Journal *of* Singularities
Volume 23 (2021), 92-106

received: 7 June 2020
in revised form: 10 May 2021
DOI: 10.5427/jsing.2021.23e

ON THE COMPARISON OF NEARBY CYCLES VIA b-FUNCTIONS

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Abstract. In this article, we give a simple proof of the comparison of nearby and vanishing cycles in the sense of Riemann-Hilbert correspondence following the idea of Beilinson and Bernstein, without using the Kashiwara-Malgrange V-filtration.

1. Introduction

The concepts of nearby and vanishing cycles can be traced back to Grothendieck and they were first introduced by Deligne [Del73]. Nearby and vanishing cycles have been widely studied from different perspectives, for instance by Beilinson [Bei87] algebraically and Kashiwara and Schapira [KS13] in the microlocal setting. They are also very useful, for instance Saito [Sai88] used nearby and vanishing cycles to give an inductive definition of pure Hodge modules.

Using the so-called Kashiwara-Malgrange filtration, as well as its refinement, the V-filtration, Kashiwara [Kas83] defined nearby and vanishing cycles for holonomic $\mathscr{D}$-modules and proved a comparison theorem via Riemann-Hilbert correspondence (see [Kas83, Theorem 2]).

Beilinson and Bernstein constructed the unipotent (or more precisely, nilpotent) nearby and vanishing cycles for holonomic $\mathscr{D}$-modules using b-functions in the algebraic setting in [BB93, §4.2] by using the complete ring $\mathbb{C}[[s]]$; see also [BG12, §2.4] by using localizations of $\mathbb{C}[s]$ without completion. One then can "glue" the open part and the vanishing cycles of a holonomic $\mathscr{D}$-module along any regular functions in the sense of Beilinson [Bei87] (see also [Gin98, Theorem 4.6.28.1] and [Lic09]).

In this article, we extend the construction of nearby and vanishing cycles of Beilinson and Bernstein to other eigenvalues. Then we give a simple and down-to-earth proof of the comparison of nearby and vanishing cycles via Riemann-Hilbert correspondence (see Theorem 1.1). Since we follow the approach of Beilinson and Bernstein, there is no need to use V-filtrations. The construction of the nearby cycles depends on a Cohen-Macaulay type result for $\mathscr{D}$-modules. We provide a proof of the global Cohen-Macaulayness (Lemma 2.7) by using relative $\mathscr{D}$-modules over $\mathbb{C}[s]$, analogous to the proof of a generic Cohen-Macaulay phenomenon for relative holonomic $\mathscr{D}$-modules in general [BVWZ19, Lemma 3.5.2]. But a local version of the Cohen-Macaulayness has been known to experts for a long time (see for instance [Gin86, Corollary 3.8.4] and [BG12, Lemma 2(a)]). Let us also remark that to make the arguments simpler, the proof of the comparison of vanishing cycles is not functorial, while on the contrary that for nearby cycles is functorial.

Let X be a smooth algebraic variety over $\mathbb{C}$ (or a complex manifold) and f a regular function on X (or a holomorphic function on X) and let $\mathcal{M}$ be a holonomic $\mathscr{D}_X$-module. For $\alpha \in \mathbb{C}$, we denote the α-nearby cycles of $\mathcal{M}$ along f by $\Psi_{f,\alpha}\mathcal{M}$ and the vanishing cycles of $\mathcal{M}$ along f by $\Phi_f\mathcal{M}$ (see §2.2 for definitions). The sheaves $\Psi_{f,0}\mathcal{M}$ and $\Phi_f\mathcal{M}$ are the same as $\Psi^{\mathrm{nil}}(\mathcal{M})$ and $\Phi^{\mathrm{nil}}(\mathcal{M})$ in [BG12, §2.4].

2010 *Mathematics Subject Classification.* 14F10; 32S30; 32S40.
Key words and phrases. $\mathscr{D}$-module; perverse sheaf; b-function; nearby cycles; vanishing cycles.

By construction, $\Psi_{f,\alpha}\mathcal{M}$ and $\Phi_f\mathcal{M}$ have the action by s, where s is the independent variable introduced in the definition of b-functions (see §2.1) and $\Psi_{f,\alpha}\mathcal{M}$ only depends on $\mathcal{M}|_U$. The b-function is also called the Bernstein-Sato polynomial. At least for $\mathcal{M} = \mathscr{O}_X$, there are algorithms to compute them with the help of computer algebra programs (for instance Singular and Macaulay2). On the contrary, Kashiwara-Malgrange filtrations are more difficult to deal with from algorithmic perspectives as far as we know. Therefore, it seems easier to deal with nearby and vanishing cycles via b-functions.

Following Beilinson's idea in [Bei87], we define the nearby and vanishing cycles for $\mathbb{C}$-perverse sheaves by using Jordan blocks. Let K be a perverse sheaf of $\mathbb{C}$-coefficients on X. When we talk about perverse sheaves on an algebraic variety over $\mathbb{C}$, we automatically use the Euclidean topology. For $\lambda \in \mathbb{C}^*$ we define the λ-nearby cycles by

$$\psi_{f,\lambda}K \coloneqq \lim_{m\to\infty} i^{-1}Rj_*(K|_U \otimes f_0^{-1}L_m^{1/\lambda}),$$

where $j : U = X \setminus D \hookrightarrow X$ and D is the divisor defined by $f = 0$, $i\colon D \hookrightarrow X$ and $f_0 = f|_U$. Here $L_m^{1/\lambda}$ is isomorphic to the local system on $\mathbb{C}^*$ given by a $m \times m$ Jordan block with eigenvalue λ (see §3 for the construction). The local systems $L_m^{1/\lambda}$, for $m \in \mathbb{Z}$, form a direct system with respect to the natural order on $\mathbb{Z}$. The vanishing cycles of K along f is then defined by

$$\phi_f K \coloneqq \mathrm{Cone}(i^{-1}K \to \psi_{f,1}K).$$

We then have a canonical morphism

$$\mathrm{can}\colon \psi_{f,1}K \to \phi_f K$$

fitting in the tautological triangle

$$i^{-1}K \to \psi_{f,1}K \xrightarrow{\mathrm{can}} \phi_f K \xrightarrow{+1}.$$

The monodromy action on $L_m^{1/\lambda}$ naturally induces the monodromy actions on both $\psi_{f,\lambda}K$ and $\phi_f K$, and they are denoted by T. By construction, $T - \lambda$ acts on $\psi_{f,\lambda}K$ nilpotently. When $\lambda = 1$, $\log T_u$ induces

$$\mathrm{Var}\colon \phi_f K \to \psi_{f,1}K.$$

The morphism Var corresponds to the "Var" morphism defined in [Kas83]. See §4 for details.

The definition of the nearby cycles above coincides with Deligne's construction (see [Bj93, Chapter VI. 6.4.6] and [Rei10, §2] for $\psi_{f,1}K$ and [Wu17, §3] for the general case). However, the definition of the vanishing cycles presented as above by means of the mapping cone is not functorial, but it is the underlying complex of the functorially defined vanishing cycles for a fixed perverse sheaf K (see [Wu17, §3] for details).

We denote by DR the de Rham functor for $\mathscr{D}$-modules (see [Bj93, Chapter I.1.2.17] for definition). The rest of this paper is mainly about the proof of the following comparison theorem.

Theorem 1.1. *Assume that $\mathcal{M}$ is a regular holonomic $\mathscr{D}_X$-module. Then we have*

$$\mathrm{DR}(\Psi_{f,\alpha}\mathcal{M}) \simeq i_*\psi_{f,\lambda}\mathrm{DR}(\mathcal{M})[-1]$$

for every $\alpha \in \mathbb{C}$ and for $\lambda = e^{2\pi\sqrt{-1}\alpha}$. Via the isomorphism above, the monodromy action T on the right-hand side corresponds to the action of

$$\mathrm{DR}(e^{-2\pi\sqrt{-1}s}) = \mathrm{DR}(\lambda \cdot e^{-2\pi\sqrt{-1}(s+\alpha)})$$

on the left-hand side (note that $s + \alpha$ acts on $\Psi_{f,\alpha}\mathcal{M}$ nilpotently), and

$$\mathrm{DR}(\Phi_f\mathcal{M}) \simeq i_*\phi_f\mathrm{DR}(\mathcal{M})[-1].$$

Furthermore, we have natural morphisms of $\mathscr{D}$-modules (or complexes of $\mathscr{D}$-modules)

$$v : \Phi_f \mathcal{M} \to \Psi_{f,0}\mathcal{M} \text{ and } c : \Psi_{f,0}\mathcal{M} \to \Phi_f \mathcal{M}$$

so that there exists an isomorphism of quivers

$$\mathrm{DR}\big(\Psi_{f,0}\mathcal{M} \underset{v}{\overset{c}{\rightleftarrows}} \Phi_f\mathcal{M}\big) \simeq i_*\big(\psi_{f,1}\big(\mathrm{DR}(\mathcal{M})\big) \underset{\mathrm{Var}}{\overset{\mathrm{can}}{\rightleftarrows}} \phi_f\big(\mathrm{DR}(\mathcal{M})\big)[-1]\big).$$

Remark 1.2. In Theorem 1.1, we see that $\Psi_{f,\alpha+k}\mathcal{M}$ correspond to a unique nearby cycles of $\mathrm{DR}(\mathcal{M})$ for all $k \in \mathbb{Z}$. Namely, the Riemann-Hilbert correspondence for nearby cycles is $\mathbb{Z}$-to-1.

Acknowledgement. The author thanks P. Zhou for useful comments. He is very grateful to a referee for providing numerous useful comments and suggestions in improving this paper.

2. Nearby and vanishing cycles for holonomic $\mathscr{D}$-modules

2.1. b-function and Localization.

We recall the construction of b-functions. Let X be a smooth algebraic variety over $\mathbb{C}$ of dimension n and let f be a regular function on X. We denote by D the divisor defined by $f = 0$, by $j : U = X \setminus D \hookrightarrow X$ the open embedding and by $i \colon D \hookrightarrow X$ the closed embedding. We assume that $\mathcal{M}_U$ is a (left) holonomic $\mathscr{D}_U$-module so that

$$\mathcal{M}_U = \mathscr{D}_U \cdot \mathcal{M}_0|_U$$

for some fixed coherent $\mathscr{O}_X$-submodule $\mathcal{M}_0 \subseteq j_*(\mathcal{M}_U)$ throughout this section. We then introduce an independent variable s and consider the free $\mathbb{C}[s]$-module

$$j_*(\mathcal{M}_U[s] \cdot f^s) = j_*(\mathcal{M}_U) \cdot f^s \otimes_{\mathbb{C}} \mathbb{C}[s].$$

The module $j_*(\mathcal{M}_U[s] \cdot f^s)$ has a natural $\mathscr{D}_X[s]$-module structure by requiring

$$v(f^s) = sv(f)f^{s-1},$$

for any vector field v on X, where $\mathscr{D}_X[s] \coloneqq \mathscr{D}_X \otimes_{\mathbb{C}} \mathbb{C}[s]$. Notice that the module $j_*(\mathcal{M}_U[s] \cdot f^s)$ is not necessarily coherent over $\mathscr{D}_X[s]$. We then consider the coherent $\mathscr{D}_X[s]$-submodule generated by $\mathcal{M}_0 \cdot f^{s+k}$

$$\mathscr{D}_X[s]\mathcal{M}_0 \cdot f^{s+k} \subseteq j_*(\mathcal{M}_U[s] \cdot f^s)$$

for every $k \in \mathbb{Z}$. It is obvious that we have inclusions

$$\mathscr{D}_X[s]\mathcal{M}_0 \cdot f^{s+k_1} \subseteq \mathscr{D}_X[\mathbf{s}]\mathcal{M}_0 \cdot f^{s+k_2}$$

when $k_1 \geq k_2$.

Definition 2.1 (b-function)**.** *The b-function of $\mathcal{M}_U$ along f, also called the Bernstein-Sato polynomial, is the monic polynomial $b(s) \in \mathbb{C}[s]$ of least degree so that*

$$b(s) \text{ annihilates } \frac{\mathscr{D}_X[s]\mathcal{M}_0 \cdot f^s}{\mathscr{D}_X[s]\mathcal{M}_0 \cdot f^{s+1}}.$$

In particular, if we pick $\mathcal{M}_U = \mathscr{O}_U$ and $\mathcal{M}_0 = \mathscr{O}_X$, then the above definition gives us the usual b-function for f (see for instance [Kas77]). By definition, the b-function (and its roots) of $\mathcal{M}_U$ depends on the choice of $\mathcal{M}_0$. However, we will see that an arithmetic set generated by the roots is independent of the choice.

Remark 2.2. In the case that X is a complex manifold and f is a holomorphic function on X, for an analytic holonomic $\mathscr{D}_X$-module $\mathcal{M}$, one can use $\mathcal{M}(*D)$, the algebraic localization of $\mathcal{M}$ along D, to replace $j_*(\mathcal{M}_U)$ and define b-functions in the analytic setting in a similar way.

Theorem 2.3 (Bernstein and Sato)**.** *The b-functions along f exist for holonomic $\mathscr{D}_U$-modules.*

The above theorem for $\mathscr{O}_U$ is due to Bernstein algebraically and Sato analytically. Björk extended it for arbitrary holonomic modules in the analytic setting (see [Bj93, Chapter VI]).

Definition 2.4 (Localization). *Assume that $\mathcal{N}$ is a (left) coherent $\mathscr{D}_X[s]$-module and q is a prime ideal in $\mathbb{C}[s]$. Then we define the localization of $\mathcal{N}$ at q by*

$$\mathcal{N}_q = \mathcal{N} \otimes_{\mathbb{C}[s]} \mathbb{C}[s]_q$$

where $\mathbb{C}[s]_q$ is the localization of $\mathbb{C}[s]$ at q. In particular, if q is the ideal generated by $0 \in \mathbb{C}[s]$ (i.e. q is the generic point of $\mathbb{C} = \mathrm{Spec}\mathbb{C}[s]$), then $\mathcal{N}_q$ becomes a coherent $\mathscr{D}_{X(s)}$-module, where $X(s)$ is the variety defined over $\mathbb{C}(s)$ of X after the base change $\mathbb{C} \to \mathbb{C}(s)$, where $\mathbb{C}(s)$ is the fractional field of $\mathbb{C}[s]$.

For a maximal ideal $\mathfrak{m} \subseteq \mathbb{C}[\mathbf{s}]$, we write the localization of $\mathscr{D}_X[s]\mathcal{M}_0 \cdot f^{s+k}$ and $j_*(\mathcal{M}_U[s] \cdot f^s)$ at $\mathfrak{m}$ by

$$\mathscr{D}_X[s]_{\mathfrak{m}}\mathcal{M}_0 \cdot f^{s+k} \text{ and } j_*(\mathcal{M}_U[s]_{\mathfrak{m}} \cdot f^s)$$

respectively and by

$$\mathscr{D}_X(s)\mathcal{M}_0 \cdot f^{s+k} \text{ and } j_*(\mathcal{M}_U(s) \cdot f^s)$$

the localization at the generic point respectively.

Definition 2.5 (Duality). *Assume that $\mathcal{N}$ is a (left) coherent $\mathscr{D}_X[s]$-module and $\mathcal{N}_q$ is a coherent $\mathscr{D}_X[s]_q$-module for a prime ideal $q \subseteq \mathbb{C}[s]$. We then define the duality by*

$$\mathbb{D}(\mathcal{N}) \coloneqq \mathcal{R}hom_{\mathscr{D}_{X[s]}}(\mathcal{N}, \mathscr{D}_{X[s]}) \otimes_{\mathscr{O}} \omega_X^{-1}[n],$$

and

$$\mathbb{D}(\mathcal{N}_q) \coloneqq \mathcal{R}hom_{\mathscr{D}_X[s]_q}(\mathcal{N}_q, \mathscr{D}_X[s]_q) \otimes_{\mathscr{O}} \omega_X^{-1}[n]$$

where ω_X is the dualizing sheaf of X. The twist by ω_X^{-1} is to make the dual of $\mathcal{N}$ (resp. $\mathcal{N}_q$) a complex of left $\mathscr{D}_X[s]$-modules (resp. $\mathscr{D}_X[s]_q$-modules).

In the case that $\mathbb{D}(\mathcal{N})$ (resp. $\mathbb{D}(\mathcal{N}_q)$) has only the zero-th cohomological sheaf non-zero, we also use $\mathbb{D}(\mathcal{N})$ (resp. $\mathbb{D}(\mathcal{N}_q)$) to denote $\mathcal{H}^0(\mathbb{D}(\mathcal{N}))$ (resp. $\mathcal{H}^0(\mathbb{D}(\mathcal{N}_q))$).

Since the variable s is in the center of $\mathscr{D}_X[s]$, one can easily check that duality and localization commute, i.e.

$$\mathbb{D}(\mathcal{N})_q \simeq \mathbb{D}(\mathcal{N}_q). \tag{1}$$

We can evaluate $\mathcal{N}$ at the residue field of a maximal ideal $m \subseteq \mathbb{C}[s]$:

$$\mathcal{N} \otimes^L_{\mathbb{C}[s]} \mathbb{C}_{\mathfrak{m}}$$

where $\mathbb{C}_{\mathfrak{m}} \simeq \mathbb{C}$ is the residue field $\mathbb{C}[s]/\mathfrak{m}$ and the $\otimes^L_{\mathbb{C}[s]}$ denotes the derived tensor functor over $\mathbb{C}[s]$; it gives a complex of coherent $\mathscr{D}_X$-modules. Furthermore, since $\mathscr{D}_X[s]$ is free over $\mathbb{C}[s]$, one can check that evaluation and duality commute (see for instance [BVWZ19, Eq. (3.5)]), i.e.

$$\mathbb{D}(\mathcal{N}) \otimes^L_{\mathbb{C}[s]} \mathbb{C}_{\mathfrak{m}} \simeq \mathbb{D}(\mathcal{N} \otimes^L_{\mathbb{C}[s]} \mathbb{C}_{\mathfrak{m}}), \tag{2}$$

where the second $\mathbb{D}$ denotes the duality functor for complexes of coherent $\mathscr{D}$-modules. Because of the evaluation functor and its commutativity with duality, we also call $\mathscr{D}_X[s]$-modules the relative $\mathscr{D}$-modules over $\mathbb{C}[s]$. See [WZ19, §5] for further discussions of relative $\mathscr{D}$-modules for the multivariate s and also [BVWZ19, §3] in general.

The following lemma is obvious to check; see also [BVWZ20, Lemma 5.3.1] for a multivariate version.

Lemma 2.6. *We have*

$$\mathbb{D}(\mathcal{M}_U[s] \cdot f^s) \simeq \mathbb{D}(\mathcal{M}_U)[s] \cdot f^{-s} \simeq \mathbb{D}(\mathcal{M}_U)[s] \cdot f^s$$

where the last isomorphism is given by substituting s by $-s$ (and hence it is not canonical).

Lemma 2.7. *The $\mathscr{D}_X[s]$-module $\mathscr{D}_X[s]\mathcal{M}_0 \cdot f^{s+k}$ is n-Cohen-Macaulay for every $k \in \mathbb{Z}$, i.e. the complex $\mathbb{D}(\mathscr{D}_X[s]\mathcal{M}_0 \cdot f^{s+k})$ only has one non-zero cohomology sheaf*

$$\mathcal{H}^0(\mathbb{D}(\mathscr{D}_X[s]\mathcal{M}_0 \cdot f^{s+k})) \simeq \mathcal{E}xt^n(\mathscr{D}_X[s]\mathcal{M}_0 \cdot f^{s+k}, \mathscr{D}_X[s]) \otimes_{\mathscr{O}} \omega_X^{-1}.$$

Proof. The idea we use here is the same as that for the proof of [BVWZ19, Lemma 3.5.2]. The only difference is that we are now over the one-dimensional ring $\mathbb{C}[s]$.

By [Mai16, Proposition 14] (taking $p = 1$), we see that $\mathscr{D}_X[s]\mathcal{M}_0 \cdot f^{s+k}$ is n-pure (see for instance [BVWZ19, Definition 4.3.4] for purity). Moreover, by [Mai16, Résultat 2], $\mathscr{D}_X[s]\mathcal{M}_0 \cdot f^{s+k}$ is relative holonomic (cf. [BVWZ19, Definition 3.2.3]). Relative holonomicity and [BVWZ19, Lemma 3.2.2 (1)] give us that the graded number of $\mathscr{D}_X[s]\mathcal{M}_0 \cdot f^{s+k}$ is n. In particular, $\mathcal{E}xt^l(\mathscr{D}_X[s]\mathcal{M}_0 \cdot f^{s+k}, \mathscr{D}_X[s]) = 0$ for $l < n$. For $l > n$, by [BVWZ19, Lemma 4.3.5 (2)], the purity of $\mathscr{D}_X[s]\mathcal{M}_0 \cdot f^{s+k}$ implies that

$$\mathcal{E}xt^l(\mathcal{E}xt^l(\mathscr{D}_X[s]\mathcal{M}_0 \cdot f^{s+k}, \mathscr{D}_X[s]), \mathscr{D}_X[s]) = 0.$$

By Auslander regularity (cf. [BVWZ19, §4.3]), we further have that for $l > n$ the graded number of $\mathcal{E}xt^l(\mathscr{D}_X[s]\mathcal{M}_0 \cdot f^{s+k}, \mathscr{D}_X[s])$ is strictly greater than l. If

$$\mathcal{E}xt^l(\mathscr{D}_X[s]\mathcal{M}_0 \cdot f^{s+k}, \mathscr{D}_X[s]) \neq 0 \text{ for some } l > n,$$

then by [BVWZ19, Lemma 3.2.2 (1)] the dimension of the relative characteristic variety of

$$\mathcal{E}xt^l(\mathscr{D}_X[s]\mathcal{M}_0 \cdot f^{s+k}, \mathscr{D}_X[s])$$

is less than n. But by [BVWZ19, Lemma 3.2.4 (2)], $\mathcal{E}xt^l(\mathscr{D}_X[s]\mathcal{M}_0 \cdot f^{s+k}, \mathscr{D}_X[s])$ is relative holonomic and hence the dimension of its relative characteristic variety is great than or equal to n, which is a contradiction. Consequently,

$$\mathcal{E}xt^l(\mathscr{D}_X[s]\mathcal{M}_0 \cdot f^{s+k}, \mathscr{D}_X[s]) = 0 \text{ for } k \neq n.$$

□

By the above lemma and the isomorphism (1), we immediately have:

Corollary 2.8. *For every prime ideal $q \subseteq \mathbb{C}[s]$, the $\mathscr{D}_X[s]_q$-module $\mathscr{D}_X[\mathbf{s}]_q\mathcal{M}_0 \cdot f^{s+k}$ is n-Cohen-Macaulay for every $k \in \mathbb{Z}$, i.e. the complex $\mathbb{D}(\mathscr{D}_X[s]_q\mathcal{M}_0 \cdot f^{s+k})$ only has one non-zero cohomology sheaf*

$$\mathcal{H}^0(\mathbb{D}(\mathscr{D}_X[s]_q\mathcal{M}_0 \cdot f^{s+k})) \simeq \mathcal{E}xt^n(\mathscr{D}_X[s]_q\mathcal{M}_0 \cdot f^{s+k}, \mathscr{D}_X[s]_q) \otimes_{\mathscr{O}} \omega_X^{-1}.$$

The above corollary is the same as [BG12, Lemma 2(a)]. But our proof (by using Lemma 2.7) is different from the approach in *loc. cit.* See also [WZ19, §5] for the multivariate generalization.

For every $\alpha \in \mathbb{C}$, we denote by $\mathfrak{m}_\alpha$ the maximal ideal of α in $\mathbb{C}[s]$, that is, the ideal generated by $s - \alpha$, and $\mathbb{C}_\alpha$ its residue field.

Lemma 2.9. *We have*

$$\mathscr{D}_X(s)\mathcal{M}_0 \cdot f^{s-k} = j_*(\mathcal{M}_U(s) \cdot f^s)$$

for every $k \in \mathbb{Z}$. Moreover, for every $\alpha \in \mathbb{C}$, there exists $k_0 > 0$ so that

$$\mathscr{D}_X[s]_{\mathfrak{m}_\alpha}\mathcal{M}_0 \cdot f^{s-k} = j_*(\mathcal{M}_U[s]_{\mathfrak{m}_\alpha} \cdot f^s)$$

for all $k > k_0$.

Proof. We denote by $b(s)$ the b-function of $\mathcal{M}_U$. Since $b(s + k)$ is invertible in $\mathbb{C}(s)$, we have $\mathscr{D}_X(s)\mathcal{M}_0 \cdot f^{s+k} = \mathscr{D}_X(s)\mathcal{M}_0 \cdot f^s$ for every $k \in \mathbb{Z}$. Hence, the first statement follows. The second statement can be proved similarly. □

We define $j_!$-extensions by

$$j_!(\mathcal{M}_U(s)\cdot f^s) \coloneqq \mathbb{D}\circ j_*\circ\mathbb{D}(\mathcal{M}(s)\cdot f^s)$$

and

$$j_!(\mathcal{M}_U[s]_{\mathfrak{m}_\alpha}\cdot f^s) \coloneqq \mathbb{D}\circ j_*\circ\mathbb{D}(\mathcal{M}[s]_{\mathfrak{m}_\alpha}\cdot f^s)$$

for every $\alpha\in\mathbb{C}$. By Lemma 2.6, Corollary 2.8 and Lemma 2.9, the $j_!$-extensions above are all sheaves (instead of complexes).

Since $\mathbb{D}\circ\mathbb{D}$ is identity, using the adjoint pair (j^{-1}, j_*), we have natural morphisms

$$j_!(\mathcal{M}_U(s)\cdot f^s)\to j_*(\mathcal{M}_U(s)\cdot f^s)$$

and

$$j_!(\mathcal{M}_U[s]_{\mathfrak{m}_\alpha}\cdot f^s)\to j_*(\mathcal{M}_U[s]_{\mathfrak{m}_\alpha}\cdot f^s)$$

for every $\alpha\in\mathbb{C}$.

For every $\alpha\in\mathbb{C}$, the multi-valued function f^α gives a local system on U. We then denote by $\mathcal{M}_U\cdot f^\alpha$ the holonomic $\mathscr{D}_U$-module twisted by the local system given by f^α. It is obvious by construction that $\mathcal{M}_U\cdot f^\alpha$ is $\mathbb{Z}$-periodic, that is,

$$\mathcal{M}_U\cdot f^\alpha = \mathcal{M}_U\cdot f^{\alpha+k} \tag{3}$$

for every $k\in\mathbb{Z}$.

Example. For some $\alpha\in\mathbb{C}$, consider the regular holonomic module

$$\mathbb{C}[t,1/t]\cdot t^\alpha$$

by assigning $t\partial_t\cdot t^\alpha=\alpha t^\alpha$, where t is the complex coordinate of the complex plane $\mathbb{C}$ and t^α is the symbol of the multivalued function "t^α". Then the multi-valued flat section on $\mathbb{C}^*$, the punctured complex plane, is $e^{-\alpha\log t}\cdot t^\alpha$. Consequently, the monodromy T of the underlying rank 1 local system (around the origin counterclockwise) is the multiplication by $e^{-2\pi\sqrt{-1}\alpha}$, by choosing different branches of $\log t$.

By using the Deligne-Goresky-MacPherson extension (or the minimal extension), the following theorem is first proved by Ginsburg in [Gin86, §3.6 and 3.8], as well as in [BG12], which is essentially due to Beilinson and Bernstein. See also [WZ19, Theorem 5.3] for the multivariate generalization.

Theorem 2.10 (Beilinson and Bernstein)**.** *We have:*

1. *the natural morphism $j_!(\mathcal{M}_U(s)\cdot f^s)\to j_*(\mathcal{M}_U(s)\cdot f^s)$ is isomorphic and they are both equal to $\mathscr{D}_X(s)\mathcal{M}_0\cdot f^{s+k}$ for every $k\in\mathbb{Z}$;*
2. *the natural morphism $j_!(\mathcal{M}_U[s]_{\mathfrak{m}_\alpha}\cdot f^s)\to j_*(\mathcal{M}_U[s]_{\mathfrak{m}_\alpha}\cdot f^s)$ is injective for every $\alpha\in\mathbb{C}$;*
3. *for every $\alpha\in\mathbb{C}$, there exists $k_0\in\mathbb{Z}_+$ so that for all $k>k_0$ we have*

$$j_*(\mathcal{M}_U[s]_{\mathfrak{m}_\alpha}\cdot f^s)=\mathscr{D}_X[s]_{\mathfrak{m}_\alpha}\mathcal{M}_0\cdot f^{s-k},$$

$$j_!(\mathcal{M}_U[s]_{\mathfrak{m}_\alpha}\cdot f^s)=\mathscr{D}_X[s]_{\mathfrak{m}_\alpha}\mathcal{M}_0\cdot f^{s+k},$$

$$j_*(\mathcal{M}_U\cdot f^\alpha)\overset{q.i.}{\simeq}\mathscr{D}_X[s]_{\mathfrak{m}_\alpha}\mathcal{M}_0\cdot f^{s-k}\otimes^L_{\mathbb{C}[\mathbf{s}]}\mathbb{C}_\alpha$$

and

$$j_!(\mathcal{M}_U\cdot f^\alpha)\overset{q.i.}{\simeq}\mathscr{D}_X[s]_{\mathfrak{m}_\alpha}\mathcal{M}_0\cdot f^{s+k}\otimes^L_{\mathbb{C}[\mathbf{s}]}\mathbb{C}_\alpha;$$

(4) *for every $\alpha \in \mathbb{C}$, if $\alpha + k$ is not a root of the b-function of $\mathcal{M}_U$ for every $k \in \mathbb{Z}$, then we have*

$$j_!(\mathcal{M}_U[s]_{\mathfrak{m}_\alpha} \cdot f^s) = j_*(\mathcal{M}_U[s]_{\mathfrak{m}_\alpha} \cdot f^s) = \mathscr{D}_X[s]_{\mathfrak{m}_\alpha}\mathcal{M}_0 \cdot f^{s+k}$$

and

$$j_!(\mathcal{M}_U \cdot f^\alpha) = j_*(\mathcal{M}_U \cdot f^\alpha) \overset{q.i.}{\simeq} \mathscr{D}_X[s]_{\mathfrak{m}_\alpha}\mathcal{M}_0 \cdot f^{s+k} \otimes^L_{\mathbb{C}[s]} \mathbb{C}_\alpha$$

for every $k \in \mathbb{Z}$, where q.i. stands for quasi-isomorphism.

2.2. **Nearby and vanishing cycles.** We now give constructions of nearby and vanishing cycles. We continue using the notation and setup in §2.1. We assume that $\mathcal{M}$ is a holonomic $\mathscr{D}_X$-module so that $\mathcal{M}|_U \simeq \mathcal{M}_U$.

Definition 2.11. *For every $\alpha \in \mathbb{C}$, the α-nearby cycles of $\mathcal{M}$ is*

$$\Psi_{f,\alpha}\mathcal{M} \simeq \Psi_{f,\alpha}\mathcal{M}_U = \frac{j_*(\mathcal{M}_U[s]_{\mathfrak{m}_{-\alpha}} \cdot f^s)}{j_!(\mathcal{M}_U[s]_{\mathfrak{m}_{-\alpha}} \cdot f^s)}.$$

The above definition needs Theorem 2.10 (2) to get the quotient. From definition, the α-nearby cycles of $\mathcal{M}$ only depends on $\mathcal{M}|_U$.

Recall that $\mathscr{D}_X[s]\mathcal{M}_0 \cdot f^s$ has a t-action given by

$$t \cdot s = s + 1.$$

The t-action induces an t-operation on $\Psi_{f,\alpha}\mathcal{M}$:

$$t \cdot \Psi_{f,\alpha}\mathcal{M} = \Psi_{f,\alpha+1}\mathcal{M}. \tag{4}$$

We define an arithmetic subset of $\mathbb{C}$,

$$\Lambda := \{k - \alpha \mid \alpha \text{ goes over roots of the } b\text{-function of } \mathcal{M}_0 \text{ while } k \text{ over all integers}\}.$$

The set Λ is independent of choices of $\mathcal{M}_0$. In fact, substituting s by $s + k$, we obtain that the set Λ for $\mathcal{M}_0$ and that for $\mathcal{M}_0 \cdot f^k$ are the same for each $k \in \mathbb{Z}$. Now if $\mathcal{M}'_0$ is another choice, then by coherence and Eq.(3) we have

$$\mathcal{M}_0 \cdot f^k \subseteq \mathcal{M}'_0 \subseteq \mathcal{M}_0 \cdot f^{-k} \text{ for some } k \gg 0.$$

Then

$$\frac{\mathscr{D}_X[s]\mathcal{M}'_0 \cdot f^s}{\mathscr{D}_X[s]\mathcal{M}'_0 \cdot f^{s+1}}$$

is a subquotient of

$$\frac{\mathscr{D}_X[s]\mathcal{M}_0 \cdot f^{s-k}}{\mathscr{D}_X[s]\mathcal{M}_0 \cdot f^{s+k}} \text{ for some } k \gg 0.$$

We denote by $b(s)$ the b-function of $\mathcal{M}_0$. Therefore,

$$\prod_{i=-k}^{k} b(s - i) \text{ for some } k \gg 0$$

annihilates $\dfrac{\mathscr{D}_X[s]\mathcal{M}'_0 \cdot f^s}{\mathscr{D}_X[s]\mathcal{M}'_0 \cdot f^{s+1}}$, which means that Λ for $\mathcal{M}'_0$ is a subset of that for $\mathcal{M}_0$ and hence that they are the same by symmetry.

Furthermore, we have

$$\Psi_{f,\alpha}\mathcal{M}_U \neq 0 \Longleftrightarrow \alpha \in \Lambda.$$

The implication "$\Rightarrow$" follows from Theorem 2.10 (4). We now prove "$\Leftarrow$". We assume $\alpha \in \Lambda$. Then $-\alpha - l$ is a root of the b-function of $\mathcal{M}_0$ for some $l \in \mathbb{Z}$. By [BVWZ19, Lemma 3.4.1], we have

$$\frac{\mathscr{D}_X[s]_{\mathfrak{m}_{-\alpha-l}}\mathcal{M}_0 \cdot f^s}{\mathscr{D}_X[s]_{\mathfrak{m}_{-\alpha-l}}\mathcal{M}_0 \cdot f^{s+1}} \neq 0.$$

Hence,

$$\Psi_{f,\alpha+l}\mathcal{M}_U = \frac{\mathscr{D}_X[s]_{\mathfrak{m}_{-\alpha-l}}\mathcal{M}_0 \cdot f^{s-k}}{\mathscr{D}_X[s]_{\mathfrak{m}_{-\alpha-l}}\mathcal{M}_0 \cdot f^{s+k}} \neq 0 \text{ for some } k \gg 0.$$

By Eq.(4), we have $\Psi_{f,\alpha}\mathcal{M}_U \neq 0$.

When $\alpha = 0$, $\Psi_{f,0}\mathcal{M}_U$ coincides with $\Psi^{\mathrm{nil}}(\mathcal{M})$ in [BG12, §2.4].

Proposition 2.12. *For every $\alpha \in \mathbb{C}$, we have*

(1) $(s+\alpha)^N$ annihilates $\Psi_{f,\alpha}\mathcal{M}_U$ for some $N \gg 0$.

(2) $\Psi_{f,\alpha}\mathcal{M}_U$ is a holonomic $\mathscr{D}_X$-module supported on D; moreover, if $j_*(\mathcal{M}_U)$ is regular holonomic, then so is $\Psi_{f,\alpha}\mathcal{M}_U$;

(3) $\mathbb{D}(\Psi_{f,\alpha}\mathcal{M}_U) \simeq \Psi_{f,-\alpha}\mathbb{D}(\mathcal{M}_U)$.

Proof. This proposition is essentially proved in [BB93, §4.2]. We give a proof here for completeness.

By Theorem 2.10 (3), we have

$$\Psi_{f,-\alpha}\mathcal{M}_U = \frac{\mathscr{D}_X[s]_{\mathfrak{m}_\alpha}\mathcal{M}_0 \cdot f^{s-k}}{\mathscr{D}_X[s]_{\mathfrak{m}_\alpha}\mathcal{M}_0 \cdot f^{s+k}}$$

Therefore, $(s-\alpha)^N$ annihilates $\Psi_{f,-\alpha}\mathcal{M}_U$ for some $N \gg 0$ by using the b-function of $\mathcal{M}_U$. The first statement is thus proved.

For the second one, it is obvious that $\Psi_{f,\alpha}\mathcal{M}_U$ is supported on D. We then prove holonomicity. Using Theorem 2.10 (3) one more time, we get a short exact sequence

$$0 \to j_*(\mathcal{M}_U \cdot f^\alpha) \to \frac{j_*(\mathcal{M}_U[s]_{\mathfrak{m}_\alpha} \cdot f^s)}{(s-\alpha)^2 \cdot j_*(\mathcal{M}_U[s]_{\mathfrak{m}_\alpha} \cdot f^s)} \to j_*(\mathcal{M}_U \cdot f^\alpha) \to 0.$$

Hence, $\dfrac{j_*(\mathcal{M}_U[s]_{\mathfrak{m}_\alpha} \cdot f^s)}{(s-\alpha)^2 \cdot j_*(\mathcal{M}_U[s]_{\mathfrak{m}_\alpha} \cdot f^s)}$ is holonomic. By induction, we then have

$$\frac{j_*(\mathcal{M}_U[s]_{\mathfrak{m}_\alpha} \cdot f^s)}{(s-\alpha)^N \cdot j_*(\mathcal{M}_U[s]_{\mathfrak{m}_\alpha} \cdot f^s)}$$

is holonomic.

By Part (1), we have

$$\Psi_{f,-\alpha}\mathcal{M}_U = \Psi_{f,\alpha}\mathcal{M}_U/(s-\alpha)^N \cdot \Psi_{f,\alpha}\mathcal{M}_U.$$

Therefore, $\Psi_{f,-\alpha}\mathcal{M}_U$ is a quotient of $\dfrac{j_*(\mathcal{M}_U[s]_{\mathfrak{m}_\alpha} \cdot f^s)}{(s-\alpha)^N \cdot j_*(\mathcal{M}_U[s]_{\mathfrak{m}_\alpha} \cdot f^s)}$, from which we have proved the holonomicity. Regularity can be proved similarly.

The third assertion follows from Lemma 2.6 and Eq.(1). □

We now give an alternative description of the α-nearby cycles.

Proposition 2.13. *For each $\alpha \in \mathbb{C}$, $\Psi_{f,\alpha}\mathcal{M}_U$ is canonically isomorphic to the generalized $-\alpha$-eigenspace of $\mathscr{D}_X[s]\mathcal{M}_0 \cdot f^{s-k}/\mathscr{D}_X[s]\mathcal{M}_0 \cdot f^{s+k}$ with respect to the s-action for $k \gg \alpha$.*

Proof. Using the b-function, we first know that the s-action on

$$\mathscr{D}_X[s]\mathcal{M}_0\cdot f^{s-k}/\mathscr{D}_X[s]\mathcal{M}_0\cdot f^{s+k}$$

admits a minimal polynomial for each $k\geq 0$. Hence, we have the generalized α-eigenspace. Since $\mathscr{D}_X[s]\mathcal{M}_0\cdot f^{s-k}/\mathscr{D}_X[s]\mathcal{M}_0\cdot f^{s+k}$, as a $\mathbb{C}[s]$-module, is supported at a finite subset of $\mathrm{Spec}\mathbb{C}[s]$ (determined by the roots of the b-function by [BVWZ19, Lemma 3.4.1]). Therefore, its α-eigenspace is naturally

$$(\mathscr{D}_X[s]\mathcal{M}_0\cdot f^{s-k}/\mathscr{D}_X[s]\mathcal{M}_0\cdot f^{s+k})_{\mathfrak{m}_\alpha}.$$

The proof is now done by Theorem 2.10(3). □

We denote by $b_f(s)$ the b-function for $\mathscr{O}_U$ along f. Since $\mathbb{D}(\mathscr{O}_U)\simeq\mathscr{O}_U$, we have the following well-known fact as an immediate corollary of Proposition 2.12 (2):

Corollary 2.14. *If α is a root of $b_f(s)$, then $-\alpha+k$ is also a root of $b_f(s)$ for some $k\in\mathbb{Z}$.*

Definition 2.15 (Beilinson)**.** *The maximal extension of $\mathcal{M}_U$ is*

$$\Xi(\mathcal{M}_U)=\frac{j_*(\mathcal{M}_U[s]_{m_0}\cdot f^s)}{s\cdot j_!(\mathcal{M}_U[s]_{m_0}\cdot f^s)}.$$

Using Theorem 2.10 (3) with $\alpha=0$, we have the following two short exact sequences

$$0\to j_!(\mathcal{M}_U)\xrightarrow{\alpha_-}\Xi(\mathcal{M}_U)\xrightarrow{\beta_-}\Psi_{f,0}\mathcal{M}_U\to 0 \tag{5}$$

and

$$0\to \Psi_{f,0}\mathcal{M}_U\xrightarrow{\beta_+}\Xi(\mathcal{M}_U)\xrightarrow{\alpha_+}j_*(\mathcal{M}_U)\to 0, \tag{6}$$

where β_+ is induced by the isomorphism

$$\Psi_{f,0}\mathcal{M}_U\simeq\frac{s\cdot j_*(\mathcal{M}_U[s]_{m_0}\cdot f^s)}{s\cdot j_!(\mathcal{M}_U[s]_{m_0}\cdot f^s)}.$$

By construction, (5) and (6) are dual to each other.

Since $\mathcal{M}|_U\simeq\mathcal{M}_U$, we have natural morphisms

$$\mathcal{M}\to j_*(\mathcal{M}_U)\text{ and }j_!(\mathcal{M}_U)\to\mathcal{M}$$

where the second one is obtained by dualizing the first one. We then have the following commutative diagram

$$\begin{tikzcd} \Xi(\mathcal{M}_U) \arrow[r, "\alpha_+"] & j_*(\mathcal{M}_U) \\ j_!(\mathcal{M}_U) \arrow[u, "\alpha_-"] \arrow[r] & \mathcal{M}. \arrow[u] \end{tikzcd} \tag{7}$$

Definition 2.16 (Beilinson)**.** *The vanishing cycles of $\mathcal{M}$ is*

$$\Phi_f\mathcal{M}\coloneqq\mathcal{H}^0([j_!(\mathcal{M}_U)\to\Xi(\mathcal{M}_U)\oplus M\to j_*(\mathcal{M}_U)]),$$

where the complex is the total complex of the above double complex, placed in degrees -1, 0 and 1.

Since the first vertical morphism is injective and the first horizontal morphism is surjective in the double complex by using the two short exact sequences (5) and (6), we know that

$$\mathcal{H}^i([j_!(\mathcal{M}_U)\to\Xi(\mathcal{M}_U)\oplus M\to j_*(\mathcal{M}_U)])=0,\ \text{for } i\neq 0.$$

One then can alternatively define the vanishing cycles complex of $\mathcal{M}$ by the complex

$$[j_!(\mathcal{M}_U)\to\Xi(\mathcal{M}_U)\oplus M\to j_*(\mathcal{M}_U)].$$

The upshot is that the definition of the vanishing cycles complex of $\mathcal{M}$ is funtorial. Therefore, by a slight abuse of notation, when we say the vanishing cycles complex $\Phi_f\mathcal{M}$, we mean

$$\Phi_f\mathcal{M} \coloneqq [j_!(\mathcal{M}_U) \to \Xi(\mathcal{M}_U) \oplus M \to j_*(\mathcal{M}_U)].$$

We define the morphisms of $\mathscr{D}_X$-modules

$$c : \Psi_{f,0}\mathcal{M} \to \Phi_f\mathcal{M} \tag{8}$$

by $c(\eta) = (\beta_+(\eta), 0)$, and

$$v : \Phi_f\mathcal{M} \to \Psi_{f,0}\mathcal{M} \tag{9}$$

by $v(\xi, m) = \beta_-(\xi)$. Then

$$v \circ c = s \text{ and } c \circ v = (s, 0).$$

One then can similarly define c and v as morphisms of complexes.

The above construction of $\Xi(\mathcal{M}_U)$ and $\Phi_f\mathcal{M}$ follows the recipe in [Bei87] and $\Phi_f\mathcal{M}$ coincides with $\Phi^{\mathrm{nil}}(\mathcal{M})$ in [BG12, §2.4].

The following corollary follows immediately from Proposition 2.12.

Corollary 2.17. *We have:*

(1) $\Xi(\mathcal{M}_U)$ *and* $\Phi_f\mathcal{M}$ *are both holonomic; moreover, if* $\mathcal{M}$ *is regular holonomic, then so are* $\Xi(\mathcal{M}_U)$ *and* $\Phi_f\mathcal{M}$*;*
(2) $\mathbb{D}(\Xi(\mathcal{M}_U)) \simeq \Xi(\mathbb{D}(\mathcal{M}_U))$*;*
(3) $\mathbb{D}(\Phi_f\mathcal{M}) \simeq \Phi_f(\mathbb{D}\mathcal{M})$.

3. Twisted $\mathscr{D}_X[s]$-module by Jordan block

We discuss $\mathscr{D}_X[s]$-modules twisted by local systems given by Jordan blocks. We first consider a key example: local systems of Jordan blocks on $\mathbb{C}^*$.

For $\alpha \in \mathbb{C}$ and $m \geq 1$, we define a free $\mathscr{O}_{\mathbb{C}}[1/t]$-module

$$K_m^\alpha = \bigoplus_{l=0}^{m-1} \mathscr{O}_{\mathbb{C}}[t^{-1}]e_l^\alpha$$

with a naturally defined connection ∇ by requiring

$$\nabla e_l^\alpha = \frac{1}{t}(\alpha e_l^\alpha + e_{l-1}^\alpha),$$

where t is the coordinate of the complex plane $\mathbb{C}$. The generator e_l^α can be understood as the formal symbol of the multi-valued function $t^\alpha \frac{\log^l t}{l!}$ and we conventionally set $e_{-1}^\alpha = 0$.

We can identify $t\nabla$ with the action of $J_{\alpha,m}$, where $J_{\alpha,m}$ is the $m \times m$ Jordan block with eigenvalue α. The nilpotent part of $t\nabla$ is then $J_{0,m}$, or more explicitly

$$(t\nabla)^{\mathrm{nil}}(e_l^\alpha) = e_{l-1}^\alpha.$$

It is then obvious that the multivalued ∇-flat sections of K_m^α (on $\mathbb{C}^*$) are the $\mathbb{C}$-span of

$$\{e^{-J_{\alpha,m}\log t} \cdot e_k^\alpha\}_{k=0,\dots,m-1}.$$

We set L_m^λ the local system of the multivalued ∇-flat sections of K_m^α, or equivalently

$$\mathrm{DR}(K_m^\alpha)|_{\mathbb{C}^*} \overset{q.i.}{\simeq} L_m^\lambda[1],$$

where $\lambda = e^{2\pi\sqrt{-1}\alpha}$.

The monodromy action T (around the origin of the complex plane counterclockwise) on L^{λ}_m is given by $e^{-2\pi\sqrt{-1}J_{\alpha,m}}$. In particular

$$\log T_u = -2\pi\sqrt{-1}J_{0,m}$$

where T_u is the unipotent part of T in the Jordan-Chevalley decomposition.

By construction, we have a direct system of $\mathscr{D}$-modules

$$\cdots \to K^{\alpha}_m \to K^{\alpha}_{m+1} \to \cdots .$$

Applying DR, we then obtain a direct system of local systems

$$\cdots \to L^{\lambda}_m \to L^{\lambda}_{m+1} \to \cdots .$$

We now define the $\mathscr{D}_X[s]$-module

$$\mathcal{N}^{\alpha,k}_m := \bigoplus_{l=0}^{m-1} \mathscr{D}_X[s]\mathcal{M}_0 \cdot f^{s-k} \otimes e^{\alpha}_l,$$

by assigning the s-action by

$$s \cdot (\eta e^{\alpha}_l) = (s+\alpha)\eta e^{\alpha}_l - \eta e^{\alpha}_{l-1}$$

for η a section of $\mathscr{D}_X[s]\mathcal{M}_0 \cdot f^{s-k}$. We therefore have a direct symstem of $\mathscr{D}_X[s]$-modules

$$\cdots \to \mathcal{N}^{\alpha,k}_m \to \mathcal{N}^{\alpha,k}_{m+1} \to \cdots .$$

Let $\iota : X \hookrightarrow Y = X \times \mathbb{C}$ be the graph embedding of f, i.e.

$$\iota(x) = (x, f(x)).$$

By identifying s with $-\partial_t t$, we have a natural injection

$$\mathcal{N}^{\alpha,k}_m \hookrightarrow \iota_+(j_*(\mathcal{M}_U)) \otimes_{\mathscr{O}_Y} p_1^* K^{-\alpha}_m, \tag{10}$$

where ι_+ denotes the $\mathscr{D}$-module direct image functor and $p_1 : Y \to \mathbb{C}$ the projection (cf. [BMS06, §2.4]). In fact, the above injection is a morphism of log $\mathscr{D}$-modules with the log structure along $X \times \{0\}$; see [WZ19, §2] for details.

Lemma 3.1. *Assume that $b(s)$ is the b-function of $\mathcal{M}_U$ along f. Then*

$$(b(s+\alpha))^m \text{ annihilates } \mathcal{N}^{\alpha,0}_m/\mathcal{N}^{\alpha,-1}_m.$$

Proof. By construction, we have a short exact sequence of $\mathscr{D}_X[s]$-modules

$$0 \to \mathcal{N}^{\alpha,k}_{m-1} \to \mathcal{N}^{\alpha,k}_m \to \mathcal{Q} \to 0.$$

We also know that

$$\mathcal{N}^{\alpha,k}_1 \simeq \mathscr{D}_X[s]\mathcal{M}_0 \cdot f^{s-k+\alpha} \simeq \mathcal{Q}.$$

By substituting $s+\alpha$ for s, we know $b(s-k+\alpha)$ annihilates

$$\mathscr{D}_X[s]\mathcal{M}_0 \cdot f^{s-k+\alpha}/\mathscr{D}_X[s]\mathcal{M}_0 \cdot f^{s-k+1+\alpha}.$$

Therefore, we obtain the required statement by induction. □

Proposition 3.2. *Assume that $j_*(\mathcal{M}_U)$ is a regular holonomic $\mathscr{D}_X$-module. For each $\alpha \in \mathbb{C}$, there exists $k_0 > 0$ so that for all $k \geq k_0$*

$$\mathrm{DR}(\mathcal{N}^{\alpha,k}_m \xrightarrow{s} \mathcal{N}^{\alpha,k}_m) \overset{q.i.}{\simeq} \iota_* Rj_*(\mathrm{DR}(\mathcal{M}_U) \otimes f_0^{-1} L^{1/\lambda}_m)$$

and

$$\mathrm{DR}(\mathcal{N}^{\alpha,-k}_m \xrightarrow{s} \mathcal{N}^{\alpha,-k}_m) \overset{q.i.}{\simeq} \iota_! j_!(\mathrm{DR}(\mathcal{M}_U) \otimes f_0^{-1} L^{1/\lambda}_m)$$

for all m, where the complex $[\mathcal{N}^{\alpha,k}_m \xrightarrow{s} \mathcal{N}^{\alpha,k}_m]$ is in degrees -1 and 0, and $\lambda = e^{2\pi\sqrt{-1}\alpha}$.

Proof. Using the inclusion (10), we have

$$\mathcal{N}_m^{\alpha,k}|_U \simeq \iota_+ j_*(\mathcal{M}_U)\otimes_{\mathscr{O}_Y} p_1^* K_m^{-\alpha}|_U \simeq \iota_+ j_*(\mathcal{M}_U)|_U \otimes_{\mathbb{C}} p_1^{-1} L_m^{1/\lambda}.$$

By using Theorem 2.10(3), we have that

$$\mathcal{N}_m^{\alpha,k} \otimes^L_{\mathbb{C}[s]} \mathbb{C}_0 \simeq (\mathcal{N}_m^{\alpha,k})_{\mathfrak{m}_0} \otimes^L_{\mathbb{C}[s]_{\mathfrak{m}_0}} \mathbb{C}_0 \simeq j_*(\mathcal{N}_m^{\alpha,k}|_U)\otimes^L_{\mathbb{C}[s]} \mathbb{C}_0,$$

where $(\mathcal{N}_m^{\alpha,k})_{\mathfrak{m}_0}$ is the localization of $\mathcal{N}_m^{\alpha,k}$ at $\mathfrak{m}_0$, the maximal ideal of $0\in\mathbb{C}$. Moreover, since we identify s with $-\partial_t t$, we further have

$$j_*(\mathcal{N}_m^{\alpha,k}|_U)\otimes^L_{\mathbb{C}[s]} \mathbb{C}_0 \overset{q.i.}{\simeq} [\iota_+ j_*(\mathcal{M}_U)\otimes_{\mathscr{O}_Y} p_1^* K_m^{-\alpha} \xrightarrow{\partial_t} \iota_+ j_*(\mathcal{M}_U)\otimes_{\mathscr{O}_Y} p_1^* K_m^{-\alpha}].$$

By using the Koszul decompostions of *de Rham* complexes (see for instance [Wu17, §4.1]), we therefore have

$$\mathrm{DR}(\mathcal{N}_m^{\alpha,k} \xrightarrow{s} \mathcal{N}_m^{\alpha,k}) \overset{q.i.}{\simeq} \mathrm{DR}_Y(\iota_+ j_*(\mathcal{M}_U)\otimes_{\mathscr{O}_Y} p_1^* K_m^{-\alpha}),$$

where the second DR is taken over the ambient space Y. Since

$$\iota_+(j_*(\mathcal{M}_U))\otimes_{\mathscr{O}_Y} p_1^* K_m^{-\alpha}$$

is regular holonomic, we also naturally have

$$\mathrm{DR}_Y(\iota_+ j_*(\mathcal{M}_U)\otimes_{\mathscr{O}_Y} p_1^* K_m^{-\alpha}) \overset{q.i.}{\simeq} Rj_{Y*}(\iota^o_*\mathrm{DR}(\mathcal{M}_U)\otimes_{\mathbb{C}} p_1^{-1} L_m^{1/\lambda}),$$

where j_Y and ι^o are as in the following diagram

$$\begin{tikzcd} U \arrow[r, hook, "\iota^o"] \arrow[d, hook, "j"] & U_Y = Y\setminus X\times\{0\} \arrow[d, hook, "j_Y"] \\ X \arrow[r, hook, "\iota"] & Y; \end{tikzcd}$$

see [Bj93, Chapter V.4]. By projection formula for local systems ([KS13, Proposition 2.5.11]), we further have

$$Rj_{Y*}(\iota^o_*\mathrm{DR}(\mathcal{M}_U)\otimes_{\mathbb{C}} p_1^{-1} L_m^{1/\lambda}) \overset{q.i.}{\simeq} \iota_*(Rj_*(\mathrm{DR}(\mathcal{M}_U)\otimes_{\mathbb{C}} f_0^{-1} L_m^{1/\lambda}))$$

and the first quasi-isomorphism is obtained.

The second quasi-isomorphism can be obtained similarly. The choice of k_0 only depends on α and the roots of the b-function for $\mathcal{N}_m^{\alpha,0}/\mathcal{N}_m^{\alpha,-1}$. We therefore can choose a uniform k_0 working for all m by Lemma 3.1. ∎

4. Nearby cycles for perverse sheaves via Jordan blocks

In this section, we define nearby and vanishing cycles via local systems given by Jordan blocks on $\mathbb{C}^*$, the punctured complex plane. We keep the notation introduced in §2.1.

Assume that K is a $\mathbb{C}$-perverse sheaf on X.

Definition 4.1. *For $\lambda\in\mathbb{C}^*$, the λ-nearby cycles of K is*

$$\psi_{f,\lambda}(K) \coloneqq \varinjlim_m i^{-1} Rj_*(j^{-1}K\otimes f_0^{-1} L_m^{1/\lambda}).$$

The vanishing cycles is

$$\phi_f K \coloneqq \mathrm{Cone}(i^{-1}K \to \psi_{f,1}(K))$$

where the morphism $i^{-1}K\to \Psi_{f,1}(K)$ is induced by the natural map

$$K\to Rj_*(j^{-1}K).$$

The monodromy action T of $L_m^{1/\lambda}$ induces the monodromy action on $\psi_{f,\lambda}(K)$ for each λ, denoted also by T. We then have the induced monodromy action T on $\phi_f K$, by requiring T acting on $i^{-1}K$ identically.

By construction, we have the tautological triangle

$$i^{-1}K \to \psi_{f,1}(K) \xrightarrow{\text{can}} \phi_f K \xrightarrow{+1}$$

with the induced canonical map

$$\text{can} : \psi_{f,1}(K) \to \phi_f K.$$

We define

$$\text{Var} : \phi_f K \to \psi_{f,1}(K)$$

by

$$\text{Var} \coloneqq (0, -\frac{\log T_u}{2\pi\sqrt{-1}}) = (0, J_{0,\infty})$$

where $J_{0,\infty} = \varinjlim_m J_{0,m}$.

Remark 4.2. Using these definitions, one can prove the perversity of $\psi_{f,\lambda}K$ and $\phi_f K$ (up to a shift of cohomological degrees) directly. Let us refer to [Rei10] for the proof of this point and other related results.

5. The proof of Theorem 1.1

In this section, we prove Theorem 1.1. Before we start, the following preliminary result about infinite Jordan blocks is needed.

Lemma 5.1. [Bj93, Lemma 6.4.5] *Let W be a $\mathbb{C}$-vector space, and let φ be a $\mathbb{C}$-linear operator on W admitting a minimal polynomial. Set $W_\infty = \bigoplus_{k=0}^{\infty} W \otimes e_k$ and define*

$$\varphi_\infty(w \otimes e_k) = (\varphi - \alpha)w \otimes e_k - w \otimes e_{k-1}$$

for $w \in W$ (assume $e_{-1} = 0$). Then φ_∞ is surjective and $\ker(\varphi_\infty) \simeq W_\alpha$, where W_α is the generalized α-eigenspace.

Proof. Define a map $W_\alpha \to \ker(\varphi_\infty)$ by

$$w \mapsto \sum_{i\geq 0} (\varphi - \alpha)^i w \otimes e_i.$$

Clearly, this map is an isomorphism.

We then prove surjectivity. If $w \in W_\alpha$, then

$$\varphi_\infty(-\sum_{i>j} (\varphi - \alpha)^{i-j-1} w \otimes e_i) = w \otimes e_j.$$

If $w \in W_\alpha^\perp$, then then

$$\varphi_\infty(\sum_{i=1}^{j} (\varphi - \alpha)^{-i} w \otimes e_{j-i+1}) = w \otimes e_j.$$

Therefore, the surjectivity follows. ∎

Using the above lemma and Proposition 2.13, we immediately have:

Corollary 5.2. *For a holonomic $\mathscr{D}_U$-module $\mathcal{M}_U$ and some $\alpha \in \mathbb{C}$, there exists $k \gg 0$ so that*

$$\Psi_{f,\alpha}\mathcal{M}_U[1] \overset{q.i.}{\simeq} \varinjlim_m \left(\frac{\mathcal{N}_m^{\alpha,k}}{\mathcal{N}_m^{\alpha,-k}} \xrightarrow{s} \frac{\mathcal{N}_m^{\alpha,k}}{\mathcal{N}_m^{\alpha,-k}} \right)$$

Proof of Theorem 1.1. We assume $\mathcal{M}$ a regular holonomic $\mathscr{D}_X$-module and write $\mathcal{M}_U = \mathcal{M}|_U$. We first prove the comparison of nearby cycles.

Since DR and the direct limit functor commute (since DR is identified with $\omega_X \otimes^L_{\mathscr{D}} \bullet$, one can apply [Wei95, Corollary 2.6.17]), we have

$$\mathrm{DR}(\Psi_{f,\alpha}\mathcal{M}_U[1]) \simeq i_*\psi_{f,\lambda}\mathrm{DR}(\mathcal{M}_U)$$

by Corollary 5.2 and Proposition 3.2.

One can check that the operator $J_{0,\infty}$ on $\varinjlim_m \Psi_{f,0}\mathcal{N}^\alpha_m$ corresponds to the action $(s+\alpha)$ on $\Psi_{f,\alpha}\mathcal{M}_U$ under the quasi-isomorphism in Corollary 5.2 by Lemma 5.1. Taking DR, the operator $J_{0,\infty}$ becomes $-\dfrac{\log T_u}{2\pi\sqrt{-1}}$ on $\psi_{f,\lambda}\mathrm{DR}(\mathcal{M}_U)$ by the construction of the monodromy operator T. Or equivalently, the monodromy operator T corresponds to

$$T \simeq \mathrm{DR}(e^{-2\pi\sqrt{-1}s}) = \mathrm{DR}(\lambda \cdot e^{-2\pi\sqrt{-1}(s+\alpha)}).$$

We thus have proved the comparison for nearby cycles.

We now prove the comparison for vanishing cycles. For simplicity, we write by $\mathcal{A}^\bullet$ the complex

$$[\Xi(\mathcal{M}_U) \to j_*(\mathcal{M}_U)]$$

in degrees 0 and 1, by $\mathcal{B}^\bullet$ the complex

$$[j_!(\mathcal{M}_U) \to \Xi(\mathcal{M}_U) \oplus M \to j_*(\mathcal{M}_U)]$$

and by $\mathcal{C}^\bullet$ the complex

$$[j_!(\mathcal{M}_U) \to \mathcal{M}]$$

in degrees -1 and 0. Then we have a triangle

$$\mathcal{C}^\bullet \to \mathcal{A}^\bullet[1] \to \mathcal{B}^\bullet[1] \xrightarrow{+1}.$$

By definition, we have $\Phi_f\mathcal{M} = \mathcal{B}^\bullet$ as complexes and

$$[\Psi_{f,0}\mathcal{M} \xrightarrow{c} \Phi_f\mathcal{M}] \overset{q.i.}{\simeq} [\mathcal{A}^\bullet \to \mathcal{B}^\bullet].$$

We hence have

$$\Phi_f\mathcal{M}[1] \overset{q.i.}{\simeq} \mathrm{Cone}(\mathcal{C}^\bullet \to \Psi_{f,0}\mathcal{M}_U[1]). \tag{11}$$

It is worth mentioning that the above quasi-isomorphism is not functorial because the mapping cone is used.

Since

$$\mathrm{DR}(\mathcal{C}^\bullet) \overset{q.i.}{\simeq} i_*i^{-1}\mathrm{DR}(\mathcal{M}),$$

we have

$$\mathrm{DR}(\Psi_{f,0}\mathcal{M}_U[1] \xrightarrow{c} \Phi_f\mathcal{M}[1]) \simeq i_*\big(\psi_{f,0}\mathrm{DR}(\mathcal{M}) \xrightarrow{\mathrm{can}} \phi_f\mathrm{DR}(\mathcal{M})\big).$$

Using the short exact sequence (6), we see that the $\Psi_{f,0}\mathcal{M}_U$ in

$$\mathrm{Cone}(\mathcal{C}^\bullet \to \Psi_{f,0}\mathcal{M}_U[1])$$

has a s-twist. Therefore, under the quasi-isomorphism (11)

$$v : \mathrm{Cone}(\mathcal{C}^\bullet \to \Psi_{f,0}\mathcal{M}_U[1]) \to \Psi_{f,0}\mathcal{M}_U[1]$$

is given by $(0, s)$. But the s action on $\Psi_{f,0}\mathcal{M}_U$ is $J_{0,\infty}$ after taking DR. Therefore, we obtain

$$\mathrm{DR}(\Phi_f\mathcal{M}[1] \xrightarrow{v} \Psi_{f,0}\mathcal{M}[1]) \simeq i_*(\phi_f\mathrm{DR}(\mathcal{M}) \xrightarrow{\mathrm{Var}} \psi_{f,0}\mathrm{DR}(\mathcal{M})).$$

$\square$

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