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REMARKS ON SEMI-SIMPLICITY OF ALEXANDER MODULES

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Abstract. We discuss examples of smooth quasi-projective manifolds with non-reduced Alexander modules, giving a non-semisimple Alexander module in the one variable case and prove a result giving sufficient conditions for semi-simplicity.

1. Introduction

Let X be a smooth quasi-projective variety such that the fundamental group $\pi_1(X)$ admits a surjection $\rho : \pi_1(X) \to \mathbb{Z}$ onto an infinite cyclic group. The i-th Alexander module of the pair (X, ρ) is defined as the homology group $H_i(\tilde{X}_\rho, \mathbb{Q})$ with closed support of the infinite cyclic cover $\tilde{X}_\rho$ corresponding to the kernel of ρ. This vector space, may or may not be finite-dimensional but it is a finitely generated $\mathbb{Q}[t, t^{-1}]$-module with the module structure given by the action of a generator t of the infinite cyclic target of ρ as the deck transformation of the cover X_ρ. In the case $i = 1$, the Alexander module is the abelianized kernel of ρ (tensored with $\mathbb{Q}$). If the Alexander module is a finite-dimensional $\mathbb{Q}$-vector space it is a torsion $\mathbb{Q}[t, t^{-1}]$-module and its order, well defined up to a unit of $\mathbb{Q}[t, t^{-1}]$, is called the Alexander polynomial. If $H_i(\tilde{X}_\rho)$ is infinite-dimensional, then the $\mathbb{Q}[t, t^{-1}]$-order of its $\mathbb{Q}[t, t^{-1}]$-torsion submodule is also a useful invariant.

If X is a complement to a divisor on a smooth projective surface, some assumptions of ampleness of irreducible components of this divisor and properties of ρ imply that the first Alexander module is a torsion module (cf.[9] Theorem 3.3 or Theorem 1.2 below for precise conditions). Again with certain ampleness assumptions on irreducible components of $\bar{X} \setminus X$, the Alexander polynomial can be calculated in terms of classes of these components in the Neron-Severi group and the superabundances of the linear systems defined by the singularities of the components.

In [7] it was pointed out that in the case when X is a complement to an irreducible plane curve $C \subset \mathbb{C}^2$, having singularities with semi-simple monodromy only and transversal to the line at infinity, one has a canonical surjection: $\rho : \pi_1(\mathbb{C}^2 \setminus C) \to H_1(\mathbb{C}^2 \setminus C, \mathbb{Z}) = \mathbb{Z}$ and the Alexander module (over $\mathbb{R}$) is isomorphic to a direct sum $\oplus \mathbb{R}[t, t^{-1}]/(\Delta_\kappa)$ (here Δ_κ are the polynomials defined in terms of local type of singularities and the number of summands given by the superabundances of the linear systems associated with the singularities). In particular, the first Alexander module of the complement is a semi-simple $\mathbb{R}[t, t^{-1}]$-module. The semi-simplicity of the torsion parts of the Alexander modules with $i \geq 1$ for the complements to hypersurfaces in affine space was systematically studied in [12] and continued in [4], [10], [13]. Semi-simplicity of the first Alexander module for general Kähler groups was shown in [1]. The Alexander invariants of solvable and nilpotent quotients of the fundamental groups of the complements to arrangements of hyperplanes were considered in [11].

A detailed study of the Hodge structures on the torsion parts of the Alexander modules of a wider class of quasi-projective manifolds was carried out in [5]. The question of semi-simplicity also was considered in this paper and it was shown that the torsion parts of the Alexander modules are semi-simple if X admits a proper holomorphic map $X \to \mathbb{C}^*$ and the surjection

ρ is the composition $\pi_1(X) \to \pi_1(\mathbb{C}^*) = \mathbb{Z}$. [5] gives also a Hodge theoretical condition for semi-simplicity. In this paper also the question was raised if non-semisimple Alexander modules exist.

In this note we prove two Theorems concerning the semi-simplicity property. Theorem 1.1 considers a multi-variable analog of non-semi-simplicity, i.e. the Alexander modules $H_i(\tilde{X}, \mathbb{C})$ over the group ring $\mathbb{C}[\mathbb{Z}^r]$ corresponding to a surjection of $\pi_1(X)$ onto $\mathbb{Z}^r$, having an annihilator which is not a radical ideal. A natural invariant of $H_i(\tilde{X}_r, \mathbb{C})$ is the characteristic subscheme of the torus *Spec* $\mathbb{C}[\mathbb{Z}^r]$ which is the affine subscheme corresponding to the annihilator in the ring of Laurent polynomials *Spec* $\mathbb{C}[\mathbb{Z}^r]$ of the i-th Alexander module. The corresponding reduced subscheme is the characteristic variety (cf. [8]) and is an analog of the Alexander polynomial in the one variable case. Calculations of characteristic varieties, not relying on presentations of the fundamental group, are based on their relation with the homology of finite abelian covers which use only the reduced part of the characteristic scheme (cf. [9]). Hence in the cases when the characteristic scheme is different from the characteristic variety, the difference cannot be detected by the Betti numbers of abelian covers. Since a module over $\mathbb{Q}[t, t^{-1}]$ is semisimple if and only if its support is reduced, the cases when characteristic schemes are different from characteristic varieties provide a multivariable counterpart of non-semi-simplicity.

Theorem 1.1 gives examples of quasi-projective varieties which have contractible universal covers, the 2-step nilpotent groups as their fundamental groups and have non-radical annihilator of the Alexander modules. Construction of [3] and [2] shows that there exist even Kähler groups with such a property. More specifically, a lattice in a $(2k+1)$-dimensional Heisenberg Lie group is Kähler if and only if $k \geq 4$ (cf. [3]) and the groups considered in Theorem 1.1 are of such type. Note that in examples of Theorem 1.1 with a non-semi-simple Alexander $\mathbb{C}[t, t^{-1}]$-module, unlike in the cases considered in [5], the surjection of the fundamental group onto $\mathbb{Z}$ is not induced by a holomorphic surjection onto $\mathbb{C}^*$. Existence of the latter is additional strong constraint on X.

Theorem 1.2 gives sufficient conditions for semi-simplicity of one variable Alexander modules. This is a generalization of the results in [4] and the argument is close to the one used in the proof of divisibility theorems for Alexander polynomials (cf. [9] for a recent account of those). The specific statements are as follows.

Theorem 1.1. *Let A be a polarized abelian variety of dimension n and $\omega \in \Lambda^2 H^1(A, \mathbb{Z})$ be a polarization. Let X_n^ω be the complement to the zero section in a corresponding positive definite line bundle. Then $\pi_1(X_n^\omega)$ is a subgroup of finite index in the Heisenberg group H_n with presentation:*

$$\{x_i, y_i, z | [x_i, x_j] = [y_i, y_j] = [x_i, z] = [y_i, z] = 1, [x_i, y_i] = z, \quad \forall i, j, \tag{1}$$
$$[x_i, y_j] = 1 (i \neq j) i, j = 1, .., n\}$$

and $\pi_1(X_n^\omega) = H_n$, if polarization is principal. Let $\rho : H_n \to \mathbb{Z}^n$ be the surjection with the kernel being the normal closure of the subgroup of H_n generated by $y_1, .., y_n, z$. Then the annihilator of the corresponding to ρ Alexander module $H_i(\tilde{X}_n^\omega, \mathbb{C})), i \geq 1$ is $\mathfrak{m}^2$ where $\mathfrak{m}$ is the maximal ideal of the identity of the torus Spec $\mathbb{C}[\mathbb{Z}^n]$. In particular for $n = 1$ one obtains quasi-projective groups with corresponding Alexander module being not semi-simple with the action of a generator of $\mathbb{Z}$ having one 2×2 Jordan block.

Theorem 1.2. *Let X be a smooth projective variety of dimension greater than one and let $D = D_1 \cup D_0$ be a reduced divisor on X. Assume that*

(i) D_0 is irreducible, smooth and ample.

(ii) D_0 intersects D_1 transversally (in particular only at smooth points of the latter).

(iii) One has surjection $\rho : \pi_1(X \setminus D) \to \mathbb{Z}$ which takes the meridian of D_0 to non-zero.

Then $H_i(\widetilde{(X \setminus D)}_\rho, \mathbb{C})$, where $\widetilde{(X \setminus D)}_\rho$ is the cyclic cover corresponding to $Ker\rho$, is a semisimple $\mathbb{C}[t,t^{-1}]$-module for $0 \le i \le \dim X - 1$.

These results are proven in the next section where also illustrating examples are given. I am grateful to L. Maxim for his comments on an earlier version of this note, to the anonymous referee for detailed comments and the editors for pointing out several typos.

2. Proofs of the Theorems 1.1 and 1.2

2.1. **Proof of Theorem 1.1.** The description of $\pi_1(X_n^\omega)$ follows immediately from the exact sequence of a locally trivial $\mathbb{C}^*$-fibration:

$$0 \to \mathbb{Z} \to \pi_1(X_n^\omega) \to \mathbb{Z}^{2n} \to 0 \tag{2}$$

since the class of this extension can be identified with the symplectic form corresponding to polarization (cf. [3]; in fact sect. 5 of this paper shows that for appropriate polarization and $n \ge 4$, a generic linear section of the total space of such a bundle gives a projective surface with fundamental group being the Heisenberg group (2); cf. [2] for another version of the argument of projectivity of these groups).

It follows from (1) that the subgroup of H_n generated by $y_1, ..., y_n, z$ is abelian and normal. Denote by $t_1, ..., t_n$ generators of the quotient corresponding to $x_1, .., x_n$. Then we have:

$$t_i y_j = y_j \quad (i \ne j), \quad t_i y_i = y_i + z \quad 1 \le i \le n. \tag{3}$$

In the case when polarization is not principal, the second part in (3) is given by $t_i y_i = y_i + \alpha_i z$ (an explicit form of α_i can be obtained in terms of elementary divisors of the symplectic form corresponding to polarization, using the matrix form of the Heisenberg group, cf. [3], Sect. 5). In particular the Alexander module is generated by $y_1, .., y_n$ satisfying the relations:

$$\alpha_j(t_i - 1)y_i = \alpha_i(t_j - 1)y_j \qquad (t_i - 1)y_j = 0, \ (i \ne j). \tag{4}$$

Therefore, the annihilator coincides with $\mathfrak{m}^2$. Finally, it follows that the abelian cover $\tilde{X}_n^\omega$ is homotopy equivalent to a torus and hence higher Alexander modules are the exterior powers of the first one. The claim follows.

2.2. **Proof of Theorem 1.2.** Note that assumptions (i),(ii),(iii) imply that the first Alexander module is a torsion (cf. [9] Theorem 3.3 [1]). Let $T(D_0)$ be a small regular neighborhood of D_0. Let D_0' be a small deformation of D_0 which is a smooth member of the linear system $L(D_0)$ and which is a smooth closed submanifold of $T(D_0)$ transversal to all components of D_1 at smooth points of the latter. It follows from the (stratified) Lefschetz hyperplane section theorem for quasi-projective varieties that the spaces $X \setminus (D_0 \cup D_1)$ and $D_0' \setminus (D_0' \cap (D_0 \cup D_1))$ have the same $(n-2)$-homotopy type i.e. $X \setminus (D_0 \cup D_1)$ is homotopy equivalent to a CW-complex which is a union of a CW-complex homotopy equivalent to $D_0' \setminus D_0' \cap (D_0 \cup D_1)$ and cells of dimension $i \ge n-1$. Hence the infinite cyclic covers of both these spaces, corresponding to $\rho : \pi_1(X \setminus D) \to \mathbb{Z}$ and surjection $\rho_{D_O'} : \pi_1(D_0' \setminus (D_0' \cap (D_0 \cup D_1))) \to \mathbb{Z}$ induced by embedding $D_0' \setminus D_0' \cap (D_0 \cup D_1) \to X \setminus D_0 \cup D_1$ and ρ, also have cellular decompositions identical up to dimension $n-2$. Hence one has isomorphism of $\mathbb{C}[t,t^{-1}]$-modules up to dimension $n-2$ and surjection for $i = n-1$:

$$H_i([D_0' \setminus \widetilde{D_0' \cap (D_0} \cup D_1)]_{\rho_{D_0}}) \to H_i([X \setminus \widetilde{D_0 \cup} D_1]_\rho, \mathbb{C})) \tag{5}$$

where $[\dot{}]_\rho$ denotes the infinite cyclic covers corresponding to the surjection onto $\mathbb{Z}$ indicated by the subscript.

[1] in fact the proof of semi-simplicity below is close to the one used in this reference to show this property.

Next notice that $T(D_0)\setminus((D_0\cup D_1)\cap T(D_0))$ is diffeomorphic to a locally trivial fibration

$$T(D_0)\setminus((D_0\cup D_1)\cap T(D_0))\stackrel{D^*}{\to} D_0\setminus(D_0\cap D_1)$$

over $D_0\setminus(D_0\cap D_1)$ having a punctured disk D^* as a fiber. Let $d=\rho(\gamma_{D_0})\in\mathbb{Z}$ where $\gamma_{D_0}\in\pi_1(T(D_0)\setminus((D_0\cup D_1)\cap T(D_0)))$ is the class meridian in the component D_0. The sequence of the fundamental groups induced by this fibration and surjections on the cyclic groups induced by the homomorphism $\rho_{T(D_0)}$, which is the composition of the homomorphism of the fundamental groups induced by embedding of open manifolds and the homomorphism ρ, gives the diagram:

$$\begin{array}{ccccccc} \mathbb{Z} & \to & \pi_1(T(D_0)\setminus((D_0\cup D_1)\cap T(D_0))) & \to & \pi_1(D_0\setminus(D_0\cap D_1)) & \to & 0\\ \downarrow & & \rho_{T(D_0)}\downarrow & & \rho'\downarrow & & \\ \mathbb{Z} & \to & \mathbb{Z} & \to & \mathbb{Z}/d\mathbb{Z} & \to & 0 \end{array} \tag{6}$$

The homomorphism ρ' here is induced by $\rho_{T(D_0)}$. It follows from Lemma 3.1 of [6] that the infinite cyclic cover

$$[T(D_0)\setminus((\widetilde{D_0\cup D_1})\cap T(D_0))]_{\rho_{T(D_0)}}$$

fibers over the d-fold cyclic cover of $D_0\setminus(D_0\cap D_1)$ corresponding to surjection ρ' in the last column of the above diagram. This fibration has contractible fiber $\tilde{\mathbb{C}}^*$ and the action of the deck transformation on $[T(D_0)\setminus((\widetilde{D_0\cup D_1})\cap T(D_0))]_{\rho_{T(D_0)}}$ factors through the action of a finite cyclic group. Hence the Alexander modules $H_i([T(D_0)\setminus((\widetilde{D_0\cup D_1})\cap T(D_0))]_{\rho_{T(D_0)}}$ are semisimple.

Finally, consider the diagram:

$$\begin{array}{ccc} H_i([D_0'\setminus \widetilde{D_0'\cap(D_0}\cup D_1)]_{\rho_{D_0'}}) & \to & H_i([T(D_0)\setminus((\widetilde{D_0\cup D_1})\cap T(D_0))]_{\rho_{T(D_0)}}\\ & \searrow & \downarrow\\ & & H_i([\widetilde{X\setminus D}]_\rho) \end{array} \tag{7}$$

in which the homomorphism $\rho_{D_0'}:\pi_1(D_0'\setminus D_0'\cap(D_0\cup D_1))\to\mathbb{Z}$ used to construct the covering space is the composition of the map induced by embedding and ρ and all the arrows are induced by respective embeddings. Since the map (5) is an isomorphism for $0\le i<n-1$ and surjective for $i=n-1$, it follows that the vertical map is surjective for $i\le n-1$. Hence $H_i([\widetilde{X\setminus D}]_\rho)$, being the quotient of a semisimple module, in the same range of i is semisimple as well. $\square$

3. Miscellaneous comments.

3.1. Non-semisimple Alexander modules and Heisenberg groups.

Proposition 3.1. *Let $\rho:G\to\mathbb{Z}$ be a surjection such that the corresponding $\mathbb{C}[t,t^{-1}]$ Alexander module $G'/G''\otimes\mathbb{C}$ has a 2×2 Jordan block corresponding to the eigenvalue 1 in a basis belonging to the lattice $(G'/G'')/Torsion\subset G'/G''\otimes\mathbb{C}$. Then G has as a quotient a subgroup of finite index in the Heisenberg group*

$$\{x,y,z|[x,z]=[y,z]=1,[x,y]=z\}. \tag{8}$$

Proof. Let $x\in G$ be such that $\rho(x)$ is the generator t of the multiplicative infinite cyclic group and $y_1,...,y_N\in G'$ be representatives of a basis of G'/G'', such that the first two elements of this basis $\bar{y}_1,\bar{y}_2\in G'/G''$ form the 2×2 Jordan block i.e. the action of t has the form $t\bar{y}_1=\bar{y}_1, t\bar{y}_2=\bar{y}_2+\alpha\bar{y}_2$. Let K be the subgroup of G' generated by representatives $y_3,...,y_N$, of remaining elements of $\mathbb{Z}$- basis of G'/G''. It follows that G/K is isomorphic to the group of 3×3 unipotent matrices over $\mathbb{Z}$ and hence is a subgroup of finite index in the group (8). $\square$

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3.2. **A group with non-semi-simple Alexander module.** While this paper does not contain examples of quasi-projective groups having nilpotent quotients with rank of the center being greater than 1, one can construct a 2-step nilpotent group obtained as extension (2) in which one replaces $\mathbb{Z}$ by a free group $\mathbb{Z}^k$. If one takes as cocycle in $H^2(\mathbb{Z}^{2n}, \mathbb{Z}^k) = \oplus_1^k H^2(\mathbb{Z}^{2n}, \mathbb{Z})$ and the collection of k integer 2-forms of rank 2, with null spaces belonging to a codimension 1 subspace (i.e. the forms $\omega \wedge \eta_i, i = 1, ..., k$ with ω, η_i being 1-forms with η_i being linearly independent), it is easy to check that such a group has the Alexander module corresponding to k Jordan blocks of size 2×2. Selecting a basis $(x_1 y_1, ...x_n, y_n)$ in $\mathbb{Z}^{2n}$ such that

$$\omega(x_1) = 1, \omega(x_i) = 0, i > 1, \omega(y_i) = 0, i \geq 1, \eta_i(y_i) = 1, \eta_i(y_j) = 0, i \neq j, \eta(x_i) = 0, \text{ for all } i$$

for the central extension:

$$0 \to \mathbb{Z}^k \to G \to \mathbb{Z}^{2n} \to 0 \tag{9}$$

corresponding to the cocycle $(\omega \wedge \eta_1, ..., \omega \wedge \eta_k) \in H^2(\mathbb{Z}^{2n}, \mathbb{Z}^k) = (\Lambda^2(\mathbb{Z}^n))^k$ one obtains the group with presentation:

$$\begin{gathered}[x_1, y_i] = z_i, i = 1, ..., k, [x_1, y_j] = 1, j > k, [x_i, y_j] = 1, i \geq 2, \forall j, \\ [x_i, x_j] = [y_i, y_j] = [x_i, z_s] = [y_i, z_s] = 1, \forall i, j, s.\end{gathered} \tag{10}$$

The abelian subgroup H generated by $(x_2, ..., x_n, y_1, .., y_n, z_1, .., z_k)$ is such that G/H is infinite cyclic and is generated by the class of x_1. The action of the generator of the corresponding Alexander module is given by

$$ty_i = y_i + z_i, tz_i = 0, ty_i = 0, i > k, tx_i = 0 \tag{11}$$

i.e. the Alexander module is isomorphic to

$$(\mathbb{C}[t, t^{-1}]/(t-1)^2)^k \oplus \mathbb{C}[t, t^{-1}]/(t-1)^{2n-k-1}. \tag{12}$$

It is not clear, at the moment of this writing, if such a group is quasi-projective.

3.3. **Example.** *A semi-simple Alexander module of a Zariski open subset of a general simply connected smooth projective surface.* This is the Alexander modules version of the example in section 4.5 of [9]. Let X be a smooth simply connected projective surface and D_0 be a smooth ample divisor with the class $[D_0] \in H^2(X, \mathbb{Z})$. Let $p, q \in \mathbb{Z}_{>0}$ be coprime and such that there exist sections $s_1 \in H^0(X, \mathcal{O}_X(pD_0)$ and $s_2 \in H^0(X, \mathcal{O}_X(qD_0)$ which are smooth, transversal to each other, and are also transversal to D_0. Let D be the zero set of the section $s_1^q + s_2^p \in H^0(X, pqD_0)$. It follows from [9] (1.6), that $H_1(X \setminus D \cup D_0, \mathbb{Z})$ has rank 1 with torsion group having order l, which is the greatest common divisor of the integers $([D_0], E), E \in H^2(X, \mathbb{Z})$. Surjection of $\pi_1(X \setminus D \cup D_0)$ onto $\mathbb{Z}$, given by abelianization followed by taking the quotient by the torsion subgroup allows us to define the corresponding Alexander module. D has pqD_0^2 singularities each having the local equation $x^p + y^q = 0$ at the subscheme $SingD$ of X which is the complete intersection $s_1 = s_2 = 0$. Calculation as in [9] section 4.6 shows that the sheaf of ideals of quasi-adjunction for $D \cup D_0$ is the ideal of the zero-dimensional reduced scheme $Sing(D)$ and that the superabundance of the linear system $H^0(X, K_X \otimes (p+q)D_0 \otimes \mathcal{I}_{Sing(D)})$ is equal to 1. It follows from Theorem 1.2, that the ideal of reduced support of the Alexander module is the annihilator of the Alexander module:

$$[\pi_1'(X \setminus D \cup D')/\pi_1''(X \setminus D \cup D')] \otimes \mathbb{Q} = \mathbb{Q}[t, t^{-1}]/(\frac{(t^{pq}-1)(t-1)}{(t^p-1)(t^q-1)}). \tag{13}$$

In the case when $X = \mathbb{P}^2, D_0 = \mathbb{P}^1$, the fundamental group $\mathbb{P}^2 \setminus D$ (D can be taken to be a curve with equation $(x^p + y^p)^q + (x^q + z^q)^p = 0$) is the free product of $\mathbb{Z}_p * \mathbb{Z}_q$ which implies the isomorphism (13). It would be interesting to calculate the fundamental groups $\pi_1(X \setminus (D \cup D_0))$ for other simply connected surfaces.

3.4. **Example.** *One variable Alexander modules of Campana-Carlson-Toledo groups.* Let us consider one variable Alexander modules of groups discussed in Theorem 1.1. The subgroup K generated by $z, y_i, i = 1, .., n, x_2, ..., x_n$ is normal and $\pi_1(X_n)/K = \mathbb{Z}$. Abelianization of K is a free abelian group of rank $2n - 1$ and the action of x_1 on it by conjugation for $n > 1$ is trivial. Hence the Alexander module is just $[\mathbb{Q}[t, t^{-1}]/(t - 1)]^{2n-1}$. As shown in Theorem 1.1, in the case $n = 1$, one obtains a non-semisimple module $\mathbb{Q}[t, t^{-1}]/(t-1)^2$. Calculation in this case was made in [15].

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