PURELY INSEPARABLE COVERINGS OF RATIONAL DOUBLE POINTS IN POSITIVE CHARACTERISTIC

YUYA MATSUMOTO

ABSTRACT. We classify purely inseparable morphisms of degree p between rational double points (RDPs) in characteristic $p > 0$. Using such morphisms, we refine a result of Artin that any RDP admits a finite smooth covering.

1. INTRODUCTION

Rational double points, RDPs for short, are perhaps the most basic singularities in dimension 2. They are precisely the canonical singularities in dimension 2 and have many other characterizations. In characteristic 0, they are characterized as the quotient singularities by the linear actions of finite subgroups of SL_2, and in particular they admit finite coverings by regular local rings. However, some of the quotient presentations fail in positive characteristic, even if we take finite group schemes into account. Nevertheless, Artin [Art77] determined étale fundamental groups of RDPs and showed the existence of finite coverings (possibly ramified and possibly inseparable) by regular local rings.

In this paper we study finite purely inseparable morphisms $\operatorname{Spec} B \to \operatorname{Spec} B'$ of degree p between RDPs, and give a classification (Theorem 1.1). As an application, we show (Theorem 4.1) that any RDP $\operatorname{Spec} B$ admits a finite covering of the form $\operatorname{Spec} k[[x, y]] \xrightarrow{f} \operatorname{Spec} \tilde{B} \xrightarrow{g} \operatorname{Spec} B$, where f is a composite of purely inseparable morphisms of degree p and g is the universal covering outside the closed point. This refines the above-mentioned result of Artin in the respect that we use a restricted class of morphisms. We moreover show that, in all but one cases, $\tilde{B}$ and all the intermediate steps of f can be taken to be RDPs.

Our classification will be used in a subsequent paper [LMM21] to show that certain RDPs in characteristic 2 and 3 are not quotient singularities by any finite group schemes. See loc. cit. for the precise definition of the quotient singularities by finite group schemes.

To state the main theorem, we first introduce some terminology and conditions.

We work over an algebraically closed field k of characteristic $p > 0$. An *at most RDP* is a complete local k-algebra of dimension 2 that is either an RDP or smooth.

Let $\pi \colon \operatorname{Spec} B \to \operatorname{Spec} B'$ be a finite purely inseparable morphism of degree p between at most RDPs. Then it is given as the quotient by a p-closed derivation D on B (Proposition 2.4(2)), and D is unique up to $\operatorname{Frac}(B)^*$-multiple. The relation between B and B' is reciprocal in the sense that knowing $\pi \colon \operatorname{Spec} B \to \operatorname{Spec} B'$ is equivalent to knowing $\pi' \colon \operatorname{Spec} B' \to \operatorname{Spec} B^{(p)}$. The morphism π' is also purely inseparable of degree p and hence given as the quotient by a p-closed derivation D' on B'. We say that the morphisms π and π' are dual to each other.

We may assume that the fixed locus $\operatorname{Fix}(D)$ of D (Definition 2.9), which is a closed subscheme of $\operatorname{Spec} B$, satisfies exactly one of the following three conditions:

(a) $\mathrm{Fix}(D) = \emptyset$.
(b) $\mathrm{Fix}(D) = \{\mathfrak{m}\}$.
(c) The divisorial part (D) of $\mathrm{Fix}(D)$ is a non-Cartier (Weil) divisor. In other words, the corresponding ideal of B is not principal.

Similarly we may assume exactly one of (a′), (b′), (c′) according to $\mathrm{Fix}(D')$. Thus there are a priori $3^2 = 9$ possibilities.

Theorem 1.1. *Suppose B and B' are complete 2-dimensional local k-algebras, both either smooth or RDP. Suppose $\mathrm{Spec}\,B \to \mathrm{Spec}\,B'$ is purely inseparable of degree p. Then, (c) holds if and only if (c′) holds. Therefore 5 possibilities among 9 remains, and in each case we have the following classification.*

(1) *If (a) and (a′) hold, then both B and B' are smooth. In this case, it is known (Rudakov–Shafarevich [RS76, Theorem 1 and Corollary] and Ganong [Gan82, Theorem]) that $B = k[[x,y]]$ and $B' = k[[x,y^p]]$ for some $x, y \in B$.*

(2) *If (a) and (b′) hold, then*
$$B \cong k[[x,y,z]]/(F) \supset B' \cong k[[x,y,z^p]]/(F)$$
and $D(x) = D(y) = 0$, where F is one of Table 1.

(3) *Dually, if (b) and (a′) hold, then either $B \cong k[[x,y]]$ or $B \cong k[[x,y,z]]/(F)$ with F and D as in one of Table 2, up to replacing D with a unit multiple. B' is generated by x^p, y^p and one more element if B is smooth and by x^p, y^p, z if B is an RDP, subject to an equation similar to the one in the dual case in Table 1.*

(4) *Suppose (b) and (b′) hold. Then $(\mathrm{Sing}(B), \mathrm{Sing}(B'))$ is one of Table 3, and there are elements $x, y, z, w \in B$ such that*
$$B = k[[x,y,z,w]]/(x^p - P(y^p, z, w), w - Q(z,y,x)) \quad \text{and}$$
$$B' = k[[w,z,y^p,x^p]]/(w^p - Q(z,y,x)^p, x^p - P(y^p, z, w)),$$
with P and Q as in the table up to terms of high degree. Up to a unit multiple, D satisfies $(D(x), D(y), D(z)) = (-Q_y, Q_x, 0)$.

(5) *Suppose (c) and (c′) hold. Let $l^{(\prime)}$ be the order of the class $[(D^{(\prime)})]$ in $\mathrm{Cl}(B^{(\prime)})$. Then $l = l'$ and $l \mid (p-1)$, and we have $B \cong k[[x,y,z]]/(F)$ with $D(z) = 0$ and $B' \cong k[[w,z,y^p]]/(F')$, where $(l, \mathrm{Sing}(B), \mathrm{Sing}(B'), F, w)$ is one of Table 4.*

In particular, if B' (resp. B) is smooth, then (a) (resp. (a′)) holds.

In Table 3, $(q)^+ := \max\{0, q\}$ denotes the positive part of a real q. In Table 4, l is an arbitrary positive integer and p is a prime satisfying $p \equiv 1 \pmod{l}$, unless specified otherwise.

Convention 1.2. Non-taut RDPs are the RDPs (over a fixed algebraically closed field k) not uniquely determined from the dual graph of the minimal resolution. Artin [Art77] classified them and introduced the notation D_n^r, E_n^r using the coindex r. We follow Artin's notation with the following exception: We say that
$$\frac{k[[x,y,z]]}{(z^2 + x^2 y + zy^n + zxy^{n-s})}, \quad n \geq 2, \ 0 \leq s \leq n-1$$
in characteristic 2 to be of type $D_{2n+1}^{s+1/2}$, instead of Artin's notation D_{2n+1}^s. Consequently, the range of r for D_{2n+1}^r is $\{\frac{1}{2}, \frac{3}{2}, \ldots, \frac{2n-1}{2}\}$. Under this notation, the RDP defined by
$$z^2 + x^2 y + zxy^m + y^n = 0 \quad (m \geq 1, \ n \geq 2)$$
is of type $D_{n+2m}^{n/2}$ regardless of the parity of n, and this makes it easier to write Table 3 and the proofs of Theorem 4.1 and Proposition 4.3.

TABLE 1. p-closed derivations on RDPs with D fixed-point-free ($m \geq 2$)

p	B	B'	equation of B
any	A_{p-1}	smooth	$xy + z^p$
any	A_{pm-1}	A_{m-1}	$xy + z^{pm}$
5	E_8^0	smooth	$z^5 + x^2 + y^3$
3	E_6^0	smooth	$z^3 + x^2 + y^4$
3	E_7^0	A_1	$x^2 + y^3 + yz^3$
3	E_8^0	smooth	$z^3 + x^2 + y^5$
2	D_{2m}^0	smooth	$z^2 + x^2y + xy^m$
2	$D_{2m+1}^{1/2}$	A_1	$x^2 + yz^2 + xy^m$
2	E_6^0	A_2	$x^2 + xz^2 + y^3$
2	E_7^0	smooth	$z^2 + x^3 + xy^3$
2	E_8^0	smooth	$z^2 + x^3 + y^5$

Also, we use the convention that $D_3 := A_3$ and that A_0 is a smooth point.

Convention 1.3. Let $X = \operatorname{Spec} B$ be the spectrum of a complete local k-algebra B, x its closed point, $U = X \setminus \{x\}$, and suppose U is smooth.

(1) $\operatorname{Cl}(B) := \operatorname{Pic}(U)$ is the (divisor) class group of B. If B is an at most RDP (which is complete by our definition), it follows from [Lip69, Proposition 17.1] that this group is determined from the Dynkin diagram as in Table 5 and is independent of the characteristic and the coindex.

(2) We call $\pi_{\mathrm{loc}}^{\mathrm{et}}(B) := \pi_1^{\mathrm{et}}(U)$ the local fundamental group of B, and we define the universal (étale) covering of RDP $X = \operatorname{Spec} B$ to be the integral closure X' of X in the universal covering U' of U. Artin [Art77] showed that if X is an RDP then $\pi_{\mathrm{loc}}^{\mathrm{et}}(B)$ is always finite and X' is an at most RDP, and determined them (Table 7). We say that X is simply-connected if U is so.

We prove Theorem 1.1 in Section 3 after some preparations on derivations in Section 2.

2. PRELIMINARIES ON p-CLOSED DERIVATIONS

2.1. **p-closed derivations and their properties.** A derivation D on an $\mathbb{F}_p$-algebra B is called *p-closed* if there exists $h \in B$ with $D^p = hD$. (See also Proposition 2.5.)

The following formula is well-known. See [Mat89, Theorem 25.5] for a proof.

Lemma 2.1 (Hochschild's formula). *Let B be an $\mathbb{F}_p$-algebra, $a \in B$ an element, and D a derivation on B. Then*

$$(aD)^p = a^p D^p + (aD)^{p-1}(a)D.$$

Corollary 2.2. *Suppose a derivation D on B is p-closed.*

- *For any $a \in B$, aD is also p-closed.*
- *If B is a domain, $D \neq 0$, $a \neq 0$, and $(aD)^p = h_1 aD$, then $h_1 \in (h) + (\operatorname{Im} D)$.*

Proof. Suppose $D^p = hD$ with $h \in B$. Then Hochschild's formula implies that $(aD)^p = h_1 aD$ with $h_1 = a^{p-1}h + D((aD)^{p-2}(a))$. $\qquad\square$

TABLE 2. p-closed derivations on smooth points and RDPs whose quotients are RDPs, with D' fixed-point-free ($m \geq 2$)

p	B	B'	equation of B	$D(x), D(y), D(z)$	h
any	smooth	A_{p-1}	—	$x, -y$	1
any	A_{m-1}	A_{pm-1}	$xy + z^m$	$x, -y, 0$	1
5	smooth	E_8^0	—	y^2, x	0
3	smooth	E_6^0	—	y^3, x	0
3	A_1	E_7^0	$x^2 + y^3 + yz$	$z, x, 0$	0
3	smooth	E_8^0	—	$y^4, -x$	$-y^3$
2	smooth	D_{2m}^0	—	$x^2 + mxy^{m-1}, y^m$	my^{m-1}
2	A_1	$D_{2m+1}^{1/2}$	$x^2 + yz + xy^m$	$z + my^{m-1}x, y^m, 0$	my^{m-1}
2	A_2	E_6^0	$x^2 + xz + y^3$	$y^2, z, 0$	0
2	smooth	E_7^0	—	$xy^2, x^2 + y^3$	y^2
2	smooth	E_8^0	—	y^4, x^2	0

TABLE 3. p-closed derivations on RDPs whose quotients are RDPs, with $\mathrm{Fix}(D) = \{\mathfrak{m}\}$ and $\mathrm{Fix}(D') = \{\mathfrak{m}'\}$ ($m, m' \geq 1$, $n \geq 2$)

p	B	B'	$x^p = P(y^p, z, w) \approx$	$w = Q(z, y, x) \approx$	$D(x), D(y) \approx$	$h \approx$
3	E_6^1	E_6^1	$z^2 + zw$	$y^2 + yx$	$x - y, -y$	1
3	E_8^1	E_6^0	$z^2 + y^3 w$	$y^2 + yx$	$x - y, -y$	1
3	E_6^0	E_8^1	$z^2 + zw$	$y^2 + zx$	y, z	0
3	E_8^0	E_8^0	$z^2 + y^3 w$	$y^2 + zx$	y, z	0
2	$D_{2m'+2}^{(2-m)^+}$	$D_{2m+2}^{(2-m')^+}$	$y^2 z + z^{m'} w$	$zy + y^m x$	$z + my^{m-1}x, y^m$	my^{m-1}
2	$D_{2n}^{(n-m)^+}$	$D_{n+2m}^{n/2}$	zw	$zy + y^n + y^m x$	$z + my^{m-1}x + ny^{n-1}, y^m$	my^{m-1}
2	$D_{n+2m}^{n/2}$	$D_{2n}^{(n-m)^+}$	$y^2 z + z^n + z^m w$	yx	x, y	1
2	$D_{2m+3}^{1/2}$	$E_7^{(3-m)^+}$	$y^2 z + z^m w$	$y^3 + zx$	y^2, z	0
2	$E_7^{(3-m)^+}$	$D_{2m+3}^{1/2}$	$z^3 + y^2 w$	$zy + y^m x$	$z + my^{m-1}x, y^m$	my^{m-1}
2	E_7^3	E_7^3	$z^3 + zw$	$y^3 + yx$	$x + y^2, y$	1
2	E_7^2	E_8^3	$z^3 + zw$	$y^3 + zx$	y^2, z	0
2	E_8^3	E_7^2	$z^3 + y^2 w$	$y^3 + yx$	$x + y^2, y$	1
2	E_8^2	E_8^2	$z^3 + y^2 w$	$y^3 + zx$	y^2, z	0

TABLE 4. p-closed derivations on RDPs whose quotients are RDPs, with non-Cartier divisorial fixed loci ($m \geq 1$) (p is any prime $\equiv 1 \pmod{l}$ unless specified)

p	B	B'	l	$(D(x), D(y))$	equation of B	w
	A_{l-1}	A_{l-1}	any	$(x, ly), (z, lx^{l-1})$	$-x^l + yz$	$xy^{(p-1)/l}$
	D_{m+2}	D_{mp+2}	2	$(yz, x), (yx, y^2 + z^m)$	$-x^2 + z(y^2 + z^m)$	$x(y^2 + z^m)^{(p-1)/2}$
	D_{mp+2}	D_{m+2}	2	$(z^2 + y^{mp}, 2x), (x, 2y)$	$-x^2 + y(z^2 + y^{mp})$	$xy^{(p-1)/2}$
3	E_7^1	E_7^1	2	$(xy, y^2 + z), ((y^3 + z^2)y, x)$	$-x^2 + (y^3 + z^2)(y^2 + z)$	$x(y^2 + z)$

TABLE 5. Class groups of complete RDPs (in any characteristic)

smooth	A_n	D_{2m}	D_{2m+1}	E_6	E_7	E_8
0	$\mathbb{Z}/(n+1)\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^2$	$\mathbb{Z}/4\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0

We list some properties of p-closed derivations and inseparable extensions. See Section 2.2 for counterexamples under weaker assumptions. Note that RDPs are normal complete F-finite Noetherian domains and hence satisfy all the assumptions. Here, we say that a domain is normal if it is integrally closed in its field of fractions.

Lemma 2.3. *Let B be an $\mathbb{F}_p$-algebra, h an element of B, and D a p-closed derivation on B with $D^p = hD$. Assume either B is reduced, or $\operatorname{Im} D$ contains a regular element of B. Then $D(h) = 0$.*

Proof. We have $hDD = D^p D = DD^p = D(hD) = D(h)D + hDD$, hence $D(h)D = 0$. If B is reduced, then $D(h)^2 = 0$ and $D(h) = 0$. If $D(a) \in \operatorname{Im} D$ is a regular element, then $D(h)D(a) = 0$ and $D(h) = 0$. $\qquad\square$

Proposition 2.4.
(1) *Let B be an F-finite normal domain and $D \neq 0$ be a p-closed derivation on B. Then $B \supset A := B^D$ is a finite extension of normal domains and $\operatorname{Frac} B / \operatorname{Frac} A$ is purely inseparable of degree p.*
(2) *Let $B \supset A$ be a finite extension of normal domains over $\mathbb{F}_p$, such that $\operatorname{Frac} B / \operatorname{Frac} A$ is purely inseparable of degree p. Then there exists a p-closed derivation D on B such that $A = B^D$. Moreover, D is unique up to $(\operatorname{Frac} B)^*$.*

Proof. (1) Since B is finite over $B^{(p)}$ (this is the definition of F-finiteness), and clearly $A \supset B^{(p)}$, it follows that B is finite over A. Normality of A follows from that of B. Now take $b \in \operatorname{Frac} B$ such that $D(b) \neq 0$. It is straightforward to show that b^i ($0 \leq i < p$) is a basis of $\operatorname{Frac} B$ over $\operatorname{Frac} A$, and that $b^p \in \operatorname{Frac} A$.

(2) There exists $t \in \operatorname{Frac} B$ such that $\operatorname{Frac} B = (\operatorname{Frac} A)[t]$ (and then $t^p \in A$). Then $D' = \frac{\partial}{\partial t}$ is a p-closed derivation on $\operatorname{Frac} B$ such that $\operatorname{Frac} A = (\operatorname{Frac} B)^{D'}$. Take $x_1, \ldots, x_n \in B$ such that $B = A[x_1, \ldots, x_n]$. Take $b \in B \setminus \{0\}$ such that

$$bD'(x_1^{i_1} \ldots x_n^{i_n}) \in B \text{ for every } i_1, \ldots, i_n \in \{0, 1, \ldots, p-1\}.$$

Then $D := bD'$ takes values in B and is also p-closed. We obtain $B^D = B \cap \operatorname{Frac} A = A$, where the last equality follows from B being integral over A and A being normal.

Suppose D'' is another derivation on B with the same property. Then the derivations $(D(t)/D''(t))D''$ and D on $\operatorname{Frac} B$ agree on $\operatorname{Frac} A$ and t, hence agree on $\operatorname{Frac} B$. $\qquad\square$

Proposition 2.5. *Let B be a normal Noetherian domain, $D \neq 0$ a derivation on B, and $h \in \operatorname{Frac} B$. Suppose $D^p = hD$. Then $h \in B$.*

Thus, under this assumption on B, the definition of p-closedness can be either $h \in B$ or $h \in \operatorname{Frac} B$.

Proof. Let $I := (\operatorname{Im} D) \neq 0$ be the ideal generated by the image of D. Then

$$hI = (\operatorname{Im} hD) = (\operatorname{Im} D^p) \subset (\operatorname{Im} D) = I.$$

Since B is Noetherian, I is finitely generated, and then the usual argument using determinant shows that h is integral over B. Since B is normal, we have $h \in B$. $\qquad\square$

2.2. **Examples of derivations.** For the sake of completeness, we give counterexamples for Lemma 2.3 and Propositions 2.4 and 2.5 under weaker assumptions. These examples are not needed for the rest of the paper. k will be a field of characteristic p.

Example 2.6 (cf. Lemma 2.3).   (1) There are examples of D such that $D^p = h_1 D = h_2 D$, $D(h_1) = 0$, but $D(h_2) \neq 0$. Let $B = k[t]/(t^p)$, $D = t\frac{\partial}{\partial t}$, $h_1 = 1$, $h_2 = 1 + t^{p-1}$. Then $D(h_1) = 0$ but $D(h_2) = (p-1)t^{p-1} \neq 0$.

   (2) There are examples of D that is p-closed but $D(h) \neq 0$ for any h satisfying $D^p = hD$.

First, let $B = k[x,y]/(x^i y^j \mid i + j = p + 1, (i,j) \neq (1,p))$, $D = (x^2 + xy)\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}$. We observe that D induces $D_l \colon \mathfrak{m}^l/\mathfrak{m}^{l+1} \to \mathfrak{m}^{l+1}/\mathfrak{m}^{l+2}$ and that D_l are injective for $1 \leq l < p$. Given $h \in B$, to check whether the derivations D^p and hD are equal, it suffices to check that they agree on x and y. Since $D^p(x) = D^p(y) = xy^p$, we obtain $D^p = y^{p-1}D$, hence h satisfies this property if and only if $h - y^{p-1} \in \mathrm{Ann}(D(x), D(y)) = (\mathfrak{m}^p + x\mathfrak{m}^{p-2})$, which implies $h \not\equiv 0 \pmod{\mathfrak{m}^p}$. By the injectivity of D_{p-1}, we have $D(h) \neq 0$ for any such h.

Another example is as follows. Let $B = k[x, y_2, \ldots, y_p]/(y_2, \ldots, y_p)^2$,

$$D = y_2 \frac{\partial}{\partial x} + \sum_{i=2}^{p-1} y_{i+1} \frac{\partial}{\partial y_i} + xy_2 \frac{\partial}{\partial y_p}.$$

We have $D^p = xD$. Then $D^p = hD$ if and only if $h \equiv x \pmod{(y_2, \ldots, y_p)}$ and then $D(h) \equiv y_2 \not\equiv 0 \pmod{(xy_2, y_3, \ldots, y_p)}$.

Example 2.7 (cf. Proposition 2.4). Let $B = k[x_1, \ldots, x_n, \ldots]$ (which is not F-finite).

   (1) Let $D = \sum_{i=1}^{\infty} x_i \frac{\partial}{\partial x_i} \colon B \to B$. Then $\mathrm{Spec}\, B \to \mathrm{Spec}\, B^D$ is not finite. Indeed, if $\mathfrak{m} \subset B$ and $\mathfrak{m}' \subset B^D$ are the maximal ideals at the origin, then $\mathfrak{m}/(\mathfrak{m}^2 + \mathfrak{m}')$ is of infinite dimension.

   (2) Let $D' = \sum_{i=1}^{\infty} x_i^{-p} \frac{\partial}{\partial x_i} \colon B \to \mathrm{Frac}\, B$. Then, although $\mathrm{Frac}\, B/\mathrm{Frac}\, B^{D'}$ is purely inseparable of degree p and $B \supset B^{D'}$ is an integral extension between normal domains, there is no p-closed derivation D on B such that $B^D = B^{D'}$. Indeed, if such D exists then $D = bD'$ for some $b \in \mathrm{Frac}\, B \setminus \{0\}$, and b satisfies $bD'(x_i) = bx_i^{-p} \in B$ for every i, but this is impossible.

Example 2.8 (cf. Proposition 2.5). Let $B = k[t^i \mid i \in \mathbb{Z}, i > p(p-1)] \subset k[t]$ and $D = t^{p+1}\frac{\partial}{\partial t}$. This is Noetherian and not normal. Then $D^p = hD$ with $h = t^{p(p-1)} \in \mathrm{Frac}\, B$, but $h \notin B$.

Let $B = k[yx^i \mid i \in \mathbb{Z}]$ or $B = k[x, yx^i \mid i \in \mathbb{Z}] \subset k[x^{\pm 1}, y]$ and $D = \frac{y}{x}\frac{\partial}{\partial y}$. This is normal and non-Noetherian. Then $D^p = hD$ with $h = x^{-(p-1)} \in \mathrm{Frac}\, B$, but $h \notin B$.

2.3. **p-closed derivation quotients.** We recall the notion of the fixed loci of derivations and how they are related to derivation quotients. Rudakov–Shafarevich [RS76] uses the term *singularity* but we do not use this, as we want to distinguish them from the singularities of the varieties.

Definition 2.9. Suppose D is a derivation on a scheme X. The *fixed locus* $\mathrm{Fix}(D)$ is the closed subscheme of X corresponding to the sheaf $(\mathrm{Im}(D))$ of ideals generated by

$$\mathrm{Im}(D) = \{D(a) \mid a \in \mathcal{O}_X\}.$$

A *fixed point* of D is a point of $\mathrm{Fix}(D)$.

Assume X is a smooth irreducible variety and $D \neq 0$. Then $\mathrm{Fix}(D)$ consists of its divisorial part (D) and non-divisorial part $\langle D \rangle$. If we write $D = f \sum_i g_i \frac{\partial}{\partial x_i}$ for some local coordinate

x_i with g_i having no common factor, then (D) and $\langle D \rangle$ correspond to the ideals (f) and (g_i) respectively.

Assume X is a smooth irreducible variety and suppose $D \neq 0$ is now a *rational* derivation, locally of the form $f^{-1}D'$ for some regular function f and (regular) derivation D'. Then we define $(D) = (D') - \mathrm{div}(f)$ and $\langle D \rangle = \langle D' \rangle$.

If X is only normal, then we can still define (D) as a Weil divisor.

Proposition 2.10 ([Mat21a, Proposition 2.14]). *Suppose $X_0 \xrightarrow{\pi_0} X_1 \xrightarrow{\pi_1} \ldots \xrightarrow{\pi_{n-1}} X_n = X_0{}^{(p)}$ is a sequence of purely inseparable morphisms of degree p between n-dimensional integral normal varieties, with each π_i given by a p-closed rational derivation D_i on X_i. Then K_{X_0} is linearly equivalent to $-\sum_{i=0}^{n-1}(\pi_{i-1} \circ \cdots \circ \pi_0)^*((D_i))$.*

Corollary 2.11. *If $\pi \colon X = \mathrm{Spec}\, B \to X' = \mathrm{Spec}\, B'$ and D, D' are as in Theorem 1.1, then the order of the class $[(D)]$ in $\mathrm{Cl}(B)$ is equal to that of $[(D')]$ in $\mathrm{Cl}(B')$.*

Proof. By applying Proposition 2.10 to the sequence $X \to X' \to X^{(p)}$ and using $K_X = 0$, we obtain $[(D)] + \pi^*([(D')]) = 0$ in $\mathrm{Cl}(B)$. Dually we have $\pi'^*([(D^{(p)})]) + [(D')] = 0$ in $\mathrm{Cl}(B')$. $\square$

Proposition 2.12 ([Mat21a, Proposition 2.12]). *Suppose $\pi \colon X \to X'$ is a purely inseparable morphism of degree p between smooth varieties of dimension m, induced by a p-closed rational derivation D such that $\Delta := \mathrm{Fix}(D)$ is divisorial. Then we have isomorphisms*

$$\Omega^m_{X/k}(\Delta) \cong (\pi^* \Omega^m_{X'/k})(p\Delta) \quad \text{and} \quad (\pi_* \Omega^m_{X/k}(\Delta))^D \cong \Omega^m_{X'/k}(\pi_*\Delta)$$

sending

$$f_0 \cdot df_1 \wedge \cdots \wedge df_{m-1} \wedge D(g)^{-1}dg \mapsto f_0 \cdot df_1 \wedge \cdots \wedge df_{m-1} \wedge D(g)^{-p}d(g^p)$$

if $f_i, g \in \mathcal{O}_X$, $D(f_i) = 0$ $(1 \leq i \leq m-1)$, and $D(g)^{-1} \in \mathcal{O}_X(\Delta)$, and for the second morphism if moreover $D(f_0) = 0$.

In particular, we obtain the Rudakov–Shafarevich formula $K_X \sim \pi^* K_{X'} + (p-1)\Delta$.

See [Mat21a, Section 2.1] for the definition of the action of D on m-forms.

Remark 2.13. This isomorphism (although it depends on the choice of D) may be considered as an analogue of the natural pullback isomorphism $\Omega^m_{X/k} \cong \pi^* \Omega^m_{X'/k}(R)$ for a finite flat morphism $\pi \colon X \to X'$ between smooth varieties, where R is the ramification divisor which is 0 if and only if π is étale. Thus, the existence of D with $\mathrm{codim}\,\mathrm{Fix}(D) \geq 2$ might be considered as an inseparable analogue of codimension 1 étaleness. However, this is not a local property, and not compatible with composition, as the following example shows. Also, if the divisor (D) is nonzero, then it is not defined canonically from π but only up to linear equivalence.

Example 2.14. Suppose $\mathrm{char}\, k = p > 2$. Let $\pi \colon A_1 \to A_2$ be a purely inseparable isogeny of degree p between abelian surfaces, $f_i \colon A_i \to X_i = A_i/\{\pm 1\}$ the quotient morphisms, and $\pi' \colon X_1 \to X_2$ the purely inseparable morphism of degree p induced by π. Clearly f_1 and f_2 are étale in codimension 1 and π is induced by a derivation D on A_1, which is unique up to scalar and fixed-point-free. This D satisfies $[-1]^*D = -D \neq D$ and hence does not descend to X_1. Hence there is no regular derivation on X_1, even on X_1^{sm}, inducing π'.

3. Proof of classification

In this section we prove Theorem 1.1. In Section 3.1 we reduce the proof to the following involved cases, which will be treated in individual sections.

- $p = 3$, (b) and (b′) hold, both B and B' are either of type E_8^0 or E_8^1, and both h and h' are non-units at every upper fixed point (see Section 3.1 for the definition) of D and D'. (Section 3.2)
- $p = 2$, (b) and (b′) hold, and both h and h' are non-units of B and B'. (Section 3.3)
- The divisorial part (D) of $\mathrm{Fix}(D)$ is not Cartier. (Section 3.4)

3.1. **Reduction part.** We first note that the conditions (a), (b), (c) are pairwise exclusive, and that we can replace D so that one of these holds. Indeed, replacing D changes $\mathrm{Fix}(D)$ precisely by a Cartier divisor. Similar for D'.

The equivalence (c) $\iff$ (c′) and the equality $l = l'$ follow from Corollary 2.11. This case will be considered in Section 3.4. Assume this is not the case.

The following assertions hold.

(1) If B' is smooth then (a) holds (by [Mat21a, Theorem 3.3(2)]).
(2) Dually, if B is smooth then (a′) holds.
(3) Suppose (a) holds and B is an RDP. Then by [Mat22, Proposition 4.7], there is $F \in k[[x, y, z^p]]$ such that $B \cong k[[x, y, z]]/(F)$, $D = \frac{\partial}{\partial z}$ up to unit multiple, and $B' = k[[x, y, z^p]]/(F)$, and moreover all possible F are classified. The result is as in Table 1.
(4) Dually, if (a′) holds and B' is an RDP, then we have a classification, as in Table 2.
(5) If (a) holds and B is an RDP, then (b′) holds. This follows from the classification of assertion (3).
(6) Dually, if (a′) holds and B is an RDP, then (b) holds.

Among 2^4 possibilities ((a) or (b), (a′) or (b′), B smooth or an RDP, B' smooth or an RDP)), 10 are proved impossible by (1), (2), (5), and (6). The case where B and B' are both smooth (and (a) and (a′) hold)) is done by the works cited in the statement of Theorem 1.1(1). If (a) and (b′) hold and B is an RDP (B' is either smooth or an RDP), then we have a classification by (3). Same for the dual cases. Thus the only remaining case is where (b) and (b′) hold and B and B' are both RDPs. Hereafter we consider this case.

If D is of multiplicative type (i.e. $h = 1$) then such derivations correspond to μ_p-actions and are classified in [Mat22, Proposition 4.9]. More generally, if $h \in B^*$, then we may replace D with $h^{-1/(p-1)}D$, which is of multiplicative type (since $D(h) = 0$ by Lemma 2.3). Dually, we have a classification for the cases where $h' \in B'^*$. Hereafter we assume $h \in \mathfrak{m}$ and $h' \in \mathfrak{m}'$.

By [Mat21a, Corollary 3.5], there are partial resolutions $f \colon \tilde{X} \to X$ and $f' \colon \tilde{X}' \to X'$ with a morphism $\tilde{\pi} \colon \tilde{X} \to \tilde{X}'$ as in the diagram

$$
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\ \tilde{\pi}\ } & \tilde{X}' \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\
X & \xrightarrow{\ \pi\ } & X',
\end{array}
$$

and a derivation $\tilde{D}$ on $\tilde{X}$ with $\tilde{D} = D$ outside the exceptional loci, satisfying the following properties:

$$\tilde{X}' = \tilde{X}^{\tilde{D}},\ (\tilde{D}) = 0,\ \mathrm{Sing}(\tilde{X}) \cap \tilde{\pi}^{-1}(\mathrm{Sing}(\tilde{X}')) = \emptyset \ (\text{in particular, } \mathrm{Sing}(\tilde{X}) \cap \mathrm{Fix}(\tilde{D}) = \emptyset),$$

and $\mathrm{Fix}(\tilde{D}) \neq \emptyset$. We will refer to the points of $\mathrm{Fix}(\tilde{D})$ as the *upper fixed points* of D.

If $h \in \mathcal{O}_{\tilde{X},w}^*$ at some upper fixed point $w \in \mathrm{Fix}(\tilde{D})$, then $h \in B \cap \mathcal{O}_{\tilde{X},w}^* = B^*$, contradicting our assumption that $h \in \mathfrak{m}$. Thus we have $h \in \mathfrak{m}_w$ for any $w \in \mathrm{Fix}(\tilde{D})$. Since B_w is smooth, we have the classification [Mat21a, Lemma 3.6(2)] of RDPs B_w' satisfying this. Suppose $p \geq 3$. Then there are very few possibilities: $p = 3$ and B_w' is either of type E_6^0 or E_8^0, or $p = 5$ and B_w'

is of type E_8^0. In other words, B' admits one of these RDPs as a partial resolution. Since no Dynkin diagram contains E_8 strictly, the only possibility is that $p = 3$ and $\mathrm{Sing}(B') \in \{E_7^r, E_8^r\}$. But E_7^1 and E_8^2 are impossible since their partial resolutions are E_6^1, not E_6^0, by [Mat21b, Lemma 4.9]. Also E_7^0 is impossible, since by replacing B and B' with their universal coverings we obtain a derivation with satisfying (a) or (b) and with quotient of type E_6^0, which is impossible by above. Hence $\mathrm{Sing}(B') \in \{E_8^0, E_8^1\}$. Applying the same argument to D', we obtain the same conclusion for $\mathrm{Sing}(B)$. Thus we are in the case to be discussed in Section 3.2.

In the remaining case we have: $p = 2$, (b) and (b′) hold, B and B' are both RDPs, $h \in \mathfrak{m}$, and $h' \in \mathfrak{m}'$. Thus we are in the case to be discussed in Section 3.3.

Before ending this section, we show the following.

Lemma 3.1. *Suppose* $\pi \colon \mathrm{Spec}\, B \to \mathrm{Spec}\, B'$ *is as in Theorem 1.1 and satisfies* (b). *Let*
$$\delta = \dim_k \mathrm{Im}(\mathfrak{m}'/\mathfrak{m}'^2 \to \mathfrak{m}/\mathfrak{m}^2).$$
Then $\delta \in \{0, 1\}$.

Proof. If B is smooth, then $\delta < \dim \mathfrak{m}/\mathfrak{m}^2 = 2$ since $\mathfrak{m}'$ cannot generate $\mathfrak{m}$.

Suppose B is an RDP. We have $\delta < \dim \mathfrak{m}/\mathfrak{m}^2 = 3$. Assume $\delta = 2$. We may assume $\mathfrak{m} = (x, y, z)$ and $\mathfrak{m}' = (x, y, z^p)$. If the ideal $(F) = \mathrm{Ker}(k[[x, y, z]] \to B)$ does not have a generator that belongs to $k[[x, y, z^p]]$, then $B' = k[[x, y, z^p]]/(F^p)$, but this cannot be normal. Hence (F) is generated by an element of $k[[x, y, z^p]]$, which we may assume to be F itself, and $B' = k[[x, y, z^p]]/(F)$. Then D is proportional to $\frac{\partial}{\partial z}$, contradicting the assumption (b). $\qquad\square$

It turns out that δ is always 1 if B is an RDP, but we can prove this only after case-by-case arguments.

3.2. Case of E_8 in $p = 3$.

Suppose $p = 3$, both B and B' are either of type E_8^0 or E_8^1, and both h and h' are non-units at every upper fixed point of D. By the arguments in Section 3.1, there is exactly one upper fixed point of D, and its quotient is of type E_6^0. Let $X_1 \to X$ be the blow-up at the closed point. Then $\mathrm{Sing}(X_1) = E_7^0$. Since there is no derivation on E_7^0 fixing precisely the closed point with non-unit h and RDP quotient, this point is not fixed, and the upper fixed point should lie above X_1^{sm}. In particular, there exists a unique fixed point on X_1^{sm}.

Suppose B is of type E_8^1. We may assume that $B = k[[x, y, z]]/(z^2 + x^3 + y^5 + x^2 y^3)$. Then the space of derivations on B are generated by the following elements D_1, D_2, D_3. Then D has at least two fixed points $(x = 1 + y_1^2 = z_1 = 0)$ on
$$\mathrm{Spec}\, B[y/x, z/x] = \mathrm{Spec}\, \frac{k[[x]][y_1, z_1]}{(z_1^2 + x + x^3 y_1^3 (y_1^2 + 1))} \subset X_1,$$
a contradiction.

Dually, B' is not of type E_8^1.

	D_1	D_2	D_3
x	0	z	y
y	z	0	$-x$
z	$-y^4$	$-xy^3$	0
x	0	xz_1	xy_1
$y_1 = y/x$	z_1	$-y_1 z_1$	$-1 - y_1^2$
$z_1 = z/x$	$-x^3 y_1^4$	$-x^3 y_1^3 - z_1^2$	$-y_1 z_1$

Now suppose B and B' are of type E_8^0.

Claim 3.2. $D|_{\mathfrak{m}/\mathfrak{m}^2}$ *is nilpotent of index* 3 *and* $\delta = 1$, *where* δ *is as in Lemma* 3.1.

Proof of Claim. Since we know that $\delta \leq 1$ by Lemma 3.1 and that $(D|_{\mathfrak{m}/\mathfrak{m}^2})^3 = 0$ since $h \in \mathfrak{m}$, it remains to show that $\delta \geq 1$ and $(D|_{\mathfrak{m}/\mathfrak{m}^2})^2 \neq 0$.

We write $B = \operatorname{Spec} k[[x,y,z]]/(z^2 + x^3 + y^5)$ during this proof. Then the space of derivations on B are generated by the following elements D_1, D_2.

	D_1	D_2
x	0	1
y	z	0
z	$-y^4$	0
x	0	1
$y_1 = y/x$	z_1	$-y_1/x$
$z_1 = z/x$	$-x^3 y_1^4$	$-z_1/x$
$x_2 = x/y$	$-x_2 z_2$	$1/y$
y	yz_2	0
$z_2 = z/y$	$-y^3 - z_2^2$	0

Write $D = f_1 D_1 + f_2 D_2$ with $f_i \in B$, and suppose $f_i \equiv f_{i0} + f_{i1} x + f_{i2} y + f_{i3} z \pmod{\mathfrak{m}^2}$ with $f_{ij} \in k$. Since D extends to X_1, we have $f_2 \in \mathfrak{m}$ ($f_{20} = 0$). Since $h \notin B^*$, we have $f_{21} = 0$. Since D does not fix the origin of $\operatorname{Spec} B[x/y, z/y] = \operatorname{Spec} k[[y]][x_2, z_2]/(z_2^2 + x_2^3 y + y^3) \subset X_1 = \operatorname{Bl}_{\mathfrak{m}} X$, we have $f_{22} \neq 0$. Let $\mathfrak{m}_1 = (y_1, z_1)$ be the maximal ideal at the origin of

$$\operatorname{Spec} B[y/x, z/x] = \operatorname{Spec} \frac{k[[x]][y_1, z_1]}{(z_1^2 + x + x^3 y_1^5)} \subset X_1,$$

which is a fixed point of D. Since the image of this point is of type E_6^0, we have $D(\mathfrak{m}_1) \not\subset \mathfrak{m}_1^2$, hence $f_{10} \neq 0$. Then $v := D(D(x)) - hx$ satisfies $D(v) = (D^3 - hD)(x) = 0$ (by Lemma 2.3), hence $v \in \operatorname{Im}(\mathfrak{m}' \to \mathfrak{m})$, and $v \equiv D(D(x)) \equiv D(f_{22} y + f_{23} z) \equiv f_{10} f_{22} z \not\equiv 0 \pmod{\mathfrak{m}^2}$.   □

We note that if $(B = k[[x,y,z]]/(F), \mathfrak{m})$ is an RDP of type D_n or E_n in characteristic ≥ 3 and D is a derivation on B with $D(\mathfrak{m}) \subset \mathfrak{m}$ and $D|_{\mathfrak{m}/\mathfrak{m}^2}$ is nilpotent of index 3, then $F \equiv l^2 \pmod{\mathfrak{m}^3}$ with $l \in \operatorname{Ker}(D|_{\mathfrak{m}/\mathfrak{m}^2})$ (otherwise $D(F)$ cannot be zero).

We will show that B admits elements x, y, z, w as in the statement of Theorem 1.1(4). We may assume $\mathfrak{m} = (x, y, z)$, $z \in B'$, and $\mathfrak{m}' = (Y, z, w)$, where $Y := y^3$. We may moreover assume $x^3 \in \mathfrak{m}'^2$ and $w \in \mathfrak{m}^2$. Write $w = Q(z, y, x) \in (z, y, x)^2 \subset k[[z, y, x]]$. Since $x^3 \in \mathfrak{m}'^2$, there exists $u \in k[[x, y, z]]$ and $P \in (Y, z, w)^2 \subset k[[Y, z, w]]$ such that $uF = -x^3 + P(Y, z, Q(z, y, x))$. If $u \notin k[[x, y, z]]^*$, then the degree 2 part of F cannot be the square of a linear term contained in $\operatorname{Ker}(D|_{\mathfrak{m}/\mathfrak{m}^2}) = kz$, contradicting the observation above. Hence $u \in k[[x, y, z]]^*$, and we may assume $u = 1$: $F = -x^3 + P(Y, z, Q(z, y, x))$, and also $F' = -w^3 + \tilde{Q}(z^3, Y, P(Y, z, w))$, where $\tilde{Q}(z^3, Y, P) := Q(z, y, x)^3$.

We will say that a polynomial *has* a monomial if the corresponding coefficient is nonzero.

Since B and B' are of type E_8, we obtain the following: P has z^2; Q has y^2; we may assume P does not have zw nor w^2; we may assume Q does not have yx nor x^2; P has Yw; Q has zx. By replacing x, y, z, w, F, F' with scalar multiples, we may assume $F = -x^3 + z^2 + y^3 w + (\dots)$, $F' = (-w + y^2 + zx + (\dots))^3$. Since $0 = D(w) = D(x)Q_x + D(y)Q_y$ and since $Q_x \equiv z \pmod{\mathfrak{m}^2}$ and $Q_y \equiv 2y \pmod{(z) + \mathfrak{m}^2}$ have no common factor, we may assume $(D(x), D(y)) = (-Q_y, Q_x)$.

3.3. **Case of characteristic** 2. Suppose $p = 2$ and both h and h' are non-units.

In this case, there are still many possibilities for the quotients of upper fixed points, hence the geometric arguments used in Section 3.2 are not effective. Instead, we will determine the equations directly (up to terms of high degree).

As in Lemma 3.1, we let

$$\delta = \dim_k \text{Im}(\mathfrak{m}'/\mathfrak{m}'^2 \to \mathfrak{m}/\mathfrak{m}^2), \quad \text{and similarly} \quad \delta' = \dim_k \text{Im}(\mathfrak{m}^{(2)}/(\mathfrak{m}^{(2)})^2 \to \mathfrak{m}'/\mathfrak{m}'^2).$$

We have $\delta, \delta' \in \{0, 1\}$. We shall show $\delta = \delta' = 1$.

Assume $\delta = 0$, that is, $\mathfrak{m}' \subset \mathfrak{m}^2$. Since $x^2, y^2, z^2 \in \text{Frac } B^{(2)} \subsetneq \text{Frac } B'$, the elements $x^2, y^2, z^2 \subset \mathfrak{m}^{(2)} \subset \mathfrak{m}'$ cannot generate $\mathfrak{m}'$. In other words, there exists a nonzero linear combination f of x^2, y^2, z^2 that belongs to $\mathfrak{m}'^2$. Since $f \in \mathfrak{m}'^2 \subset \mathfrak{m}^4$, (a unit multiple of) F is of the form $f + G$ with $f \in \mathfrak{m}^{(2)}$ and $G \in \mathfrak{m}^4$. But such a polynomial cannot define an RDP. Hence $\delta = 1$. Dually $\delta' = 1$.

We may assume that $\mathfrak{m} = (x, y, z)$ and $\mathfrak{m}' = (w, y^2, z)$, and that $x^2 \in \mathfrak{m}'^2$ and $w^2 \in (\mathfrak{m}^{(2)})^2$.

Since $x^2 \in \mathfrak{m}'^2$, we may assume

$$F = x^2 + P_0(y^2, z) + P_1(y^2, z)w$$

with $P_0 \in \mathfrak{n}'^2$ and $P_1 \in \mathfrak{n}'$, where $\mathfrak{n}'$ is the maximal ideal (y^2, z) of $k[[y^2, z]]$. Similarly we may assume $w = Q_0(z, y) + Q_1(z, y)x$ with $Q_0 \in \mathfrak{n}^2$ and $Q_1 \in \mathfrak{n}$, where $\mathfrak{n}$ is the maximal ideal (z, y) of $k[[z, y]]$. Hence we have

$$F = x^2 + P_0(y^2, z) + P_1(y^2, z)(Q_0(z, y) + Q_1(z, y)x),$$
$$F' = w^2 + \tilde{Q}_0(z^2, Y) + \tilde{Q}_1(z^2, Y)(P_0(Y, z) + P_1(Y, z)w),$$

where $Y := y^2$, and $\tilde{Q}_i(z^2, Y) := Q_i(z, y)^2$.

As in Section 3.2, we will say that a polynomial *has* a monomial if the corresponding coefficient is nonzero.

We may assume P_0 does not have z^2, and $\tilde{Q}_0$ does not have Y^2.

We show that Q_1 does not have y. For this, consider the derivation D_1 defined by

$$D_1(x, y, z) = ((Q_0)_y + (Q_1)_y x, Q_1, 0).$$

Then it satisfies $B^{D_1} = B'$, hence $D_1 = aD$ for some $a \in B \setminus \{0\}$, and $D_1^2 = (Q_1)_y D_1$. By Corollary 2.2 we obtain $(Q_1)_y \in \mathfrak{m}$. Similarly, P_1 does not have z.

For F to define an RDP, we need either

- P_0 has $y^2 z$ (then we may assume P_0 does not have z^3), P_1 has z^k, and either Q_0 has $z^m y$ or Q_1 has z^l, or
- P_0 does not have $y^2 z$ but has z^3, P_1 has y^2, and Q_0 has zy or y^3.

Similarly, for F' to define an RDP, we need either

- $\tilde{Q}_0$ has $z^2 Y$ (then we may assume $\tilde{Q}_0$ does not have Y^3), $\tilde{Q}_1$ has $Y^{k'}$, and either P_0 has $Y^{m'} z$ or P_1 has $Y^{l'}$, or
- $\tilde{Q}_0$ does not have $z^2 Y$ but has Y^3, $\tilde{Q}_1$ has z^2, and P_0 has Yz or z^3.

Combining these conditions, one of the following holds.

- P_0 has $y^2 z$ and does not have z^3, P_1 has $z^{m'}$, Q_0 has zy and does not have y^3, and Q_1 has y^m. In this case, B is of type $D_{2m'+2}^{(2-m)^+}$ and B' is of type $D_{2m+2}^{(2-m')^+}$.
- P_0 does not have $y^2 z$ but has z^3, P_1 has y^2, Q_0 has zy and does not have y^3, and Q_1 has y^m. In this case, B is of type $E_7^{(3-m)^+}$ and B' is of type D_{2m+3}^0.
- Dual of the previous case: B is of type D_{2m+3}^0 and B' is of type $E_7^{(3-m)^+}$.

- P_0 does not have y^2z but has z^3, P_1 has y^2, Q_0 does not have zy but has y^3, and Q_1 has z. In this case, B and B' are of type E_8^2.

3.4. **Non-Cartier case.** Suppose the divisor (D) is not Cartier. Let l be the order of the class $[(D)]$ in $\mathrm{Cl}(B)$. We have $l \mid (p-1)$ by the Rudakov–Shafarevich formula (Proposition 2.12). Let $f\colon \bar{X} = \mathrm{Spec}\,\bar{B} \to X = \mathrm{Spec}\,B$ be the covering (étale outside the closed point) trivializing the class, and let $\bar{\mathfrak{m}} \subset \bar{B}$ be the maximal ideal. Let $\bar{D}$ be a derivation proportional to f^*D with $(\bar{D}) = 0$ (which exists since $[(f^*D)] = 0$). Here the pullback f^*D is first defined on the étale locus of f, and then extended to $\bar{X}$ since f is étale in codimension 1. Let $\bar{X}' = \mathrm{Spec}\,\bar{B}' = \bar{X}^{\bar{D}}$.

Let $G = \mathrm{Gal}(\bar{X}/X) = \mathrm{Gal}(\bar{X}'/X') \cong \mathbb{Z}/l\mathbb{Z}$. For $g \in G$, the derivation $g^*\bar{D}$ can be written as $g^*\bar{D} := g^\circ \bar{D}(g^\circ)^{-1}$, where $g^\circ\colon \bar{B} \to \bar{B}$ corresponds (contravariantly) to $g\colon \bar{X} \to \bar{X}$. Since $\bar{B}^{g^*\bar{D}} = \bar{B}^{\bar{D}}$ and $(g^*\bar{D}) = 0 = (\bar{D})$, there exists $\beta_g \in \bar{B}^*$ such that $g^*\bar{D} = \beta_g \cdot \bar{D}$. We have $\beta_{hg} = g^\circ(\beta_h)\beta_g$. This shows that ($\beta$ is a 1-cocycle and), since $G \curvearrowright \bar{B}/\bar{\mathfrak{m}} = k$ is trivial, the map $\rho\colon G \xrightarrow{\beta} \bar{B}^* \to (\bar{B}/\bar{\mathfrak{m}})^* = k^*$ is a homomorphism.

We will show that this $\rho\colon G \to k^*$ is injective. Let $G_1 := \mathrm{Ker}\,\rho$. Let

$$\bar{D}_1 := \sum_{h \in G_1} h^*\bar{D} = \big(\sum_{h \in G_1} \beta_h\big) \cdot \bar{D}.$$

Then $\bar{D}_1$ is G_1-invariant, hence descends to a derivation on $\bar{B}^{G_1}$. Since $\sum_{h \in G_1} \beta_h \in \bar{B}^*$ (since $\sum_{h \in G_1} \beta_h \equiv \sum 1 = |G_1| \not\equiv 0 \pmod{\bar{\mathfrak{m}}}$), $\bar{D}_1$ has no divisorial fixed locus. In other words, the pullback of the class $[(D)]$ to $\bar{X}/G_1$ is trivial. By the definition of $\bar{X}$, we obtain $G_1 = \{1\}$.

Summarizing the arguments so far, the at most RDP $\bar{X} = \mathrm{Spec}\,\bar{B}$, the action of the cyclic group G, and the derivation $\bar{D}$ satisfy the following properties.

- $\bar{X} \to \bar{X}/G = X$ is induced by a non-trivial divisor class of order dividing $p-1$ on an at most RDP X. Such $\bar{X} \to X$ can be listed from the list of RDPs and their class groups.
- $\bar{D}$ is p-closed with $(\bar{D}) = 0$, and $\bar{X}^{\bar{D}}$ is at most RDP. Such $\bar{X} \to \bar{X}^{\bar{D}}$ is classified in the cases (a) and (b) of Theorem 1.1.
- For each $g \in G$ we have $g^*\bar{D} = \beta_g \cdot \bar{D}$ for some unit $\beta_g \in \bar{B}^*$, and the homomorphism $\rho\colon G \to k^*\colon g \mapsto (\beta_g \bmod \bar{\mathfrak{m}})$ is injective.

The first two conditions imply that $(\mathrm{Sing}(B), \mathrm{Sing}(\bar{B}), \mathrm{Sing}(\bar{B}^{\bar{D}}))$ is one in Table 6 (with $n \geq 1$), that $(\bar{B}, G) \cong (k[[x, y, z]]/\bar{F}, \langle g \rangle)$ as in the table, and that $(\bar{B}, \bar{D}) \cong (k[[x, y, z]]/(\bar{F}), u\bar{D})$ as in the table with $u \in \bar{B}^*$, but not necessarily under the same isomorphism $\bar{B} \cong k[[x, y, z]]/(\bar{F})$. We show that:

- Among the candidates in Table 6, those without check marks are inappropriate.
- In the other cases (with check marks), we have an isomorphism $\bar{B} = k[[x, y, z]]/(\bar{F})$ with $\bar{F}$ and the both actions of $\bar{D}$ and G as in the table, up to replacing $\bar{D}$ by a unit multiple.

Then it is straightforward to compute $B = \bar{B}^G$ and the restrictions to B of suitable multiples of $\bar{D}$, which gives the formulas as in Table 4.

First consider the first 4 cases of Table 6, where $\bar{D}$ satisfies (a). Let $V = \bar{\mathfrak{m}}/\bar{\mathfrak{m}}^2$ and $V' = \mathrm{Im}(\bar{\mathfrak{m}}' \to V)$. These spaces are stable under G, hence G acts on V/V', which is 1-dimensional by the classification. The linear map $V/V' \to \bar{B}/\bar{\mathfrak{m}} = k$ induced by $\bar{D}$ is nonzero (since (a)), and is G-equivariant where the action on k is ρ^{-1}.

In the case where $\bar{B}$ is smooth, a generator g of G acts on V/V' by a primitive l-th root ζ of 1 and on V' by ζ^{-1}. We obtain lifts $y \in \bar{\mathfrak{m}}'$ and $x \in \bar{\mathfrak{m}}$ with $g(y) = \zeta^{-1}y$ and $g(x) = \zeta x$.

In the other 3 cases where $\bar{B}$ is of type A_{np-1} there is, since $np \geq p \geq 3$, a canonical 2-element set $\{kx, ky\} \subset \mathbb{P}(V)$ characterized by the property that $xy \in \bar{\mathfrak{m}}^3$. Then the order of the action of G on V/V' is $1, 2, 2$ respectively, by the classification. Since this should be equal to $l = l, 2, 4$, the first and the third cases are inappropriate. Consider the case where $\bar{B}$ is of type A_{np-1} and

TABLE 6. Candidates for $(B, \bar{B}, \bar{B}^{\bar{D}})$

	p	l	B	$\bar{B}$	$\bar{B}^{\bar{D}}$	$\bar{F}$	$\bar{D}$	g
✓	any	l	A_{l-1}	smooth	smooth		$0,1$	$\zeta x, \zeta^{-1}y$
	any	l	A_{lnp-1}	A_{np-1}	A_{n-1}	$xy - z^{np}$	$0,0,1$	$\zeta x, \zeta^{-1}y, z$
✓	any	2	$D_{np/2+2}$ (n even)	A_{np-1}	A_{n-1}	$xy - z^{np}$	$0,0,1$	$y, x, -z$
	any	4	D_{np+2} (n odd)	A_{np-1}	A_{n-1}	$xy - z^{np}$	$0,0,1$	$y, -x, -z$
	any	l	A_{ln-1}	A_{n-1}	A_{np-1}	$xy - z^{n}$	$x,-y,0$	$\zeta x, \zeta^{-1}y, z$
✓	any	2	$D_{n/2+2}$ (n even)	A_{n-1}	A_{np-1}	$xy - z^{n}$	$x,-y,0$	$y, x, -z$
	any	4	D_{n+2} (n odd)	A_{n-1}	A_{np-1}	$xy - z^{n}$	$x,-y,0$	$y, -x, -z$
✓	3	2	E_7^1	E_6^1	E_6^1	$z^2 + x^3 + y^3 + x^2y^2$	$x,-y,0$	$y, x, -z$

B is of type $D_{np/2+2}$, with n even. We sketch how to find x, y, z. We may assume the nontrivial element $g \in G$ interchanges x and y. We take z such that x, y, z generate $\bar{\mathfrak{m}}$ and $g(z) = -z$. Since $\bar{B}$ is of type A_{np-1}, we have $xy = cz^{np}$ ($c \in k^*$) up to terms of high degree. By a standard argument we can modify x, y, z so that the actions of G and $\bar{D}$ are as above and they satisfy $xy = z^{np}$. We omit the details.

The next 3 cases of Table 6, where $\bar{B}$ is of type A_{n-1} and $\bar{B}'$ is of type A_{pn-1}, are simply the dual of the previous cases.

Finally, consider the case $\bar{B}$ is of type E_6^1 in characteristic 3. As before, let $V = \bar{\mathfrak{m}}/\bar{\mathfrak{m}}^2$. The line $kz \in \mathbb{P}(V)$ is characterized by $z^2 \in \bar{\mathfrak{m}}^3$, and we can take a lift $z \in \bar{\mathfrak{m}}$ satisfying both $g(z) = -z$ and $D(z) = 0$. We take $x, y \in \bar{\mathfrak{m}}$ such that x, y, z generate $\bar{\mathfrak{m}}$, $g(x, y) = (y, x)$, and $D(x, y) = (ix, -iy)$. Since $\bar{B}$ is of type E_6^1, we have $z^2 + c(x^3 + y^3) + c'x^2y^2 + $ (higher) $= 0$ for some $c, c' \in k^*$. We may assume $c = c' = 1$. Again we omit the details to modify x, y, z.

4. Smooth coverings of RDPs

In this section we construct smooth coverings of a certain type for every RDPs (Theorem 4.1), which refines the result of Artin. We also prove some variants (Corollary 4.2 and Proposition 4.3).

Theorem 4.1. *Suppose B is a complete local RDP in characteristic $p > 0$. Then there exists a sequence $B \subset C_0 \subset C_1 \subset \cdots \subset C_n$ ($n \geq 0$) of finite extensions of complete local algebras such that*

- *C_n is smooth, and all other C_i are RDPs,*
- *$\operatorname{Spec} C_0 \to \operatorname{Spec} B$ is the universal covering, and*
- *$\operatorname{Spec} C_{i+1} \to \operatorname{Spec} C_i$ are purely inseparable of degree p, of type (b),*

if and only if $(p, \operatorname{Sing}(B)) \neq (2, E_8^1)$. If B is of type E_8^1 in characteristic 2, then we only have a weaker assertion where C_i are not assumed to be RDPs.

Proof. First suppose B is not of type E_8^1 in characteristic 2. As mentioned in Convention 1.3(2), the universal covering of an at most RDP is again an at most RDP. Hence it suffices to consider the case B is simply-connected. See Table 7 (due to [Art77]) for the fundamental groups and the universal coverings of RDPs. By Theorem 1.1 we have the following sequences.

$A_{p^e-1} \leftarrow A_{p^{e-1}-1} \leftarrow \ldots \leftarrow A_0$.

For $p = 5$: $E_8^0 \leftarrow A_0$.

For $p = 3$: $E_8^0 \leftarrow A_0$, $E_8^1 \leftarrow E_6^0 \leftarrow A_0$.

For $p = 2$: $E_8^3 \leftarrow E_7^2 \leftarrow D_5^{1/2} \leftarrow A_1 \leftarrow A_0$, $E_8^0 \leftarrow A_0$, $E_7^1 \leftarrow D_7^{1/2} \leftarrow A_0$, $E_7^0 \leftarrow A_0$, $D_{2m}^0 \leftarrow A_0$, $D_{2m+1}^{1/2} \leftarrow A_1 \leftarrow A_0$, $D_{4k-l}^{k-l/2} \leftarrow D_{4k-2l}^{(k-l)^+} \leftarrow \ldots \leftarrow D_{4k-2^n l}^0 \leftarrow A_0$ if $l > 0$ and $k - l/2 > 1/2$, where n is the minimum integer with $4k - 2^{n-1}l \leq 0$.

Note that the correspondence $(r, N) = (k - l/2, 4k - l)$ gives a bijection between the sets

$$\{(r, N) \in \frac{1}{2}\mathbb{Z} \times \mathbb{Z} \mid N \geq 4, 2r - N \in \mathbb{Z}, 0 \leq r \leq (N/2) - 1\}$$

and

$$\{(k, l) \in \mathbb{Z} \times \mathbb{Z} \mid k \geq 1, k \geq l/2, (k, l) \neq (1, 1), (1, 2)\},$$

and $4r < N$ if and only if $l > 0$.

Now suppose B is of type E_8^1 in characteristic 2. Then $C_0 = B$ is not smooth and, by Theorem 1.1, it does not admit any purely inseparable covering of degree p that is at most RDP. Hence the assertion with the RDP assumption does not hold. To prove the weaker assertion (without the RDP assumption), we will give a sequence $B = C_0 \subset C_1 \subset C_2$ of purely inseparable coverings of degree p of type (b) such that (C_1 is a non-RDP and) C_2 is an RDP of another type. Take $c \in k$ and let

$$C_2 = k[[t, v, y]]/(v^2(1 + ty^2) + (c + y)(y^4 v + t^2))$$
$$\supset C_1 = k[[x, y, z]]/(x^2 + z^3 + zy^2(y^2(c + y) + zx)),$$
$$\supset C_0 = k[[z, Y, w]]/(w^2 + Y^3 + z^2(z^3 + zYw + czY^2)),$$

where

$$x = \frac{y^2 v + t^3}{1 + ty^2}, \quad z = \frac{y^4 v + t^2}{1 + ty^2}, \quad w = y^3 + xz, \quad Y = y^2.$$

Define derivations $\delta_1 \colon C_2 \to C_2$ and $\delta_2' \colon C_1 \to C_1$ by

$$\delta_1(t, v, y) = (y^2, z, 0), \qquad \delta_2'(x, y, z) = (y^2, z, 0).$$

Then it is straightforward to check that C_2 is an RDP of type D_8^0 or $D_{11}^{1/2}$ if $c \neq 0$ or $c = 0$ respectively, C_0 is an RDP of type E_8^1, both δ_1 and δ_2' are of additive type, both $\text{Fix}(\delta_1)$ and $\text{Fix}(\delta_2')$ consist precisely of the closed point, $C_1 = C_2^{\delta_1}$, and $C_0 = C_1^{\delta_2'}$.

For the (non-RDP) singularity of C_1, see Remark 4.5. $\square$

Corollary 4.2. *Suppose B_1 and B_2 are at most RDPs (over the same algebraically closed field k) in characteristic $p > 0$ having isomorphic local fundamental groups $\pi_{\text{loc}}^{\text{et}}(B_1) \cong \pi_{\text{loc}}^{\text{et}}(B_2)$. If $(p, \pi_{\text{loc}}^{\text{et}}) \neq (2, \text{Dih}_n)$ for any $n \geq 1$, then B_1 and B_2 are connected by a finite purely inseparable morphism.*

Here Dih_n is the dihedral group (of order $2n$). In particular, $\text{Dih}_1 = C_2$. Again, see Table 7 (due to [Art77]) for the fundamental groups and the universal coverings of RDPs.

Proof. If $\pi_{\text{loc}}^{\text{et}} = 0$, then both B_1 and B_2 are connected to $k[[x, y]]$ by Theorem 4.1.

Suppose $\pi_{\text{loc}}^{\text{et}} \neq 0$. Theorem 1.1 shows that B_1 and B_2 are connected by a purely inseparable morphism of degree p if (B_1, B_2) is one of the following:

$$(A_{np^e-1}, A_{np^{e'}-1}), \ (D_{np^e+2}, D_{np^{e'}+2}) \ (p \neq 2), \ (A_1, E_7^0) \ (p = 3), \ (A_2, E_6^0) \ (p = 2).$$

According to Table 7, general cases (with $(p, \pi_{\text{loc}}^{\text{et}}) \neq (2, \text{Dih}_n)$) follow from these cases. $\square$

Also, in some (not all) cases the order of the étale and purely inseparable coverings in Theorem 4.1 can be reversed.

TABLE 7. Etale fundamental groups and the universal coverings of RDPs

char	univ. cov.	RDP		π_1
any	A_{p^e-1}	A_{np^e-1}	$(p\nmid n)$	C_n: cyclic (of order n)
$\neq 2$	A_{p^e-1}	D_{np^e+2}	$(p\nmid n)$	$\widetilde{\mathrm{Dih}}_n$: binary dihedral (of order $4n$)
$\neq 2,3$	smooth	E_6		$\widetilde{\mathrm{Tet}}$: binary tetrahedral (of order 24)
$\neq 2,3$	smooth	E_7		$\widetilde{\mathrm{Oct}}$: binary octahedral (of order 48)
$\neq 2,3,5$	smooth	E_8		$\widetilde{\mathrm{Ico}}$: binary icosahedral (of order 120)
5	E_8^0	E_8^0		0
5	smooth	E_8^1		C_5
3	E_6^0	E_6^0		0
3	smooth	E_6^1		C_3
3	E_6^0	E_7^0		C_2
3	smooth	E_7^1		C_6
3	E_8^r	E_8^r	$(r=0,1)$	0
3	smooth	E_8^2		$\widetilde{\mathrm{Tet}}$: binary tetrahedral (of order 24)
2	$A_{2^{e+1}-1}$	D_N^r	$(4r>N)$	$\mathrm{Dih}_{(4r-N)'}$, $4r-N=2^e(4r-N)'$, $2\nmid(4r-N)'$: dihedral (of order $2\cdot(4r-N)'$)
2	smooth	D_N^r	$(4r=N)$	C_2
2	D_N^r	D_N^r	$(4r<N)$	0
2	D_4^0	E_6^0		C_3
2	smooth	E_6^1		C_6
2	E_7^r	E_7^r	$(r=0,1,2)$	0
2	smooth	E_7^3		C_4
2	E_8^r	E_8^r	$(r=0,1,3)$	0
2	smooth	E_8^2		C_2
2	smooth	E_8^4		metacyclic of order 12

Proposition 4.3. *Suppose B is as in Theorem 4.1. Then there exists a sequence*

$$B = B_0 \subset B_1 \subset \cdots \subset B_n \subset C \ (n \geq 0)$$

of finite extensions of complete local algebras such that

- *C is smooth, and all other B_i are RDPs,*
- *$\mathrm{Spec}\, C \to \mathrm{Spec}\, B_n$ is the universal covering, and*
- *$\mathrm{Spec}\, B_{i+1} \to \mathrm{Spec}\, B_i$ are purely inseparable of degree p, of type (b) or (c),*

if and only if $(p, \mathrm{Sing}(B)) \neq (2, E_8^1), (2, D_N^r)\ (4r > N)$.

Proof. Again, this is impossible for E_8^1 in characteristic 2.

Next, if $p = 2$ and $\mathrm{Sing}(B)$ belongs to the set $\{D_N^r \mid 4r > N\}$, then by Theorem 1.1 so does any RDP that is a purely inseparable covering of B of degree p, and no member of this family has smooth universal covering.

Suppose B is none of the above. We shall describe $\mathrm{Spec}\, B_0 \leftarrow \mathrm{Spec}\, B_1 \leftarrow \ldots \leftarrow \mathrm{Spec}\, B_n$ with B_n having smooth universal covering. By Theorem 4.1, it remains to consider the cases with $\pi_{\mathrm{loc}}^{\mathrm{et}}(B) \neq 0$.

$A_{np^e-1} \leftarrow A_{np^{e-1}-1} \leftarrow \ldots \leftarrow A_{n-1}$ $(p \nmid n)$.

$D_{np^e+2} \leftarrow D_{np^{e-1}+2} \leftarrow \ldots \leftarrow D_{n+2}$ $(p \neq 2, p \nmid n)$. (Here, the corresponding derivations satisfy (c).)

For $p = 3$: $E_7^0 \leftarrow A_1$.

For $p = 2$: $E_6^0 \leftarrow A_2$. $\hfill \square$

Remark 4.4. In fact, we can show that $\operatorname{Spec} C_2 \to \operatorname{Spec} C_0$ we gave in the case of E_8^1 of the proof of Theorem 4.1 is the quotient by an action of the group scheme α_4. The existence of such a quotient morphism will be used in [LMM21, Remark 9.32].

Here $\alpha_{p^e} = \operatorname{Spec} R$, $R = k[\varepsilon]/(\varepsilon^{p^e})$, is the group scheme with comultiplication given by $R \to R \otimes R \colon \varepsilon \mapsto \varepsilon \otimes 1 + 1 \otimes \varepsilon$. Just as α_p-actions correspond to derivations of additive type, α_{p^e}-actions can be described by using so-called higher derivations. For simplicity, we consider only the case $p^e = 4$. Giving an α_4-action on $\operatorname{Spec} A$ is equivalent to giving two k-linear maps $\delta_1, \delta_2 \colon A \to A$ satisfying the following properties:

- $\delta_1^2 = \delta_2^2 = 0$ and $\delta_1 \delta_2 = \delta_2 \delta_1$,
- δ_1 is a derivation, and
- δ_2 satisfies $\delta_2(ab) = \delta_2(a)b + \delta_1(a)\delta_1(b) + a\delta_2(b)$.

The corresponding action is

$$A \to R \otimes A \colon a \mapsto \sum_{0 \leq i < 4} \varepsilon^i \otimes \delta_i(a),$$

where $\delta_0 := \operatorname{id}$ and $\delta_3 := \delta_1 \delta_2 = \delta_2 \delta_1$. The quotient of $\operatorname{Spec} A$ by α_4 (resp. its subgroup scheme α_2) is the spectrum of the subalgebra $\operatorname{Ker} \delta_1 \cap \operatorname{Ker} \delta_2$ (resp. $\operatorname{Ker} \delta_1$).

Now let δ_1 be the one given in the proof and define $\delta_2 \colon C_2 \to C_2$ by

$$\delta_2(t, v, x, y, z) = (0, 1, y^2, z, 0)$$

and the Leibniz-like formula. Note that δ_2 extends $\delta_2' \colon C_1 \to C_1$ given in the proof. It is straightforward to check that this (δ_1, δ_2) satisfy the required conditions.

Remark 4.5. The equation of C_1 in the case of E_8^1 of the proof of Theorem 4.1 is, after a coordinate change, of the form $x^2 + z^3 + zy^5 + xy^i +$ (higher), with $i = 6$ if $c \neq 0$ and $i = 7$ if $c = 0$. In characteristic 0, $k[[x, y, z]]/(x^2 + z^3 + zy^5)$ is an exceptional unimodal singularity usually denoted by the symbol E_{13}, where the index 13 stands for the Milnor number $(\dim_k k[[x, y, z]]/(F_x, F_y, F_z)$ for $k[[x, y, z]]/(F))$ in characteristic 0, although in characteristic 2 it is not equal to the Milnor number nor the Tjurina number $(\dim_k k[[x, y, z]]/(F, F_x, F_y, F_z))$. It is straightforward to check that our C_1 has Tyurina number 20 if $c \neq 0$ and 22 if $c = 0$.

References

[Art77] Michael Artin, *Coverings of the rational double points in characteristic p*, Complex analysis and algebraic geometry, Iwanami Shoten, Tokyo, 1977, pp. 11–22. DOI: 10.1017/cbo9780511569197.003.

[Gan82] Richard Ganong, *Plane Frobenius sandwiches*, Proc. Amer. Math. Soc. **84** (1982), no. 4, 474–478. DOI: 10.1090/s0002-9939-1982-0643732-1.

[LMM21] Christian Liedtke, Gebhard Martin, and Yuya Matsumoto, *Torsors over the rational double points in characteristic p.* arχiv: 2110.03650v1.

[Lip69] Joseph Lipman, *Rational singularities, with applications to algebraic surfaces and unique factorization*, Inst. Hautes Études Sci. Publ. Math. **36** (1969), 195–279. DOI: 10.1007/bf02684604.

[Mat22] Yuya Matsumoto, *On μ_n-actions on K3 surfaces in positive characteristic.* arχiv: 1710.07158v4.

[Mat21a] _____, μ_p- and α_p-actions on K3 surfaces in characteristic p, J. Algebraic Geom., to appear. arχiv: 1812.03466v5.

[Mat21b] _____, Inseparable maps on W_n-valued Ext groups of non-taut rational double point singularities and the height of K3 surfaces. arχiv: 1907.04686v3.

[Mat89] Hideyuki Matsumura, Commutative ring theory, 2nd ed., Cambridge Studies in Advanced Mathematics, vol. 8, Cambridge University Press, Cambridge, 1989. DOI: 10.1017/s0013091500006702.

[RS76] Aleksei Nikolaevich Rudakov and Igor Rostislavovich Shafarevich, Inseparable morphisms of algebraic surfaces, Izv. Akad. Nauk SSSR Ser. Mat. **40** (1976), no. 6, 1269–1307, 1439 (Russian). DOI: 10.1070/im1976v010n06abeh001833.

Yuya Matsumoto, Department of Mathematics, Faculty of Science and Technology, Tokyo University of Science, 2641 Yamazaki, Noda, Chiba, 278-8510, Japan

Email address: matsumoto_yuya@ma.noda.tus.ac.jp

Email address: matsumoto.yuya.m@gmail.com