$ stable map $\Sigma_g \to S^2$ whose contour consists of $i+1$ components induces the inequality (4.2).

Let us consider the case that $d = i = 1$. Then, the formula (4.2) shows

$$2 \leq (N^+ - N^-) + \frac{c}{2}. \tag{4.3}$$

If the contour has a node, by Lemma 2.5, then $c + n \geq 4$. Otherwise, then $c \geq 4$. On the other hand, in the case that $d = i = 2$, the formula (4.2) also induces inequality (4.3). Then, by the similarly argument as the above, the number $c+n$ of the contour of a degree two stable map $T^2 \to S^2$ whose contour consists of two components is greater than or equal to four. Thus, the contour of the degree two stable map $T^2 \to S^2$ which is obtained by applying Lemma 4.1 to by the degree one stable map $T^2 \to S^2$ whose contour is 1-minimal is 2-minimal.

In general, we obtain the following proposition.

Proposition 4.4. Let f be a degree d stable map $\Sigma_g \to S^2$ whose contour consists of i components and f' be a degree $d+1$ stable map obtained by applying Lemma 4.1 to f. If the contour $\gamma(f)$ is i-minimal and the number $c+n$ for $\gamma(f)$ is the smallest with respect to the inequality induced by (4.2), then $\gamma(f')$ is $(i+1)$-minimal.

Remark **4.5.** The degree one stable map $f' \colon T^2 \to S^2$ obtained by applying Lemma 4.1 to a degree zero $f \colon T^2 \to S^2$ whose contour is 1-minimal is not 2-minimal. The number $c+n$ of $\gamma(f)$ is equal to four. The number $c+n$ of a 2-minimal contour of a degree one C^∞ map $\Sigma_g \to S^2$ is two, see Proposition 4.3(2).

Note that for each $d \geq 1$, the number $c+n$ of a degree d stable map $T^2 \to S^2$ whose contour is 1-minimal is the minimal with respect to the inequality induced by (4.2), see [2] for details. Hence, the case (1-i) of Theorem 1.2 can be proven inductively by using Theorem 1.1 and Proposition 4.4.

4.3. **The case of** $g \geq 2$**.** Let us consider the cases (2-iv) and (g-iv). Let $f \colon \Sigma_g \to S^2$ be a degree zero stable map whose contour consists of g components. Note that Lemma 4.2 shows the contour $\gamma(f)$ has at least two cusps. We divide this case into the following cases (i) and (ii).

(i) $n_2(f) = 0$: Assume (i^+, i^-) for $\gamma(f)$ is equal to $(g-1, 1)$. Then, by the formula (2.1), we have $1 + m(f) - c/2 = \ (N^+ - N^-)$. Thus, we have

$$n_1(f) = 1 + m(f) + 2N^- - \frac{c}{2}. \tag{4.4}$$

If $\gamma(f)$ has a node, then by the inequality (4.4) and Corollary 2.7,

$$c+n=c+n_1(f)\geq c+\left(1+m(f)+2N^- -\frac{c}{2}\right)\geq 1+2+1=4. \tag{4.5}$$

Note that there is no degree zero stable map $f\colon \Sigma_g\to S^2$ with $m(f)=0$ whose pair (c,n) is equal to $(2,0)$ by the geometrical meaning of cusps. Thus, if $\gamma(f)$ has no node, then $m(f)\geq 2$. Then, by (4.4), we have

$$c+n\geq 2(1+m(f))\geq 6. \tag{4.6}$$

Assume (i^+,i^-) for $\gamma(f)$ is equal to $(g-\lambda,\lambda)$, where $\lambda=2,\dots,g+d$. Then, by the formula (2.1), we have $3-c/2\leq (N^+-N^-)$. Thus, we have

$$n_1(f)\geq 3+2N^- -\frac{c}{2}\geq 3-\frac{c}{2}.$$

Therefore, we have

$$c+n=c+n_1(f)\geq c+\left(3-\frac{c}{2}\right)\geq 3+1=4. \tag{4.7}$$

(ii) $n_2(f)\neq 0$: Put (i^+,i^-) for $\gamma(f)$ is equal to $(g-\lambda,\lambda)$, where $\lambda=1,\dots,g$. Then, by the formula (2.1), we have $1-c/2\leq (N^+-N^-)$. Thus,

$$n_1(f)\geq 1-\frac{c}{2}.$$

Therefore, we have

$$c+n=c+n_1(f)+n_2(f)\geq c+\left(1-\frac{c}{2}\right)+2\geq 1+1+2=4. \tag{4.8}$$

The inequalities (4.5), (4.6), (4.7) and (4.8) shows that the pair (c,n) of a g-minimal contour of a degree zero stable map $\Sigma_g\to S^2$ is equal to $(2,2)$.

Thus, the contour $\gamma(f_{0,g,g})$, $f_{0,g,g}$ is in Proposition 3.2(a) with $i=g$, is g-minimal for each number $g\geq 2$.

By the similar argument as the cases (2-iv) and (g-iv), we can prove the contour $\gamma(f_{0,g,i})$, $f_{0,g,i}$ is in Proposition 3.2(a) and (b), is i-minimal. The contours of the stable maps $\Sigma_g\to S^2$ which are obtained by applying Lemma 4.1 inductively to the stable maps in Proposition 3.2(a), (b) and Theorem 1.1 with $(d,g)=(1,2)$ are also i-minimal. We omit the proof here. The cases (2-ii), (2-iii), (g-ii) and (g-iii) are proved.

Note that for each $d\geq 0$, the number $c+n$ of a degree d stable map $\Sigma_g\to S^2$ whose contour is 1-minimal is the minimal with respect to the inequality induced by (4.2), see [2] for details. Hence, the cases (2-i) and (g-i) of Theorem 1.2 can be proven inductively by using Theorem 1.1 and Proposition 4.4.

This completes the proof of Theorem 1.2. □

5. FOLD MAP CASE

Let M be a connected and closed surface, and N be a connected surface. A stable map $f\colon M\to N$ which has no cusp is called a *fold map*.

Let $\varphi_0\colon M\to S^2$ be a C^∞ map and $\varphi\colon M\to S^2$ be a fold map which is homotopic to φ_0 and whose contour consists of i components. Then, call the contour $\gamma(\varphi)$ a *regular i-minimal contour* of φ_0 if the number $c+n$ for $\gamma(\varphi)$ is the smallest among the contours of fold maps which are homotopic to φ_0 and whose contours consist of i components.

Note that by Lemma 4.2 if $d+g+i$ is even, then there is no degree d fold map $\Sigma_g\to S^2$ whose contour consists of i components.

Then, as a corollary of Theorem 1.2, we obtain the following.

Theorem 5.1. Assume $d+g+i$ be an odd number. Let $f\colon \Sigma_g \to S^2$ be a degree d fold map whose contour consists of i components. Then, $\gamma(f)$ is a regular i-minimal contour if and only if the number of nodes n for $\gamma(f)$ is one of the items below:

$g=0$**:**

$$n = 0 \quad \text{if } i \geq |d|+1 \text{ and } i \not\equiv d \pmod 2$$

$g \geq 1$**:**

$$n = \begin{cases} 2+2g & \text{if } i = |d|-g+1, \\ |d|+g-i+3 & \text{if } |d|-g+2 \leq i \leq |d|+g-1 \text{ and } i \not\equiv d+g \pmod 2), \\ 0 & \text{if } i \geq |d|+g,\ i \not\equiv |d|+g \pmod 2). \end{cases}$$

6. Problems

In this section, we pose some problems with respect to the apparent contour of a stable map $M \to N$ between surfaces.

Kamenosono and the second author studied a 1-minimal contour of a C^∞ map $F \to S^2$ of a non-orientable surface. Then, there are the following problems.

***Problem* 6.1.** Study an i-minimal contour and a regular i-minimal contour of a C^∞ map $F \to S^2$ of a non-orientable closed surface into the sphere for each $i \geq 2$.

Let $\varphi_0\colon M \to N$ be a C^∞ map between surfaces and $\varphi\colon M \to N$ a stable map which is homotopic to φ_0 and whose contour consists of i components. Then, the contour $\gamma(f)$ is an i-*essential contour* if the pair (c,n) is the smallest with respect to the lexicographic order, among the stable maps $M \to N$ which are homotopic to φ_0 and whose contour consists of i components. Then, Theorem 1.2 yields the following Theorem.

Theorem 6.2. Let $f\colon \Sigma_g \to S^2$ be a degree d stable map whose contour consists of i components. Then, $\gamma(f)$ is i-essential if and only if the pair (c,n) for $\gamma(f)$ is one of the items below:

$$(c,n) = \begin{cases} (2|d|-i, 4) & \text{if } g=1 \text{ and } 1 \leq i \leq |d|, \\ (2,4) & \text{if } g=2 \text{ and } i=|d|. \end{cases}$$

In the other case, the pair (c,n) is of an i-minimal contour.

Corollary 6.3. Let $f_0\colon \Sigma_g \to S^2$ be a C^∞ map whose contour consists of i components. An i-essential contour of f_0 is an i-minimal contour of f_0.

Note that for a C^∞ map $h_0\colon \mathbb{R}P^2 \to S^2$ of modulo two degree one, a 1-minimal (or 1-essential) contour of h_0 is not 1-essential (resp. 1-minimal), see [2] for details. Thus, we pose the following problem.

***Problem* 6.4.** Study the i-essential contours of C^∞ maps from non-orientable surfaces into S^2. Then, compare an i-minimal contour of h_0 and an i-essential contour of h_0.

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ASYMMETRY IN SINGULARITIES OF TANGENT SURFACES IN CONTACT-CONE LEGENDRE-NULL DUALITY

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Abstract. We give the generic classification on singularities of tangent surfaces to Legendre curves and to null curves by using the contact-cone duality between the contact 3-sphere and the Lagrange-Grassmannian with cone structure of a symplectic 4-space. As a consequence, we observe that the symmetry on the lists of such singularities is breaking for the contact-cone duality, compared with the ordinary projective duality.

1. Introduction

Let $V = (V, \Omega)$ be a real symplectic vector space of dimension 4 with a symplectic form Ω. We consider the Lagrange flag manifold $\mathcal{F} = \mathcal{F}_{1,2}^{\rm Lag}(V)$ consisting of pairs (ℓ, L) of lines ℓ and Lagrange planes L in V containing ℓ. Then there are natural projections $\pi_1 : \mathcal{F} \to P(V)$ to the projective 3-space and $\pi_2 : \mathcal{F} \to {\rm LG}(V)$ to the Grassmannian of Lagrange planes in V:

$$P(V) \xleftarrow{\pi_1} \mathcal{F} \xrightarrow{\pi_2} {\rm LG}(V).$$

Note that $\dim \mathcal{F} = 4, \dim P(V) = \dim {\rm LG}(V) = 3$ and both π_1 and π_2 are fibrations with S^1 as fibers.

There exist the projective Engel structure on $\mathcal{F}$, the projective contact structure on $P(V)$ and the projective indefinite conformal structure on ${\rm LG}(V)$ of signature $(1, 2)$, such that both π_1-fibers and π_2-fibers are projective lines in $\mathcal{F}$, and that each π_1-fiber (resp. π_2-fiber) projects to a projective line in ${\rm LG}(V)$ (resp. $P(V)$) by π_2 (resp. by π_1). We give precise coordinate charts on $\mathcal{F}$, $P(V)$ and ${\rm LG}(V)$ in §3. A projective Legendre line through $\ell \in P(V)$ is given by $\pi_1(\pi_2^{-1}(L))$ for some $L \in {\rm LG}(V)$. On the other hand, a null (lightlike) line through $L \in {\rm LG}(V)$ is given by $\pi_2(\pi_1^{-1}(\ell))$ for some $\ell \in P(V)$.

Let $f : I \to \mathcal{F}$ be an integral curve to the Engel structure of $\mathcal{F}$ from an open interval I. Then $\pi_1 \circ f : I \to P(V)$ is a Legendre curve and $\pi_2 \circ f : I \to {\rm LG}(V)$ is a null curve for the null cone field on ${\rm LG}(V)$.

For a curve $c : I \to M$ in a 3-dimensional space M with a projective structure, its *tangent surface* (or, *tangent developable*) is defined as the ruled surface by the tangent lines ([15],[16],[18],[10],[11]).

An *associated variety* to a curve in $P(V)$ (resp. ${\rm LG}(V)$) is the subset of ${\rm LG}(V)$ (resp. $P(V)$) consisting of $L \in {\rm LG}(V)$ (resp. $\ell \in P(V)$) corresponding to a Legendre line (resp. a null line) which intersects with the curve (cf. [8]). Then we see that the associated variety to $\pi_1 \circ f$ (resp. $\pi_2 \circ f$) is the tangent surface to $\pi_2 \circ f$ (resp. $\pi_1 \circ f$) if $\pi_2 \circ f$ (resp. $\pi_1 \circ f$) is an immersion. In fact it is given by $\pi_2(\pi_1^{-1}(\pi_1(f(I)))$ (resp. $\pi_1(\pi_2^{-1}(\pi_2(f(I))))$, see §2.

Key words: tangent developable, null curve, Legendre curve, Lagrangian-Grassmannian, projective structure, Engel structure.

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Then the main purpose of this paper is to prove the following result:

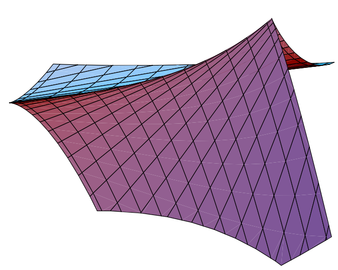
Theorem 1.1. *For a generic Engel integral curve $f : I \to \mathcal{F}$ from an open interval I to the Lagrange flag manifold $\mathcal{F}$ in C^∞ topology, we have that, for any $t_0 \in I$, the pair of singularities of tangent surfaces to $\pi_1 \circ f$ and to $\pi_2 \circ f$ is given by one of the following three cases:*

$$\begin{array}{ccl} \mathrm{I} & : & \text{(cuspidal edge, cuspidal edge)}, \\ \mathrm{II} & : & \text{(Mond surface, swallowtail)}, \\ \mathrm{III} & : & \text{(generic folded pleat, Shcherbak surface)}. \end{array}$$

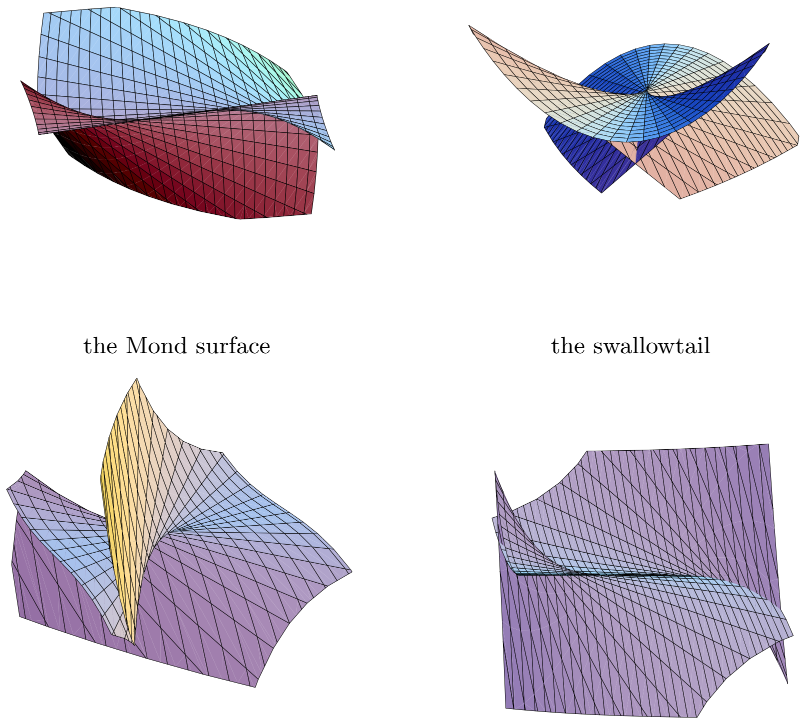
In fact, there exists a residual subset $\mathcal{R}$ in the space $C^\infty_E(I, \mathcal{F})$ of Engel integral curves with C^∞-topology, such that any $f \in \mathcal{R}$ enjoys the properties stated in Theorem 1.1. The usage of the C^∞ topology on an open interval is essential for our classification, see Remark 4.3.

The singularities appeared in Theorem 1.1 have the following parametric normal forms respectively, see Figure 1: A cuspidal edge (resp. Mond surface, swallowtail, generic folded pleat, Shcherbak surface) is locally *diffeomorphic* to the germ of parametrized surface $(\mathbf{R}^2, 0) \to (\mathbf{R}^3, 0)$ explicitly given by

$$\begin{array}{rl} \text{cuspidal edge}: & (x,t) \mapsto (x,\ -\frac{1}{2}t^2 + xt,\ \frac{1}{3}t^3 - \frac{1}{2}xt^2), \\ \text{Mond surface}: & (x,t) \mapsto (x,\ -\frac{1}{3}t^3 + \frac{1}{2}xt^2,\ \frac{1}{4}t^4 - \frac{1}{3}xt^3), \\ \text{swallowtail}: & (x,t) \mapsto (x,\ \frac{1}{6}t^3 - xt,\ -\frac{1}{4}t^4 + xt^2), \\ \text{generic folded pleat}: & (x,t) \mapsto (x,\ -\frac{1}{6}t^3 + xt - \frac{1}{8}t^4 + \frac{1}{2}xt^2, \\ & \qquad \frac{1}{20}t^5 - \frac{1}{6}xt^3 + \frac{1}{24}t^6 - \frac{1}{8}xt^4), \\ \text{Shcherbak surface}: & (x,t) \mapsto (x,\ \frac{1}{3}t^3 - \frac{1}{2}xt^2,\ -\frac{1}{5}t^5 + \frac{1}{4}xt^4). \end{array}$$



the cuspidal edge



the Mond surface the swallowtail

the generic folded pleat the Shcherbak surface

Figure 1.

The above normal forms of singularities are written in the projective coordinates which are given in §3 and therefore they look different from, for example, those given in [10], [11]. Mond surfaces are called also *cuspidal beaks* and they appear as singularities on wave-fronts of codimension one, see for instance [1], [3].

Singularities of the tangent developable to a curve of type $(2, 3, 5)$ was called *folded pleats* in [12]. It was known that the local differential classes of folded pleats are not unique [10] while the local homeomorphism class of them is unique [11]. Any folded pleat is locally homeomorphic to the plane and it has singular locus along the original curve. In this paper, we show the folded pleat singularities form exactly two classes of local diffeomorphism equivalence and the folded pleat singularities arising from *generic* Engel integral curves have a unique diffeomorphism class, see §6. We call it the *generic folded pleat*.

The generic appearance of Shcherbak surfaces is observed in the classification of lightlike developables in Minkowski 3-space earlier in [6]. However the meaning of genericity of null curves in [6] is different from that of our paper.

In the context of the ordinary projective duality, the role of projective space and that of dual projective space are completely equal. Therefore the lists of singularities must be symmetric because of the symmetry on the underlying geometric structures. Compared with it, the contact-cone Legendre-null duality is naturally supposed to be asymmetric for the list of singularities on tangent surfaces, because of the asymmetry on the underlying geometric structures, see

Proposition 5.1. As we see clearly in Theorem 1.1, the list of singularities is never symmetric in fact.

The singularities of tangent surfaces to null curves are regarded as singularities of "null surfaces" in the Lagrange-Grassmannian $\mathrm{LG}(V)$. A surface in $\mathrm{LG}(V)$ is called a *null surface*, if it is tangent to the null-cone C_L at any point L of the surface. Typical examples of null surfaces in the Lagrange-Grassmannian $\mathrm{LG}(V)$ are given by Schubert varieties $S_L = \{L' \in \mathrm{LG}(V) \mid L \cap L' \neq \{0\}\}$ ($L \in \mathrm{LG}(V)$) and tangent surfaces to null curves. (Schubert varieties are called *trains* in [19]). They are associated varieties to Legendre curves in $\mathrm{Gr}(1, V)$. In fact any null surface in $\mathrm{LG}(V)$ is locally a part of the associated variety to a Legendre curve in $\mathrm{Gr}(1, V)$, see Proposition 2.3.

The double fibration treated in this paper is a prototype of various constructions appeared in twistor theory, where one geometric structure is related to another geometric structure via a double fibration. In our case, one is the contact structure and another is the conformal (or cone) structure. Moreover tangent surfaces and associated varieties to Legendre curves and to null curves turn out to be important objects in the geometric study of differential equations. For instance, the contact space $P(V)$ (resp. the Engel space $\mathcal{F}$) is regarded as the compactification of 1-jet space $J^1(\mathbf{R}, \mathbf{R}) = \mathbf{R}^3$ (resp. $J^2(\mathbf{R}, \mathbf{R}) = \mathbf{R}^4$), and tangent surfaces to Legendre curves appear naturally in the study on certain type of third order ordinary differential equations. Further, $\mathrm{LG}(V)$ can be identified with the compactification of $J^0(\mathbf{R}^2, \mathbf{R}) = \mathbf{R}^3$ and tangent surfaces to null curves appear as the first order partial differential equations called eikonal equations. See [7] as a related work. Furthermore, if we regard $\mathrm{LG}(V)$ as the compactification of the space of second derivatives (the space of 2 by 2 symmetric matrices), then tangent surfaces to null curves appear as second order partial differential equations associated with Lagrange cone fields. The cuspidal edge singularities of tangent surfaces were appeared in E. Cartan's classical work (see [13]). Therefore it is an interesting open problem to study the differential equations corresponding to the complicated generic singularities of tangent varieties, which we have classified in this paper, beyond the Cartan's case.

In §2, we introduce the Lagrange flag manifold and explain the duality between the projective contact 3-space and the Lagrange-Grassmannian of a symplectic 4-space. Mainly we provide the descriptions for the oriented case. Those for the non-oriented case can be obtained easily by just taking coverings or by the exactly same manner. In §3, we provide the exact projective coordinates of the Lagrange flag manifold, the contact 3-sphere and the Lagrange-Grassmannian, which are suitable to obtain normal forms of tangent surfaces. In §4, we formulate the transversality theorem in our case and prove it. It is necessary to make the meaning of the "generic" Engel integral curves clear. In §5, we introduce the notion of types for curves in a space with a projective structure and give the codimension formula and the duality formula for the set of Engel integral jets which have given types under the projections. In §6, we determine the diffeomorphism class of "generic" folded pleats and finally we give the proof of the main theorem.

2. The contact-cone Legendre-null duality

We explain the contact-cone, or, Legendre-null duality via the Lagrange flag manifold.

Let (V^4, Ω) be a symplectic 4-dimensional real vector space with a symplectic form Ω. See [3] on the symplectic geometry. Consider the *oriented Lagrange flag manifold* $\widetilde{\mathcal{F}} = \widetilde{\mathcal{F}}^{\mathrm{Lag}}_{1,2}(V)$ which consists of pairs (ℓ, L) of oriented lines ℓ and oriented Lagrange planes L containing ℓ in V :

$$\widetilde{\mathcal{F}} = \{(\ell, L) \mid \ell \subset L \subset V,\ \dim(\ell) = 1,\ \dim(L) = 2,\ \Omega|_L = 0,\ \ell, L \text{ are oriented}\}.$$

Note that $\widetilde{\mathcal{F}} \cong \mathrm{U}(2) \cong S^1 \times S^3$ via any isomorphism $V^4 \cong \mathbf{C}^2$ with the standard Hermitian form, see [2], [9]. Note that $\widetilde{\mathcal{F}}$ covers $\mathcal{F}$ in degree 4.

There are natural projections $\pi_1 : \widetilde{\mathcal{F}} \to \widetilde{\mathrm{Gr}}(1, V) \cong \mathrm{U}(2)/\mathrm{U}(1) \cong S^3$ and $\pi_2 : \widetilde{\mathcal{F}} \to \widetilde{\mathrm{LG}}(V) \cong \mathrm{U}(2)/\mathrm{SO}(2) \cong S^1 \times S^2$. Here $\widetilde{\mathrm{Gr}}(1, V)$ is the Grassmannian of oriented lines through 0 in V, the double cover of the projective 3-space $P(V)$, and $\widetilde{\mathrm{LG}}(V)$ is the Grassmannian of oriented Lagrange planes through 0 in V, the double cover of $\mathrm{LG}(V)$.

A point $(\ell, L) \in \widetilde{\mathcal{F}}$ defines an oriented projective line $[L] \subset \widetilde{\mathrm{Gr}}(1, V)$ through $\ell \in \widetilde{\mathrm{Gr}}(1, V)$, as well as an oriented line $[[L]] = T_\ell[L] \cong L/\ell$ in the tangent space $T_\ell\widetilde{\mathrm{Gr}}(1, V)$.

The contact distribution $D \subset T\widetilde{\mathrm{Gr}}(1, V)$ at $\ell \in \widetilde{\mathrm{Gr}}(1, V)$ is obtained by

$$D_\ell = [[\ell^s]] \subset T_\ell\widetilde{\mathrm{Gr}}(1, V),$$

where $\ell^s = \{v \in V \,|\, \Omega(v, w) = 0 \text{ for any } w \in \ell\}$. For $(\ell, L) \in \widetilde{\mathcal{F}}$, we have $[[L]] \subset D_\ell$. The canonical (or tautological) sub-bundle $E \subset T\widetilde{\mathcal{F}}$ over $\widetilde{\mathcal{F}}$ is defined by

$$E_{(\ell,L)} = \{\boldsymbol{v} \in T_{(\ell,L)}\widetilde{\mathcal{F}} \mid \pi_{1*}\boldsymbol{v} \in [[L]]\}.$$

Then E is an Engel distribution over $\widetilde{\mathcal{F}}$. In fact $\widetilde{\mathcal{F}}$ is identified with the manifold of oriented tangent lines in D, and E is obtained as the prolongation of the contact structure on $\widetilde{\mathrm{Gr}}(1, V) \cong S^3$ ([5]). Moreover, we have $E_{(\ell,L)} = T_{(\ell,L)}\pi_1^{-1}(\ell) \oplus T_{(\ell,L)}\pi_2^{-1}(L)$.

The natural structure on $\widetilde{\mathrm{LG}}(V)$ is not given by a vector sub-bundle of $T\widetilde{\mathrm{LG}}(V)$ but by a cone-bundle $C \subset T\widetilde{\mathrm{LG}}(V)$ which is defined as follows: For each $L \in \widetilde{\mathrm{LG}}(V)$, we consider the Schubert variety

$$S_L = \{L' \in \widetilde{\mathrm{LG}}(V) \mid L' \cap L \neq \{0\}\} = \pi_2(\pi_1^{-1}(\pi_1(\pi_2^{-1}(L)))).$$

Then the cone $C_L \subset T_L\widetilde{\mathrm{LG}}(V)$ is defined as the tangent cone of S_L at L. We regard the flag manifold $\widetilde{\mathcal{F}}$ as the oriented projective bundle $\widetilde{P}D = (D - Z)/\mathbf{R}_{>0}$, where Z is the zero-section, for the contact structure $D \subset T\widetilde{\mathrm{Gr}}(1, V)$ as well as $\widetilde{P}C$, the set of oriented lines in C, for the cone structure $C \subset T\widetilde{\mathrm{LG}}(V)$.

Note that, for any $\ell \in \widetilde{\mathrm{Gr}}(1, V)$, $\pi_1(\pi_2^{-1}(\pi_2(\pi_1^{-1}(\ell)))) \subset \widetilde{\mathrm{Gr}}(1, V)$ is the projective plane which is associated to $\ell^s \subset V$ and its tangent cone coincides with the contact plane $D_\ell \subset T_\ell\widetilde{\mathrm{Gr}}(1, V)$. Moreover, note that for the Engel structure $E \subset T\widetilde{\mathcal{F}}$, we can write as

$$\begin{aligned} E_{(\ell,L)} &= T_{(\ell,L)}\pi_1^{-1}(\ell) \oplus T_{(\ell,L)}\pi_2^{-1}(L) \\ &= T_{(\ell,L)}(\pi_2^{-1}(\pi_2(\pi_1^{-1}(\ell)))) = T_{(\ell,L)}(\pi_1^{-1}(\pi_1(\pi_2^{-1}(L)))). \end{aligned}$$

Let $E^2 = E + [E, E]$ be the derived system from the Engel structure E. Then E^2 is a sub-bundle of $T\widetilde{\mathcal{F}}$ of rank 3 and $E^2 = \pi_{1*}^{-1}(D)$ ([5]). Moreover, we have the following lemma.

Lemma 2.1. *Let $v \in T_{(\ell,L)}\widetilde{\mathcal{F}}$ for $(\ell, L) \in \widetilde{\mathcal{F}}$. Then $v \in (E^2)_{(\ell,L)}$ if and only if $\pi_{2*}(v) \in (T_L[\ell])^\perp \subset T_L(\widetilde{\mathrm{LG}}(V))$. Here $(T_L[\ell])^\perp$ means the pseudo-orthogonal space to $T_L[\ell]$ (the tangent line at L of the null line $[\ell]$ determined by ℓ) for the conformal structure defined by the null-cone field C.*

The proof is given in §3 by using a local coordinate.

A C^∞ map $f : I \to (\widetilde{\mathcal{F}}, E)$ is called an *Engel integral curve* if $f_*(TI) \subset E(\subset T\widetilde{\mathcal{F}})$. A C^∞ map $g : I \to (\widetilde{\mathrm{Gr}}(1,V), D)$ is called a *Legendre curve* if $g_*(TI) \subset D(\subset T\widetilde{\mathrm{Gr}}(1,V))$. A C^∞ map $h : I \to (\widetilde{\mathrm{LG}}(V), C)$ is called a *null curve* if $h_*(TI) \subset C(\subset T\widetilde{\mathrm{LG}}(V))$.

Lemma 2.2. *For any Engel integral curve f, the projection $\pi_1 \circ f$ by π_1 is a Legendre curve and the projection $\pi_2 \circ f$ by π_2 is a null curve.*

Proof: We have $(\pi_1 \circ f)_*(T_tI) \subseteq (\pi_1)_*(E_{f(t)}) \subseteq D_{(\pi_1 \circ f)(t)}$ and

$$\begin{aligned} (\pi_2 \circ f)_*(T_tI) &\subseteq (\pi_2)_*(E_{f(t)}) \\ &= (\pi_2)_*(T_{f(t)}(\pi_2^{-1}(\pi_2(\pi_1^{-1}(\pi_1(f(t)))))) \\ &= T_{(\pi_2\circ f)(t)}(\pi_2(\pi_1^{-1}(\pi_1(f(t))))) \subseteq C_{(\pi_2\circ f)(t)}. \end{aligned}$$

□

There are natural classes of embedded Legendre curves in $\widetilde{\mathrm{Gr}}(1,V)$ and embedded null curves in $\widetilde{\mathrm{LG}}(V)$. Let $L \in \widetilde{\mathrm{LG}}(V)$. Then $\pi_1(\pi_2^{-1}(L))$ is a Legendre curve and is called a *Legendre straight line* or simply a *Legendre line* associated to L. Let $\ell \in \widetilde{\mathrm{Gr}}(1,V)$. Then $\pi_2(\pi_1^{-1}(\ell))$ is a null curve and is called a *null straight line* or simply a *null line* associated to ℓ. In fact we will give a projective structure on $\widetilde{\mathcal{F}}$ (resp. $\widetilde{\mathrm{Gr}}(1,V), \widetilde{\mathrm{LG}}(V)$) in §3. Then Legendre lines $\pi_1(\pi_2^{-1}(L))$, null lines $\pi_2(\pi_1^{-1}(\ell))$ and also $\pi_2^{-1}(L), \pi_1^{-1}(\ell)$ are actually "lines" for those projective structures. Same definitions are applied to $\mathcal{F}$ (non-oriented case).

Proposition 2.3. *Let $N \subset \widetilde{\mathcal{F}}$ be a null surface (see Introduction for the definition). Then locally (in a neighbourhood of any point of N), N is contained in the associated variety to a Legendre curve in $\widetilde{\mathrm{Gr}}(1,V)$.*

Proof: Let N be a null surface in $\widetilde{\mathrm{LG}}(V)$. Then N has the null direction field $C_L \cap T_LN$ $(L \in N)$ which lifts to a surface $\widetilde{N} \subset \widetilde{\mathcal{F}}$ via π_2. (If the direction field $C_L \cap T_LN$ $(L \in N)$ is not orientable, then $\pi_2|_{\widetilde{N}} : \widetilde{N} \to N$ is a double covering.) Then $\widetilde{N}$ is an integral surface to $E^2 = \pi_{1*}^{-1}(D)$. In fact, for any $\widetilde{x} \in \widetilde{N}$,

$$\pi_{2*}(T_{\widetilde{x}}\widetilde{N}) = T_{\pi_2(\widetilde{x})}N$$

is pseudo-orthogonal to the null direction $C_{\pi_2(\widetilde{x})} \cap T_{\pi_2(\widetilde{x})}N$, which is equal to $\pi_{2*}((E^2)_{\widetilde{x}})$ by Lemma 2.1. Since $\mathrm{Ker}(\pi_{2*}) \subset E$, we have

$$T_{\widetilde{x}}\widetilde{N} \subset (E^2)_{\widetilde{x}} + E_{\widetilde{x}} = (E^2)_{\widetilde{x}}.$$

Now $\pi_1|\widetilde{N}$ is an integral mapping to the contact distribution D. Therefore the rank of $\pi_1|\widetilde{N}$ is at most one, while at least one, hence the rank is identically one. Thus $\widetilde{N}$ is foliated by π_1-fibers. Take the local image γ of $\widetilde{N}$ by π_1. Then γ is a Legendre curve and, locally, $\widetilde{N} \subset \pi_1^{-1}(\gamma)$. Therefore we have $N \subset \pi_2(\pi_1^{-1}(\gamma))$, the associated variety to γ. □

Remark 2.4. The associated variety in $\widetilde{\mathrm{Gr}}(1,V)$ to a null curve in $\widetilde{\mathrm{LG}}(V)$ is characterized, in its smooth part, as a surface foliated by Legendre straight lines which lifts to an integral surface to the 3-dimensional cone field

$$\pi_{2*}^{-1}(C) = \{v \in T\widetilde{\mathcal{F}} \mid \pi_{2*}(v) \in C\}$$

on $\widetilde{\mathcal{F}}$. Typical examples are provided by tangent surfaces to Legendre curves and the "great spheres" given by

$$\widetilde{\mathrm{Gr}}(1,\ell^s) = \pi_1(\pi_2^{-1}(\pi_2(\pi_1^{-1}(\ell)))) \subset \widetilde{\mathrm{Gr}}(1,V), \ (\ell \in \widetilde{\mathrm{Gr}}(1,V)).$$

Thus naturally we are treating integral surfaces to derived systems $E^2 = \pi_{1*}^{-1}(D)$ or to $\pi_{2*}^{-1}(C)$ on the flag manifold $\widetilde{\mathcal{F}}$.

3. Projective Engel structure on the flag manifolds

We introduce systems of coordinates of $\widetilde{\mathcal{F}}$ which define the projective Engel structure on $\widetilde{\mathcal{F}}$. For projective structures, see [17] for instance.

Recall that (V, Ω) is a symplectic vector space of dimension 4 and $\widetilde{\mathcal{F}}$ the oriented Lagrange flag manifold consisting pairs (ℓ, L) of oriented lines ℓ and oriented Lagrangian planes $L \supset \ell$. Fix $(\ell_0, L_0) \in \widetilde{\mathcal{F}}$. Then the flag

$$\ell_0 \subset L_0 \subset {\ell_0}^s \subset V$$

is induced. Recall that ${\ell_0}^s$ denotes the skew-orthogonal space to ℓ_0 for Ω. We give a chart on the open subset

$$U = \{(\ell, L) \in \widetilde{\mathcal{F}} \mid L \cap L_0 = \{0\}, \ell \cap {\ell_0}^s = \{0\}\}.$$

Fix $(\ell_1, L_1) \in U$. Then we have the canonical direct sum decomposition

$$V = \ell_1 \oplus (L_1 \cap {\ell_0}^s) \oplus \ell_0 \oplus (L_0 \cap {\ell_1}^s).$$

Take a basis (e_1, e_2, f_1, f_2) of V such that

$$e_1 \in \ell_1,\ e_2 \in L_1 \cap {\ell_0}^s,\ f_1 \in \ell_0,\ f_2 \in L_0 \cap {\ell_1}^s,$$

and that

$$\Omega(e_1, f_1) = 1,\ \Omega(e_2, f_1) = 0,\ \Omega(e_1, f_2) = 0,\ \Omega(e_2, f_2) = 1.$$

Let $(\ell, L) \in U$. Since $L \cap L_0 = \{0\}$, there exists the unique basis g_1, g_2 of L of form

$$g_1 = e_1 + x f_1 + y f_2, \quad g_2 = e_2 + y f_1 + z f_2,$$

where $x, y, z \in \mathbf{R}$. Since $\ell \cap {\ell_0}^s = \{0\}$, there exists the unique basis h of ℓ of form $h = g_1 + \lambda g_2$, where $\lambda \in \mathbf{R}$. Then

$$h = e_1 + \lambda e_2 + (x + \lambda y) f_1 + (y + \lambda z) f_2.$$

Thus we have a chart $(\lambda, x, y, z) : U \to \mathbf{R}^4$. Then the Engel structure E on $\widetilde{\mathcal{F}}$ is described as follows: A curve $f(t) = (\lambda(t), x(t), y(t), z(t))$ in U through $(\ell, L) = (\lambda, x, y, z)$ at $t = 0$ defines a vector in $E_{(\ell,L)}$ if and only if the velocity vector $\dfrac{df}{dt}|_{t=0} \in L$. The condition is equivalent to that

$$\begin{pmatrix} 0 \\ \lambda' \\ (x+\lambda y)' \\ (y + \lambda z)' \end{pmatrix} = p \begin{pmatrix} 1 \\ 0 \\ x \\ y \end{pmatrix} + q \begin{pmatrix} 0 \\ 1 \\ y \\ z \end{pmatrix}$$

for some $p, q \in \mathbf{R}$. Then $p = 0$ and $q = \lambda'$. Therefore we have

$$(x + \lambda y)' = \lambda' y, \quad (y + \lambda z)' = \lambda' z.$$

Thus E is defined by the differential system

$$dx + \lambda dy = 0, \quad dy + \lambda dz = 0,$$

via the chart (λ, x, y, z).

In particular, any Engel integral curve $f(t) = (\lambda(t), x(t), y(t), z(t))$ in $U \subset \widetilde{\mathcal{F}}$ is given by

$$x(t) = \int \lambda(t)^2 z'(t) dt, \quad y(t) = -\int \lambda(t) z'(t) dt.$$

from any C^∞ functions $\lambda(t), z(t)$.

We describe the Engel structure E and its square E^2 in terms of frames (vector fields) on the coordinate neighbourhood $U \subset \widetilde{\mathcal{F}}$ introduced above. Moreover we give the coordinate expression of the cone field C and the conformal indefinite metric uniquely defined from C on the coordinate neighbourhood of $\widetilde{\mathrm{LG}}(V)$. Then we show the geometric interpretation of $\pi_{2*}(E_{(\ell,L)})$ and $\pi_{2*}(E^2_{(\ell,L)})$ for any $(\ell, L) \in U$, which shows Lemma 2.1.

Let $(\ell, L) \in \widetilde{\mathcal{F}}$. We fix an $(\ell_0, L_0) \in \widetilde{\mathcal{F}}$ satisfying $L \cap L_0 = \{0\}, \ell \cap \ell_0^s = \{0\}$, and, setting $(\ell_1, L_1) = (\ell, L)$, we consider the local coordinate system (λ, x, y, z) of $\widetilde{\mathcal{F}}$ centered at (ℓ, L) as above.

The local frame of $E \subset T\widetilde{\mathcal{F}}$ is given by

$$\Lambda = \frac{\partial}{\partial \lambda}, \quad X = \lambda^2 \frac{\partial}{\partial x} - \lambda \frac{\partial}{\partial y} + \frac{\partial}{\partial z},$$

under the coordinates λ, x, y, z. The square E^2 is spanned by Λ, X and $Y = 2\lambda \frac{\partial}{\partial x} - \frac{\partial}{\partial y}$. In terms of co-frame, E^2 is given by the 1-form

$$dx + 2\lambda dy + \lambda^2 dz = (dx + \lambda dy) + \lambda(dy + \lambda dz) = 0.$$

The condition that a Lagrange plane $\langle e_1 + xf_1 + yf_2, e_2 + yf_1 + zf_2 \rangle_{\mathbf{R}}$ belongs to the Schubert variety S_L is given by

$$S_L : xz - y^2 = 0.$$

The tangent cone C_L at L of S_L is given by

$$C_L : \xi\zeta - \eta^2 = 0.$$

for $v = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \zeta \frac{\partial}{\partial z} \in T_L\widetilde{\mathrm{LG}}(V)$. Using symmetric tensors, C is defined by $dxdz - (dy)^2 = 0$. The induced conformal metric g on $\widetilde{\mathrm{LG}}(V)$ is given by the bilinear form on $T_L\widetilde{\mathrm{LG}}(V)$ defined by

$$g(v_1, v_2) = \frac{1}{2}(\xi_1\zeta_2 + \xi_2\zeta_1) - \eta_1\eta_2,$$

for $v_i = \xi_i \frac{\partial}{\partial x} + \eta_i \frac{\partial}{\partial y} + \zeta_i \frac{\partial}{\partial z}$ $(i = 1, 2)$.

The projection $\pi_{2*}(E^2_{(\ell,L)})$ of the derived $E^2_{(\ell,L)}$ is given by the plane

$$\xi + 2\lambda\eta + \lambda^2\zeta = 0,$$

in $T_L\widetilde{\mathrm{LG}}(V)$ for a fixed λ. Regarding λ as a parameter, we have one-parameter family of planes, which envelopes C_L. The projection $\pi_{2*}(E_{(\ell,L)}) = T_L[\ell]$ of $E_{(\ell,L)}$ itself is given by the line

$$\xi + \lambda\eta = 0, \quad \eta + \lambda\zeta = 0,$$

while the null-vector $v = \pi_{2*}X = \lambda^2 \frac{\partial}{\partial x} - \lambda \frac{\partial}{\partial y} + \frac{\partial}{\partial z}$ provides the direction of the null straight line $[\ell]$.

Proof of Lemma 2.1: Note that $\pi_{2*}((E^2)_{(\ell,L)})$ is spanned by v and $u = 2\lambda \frac{\partial}{\partial x} - \frac{\partial}{\partial y}$ and that $g(v, u) = 0$. Therefore, by counting the dimension, we see that $\pi_{2*}((E^2)_{(\ell,L)})$ coincides with the pseudo-orthogonal space to $T_L[\ell]$. $\square$

Remark 3.1. The contact structure D on $\widetilde{\mathrm{Gr}}(1, V)$ is expressed by

$$D : \ d\mu = \nu d\lambda - \lambda d\nu,$$

under the local coordinates $\lambda, \mu = x + \lambda y$ and $\nu = y + \lambda z$ of $\widetilde{\mathrm{Gr}}(1, V)$.

Remark 3.2. Let $J^1(\mathbf{R},\mathbf{R})$ be the projective contact manifold with coordinates t, u, p and the contact structure $D_0 : du - pdt = 0$. Then the projective contact structure (S^3, D) is not isomorphic to $(J^1(\mathbf{R},\mathbf{R}), D_0)$ as projective contact structures locally. In fact, there are just two Legendre straight lines through a given point (t_0, u_0, p_0) in $J^1(\mathbf{R},\mathbf{R})$:

$$(s + x_0, p_0 s + y_0, p_0), \quad (x_0, y_0, s + p_0),$$

up to right equivalence, s being the parameter of straight line. On the other hand, on (S^3, D), there exists a Legendre straight line though any point with any direction of D in S^3.

Let $J^2(\mathbf{R},\mathbf{R})$ be the projective Engel manifold with coordinates t, u, p, q and the Engel structure $E_0 : du-pdt = 0, dp-qdt = 0$. Then the projective Engel structure $(\widetilde{\mathcal{F}}, E)$ with coordinates λ, x, y, z is not isomorphic to $(J^2(\mathbf{R},\mathbf{R}), E_0)$ as projective Engel structures locally. In fact, there is just one Engel integral straight line $(t_0, u_0, p_0, s + q_0)$ through a given point (t_0, u_0, p_0, q_0) in $J^2(\mathbf{R},\mathbf{R})$, if $q_0 \neq 0$. On the other hand, on $(\widetilde{\mathcal{F}}, E)$, there exist exactly two Engel straight lines, the π_1-fiber and the π_2-fiber, through any given point of $\widetilde{\mathcal{F}}$.

For the projective coordinate neighbourhood U, there exists the explicit diffeomorphism between $(U, E|_U)$ and $(J^2(\mathbf{R},\mathbf{R}), E_0)$ of Engel manifolds, given by

$$\begin{aligned} (\lambda, x, y, z) &\mapsto (t, u, p, q) = (\lambda,\ \tfrac{1}{2}\{(x + \lambda y) + \lambda(y + \lambda z)\},\ y + \lambda z,\ z), \\ (t, u, p, q) &\mapsto (\lambda, x, y, z) = (t,\ 2u - 2tp + t^2 q,\ p - tq,\ q), \end{aligned}$$

the "Engel-Legendre transformation".

Remark 3.3. For any $p_0 = (\lambda_0, x_0, y_0, z_0) \in \mathbf{R}^4$, there is a linear Engel transformation $T : (\mathbf{R}^4, p_0) \to (\mathbf{R}^4, 0)$ defined by

$$T(\lambda, x, y, z) = (\lambda - \lambda_0, x + 2\lambda_0 y + \lambda_0^2 z - x_0 - 2\lambda_0 y_0 - \lambda_0^2 z_0, y + \lambda_0 z - y_0 - \lambda_0 z_0, z - z_0).$$

4. Engel integral jet space and transversality

We introduce the jet-spaces of Engel integral curves.

Let I be an open interval. In the jet-space $J^r(I, \widetilde{\mathcal{F}})$ we consider the Engel integral jet-space:

$$J^r_E(I, \widetilde{\mathcal{F}}) = \{j^r f(t_0) \mid t_0 \in I,\ f : (\mathbf{R}, t_0) \to \widetilde{\mathcal{F}} \text{ is Engel integral}\}.$$

Lemma 4.1. *$J^r_E(I, \widetilde{\mathcal{F}})$ is a subbundle of $J^r(I, \widetilde{\mathcal{F}})$ for the projection $\Pi : J^r(I, \widetilde{\mathcal{F}}) \to I \times \widetilde{\mathcal{F}}$ of codimension $2r$.*

Proof: By Remark 3.3, it is sufficient to show that

$$J^r_E(1, 4) = \{j^r f(0) \mid f : (\mathbf{R}, 0) \to (\mathbf{R}^4, 0) \text{ is Engel integral}\}$$

is a submanifold of $J^r(1, 4)$ of codimension $2r$. To show it, define the mapping $\Phi : J^r(1, 4) \to \Lambda_1^{r-1} \times \Lambda_1^{r-1} \cong \mathbf{R}^{2r}$ by

$$\Phi(j^r(\lambda, x, y, z)(0)) = (j^{r-1}(dx + \lambda dy)(0), j^{r-1}(dy + \lambda dz)(0)).$$

Here Λ_1^{r-1} denotes the $(r - 1)$-jet space of 1-forms on $(\mathbf{R}, 0)$. Then Φ is a submersion. In fact any deformation $(B_1(t, s), B_2(t, s))$ with parameter s of the pair $(b_1(t), b_2(t)) = (x'(t) + \lambda(t)y'(t), y'(t) + \lambda(t)z'(t))$ is lifted to $(\lambda(t), x(t, s), y(t, s), z(t))$ by setting

$$x(t, s) = \int \{\lambda(t)^2 z'(t) + B_1(t, s)\}dt, \quad y(t, s) = \int \{-\lambda(t)z'(t) + B_2(t, s)\}dt,$$

$x(0,s)=0, y(0,s)=0$. Therefore $\Phi^{-1}(0)=J^r_E(1,4)$ is a submanifold of $J^r(1,4)$ of codimension $2r$. □

Proposition 4.2. (Engel transversality theorem on open intervals) *Let $Q \subset J^r_E(I,\widetilde{\mathcal{F}})$ be a submanifold. Then any Engel integral curve $f : I \to \widetilde{\mathcal{F}}$ is approximated in C^∞-topology by an Engel integral curve $f' : I \to \widetilde{\mathcal{F}}$ for which $j^r f' : I \to J^r_E(I,\widetilde{\mathcal{F}})$ is transverse to Q.*

Proof: For any open sub-interval $V \subset I$ and for any coordinate neighbourhood $U \subset \widetilde{\mathcal{F}}$ introduced in §3, we define a diffeomorphism

$$\varphi = \varphi_{(V,U)} : J^r_E(V,U) \to V \times U \times J^r(1,2)$$

by $\varphi(j^r f(t_0)) = (t_0, f(t_0), j^r((\lambda, z)\circ T \circ f(t+t_0))(0))$, using the linear Engel transformation T with $T(f(t_0)) = 0$.

Now let $f : I \to \widetilde{\mathcal{F}}$ be an Engel integral curve. Suppose, as a special case, $f(I)$ is in some projective coordinate neighbourhood U introduced in §3. Then, by the ordinary transversality theorem, (λ, z)-components of f are perturbed so that, for a perturbed f', $\varphi \circ j^r f'$ is transverse to $\varphi(Q \cap J^r_E(I,U)) \subset I \times U \times J^r(1,2)$. Then $j^r f'$ is transverse to Q.

In general case, there is a strictly increasing sequence $\{t_i\}_{i\in\mathbf{Z}}$ of points in I such that $f([t_i, t_{i+1}])$ is contained in some projective coordinate neighbourhood U_i. We set $K_i = [t_i, t_{i+1}]$ and take open intervals $W_i \supset K_i$ such that also $f(W_i) \subset U_i$ and that $W_i \cap W_j = \emptyset$ if $|i-j| \geq 2$.

First we perturb f over W_0 into an Engel integral curve $f_0 : W_0 \to \widetilde{\mathcal{F}}$ such that $j^r f_0$ is transverse to Q over W_0. In fact, similarly as in the special case, by the ordinary transversality theorem via $\varphi = \varphi_{(W_0,U_0)}$, (λ, z)-components of $f|_{W_0}$ are perturbed so that, for the perturbed f_0, $\varphi \circ j^r f_0$ is transverse to $\varphi(Q \cap J^r_E(W_0,U_0)) \subset W_0 \times U_0 \times J^r(1,2)$. Then $j^r f_0$ is transverse to Q over W_0.

Second we perturb f over $W_0 \cup W_1$ into an Engel integral curve $f_1 : W_0 \cup W_1 \to \widetilde{\mathcal{F}}$ such that $j^r f_1$ is transverse to Q and $f_1|_{K_0} = f_0|_{K_0}$. This is achieved, under the coordinates on U_1, by

$$x(t) = \int_{t_1}^{t} \lambda(t)^2 z'(t)dt + x(t_1), \quad y(t) = -\int_{t_1}^{t} \lambda(t) z'(t)dt + y(t_1),$$

perturbing $\lambda(t), z(t)$ over W_1 just outside of $K_0 \cap W_1$ and setting $f_1(t_1) = f_0(t_1)$.

Third we perturb f over $W_0 \cup W_1 \cup W_2$ into an Engel integral curve $f_2 : W_0 \cup W_1 \cup W_2 \to \widetilde{\mathcal{F}}$ such that $j^r f_2$ is transverse to Q and $f_2|_{K_0\cup K_1} = f_1|_{K_0\cup K_1}$. Thus, by continuing this procedure, we have a perturbation $f' : \cup_{0\leq i} W_i \to \widetilde{\mathcal{F}}$ of f such that $j^r f'$ is transverse to Q.

Finally we perturb f backward to an Engel integral curve $f'' : I = \cup_{i\in\mathbf{Z}} W_i \to \widetilde{\mathcal{F}}$ such that $j^r f''$ is transverse to Q, by perturbing $\lambda(t), z(t)$ and using, for $i \leq 0$,

$$x(t) = -\int_{t}^{t_i} \lambda(t)^2 z'(t)dt + x(t_i), \quad y(t) = \int_{t}^{t_i} \lambda(t) z'(t)dt + y(t_i).$$

Note that, on any compact $K \subset \cup_{i\in\mathbf{Z}} W_i$, the perturbation is achieved just by a finite number of steps. Therefore we can take transversal perturbations of f to Q which are arbitrarily small in C^∞ topology. □

Remark 4.3. The transversality theorem does not hold for Engel integral curves by perturbations with *compact supports* (or for Engel integral curves on closed interval by perturbations with *fixed ends*). In fact it is known that the abnormal (singular) curves for Engel structures are rigid and have no essential perturbations with fixed ends ([5]).

5. Codimension formula, duality, and generic Engel integral curves

For the local coordinates (λ, x, y, z) of $\widetilde{\mathcal{F}}$ introduced in §3, the double fibration

$$\widetilde{\mathrm{Gr}}(1, V) \xleftarrow{\pi_1} \widetilde{\mathcal{F}} \xrightarrow{\pi_2} \widetilde{\mathrm{LG}}(V)$$

are given by

$$\pi_1(\lambda, x, y, z) = (\lambda, x + \lambda y, y + \lambda z), \quad \pi_2(\lambda, x, y, z) = (x, y, z).$$

Let $c : I \to M^3$ be a C^∞ curve in a 3-space with a projective structure. We say that c is *of finite type* at $t = t_0 \in I$ if there exists a local projective coordinates (x_1, x_2, x_3) of M centred at $c(t_0)$ such that

$$x_1 \circ c(t) = t^{a_1} + O(t^{a_1+1}),\ x_2 \circ c(t) = t^{a_2} + O(t^{a_2+1}),\ x_3 \circ c(t) = t^{a_3} + O(t^{a_3+1}),$$

for some increasing sequence of positive integers $1 \le a_1 < a_2 < a_3$. Then (a_1, a_2, a_3) is uniquely determined from the projective class of the germ of c at $t = t_0$, and we say that c is *of type* (a_1, a_2, a_3) at $t = t_0$. If we consider the Wronski matrices

$$W_i(t) = \begin{pmatrix} x_1'(t) & x_1''(t) & \cdots & x_1^{(i)}(t) \\ x_2'(t) & x_2''(t) & \cdots & x_2^{(i)}(t) \\ x_3'(t) & x_3''(t) & \cdots & x_3^{(i)}(t) \end{pmatrix}, \quad i = 1, 2, \dots,$$

then we have

$$a_1 = \min\{i \mid \text{rank } W_i(t_0) = 1\}, \quad a_2 = \min\{i \mid \text{rank } W_i(t_0) = 2\},$$
$$a_3 = \min\{i \mid \text{rank } W_i(t_0) = 3\}.$$

Let $\mathbf{A} = (a_1, a_2, a_3)$ and $\mathbf{B} = (b_1, b_2, b_3)$ be increasing sequences of positive integers, $1 \le a_1 < a_2 < a_3$, $1 \le b_1 < b_2 < b_3$. We set, for a sufficiently large r,

$$\Sigma_{\pi_1, \mathbf{A}} = \{j^r f(t_0) \in J^r_E(I, \widetilde{\mathcal{F}}) \mid \pi_1 \circ f : I \to \widetilde{\mathrm{Gr}}(1, V) \text{ is of type } \mathbf{A}\},$$
$$\Sigma_{\pi_2, \mathbf{B}} = \{j^r f(t_0) \in J^r_E(I, \widetilde{\mathcal{F}}) \mid \pi_2 \circ f : I \to \widetilde{\mathrm{LG}}(V) \text{ is of type } \mathbf{B}\}.$$

Proposition 5.1.
(1) Codimension formula for π_1:
We have, for $r \ge a_3$, $\Sigma_{\pi_1, \mathbf{A}} \ne \emptyset$ if and only if $a_3 = a_1 + a_2$. Then we have $\Sigma_{\pi_1, \mathbf{A}}$ is a submanifold of $J^r_E(I, \widetilde{\mathcal{F}})$ of codimension $a_2 - 2$.

(2) Codimension formula for π_2:
We have, for $r \ge b_3$, $\Sigma_{\pi_2, \mathbf{B}} \ne \emptyset$ if and only if $b_3 = 2b_2 - b_1$. Then we have $\Sigma_{\pi_2, \mathbf{B}}$ is a submanifold of $J^r_E(I, \widetilde{\mathcal{F}})$ of codimension $b_2 - 2$.

(3) The duality formula:
Let $f : I \to \widetilde{\mathcal{F}}$ be an Engel integral curve of finite type. Then the type $\mathbf{A}$ of $\pi_1 \circ f$ and the type $\mathbf{B}$ of $\pi_2 \circ f$ are related by

$$(b_1, b_2, b_3) = (a_2 - a_1, a_2, a_3), \quad (a_1, a_2, a_3) = (b_2 - b_1, b_2, b_3).$$

Proof: Let $f : (\mathbf{R}, 0) \to (\mathbf{R}^4, 0)$, $f(t) = (\lambda(t), x(t), y(t), z(t))$ be an Engel integral curve-germ. If $\lambda(t)$ or $z(t)$ is infinitely flat at $t = 0$, then both $x(t)$ and $y(t)$ are infinitely flat at $t = 0$ by the Engel condition. Then both $\pi_1 \circ f$ and $\pi_2 \circ f$ are not of finite type. Now let $u = \mathrm{ord}\lambda(t) < \infty, v = \mathrm{ord}z(t) < \infty$. Here $\mathrm{ord}\varphi(t)$ denotes the order of a function $\varphi(t)$ at $t = 0$. Then

$$\mathrm{ord}y(t) = \mathrm{ord}\lambda(t) + \mathrm{ord}z(t) = u + v \quad \mathrm{ord}x(t) = \mathrm{ord}\lambda(t) + \mathrm{ord}y(t) = 2u + v.$$

Since

$$(x+\lambda y)'(t) = y(t)\lambda'(t), \quad (y+\lambda z)'(t) = z(t)\lambda'(t),$$

we have

$$\mathrm{ord}(x(t)+\lambda(t)y(t)) = 2u+v, \quad \mathrm{ord}(y(t)+\lambda(t)z(t)) = u+v.$$

Suppose the type of $\pi_1 \circ f$ at $t=0$ is $\mathbf{A} = (a_1, a_2, a_3)$. Then we have $a_1 = u, a_2 = u+v, a_3 = 2u+v$. This is realized for some $u, v \geq 1$ if and only if $a_3 = a_1 + a_2$. Then the codimension of $\Sigma_{\pi_1,\mathbf{A}}$ is given by $u+v-2 = a_2-2$. This shows (1). On the other hand, suppose the type of $\pi_2 \circ f$ at $t=0$ is $\mathbf{B} = (b_1, b_2, b_3)$. Then $b_1 = v, b_2 = v+u, b_3 = v+2u$. This is realized for some $v, u \geq 1$ if and only if $b_3 = 2b_2 - b_1$. Then the codimension of $\Sigma_{\pi_2,\mathbf{B}}$ is given by $u+v-2 = b_2-2$. This shows (2). Moreover $b_1 = v = a_2 - a_1, b_2 = v+u = a_2, b_3 = v+2u = a_3$. Thus we see (3). □

Remark 5.2. The conditions $\mathrm{ord}\lambda(t) = u$ and $\mathrm{ord}z(t) = v$ give a submanifold of $J^r(1,2)$ of codimension $(u-1)+(v-1) = u+v-2$.

Proposition 5.3. *For any generic Engel integral curve $f : I \to \widetilde{\mathcal{F}}$ and for any point $t_0 \in I$, the type of $\pi_1 \circ f : I \to \widetilde{\mathrm{Gr}}(1,V)$ is $(1,2,3), (1,3,4)$ or $(2,3,5)$. Moreover the type of $\pi_2 \circ f : I \to \widetilde{\mathrm{LG}}(V)$ is $(1,2,3), (2,3,4)$ or $(1,3,5)$ correspondingly.*

Proof: For a sufficiently large r, we set

$$\Sigma = \overline{(\cup_{a_2 \geq 4}\ \Sigma_{\pi_1,\mathbf{A}}) \cup (\cup_{b_2 \geq 4}\ \Sigma_{\pi_2,\mathbf{B}})} \subset J^r_E(I, \widetilde{\mathcal{F}}).$$

Then Σ is fibered over $I \times \widetilde{\mathcal{F}}$ by a real algebraic set in $J^r_E(1,4)$ of codimension ≥ 2. In fact the fiber of Σ is defined in $J^r(1,4)$ by the vanishing of some minors of the Wronski matrices for the curves $\pi_1 \circ f$ and $\pi_2 \circ f$. Note that Σ contains curve-jets $j^r f(t_0)$ for which the type of $\pi_1 \circ f$ or $\pi_2 \circ f$ at t_0 is not determined by the jet $j^r f(t_0)$. However they form a subset of codimension $\geq r-2$, which does not affect the codimension calculus.

Let $\mathcal{R}$ be the set of $f \in C^\infty_E(I, \widetilde{\mathcal{F}})$ such that $j^r f : I \to \widetilde{\mathcal{F}}$ is transversal to all $\Sigma_{\pi_1,\mathbf{A}}$ with $a_2 \leq 3$ and to all $\Sigma_{\pi_2,\mathbf{B}}$ with $b_2 \leq 3$ and moreover to (all strata of a stratification of) Σ. By Proposition 4.2, $\mathcal{R}$ is dense in $C^\infty_E(I, \widetilde{\mathcal{F}})$ for the C^∞-topology. By Proposition 5.1, $f \in \mathcal{R}$ is equivalent to that $j^r f$ is transversal to $\Sigma_{\pi_1,\mathbf{A}}$ with $a_2 = 3$ and $\Sigma_{\pi_2,\mathbf{B}}$ with $b_2 = 3$ at isolated points in I and that $j^r f(I) \cap \Sigma = \emptyset$. Therefore $\mathcal{R}$ is residual in $C^\infty_E(I, \widetilde{\mathcal{F}})$ for the C^∞-topology. Let $f \in \mathcal{R}$ and $t_0 \in I$. Let $\mathbf{A}$ be the type of $\pi_1 \circ f$ and $\mathbf{B}$ the type of $\pi_2 \circ f$. Then we have $a_2 \leq 3$. So $a_1 \leq 2$. If $a_1 = 1$, then $(a_1, a_2, a_3) = (1,2,3)$ or $(1,3,4)$ by Propositions 5.1 (1). If $a_1 = 2$, then $(a_1, a_2, a_3) = (2,3,5)$. Then the rest is proved by the formula $(b_1, b_2, b_3) = (a_2 - a_1, a_2, a_3)$ of Proposition 5.1 (3). □

In particular we have:

Corollary 5.4. *Generic Engel integral curves are immersions. In fact, for any generic Engel integral curve $f : I \to \widetilde{\mathcal{F}}$, and for any point $t_0 \in I$, either $\pi_1 \circ f$ or $\pi_2 \circ f$ is an immersion.*

Remark 5.5. Under the ordinary projective duality of space curves, the duality formula between a space curve and its projective dual curve is given by

$$(b_1, b_2, b_3) = (a_3 - a_2, a_3 - a_1, a_3), \quad (a_1, a_2, a_3) = (b_3 - b_2, b_3 - b_1, b_3),$$

see [18]. Then the cuspidal edges, Mond surfaces and folded pleats are self-dual, the swallowtails are dual to the folded umbrellas (the cuspidal cross-caps), and the Shcherbak surfaces are dual to the butterflies as singularities of tangent surfaces, see the survey article [11].

6. Normal forms on singularities of tangent surfaces

First we show the procedure to obtain normal forms of tangent surfaces to space curves in $P(V)$ or in $\widetilde{\mathrm{Gr}}(1,V)$. Then we give the differential classification of tangent surfaces to curves of type $(2,3,5)$ and prove all statements in Theorem 1.1.

Let $f=(\lambda,x,y,z):(\mathbf{R},0)\to(\mathbf{R}^4,0)$ be an Engel integral curve satisfying $dx+\lambda dy=0$ and $dy+\lambda dz=0$ (see §3).

For example, let $\lambda=t, z=t$. Then

$$y=-\tfrac{1}{2}t^2,\ x=\tfrac{1}{3}t^3,\ x+\lambda y=-\tfrac{1}{6}t^3,\ y+\lambda z=\tfrac{1}{2}t^2.$$

Then

$$\pi_1(f(t))=(\lambda,x+\lambda y,y+\lambda z)=(t,-\tfrac{1}{6}t^3,\tfrac{1}{2}t^2),$$
$$\pi_2(f(t))=(x,y,z)=(\tfrac{1}{3}t^3,-\tfrac{1}{2}t^2,t).$$

The tangent surface in $\widetilde{\mathrm{Gr}}(1,V)$ is parametrized by

$$\begin{pmatrix} t \\ -\frac{1}{6}t^3 \\ \frac{1}{2}t^2 \end{pmatrix}+s\begin{pmatrix} 1 \\ -\frac{1}{2}t^2 \\ t \end{pmatrix} = \begin{pmatrix} t+s \\ -\frac{1}{6}t^3-\frac{1}{2}st^2 \\ \frac{1}{2}t^2+st \end{pmatrix}.$$

Introducing a new parameter $X=t+s$, we have the parametrization

$$(X,\ -\tfrac{1}{2}t^2+Xt,\ \tfrac{1}{3}t^3-\tfrac{1}{2}Xt^2)$$

of the tangent surface in $\widetilde{\mathrm{Gr}}(1,V)$ to a curve of type $(1,2,3)$.

In general, the velocity vector of $\pi_1\circ f$ is given by

$$(\lambda',(x+\lambda y)',(y+\lambda z)')=\lambda'(1,y,z).$$

Therefore the parametrization of the tangent surface to $\pi_1\circ f$ is diffeomorphic to

$$(\lambda,\ y+\lambda z,\ x+\lambda y)+s(1,z,y)=(\lambda+s,\ y+(\lambda+s)z,\ x+(\lambda+s)y).$$

If we set $X=\lambda+s$, then we have the parametrization

$$(X,t)\mapsto(X,\ y(t)+Xz(t),\ x(t)+Xy(t)).$$

Now for a given Engel integral curve, suppose that $\mathrm{ord}\lambda(t)=2$ and $\mathrm{ord}z(t)=1$ at $t=0$. Then after a re-parametrization of t, we may suppose that $\lambda=\frac{1}{2}t^2$ and $z=at+\frac{b}{2}t^2+O(t^3)$ for some $a,b\in\mathbf{R}, a\neq 0$. Then we have the parametrization

$$x=\tfrac{a}{20}t^5+\tfrac{b}{24}t^6+O(t^7),\quad y=-\tfrac{a}{6}t^3-\tfrac{b}{8}t^4+O(t^5).$$

The parametrization of $\pi_1\circ f$ is given by

$$(\tfrac{1}{2}t^2,\ \ \tfrac{a}{3}t^3+\tfrac{b}{8}t^4+O(t^5),\ \ -\tfrac{a}{30}t^5-\tfrac{b}{48}t^6+O(t^7)).$$

We obtain the parametrization $F:(\mathbf{R}^2,0)\to(\mathbf{R}^3,0)$, $(X,t)\mapsto(\lambda,\mu,\nu)$ of the tangent surface in $\widetilde{\mathrm{Gr}}(1,V)$ to the curve $\pi_1\circ f$ given in a form

$$\big(X,\ \ a(-\tfrac{1}{6}t^3+Xt)+b(-\tfrac{1}{8}t^4+\tfrac{1}{2}Xt^2)+\psi(X,t),$$
$$a(\tfrac{1}{20}t^5-\tfrac{1}{6}Xt^3)+b(\tfrac{1}{24}t^6-\tfrac{1}{8}Xt^4)+\rho(X,t)\big).$$

Here we give the natural weights $w(X)=2, w(t)=1$. Then the order of ψ (resp. ρ) is higher than 4 (resp. 6) with respect to the given weights. Moreover, we have that $\frac{\partial\psi}{\partial t}$ is a multiple of $-\frac{1}{2}t^2+X$ by some function, and that $\frac{\partial\rho}{\partial t}=-\frac{t^2}{2}\frac{\partial\psi}{\partial t}$.

Proposition 6.1. *If $b \neq 0$, then F is locally diffeomorphic to*
$$(X, \quad -\tfrac{1}{6}t^3 + Xt - \tfrac{1}{8}t^4 + \tfrac{1}{2}Xt^2, \quad \tfrac{1}{20}t^5 - \tfrac{1}{6}Xt^3 + \tfrac{1}{24}t^6 - \tfrac{1}{8}Xt^4).$$
If $b = 0$, then F is locally diffeomorphic to
$$(X, \quad -\tfrac{1}{6}t^3 + Xt, \quad \tfrac{1}{20}t^5 - \tfrac{1}{6}Xt^3).$$
The two map-germs are not diffeomorphic to each other.

Remark 6.2. Let $\Sigma'_{\pi_1,(2,3,5)}$ be set of jets $j^r f(t_0)$ such that $\pi_1 \circ f$ is of type $(2,3,5)$ at t_0 and $z \circ f(t+t_0) = f(t_0) + at + O(t^3)$, for some $a \neq 0$, in a projective chart introduced in §3. Then $\Sigma'_{\pi_1,(2,3,5)}$ has codimension ≥ 2. Therefore the Engel integral transversality theorem (Proposition 4.2) yields that generically we have $b \neq 0$.

Remark 6.3. The proof of Proposition 6.1 can be applied also to the differential classification of singularities for tangent developables to curves of type $(2,3,5)$: There exists exactly two diffeomorphism classes as in Proposition 6.1.

To show Proposition 6.1, we follow the standard infinitesimal method of singularity theory ([14], [4], [20]). Because we treat a specialized class of map-germs, we need also an additional algebraic method as in [10]. The proof goes similarly to that for the classification, for instance, in case $(1,3,5)$ of [10]. However, in our case $(2,3,5)$, the terms next to the leading terms turn to be regarded as well, and the proof must be modified accordingly.

Introducing an additional parameter s, we set
$$\begin{aligned} F_s(X,t) = {}^T\big(X, \quad & a(-\tfrac{1}{6}t^3 + Xt) + b(-\tfrac{1}{8}t^4 + \tfrac{1}{2}Xt^2) + s\psi, \\ & a(\tfrac{1}{20}t^5 - \tfrac{1}{6}Xt^3) + b(\tfrac{1}{24}t^6 - \tfrac{1}{8}Xt^4) + s\rho\big)\,. \end{aligned}$$
We are going to show that this family is trivialized under diffeomorphism equivalence (i.e. C^∞-right-left equivalence). Strictly we see that it is trivialized, preserving the tangent lines to the base point.

Proposition 6.4. *For any $s_0 \in \mathbf{R}$, we can solve the infinitesimal equation*
$$\begin{pmatrix} 0 \\ \psi \\ \rho \end{pmatrix} = \left(A\frac{\partial}{\partial X} + Bt\frac{\partial}{\partial t}\right) F_s - \begin{pmatrix} C(F_s) \\ D(F_s) \\ E(F_s) \end{pmatrix}$$
near $(0,0,s_0)$, for some C^∞ functions $A = A(X,t,s)$, $B = B(X,t,s)$ and $C(\lambda,\mu,\nu), D(\lambda,\mu,\nu), E(\lambda,\mu,\nu)$ satisfying that
$$A(0,0,s) = 0, C(0,0,0) = D(0,0,0) = E(0,0,0) = 0.$$

Proof: The form of the vector field $A\frac{\partial}{\partial X} + Bt\frac{\partial}{\partial t}$ is essential to apply our algebraic method.

By the first row of the equation, necessarily we have $A = C(F_s)$.

We set $U = U(X,t) = a(-\frac{1}{6}t^3 + Xt) + b(-\frac{1}{8}t^4 + \frac{1}{2}Xt^2)$. First we solve the equation of second row:
$$\psi = (C(F_s))\frac{\partial(U + s\psi)}{\partial X} + Bt\,\frac{\partial(U + s\psi)}{\partial t} - D(F_s). \tag{1}$$

Lemma 6.5. *The equation* (1) *is solved for some $B(X,t,s), C(\lambda,\mu,\nu), D(\lambda,\mu,\nu)$ with the condition $C(0,0,0) = 0, D(0,0,0) = 0$.*

To show Lemma 6.5, we define, additionally, the map-germ

$$G : (\mathbf{R}^3, (0,0,s_0)) \to (\mathbf{R}^3, (0,0,s_0))$$

by $G(X,t,s) = (X, U(X,t) + s\psi(X,t), s)$, and we denote by $\mathcal{E}_{X,t,s}$ (resp. $\mathcal{E}_{\lambda,\mu,s}$) the algebra of function-germ $(\mathbf{R}^3, (0,0,s_0)) \to \mathbf{R}$ on the source (resp. target) of G and by $\mathfrak{m}_{X,t,s}$ (resp. $\mathfrak{m}_{\lambda,\mu,s}$) its maximal ideal. Moreover we set, for $\ell = 0,1,2,\dots,$

$$\mathfrak{m}^{(\ell)}_{X,t,s} = \{h \in \mathcal{E}_{X,t,s} \mid \operatorname{ord}(h) \geq \ell\},$$

with respect to the weights $w(t) = w(s) = 1, w(X) = 2$. Note that $\psi \in \mathfrak{m}^{(5)}_{X,t,s}$.

We define the $\mathcal{E}_{\lambda,\mu,s}$-submodule, for $r = 0,1,2,\dots,$

$$M^{(r)} := G^*\mathfrak{m}_{\lambda,\mu,s} + \frac{\partial(U+s\psi)}{\partial X} G^*\mathfrak{m}_{\lambda,\mu,s} + t\frac{\partial(U+s\psi)}{\partial t}\mathfrak{m}^{(r)}_{X,t,s}$$

of $\mathcal{E}_{X,t,s}$ via $G^* : \mathcal{E}_{\lambda,\mu,s} \to \mathcal{E}_{X,t,s}$.

Lemma 6.6. *If $\ell \geq 5$, then $\mathfrak{m}^{(\ell)}_{X,t,s} \subset M^{(\ell-3)}$.*

Proof: In fact, using the initial part of U, we obtain that, if $\ell \geq 5$, then

$$\mathfrak{m}^{(\ell)}_{X,t,s} \subset M^{(\ell-3)} + \mathfrak{m}^{(\ell+1)}_{X,t,s}.$$

For example, in the case $\ell = 5$, we have $t^5 + 2Xt^3 \equiv 0, Xt^3 + 2X^2t \equiv 0, -\frac{1}{6}Xt^3 + X^2t \equiv 0$ modulo $M^{(2)} + \mathfrak{m}^{(6)}_{X,t,s}$, which implies $t^5 \equiv Xt^3 \equiv X^2t \equiv 0$.

Note that G is a finite map-germ, namely that $\mathcal{E}_{X,t,s}$ is a finite $\mathcal{E}_{\lambda,\mu,s}$-module via G^*. Then, for any ℓ and for a sufficiently large N, we have

$$\mathfrak{m}^{(N)}_{X,t,s} \subset G^*\mathfrak{m}_{\lambda,\mu,s} \cdot \mathfrak{m}^{(\ell)}_{X,t,s}.$$

Therefore we have

$$\mathfrak{m}^{(\ell)}_{X,t,s} \subset M^{(\ell-3)} + G^*\mathfrak{m}_{\lambda,\mu,s} \cdot \mathfrak{m}^{(\ell)}_{X,t,s}.$$

Since $\mathfrak{m}^{(\ell)}_{X,t,s}$ is a finite $\mathcal{E}_{\lambda,\mu,s}$-module via G^*, we have $\mathfrak{m}^{(\ell)}_{X,t,s} \subset M^{(\ell-3)}$ by Nakayama's lemma. □

Proof of Lemma 6.5: Since $\psi \in \mathfrak{m}^{(5)}_{X,t,s}$, Lemma 6.6 implies Lemma 6.5. □

Since we can solve the infinitesimal equation for the first and second rows in Proposition 6.4, we have a diffeomorphism germ $\sigma : (\mathbf{R}^2, 0) \to (\mathbf{R}^2, 0)$ of form $\sigma(X,t) = (\sigma_1(X,t), t\sigma_2(X,t))$ and a diffeomorphism germ $\tau : (\mathbf{R}^2, 0) \to (\mathbf{R}^2, 0)$ such that $\tau \circ (X, U+\psi) \circ \sigma^{-1} = (X, U)$. This construction is needed just to guarantee the properties of the following algebraic objects.

As in [10], we set, for $k = 0,1,2,\dots,$

$$\begin{aligned}\mathcal{H}_k &:= \left\{ h \in t^k\mathcal{E}_{X,t,s} \;\middle|\; \frac{\partial h}{\partial t} \in t^k \frac{\partial U}{\partial t}\mathcal{E}_{X,t,s} \right\} \\ &= \left\{ h \in t^k\mathcal{E}_{X,t,s} \;\middle|\; \frac{\partial h}{\partial t} \in t^k(-\frac{1}{2}t^2 + X)\mathcal{E}_{X,t,s} \right\}.\end{aligned}$$

Note that $G^*\mathcal{E}_{\lambda,\mu,s} \in \mathcal{H}_0$ and $\rho \in \mathcal{H}_4$. Also note that $\frac{\partial U}{\partial t} = (a+bt)(-\frac{1}{2}t^2 + X)$.

We have a sequence of $G^*\mathcal{E}_{\lambda,\mu,s}$-modules:

$$\mathcal{E}_{X,t,s} \supset \mathcal{H}_0 \supset \mathcal{H}_1 \supset \cdots \supset \mathcal{H}_k \supset \cdots.$$

Then we have

Lemma 6.7. (Lemma 2.3 of [10]) *Let a vector field of form $\xi = A\frac{\partial}{\partial X} + Bt\frac{\partial}{\partial t}$ satisfy $A \in \mathcal{H}_0$ and $\xi(U + s\psi) \in \mathcal{H}_0$. Then, for any $k \geq 0$ and for any $h \in \mathcal{H}_k$, we have $\xi h \in \mathcal{H}_k$.*

We set $U_k = \int_0^t \frac{t^k}{k!}\frac{\partial U}{\partial t}dt$. Then $U_k \in \mathcal{H}_k$. Note that the leading term of the third component of F_s is equal to $-U_2$. Moreover $U_k(0,t)$ is of order $k+3$. Then we have

Lemma 6.8. (1) *$\mathcal{H}_k$ is generated as $G^*\mathcal{E}_{\lambda,\mu,s}$-module by $U_k, U_{k+1}, U_{k+2}, U_{k+3}$.*
(2) *$\mathcal{H}_k$ is generated as $G^*\mathcal{E}_{\lambda,\mu,s}$-module by those elements generating the vector space $t^{k+3}\mathcal{E}_t/t^{k+7}\mathcal{E}_t$ over $\mathbf{R}$ via the inclusion $i : (\mathbf{R}, 0) \to (\mathbf{R}^3, (0, 0, s_0))$, $i(t) = (0, t, s_0)$.*

Proof: The proof is achieved by applying the method used in the proof of Lemma 2.4 of [10], to the case $m = 3$ and $U = a(-\frac{1}{6}t^3 + Xt) + b(-\frac{1}{8}t^4 + \frac{1}{2}Xt^2)$. Note that we need more generators in (1) than in the case treated in [10], since U may not be taken to be quasi-homogeneous in our case. □

To complete the proof of Proposition 6.1, we modify the vector field $\xi = A\frac{\partial}{\partial X} + Bt\frac{\partial}{\partial t}$ and $D(F_s)$ such that also the equation of third row holds, for some $E(F_s)$. Since $\rho, \xi(-U_2+s\rho) \in \mathcal{H}_4$, it is sufficient, for the solvability of our infinitesimal equation, to find C_1, B_1, D_1, E_1 satisfying that $\xi = C_1(G)\frac{\partial}{\partial X} + B_1 t\frac{\partial}{\partial t}$ satisfies that $\xi(U + s\psi) - D_1(F_s) = 0$, and that $h = \xi(-U_2 + s\rho) - E_1(F_s)$ is of order $7, 8, 9, 10$ when restricted to $\{X = 0, s = s_0\}$, by Lemma 6.8.

Note that $h_{10} := (-U_2 + s\rho)^2 \in \mathcal{H}_4$ is a composite function of F_s and that $h_{10}(0,t,s_0)$ is of order 10. In fact any order ≥ 10 is realizable by a composite function of F_s which belongs to $\mathcal{H}_4$. Then we take it as $E_1(F_s)$ and set $C_1(G) = 0, B_1 = 0, D_1(G) = 0$.

To produce elements of order $7, 8, 9$, we use Lemma 6.6 again.

We choose $c_1, c_2, c_3, c_4 \in \mathbf{R}$ with $c_2 \neq 0$ such that the terms of weight 5 of

$$\theta_7 = c_1X^2\frac{\partial(U+s\psi)}{\partial X} + (c_2t^3 + c_3Xt)\frac{\partial(U+s\psi)}{\partial t} + c_4X(U+s\psi).$$

vanish and so that θ_7 belongs to $\mathfrak{m}^{(6)}_{X,t,s} \subset M^{(3)}$. Then we have, for some C_2, B_2, D_2,

$$\theta_7 = C_2(G)\frac{\partial(U+s\psi)}{\partial X} + B_2t\frac{\partial(U+s\psi)}{\partial t} + D_2(G)$$

with $C_2(0) = D_2(0) = 0$ and $B_2 \in \mathfrak{m}^{(3)}_{X,t,s}$. We set

$$\xi_2 = (c_1X^2 - C_2(G))\frac{\partial}{\partial X} + (c_2t^3 + c_3Xt - B_2t)\frac{\partial}{\partial t},$$

and set $h_7 := \xi_2(-U_2 + s\rho)$. Then we see that $\xi_2(U + s\psi) - D_2'(F_s) = 0$ where $D_2'(F_s) = c_4X(U + s\psi) - D_2(G)$. Moreover we have $h_7 \in \mathcal{H}_4$. By comparing orders, we see also that $h_7(0,t,s_0)$ is of order 7.

Similarly choose $c_1, c_2, c_3, c_4 \in \mathbf{R}$ with $c_2 \neq 0$ such that

$$\theta_8 = c_1X(U+s\psi)\frac{\partial(U+s\psi)}{\partial X} + (c_2t^4 + c_3Xt^2)\frac{\partial(U+s\psi)}{\partial t} + c_4(U+s\psi)^2$$

belongs to $\mathfrak{m}^{(7)}_{X,t,s} \subset M^{(4)}$. Then we have, for some C_3, B_3, D_3,

$$\theta_8 = C_3(G)\frac{\partial(U+s\psi)}{\partial X} + B_3t\frac{\partial(U+s\psi)}{\partial t} + D_3(G)$$

with $C_3(0) = D_3(0) = 0$ and $B_3 \in \mathfrak{m}^{(4)}_{X,t,s}$. We set

$$\xi_3 = (c_1X(U+s\psi) - C_3(G))\frac{\partial}{\partial X} + (c_2t^4 + c_3Xt^2 - B_3t)\frac{\partial}{\partial t},$$

and set $h_8 := \xi_3(-U_2+s\rho)$. Then $\xi_3(U+s\psi)-D_3'(F_s)=0$ where $D_3'(F_s)=c_4(U+s\psi)^2-D_3(G)$. Moreover we have $h_8 \in \mathcal{H}_4$ and $h_8(0,t,s_0)$ is of order 8.

Lastly choose c_1, c_2, c_3, c_4, c_5 with $c_1 \neq 0$ such that

$$\theta_9 = (c_1t^5 + c_2Xt^3 + c_3X^2t)\frac{\partial(U+s\psi)}{\partial t} + c_4X(-U_2+s\rho) + c_5X^2(U+s\psi).$$

belongs to $\mathfrak{m}^{(8)}_{X,t,s} \subset M^{(5)}$. Then we can write as

$$\theta_9 = C_4(G)\frac{\partial(U+s\psi)}{\partial X} + B_4t\frac{\partial(U+s\psi)}{\partial t} + D_4(G),$$

for some C_4, D_4, B_4 with $C_4(0) = D_4(0) = 0$ and $B_4 \in \mathfrak{m}^{(5)}_{X,t,s}$. Then we set $\xi_4 = (c_1t^5 + c_2Xt^3 + c_3X^2t - B_4)t\frac{\partial}{\partial t}$ and $h_9 := \xi_4(-U_2+s\rho)$. Then we see that $\xi_4(U+s\psi) - D_4'(F_s) = 0$, where $D_4'(G) = c_4X(-U_2+s\rho) + c_5X^2(U+s\psi) - D_4(G)$. Moreover we have $h_9 \in \mathcal{H}_4$ and $h_9(0,t,s_0)$ is of order 9.

By Lemma 6.8, we see that $h_7(0,t,s_0), h_8(0,t,s_0), h_9(0,t,s_0), h_{10}(0,t,s_0)$ from a basis of $t^7\mathcal{E}_t/t^{11}\mathcal{E}_t$ and therefore $1, h_7, h_8, h_9, h_{10}$ generate $\mathcal{H}_4$ as $G^*\mathcal{E}_{\lambda,\mu,s}$-module. Hence we have

$$\begin{aligned}\rho - \xi(-U_2+s\rho) = {} & A_1(G) + (A_2(G)\xi_2 + A_3(G)\xi_3 + A_4(G)\xi_4)(-U_2+s\rho)\\ & + A_5(G)(-U_2+s\rho)^2,\end{aligned}$$

for some A_1, A_2, A_3, A_4, A_5. We set $\widetilde{\xi} = \xi + A_2(G)\xi_2 + A_3(G)\xi_3 + A_4(G)\xi_4$, then we have

$$\rho = \widetilde{\xi}(-U_2+s\rho) + A_1(G) + A_5(G)(-U_2+s\rho)^2,$$

while

$$\psi = \widetilde{\xi}(U+s\psi) - (D(F_s) + A_2(G)D_2'(F_s) + A_3(G)D_3'(F_s) + A_4(G)D_4'(F_s)).$$

Thus we have solved the infinitesimal equation as required. This complete the proof of Proposition 6.4. □

Proof of Theorem 6.1: By Proposition 6.4, F_s is trivialized under the diffeomorphism equivalence. Hence we have that $F = F_1$ is diffeomorphic to $F_0 = F_{a,b}$ namely to

$$\left(X,\quad a(-\tfrac{1}{6}t^3 + Xt) + b(-\tfrac{1}{8}t^4 + \tfrac{1}{2}Xt^2),\quad a(\tfrac{1}{20}t^5 - \tfrac{1}{6}Xt^3) + b(\tfrac{1}{24}t^6 - \tfrac{1}{8}Xt^4)\right).$$

Then we easily see that $F_{a,b}$ is diffeomorphic to $F_{1,1}$ if $b \neq 0$ and to $F_{1,0}$ if $b = 0$, by a linear change of coordinates.

Finally $F_{1,1}$ and $F_{1,0}$ are not diffeomorphic. In fact, for $F_{1,0}$, we see that the infinitesimal equation

$$\begin{pmatrix} 0 \\ -\frac{1}{8}t^4 + \frac{1}{2}Xt^2 \\ \frac{1}{24}t^6 - \frac{1}{8}Xt^4 \end{pmatrix} = \left(A\frac{\partial}{\partial X} + B\frac{\partial}{\partial t}\right)\begin{pmatrix} X \\ -\frac{1}{6}t^3 + Xt \\ \frac{1}{20}t^5 - \frac{1}{6}Xt^3 \end{pmatrix} - \begin{pmatrix} C(F) \\ D(F) \\ E(F) \end{pmatrix}$$

has no solution. This complete the proof of Proposition 6.1. □

Proof of Theorem 1.1: We combine Proposition 5.3 and the known results on singularities of tangent surfaces (tangent developables) ([10], [11], [12]). It was proved that the tangent surface to a curve of type $(1,2,3)$ (resp. $(1,3,4)$, $(2,3,4)$, $(1,3,5)$) is locally diffeomorphic to the *cuspidal edge* (resp. *Mond surface*, *swallowtail*, *Shcherbak surface*) respectively (Theorem 1 of [10]). Moreover it is known that the local differential types (resp. the local topological type) of the tangent surface to a curve of type $(2,3,5)$ are not unique (resp. is unique) ([10],[11]). Then by above Proposition 6.1 and Remark 6.2, generically the local differential type is unique and diffeomorphic to the *generic folded pleat*. Thus we complete the proof of Theorem 1.1. □

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MILNOR FIBRATIONS AND THE THOM PROPERTY FOR MAPS $f\bar{g}$

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Abstract. We prove that every map-germ $f\bar{g} : (\mathbb{C}^n, \underline{0}) \to (\mathbb{C}, 0)$ with an isolated critical value at 0 has the Thom $a_{f\bar{g}}$-property. This extends Hironaka's theorem for holomorphic mappings to the case of map-germs $f\bar{g}$ and it implies that every such map-germ has a Milnor-Lê fibration defined on a Milnor tube. One thus has a locally trivial fibration $\phi : \mathbb{S}_\varepsilon \setminus K \to \mathbb{S}^1$ for every sufficiently small sphere around $\underline{0}$, where K is the link of $f\bar{g}$ and in a neighbourhood of K the projection map ϕ is given by $f\bar{g}/|f\bar{g}|$.

Introduction

Soon after J. Milnor published his book [14], there were several interesting articles about Milnor fibrations for real singularities published by various people, as for instance by E. Looijenga, P. T. Church and K. Lamotke, N. A'Campo, B. Perron, L. Kauffman and W. Neumann, A. Jacquemard and others. More recently, there has been a new wave of interest in the topic and a number of articles have been published by various authors (see for instance [1, 2, 3, 5, 7, 13, 15, 17, 18, 19, 20, 22]).

Unlike the fibration theorem for complex singularities, which holds for every map-germ $(\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$, in the real case one needs to impose stringent conditions to get a fibration on a "Milnor tube", or a fibration on a sphere, as in the holomorphic case.

In [18] we observed that Lê's arguments in [10] for holomorphic mappings extend to every real analytic map germ $(\mathbb{R}^n, \underline{0}) \to (\mathbb{R}^p, 0)$, $n > p$, with an isolated critical value, provided it has the Thom a_f-property and $V := f^{-1}(0)$ has dimension more than 0. Hence one has in that setting a Milnor-Lê fibration:

$$f : N(\varepsilon, \delta) \to \mathbb{D}_\delta \setminus \{0\} \ .$$

Here $N(\varepsilon, \delta)$ denotes a "solid Milnor tube": it is the intersection $f^{-1}(\mathbb{D}_\delta \setminus \{0\}) \cap \mathbb{B}_\varepsilon$, where $\mathbb{B}_\varepsilon$ is a sufficiently small ball around $\underline{0} \in \mathbb{R}^n$ and $\mathbb{D}_\delta$ is a ball in $\mathbb{R}^p$ of radius small enough with respect to ε. This was later completed in [5] (see also [7]), giving necessary and sufficient conditions for one such map-germ to define a Milnor fibration on every small sphere around the origin, with projection map $f/|f|$.

Then, an interesting problem is finding families of map germs $(\mathbb{R}^n, \underline{0}) \to (\mathbb{R}^p, 0)$, $n > p$, having an isolated critical value and the Thom property. This is even better when the given families further have a rich geometry one can use in order to study the topology of the corresponding Milnor fibrations (cf. [3]).

In this article we prove:

Theorem. *Let f, g be holomorphic map germs $(\mathbb{C}^n, \underline{0}) \to (\mathbb{C}, 0)$ such that the map $f\bar{g}$ has an isolated critical value at $0 \in \mathbb{C}$. Then $f\bar{g}$ has the Thom $a_{f\bar{g}}$-property.*

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In fact our proof is of a local nature and therefore extends, with same proof, to the case of holomorphic map-germs defined on a complex analytic variety X with an isolated singularity. This result generalizes to higher dimensions the corresponding theorem in [18] for $n = 2$, and it has the following corollaries:

Corollary 1. *Let f, g be holomorphic map-germs defined on a complex analytic variety X with an isolated singularity at a point $\underline{0}$, such that the germ $f\bar{g}$ has an isolated critical value at 0. Then one has a locally trivial fibration*

$$N(\varepsilon,\delta) \xrightarrow{f} \mathbb{D}_\delta \setminus \{0\}\,, \quad \varepsilon \gg \delta > 0 \textit{ sufficiently small},$$

where $N(\varepsilon,\delta) := [(f\bar{g})^{-1}(\mathbb{D}_\delta \setminus \{0\}) \cap \mathbb{B}_\varepsilon]$ is a solid Milnor tube for $f\bar{g}$.

Corollary 2. *Let $\mathcal{L}_X := X \cap \mathbb{S}_\varepsilon$ be the link of X, $V := (f\bar{g})^{-1}(0)$ and $\mathcal{L}_V := \mathcal{L}_X \cap V$ be the link of V. Then one has a locally trivial fibration,*

$$\phi : \mathcal{L}_X \setminus \mathcal{L}_V \longrightarrow \mathbb{S}^1\,,$$

which restricted to $\mathcal{L}_X \cap N(\varepsilon,\delta)$ is the natural projection $\phi = \frac{f\bar{g}}{|f\bar{g}|}$.

In fact we know from [18] that for $n = 2$ the projection map ϕ in Corollary 2 can be taken to be $\frac{f\bar{g}}{|f\bar{g}|}$ everywhere on $\mathcal{L}_X \setminus \mathcal{L}_V$, not only near the link of V. It would be interesting to know whether or not this statement holds also in higher dimensions. By [5], this is equivalent to asking whether all germs $f\bar{g}$ are d-regular (we refer to [5] for the definition); this is so when $n = 2$, by [18] and [2].

We notice too that holomorphic map-germs actually have the stronger strict Thom w_f-property, by [16] and [4, Theorem 4.3.2], even for functions defined on spaces with non-isolated singularities. We do not know whether or not these statements extend to map-germs $f\bar{g}$ in general. Perhaps this can be proved using D. Massey's work [13] about real analytic Milnor fibrations and a Łojasiewicz inequality.

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1. The theorem

Let U be an open neighbourhood of the origin $\underline{0}$ in $\mathbb{R}^m$ and let $X \subset U$ be a real analytic variety of dimension $n > 0$ with an isolated singularity at 0. Let $\tilde{f} : (U, \underline{0}) \to (\mathbb{R}^k, 0)$ be a real analytic map-germ which is generically a submersion, *i.e.*, its Jacobian matrix $D\tilde{f}$ has rank k on a dense open subset of U. We denote by f the restriction of $\tilde{f}$ to X. As usual, we say that $x \in X \setminus \{0\}$ is *a regular point* of f if $Df_x : T_xX \to \mathbb{R}^k$ is surjective, otherwise x is *a critical point*. A point $y \in \mathbb{R}^k$ is *a regular value* of f if there is no critical point in $f^{-1}(y)$; otherwise y is *a critical value*. We say that f has *an isolated critical value* at $0 \in \mathbb{R}^k$ if there is a neighbourhood $\mathbb{D}_\delta$ of 0 in $\mathbb{R}^k$ so that all points $y \in \mathbb{D}_\delta \setminus \{0\}$ are regular values of f.

Now let U and X be as before, and let $\tilde{f} : (U, \underline{0}) \to (\mathbb{R}^k, 0)$ be a real analytic map-germ such that $f = \tilde{f}|_X$ has an isolated critical value at $0 \in \mathbb{R}^k$. We set $V = f^{-1}(0) = \tilde{f}^{-1}(0) \cap X$. According to [9, 11], there exist Whitney stratifications of U adapted to X and V. Let $(V_\alpha)_{\alpha\in A}$ be such an stratification.

Definition 1.1. The Whitney stratification $(V_\alpha)_{\alpha\in A}$ satisfies *the Thom a_f-condition* with respect to f if for every pair of strata S_α, S_β such that $S_\alpha \subset \overline{S}_\beta$ and $S_\alpha \subset V$, one has that for every sequence of points $\{x_k\} \in S_\beta$ converging to a point x such that the sequence of tangent spaces $T_{x_k}(f^{-1}(f(x_k)) \cap S_\beta)$ has a limit T, one has that T contains the tangent space of S_α at x. We say that f has *the Thom property* if such an stratification exists.

Notice that this condition is automatically satisfied for pairs of strata contained in V, since in that case this regularity condition simply becomes Whitney's (a)-regularity.

Thom's property for complex analytic maps was proved by Hironaka in [9, Section 5 Corollary 1] for all holomorphic maps into $\mathbb{C}$ defined on arbitrary complex analytic varieties. We remark that the critical values of holomorphic maps are automatically isolated, while for real analytic maps into $\mathbb{R}^2$ this is a hypothesis we need to impose. We refer to [18] for examples of maps $f\bar{g}$ with isolated critical values, and also for examples with non-isolated critical values. Hironaka's theorem was an essential ingredient for Lê's fibration theorem in [10]. The corresponding statement was shown by Lê Dũng Tráng to be false in general for complex analytic mappings into $\mathbb{C}^2$ (see Lê's example, for instance in [21, p. 290]). Similarly, there are real analytic map-germs into $\mathbb{R}^2$ which do not have the Thom Property. Here we prove:

Theorem 1.2. *Let $(X, \underline{0})$ be a germ of an n-dimensional complex analytic set with an isolated singularity and let $f, g : (X, \underline{0}) \to (\mathbb{C}, 0)$ be holomorphic map-germs such that $f\bar{g}$ has an isolated critical value at $0 \in \mathbb{C}$. Then the real analytic germ $f\bar{g}$ has the Thom $a_{f\bar{g}}$-property.*

Proof. The proof is inspired by the proof of Pham's theorem given in [8] (Theorem 1.2.1), which concerned holomorphic germs of functions defined on complex analytic subsets of $\mathbb{C}^m$.

We first prove the theorem in the case when the germ of X at $\underline{0}$ is smooth, *i.e.*, $X \cong \mathbb{C}^n$.

Let U be an open neighbourhood of the origin $\underline{0}$ in $\mathbb{C}^n$ so that $f, g : U \to \mathbb{C}$ represent the germs f and g. We identify $\mathbb{C}^{n+1} \cong \mathbb{C}^n \times \mathbb{C}$ and denote by $(z_1, \dots, z_{n+1})$ the coordinates in $\mathbb{C}^{n+1}$.

Let us denote by V the subset in $\mathbb{C}^n$ with equation $fg = 0$ and by $Sing(f\bar{g})$ the critical locus of $f\bar{g}$. Since $f\bar{g}$ has an isolated critical value at 0, $Sing(f\bar{g})$ is contained in V.

We need the following lemma:

Lemma 1.3. *For each integer $N \geq 1$, let $G = G(N)$ be the subset of $U \times \mathbb{C}$ defined by the equation*

$$F_N(z_1, \cdots z_{n+1} := (f\bar{g})(z_1, \dots, z_n) - z_{n+1}^N = 0\,.$$

Then the singular locus of G is contained in $Sing(f\bar{g}) \times \{0\}$.

Proof. This follows by a straightforward computation of the $2 \times 2(n+1)$ Jacobian matrix of $f\bar{g} - z_{n+1}^N$. □

Therefore, according to [24] (just as in [8, 1.2.4] for the real analytic case), there exists a Whitney stratification σ_N of G such that $G \cap (\mathbb{C}^n \times \{0\}) = V \times \{0\}$ is a union of strata and such that $G \setminus (V \times \{0\})$ is the union of the strata having dimension $2n$. We assume further that $\underline{0}$ is itself a stratum and that U is chosen small enough so that every other stratum contains $\underline{0}$ in its closure.

Let S_N be the stratification induced by σ_N on $V \times \{0\}$. Adapting the arguments of [8], we will prove that for N sufficiently large, S_N has the Thom condition with respect to $f\bar{g}$. For this we must show that given a sequence of points in $\mathbb{C}^n \setminus V$ which converges to a point x in a stratum in S_N, such that the corresponding sequence of spaces tangent to the fibers of $f\bar{g}$ has a limit T, then T contains the tangent space at x of the corresponding stratum.

For this we will prove that whenever we have a sequence of points $(x_k) = (z_1^{(k)}, \dots, z_{n+1}^{(k)})$ in $G \setminus (V \times \{0\})$ such that:

(1) $\lim_{k\to\infty} x_k = x \in V \times \{0\}$, and
(2) if we set $t_k = (f\bar{g})(z_1^{(k)}, \dots, z_n^{(k)})$, we have that the sequence of $(2n-2)$-planes $T_{x_k}\big((f\bar{g})^{-1}(t_k) \times \{z_{n+1}^{(k)}\}\big)$ converges to a limit T,

then, if N is sufficiently large, the space T must contain the tangent space $T_x V_\alpha$ to the strata V_α of S_N containing x. It is clear that this will imply Theorem 1.2 since $G \subset \mathbb{C}^{n+1}$ is a union of fibers of $f\bar{g}$.

We will prove this claim by contradiction. In other words, we assume that there is a sequence (x_k) as above, such that the limit T does not contain the tangent space $T_x V_\alpha$, then we will show that when N is large enough, we necessarily come to a contradiction.

Notice that we can assume that the sequence of $2n$-planes $T_{x_k} G$ converges to a limit τ since the Grassmanian of $2n$-planes in the Euclidian space is a compact manifold.

For each k one has

$$T_{x_k}\big((f\bar{g})^{-1}(t_k) \times \{z_{n+1}^{(k)}\}\big) \subset T_{x_k} G,$$

therefore $T \subset \tau$ and the intersection $\tau \cap (\mathbb{C}^n \times \{0\})$ has real dimension at least $2n-2$. Moreover, as $T_x V_\alpha \not\subset T$, one gets $T_x V_\alpha \neq \tau \cap \mathbb{C}^n \times \{0\}$.

But, since σ_N satisfies Whitney's condition (a), one has $T_x V_\alpha \subset \tau$. This implies that in fact the intersection $\tau \cap (\mathbb{C}^n \times \{0\})$ has real dimension at least $2n-1$. We will show that this is not possible if N is sufficiently large.

According to [12], there exists an open neighbourhood of $\underline{0}$ in $\mathbb{C}^n$ and a real number θ, $0 < \theta < 1$, such that for each $z = (z_1, \ldots, z_n) \in \Omega$ one has :

$$\|(\mathrm{grad} f)(z)\| \geq |f(z)|^\theta \quad \text{and} \quad \|(\mathrm{grad} g)(z)\| \geq |g(z)|^\theta$$

The Jacobian matrix of the map $f\bar{g} - z_{n+1}^N$ with respect to the coordinates $(z_1, \bar{z}_1, z_2, \bar{z}_2, \cdots, z_{n+1}, \overline{z_{n+1}})$ in $\mathbb{R}^{2(n+1)}$ is the $2 \times 2(n+1)$ matrix given in blocks by

$$D(f\bar{g})(z_1, \bar{z_1}, \ldots, z_{n+1}, \overline{z_{n+1}}) = \begin{pmatrix} M_1 & \ldots & M_i & \ldots & M_{n+1} \end{pmatrix},$$

where for each $i = 1, \ldots, n$ the block M_i is:

$$M_i = \begin{pmatrix} \frac{\partial(\Re(f\bar{g}))}{\partial z_i} & \frac{\partial(\Re(f\bar{g}))}{\partial \bar{z}_i} \\ \\ \frac{\partial(\Im(f\bar{g}))}{\partial z_i} & \frac{\partial(\Im(f\bar{g}))}{\partial \bar{z}_i} \end{pmatrix},$$

and

$$M_{n+1} = -\frac{N}{2} \begin{pmatrix} z_{n+1}^{N-1} & \overline{z_{n+1}}^{N-1} \\ \\ \frac{1}{i} z_{n+1}^{N-1} & -\frac{1}{i}\overline{z_{n+1}}^{N-1} \end{pmatrix}.$$

Then an easy computation leads to the following equation for the tangent space $T_{x_k} G$ at $x_k = (z, z_{n+1}) \in G$ (we omit the k in the coordinates in order to simplify the notations) :

$$\sum_{i=1}^{n} \left(\frac{\partial f}{\partial z_i}(z)\bar{g}(z) v_i + \overline{\frac{\partial g}{\partial z_i}}(z) \bar{f}(z) \overline{v_i} \right) - N z_{n+1}^{N-1} v_{n+1} = 0\,.$$

We consider the $2n$-vector appearing in the equation :

$$w_k(z) = \left(\frac{\partial f}{\partial z_1}(z)\bar{g}(z), \overline{\frac{\partial g}{\partial z_1}}(z), \ldots, \frac{\partial f}{\partial z_n}(z)\bar{g}(z), \overline{\frac{\partial g}{\partial z_n}}(z) \right).$$

For simplicity we omit to write that the functions below are evaluated at (z). We have:

$$\left(\frac{\|w_k\|}{N|z_{n+1}|^{N-1}} \right)^2 = \frac{|\bar{g}|^2 \sum_{i=1}^n \left|\frac{\partial f}{\partial z_i}\right|^2 + |\bar{f}|^2 \sum_{i=1}^n \left|\frac{\partial g}{\partial z_i}\right|^2}{N^2 |f\bar{g}|^{2\frac{N-1}{N}}} = \frac{|\bar{g}|^2 \|\mathrm{grad} f\|^2 + |\bar{f}|^2 \|\mathrm{grad} g\|^2}{N^2 |f\bar{g}|^{2\frac{N-1}{N}}}\,.$$

Thus,

$$\left(\frac{\|w_k\|}{N|z_{n+1}|^{N-1}} \right)^2 = \frac{(|\bar{g}||\bar{f}|^\theta)^2 + (|\bar{f}||\bar{g}|^\theta)^2}{N^2 |f\bar{g}|^{2\frac{N-1}{N}}} \geq \frac{2}{N^2} |f\bar{g}|^{\theta - \frac{N-1}{N}}\,.$$

When N is sufficiently large, *i.e.*, $\theta - \frac{N-1}{N} < 0$, one has :

$$\lim_{k\to\infty} \frac{\|w_k\|}{N|z_{n+1}|^{N-1}} = +\infty\,.$$

Therefore the normalized limit of the vector $(w_k, -N(z_{n+1}^{(k)})^{N-1})$ when $k \to \infty$, is a vector contained in $\mathbb{C}^n \times \{0\}$. Then the $2n$-plane τ contains the complex line $\{\underline{0}\} \times \mathbb{C} \subset \mathbb{C}^n \times \mathbb{C}$. This contradicts the fact that $\tau \cap \mathbb{C}^n \times \{0\}$ has dimension at least $2n-1$. Thus, if we set $t_k = (f\bar{g})(z_1^{(k)}, \ldots, z_n^{(k)})$, then every sequence of $(2n-2)$-planes $T_{x_k}\big((f\bar{g})^{-1}(t_k) \times \{z_{n+1}^{(k)}\}\big)$ that converges to a limit T contains the tangent space $T_x V_\alpha$ to the strata V_α of S_N containing x. This completes the proof of the theorem when X is smooth at $\underline{0}$.

When $X \hookrightarrow \mathbb{C}^m$ has an isolated singularity at the origin, we take a Whitney stratification of a neighbourhood U of X in $\mathbb{C}^m$ adapted to X and to $V := (f\bar{g})^{-1}(0)$, and such that $\underline{0}$ is a stratum. We choose U small enough such that any other stratum contains $\overline{0}$ in its closure. Now we consider a sequence of points (x_k) in $X \setminus V$ converging to a point $x \in V$ and such that there is a limit T of the corresponding sequence of spaces tangent to the fibers. If $x = \underline{0}$, then there is nothing to prove since T contains the space tangent to this 0-dimensional stratum. If $x \neq \underline{0}$, then we consider a coordinate chart U_1 for X around x and argue exactly as in the previous case, when X was assumed to be smooth. □

We now look at the corollaries. We know, by Bertini-Sard's theorem in [23], that there is $\varepsilon > 0$ such that all spheres in $\mathbb{R}^m$ centered at $\underline{0}$ with radius $\leq \varepsilon$ meet transversally each stratum in $\{f\bar{g} = 0\}$. Since $f\bar{g}$ has Thom's $a_{f\bar{g}}$-property, by Theorem 1.2, we get that given $\varepsilon > 0$ as above, there exists $\delta > 0$ sufficiently small with respect to ε such that all fibers $(f\bar{g})^{-1}(t)$ with $|t| \leq \delta$ are transversal to the link $\mathcal{L}_X$. As usual, following the proof of Ehresmann's fibration theorem (see for instance [14, 10, 18]), this implies that one has a locally trivial fibration $N(\varepsilon, \delta) \xrightarrow{f} \text{Image}(f\bar{g}) \subset \mathbb{D}_\delta \setminus \{0\}$, where $N(\varepsilon, \delta) := [(f\bar{g})^{-1}(\mathbb{D}_\delta \setminus \{0\})] \cap \mathbb{B}_\varepsilon$ is a solid Milnor tube for $f\bar{g}$. Thus to complete the proof of Corollary 1 we must show that the image of $f\bar{g}$ covers all of $\mathbb{D}_\delta \setminus \{0\}$. This follows from the lemma below.

Lemma 1.4. *Let X, f and g be as above, so that $f\bar{g}$ is not constant and it has an isolated critical value at $0 = f\bar{g}(\underline{0})$. Then the germ $f\bar{g}$ is locally surjective at $\underline{0}$.*

Proof. If either f or g is constant, the statement in this lemma is a well-known property of holomorphic mappings. So we assume none of these maps is constant, neither is constant the map $f\bar{g}$. We may further assume that f, g have no common factor, for otherwise we may divide both map-germs by that common factor and this will not change the image of the map $f\bar{g}$. We claim that since f and g are both holomorphic, we have that the map-germ

$$(f,g) : \mathbb{C}^n \to \mathbb{C} \times \mathbb{C}$$

is locally surjective for all $n \geq 2$. That is, the image of every neighbourhood of $\underline{0} \in \mathbb{C}^n$ covers a neighbourhood of $(0,0) \in \mathbb{C} \times \mathbb{C}$. In fact, for $n = 2$ the map germ (f,g) is a finite morphism, which is a finite covering map with ramification locus the discriminant curve; so it is locally surjective. When $n \geq 3$ we may consider a generic complex 2-plane $\mathcal{P}$ in $\mathbb{C}^n$ which is transversal to the fibers of (f,g) and apply the above arguments. Hence $(f,\bar{g})$ is locally surjective, and so is $f\bar{g}$. □

There are in [6] examples of analytic map-germs $(\mathbb{R}^n, \underline{0}) \xrightarrow{h} (\mathbb{R}^2, 0)$ with an isolated critical value at 0 which are not surjective. The image of h misses a neighbourhood of an arc converging to 0.

The proof of Corollary 2 is just as that of Theorem 1.3 in [18], replacing the Milnor tube $[(f\bar{g})^{-1}(\partial\mathbb{D}_\delta)] \cap \mathbb{B}_\varepsilon$ by the solid Milnor tube $[(f\bar{g})^{-1}(\mathbb{D}_\delta \setminus \{0\})] \cap \mathbb{B}_\varepsilon$, so we leave the details to the reader. (Compare with the first part of the proof of Theorem 1 in [5]).

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