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QUASI-PERIODIC MOTIONS ON SYMPLECTIC TORI

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Abstract. The results of Kolmogorov, Arnold, and Moser on the stability of quasi-periodic motions spanning lagrangian tori in Hamiltonian systems are of fundamental importance and led to the development of KAM theory. Over the years, many variations of these results on quasi-periodic motions have been considered. In this paper, we present a more conceptual way of attacking such problems by considering the particular case of quasi-periodic motions on symplectic tori.

Introduction

The theorems of Kolmogorov, Arnold, and Moser guarantee, under appropriate conditions, the stability of quasi-periodic motions on Lagrangian invariant tori [1, 6, 17, 19, 20, 23, 24]. The phase space is then a symplectic manifold of dimension $2d$ and these KAM-tori have dimension d. One may ask about the stability of other types of quasi-periodic motions that fill out tori of dimension $k \neq d$. The case of such movements on a lower dimensional torus ($k < d$) was already considered by Moser in 1967 [18]. But it was only in 1991 that Herman exhibited examples of Hamiltonian systems with stable quasi-periodic motion on a $k = 2d - 2$ dimensional co-isotropic torus [14, 28]. More recently, quasi-periodic motions on tori that are neither isotropic nor co-isotropic were discovered (see [25] and references therein).

Quasi-periodic motions are also observed for more general families of vector fields. For the isochore case and in the reversible context we refer to [2, 3, 4]. It appears that there are many theorems one can derive along the lines of proof indicated by the founders of KAM theory, although sometimes new unexpected difficulties appear. The first author formulated in [7] a result that could be seen as a first step to gathering all these results in a single theorem. This led us to develop a general framework, based on certain systems of Banach spaces [9]. We present here an application of that theory in one particular situation: motions on *symplectic tori*. These correspond to the case $k = 2d$, which were not considered in the works quoted above. As will be made clear in this paper, our proof is based on general arguments, which may be used in many other situations involving quasi-periodic motions, and beyond. We will try nevertheless to remain as explicit and self-contained as possible; a presentation of the development of the abstract theory will appear elsewhere.

The structure of the paper is as follows.
In the first section, we formulate our main theorem on the stability of real analytic symplectic vector fields on a symplectic torus under an arithmetic condition on the frequencies, which is much weaker than the usual Diophantine condition.

As usual in a real analytic context, we will use holomorphic tools on a complex analytic neighborhood of the torus. This is explained in the second section and serves to fix some of the notations we use. We introduce parameters that describe the *detuning* of the frequencies and *perturbation* of the symplectic form.

In the third section, we give an algebraic description of an almost quadratic iteration a certain ring R, that brings our vector field formally into normal form. In section six, this iteration scheme is lifted to an iteration scheme in a functional analytic context, so that the issue of convergence can be addressed. In order to do so, we have to make some careful preparations.

In section four we sketch out an abstract framework for handling families of Banach spaces parametrized by partially ordered sets (i.e. Banach scales) that we call *Kolmogorov spaces*. These naturally arise in situations where one has to deal with functions and vector fields defined over domains that shrink during an iteration process, for example to avoid to the appearance of small denominators. Most relevant is the general theorem formulated in 4.5.2, which is crucial in the later part of the paper to control the norm estimates, in particular for the exponentials of vector fields that we use.

In section five, we describe the neighborhoods of the resonance hyperplanes that need to be removed in order to control the small denominators and we introduce the precise Kolmogorov spaces of holomorphic functions that we will use in our proof.

In section six, we lift the iteration scheme on the level of Kolmogorov spaces. The required estimates are all automatic now and the proof is completed with relative ease, by comparison with a simple one dimensional model iteration.

1. Quasi-periodic movement on a symplectic torus

1.1. We will describe our basic setup and formulate the main result of this paper. By *quasi-periodic motion* on a torus

$$\mathbb{T}^n := (\mathbb{R}/\mathbb{Z})^n$$

with coordinates $x_1, \dots, x_n$ we mean the flow of a constant vector field

$$X := \sum_{i=1}^{n} \nu_i \frac{\partial}{\partial x_i}.$$

If the components of the *frequency vector*

$$\nu := (\nu_1, \nu_2, \dots, \nu_n) \in \mathbb{R}^n$$

are independent over $\mathbb{Q}$, the orbits of X are dense in $\mathbb{T}^n$ and we say ν is *non-resonant*. We will refer to these constant vector fields as *quasi-periodic vector fields*.

It is clear that with an arbitrarily small perturbation S of the vector field X, we can create a vector field $X + S$ that is not conjugate to X, i.e. can not be transformed back to X by an appropriate change of coordinates. So in this sense, the dynamics generated by the vector field X is not stable. KAM type theorems state that in a family of quasi-periodic motions, a large part of them are preserved under perturbation.

1.2. Consider the case where $n = 2d$ and equip the torus $\mathbb{T} := \mathbb{T}^{2d}$ with a constant symplectic form

$$\omega = \sum_{1 \le i < j \le 2d} \omega_{ij} dx_i \wedge dx_j, \ \omega_{ij} \in \mathbb{R}.$$

A C^∞ vector field X on $\mathbb{T}$ is called a *symplectic vector field* if the time t flow Φ_t of X preserves the symplectic form:

$$\Phi_t^*(\omega) = \omega.$$

This is equivalent to the infinitesimal condition

$$L_X(\omega) = 0,$$

where

$$L_X = d\iota_X + \iota_X d$$

denotes the *Lie-derivative*. As $d\omega = 0$, this is equivalent to the statement that the one-form $\iota_X \omega$, symplectically dual to X, is *closed*. In the particular case where the form is *exact*, the corresponding vector field X satisfies

$$\iota_X \omega = dh$$

for some function h on $\mathbb{T}$ and we say that the field is *Hamiltonian*, or more precisely that the vector field is *associated to the Hamiltonian function h.* Obviously, constant vector fields are symplectic, but not Hamiltonian.

Now we will seek a statement expressing the *stability of quasi-periodic motions under perturbation with symplectic vector fields.* For this to work properly, we will also do allow for perturbations of the chosen symplectic form.

1.3. The space $\mathbf{S}$ of constant symplectic forms on the torus $\mathbb{T}$ can be identified with those on $\mathbb{R}^{2d}$ and thus forms an open subset of non-degenerate skew-symmetric 2-forms

$$\mathbf{S} \subset \mathbf{\Lambda^2}(\mathbb{R}^{\mathbf{2d}})^* \simeq \mathbb{R}^{\mathbf{d(2d-1)}}.$$

The fibre of the fibration

$$\pi : \mathbb{T} \times \mathbf{S} \longrightarrow \mathbf{S}$$

over the point

$$\omega = \sum_{i<j} \omega_{ij} dx_i \wedge dx_j \in \mathbf{S}$$

describes the torus $\mathbb{T}$ with symplectic form ω. Such forms can be seen as relative differential two forms, i.e., elements of $\Omega^2_\pi := \Omega^2_{\mathbb{T}\times\mathbf{S}}/\pi^*\Omega^1_S \wedge \Omega^1_{\mathbb{T}\times\mathbf{S}}$.

In this situation, the module of relative 1-forms Ω^1_π defined as $\Omega^1_{\mathbb{T}\times\mathbf{S}}/\pi^*\Omega^1_S$ can be thought as the module of 1-forms which can be written as

$$\sum_{i=1}^{2d} a_i(x,\omega) dx_i \in \Omega^1_\pi,$$

Dually, elements of the module Θ_π of relative vector fields are of the form

$$\sum_{i=1}^{2d} b_i(x,\omega) \partial_{x_i} \in \Theta_\pi.$$

The interior product with the symplectic form ω induces an isomorphism

$$\Theta_\pi \longrightarrow \Omega^1_\pi, \quad X \mapsto \iota_X \omega := \omega(X, -).$$

1.4. Now let $V \subset \mathbf{S}$ be a smooth real analytic submanifold and consider a map

$$\nu := (\nu_1, \dots, \nu_{2d}) : V \longrightarrow \mathbb{R}^{2d}$$

Definition. *The vector field defined on $\mathbb{T} \times V$ by*

$$X_\nu = \sum_{i=1}^{2d} \nu_i(\omega) \partial_{x_i}$$

is called the quasi-periodic vector field *with frequency map ν.*

Such a vector field generates indeed quasi-periodic movements on the fibres $\mathbb{T} \times \{\omega\}$, for all $\omega \in V$, with frequencies that may depend on ω.

1.5. So in our context, the space of symplectic forms plays the role of the parameter space. A typical example which is close to Herman's original construction would be the following. Assume $d = 2$ and consider the symplectic form associated to the bivector

$$\alpha\partial_{x_1} \wedge \partial_{x_2} + \beta\partial_{x_1} \wedge \partial_{x_3} + \gamma\partial_{x_1} \wedge \partial_{x_4} + \partial_{x_3} \wedge \partial_{x_4}, \quad (\alpha, \beta, \gamma \neq 0)$$

The symplectic vector field dual to the relative one-form $dx_1 + \delta dx_2$ ($\delta \neq 0$)) is

$$X = -\alpha\delta\partial_{x_1} + \alpha\partial_{x_2} + \beta\partial_{x_3} + \gamma\partial_{x_4}$$

In particular, any four dimensional quasi-periodic motion can be obtained in this way.

1.6. Consider now a symplectic form $\omega^0 \in V$. Our aim is to prove the following *Stability Theorem for symplectic quasi-periodic motions:*

Theorem. *Consider a quasi-periodic motion defined by a real analytic vector field*

$$X_\nu = \sum_{i=1}^{2d} \nu_i(\omega)\partial_{x_i}$$

defined on a neighbourhood $\mathbb{T} \times V$ of a symplectic torus $\mathbb{T} \times \{\omega^0\}$. Assume that

(A) the vector $\nu(\omega^0)$ satisfies a subquadratic arithmetic condition.
(B) the map

$$\nu : V \longrightarrow \mathbb{R}^{2d},\ \omega \mapsto (\nu_1(\omega), \dots, \nu_{2d}(\omega))$$

is a submersion.

Then: For any real analytic symplectic vector field X sufficiently close to X_ν, there exists a set $M \subset V$ of positive measure parametrising tori that carry a motion conjugate by a symplectomorphism to a quasi-periodic one.

The topology on the space of analytic vector fields will be reviewed in later sections of the paper.

Remark 1. Condition (B) can be weakened and replaced by the Kleinbock-Margulis condition [16, 22]:

(B') the partial derivatives of the map

$$\nu : V \longrightarrow \mathbb{R}^{2d},\ \omega \mapsto (\nu_1(\omega), \dots, \nu_{2d}(\omega))$$

generate $\mathbb{R}^{2d}$.

An even weaker condition can be formulated: we will construct a formal normal form and a formal frequency map $\hat{\nu}$ analogous to the Birkhoff normalform. We denote by E_V the minimal vector space for which $\omega^0 + E_V$ contains the image of V under this formal map . Then Condition (B') can be weakened and replaced by the condition:

(B'') the partial derivatives of the formal frequency map $\hat{\nu}$ evaluated at the origin generate E_V.

For instance if $V = \{0\}$, the vector field turns out to be integrable and we get a variant of a classical result due to Rüssmann [21]. For details we refer to [8, 12, 11]. For details we refer to [8, 12].

Remark 2. A direct generalisation of the theorem obtained by omitting condition (B) or (B') would fail.

To see this, consider for instance the case of ordinary vector fields (no symplectic structure) vector field

$$X = \partial_x + \sqrt{2}\partial_y,$$

defining a quasi-periodic motion with constant frequency. Now, let (p_n/q_n) be a sequence of rational numbers converging to $\sqrt{2}$. The vector fields

$$X_n = \partial_x + \frac{p_n}{q_n}\partial_y + \frac{1}{n}\sin(p_n x - q_n y)\partial_x$$

approach X, as n goes to infinity. The curve $p_n x - q_n y = 0$ is a periodic orbit of X_n, indeed:

$$X_n(p_n x - q_n y) = \frac{p_n}{n}\sin(p_n x - q_n y).$$

The equations of motions are:

$$\begin{cases} \dot{x} &= 1 + \frac{1}{n}\sin(p_n x - q_n y), \\ \dot{y} &= \frac{p_n}{q_n}. \end{cases}$$

Linearisation of the equations $(x \mapsto x + \xi, y \mapsto y + \eta)$ along this periodic orbit shows that it is isolated:

$$\begin{cases} \dot{\xi} = \frac{1}{n}(p_n \xi - q_n \eta) \\ \dot{\eta} = 0 \end{cases}$$

Therefore the motion defined by X_n is not quasi-periodic, although the sequence X_n approaches X.

We may adapt this example in the symplectic situation. Let us consider the 4-torus with coordinates x_1, x_2, y_1, y_2 and symplectic form $dx_1 \wedge dy_1 + dx_2 \wedge dy_2$. We consider the vector field

$$X = \partial_{x_1} + \sqrt{2}\partial_{x_2} + \sqrt{3}\partial_{y_1} + \sqrt{5}\partial_{y_2}.$$

As before, let (p_n/q_n) be a sequence of rational numbers converging to $\sqrt{2}$. The symplectic vector fields

$$X_n = \partial_{x_1} + \frac{p_n}{q_n}\partial_{x_2} + \frac{1}{n}\sin(p_n x_1 - q_n x_2)\partial_{y_1} - \frac{1}{n}\sin(p_n x_1 - q_n x_2)\partial_{y_2} + \sqrt{3}\partial_{y_3} + \sqrt{5}\partial_{y_4}$$

approach X as $n \longrightarrow +\infty$. Moreover

$$X_n(p_n x_1 - q_n x_2) = 0.$$

In particular the 3-torus $p_n x_1 = q_n x_2$ is invariant under the flow of X_n. The equations of motions are:

$$\begin{cases} \dot{x}_1 &= 1 \\ \dot{x}_2 &= \frac{p_n}{q_n} \\ \dot{y}_1 &= \frac{1}{n}\sin(p_n x_1 - q_n x_2), \\ \dot{y}_2 &= -\frac{1}{n}\sin(p_n x_1 - q_n x_2), \end{cases}$$

Linearisation of the Hamilton equations $(x_i \mapsto x_i + \xi_i, y \mapsto y_i + \eta_i)$ along this 3-torus shows that it is isolated:

$$\begin{cases} \dot{\xi}_i &= 0 \\ \dot{\eta}_i &= \pm\frac{1}{n}(p_n \xi_1 - q_n \xi_2) \quad = \text{constant} \end{cases}$$

Remark 3. There are nevertheless ways to formulate a statement without condition (B) in the spirit of the Herman invariant tori conjecture [12]. For instance, if instead of considering a neighbourhood of X we consider a deformation X_t, then one can suppress assumption (B) and the theorem remains valid as, in this case, the type of normal form we will consider automatically satisfies such a condition.

1.7. The structure of the proof runs along the following lines:

1) We consider a *versal deformation* of the quasi-periodic vector field, depending on additional parameters ϕ_i:

$$\mathcal{V} := X_\nu + \sum_i \phi_i \theta_i.$$

Similarly, to a perturbation $X_\nu + S$, we associate the perturbation $\mathcal{V} + S$ of the versal family $\mathcal{V}$.

2) We prove theorem of *stability of the versal deformation* $\mathcal{V}$. That is, we show that for *good frequencies* there exists a Poisson morphism ψ, which maps the vector fields $\mathcal{V} + S$ to $\mathcal{V}$. Here *good* means that the frequencies satisfy a certain arithmetic condition.

3) As X_ν is the restriction of $\mathcal{V} + S$ to $\phi_1 = \cdots = \phi_{2d} = 0$, the Poisson morphism ψ maps $X_\nu + S$ to the restriction of $\mathcal{V}$ to $\psi(\phi_1) = \cdots = \psi(\phi_{2d}) = 0$.

4) By the Ehresmann lemma, assumption (B) shows that that the map which sends symplectic forms to the corresponding frequencies of motion is a local fibration. It is then easy to show that the set of "good frequencies" form a positive measure set and therefore so does the set of corresponding one-forms. In case we consider assumptions (B') or (B'') instead of the Ehresmann lemma, we would have to invoke the arithmetic density theorem of [8].

1.8. Now we spell out the arithmetic condition of the theorem. By definition, a frequency vector $\nu = (\nu_1, \nu_2, \ldots, \nu_{2d})$ is called *non-resonant* if the scalar product

$$(\nu, J) := \sum_{k=1}^{2d} \nu_k J_k, \quad J = (J_1, J_2, \ldots, J_{2d}),$$

is non-zero for all $J \in \mathbb{Z}^{2d} \setminus \{0\}$. But although non-zero, this quantity can become arbitrarily small, if we allow $|J|$ to become large:

$$\inf_{J \neq 0} |(\nu, J)| = 0.$$

As during the iteration one has to divide by such quantities, these *small denominators* have a dangerous effect on the convergence and must be controlled.
A convenient way to quantify such small denominators is by the so-called *arithmetic sequence*

$$\sigma(\nu) = (\sigma(\nu)_k)$$

attached to a vector $\nu \in \mathbb{C}^{2d}$. This falling sequence is defined by setting

$$\sigma(\nu)_k := \min\{|(\nu, J)| : J \in \mathbb{Z}^{2d} \setminus \{0\}, \|J\| \leq 2^k\}.$$

If we collect all frequency vectors ν for which this arithmetic sequence is bounded from below by a given falling sequence $a = (a_k)$, we obtain what we call the *arithmetic class* of a, defined as

$$\mathcal{C}(a) := \cap_{m=0}^{\infty} \mathcal{C}_m(a), \quad \mathcal{C}_m(a) := \{\nu \in \mathbb{C}^d \;\mid\; \sigma(\nu)_k \geq a_k, k = 1, 2, \ldots, m\}.$$

The Cantor-like set $\mathcal{C}(a)$ could be called a *Swiss cheese set*, as it is obtained by removing smaller and smaller neighbourhoods around the dense collection of hyperplanes $(\omega, J) = 0$, $0 \neq J \in \mathbb{Z}^{2d}$. Obviously, one has

$$a' \leq a, \quad \Longrightarrow \mathcal{C}(a) \subset \mathcal{C}(a').$$

It is an elementary fact that, for any $\nu \in \mathcal{C}(a)$, we can find $a' \leq a$ such that $\mathcal{C}(a')$ has density $= 1$ at the point ν, see e.g. [8].

One says that ν is *Diophantine* if it satisfies a *Diophantine condition*, meaning that there exist constants C and N such that

$$|(\nu, J)| \geq \frac{C}{|J|^N}.$$

This means that $\nu \in \mathcal{C}(a)$, where a is a falling geometrical sequence. Diophantine conditions appear often in dynamical systems, but after the work of Bruno, it became apparent that in many cases this condition can be relaxed [5]. In our theorem, we will need a much weaker condition than the Diophantine one: the *subquadratic arithmetic condition* that we explain now. The sequence $a_n = q^{\alpha^n}$ has the property that it solves the iteration

$$a_{n+1} = (a_n)^\alpha$$

and this motives that the following definition.

Definition. *A strictly increasing sequence of positive numbers $a = (a_n)$ is called* positively subquadratic with exponent α, *if $\alpha \in]1, 2[$ and there exist $A, B \in \mathbb{R}_{>0}$ such that for all $n \in \mathbb{N}$ one has*

$$a_n \leq A e^{B\alpha^n}.$$

We denote the set of such sequences by $\mathbb{S}^+(\alpha)$. Similarly, a strictly decreasing sequence of positive numbers $a = (a_n)$ is called negatively subquadratic with exponent α *if $\alpha \in]1, 2[$ and there exist $A, B \in \mathbb{R}_{>0}$, such that for all $n \in \mathbb{N}$ one has*

$$a_n \geq A e^{-B\alpha^n}.$$

We denote the set of such sequences by $\mathbb{S}^-(\alpha)$.

Clearly, one has $\mathbb{S}^+(\alpha) \subset \mathbb{S}^+(\beta)$ if $\alpha \leq \beta$, so that the set

$$\mathbb{S}^+ := \bigcup_{\alpha \in]1,2[} \mathbb{S}^+(\alpha)$$

is filtered by the sets $\mathbb{S}^+(\alpha)$. Note also that the product of such subquadratic sequences is again subquadratic and one has

$$\mathbb{S}^+(\alpha) \cdot \mathbb{S}^+(\beta) \subset \mathbb{S}^+(\max(\alpha, \beta))$$

Given $a \in \mathbb{S}^+$, we call the *order of a*, denoted $\operatorname{ord}(a)$, the infimum of the exponents α appearing in the definition. Taking the multiplicative inverse $(a_n) \mapsto (1/a_n)$ interchanges $\mathbb{S}^+(\alpha)$ and $\mathbb{S}^-(\alpha)$, so that one has corresponding properties.

$$\mathbb{S}^-(\alpha) \cdot \mathbb{S}^-(\beta) \subset \mathbb{S}^-(\max(\alpha, \beta))$$

Definition. *A frequency vector $\nu \in \mathbb{C}^n$ is said satisfy a* subquadratic condition, *or* is *subquadratic, if $\sigma(\nu) \in \mathbb{S}^-$.*

For a subquadratic sequence the infinite product

$$a_\Pi := a_0 a_1^{1/2} a_2^{1/2^2} \ldots = \prod_{k=0}^{\infty} a_k^{1/2^k}$$

converges to a strictly positive number, or equivalently, one has:

$$\sum_{k \geq 0} \left| \frac{\log a_k}{2^k} \right| < +\infty.$$

The sequences satisfying this last condition are called *Bruno sequences*. So subquadratic sequences form a subset of Bruno sequences. Clearly, if ν is Diophantine, then $\sigma(\nu)$ is bounded by a geometric sequence, so these are in particular subquadratic sequences with order equal to 1.

2. The analytic torus

As usual in KAM theory, our theorem is proved using *normal form* techniques and more precisely we will use ideas from *parametrised KAM theory* [3, 4, 2] (see also [13]). Our normal form is achieved using a specific iteration scheme. As usual in a real analytic context, we use complexification and construct the iteration in an appropriate open subset in the complex domain. As all constructions can be done compatibly with the underlying real structure, nothing is lost and much is gained by doing so. Before we discuss the iteration itself, we set up the basic analytic notions on the torus.

2.1. **Analytic functions on the torus.**

2.1.1. The exponential map

$$\mathbb{R}^n \longrightarrow (\mathbb{C}^*)^n, \quad (x_1, x_2, \dots, x_n) \mapsto (z_1, z_2, \dots, z_n)$$

with

$$z_j = e^{2\pi i x_j}, \quad j = 1, 2, \dots, n,$$

defines an embedding of the real torus $\mathbb{T}^n = (\mathbb{R}/\mathbb{Z})^n$ in the algebraic torus $(\mathbb{C}^*)^n$ as the product of unit circles $|z_i| = 1$. We will identify $\mathbb{T}^n$ with this subset of $(\mathbb{C}^*)^n$ and describe functions, vector fields, differential forms on the torus using the complex coordinates z_j.

2.1.2. The simplest functions on the torus $\mathbb{T}^n$ are the *Laurent-polynomials*, which are of the form

$$f = \sum_{I \in \mathbb{Z}^n} a_I z^I,$$

where only finitely many $a_I \neq 0$. Here and in the sequel, we use the *multi-index notation* and write

$$z^I = z_1^{i_1} z_2^{i_2} \dots z_n^{i_n}, \;\; I = (i_1, i_2, \dots, i_n), \quad |I| = |i_1| + |i_2| + \dots + |i_n|, \;\; \text{etc.}$$

The set of all Laurent-polynomials forms a ring denoted by $\mathbb{C}[z, z^{-1}]$ and correspond precisely to trigonometric polynomials when written in the variable x.

2.1.3. The analytic functions on $\mathbb{T}^n$ are identified with the algebra

$$A := \mathbb{C}\{z, z^{-1}\}$$

of *analytic Fourier series* and consist of functions which are holomorphic on an open neighborhood of the real torus $\mathbb{T}^n$. A fundamental system of such neighborhoods is provided by the sets

$$T_r := \{z \in \mathbb{C}^n \mid e^{-r} < |z_i| < e^r\}, \quad r \in]0, \infty].$$

So $\mathbb{C}\{z, z^{-1}\}$ is the union of Banach spaces $\mathcal{O}^b(T_r)$, the space of *bounded holomorphic functions* on T_r. As a result, the algebra A carries a natural LB-space structure (direct limit of Banach spaces).

Elements of $\mathbb{C}\{z, z^{-1}\}$ are represented by power series of the form

$$\sum_{I \in \mathbb{Z}^n} a_I z^I \text{ such that } \sum_{I \in \mathbb{Z}^n} |a_I| e^{r|I|} < +\infty$$

for some $r > 0$. Alternatively, $\mathbb{C}\{z, z^{-1}\}$ can be seen as the union of Banach spaces $\mathcal{O}^k(T_r)$ of holomorphic functions on T_r with C^k- extension to $\overline{T_r}$, or of the Hilbert spaces $\mathcal{O}^h(T_r)$ of square integrable holomorphic functions on T_r.

2.1.4. One has inclusions

$$\mathbb{C}[z, z^{-1}] \subset \mathbb{C}\{z, z^{-1}\} \subset \mathbb{C}[[z, z^{-1}]],$$

where on the right-hand side we have the vector space of *formal Laurent series*

$$F := \sum_{I \in \mathbb{Z}^n} a_I z^I,$$

where the coefficients a_I are completely arbitrary. The Cauchy product of two such formal Laurent series is usually not defined, so the ring structure of $\mathbb{C}\{z, z^{-1}\}$ does not extend to $\mathbb{C}[[z, z^{-1}]]$. However, the *Hadamard-product* $\star$ obtained by coefficient-wise multiplication

$$\sum_{I \in \mathbb{Z}^n} a_I z^I \star \sum_{I \in \mathbb{Z}^n} b_I z^I = \sum_{I \in \mathbb{Z}^n} a_I b_I z^I$$

defines an operation on formal Laurent series:

$$\star : \mathbb{C}[[z, z^{-1}]] \times \mathbb{C}[[z, z^{-1}]] \longrightarrow \mathbb{C}[[z, z^{-1}]].$$

For a formal Laurent series

$$h = \sum_{I \in \mathbb{Z}^n} h_I z^I \in \mathbb{C}[[z, z^{-1}]],$$

the operation $h\star : \mathbb{C}[[z, z^{-1}]] \longrightarrow \mathbb{C}[[z, z^{-1}]]$ maps $\mathbb{C}[z, z^{-1}]$ to itself, but in general does not preserve $\mathbb{C}\{z, z^{-1}\}$ if the coefficient h_I grows too fast for $|I| \longrightarrow \infty$. However, if the coefficients satisfy an estimate of the form

$$|h_I| \leq C|I|^N,$$

for some C, N, then $h\star$ maps $\mathbb{C}\{z, z^{-1}\}$ to itself.

2.2. **Analytic vector fields on the torus.**

2.2.1. By an analytic vector field on $\mathbb{T}^n$ we mean a vector field

$$X = \sum_{j=1}^{n} a_j(x)\partial_{x_j},$$

where the coefficients $a_j(x) \in A$ are analytic functions on the torus. Expanding these coefficients in Fourier series and using

$$\frac{1}{2\pi i}\partial_{x_j} = z_j\partial_{z_j} =: \theta_j,$$

we can write the vector field in the form

$$X = \sum_{j=1}^{n} b_j(z)\theta_j, \quad b_j(z) \in A = \mathbb{C}\{z, z^{-1}\}.$$

In particular, the constant vector field

$$\sum_{j=1}^{n} \nu_j \partial_{x_j}$$

becomes, up to a factor $2\pi i$, the linear vector field

$$\sum_{j=1}^{n} \nu_j z_j \partial_{z_j} = \sum_{j=1}^{n} \nu_j \theta_j$$

in the z-variables.

2.2.2. We denote the set of all analytic vector fields by $\Theta(A)$ and have an isomorphism of A-modules

$$A^n \xrightarrow{\simeq} \Theta(A), \quad (a_1(z), \dots, a_n(z)) \mapsto \sum_{j=1}^{n} a_j(z)\theta_j$$

and algebraically one may identify $\Theta(A)$ with the module of *derivations* of the ring A:

$$\Theta(A) \xrightarrow{\approx} \mathrm{Der}_{\mathbb{C}}(A), \quad X \mapsto (f \mapsto X(f)).$$

The commutation of derivations gives $\Theta(A)$ a natural structure of a Lie-algebra.

2.2.3. As the torus $\mathbb{T}^n$ is compact, a real analytic vector field X has a globally defined flow

$$\Phi_t : \mathbb{T}^n \longrightarrow \mathbb{T}^n, \quad t \in \mathbb{R},$$

consisting of real analytic automorphisms of the torus. These automorphisms act on the ring $A = \mathbb{C}\{z, z^{-1}\}$ by composition. This action can be described formally as the exponentiation of the vector field:

$$f \mapsto f \circ \Phi_t = e^{tX}(f) = f + tX(f) + \frac{t^2}{2!}X(X(f)) + \dots$$

In section 4 we give a functional analytic treatment of the exponential of a vector field with explicit norm-estimates. This leads to direct proof of the convergence of the series and the existence of the flow.

2.2.4. The above exponential series e^{tX} defines an automorphism of $A = \mathbb{C}\{z, z^{-1}\}$. There is a corresponding induced adjoint action on the module $\Theta(A)$ of derivations

$$Y \mapsto \Phi_t \circ Y \circ \Phi_t^{-1},$$

that we denote by

$$(\Phi_t)_* = e^{[tX,-]} = 1 + t[X,-] + \frac{t^2}{2!}[X,[X,-]] + \dots \in Aut(\Theta(A)),$$

so that

$$(\Phi_t)_*(Y) = e^{[tX,-]}(Y) = Y + t[X,Y] + \frac{t^2}{2!}[X,[X,Y]] + \dots$$

2.3. **Symplectic vector fields on the torus.**

2.3.1. Via the embedding $\mathbb{T} := \mathbb{T}^{2d} \subset (\mathbb{C}^*)^{2d}$ the symplectic form

$$\omega = \sum_{1 \le i < j \le 2d} \omega_{ij} dx_i \wedge dx_j$$

on $\mathbb{T}$ can be seen as the restriction of the (complex) holomorphic symplectic form

$$\frac{1}{(2\pi i)^2} \sum_{1 \le i < j \le 2d} \omega_{ij} \frac{dz_i}{z_i} \wedge \frac{dz_j}{z_j}$$

on $(\mathbb{C}^*)^{2d}$. Dual to the symplectic form, we have a Poisson bivector

$$\frac{1}{(2\pi i)^2} \sum_{1 \le i < j \le 2d} \omega^{ij} z_i \partial_{z_i} \wedge z_j \partial_{z_j} = \frac{1}{(2\pi i)^2} \sum_{1 \le i < j \le 2d} \omega^{ij} \theta_i \wedge \theta_j$$

which gives the ring $A = \mathbb{C}\{z, z^{-1}\}$ the structure of a *Poisson-algebra*, with Poisson-bracket

$$\{f, g\} := \sum_{i,j=1}^{2d} \omega^{ij} \theta_i(f) \theta_j(g).$$

It coincides, up to a factor $(2\pi i)^2$, with the usual Poisson-bracket

$$\sum_{i,j=1}^{2d} \omega^{ij}(\partial_{x_i} f)(\partial_{x_j} g),$$

when written in the original variables $x_1, x_2, \ldots, x_{2d}$.

2.3.2. The (complex) symplectic vector fields on $\mathbb{T}$ are denoted by

$$\mathcal{S}(A) \subset \Theta(A)$$

and are in one-to-one correspondence with closed one-forms. In particular the interior products of the symplectic form ω with the fields $\theta_i = z_i \partial_{z_i}$ give constant and therefore non-exact closed one-forms

$$\alpha_i := \iota_{\theta_i}(\omega), \quad i = 1, 2, \ldots, 2d.$$

As

$$\partial_{x_1} \wedge \partial_{x_2} \wedge \ldots \wedge \partial_{x_{2d}} \neq 0,$$

these one-forms generate the De Rham cohomology group $H^1_{dR}(\mathbb{T}, \mathbb{C})$ of the torus. Consequently, any closed 1-form α can be written as

$$\alpha = \sum_{i=1}^{2d} c_i \alpha_i + dh.$$

From the dual perspective, this means that any symplectic vector field S can be written as the sum of a quasi-periodic field and a Hamiltonian part:

$$S = \sum_{i=1}^{2d} c_i \theta_i + \{-, h\}.$$

Note that this representation is essentially unique; the function h is determined up to a constant. We will always choose h to have *vanishing constant term* when written as an analytic Fourier series.

2.3.3. One can summarise the above discussion by saying that the cokernel of the map

$$A \longrightarrow \mathcal{S}(A), \quad f \mapsto \{-, f\}$$

is identified with the De Rham cohomology group $H^1_{dR}(\mathbb{T}, \mathbb{C})$, whereas the kernel consists of the constants. Hence there is a natural exact sequence

$$0 \longrightarrow \mathbb{C} \longrightarrow A \longrightarrow \mathcal{S}(A) \longrightarrow H^1_{dR}(\mathbb{T}, \mathbb{C}) \longrightarrow 0$$

and the image of the map in the middle consists precisely of the Hamiltonian vector fields.

2.4. **The torus with parameters.** We will pick a reference symplectic form ω^0 and add parameters that describe the variation of the symplectic form on the torus. Furthermore we add parameters that detune the frequencies. The reason for this is, that it makes it easier to formulate the normal form iteration. By the implicit function theorem we will later express the perturbed frequencies in terms of the perturbed symplectic form. However, initially we will consider these as formal, independent parameters.

2.4.1. *The ring R.* We add variables

$$\phi_1, \dots, \phi_{2d}$$

to parametrise the frequencies and

$$\delta_1, \delta_2, \dots, \delta_l,$$

to parametrise the manifold V of symplectic forms in the neighbourhood of U of ω^0:

$$U \longrightarrow \mathbf{S},\ \delta \mapsto \omega_\delta$$

The elements of the ring

$$R := \mathbb{C}\{\delta, \phi, z, z^{-1}\}$$

are analytic series:

$$f = \sum_{J,K,L} f_{J,K,L} z^J \delta^K \phi^L,\ f_{J,K,L} \in \mathbb{C}$$

which are convergent in a neighbourhood of $\{0\} \times \mathbb{T}$.

2.4.2. The ring R has a natural Poisson structure that can be written as

$$\{f, g\} := \sum_{i,j=1}^{2d} \omega_\delta^{ij} \theta_i(f) \theta_j(g).$$

The sub-ring $R_0 := \mathbb{C}\{\delta, \phi\}$ is the centre of this Poisson algebra. From the point of view of algebraic geometry we are dealing with a *relative symplectic structure*, which means that we get a family of symplectic structures parametrised by R_0. Like in the absolute case (i.e. when they are no parameters), interior product with the form ω_δ induces an isomorphism:

$$\Theta_{R/R_0} \xrightarrow{\sim} \Omega^1_{R/R_0}$$

between the modules of relative vector fields and relative one-forms. Both are free R-modules and more precisely

$$\Theta_{R/R_0} = \bigoplus_{i=1}^{2d} R\theta_i, \quad \Omega^1_{R/R_0} = \bigoplus_{i=1}^{2d} R\frac{dz_i}{z_i}.$$

So there is a natural relative notion of symplectic vector fields: these are the fields which correspond to *relative closed one-forms*. This defines the module $\mathcal{S}(R)$ of *relative symplectic vector fields*. In the context of differential geometry, these are called *tangential Poisson fields* (see for instance [26]).

Note that by Cartan's formula, relative symplectic vector fields coincide with vector fields which preserve the symplectic form. Indeed for $X \in \Theta_{R/R_0}$:

$$L_X \omega_\delta = d i_X \omega_\delta + i_X d\omega_\delta = 0$$

in Ω^2_{R/R_0}. The form ω_δ being relatively closed, this shows that the form $i_X\omega_\delta$ is closed and therefore X is symplectic.

Lemma. *Relative symplectic vector fields can be written in the form:*

$$\sum_{i=1}^{2d} c_i \theta_i + \{-, h\},\ c_i \in R_0,\ \ h \in R.$$

Proof. Using symplectic duality, the cokernel of the map

$$R \mapsto \mathcal{S}(R),\ h \mapsto \{-, h\}$$

is isomorphic to the R_0-module $H^1(\Omega^\bullet_{R/R_0})$, the relative first de Rham cohomology. We need to prove that this is a free R_0-module generated by the classes of the θ_i's:

$$H^1(\Omega^\bullet_{R/R_0}) = \bigoplus_{i=1}^{2d} R_0[\theta_i].$$

This is a relative variant of the fact that the classes θ_i's generate the De Rham cohomology of the torus.

Take a closed relative one-form $\alpha \in \Omega^1_{R/R_0}$, decompose it into homogeneous parts (with respect to the degree in the z_i variables) and isolate the degree 0 part:

$$\alpha = \sum_{i \in \mathbb{Z}\setminus\{0\}} \alpha_i + \sum_{i=1}^{2d} c_i\theta_i,\ \deg(\alpha_i) = i, c_i \in R_0.$$

Let $X = \sum_{i=1}^{2d} \theta_i$ be the Euler field and define

$$\beta = \sum_{i \in \mathbb{Z}\setminus\{0\}} \frac{\alpha_i}{i}.$$

By Cartan's formula we have:

$$L_X\beta = di_X\beta = \sum_{i \in \mathbb{Z}\setminus\{0\}} \alpha_i,$$

which shows that the non-degree 0 part is exact. Therefore:

$$[\alpha] = \sum_{i=1}^{2d} c_i[\theta_i] \in H^1(\Omega^\bullet_{R/R_0}).$$

This proves the lemma. □

Relative symplectic vector fields preserve the Poisson structure, but general Poisson vector fields can have an additional component:

Lemma. *The Poisson vector fields of R are of the form:*

$$\sum_{i=1}^{2d} b_i\partial_{\phi_i} + \sum_{i=1}^{2d} c_i\theta_i + \{-,h\}, \quad b_i, c_i \in R_0, \ \ h \in R.$$

Proof. We have a direct sum decomposition

$$Der_{\mathbb{C}}(R) = Der_{R_0}(R) \oplus Der_{\mathbb{C}}(R_0),\ \Theta_{R/R_0} = Der_{R_0}(R)$$

which simply means that a vector field X can be decomposed in the form

$$X = Y + Z,\ Y = \sum_{i=1}^{l} a_i\partial_{\delta_i} + \sum_{i=1}^{2d} b_i\partial_{\phi_i},\ Z = \sum_{i=1}^{2d} \alpha_i\theta_i.$$

Denote by

$$\pi = \sum_{i<j} \omega_\delta^{ij}\theta_i \wedge \theta_j$$

the Poisson bivector. We have

$$L_X\pi = \sum_{i<j} L_Y(\omega_\delta^{ij})\theta_i \wedge \theta_j + \sum_{i<j} \omega_\delta^{ij} L_X(\theta_i \wedge \theta_j)$$

We decompose each component of the equation

$$L_X\pi = 0$$

in the z-degrees, that is, we look for each coefficients $z^a\theta_i \wedge \theta_j$ where $|a|$ is the degree. The terms $L_Y(\omega_\delta^{ij})\theta_i \wedge \theta_j$ have coefficients in R_0 and have therefore degree 0 while the coefficients of

$$L_X(\theta_i \wedge \theta_j) = L_X\theta_i \wedge \theta_j + \theta_i \wedge L_X\theta_j$$

all have positive degree. This shows that $L_X\pi = 0$ occurs if and only if

$$L_Y\pi = L_Z\pi = 0$$

In particular the vector field Z is symplectic.

Now write

$$Y = \sum_{i=1}^{l} a_i\partial_{\delta_i} + \sum_{i=1}^{2d} b_i\partial_{\phi_i}.$$

By assumption, the δ_i parametrise the symplectic forms ω_δ which means that the map

$$\partial_{\delta_i} \mapsto \partial_{\delta_i}\omega_\delta$$

is injective and, as ω_δ does not depend on ϕ this implies that the a_i's are all zero. This proves that $Y \in \bigoplus_{i=1}^{2d} R_0\partial_{\phi_i}$. □

We denote by $\mathcal{P}(R)$ the R_0-module of Poisson vector fields. Note that for any $h \in R$ and $v \in \mathcal{P}(R)$, we have

$$[v, \{-, h\}] = \{-, v(h)\}$$

In particular, both modules $\mathcal{S}(R), \mathcal{P}(R)$ are closed under Lie bracket and we have inclusions of Lie algebras:

$$\mathcal{S}(R) \subset \mathcal{P}(R) \subset \mathrm{Der}_{\mathbb{C}}(R).$$

2.5. **The versality theorem.**

2.5.1. For a given set $X \subset \mathbb{C}^{2d} \times \mathbb{C}^k$, we use the notation

$$X(a) = (\mathcal{C}(a) \times \mathbb{C}^k) \cap X$$

for the subset with "good frequencies".

Let us first formulate the following versality theorem and then discuss the various notions involved:

Theorem (Versality Theorem)**.** *Let $a = (a_n)$ be a subquadratic sequence and let*

$$U \subset \mathbb{C}^{2d} \times \mathbb{C}^l \times (\mathbb{C}^*)^{2d}$$

be a neighbourhood of $\{0\} \times \mathbb{T}$ with coordinates ϕ, δ, z. Assume $\nu(0) \in \mathbb{C}^{2d}(a)$. then

$$\mathcal{V} = \sum_{i=1}^{2d} (\nu_i + \phi_i)\theta_i$$

is stable *in the following sense: for any $k, \varepsilon > 0$, there exists $\delta > 0$, such that for any $S \in \mathcal{S}^b(U)$, δ-close to zero, there exists an open neighborhood U' of $\{0\} \times \mathbb{T}$ and a Poisson morphism*

$$\psi : \mathcal{O}^b(U) \longrightarrow \mathcal{O}^k(U'(a)),$$

ε-close to the identity, such that

$$\psi_*(\mathcal{V} + S) = \mathcal{V}.$$

2.6. For an open subset $U \in \mathbb{C}^k$, the vector space $\mathcal{O}(U)$ of holomorphic functions on U has only a Fréchet space structure. However if we consider functions which are bounded, we get a Banach space $\mathcal{O}^b(U)$ for the supremum norm. The Banach space of symplectic vector fields with coefficients in $\mathcal{O}^b(U)$ are denoted by $\mathcal{S}^b(U)$.

2.6.1. The set $\mathcal{C}(a)$ of good frequencies has in general an empty interior, so in the statement of the theorem we used the notion of holomorphy in the Whitney differentiable context. Recall that a function $f : X \longrightarrow \mathbb{R}$ defined on a closed subset $X \subset \mathbb{R}^d$ of Euclidean space is called C^k *Whitney differentiable* at $x \in X$, if there exists continuous functions $x \mapsto D^I f(x)$ called the *Whitney derivatives* of f at x such that:

$$f(y) = \sum_{|I| \leq k} \frac{D^I f(x)}{|I|!}(y-x)^I + o(\|y-x\|^k).$$

So the sole difference with the standard definition of differentiability is the requirement of uniformity of the limit in the x, y variables. The *Whitney extension theorem* says that any C^k-Whitney function is the restriction of a C^k-function [27].
If we consider complex valued functions with Whitney C^k real and imaginary part which satisfy the Cauchy-Riemann equations, then we get the definition of a C^k holomorphic function on a closed set. (These were already considered by Herman back in 1985 [15].)
These C^k-Whitney holomorphic functions define a presheaf of Banach space $\mathcal{O}^k_{U(a)}$ and we simply write $\mathcal{O}^k(U(a))$ for $\mathcal{O}^k_{U(a)}(U(a))$ for the functions with continuous and bounded extension to the closure of $U(a)$.

Note that the Poisson morphism ψ maps holomorphic functions to Whitney holomorphic functions. This express the fact that only the quasi-periodic motions with "good frequencies" are preserved. We also extended our definitions of $\mathcal{S}(R)$ and $\mathcal{P}(R)$ from germs to open sets U containing $\{0\} \times \mathbb{T}$.

2.7. We will use finite differentiability but it is easy to see that the Poisson morphism we will construct is in fact Whitney C^∞. This is due to the fact that like for the one dimensional Cauchy inequalities:

$$\sup_{|z| \leq r} |f^{(k)}(z)| \leq \frac{k!}{\varepsilon^k} \sup_{|z| \leq r+\varepsilon} |f(z)|,$$

C^0-estimates imply C^k-estimates after shrinking.

2.7.1. Having clarified the statement of the theorem, let us now show that, as announced, the Versality Theorem implies our result on the stability of quasi-periodic motions.
The frequency $\nu(0)$ is assumed to belong to $\mathcal{C}(a)$. We choose $a' < a$ such that the set $\mathcal{C}(a')$ has positive measure in any neighbourhood of $\nu(0)$ (see [8]). Assume that ν is a submersion. Then the map $\nu + g$ is a submersion for any g in a sufficiently small C^1 ε-neighbourhood. We apply the theorem above with the sequence a' and $k = 1$. We get a Poisson mapping ψ such that

$$\psi_*(\mathcal{V} + S) = \mathcal{V}$$

for any S in a δ-neighbourhood. So if we perturb the quasi-periodic motion

$$X = X_\nu + S = \sum_{i=1}^{2d} \nu_i(\delta)\theta_i + S,$$

then the image of $X_\nu + S$ under ψ is the restriction of $\mathcal{V}$ to

$$R_1 = \cdots = R_{2d} = 0.$$

where

$$R_i(\delta, \phi) := \psi(\phi_i), \; i = 1, \dots, 2d.$$

Note that these are only defined on the set $U'(a')$. We choose C^1 Whitney extensions $r_1, \dots, r_{2d}$ on U'. By the implicit function theorem, we may find a smaller neighborhood U'' of the origin and a C^1-map $g = (g_1, \dots, g_{2d})$, defined on U'' such that

$$r(\delta, \phi) = 0 \iff \phi = g(\delta).$$

This means that

$$(*) \quad \psi_*(X_\nu + S_0) = \sum_{i=1}^{2d} (\nu_i(\delta) + g_i(\delta))\theta_i$$

provided that

$$(**) \quad (\delta, \nu(\delta) + g(\delta)) \in U'(a').$$

By Ehresmann's lemma, the map

$$\delta \mapsto \nu(\delta) + g(\delta)$$

defines a local fibration above its image. The preimage of $\mathcal{C}(a')$ has therefore positive measure.

3. KAM iterative procedures

In this section, we prove a formal version of the Versality Theorem and outline the iteration process that brings a perturbed vector field back to normal on the level of power series. In later sections, we lift this iteration to the level of Banach spaces and give the estimates that lead to a proof of convergence. We start with a discussion of the associated *homological equations*, i.e. the linear operators that we will need to invert.

3.1. Formal variant of the versality theorem.

3.1.1. In the analytic case, we consider a neighbourhood of a given vector field for the LB-topology, while in the formal case we add a formal parameter t to our Poisson algebra R and get a new Poisson algebra $R[[t]]$ where t is a central element. There is now a t-adic topology on $R[[t]]$, defined 0-neighbourhood $t^k R[[t]]$ with $k \in \mathbb{N}$ as basis. A sequence (f_n) is convergent, if its projection to $R[[t]]/t^k R[[t]]$ becomes a constant for all n bigger than some index N_k.

We have a formal version of the versality theorem:

Theorem. *The deformation $\mathcal{V} \in \mathcal{P}(R[[t]])$, $R = \mathbb{C}\{\phi, \delta, z, z^{-1}\}$ is stable in the sense that for any deformation $\widetilde{\mathcal{V}} = \mathcal{V} + tS \in \mathcal{P}(R[[t]])$ of $\mathcal{V}$ there exists a Poisson automorphism ψ such that*

$$\psi_* \widetilde{\mathcal{V}} = \mathcal{V}$$

The theorem is proved by a variant of the standard iteration already used by Kolmogorov in his 1954 paper [17].

3.1.2. Put $S = S_0$ and assume we can find $Y_0 \in \mathcal{P}(R)$ that solves the *homological equation*:

$$[Y_0, \mathcal{V}] = S_0$$

The vector field Y_0 integrates to an automorphism e^{-tY_0} of $R[[t]]$. By the adjoint action, the vector field X_0 is transformed into

$$X_1 := e^{-t[Y_0,-]} X_0 = X_0 - t[Y_0, X_0] + \dots$$

and define the next perturbation S_1 by setting

$$X_1 = \mathcal{V} + t^2 S_1.$$

We repeat this operation and get an iteration scheme of the form

$$\begin{aligned} X_n &= \mathcal{V} + t^n S_n, \\ [Y_n, \mathcal{V}] &= S_n, \\ X_{n+1} &= e^{-t^{n+1}[Y_n,-]} X_n. \end{aligned}$$

This iteration converges in the t-adic topology (the orders become higher in t). So to prove the formal variant of our theorem, we only need to explain how to solve the homological equation, that we will do now.

3.2. **The homological equation for functions.**

3.2.1. The Lie-derivative

$$D : R \longrightarrow R,\ f \mapsto L_{\mathcal{V}} f = \mathcal{V}(f)$$

is diagonal in the monomial basis:

$$D : z^I \mapsto (\nu(\delta) + \phi, I) z^I,$$

where $(-,-)$ denotes the Euclidean scalar product.
Hence the operator D is equal to taking Hadamard product with

$$g(z) := \sum_{I \in \mathbb{Z}^{2d}} (\nu(\delta) + \phi, I) z^I.$$

so that

$$D(f) = g \star f.$$

The kernel of the map D is the centre $R_0 = \mathbb{C}[[\delta, \phi]]$ of the Poisson algebra R.

3.2.2. As we assumed $\nu(0)$ to be non-resonant, the functions

$$(\nu(\delta) + \phi, I), \quad I \neq 0,$$

are invertible elements of $R_0 = \mathbb{C}[[\phi, \delta]]$, and thus we can consider the formal power series:

$$H(z) := \sum_{I \in \mathbb{Z}^{2d} \setminus \{0\}} (\nu(\delta) + \phi, I)^{-1} z^I \in R_0[[z, z^{-1}]],$$

which we call the *resolvente* of $\mathcal{V}$. Note that if we try to interpret this formal series as a function, it would have, in general, poles of order 1 along a dense set of hyperplanes defined by the resonance conditions $(\nu(\delta) + \phi, I) = 0$, $I \in \mathbb{Z}^{2d}$.

If P is a Laurent polynomial in z *with vanishing constant coefficient*, then

$$D(H \star P) = g \star H \star P = P,$$

so we get an inverse to the operator D, because the Hadamard product contains only a finite number of elements. However if we want to write a similar formula for more general elements of R, it is readily shown that the Diophantine condition on $\nu(0)$ implies that the operator $H\star$ maps R to itself, and provides an inverse to the Lie-derivative operator.

$$D(H \star f) = f, \quad f \in R.$$

We note that the Lie-derivative $L_{\mathcal{V}}$ and therefore the map D extends to arbitrary tensors. The Lie-derivative commutes with the exterior derivative and therefore

$$Dd\,(H \star P) = dD\,(H \star P) = dP.$$

3.3. The homological equation for vector fields.

3.3.1. The action of the Lie-derivative on vector fields is given by the Lie bracket or commutator of vector fields:

$$L_{\mathcal{V}}Y = [\mathcal{V}, Y] = -L_Y\mathcal{V}.$$

For a given symplectic vector field S, we want to solve the *homological equation*

$$L_{\mathcal{V}}Y = S$$

for the vector field Y. Somewhat surprisingly, this can be done quite explicitly.

3.3.2. If we assume a Diophantine condition on $\nu(0)$, we have the following:

Proposition. *Decompose S into Hamiltonian and non-Hamiltonian parts:*

$$S = \{-, f\} + \sum_{i=1}^{2d} c_i\theta_i, \quad c_i \in R_0.$$

Then the equation

$$L_{\mathcal{V}}Y = S$$

is solved by

$$Y := \{-, H \star f\} + \sum_{i=1}^{2d} c_i\partial_{\phi_i},$$

where $H = H(z)$ is the resolvente introduced above.

Proof. Write $D = L_{\mathcal{V}}$ for the operation of Lie-derivative with respect to $\mathcal{V}$. As the symplectic form ω_δ is invariant under the flow of $\mathcal{V}$, one has $D\omega_\delta = 0$ and consequently

$$D(\{f, g\}) = \{Df, g\} + \{f, Dg\}.$$

The commutator of the vector fields $\mathcal{V}$ and $\{-, g\}$ acting on f is

$$[\mathcal{V}, \{-, g\}](f) = \mathcal{V}(\{f, g\}) - \{\mathcal{V}(f), g\} = D(\{f, g\}) - \{Df, g\} = \{f, Dg\},$$

which means that

$$D(\{-, g\}) = \{-, Dg\}.$$

So we get

$$\begin{aligned} DY &= \{-, D(H \star f)\} + [\mathcal{V}, \sum_{i=1}^{2d} c_i\partial_{\phi_i}] \\ &= \{-, f\} + \sum_{i=1}^{2d} c_i\theta_i \\ &= S \end{aligned}$$

□

3.3.3. The decomposition of S into Hamiltonian and non-Hamiltonian parts are unique. Therefore, still under a Diophantine condition on the frequency $\nu(0)$, the explicit solution to homological equation determines a map

$$\begin{array}{rcl} j : \mathcal{S}(R) & \longrightarrow & \mathcal{P}(R)) \\ S & \mapsto & Y = \{-, H \star f\} + \sum_{i=1}^{2d} c_i\partial_{\phi_i}. \end{array}$$

This concludes the proof of Theorem 3.1.1. In the following section, we will discuss the analytic case.

3.4. **The non-formal iterative procedure.**

3.4.1. In the purely analytic case, we have no deformation parameter t to our disposal, in other words, we have $t=1$ and one can not argue by powers of t. Instead, we have to consider the z-degrees. Let us also forget the Diophantine condition on $\nu(0)$. Then the operation of taking the Hadamard product with the resolvente H no longer maps R to itself. As a result, the map j defined above does not make sense.

However, it turns out that it is sufficient to consider an *approximate inverse* and therefore we introduce a weight filtration in the ring $R = \mathbb{C}\{\phi, \delta, z, z^{-1}\}$..

3.4.2. For each subset $A \subset \mathbb{Z}^n$ there is a canonical *truncation operator*

$$[-]_A : \mathbb{C}[[z, z^{-1}]] \longrightarrow \mathbb{C}[[z, z^{-1}]], f \mapsto [f]_A,$$

where we only keep the the monomials of f whose exponent appears in A:

$$[f]_A := \sum_{I \in \mathbb{Z}^n \cap A} a_I z^I.$$

So we are dealing with a special case of the Hadamard product and so these truncation operators map $\mathbb{C}\{z, z^{-1}\}$ to itself. If A is a finite set, $[f]_A$ belongs to $\mathbb{C}[z, z^{-1}]$. For $j, k \in \mathbb{N}$ we put

$$[f]_j^k := [f]_{A_{j,k}}, \quad A_{j,k} := \{I \in \mathbb{Z}^n \mid \ \ j \leq \ |I| < k\}.$$

We also write $[f]_j$ in case k is infinite and $[f]^k$ in case $j=0$. This corresponds to the filtration by the degrees of the Fourier harmonics.

Now we assign the weight one to the variables δ_i and weight zero to ϕ_i. We can naturally extend the truncation from the ring A to the ring R, so that

$$[f]^k := \sum_L \sum_{|K| \leq k} [f_{K,L}]^{k-|K|} \delta^K \phi^L$$

with $f_{K,L} \in \mathbb{C}\{z, z^{-1}\}$.

3.5. We may now define our approximated inverse. If $S = \{-, f\} + \sum_{i=1}^{2d} c_i \theta_i$ and $A \subset \mathbb{Z}^n$ a finite subset, we set

$$[S]_A := \{-, [f]_A\} + \sum_{i=1}^{2d} [c_i]_A \theta_i$$

and can define maps

$$j_A : \mathcal{S}(R) \longrightarrow \mathcal{P}(R))$$

by setting

$$j_A(S) := j([S]_A) = \{-, H \star [f]_A\} + \sum_{i=1}^{2d} [c_i]_A \partial_{\phi_i}.$$

We also use the notations

$$[S]^m := \{-, [f]^m\} + \sum_{i=1}^{2d} [c_i]^m \theta_i, \ \ [S]_m := \{-, [f]_m\} + \sum_{i=1}^{2d} [c_i]_m \theta_i$$

for the truncations of a vector field in lower and higher Fourier modes.

3.5.1. Now, the iteration runs as follows. We decompose the perturbation into two parts:

$$S_0 = [S_0]^2 + [S_0]_2.$$

Using proposition 3.3.2 we can find $Y_0 \in \mathcal{P}(R))$ that solves the equation:

$$[Y_0, \mathcal{V}] = [S_0]^2$$

The vector field Y_0 integrates to an automorphism e^{-Y_0} of R. By the adjoint action, the vector field X_0 is transformed into

$$X_1 := e^{-[Y_0,-]}X_0 = X_0 - [Y_0, X_0] + \dots$$

and define the next perturbation S_1 by setting

$$X_1 = \mathcal{V} + S_1.$$

3.5.2. We repeat this operation by taking terms up to order 2, 2^2, 2^3 and so on. We obtain an iteration scheme of the form

$$\begin{aligned} X_n &= \mathcal{V} + S_n, \\ S_n &= [S_n]^{2^{n+1}} + [S_n]_{2^{n+1}}, \\ [Y_n, \mathcal{V}] &= [S_n]^{2^{n+1}}, \\ X_{n+1} &= e^{-[Y_n,-]}X_n. \end{aligned}$$

3.5.3. As before, starting from a perturbed $X = X_\nu + S_0$ quasi-periodic motion, we consider the corresponding perturbation of the versal unfolding

$$X_0 = \mathcal{V} + S_0.$$

The iteration produces an adjoint automorphism

$$\psi_n := e^{[Y_n,-]} \dots e^{[Y_0,-]} \in Aut(\Theta(R))$$

$$X_n = \mathcal{V} + S_n = \psi_n(\mathcal{V} + S_0).$$

and maps the element ϕ_k to certain power series

$$\varphi_n(\phi_k) =: R_{n,k}(\phi_1, \dots, \phi_{2d}, \delta_1, \dots, \delta_l) =: R_{n,k}(\phi, \delta), \quad k = 1, 2, \dots, 2n.$$

with

$$R_{n,k}(\phi, \delta) = \phi_k + O(2),$$

Again using the formal implicit function theorem, we solve the equations $R_{n,k} = 0$. The adjoint automorphism ψ_n transforms the vector field $X_\nu + S_0$ into

$$\psi_n(X_\nu + S_0) = (\mathcal{V})_{\phi_k = g_{n.k}(\delta)} + (S_n)_{\phi_k = g_{n,k}(\delta)}.$$

So in the limit $n \longrightarrow \infty$, we expect to get a relation of the form

$$\psi_\infty(X_\nu + S_0) = (\mathcal{V})_{\phi_k = g_{\infty,k}(\delta)} = \sum_{i=1}^{2d} (\nu_i(\delta) + g_{\infty,i}(\delta))\theta_i,$$

hence we produce a coordinate transformation that conjugates the symplectic perturbation $X_\nu + S$ of a quasi-periodic vector field X_ν to a nearby quasi-periodic vector field, with a frequency that depends on the perturbation of the symplectic form.

3.5.4. This new iteration is foolish from a formal point of view: as the vector fields Y_n will contain all terms of degrees up to 2^n, its exponential will reintroduce monomials that one tries to remove. One is reminded of the mythos of SISYPHOS, but we will show later by a direct estimate that the norm of the remainder decreases quadratically in appropriate Banach spaces, so that his burden is decreasing quickly, although the removal of even the first term will require an infinite number of iterations and keeps him busy forever.

3.6. **Almost quadratic nature of the iteration.**

3.6.1. Before going into the details of functional analytic aspects, we discuss the quadratic nature of the iteration. It is defined by first writing

$$X_n = \mathcal{V} + S_n$$

and then recursively

$$\begin{cases} Y_n &= j_n(S_n) \\ X_{n+1} &= e^{-[Y_n,-]} X_n \end{cases}$$

Here we use the notation

$$j_n(-) = j_{A_n}(-)$$

where A_n is the set of monomials, whose absolute value of weight is smaller than 2^{n+1}.

3.6.2. The iteration is quadratic with the remainder term in the following sense:

$$\begin{aligned} X_{n+1} &= e^{-[Y_n,-]}\left(\mathcal{V} + [S_n]^{2^n}\right) + e^{-[Y_n,-]}\left([S_n]_{2^n}\right) \\ &= e^{-[Y_n,-]}\left(\mathcal{V} + [Y_n, \mathcal{V}]\right) + e^{-[Y_n,-]}\left([X_n]_{2^n}\right) \\ &= \mathcal{V} + (e^{-[Y_n,-]}(\mathrm{Id} + [Y_n,-]) - \mathrm{Id}\,)\mathcal{V} + e^{-[Y_n,-]}\left([X_n]_{2^n}\right) \end{aligned}$$

For a power series in a single variable x

$$f(x) = \sum_{i=1}^{\infty} a_i x^i$$

and a vector field X, we put

$$f_*(X) := \sum_{i=1}^{\infty} a_i (L_X)^i$$

So if we write

$$f(x) = e^{-x}(1+x) - 1 = -\frac{x^2}{2} + o(x^2),$$

the iteration can be written in the form

$$S_{n+1} = f_*(j_n(S_n))\mathcal{V} + e^{-[j_n(S_n),-]}\left([S_n]_{2^n}\right)$$

So in a formal sense, the iteration has a quadratic term $f_*(j_n(S_n))$ and a remainder part $e^{-[j_n(S_n),-]}([S_n]_{2^n})$. Although terms of low degree remain at each step of the iteration, it might be expected that, because of this quadraticity, their coefficients rapidly tend to zero. We will see that this is, under certain conditions, indeed the case.

3.6.3. The above can be seen as an iteration in the LB-space $\mathcal{S}(R)$. Now we will formulate a version of the iteration in terms of a system of Banach spaces of holomorphic functions attached to neighborhoods T_r of $\mathbb{T}$ in $(\mathbb{C}^*)^{2d}$. We will keep track of the norms of S_n during the iteration. This will show that the above process converges over a non-trivial Cantor-like set, defined by the condition that the norm remains sufficiently small.

4. A short review on functorial analysis

In the proof of the Versality Theorem, we will have to work with many different Banach spaces, usually called a Banach scale or a Banach chain. The important feature is not only these Banach spaces as such, but rather the various maps between them that are 'compatible' in various ways. These maps result from restriction maps, that appear in the shrinking of domains during the iteration process, or changes in the type of Banach space considered. Of course, one needs to have explicit control over all the norms of these maps. To keep track of all these, we found it convenient to use an abstract framework that was developed in [9], to which we refer for more details.

4.1. **Relative Banach spaces.** We give a quick overview of the formalism of Banach spaces parametrized by ordered sets that one encounters often in dealing with function spaces over shrinking domains of definition.

4.1.1. Let $(B, \geq)$ be a partially ordered set.

Definition: By a *Banach space E over B* we mean a collection of Banach spaces space $E_t, t \in B$, and for each $t \geq s$ compatible continuous linear maps

$$\varepsilon_{st} : E_t \longrightarrow E_s,$$

called *restriction maps*, where compatibility means

$$(\star) : e_{ss} = Id, \ \ e_{st} \circ e_{tu} = e_{su} \ \ \text{for all} \ \ u \geq t \geq s.$$

We emphasize that all Banach spaces E_t are equipped with a specific norm $|-|_t$.

Examples come readily to mind. The chain of Banach spaces

$$\cdots \subset C^{k+1}([0,1], \mathbb{R}) \subset C^k([0,1], \mathbb{R}) \subset \cdots$$

can be seen as a Banach space over $(\mathbb{N}, \geq)$, where the inclusion maps take the role of restrictions. Another example is obtained as follows: for an open subset $U \subset \mathbb{C}^N$, let $\mathcal{O}^b(U)$ denote the Banach space of all bounded holomorphic functions on U. For $V \subset U$ there are compatible restriction maps $\varphi_{VU} : \mathcal{O}^b(U) \longrightarrow \mathcal{O}^b(V)$. So we obtain a Banach space over the partially ordered set of open subsets of $\mathbb{C}^N$, which is just a *presheaf of Banach spaces*. By further restriction to open balls of radius B_r, we get a Banach space E over $(\mathbb{R}_{>0}, \geq)$, with $E_t := \mathcal{O}^b(B_t)$.

4.1.2. One can describe Banach spaces over B in the geometric language of *fibre bundles*. One forms the *total space* by setting

$$E := \bigsqcup_{b \in B} E_b,$$

and there is a natural map $p : E \longrightarrow B$, which maps the elements of E_b to b. We sometimes use the notation $(b, x), x \in E_b$ for the elements of E and we use the generic name $|-|_b$ for the norm on the Banach space E_b. For an element $x_b \in E_b$ we often write $|x_b|$ instead of $|x_b|_b$, etc.

Definition: A *section of E over $A \subset B$* is a map $x : A \longrightarrow E$, such that $p \circ \sigma = Id_{|A}$, i.e. a choice of vectors $x(b) =: x_b \in E_b$ for all $b \in A$, like in the theory of vector bundles or sheaves. The set of all such sections over A form a vector space

$$\Gamma(A, E) := \prod_{b \in A} E_b.$$

A section x is called *horizontal*, if it is *compatible with the restriction mappings*: for all $s, t \in A$ with $s < t$, we have

$$e_{st} x_t = x_s.$$

A section over A is called *bounded*, if the function $b \mapsto |x_b|$ is bounded on A. A simple but fundamental fact is the following

Proposition. *The vector space*

$$\Gamma^\infty(A,E) = \{x \in \Gamma(A,E) \mid \ x \text{ is horizontal and bounded}\}$$

is a Banach space with norm

$$|x|_A := \sup_{a \in A} |x_a|_a.$$

Proof. The only non-trivial fact to check is the completeness of the vector space $\Gamma^\infty(A,E)$ with respect to the norm $|-| := |-|_A$. Let $(x_n) \subset \Gamma^\infty(A,E)$ be a Cauchy sequence of sections. For any $b' > b$, the sequence $x_n(b')$ is a Cauchy sequence in $E_{b'}$ and therefore converges to a limit $x(b')$. We need to show that the norm of this limit section is finite. The norms $|x_n|$ form a Cauchy sequence of real numbers and therefore converges to a limit M.

$$\begin{aligned} |x(b')| &\leq |x(b') - x_n(b')| + |x_n(b')| \\ &\leq |x(b') - x_n(b')| + M. \end{aligned}$$

So, passing to the limit, we see that the norm of x is bounded by M and in fact equal to M (because the norm is a continuous map). □

4.1.3. One can give a categorical definition of relative Banach spaces that is more general and useful in many situations. Denote by **Ban** the category whose objects are Banach spaces, and whose morphisms are bounded linear operators. For a small category B, we mean by a *relative Banach space over* B, a covariant functor

$$F : B \longrightarrow \mathbf{Ban}.$$

Any *partially ordered set* $(B, \geq)$ is naturally such a small category, with spaces of morphism $Mor(t,s)$ consisting of a single element if $t \geq s$. If we apply the functor F to $t \in B$, we obtain a Banach space E_t and applied to any morphism $t \geq s$, we obtain a continuous linear mappings $e_{st} : E_t \longrightarrow E_s$; the functor property is precisely the compatibility conditions $(\star)$ between these maps.

4.1.4. **Definition:** A Banach space E over B is called a *Kolmogorov space*, if all the restriction mappings $e_{st} \in \operatorname{Hom}(E_t, E_s)$ have norm ≤ 1, where we put the operator norm $\|-\|$ on the space $\operatorname{Hom}(E_t, E_s)$ of continuous linear maps; one always has:

$$|e_{st}x|_s \leq \|e_{s,t}\| |x|_t \leq |x|_t \quad \text{for all } s \leq t.$$

The relative Banach space E over $\mathbb{R}_{>0}$ with $E_t := C^0([0,t],\mathbb{R})$ and F with $F_t = \mathcal{O}^b(B_t)$ are Kolmogorov spaces, because the norms on these spaces are defined by the supremum over a set, which can only become smaller if $s \leq t$ after restriction to a smaller set.

A global horizontal section of E over $\mathbb{R}$ is uniquely determined by the choice of a function $f \in C^0(\mathbb{R},\mathbb{R})$: the value of the section above a point $t \in \mathbb{R}_{>0}$ is simply the restriction of f to the interval $[0,t]$. The section is bounded precisely if the function f is bounded. Similarly the relative Banach space F with $F_t := \mathcal{O}^b(D_t)$, $D_t = \{z \in \mathbb{C} \mid |z| \leq t\}$ is a Kolmogorov space over $\mathbb{R}_{>0}$.

4.1.5. If $(B, \geq)$ is a partially ordered set, and $b \in B$, then the *down-set* of b is defined as

$$]-\infty, b] := \{b' \in B \mid b' \leq b\}.$$

If $x_b \in E_b$, we obtain be restriction $x_{b'} = e_{b'b} x_b$, and thus a horizontal section $x \in \Gamma(]-\infty, b], E)$. If E is a Kolmogorov space over B, then we also have $|x|_{b'} \leq |x|_b$, so that this section is also bounded. Trivially we have for Kolmogorov spaces a norm preserving isomorphisms of Banach spaces

$$\Gamma^\infty(]-\infty, b], E) = E_b.$$

This leads to the following idea: to *any* Banach space E over B we can associate in a natural way a Kolmogorov space EK over B, by setting

$$EK_b := \Gamma^\infty(]-\infty, b], E),$$

where the norm of such a section is given by the supremum norm. Clearly, if $b' \leq b$, then we have $]-\infty, b'] \subset]-\infty, b]$, so that indeed

$$|x|_{b'} \leq |x|_b,$$

as we are taking the supremum over a smaller set and so the natural restriction mappings $EK_b \longrightarrow EK_{b'}$ have norm ≤ 1. We see:

Proposition ([9])**.** *Given a relative Banach space E over B, the associated space EK over B is a Kolmogorov space.*

For this reason we call EK the *Kolmogorification* of E; if E is already Kolmogorov, then $EK = E$, so we get back the original space[1].

In practice, Kolmogorification tells us that there is a right norm to be considered, without having to guess it, or even to write it in explicit terms. There is a simple moto: if any space appears to be a relative Banach space that is not a Kolmogorov space, then we must take its Kolmogorification.

4.1.6. If B is a partially ordered set, we denote by B^{op} the partially ordered set with the reversed order. The underlying set B^{op} is the same as B, but the order relation on B^{op} is *opposite* to that of B: if $t \geq s$ in B, then $s \geq t$ in B^{op}.

If E is a Banach space over B, it can no longer be considered as a Banach space over B^{op}, as the maps go in the wrong direction. Nevertheless, we can form an *opposite Kolmogorov space.*

Definition. *Given a Banach space E over B, the* opposite Kolmogorov space

$$E^{op} \longrightarrow B^{op}$$

is defined by defining

$$E^{op}_b := \Gamma^\infty([b, +\infty[, E),$$

the Banach space of horizontal bounded sections over the up-set *of $b \in B$:*

$$[b, +\infty[= \{b' \in B \mid b' \geq b\},$$

(which is the down-set of $b \in B^{op}$), with the supremum norm.

Note there are natural restriction mappings of norm ≤ 1

$$E^{op}_b \longrightarrow E^{op}_{b'},$$

for $b' \geq b$ in B, i.e. $b' \leq b$ in B^{op}.

[1]The process of Kolmogorification is somewhat analogous to sheafification of a presheaf.

We remark that such reversions of ordering appear naturally in dealing with spaces of homomorphisms. For example, given two relative Banach spaces E and F over an interval $B =]0, S]$ with the usual ordering $\geq$, it seems we can form the relative Banach space

$$\mathrm{Hom}(E, F) \longrightarrow B \times B,$$

whose fibre over (t, s) is

$$\mathrm{Hom}(E_t, E_s),$$

the space of continuous linear maps from E_t to F_s, with the operator norm $\|\cdot\|$:

$$\|u_{st}\| := \sup_{x \in E_t \setminus \{0\}} \frac{|u(x)|_s}{|x|_t}.$$

It seems we obtain a relative Banach space: the restriction maps from $\mathrm{Hom}(E_t, E_s)$ to $\mathrm{Hom}(E_{t'}, F_{s'})$ are obtained as composition with the restriction maps of E and F:

$$u_{s't'} = f_{s's} u_{st} e_{tt'},$$

which forces to have $s \geq s'$, but $t' \geq t$. But note that the partial order relation needs to be reversed in the first factor! So $\mathrm{Hom}(E, F)$ really is a Banach space over $B^{op} \times B$ and not over $B \times B$.

4.2. **Local operators.** One can abstract and generalise the concept of *differential operator* in our context.

4.2.1. For two relative Banach space E and F over $B =]0, S]$ with the usual ordering $\geq$, we described above the relative Banach $\mathrm{Hom}(E, F)$ over $B^{op} \times B$. By restriction to the triangle

$$\Delta := \{(t, s) \in B^2 : t > s\} \subset B^{op} \times B$$

and a rescaling of the norm we obtain a Kolmogorov space

$$\mathcal{H}om^k(E, F) \longrightarrow \Delta,$$

whose fibre is the space $\mathrm{Hom}(E_t, F_s)$ of continuous linear mappings with *rescaled operator norm*

$$|t - s|^k \, \|u_{st}\|.$$

4.2.2. We also consider the *opposite Kolmogorov space*

$$\mathcal{H}om^k(E, F)^{op} \longrightarrow \Delta^{op}, \;\; \Delta^{op} \subset B \times B^{op},$$

which we denote by

$$\mathcal{L}^k(E, F) := \mathcal{H}om^k(E, F)^{op},$$

and call the *space of k-local operators.* Let us look what the elements of this space are, and how its norm is defined. Unravelling the definitions, we have

$$\mathcal{L}^k(E, F)_{s,t} := \Gamma^\infty(\Delta(t, s), \mathcal{H}om^k(E, F)),$$

so elements of this Banach space are bounded horizontal *sections* $u = (u_{a,b})$:

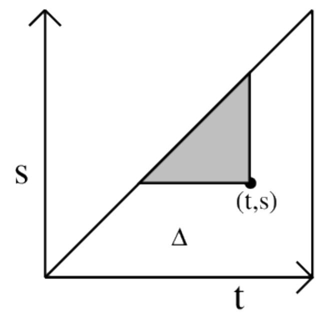
$$u_{a,b} \in \mathrm{Hom}(E_b, F_a), \quad (b, a) \in \Delta(t, s),$$

where

$$\Delta(t, s) := \{(b, a) \in B^2 : t \geq b > a \geq s\} \subset B^{op} \times B,$$

for which

$$|u|_{s,t} := \sup_{(b,a) \in \Delta(t,s)} |b - a|^k \, \|u_{a,b}\| < \infty.$$



$\mathcal{L}^k(E,F)_{s,t}$ is a Banach space that arises naturally in our formalism, but which is hard to describe in simpler terms.

4.2.3. A simple example may help to clarify the above construction. Consider the map sending a convergent power series to its derivative

$$\mathbb{C}\{z\} \longrightarrow \mathbb{C}\{z\},\ f \mapsto f' = \partial_z f,$$

and let $\mathcal{O}^b(D)$ be the Kolmogorov space over $]0,S]$ defined by

$$\mathcal{O}^b(D)_t := \mathcal{O}^b(D_t), \qquad D_t := \{z \in \mathbb{C} \mid |z| < t\}.$$

If $f \in \mathcal{O}^b(D_t)$, its derivative f' needs not to be bounded on D_t, and therefore is, in general, not an element of $\mathcal{O}^b(D_t)$. However, the function is certainly holomorphic and bounded inside any disc D_s of radius $s < t$. This peculiar property shows that the derivative is a *horizontal section of the opposite Kolmogorov space* $Hom(\mathcal{O}^b(D), \mathcal{O}^b(D))^{op}$ over Δ^{op}. Is this section bounded? It is not, as to be expected of a derivative operator. But the Cauchy inequality

$$|f'|_{s'} \leq \frac{1}{t'-s'}|f|_{t'},\ s' \geq s,\ t' \leq t$$

shows that it becomes *bounded after rescaling* by a factor $(t-s)$: the derivative is 1-local with norm ≤ 1. Similarly, a differential operator of order k will be k-local.

Differential operators and Hadamard products provide examples of local operators, but these examples are by no means exhaustive. For instance, some changes in scales provide an important further class of examples of local operators, as we shall see.

4.2.4. Composing partial differential operators of order k and l results in a partial differential operator of order $k+l$. This holds more generally for local operators.

Proposition. *Let E, F, G be Kolmogorov spaces over $]0,S]$. If $u \in \mathcal{L}^k(E,F)$ and $v \in \mathcal{L}^m(F,G)$ then*

$$v \circ u \in \mathcal{L}^{k+m}(E,G).$$

Moreover, one has the norm estimate:

$$|v \circ u| \leq \frac{(k+m)^{k+m}}{k^k m^m}|v||u|.$$

Proof. For $s < s' < t$ we have $\|(v\circ u)_{st}\| \leq \|v_{ss'}\|\,\|u_{s't}\|$. From the definition of locality we have:

$$\|v_{ss'}\| \leq \frac{|v|}{(s'-s)^k}, \quad \|u_{s't}\| \leq \frac{|u|}{(t-s')^m}.$$

We take the point s' such that

$$(s'-s) = \frac{m}{k+m}(t-s), \quad (t-s') = \frac{k}{k+m}(t-s),$$

and find the estimate

$$\|v_{ss'}\|\,\|u_{s't}\| \leq \frac{(k+m)^{k+m}}{k^k m^m}\frac{|v||u|}{(t-s)^{k+m}},$$

thus

$$|v \circ u| \leq \frac{(k+m)^{k+m}}{k^k m^m}|v||u|.$$

□

So composing local operators of order k and l results a local operator of order $k+l$.

4.2.5. *Cauchy-Lipschitz theorem for local operators.* We now formulate a Cauchy-Lipschitz theorem for local operators which implies the existence of the exponential of a 1-local operator and, more generally, one may define a functional calculus in $\mathcal{L}^1(E,E)$. Most proofs become elementary, using the language of Banach functors. We refer to [9, 10] for more details. Note that a vector field is a particular case of local operator and its flow will be simply defined by its exponential series. More generally we consider a power series

$$f = \sum_{n \geq 0} a_n z^n.$$

We use the notation

$$|f| = \sum_{n \geq 0} |a_n| z^n.$$

The *Borel transform* of a formal power series is defined by

$$B : \mathbb{C}[[z]] \longrightarrow \mathbb{C}[[z]],\ \sum_{n \geq 0} a_n z^n \mapsto \sum_{n \geq 0} \frac{a_n}{n!} z^n.$$

Consider a E over $]0,S]$ and its associated space $\mathcal{L}^1(E,E)$ of 1-local operators. We define the set

$$\mathcal{X}(R) := \{(t,s,u) \in \mathcal{L}^1(E,E) \mid \ ||u|| < R(t-s)\}$$

Theorem. *Let $f = \sum_{n \geq 0} a_n z^n \in \mathbb{C}\{z\}$ be a convergent power series with R as radius of convergence. Then there is a well-defined map of spaces over $\Delta \subset \mathbb{R}^2$, that we call the* Borel map*:*

$$\mathcal{B}f : \mathcal{X}(R) \longrightarrow \mathcal{H}om(E,E),\ (t,s,v) \mapsto (t,s,\sum_{n=0} \frac{a_n}{n!} v^n)$$

and one has the estimate

$$\|\mathcal{B}f(u)\| \leq |f| \left(\frac{||u||}{t-s} \right).$$

We also have a criterion for the convergence of change of variables, which was one of the missing point of classical KAM theory:

Theorem. *Let E over $]0,S]$ be a Kolmogorov space and let $(u_n) \subset \mathcal{L}^1(E,E)$ be a sequence of 1-local operators such that their norms $|u_n|$ define a summable sequence. Then the products*

$$e^{u_n} e^{u_{n-1}} \dots e^{u_0}$$

are well-defined and converge to a limit $\varphi \in \Gamma(U, Hom(E,E))$ with

$$U = \{(t,s) \in \mathbb{R}^2_{>0} : t-s \geq \sum_{i=1} |u_i|\}.$$

4.3. **Arnold spaces.** In applications to iterations one often encounters Banach-spaces $E_{n,t}$ indexed by a discrete iteration variable $n \in \overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$ and continuous variables t controlling the size of some neighborhoods. There are restriction mappings $E_{n,t} \longrightarrow E_{m,t}$ and $E_{n,t} \longrightarrow E_{n,s}$ for $n \leq m$ and $t \geq s$, making commutative diagrams

$$\begin{array}{ccc} E_{n,t} & \longrightarrow & E_{n,s} \\ \downarrow & & \downarrow \\ E_{m,t} & \longrightarrow & E_{m,s}. \end{array}$$

Such a structure can be seen as a Kolmogorov space over a base of the form $\overline{\mathbb{N}} \times B$, where of course we have to use the opposite ordering on the first variable. We call this structure an *Arnold space*. It can also be seen as a (compatible) sequence E_n, indexed by $n \in \overline{\mathbb{N}}$ of ordinary Kolmogorov spaces. In particular, there is fibre-wise notion of locality: a local map $u \in \mathcal{L}^k(E,F)$ is a family of local maps

$$u_n \in \mathcal{L}^k(E_n, F_n)$$

and thus defines a *norm sequence* $(|u_n|)$ which in applications needs to be controlled. A decreasing sequence $n \mapsto s_n$ defines a map

$$\sigma : \overline{\mathbb{N}} \longrightarrow \overline{\mathbb{N}} \times B, \quad n \mapsto (n, s_n),$$

that can be used to 'pull back' an Arnold space E and form a Kolmogorov space $E' := \sigma^* E$ over $\overline{\mathbb{N}}$ with fibre E_{n,s_n}. The consideration of Arnold spaces is useful in situations where one wants to postpone the choice of an appropriate sequence (s_n) as long as possible.

In this paper, we consider Banach spaces of holomorphic functions $\mathcal{O}^k$ on sets $W_{n,s}$. These sets combine into a relative open set $W \longrightarrow \overline{\mathbb{N}} \times B$ and the Banach spaces combine into an Arnold space $\mathcal{O}^k(W)$.

Proposition. *Let $u \in \mathcal{L}^m(E,F)$, $v \in \mathcal{L}^n(F,G)$ be local maps with norm subquadratic norm sequences then the norm sequence $u \circ v \in \mathcal{L}^{m+n}(E,G)$ is subquadratic with order at most*

$$\mathrm{ord}\,(|(u \circ v)_n|) \leq \max(\mathrm{ord}\,(|u_n|), \mathrm{ord}\,(|v_n|)).$$

The proof is obvious.

5. Kolmogorov spaces in the analytic context

In the previous section, we illustrated our concepts with the simple examples of one variable holomorphic functions. Now we will describe the analytic spaces that we use in our iteration.

5.1. **Spaces of holomorphic functions.**

5.1.1. Let U be a relative compact open subset of $\mathbb{C}^n$, and $\mathcal{O}(U)$ the ring of holomorphic functions on U. One can attach to U various Banach spaces of holomorphic functions. The space of *bounded holomorphic functions* is denoted by

$$\mathcal{O}^b(U) := \{f \in \mathcal{O}(U) \mid \ f \text{ is bounded}\}$$

is a Banach space with

$$|f| = \sup_{z \in U} |f(z)|$$

as norm. The space of *square integrable holomorphic functions*, denoted by

$$\mathcal{O}^h(U) := \{f \in \mathcal{O}(U) \mid \ \int_{\overline{U}} |f|^2 dV < +\infty\},$$

is a Hilbert space with the L^2-norm as norm. We denote by

$$\mathcal{O}^k(U)$$

the Banach space of complex valued (Whitney) *C^k-functions on the closure of $\overline{U}$*, which are holomorphic on the interior of U with

$$|f| = \max_{|I|\le k} \sup_{z\in U} |\partial^I f(z)|$$

as norm. Note that $\mathcal{O}^0(U) =: \mathcal{O}^c(U)$ is the same as the space of holomorphic functions that extend continuously to the boundary. As any C^k-function is bounded and any bounded function on a relative compact set is square-integrable, there are natural inclusions

$$\mathcal{O}^k(U) \subset \mathcal{O}^b(U) \subset \mathcal{O}^h(U).$$

If we generalize this to the context of relative open sets over some base, we obtain the most important class of Kolmogorov spaces.

5.2. Let $\mathcal{S}$ denote the set of subsets of $\mathbb{C}^n$, partially ordered by inclusion.

Definition. *Let $(B, \ge)$ be a partially ordered set. A* set over B *in $\mathbb{C}^n$ is an order reversing map*

$$B \longrightarrow \mathcal{S}$$

So a set over B consists of sets U_t for $t \in B$ such that for $t \ge s$ we have an inclusion $U_s \hookrightarrow U_t$. (It can be seen as a contravariant functor if we consider B and $\mathcal{S}$ as categories.) If all sets U_t are open/closed, we call it an *open/closed set over B*. We will write such an open set over B as $U \longrightarrow B$, with fibres U_t over $t \in B$.

As a simple example, the *relative unit polydisc* $D \longrightarrow \mathbb{R}_{>0}$ with fibres

$$D_t = \{z \in \mathbb{C}^d : |z_1| \le t, \dots, |z_d| \le t\}.$$

Given sets $U \longrightarrow B,\ U' \longrightarrow B$ we may perform many of the usual operations fibre-wise. For example, we may form their fibred product

$$U \times_B U' \longrightarrow B,$$

with fibres the Cartesian product of the fibres of U and U', etc.

5.2.1. Using such relative open sets we can create a plethora of Kolmogorov spaces. For an open set $U \longrightarrow B$ over B we may for each $t \in B$ form the Banach space $\mathcal{O}^b(U_t)$ of *bounded holomorphic functions* on U, with the sup-norm as norm. There are for $s \le t$ obvious restriction maps $\mathcal{O}^b(U_t) \longrightarrow \mathcal{O}^b(U_s)$ of norm < 1, and hence we obtain a Kolmogorov-space $\mathcal{O}^b(U)$ over B, with $\mathcal{O}^b(U)_t := \mathcal{O}^b(U_t)$. Similarly, the spaces $\mathcal{O}^h(U_t)$ of square integrable and $\mathcal{O}^k(U_t)$ of C^k-functions for Kolmogorov spaces $\mathcal{O}^h(U)$ and $\mathcal{O}^k(U)$. There is natural Kolmogorov space morphism:

$$\mathcal{O}^k(U) \longrightarrow \mathcal{O}^b(U), \quad \mathcal{O}^b(U) \longrightarrow \mathcal{O}^h(U).$$

5.2.2. *Cauchy-Nagumo estimate.* Let $D_t \subset \mathbb{C}$ be the disc of radius t. For a holomorphic function $f \in \mathcal{O}^c(U_t)$ one has the following elementary estimate

$$|f^{(m)}|_s \le \frac{m!}{(t-s)^m} |f|_t$$

for $s < t$, which is a straightforward consequence of the Cauchy integral formula and differentiation under the integral sign.

This simple idea can be extended to general partial differential operators on appropriate relative open sets.

Definition. *We say that an open and relatively compact set U of $\mathbb{C}^n$ over $B =]0, S]$ is a* Huygens set, *if for some $a > 0$ the following condition holds*

$$\forall x \in U_s,\ x + D_{a(t-s)} \subset U_t,$$

for any $s < t \leq S$.

The proof of the 1-variable case has an immediate generalization to

Proposition. *(Cauchy-Nagumo) Let $B =]0, S]$ and U a Huygens set over B, then any partial differential operator*

$$P = \sum_{|I| \leq m} a_I(z)\partial^I, a_I \in \mathcal{O}^k(U_S)$$

of order m defines an m-local operator of the Kolmogorov space $\mathcal{O}^k(U)$ over B:

$$P \in \mathcal{L}^m(\mathcal{O}^k(U), \mathcal{O}^k(U))$$

In applications one encounters often slightly more general situations, like the following.

Definition. *Let $(a_n) \in \mathbb{R}_{>0}$ be a falling positive sequence. We say that an open and relatively compact set U of $\mathbb{C}^n$ over $B =]0, S] \times \overline{\mathbb{N}}$ is an a-*Huygens set, *if the following condition holds*

$$\forall x \in U_{n,s},\ x + D_{a_n(t-s)} \subset U_{n,t}$$

for any $s < t \leq S$ and all $n \in \mathbb{N}$.

The sequence may very well go to 0, making an uniform choice for a impossible.

The above proposition admits a straightforward variant:

Proposition. *(Cauchy-Nagumo II) Let $a = (a_n)$ be a falling subquadratic sequence. Let $B =]0, S] \times \overline{\mathbb{N}}$ and U an (a_n)-Huygens set over B, then any partial differential operator*

$$P = \sum_{|I| \leq m} a_I(z)\partial^I, a_I \in \mathcal{O}^k(U_S)$$

of order m defines an m-local operator of the Kolmogorov space whose norm sequence $|P|_n$ is subquadratic with order bounded by that of (a_n):

$$\mathrm{ord}\,(|P|_n) \leq \mathrm{ord}\,(a_n).$$

5.2.3. A function $f \in \mathcal{O}^h(U)_t$ is by definition holomorphic in U_t. If U is Huygens, then given any $s < t$, its restriction to $\overline{U}_s$ is a C^k-function for any k. So we have 'restriction mappings'

$$J_{st} : \mathcal{O}^h(U_t) \longrightarrow \mathcal{O}^k(U_s)$$

These maps are compatible with the restrictions on $\mathcal{O}^h(U)$ and $\mathcal{O}^k(U)$, so combine into an element

$$J \in Hom_{\Delta}(\mathcal{O}^h(U), \mathcal{O}^k(U))$$

Proposition. *If U is an $a = (a_n)$-Huygens set over B, then*

$$J : \mathcal{O}^h(U) \longrightarrow \mathcal{O}^k(U)$$

is a local map. Moreover, if a is a falling subquadratic sequence, the norm sequence $|J_n|$ is bounded by an increasing subquadratic sequence of the same order.

Proof. Consider a function $f \in \mathcal{O}^h(U)_{n,t}$ and let $s < t$. The Taylor expansion of f at a point $w \in U_{n,s}$ reads:

$$f(z) = \sum_{J \in \mathbb{N}^d} \alpha_J (z-w)^J,\ \alpha_J \in \mathbb{C},$$

by assumption, U is an (a_n)-Huygens set, so the polydisc D_w centred at w with radius $\sigma = a_n(t-s)$ is contained in U_t. We then have

$$\int_{D_w} |f(z)|^2 dV = \sum_{J \in \mathbb{N}^d} C(J)|\alpha_J|^2 \sigma^{2|J|+2d},\ \ C(J) = \prod_{k=1}^{d} \frac{\pi}{j_k+1},$$

where dV is the Lebesgue measure.

So we obtain

$$C(0)|\alpha_0|^2\sigma^{2d} \leq \int_{D_w} |f(z)|^2 dV \leq \int_{U_{n,t}} |f(z)|^2 dV = |f|_t^2.$$

This shows that

$$|f(w)| = |\alpha_0| \leq \frac{c}{\sigma^d}\left(\int_{D_w} |f(z)|^2 dV\right)^{1/2} \leq \frac{c}{a_n^d(t-s)^d}|f|_t$$

for any $w \in U_{n,s}$ and $c := \sqrt{\frac{1}{C(0)}}$. When we apply the same argument to the derivatives of f and combine it with the Cauchy-Nagumo estimate, we find an estimate of the form

$$|Jf|_s = \max_{|I| \leq k} \sup_{w \in U_{n,s}} |\partial^I f(w)| \leq \frac{c'}{a_n^{d+k}(t-s)^{d+k}}|f|_t.$$

The proposition follows. □

5.3. **Kolmogorov spaces attached to the torus $\mathbb{T}$.**

5.3.1. As before, we denote by

$$T_t := \{z \in (\mathbb{C}^*)^n :\ e^{-t} < |z_i| < e^t\}$$

the neighbourhood of $\mathbb{T} \subset \mathbb{C}^n$. The space of holomorphic functions $\mathcal{O}(T_t)$ can be identified with the analytic Fourier series

$$f = \sum a_I z^I \in \mathbb{C}\{z, z^{-1}\}$$

for which

$$|a_I| = O(e^{-|I|t}).$$

The sets T_t can be seen as fibres of an open set T over $\mathbb{R}_{>0}$. When we restrict it to $B =]0, s_0]$, it is an a-Huygens set, for an appropriate a (which goes to 0 if $s_0 \longrightarrow \infty$). We will consider the corresponding Kolmogorov spaces

$$\mathcal{O}^k(T),\quad \mathcal{O}^b(T),\quad \mathcal{O}^h(T)$$

over $]0, s_0]$. Clearly, the elements of each of the underlying Banach spaces $\mathcal{O}^h(T_t)$ can be seen a special elements of $\mathbb{C}\{z, z^{-1}\}$.

5.3.2. *The Arnold-Moser lemma.*

Lemma. *Assume that a function $f \in \mathcal{O}^h(T)_s$ depends only on harmonics of degree $\geq m$, then for $s \leq t$ we have the estimate*

$$|f|_s \leq \left(\frac{e^s}{e^t}\right)^{m/2} |f|_t$$

Proof. The Cartesian product

$$S_t := S_t(1) \times S_t(2) \times \cdots \times S_t(2d)$$

of coordinate strips

$$S_t(j) := \{x_j = \xi_j + i\eta_j \mid 0 < \xi_j \leq 2\pi, \quad -t < \eta_j < t\}$$

parametrises the torus neighborhood T_t via the maps

$$x_j \mapsto z_j = e^{ix_j}.$$

We use the L^2-norm on $\mathcal{O}^h(T_t)$, obtained by integration of the pull-back of $f(z)\overline{f(z)}$ over the strip S_t. The monomials z^I then form an orthogonal basis. As in one variable we have

$$\int_{S_r} e^{inx}\overline{e^{inx}} d\xi d\eta = \int_{S_r} e^{-2n\eta} d\xi d\eta = \begin{cases} 2\pi \cdot 2r \text{ if } n = 0 \\ 2\pi \frac{\sinh(2nr)}{n} \text{ if } n \neq 0 \end{cases}$$

we find that for $I = (i_1, i_2, \ldots, i_{2d})$

$$|z^I|_t^2 = (2\pi)^{2d} \prod_{k=1}^{2d} \frac{\sinh(2i_k t)}{i_k}$$

By the Pythagorean theorem, for $f \in \mathcal{O}^h(T_t)$, we have:

$$\begin{aligned} |f|_s^2 &= \sum_{|I| \geq m} |a_I|^2 |z^I|_s^2 \\ &= \sum_{|I| \geq m} \left(|a_I| \frac{|z^I|_s}{|z^I|_t} |z^I|_t\right)^2 \\ &\leq \frac{\sinh(2ms)}{\sinh(2mt)} |f|_t^2. \end{aligned}$$

Here we used the two inequalities

$$\frac{|z^I|_s^2}{|z^I|_t^2} \leq \frac{\sinh(2|I|s)}{\sinh(2|I|t)} \leq \frac{\sinh(2ms)}{\sinh(2mt)}.$$

The first one is implied by the fact that for fixed positive numbers $a, b, \ldots, z > 0$ the function

$$x \mapsto \frac{\sinh(ax)\sinh(bx)\ldots\sinh(zx)}{\sinh((a+b+\ldots+z)x)}$$

is monotonous increasing in x. The second inequality follows because all I appearing in the sum have $|I| \geq m$ and the function

$$x \mapsto \frac{\sinh(ax)}{\sinh(bx)}$$

is monotonous increasing in x for $a \geq b > 0$. Finally, as $t > s$, one has also:

$$\frac{\sinh 2ms}{\sinh 2mt} = \frac{e^{ms}(1 - e^{-2ms})}{e^{mt}(1 - e^{-2mt})} \leq \frac{e^{ms}}{e^{mt}}$$

$\square$

5.3.3. *Differential forms.* We can define similarly Kolmogorov spaces of relative one-forms by putting

$$\Omega^{k,1}(T) := \mathcal{O}^k(T)\frac{dz_1}{z_1} \oplus \mathcal{O}^k(T)\frac{dz_2}{z_2} \oplus \ldots \oplus \mathcal{O}^k(T)\frac{dz_n}{z_n}$$

where we define the norm of a form $\alpha = \sum a_i \frac{dz_i}{z_i}$ to be

$$|\alpha|_t := \sup_{1 \le i \le n} \{|a_i|_t\}.$$

By setting

$$\Omega^{k,l}(T) := \wedge^l \Omega^{k,1}(T),$$

it can be extended to higher values of l, and there are similar versions for b, h instead of k.

5.3.4. *De Rham Complex.* By 5.2.2, the exterior derivative

$$d : \Omega^{k,l}(T) \longrightarrow \Omega^{k,l+1}(T)$$

is a 1-local morphism. From this, it follows that the space of closed forms

$$Z^{k,l}(T) \subset \Omega^{k,l}(T)$$

form a Kolmogorov subspace.

The 1-forms $\alpha_k = \iota_{\theta_k}\omega$ are De Rham dual to 1-cycles

$$\gamma_1, \ldots, \gamma_{2d} \in H_1(T, \mathbb{C}) = H_1(\mathbb{T}, \mathbb{C}).$$

On $Z^{k,1}$ we define linear forms

$$c_k : Z^{k,1}(T) \longrightarrow \mathbb{C}, \ \alpha \mapsto \int_{\gamma_k} \alpha.$$

The subspace $B^{k,1}(T) \subset Z^{k,1}(T)$ of exact 1-forms coincides with the forms with vanishing period integrals, so the 1-form

$$\beta = \alpha - \sum_{k=1}^{2d} c_k(\alpha)\alpha_k$$

belongs to the space $B^{k,1}$. Consider the path γ_z connecting $|z| := (|z_1|, |z_1|, \ldots, |z_n|)$ to $z \in T_s$ by changing only the arguments:

$$\gamma_z : [0,1] \longrightarrow T_s, \ \ t \mapsto (|z_1|e^{it\theta_1}, \ldots, |z_n|e^{it\theta_n})$$

where $\theta_i = arg(z_i) \in]0, 2\pi]$. Integration over γ_z defines a map of Kolmogorov spaces

$$\int : B^{k,1}(T) \longrightarrow \mathcal{O}^k(T), \ \beta \mapsto [z \mapsto \int_{\gamma_z} \beta],$$

as the exactness of the form β guarantees that the function $\int \beta$ is continuous. The obvious estimate

$$|\int_{\gamma_z} \beta| \le (2\pi e^s)^{2d} |\beta| \le (2\pi e^{s_0})^{2d} |\beta|$$

guarantees the boundedness of the map.

We have shown the

Proposition. *The maps c_k, $\int$ define a morphism of Kolmogorov spaces:*

$$Z^{k,1}(T) \longrightarrow \mathbb{C}^{2d} \oplus \mathcal{O}^k(T),$$

$$\alpha \mapsto (c_1(\alpha), \dots, c_{2d}(\alpha), \int \beta)$$

where

$$\beta := \alpha - \sum_{i=1}^{2d} c_i(\alpha)\alpha_i$$

5.3.5. *Symplectic vector fields.* The Kolmogorov space of relative vector fields is defined to be dual to $\Omega^{k,1}$:

$$\Theta^k(T) := \mathcal{O}^k(T)\theta_1 \oplus \mathcal{O}^k(T)\theta_2 \oplus \ldots \oplus \mathcal{O}^k(T)\theta_n,$$

with $\theta_j := z_j \partial_{z_j}$ By 5.2.2, there is an embedding of Kolmogorov spaces

$$\Theta^k(T) \hookrightarrow \mathcal{L}^1(\mathcal{O}^k(T), \mathcal{O}^k(T)) \ .$$

As the Lie bracket is a first order differential operator in the coefficients, 5.2.2 also implies a functorial analytic version of the adjoint representation:

Proposition. *The Lie bracket defines a 1-local map*

$$ad : \Theta^k(T) \longrightarrow \mathcal{L}^1(\Theta^k(T), \Theta^k(T)), \ X \mapsto [X, -].$$

By the standard symplectic duality isomorphism,

$$\Theta \longrightarrow \Omega^1, \quad X \mapsto \iota_X(\omega)$$

the closed forms correspond to symplectic vector fields and the exact forms to Hamiltonian fields.

The decomposition into exact and non-exact parts of a closed 1-form translates into the following statement

Proposition. *The decomposition of symplectic derivations into exact and non-exact parts:*

$$\mathcal{S}^k(T) \longrightarrow \mathcal{O}^k(T) \oplus \bigoplus_{j=1}^{2d} \mathbb{C}\theta_j,$$

$$X \mapsto \int \iota_X \omega + \sum_{j=1}^{2d} c_j(\iota_X \omega)\theta_j$$

is a locally bounded morphism of Kolmogorov spaces.

5.4. **Functional spaces involved in the iteration.**

5.4.1. *Frequencies.* We use the following local variant of arithmetic classes:

Definition. *For a fixed ν and falling sequence $a = (a_n)$ and $s_0 \in \mathbb{R}_{>0}$ we define a sets*

$$Z_{n,s} := Z_{n,s}(\nu, a, s_0) := \{\phi \in D_s^{2d} : \forall k \le n, \sigma(\nu + \phi)_k \ge a_k(s_0 - s)\}$$

It is readily checked that for $n \le m$ and $s \le t \le s_0$ one has:

$$Z_{m,s} \subset Z_{n,s}, \quad Z_{n,s} \subset Z_{n,t},$$

so that the $Z_{n,s}$ are fibres over a relative set

$$Z(a) \longrightarrow \overline{\mathbb{N}} \times]0, s_0].$$

Lemma. *The set $Z(a)$ is an a^*-Huygens set:*

$$Z_{n,s} + D_{a_n^*(t-s)} \subset Z_{n,t},$$

where

$$a_n^* := \frac{a_n}{2^n}$$

Proof. Let $\phi \in Z_{n,s}$ and take $x \in \mathbb{C}^{2d}$ satisfying

$$\|x\| \leq \frac{a_n}{2^n}(t-s).$$

For any $k \leq n$ and $\|J\| \leq 2^k$ we then have:

$$|(x,J)| \leq \|x\|\|J\| \leq \frac{a_n}{2^n}(t-s)\cdot 2^n = a_n(t-s) \leq a_k(t-s).$$

So we obtain

$$\begin{aligned} |(\nu+\phi+x,J)| &\geq |(\nu+\phi,J)| - |(x,J)| \\ &\geq a_k(s_0-s) - a_k(t-s) = a_k(s_0-t). \end{aligned}$$

This shows that $\phi + x \in Z_{n,t}$ and thus proves the lemma. □

Corollary. *Assume $a = (a_n)$ is a falling subquadratic sequence then any partial differential operator with coefficients in $\mathcal{O}^c(Z)$ defines a local map whose norm sequence is bounded by a rising subquadratic sequence with the same index.*

5.4.2. *Functional spaces with parameters.* We have considered the relative neighborhood of $\mathbb{T}$

$$T \longrightarrow \mathbb{R}_{>0}$$

with fibre

$$T_s = \{z \in (\mathbb{C}^*)^{2d} : \forall i,\ e^{-s} < |z_i| < e^s\}$$

For the formulation of the iteration we introduced detuning variables

$$\phi_1, \phi_2, \ldots, \phi_{2d}$$

and variables

$$\delta_1, \delta_2, \ldots, \delta_l$$

describing the perturbation of the symplectic form. We have to introduce appropriate neighbourhoods in the space with variables ϕ, δ, z.

Definition. *For fixed decreasing sequence a and s_0 we set*

$$W(a) \longrightarrow \overline{\mathbb{N}} \times \mathbb{R}_{>0},\ V(a) \longrightarrow \overline{\mathbb{N}} \times \mathbb{R}_{>0}$$

by putting

$$W(a) := Z(a) \times_{\mathbb{R}_{>0}} D^l \times_{\mathbb{R}_{>0}} T, \quad V(a) := Z(a) \times_{\mathbb{R}_{>0}} D^l$$

where $D \longrightarrow \mathbb{R}_{>0}$ is the relative unit polydisc.

The coordinates on $W_{n,t}$ are

$$\phi_1, \ldots, \phi_{2d},\ \delta_1, \ldots, \delta_l,\ z_1, \ldots, z_{2d},$$

and there are projection maps

$$W(a) \longrightarrow V(a), \quad (\phi, \delta, z) \mapsto (\phi, \delta).$$

The functional spaces we consider are the Arnold spaces

$$\mathcal{O}^k(W(a)), \mathcal{O}^h(W(a)),\ \mathcal{O}^k(V(a)),\ \mathcal{O}^h(V(a)),\ \Theta^k(W(a)),\ \mathcal{S}^k(W(a)),\ etc.$$

over $\overline{\mathbb{N}} \times \mathbb{R}_{>0}$.

5.4.3. *Model iteration.* Bruno sequences appear naturally in connection with quadratic iterations of the type

$$x_{n+1} = a_n x_n^2.$$

As solution one has

$$x_1 = a_0 x_0, \quad x_2 = a_1 x_1^2 = a_1 a_0^2 x_0^2 = (a_0 a_1^{1/2} x_0)^2$$

and is solved in general by

$$x_n = (a_0 a_1^{1/2} \dots a_n^{1/2^n} x_0)^{2^n},$$

so that the sequence (x_n) converges quadratically to 0 if $x_0 < 1/a_\Pi$ with $a_\Pi := \prod_{k=0}^{\infty} a_k^{1/2^k}$

Our aim is to investigate a slightly more general iteration of the form

$$x_{n+1} = \frac{1}{2}\left(a_n x_n^2 + b_n x_n\right).$$

Proposition. *Let $a = (a_n), a_n \geq 1$ and $b = (b_n)$ be sequences of positive numbers. Assume that for some $N \in \mathbb{N}$ one has*

$$(\star)_N \qquad n \geq N \implies b_n^2 a_n \leq b_{n+1}.$$

Then for any $0 < \varepsilon$ there exists δ such that $x_0 \leq \delta$ the real sequence

$$x_{n+1} = \frac{1}{2}\left(a_n x_n^2 + b_n x_n\right)$$

one has the estimate

$$x_n \leq \varepsilon\, b_n.$$

Proof. Without loss of generality, we may assume that $\varepsilon \leq 1$. Let us start by remarking if we have $x_n \leq \varepsilon b_n$ for some $n \geq N$, then we find, using $(\star)_N$, $a_n \geq 1$ and $\varepsilon^2 \leq \varepsilon$

$$x_{n+1} = \frac{1}{2}\left(a_n x_n^2 + b_n x_n\right) \leq \frac{1}{2}\left(a_n \varepsilon^2\, b_n^2 + \varepsilon\, b_n^2\right) \leq \varepsilon\, a_n b_n^2 \leq \varepsilon\, b_{n+1}$$

so that if $x_N \leq \varepsilon b_N$ holds, then it holds for all $n \geq N$.

The map $x_0 \mapsto x_N$ is a polynomial map hence continuous thus for δ small enough one has $x_N < \varepsilon b_N$. This concludes the proof of the proposition. □

We will use the above Proposition for subquadratic sequences $a = (a_n)$ and $b = (b_n)$ of the form

$$a_n = A e^{B\alpha^n}, \quad b_n = C e^{-D\beta^n}, \ \beta > \alpha \text{ and } \beta \in]1, 2[$$

and where $A, B, C, D \in \mathbb{R}_{>0}$. Condition $(\star)_N$ is obviously fulfilled.

6. Analytic Stability Theorem

6.1. **Final preparations.** The iteration scheme in the ring R for the normal form of §3 was based on maps

$$j_n : \mathcal{S}(R) \longrightarrow \mathcal{P}(R)),$$

which were defined in terms of decomposition and truncation. It is not difficult to lift these maps to the functional analytic level.

We fix a sequence a and consider the Arnold-space $\mathcal{S}^k(W(a))$. Its components $\mathcal{S}^k(W(a))_n$ are Kolmogorov spaces of symplectic vector fields X on sets $W_{n,t}$. For these we first make a *decomposition* as in 5.3.5:

$$X = \{-, f\} + \sum_{i=1} c_i \theta_i,$$

with

$$f \in \mathcal{O}^k(W_{n,t}), \quad c_i \in \mathcal{O}^k(V_{n,t})$$

and define the A-truncation as before:

$$[X]_A := \{-, [f]_A\} + \sum_{i=1}^{2d} [c_i]_A \theta_i \in \mathcal{S}^k(W_{n,s}).$$

where $s < t$. In the iteration, we will use maps

$$\sigma(n,t,s) : \mathcal{S}^k(W_{n,t}) \longrightarrow \mathcal{S}^k(W_{n,s})$$
$$j(n,t,s) : \mathcal{S}^k(W_{n,t}) \longrightarrow \mathcal{L}^1(\Theta(W), \Theta(W))_{n,s}$$

by setting :

$$\sigma(n,t,s)(X) := [X]^{2^n} = \{-, [f]^{2^n}\} + \sum_{i=1}^{2d} [c_i]^{2^n} \theta_i$$
$$j(n,t,s)(X) := ad(\{-, [f]^{2^n} \star H\} + \sum_{i=1} [c_i]^{2^n} \partial_{\phi_i}).$$

where H is the resolvente and ad is the adjoint action, which embeds the space $\mathcal{Q}(W)$ inside $\mathcal{L}^1(\Theta(W), \Theta(W))$ via a local map (see 5.3.5).

It is easy to see that norm sequence of $|j_n|$ is subquadratic and satisfies

$$\mathrm{ord}\,(|j_n|) \leq \mathrm{ord}\,(a).$$

By the Cauchy-Nagumo lemma (see 5.2.2), the norm sequences $|\partial_{\phi_i}|$, as well as the non-exact term, have the same property:

$$\mathrm{ord}\,(|\partial_{\phi_i}|) \leq \mathrm{ord}\,(a).$$

Due to local equivalence taking the convolution with H has again the same property. The Poisson bracket being a biderivation, due to the Cauchy-Nagumo lemma, the exact term is a local operator with bounded norm.

These maps form compatible systems and combine to form local maps (we take the same index for locality p to simplify further estimates):

$$\sigma \in \mathcal{L}^p(\mathcal{S}^k(W(a)), \mathcal{S}^k(W(a)))$$
$$j \in \mathcal{L}^p(\mathcal{S}^k(W(a)), \mathcal{L}^1(\Theta^k(W(a)), \Theta^k(W(a))))$$

If the sequence a that goes in the definition of $W(a)$ is subquadratic, then $W(a)$ is an a^*-Huygens set, the norm sequence of the map j is subquadratic with the index smaller than that of a, as it is the composition of such maps.

6.2. **Pulling back.** We choose $\rho = (e^{-\beta^n}) \in \mathbb{S}^-$ with order $\beta \in]1,2[$ higher than that of (a_n) and therefore of $|j_n|$. Then we define a sequence (s_n) indexed by half-integer by putting

$$s_{n+1/2} := \rho_n^{1/2^{n+1}} s_n, \quad s_0 = t$$

As ρ is subquadratic, the sequence (s_n) converges to a positive limit $s_\infty(\rho)$ and

$$s_n - s_{n+1/2} \sim \frac{1}{2}\left(\frac{\beta}{2}\right)^n s_n \sim \frac{1}{2}\left(\frac{\beta}{2}\right)^n s_\infty$$

Pulling-back an Arnold space

$$E \longrightarrow \overline{\mathbb{N}} \times \mathbb{R}_{>0}$$

via the map

$$i : \frac{1}{2}\overline{\mathbb{N}} \longrightarrow \overline{\mathbb{N}} \times \mathbb{R}_{>0}, \; n \mapsto (\lfloor n \rfloor, s_n)$$

defines a Kolmogorov space that we denote by $\rho^* E$ over $\frac{1}{2}\overline{\mathbb{N}}$. Pulling back the maps j and σ, we obtain maps whose norms sequence belong respectively to $\mathbb{S}^+$ and $\mathbb{S}^-$. From the Arnold-Moser lemma (5.3.2), we deduce the estimate:

$$|\sigma(n,t,s)| \leq \left(\frac{e^s}{e^t}\right)^{2^{n-1}} \frac{C}{a_n^p (t-s)^p}$$

we deduce that:

$$|\sigma(n,s_n,s_{n+1/2})| \leq \left(\frac{e^{s_{n+1/2}}}{e^{s_n}}\right)^{2^{n-1}} \frac{C}{a_n^p (s_n - s_{n+1/2})^p} \sim \frac{C 2^{(n+1)p} e^{-\beta^n s_\infty/4}}{a_n^p s_\infty^p \beta_n^p}$$

Therefore the norm sequence $|\sigma(n,s_n,s_{n+1/2})|$ is a falling subquadratic sequence of order β.

6.3. **The Convergence Theorem.**

Theorem. *Consider a symplectic vector field on the complex torus*

$$\mathcal{V} = \sum_{i=1}^{n} \nu_i z_i \partial_{z_i}$$

and assume that $\sigma(\nu) \in \mathbb{S}^-(\alpha), \alpha \in]1,2[$. Consider a subquadratic sequences $a \in \mathbb{S}^-(\alpha)$ such that $a \leq \sigma(\nu)$ and define $\rho = (e^{-\beta^n})$ with $\beta \in]\alpha, 2[$. Then for any $k, \varepsilon > 0$, there exists $\delta > 0$ such that for $|S_0| \leq \delta$: there is an iteration in $\rho^* \mathcal{S}^k(W(a))$ defined by

$$S_{n+1} = f_*(j_n(S_n))\mathcal{V} + e^{-[j_n(S_n),-]}\sigma_n(S_n)$$

and it satisfies the estimate $|S_n| < \varepsilon e^{-\beta^n}$.

As an immediate corollary, we get that

Corollary. *For any $k > 0$, there exists δ, such that for any S satisfying $|S| < \delta$ there is a symplectic morphism*

$$\psi : \mathcal{O}^k(W)_{0,s_0} \longrightarrow \mathcal{O}^k(W)_{\infty,s_\infty}$$

such that

$$\psi_*(\mathcal{V} + S) = \mathcal{V}$$

6.4. **Proof of the theorem.**

The analytic series

$$f(z) = e^{-z}(1+z) - 1 \in \mathbb{C}\{z\}$$

is the Borel transform of

$$g(z) = -\frac{z^2}{(1+z)^2} \in z^2\mathbb{C}\{z\}.$$

which has radius of convergence equal to 1 and, choosing $r = 1/2 < 1$, we get that:

$$\left|\frac{z^2}{(1+z)^2}\right| \leq \frac{|z|^2}{1-r^2} < 2|z|^2$$

for $|z| \leq r$. Therefore assuming

$$|S_n| \leq \frac{s_{n+1/2} - s_{n+1}}{2\|j_n\|}$$

by 4.2.5, we may deduce that:

$$\begin{aligned} |f_*(j_n(S_n))| &\leq |g| \left(\frac{|j_n(S_n)|}{(s_{n+1/2} - s_{n+1})}\right) \leq 2\frac{|j_n(S_n)|^2}{(s_{n+1/2} - s_{n+1})^2} \\ &\leq 2e^{2p} \frac{\|j_n\|^2 |S_n|^2}{(s_n - s_{n+1/2})^{2p}(s_{n+1/2} - s_{n+1})^2} \end{aligned}$$

The sequence a' with terms

$$a'_n := 2e^{2p} \frac{\|j_n\|^2}{(s_n - s_{n+1/2})^{2p}(s_{n+1/2} - s_{n+1})^2}$$

is subquadratic with order α. Therefore the right-hand side of our estimate is of the form $a'_n|S_n|^2$ with $a' = (a'_n) \in \mathbb{S}^+(\alpha)$.

The power-series:

$$e^{-z} \in \mathbb{C}\{z\}$$

is the Borel transform of

$$\frac{1}{1+z} \in \mathbb{C}\{z\},$$

therefore the remainder term of the iteration satisfies the estimate:

$$|e^{-[j_n(S_n),-]}\sigma_n(S_n)| < 2\frac{\|j_n\|\,\|\sigma_n\|\,|S_n|}{(s_{n+1/2} - s_{n+1})^{p+1}}.$$

The sequence with terms

$$b_n = 2\frac{\|j_n\|\,\|\sigma_n\|}{(s_{n+1/2} - s_{n+1})^{p+1}}$$

is now subquadratic with order $\beta > \alpha$. Combining the two estimates we obtain an estimate of the form:

$$|S_{n+1}| < a'_n|S_n|^2 + b_n|S_n|.$$

where $a' \in \mathbb{S}^+$ and $b \in \mathbb{S}^-$ are subquadratic with orders $\alpha < \beta$. The sequence of terms

$$a''_n = \frac{s_{n+1/2} - s_{n+1}}{2\|j_n\|}$$

is a falling subquadratic sequence with order α. Thus we may choose ε such that for any $n \in \mathbb{N}$:

$$\varepsilon b_n \le a''_n.$$

The sequence (x_k) defined by

$$x_0 = |S_0|$$
$$x_{n+1} = a'_n x_n^2 + b_n x_n$$

majorates $|S_n|$ and, according to 5.4.3, there exists δ such that for $x_0 < \delta$ the sequence is bounded by εb. In particular, the iteration (S_n) is well defined and smaller than εb. This concludes the proof of the convergence theorem.

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