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HOMOLOGICALLY TRIVIAL INTEGRABLE DEFORMATIONS OF GERMS OF HOLOMORPHIC FUNCTIONS

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Abstract. We study analytic deformations by integrable 1-forms a germ of a holomorphic function at the origin of the complex affine space in dimension three or higher. We prove that, under some mild nondegeneracy conditions on the function germ, the existence of a simple normal form for the deformation is equivalent to a homological condition: the annihilation of the deformed one-form in the first homology group of the non-singular fibers of the function germ. In many cases this implies the existence of a holomorphic first integral for the deformation.

1. Introduction and main results

Let ω be an integrable one-form defined in a neighborhood $U \subset \mathbb{C}^n$, $n \geq 2$ of the origin. Denote by $\operatorname{sing}(\omega)$ the singular set of ω in U. The integrability condition $\omega \wedge d\omega = 0$, assures that there is a holomorphic foliation $\mathcal{F}$ of codimension one, defined in $U \setminus \operatorname{sing}(\omega)$, with tangent space given by $T_p(\mathcal{F}) = \ker(\omega(p))$, for all $p \in U$. It is well-known that we may assume that $\operatorname{cod} \operatorname{sing}(\omega) \geq 2$ ([9]). In this case we write $\operatorname{sing}(\mathcal{F}) = \operatorname{sing}(\omega)$ for the singular set of $\mathcal{F}$. The pair $(\mathcal{F}, \operatorname{sing}(\mathcal{F}))$ is then called a codimension one holomorphic foliation with singularities in U.

One of the main questions about holomorphic foliations with singularities is whether they admit a holomorphic first integral, i.e., a nonconstant function germ $F \in \mathcal{O}_n$ which is constant in the leaves of the foliation. In terms of the 1-form ω this is equivalent to say that $dF \wedge \omega = 0$. The existence of a holomorphic first integral is a very classical question. We mention the works of Mattei-Moussu ([12]) and Malgrange ([10]).

Our standpoint is a bit different. We focus on the search for verifiable conditions, i.e., conditions that we can try by computations and decide whether the first integral exists or not. For this sake, as a first step, we consider deformations of a foliation with a holomorphic first integral. More precisely, we shall consider one-parameter *analytic families* of 1-forms $\{\omega^t\}_{t\in\mathbb{C}_t}$, depending analytically on a parameter $t \in \mathbb{C}_t$ where $\mathbb{C}_t$ is the germ of space $(\mathbb{C}, 0)$. The 1-form ω^t is holomorphic in a neighborhood $U \subset \mathbb{C}^n$ of the origin, depends analytically on t and satisfies the integrability condition $\omega^t \wedge d\omega^t = 0$. Finally, we assume that $\omega^0 = df$, *i.e.*, $\omega^t = df + \sum\limits_{j=1}^{\infty} t^j \omega_j$ where $f\colon U \to \mathbb{C}$ is a nonconstant holomorphic function with $f(0) = 0$. For $n = 2$ the integrability condition plays no role and there is too much freedom for the deformation. This case has been addressed in [8] for $f = xy \in \mathcal{O}_2$. Throughout this paper we are mainly concerned with the case $n \geq 3$. The idea is to use as a frame the basic singular fibration given by the level hypersurfaces of f in U. Given $c \in \mathbb{C}$ we shall denote by $(f_c) \subset U$ the level hypersurface $(f = c) := \{z \in U, f(z) = c\}$. By a *1-cycle* in (f_c) we shall mean an element γ_c of the first homology group $H_1((f_c), \mathbb{Z})$.

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Definition 1.1. The deformation ω^t of $\omega^0 = df$ is *homologically trivial* if for every t the 1-form ω^t vanishes in the 1-homology group of the leaf $L_c : (f = c)$, for all $c \neq 0$, i.e., $\oint_{\gamma_c} \omega^t = 0$, for all $\gamma_c \in H_1(L_c, \mathbb{Z})$, for all $0 \neq c \approx 0$.

Let us recall a notion from [2] and [7]. A germ of holomorphic function $f \in \mathcal{O}_n, n \geq 2$ with $f(0) = 0$ will be called *LS-type* (for *Lê-Saito type*) if it writes $f = f_1 \dots f_{r+1}$ as a product of irreducible reduced germs $f_j \in \mathcal{O}_n$, $n \geq 3$, $j = 1, ..., r+1$, pairwise relatively prime, such that the hypersurface germ $X_f : f = 0$ induced by f at the origin, has only normal crossings singularities except for a codimension ≥ 3 analytic subset.

In the course of this work we shall deal with *integrable* 1-forms that write as $\omega = adf + dh$ for some holomorphic germs $a, f, h \in \mathcal{O}_n, n \geq 2$. The integrability condition $\omega \wedge d\omega = 0$ is equivalent to $da \wedge df \wedge dh = 0$. We shall refer to such a 1-form as a *adfdh-form*.

In this work we obtain the following characterization of *adfdh*-forms in a deformation of a local fibration, in terms of the vanishing of the forms of the deformation in the first holomogy group of the fibers of the original fibration:

Theorem 1.2. *Let $f \in \mathcal{O}_n$ be a LS-type germ. Then any homologically trivial deformation $\{\omega^t\}_{t\in\mathbb{C}}$ of df is by adfdh-forms, i.e., ω^t is a adfdh-form for each t.*

In dimension two, a *singular adfdh-form* germ at the origin $0 \in \mathbb{C}^2$ is, in suitable local coordinates $(x, y) \in (\mathbb{C}^2, 0)$, of the form $\omega = xdy + d\alpha(x, y)$ for some holomorphic germ $\alpha \in \mathcal{O}_2$ such that $d\alpha(0, 0) = 0$.

Two germs of holomorphic functions $g, h \in \mathcal{O}_n, n \geq 2$ are in *general position* if and only if $\operatorname{cod}\operatorname{sing}(dg \wedge dh) \geq 2$. This means that the map germ

$$(g, h)\colon (\mathbb{C}^n \times \mathbb{C}^n, (0, 0)) \to (\mathbb{C}^2, 0),\ (x, y) \mapsto (f(x), g(y))$$

is a submersion off a codimension ≥ 2 analytic subset germ.

A germ of *adfdh*-form $\omega = adf + dh$ is called *generic* if at least two of the functions a, f, h are in general position.

Theorem 1.3. *Let ω be a germ of a generic adfdh-form at the origin $0 \in \mathbb{C}^n$, $n \geq 3$. Then there are two possibilities:*

(1) ω admits a holomorphic first integral.
(2) ω is a holomorphic pull-back of a singular adfdh-form in dimension two.

In general, for nongeneric *adfdh*-forms in dimension $n \geq 3$ we have:

Theorem 1.4. *Let $\omega = adf + dh$ be a germ of adfdh-form at the origin $0 \in \mathbb{C}^n, n \geq 3$. There is a germ of analytic subset $Z \subset \mathcal{U}^n$ where $\mathcal{U}^n$ is a disc type neighborhood of the origin $0 \in \mathbb{C}^n$, a galoisian covering space $P\colon \widetilde{\mathcal{U}^n \setminus Z} \to \mathcal{U}^n \setminus Z$ such that: the lifted 1-form $\tilde{\omega} = P^*(\omega)$ is the holomorphic pull-back $\tilde{\omega} = \tilde{\psi}^*(xdy + \alpha(x, y))$ of a dimension two adfdh-form by the holomorphic submersion*

$$\tilde{\psi} = (\tilde{f}, \tilde{a}) = (f \circ P, a \circ P)\colon \widetilde{\mathcal{U}^n \setminus Z} \to \mathbb{C}^2.$$

In the above statement we may assume that either Z is empty, in which case ω is already generic, or it is of pure dimension one: indeed, the codimension ≥ 2 components of Z shall be no obstruction to our construction.

Theorem 1.2 is a version for $n \geq 3$ of previous results found in [8]. It is also related to the pioneering work of Ilyashenko ([6]) and Muciño-Raymundo ([13]) on integrability of polynomial differential 1-forms.

2. Stein factorization property and Malgrange's theorem

As it is well-known, the ring $\mathcal{O}_n$ of germs at $0 \in \mathbb{C}^n$ of is a local ring and a unique factorization domain (cf. Theorem A.7 page 7 [3]). An element $f \in \mathcal{O}_n$ is *reducible* over $\mathcal{O}_n$ if it can be written as a product $f = gh$ where $g, h \in \mathcal{O}_n$ are nonunits of $\mathcal{O}_n$. Elements not having this property are called *irreducible* over $\mathcal{O}_n$ (see [4] page 71 Definition 4). A germ $f \in \mathcal{O}_n$ is *reduced* if its unique factorization in the unique factorization domain $\mathcal{O}_n$ has no repeated factors. This is equivalent to assuming that the critical locus $\Sigma(f)$ which is defined as the variety given by $\frac{\partial f}{\partial z_j} = 0, j = 1, ..., n,$ has finite dimension (as a germ at the origin) ([11] pages 1-2). A holomorphic function germ $f \in \mathcal{O}_n$ is called *primitive* if given a representative $f_D \colon D \to \mathbb{C}$ defined in a small disc $D \subset \mathbb{C}^n$ centered at the origin, then f_D has connected fibers. In view of the local version of Stein factorization theorem (see the factorization theorem in [12] page 472) this means that if we have another function $\tilde{f} \colon D \to \mathbb{C}$ such that $\tilde{f}$ is constant on each fiber $f_D^{-1}(c), c \in \mathbb{C}$ of f_D, then $\tilde{f}$ can be factorized as $\tilde{f} = \xi \circ f_D$ for some holomorphic map $\xi \colon U \subset \mathbb{C} \to V \subset \mathbb{C}$.

Classical Stein factorization theorem ([5] pages 276, 280 and 366) states that a proper morphism can be factorized as a composition of a finite mapping and a proper morphism with connected fibers. Thus such a proper morphism has the factorization property *if* it has connected fibers.

We shall need a definition:

Definition 2.1. A holomorphic function $f \colon (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ $f = (f_1, ..., f_p)$ *satisfies the factorization property* if:

For every holomorphic function $h \colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ such that $dh \wedge df_1 \wedge \ldots \wedge df_p = 0$ there is a holomorphic function $\varphi \colon (\mathbb{C}^p, 0) \to (C, 0)$ such that $h = \varphi \circ f$.

We have the following condition assuring the factorization property due to Malgrange, not related to whether the map f has connected fibers:

Theorem 2.2 ([10])**.** *Assume that* $\operatorname{cod}\operatorname{sing}(df_1 \wedge ... \wedge df_p) \geq 2$. *Then* $f = (f_1, ..., f_p)$ *satisfies the factorization property.*

3. LS-type germs

We recall that a germ $f \in \mathcal{O}_n, n \geq 2$ with $f(0) = 0$ is of *LS-type* (*Lê-Saito type*) if the corresponding hypersurface germ $X_f : f = 0$ at the origin $0 \in \mathbb{C}^n$, has, outside of an analytic subset $(Y, 0) \subset (X_f, 0)$ of dimension at most $n - 3$, only singularities of normal crossing type. We have the following general statement below, due to Lê-Saito:

Theorem 3.1 (Lê-Saito, [7] Main Theorem page 1)**.** *Let* $f \in \mathcal{O}_n$, $n \geq 3$ *be a LS-type germ. Then the local fundamental group of the complement of* $(X_f, 0)$ *in* $(\mathbb{C}^n, 0)$ *is abelian. If* f *is reduced then the Milnor fiber of* f *has a fundamental group which is free abelian of rank the number of analytic components of* X_f *at* 0, *minus one.*

3.1. Local topology and homology of the fibers.

Given $f \in \mathcal{O}_n$ a LS-type germ, write $f = f_1 \ldots f_{r+1}$ in terms of germs $f_j \in \mathcal{O}_n$ such that each irreducible component of X_f corresponds to one and only one of the sets $(f_j = 0)$. We shall consider logarithmic 1-forms

$$\theta_\nu = \sum_{j=1}^{r+1} \lambda_j^\nu df_j/f_j, \ \nu = 1, ..., r \geq 1$$

with the following property: the system $\{\theta_1, \dots, \theta_r\}$ is completely independent with respect to $df/f = \sum\limits_{j=1}^{r+1} df_j/f_j$ in the following sense: if $\sum\limits_{\nu=1}^{r} a_\nu\theta_\nu + bdf/f = 0$ for some constants $a_\nu, b \in \mathbb{C}$ then $a_\nu = b = 0$.

This occurs if the following $(r+1)\times(r+1)$ matrix is nonsingular

$$\begin{pmatrix} 1 & 1 & \dots & 1 \\ \lambda_1 & \lambda_2 & \dots & \lambda_{r+1} \\ \\ \lambda_1^2 & \lambda_2^2 & \dots & \lambda_{r+1}^2 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^r & \lambda_2^r & \dots & \lambda_{r+1}^r \end{pmatrix}.$$

Lemma 3.2 ([2])**.** *For each $c \in \mathbb{C}\setminus\{0\}$ the 1-cohomology of the local fiber $L_c : (f = c) \subset (\mathbb{C}^n, 0)$ is generated by the restrictions $\theta_j\big|_{L_c}$, $j = 1, ..., r$.*

As already observed in [2], for most applications we shall take $\theta_j = \frac{df_j}{f_j}$, $j = 1, ..., r$.

The above lemma then shows that the homology of the fibers $L_c, c \neq 0$ is generated by restrictions of a same system of forms to these fibers.

Proposition 3.3 ([2])**.** *Let ω_1 be a germ of a holomorphic 1-form at $0 \in \mathbb{C}^n$, $n \geq 3$ and assume that $d\omega_1 \wedge df = 0$ where $f = f_1 \dots f_{r+1}$ is as in Lemma 3.2 above.*

Then there are $a_1, h_1 \in \mathcal{O}_n, \psi_j \in \mathcal{O}_1, \psi_j(0) = 1$ and $\lambda_j \in \mathbb{C}, j = 1, ..., r$ such that

$$\omega_1 = a_1 df + dh_1 + \sum_{j=1}^{r} \lambda_j f \psi_j(f)\theta_j.$$

The next result states a normal form in the case of a 1-form that vanishes in the first homology group of the fibers of a map $f \in \mathcal{O}_n, n \geq 3$.

Corollary 3.4. *Let ω_1 be a germ of a holomorphic 1-form at $0 \in \mathbb{C}^n$, $n \geq 3$ and assume that $d\omega_1 \wedge df = 0$ where $f = f_1 \dots f_{r+1}$ is as in Lemma 3.2 above. Assume that ω_1 vanishes in the 1-homology of the fibers $(f = c)$ for all $c \neq 0$ close to 0. Then there are $a_1, h_1 \in \mathcal{O}_n$ such that*

$$\omega_1 = a_1 df + dh_1.$$

Proof. From Proposition 3.3 we have $\omega_1 = a_1 df + dh_1 + \sum\limits_{j=1}^{r} \lambda_j f \psi_j(f)\theta_j$. Given any 1-cycle $\gamma_c \subset (f = c)$ for $c \neq 0$ we have

$$\begin{aligned} 0 &= \oint_{\gamma_c} \omega_1 = \oint_{\gamma_c} \big(a_1 df + dh_1 + \sum_{j=1}^{r} \lambda_j f \psi_j(f)\theta_j\big) \\ &= \sum_{j=1}^{r} \lambda_j c \psi_j(c) \oint_{\gamma_c} \theta_j = c \sum_{j=1}^{r} \psi_j(c)\lambda_j \end{aligned}$$

for all $c \neq 0$ with $|c|$ small enough. Since the λ_j are linearly independent over $\mathbb{C}$ this implies that $\psi_j(c) = 0$, for all j for all $c \neq 0$ with $|c|$ small enough. Hence we conclude that $\psi_j \equiv 0$, for all $j = 1, ..., r$. This proves the corollary. □

Corollary 3.4 can also be obtained from Theorem II pages 405/406 in [1].

4. Homologically trivial deformations and $adfdh$-forms

Let ω^t be a *deformation* of ω_0 a germ of holomorphic 1-form at the origin $0 \in \mathbb{C}^n$, i.e., ω^t is a one-parameter analytic family of germs at the origin $0 \in \mathbb{C}^n$ of holomorphic 1-forms parametrized by $t \in \mathbb{D} \subset \mathbb{C}$. We shall assume that ω^t is integrable for each t, i.e., $\omega^t \wedge d\omega^t = 0$, for all $t \in \mathbb{D}$. We also write

$$\omega^t = \omega_0 + \sum_{j=1}^{\infty} t^j \omega_j \, .$$

The integrability condition $\omega^t \wedge d\omega^t = 0$ gives:

$$\begin{gathered}\omega_0 \wedge d\omega_0 = 0 \\ \omega_0 \wedge d\omega_1 + \omega_1 \wedge d\omega_0 = 0 \\ \omega_2 \wedge d\omega_0 + \omega_1 \wedge d\omega_1 + \omega_0 \wedge d\omega_2 = 0 \\ \vdots\end{gathered}$$

We shall consider the case where ω_0 admits a first integral, more precisely $\omega_0 = df$ for some holomorphic function f. In this case

$$df \wedge d\omega_1 = 0$$

and

$$\begin{gathered}\omega_1 \wedge d\omega_1 + df \wedge d\omega_2 = 0 \\ df \wedge d\omega_3 + \omega_1 \wedge d\omega_2 + \omega_2 \wedge d\omega_1 = 0, \ldots\end{gathered}$$

These are called *equations of the deformation* in the case where $\omega_0 = df$. Notice that $df \wedge d\omega_1 = 0$ means that the 1-form ω_1 is closed in the fibers of f (see for instance [2]).

4.1. **Proof of Theorem 1.2.** We are now in conditions to prove Theorem 1.2. We consider integrable deformations of $\omega_0 = df$ where $f = f_1 \ldots f_{r+1} \in \mathcal{O}_n$, $n \geq 2$ is *reducible* and as in Lemma 3.2. We have

$$\omega^t = df + \sum_{j=1}^{\infty} t^j \omega_j$$

Claim 4.1. *There are families of functions $\{a_t\}_{t \in \mathbb{C}_t}, \{h_t\}_{t \in \mathbb{C}_t}$ such that $a_t, h_t \in \mathcal{O}_n$, for all $t \in \mathbb{C}_t$ and*

$$\omega^t = a_t df + dh_t$$

where $a_0 = 1$.

Proof. We write $\omega^t = df + t\Omega_t$ for $\Omega_t = \sum\limits_{j=1}^{\infty} t^{j-1}\omega_j$. The integrability condition $\omega^t \wedge d\omega^t = 0$ then implies $d\Omega_t \wedge df = 0$. By hypothesis we have $\oint_{\gamma_c} \omega^t = 0$, for all t, for all γ_c in the first homology group of $(f = c)$, for all $c \neq 0$ with $|c|$ small enough. Since $\oint_{\gamma_c} \omega^t = \oint_{\gamma_c} df + t \oint_{\gamma_c} \Omega_t$, for all t we conclude that $\oint_{\gamma_c} \Omega_t = 0$, for all t, for all γ_c in the first homology group of $(f = c)$ as above. According to Corollary 3.4 we conclude that there are germs $b_t, g_t \in \mathcal{O}_n$ such that $\Omega_t = b_t df + dg_t$. Thus we have

$$\begin{aligned}\omega^t &= df + t\Omega_t = df + tb_t df + tdg_t \\ &= (1 + tb_t)df + d(tg_t) = a_t df + dh_t\end{aligned}$$

for $a_t = 1 + tb_t, h_t = tg_t$. $\square$

The next example illustrates Theorem 1.2.

Example 4.2. Let us consider a deformation $\{\omega^t\}_{t\in\mathbb{C}}$ in $(\mathbb{C}^n,0)$ given in coordinates

$$(x,y,z_1,...,z_{n-2})\in\mathbb{C}^2\times\mathbb{C}^{n-2}$$

by $\omega^t=d(xy)+t(xdy-\lambda ydx)$ where $\lambda\in\mathbb{C}$. Then ω^t is integrable and $\omega_0=d(xy)$. Put $f=xy$ then for $c\neq 0$ we have the fiber $L_c:(f=c)\subset\mathbb{C}^2\times\mathbb{C}^{n-2}$ given by $xy=c$ and admitting a 1-homology generator $\gamma_c(s)=(e^{is},ce^{-is},0...,0)$, $0\leq s\leq 2\pi$. We have

$$\int_{\gamma_c}\omega^t/f=t\int_{\gamma_c}\frac{(xdy-\lambda ydx)}{xy}\Big|_{xy=c}=t\int_{\gamma_c}\Big(\frac{dy}{y}-\lambda\frac{dx}{x}\Big)\Big|_{xy=c}=t\int_0^{2\pi}(-i-\lambda i)ds=-ti(1+\lambda)2\pi.$$

Then $\int_{\gamma_c}\omega^t/f=0\Leftrightarrow\lambda=-1\Leftrightarrow\omega^t=(1+t)d(xy)$.

5. On $adfdh$-forms

We now give a proof of our remanning result:

Proof of Theorem 1.3. First we assume that f and h are in general position. In this case from $da\wedge df\wedge dh=0$ we obtain from Malgrange's results in [10] that $a=\alpha(f,h)$ for some germ $\alpha(x,y)\in\mathcal{O}_2$. Then $\omega=\alpha(f,h)df+dh$. This shows that ω is the pull-back $\omega=\sigma^*(\alpha(x,y)dx+dy)$ by the holomorphic map $\sigma\colon(\mathbb{C}^3,0)\to(\mathbb{C}^2,0)$, $z\mapsto(f(z),h(z))$. Since the 1-form $\alpha(x,y)dx+dy$ is (integrable and) nonsingular, it admits a holomorphic first integral. This shows that ω also admits a holomorphic first integral.

Now we assume that a and h are in general position. In this case we have also from Malgrange's theory that $f=\alpha(a,h)$ for some holomorphic function $\alpha(x,y)$. This gives $\omega=ad\alpha(a,h)+dh$ and therefore $\omega=\sigma^*(xd\alpha(x,y)+dy)$ and similarly to the case above we have a holomorphic first integral for ω.

Finally, assume that a and f are in general position. In this case we write $h=\alpha(a,f)$ as above and obtain $\omega=\sigma^*(xdy+d\alpha(x,y)$ for the map $\sigma=(a,f)$. In this case we have a pull-back of the $adfdh$-form $\eta=xdy+d\alpha(x,y)$ in dimension two. If η is nonsingular at the origin then we have a first integral for ω. This shows that the alternatives indicated are the only possible cases. □

About the last case in the proof above we observe that this is equivalent to say that the singular set of $d\omega$ has codimension ≥ 2 at the origin $0\in\mathbb{C}^n$. Indeed, for $\omega=adf+dh$ we have $d\omega=da\wedge df$. This can be seen as a type of *Kupka phenomena* in classical the sense of I. Kupka.

We point-out that not all singular $adfdh$-forms in dimension two admit holomorphic first integrals. Indeed, the 1-form $\omega=xdy+\lambda d(xy)=(1+\lambda)xdy+\lambda ydx$ does not admit a meromorphic first-integral if $\lambda/(1+\lambda)\notin\mathbb{Q}$ (see also Example 4.2).

6. Proof of Theorem 1.4

Let us consider $a,f,h\in\mathcal{O}_n$, $n\geq 3$ such that $da\wedge dh\wedge df=0$ and the corresponding adhdf-form $\omega=adf+dh$. We let Z be the germ of analytic set at the origin $0\in\mathbb{C}^n$ given by

$$Z=\{p\in(\mathbb{C}^n,0),\,(da\wedge df)(p)=0)\}.$$

This means that, for a suitable disc type neighborhood $U\subset\mathbb{C}^n$ of the origin, we can consider representatives $a,f,h\colon U\to\mathbb{C}$ and $Z=\{p\in U,\,da(p)\wedge df(p)=0\}$. Let $p\in Z\setminus U$ be given. According to Malgrange's factorization theorem applied to the germs of a,f,h at p we conclude that there is a germ of holomorphic function $\alpha_p\in\mathcal{O}_2(p)$, such that $h=\alpha_p(a,f)$ as germs at p. This means that in any small neighborhood $U_p\subset U$ of p we have $h=\alpha_p(a,f)$ and

$\omega = adf + d\alpha_p(a, f)$. Given points $p, q \in U \setminus Z$ such that $U_p \cap U_q \neq \emptyset$ then in this intersection we have $h = \alpha_p(a, f) = \alpha_q(a, f)$. By analytic continuation we obtain a galoisian covering $P\colon \widetilde{(U \setminus Z)} \to U \setminus Z$ such that the lift $\tilde{h} = \alpha(\tilde{a}, \tilde{f})$ where for some two variables holomorphic function $\alpha\colon V \subset \mathbb{C}^2 \to \mathbb{C}$. This proves Theorem 1.4.

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