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ZARISKI INVARIANT FOR NON-ISOLATED SEPARATRICES THROUGH JACOBIAN CURVES OF PSEUDO-CUSPIDAL DICRITICAL FOLIATIONS

OZIEL GÓMEZ-MARTÍNEZ

Abstract. In this work, we consider a sequence π of blowing-up morphisms in $(\mathbb{C}^2, 0)$ corresponding to the reduction of singularities of an (n, m)-cuspidal branch and we consider as well the family $\mathfrak{F}_\pi$ of pseudo-cuspidal dicritical foliations, which consists of those dicritical foliations that have exactly one dicritical component in the last divisor of π. This family contains the family of (n, m)-cuspidal dicritical foliations that we denote by $\mathfrak{F}_\pi^C$; these foliations are given by a vector field with non-degenerate linear part. We prove that the Zariski invariant of every pair of non-isolated separatrices of any foliation in $\mathfrak{F}_\pi$ that has polar transversality with $\mathfrak{F}_\pi^C$, coincides.

1. Introduction

Throughout this paper we consider dicritical foliations in $(\mathbb{C}^2, 0)$. We focus our attention on those that we call pseudo-cuspidal dicritical foliations and we study the first analytic invariant of their non-isolated branches. We recall that a plane analytic curve in $(\mathbb{C}^2, 0)$ is defined as the zero locus of an analytic function $f \in \mathbb{C}\{x, y\}$. It is known that f can be factored into a product of a finite number of irreducible power series. Each irreducible factor defines a so-called *branch* of a plane curve. The set of branches with the same topological type is known as the equisingularity class of a given *branch*. Recall that a separatrix is a leaf of the foliation which admits an analytic extension to the singular point, i.e., in a neighborhood of the singular point the leaf is the locus of zeros of a convergent power series. It is known by Camacho-Sad's theorem, that for every singular holomorphic foliation of $(\mathbb{C}^2, 0)$ there exists at least one separatrix (see [9],[28]). Hence, a singular holomorphic foliation can be seen as the implicit equation of its separatrices. For non-dicritical foliations the equisingular type of their invariant branches is finite, since there are only a finite number of invariant analytic branches having the singular point in its closure. When the foliation is dicritical there exists an infinite number of separatrices; however as a direct consequence of Seidenberg's Theorem about reduction of singularities of foliations (see [31]), we obtain the same situation as in the case of non-dicritical foliations, that is to say, the equisingular type of the non-isolated separatrices is finite. The notion of non-isolated branch of a singular holomorphic foliation was firstly introduced in [8] (see also [12]).

The aim of the present work is to study the non-isolated separatrices which are precisely those contained in a dicritical component. For this purpose, the tangency order between a smooth curve and a singular foliation at the origin is introduced. This tangency order can be considered as a "measure" of how close the smooth curve is for being a separatrix of the foliation. Given two foliations $\mathcal{F}$ and $\mathcal{G}$ of $(\mathbb{C}^2, 0)$ the *polar* or *Jacobian* curve is the tangency locus between the two foliations. This geometrical locus appears naturally in the present work; it has been the subject of study since the 19th century and has been intensively studied for the last 40 years (see [2],

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[13], [15], [24], [26], [27], [29], among others). It can be shown (see [23]) that the tangency order at the origin between a smooth curve $\mathcal{L}$ and a holomorphic non-singular foliation is equal to the multiplicity of intersection between the smooth curve and the *polar* or *Jacobian* curve between the foliation and the foliation given by the differential of the function defining the curve $\mathcal{L}$.

An (n, m)-cuspidal dicritical foliation on $(\mathbb{C}^2, 0)$ is any foliation analytically equivalent to the one given by $nxdy - mydx = 0$, where $1 < n < m$ and g.c.d.$(n, m) = 1$. An (n, m) pseudo-cuspidal dicritical foliation on $(\mathbb{C}^2, 0)$ is any foliation $\mathcal{G}$ that has exactly one dicritical component that appears after a sequence of blowing-up morphism determined by the equisingularity type (n, m). By simplicity throughout this work we call these foliations pseudo-cuspidal dicritical foliations.

In 1965 O. Zariski proved (see [22], [35], [36]) that if the branch $\mathcal{C}$ is not analytically equivalent to the cusp given by the parameterization $\varphi(t) = (t^n, t^m)$, then the *branch* admits a Puiseux series parameterization,

$$\varphi(t) = \left(t^n, t^m + t^\lambda + \sum_{j>\lambda} a_j t^j \right).$$

The value λ is an analytic invariant of the branch and it is known as the *Zariski invariant* of the branch $\mathcal{C}$. In the case where the branch $\mathcal{C}$ is analytically equivalent to a branch parametrized by $\varphi(t) = (t^n, t^m)$, the Zariski invariant is defined as $\lambda = \infty$.

Let π be the sequence of blowing-up morphisms in $(\mathbb{C}^2, 0)$ corresponding to the reduction of singularities of an (n, m)-cuspidal branch, let us denote by $\mathfrak{F}_\pi$ the family of (n, m) pseudo-cuspidal foliations and by $\mathfrak{F}_\pi^C \subset \mathfrak{F}_\pi$ the cuspidal ones. Let $\mathcal{F}$ be a pseudo-cuspidal dicritical foliation. We say that $\mathcal{F}$ has *polar transversality* with the family of cuspidal dicritical foliations $\mathfrak{F}_\pi^C$ if and only if for every $\mathcal{G} \in \mathfrak{F}_\pi^C$ the strict transform of the *polar* or *Jacobian* curve $\mathcal{J}(\mathcal{F}, \mathcal{G})$ does not intersect the dicritical component E_{dic}; this irreducible component of the exceptional divisor $D = \pi^{-1}(0)$ is the irreducible component that appears at the last blowing-up morphism of the sequence π. We denote by $\mathfrak{F}_\pi^\star$ the subfamily of foliations $\mathcal{F} \in \mathfrak{F}_\pi$ having polar transversality with the family $\mathfrak{F}_\pi^C$. The main result in this paper is the following.

Theorem 4.3. *Let $\mathcal{F}$ be a pseudo-cuspidal dicritical foliation that has the polar transversality property with $\mathfrak{F}_\pi^C$, $\mathcal{F} \in \mathfrak{F}_\pi^\star$. Then all the non-isolated separatrices of $\mathcal{F}$ have coinciding Zariski invariants.*

Remark 1.1. Example 4.7 given in Section 4 shows that not all the non-isolated separatrices of a pseudo-dicritical cuspidal foliation have the same Zariski invariant.

The previous theorem shows that for every element $\mathcal{F} \in \mathfrak{F}_\pi^\star$ the Zariski invariant of the non-isolated branches of $\mathcal{F}$ is fixed.

Finally, we rely on the results given in [30], to present an explicit family of foliations having the property of polar transversality with the family $\mathfrak{F}_\pi^C$.

Historical Context. The development of the theory of holomorphic foliations goes back to the works of J. C. Bouquet, C. Briot and H. Poincaré at the end of the 19th century and beginning of the 20th century.

The existence of a separatrix for a singular holomorphic foliation of $(\mathbb{C}^2, 0)$ is a result given by C. Camacho and P. Sad. in [9] (see also [10] and [28]). As a consequence of this result, singular curves appear naturally in the study of singular holomorphic foliations.

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The topological study of singular curves was initiated at the beginning of the twentieth century, however Newton in 1676 had already begun with the study of singular curves from an "analytic" point of view, namely, Newton shows that there is always a formal parameterization of a singular curve. It was until 1850 that Puiseux showed that the power series is convergent. From the parameterization it is possible to introduce the *characteristic* and the *Puiseux pairs* of a branch which are invariants to the equisingularity class of the branch. Two germs of curves $\mathcal{C}, \mathcal{D}$ in $(\mathbb{C}^2, 0)$ are said to be *equisingular* or that they have the same *topological type* if there exists a homeomorphism germ of the ambient space mapping the curve $\mathcal{C}$ to the curve $\mathcal{D}$. The earlier results about the equisingularity of plane curves were obtained in the late 1920's and early 1930's. These topological classification contributions were developed mainly by K. Brauner [3], E. Kähler [25], O. Zariski [34] and W. Burau [7].

The analytic classification of plane branches was started by Oscar Zariski in the 1960's. The objects of study are the germs of irreducible singular curves known as *branches*, belonging to the same equisingularity class. The moduli problem of a plane branch consists in the description of the quotient space of the equisingularity class of a branch under the analytic equivalence relation. This problem was formally established by O. Zariski in 1973 (see [36]).

In 1965, S. Ebey (see [18]) gave a complete description of the moduli space for the equisingularity class $(5, 9)$ and introduced some ideas that were later developed by O. Zariski in an attempt to solve the problem. Later, in 1966, O. Zariski in [35] gave a characterization of the irreducible singular curves that are analytically equivalent to the curve given by the equation $y^n - x^m = 0$; it was given in terms of the set of Kähler differential values. Moreover, in this work the first analytic invariant of plane branches was introduced; this invariant is now known as the *Zariski invariant*. However, the moduli problem for plane branches remained open for approximately 40 years. In that period many works were carried out to try to give a solution to the problem (see for example [4], [17], [21] and [30]). Finally in 2011, A. Hefez and M.E. Hernandes in [22] gave a complete solution to the moduli problem of plane branches. Recently a paper was published by M.E. Hernandes and M.E. Rodrigues (see [19]), in which the analytic classification problem of curves with several branches was studied and solved, following the ideas developed by O. Zariski in [22] and [36].

As we previously mentioned, the existence of separatrices of singular holomorphic foliations naturally relates the theory of singular holomorphic foliations and the theory of singular curves. Some of the results for singular curves have an equivalent result in the context of singular holomorphic foliations in $(\mathbb{C}^2, 0)$. The resolution of singularities for singular holomorphic foliations in $(\mathbb{C}^2, 0)$, is an analogous of the result of resolution of singularities of analytic curves. It states that singularities of singular holomorphic foliations can be transformed under a sequence of blowing-up morphisms into "simpler" singularities. This result was established by A. Seidenberg in 1968, see [31]. It is natural to ask what the relation between the resolution of singularities of the separatrices and the resolution of singularities of the foliation is. In [8] C. Camacho, L. Neto and P. Sad solved this question and proved that the desingularization of a singular holomorphic foliation and the desingularization of its separatrices is the same in the case where the foliation is non-dicritical and it has no *saddle-node* singularities. Such foliations are known as *generalized curves*. In the same work the notion of an isolated separatrix was introduced (see also [12] and [16]); these separatrices are those that belong to a dicritical component of the foliation after a resolution of singularities.

Structure of the work and acknowledgements. In Section 2 we introduce the pseudo-cuspidal dicritical foliations and, in particular, the cuspidal dicritical ones. We show that the Zariski invariant of a plane branch of the equisingularity class (n, m) can be obtained in terms of the tangency order between the branch and a cuspidal dicritical foliation (see Theorem 2.22).

An explicit relation between the tangency order of a branch of a fixed pseudo-cuspidal dicritical foliation and one cuspidal foliation in terms of its corresponding polar or Jacobian curve is given in Theorem 3.8; for this purpose we use the divisorial valuation associated to one irreducible component of the sequence of blowing-up morphism π. In Section 4 we prove Theorem 4.3 and we give an explicit family of pseudo-cupidal foliations having the property of polar transversality with the family of cuspidal dicritical foliations (see Theorem 4.13).

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2. Pseudo-Cuspidal Dicritical Foliations

Let $\mathcal{F}$ be a germ of a holomorphic foliation in $(\mathbb{C}^2, 0)$ defined by a holomorphic vector field

$$v = Q(x,y)\frac{\partial}{\partial x} - P(x,y)\frac{\partial}{\partial y},$$

with $P, Q \in \mathbb{C}\{x, y\}$, or by the zero locus of its induced Pfaffian 1-form

$$\omega = P(x,y)dx + Q(x,y)dy.$$

The foliation $\mathcal{F}$ has an isolated singularity at the origin if $P(0,0) = Q(0,0) = 0$, and either $P(x,y) \neq 0$ or $Q(x,y) \neq 0$ for $(x,y) \neq (0,0)$ in a neighborhood of $(0,0)$; at such points it is said that $\mathcal{F}$ is regular. Throughout this work we always assume that we have isolated singularities.

Definition 2.1. The multiplicity of a singular holomorphic foliation $\mathcal{F}$ at the origin in $(\mathbb{C}^2, 0)$ is defined as

$$\mathrm{m}_0(\mathcal{F}) := \min\left\{\mathrm{ord}_0(P(x,y)), \mathrm{ord}_0(Q(x,y))\right\}. \tag{1}$$

Definition 2.2. Let $\mathcal{F}$ be a germ of a holomorphic foliation in $(\mathbb{C}^2, 0)$ with isolated singularity and let p be a regular point of $\mathcal{F}$. By the Rectification Theorem of Differential Equations it is known that, locally, in a neighborhood of p, the solutions can be seen as level sets of the coordinate function x. To each local level set corresponds a local solution of the differential equation and the phase curve thus defined is called a *leaf* of the foliation.

Definition 2.3. Let $\mathcal{F}$ be a germ of a holomorphic foliation in $(\mathbb{C}^2, 0)$ with isolated singularity defined by a holomorphic vector field v. We say that a germ of branch $\mathcal{C}$ is a *separatrix* of the foliation $\mathcal{F}$ if $\mathcal{C} = \mathcal{L} \cup \{0\}$, where $\mathcal{L}$ is a leaf of the foliation. In terms of the induced Pfaffian form ω the following equation holds

$$\omega \wedge df = f\eta, \tag{2}$$

where η is a holomorphic 2-form and $\{f = 0\}$ is a reduced equation (free of squares) of $\mathcal{C}$.

Definition 2.4. Let $\mathcal{F}$ be a germ of a holomorphic foliation in $(\mathbb{C}^2, 0)$ with isolated singularity. We say that 0 is a *non-dicritical singularity* of the foliation if $\mathcal{F}$ has, at most, a finite number of invariant curves having the singular point in its closure. Otherwise, we say that 0 is a *dicritical singularity*; correspondingly, $\mathcal{F}$ is said to be a *non-dicritical* or a *dicritical foliation*.

Remark 2.5. If $\mathcal{F}$ is a non-dicritical foliation, then each irreducible component of the exceptional divisor $D = \pi^{-1}(0)$, where $\pi : (M, D) \to (\mathbb{C}^2, 0)$ is the well known morphism of resolution of singularities, is an invariant curve of the strict transform of $\mathcal{F}$. Otherwise, if $\mathcal{F}$ is a dicritical foliation, there exists at least one component of D that is transverse at every point to $\mathcal{F}$ except for a finite number of points (singular or tangency points). Such a component will be called *dicritical component*.

Definition 2.6. We say that $\mathcal{C}$ is an *isolated separatrix* for the holomorphic foliation $\mathcal{F}$ with isolated singularity in $(\mathbb{C}^2, 0)$ if $\mathcal{C}$ does not intersect any dicritical component of $\mathcal{F}$. Otherwise, we say that $\mathcal{C}$ is a *non-isolated separatrix*.

Recall that the blowing-up morphism of one point $p \in M$, where M is a complex manifold of dimension 2, can be described by two standard coordinates charts given by,

$$\begin{aligned} \pi_1 : \tilde{U} \subset \tilde{M} &\to U \subset M & \pi_1 : \tilde{V} \subset \tilde{M} &\to U \subset M \\ (x_1, y_1) &\mapsto \pi_1(x_1, y_1) = (x_1, x_1 y_1), & (\tilde{x}_1, \tilde{y}_1) &\mapsto \pi_1(\tilde{x}_1, \tilde{y}_1) = (\tilde{x}_1 \tilde{y}_1, \tilde{y}_1), \end{aligned}$$

where $\tilde{U}, \tilde{V}$ are open sets such that $\tilde{M} = \tilde{U} \cup \tilde{V}$ and U is an open neighboorhod of p. We say that $(x_1, y_1) = (0, 0)$ is the origin in the first coordinate chart and $(\tilde{x}_1, \tilde{y}_1) = (0, 0)$ is the origin in the second coordinate chart. Let $\pi : (M, D) \to (\mathbb{C}^2, 0)$ be a finite sequence of blowing-up morphisms, we denote by $D := \pi^{-1}(0)$ the *exceptional divisor*. Let us recall that $D = \bigcup_{i=1}^{k} E_i$, where E_i is a rational smooth curve $E_i = \mathbb{P}$ for all i. Given a point $p \in D$, we denote by $e_p(D)$ the number of irreducible components of D passing through p, thus $e_p(D) \in \{1, 2\}$. Let $\pi:(M, D) \to (\mathbb{C}^2, 0)$ be a finite sequence of blowing-up morphisms, that is, $\pi = \pi_1 \circ \pi_2 \circ \cdots \circ \pi_s$, determined by the Euclidean algorithm of the pair (n, m), $1 < n < m$, g.c.d$(n, m) = 1$. The transformation π is determined by the algorithm as follows:

$$\begin{aligned} m &= \alpha_0 n + r_0, \quad 1 \le r_0 \le n-1 \\ n &= \alpha_1 r_0 + r_1, \quad 0 \le r_1 \le r_0 - 1 \\ r_0 &= \alpha_2 r_1 + r_2, \quad 0 \le r_2 \le r_1 - 1 \\ &\vdots \\ r_{k-1} &= \alpha_{k+1} r_k + 1, \\ r_k &= \alpha_{k+2}, \quad \text{where} \quad \alpha_{k+2} = r_k. \end{aligned}$$

Let $\{p_0, p_1, ..., p_{s-1}\}$ be the collection of points corresponding to each blowing-up morphism, where $p_0 = 0 \in \mathbb{C}^2$. Every point p_i satisfies the following properties,

(i) $p_i \in E_i = \pi_i^{-1}(p_{i-1})$ for all $i \in \left\{1, 2, ..., \sum_{j=0}^{k+2} \alpha_j\right\}$.

(ii) $e_{p_i}(E^i) = 1$ for $i \in \{1, 2, ..., \alpha_0\}$ where $E^i = E_1 \cup E_2 \cup \cdots \cup E_i$; and p_i is not the origin of the second chart for all $i \in \{1, 2, ..., \alpha_0\}$.

(iii) $e_{p_i}(E^i) = 2$ for every $i \in \left\{\alpha_0 + 1, \alpha_0 + 2, ..., \sum_{j=0}^{k+2} \alpha_j\right\}$, that is, p_i is a corner point.

(iv) The point p_i is the origin of the first coordinate chart for all i in the set

$$\left\{\sum_{j=0}^{l} \alpha_j, \sum_{j=0}^{l} \alpha_j + 1, \cdots, \sum_{j=0}^{l+1} \alpha_j\right\}$$

if l is odd and p_i is the origin of the second chart for every i in the set

$$\left\{\sum_{j=0}^{l} \alpha_j, \sum_{j=0}^{l} \alpha_j + 1, \cdots, \sum_{j=0}^{l+1} \alpha_j\right\}$$

if l is even.

The collection of points $\{p_0, p_1, p_2, ..., p_{s-1}\}$ satisfiying the conditions (i)-(iv) will be called the *centers* of the blowing-up morphisms or, simply, *centers*.

Definition 2.7. Let $\mathcal{L}$ be a smooth plane curve in $(\mathbb{C}^2, 0)$. We say that $\mathcal{L}$ has π *maximal-contact* if the collection of centers $p_0, p_1, ..., p_{\alpha_0}$ lies inside of the set of infinitely near points of $\mathcal{L}$, $\{p_0, p_1, ..., p_{\alpha_0}\} \subset \{\text{Infinitely near points of } \mathcal{L}\}$.

Lemma 2.8. *Let $\pi : (M, D) \to (\mathbb{C}^2, 0)$ be a finite sequence of blowing-up morphisms determined by the Euclidean algorithm of the pair (n, m), with $D = \pi^{-1}(0)$. Then, there exists a coordinate system (x, y) of $(\mathbb{C}^2, 0)$ such that $\{y = 0\}$ has π maximal-contact.*

Proof. Let $p_0, p_1, p_2, ..., p_{\alpha_0}$ be the respective centers of blowing-up morphisms

$$\pi_i : (M_i, \pi_i^{-1}(p_{i-1})) \to (M_{i-1}, p_{i-1})$$

for $i = 1, 2, ..., \alpha_0$. By hypothesis $e_{p_i}(E^i) = 1$ for $i = 1, 2, ..., \alpha_0$. Since p_1 is not the origin of the second chart, the blowing-up morphism π_1 is given in local coordinates x_1, y_1 at p_1 by the equation $\pi_1(x_1, y_1) = (x_1, x_1(y_1 + c_1))$, $c_1 \in \mathbb{C}$. The blowing-up morphism π_2 is given, in local coordinates x_2, y_2 at p_2, by one of the following equations

$$T_{1,c_2}(x_2, y_2) = (x_2, x_2(y_2 + c_2)) \quad \text{or} \quad T_2(x_2, y_2) = (x_2 y_2, y_2). \tag{3}$$

If we have T_{1,c_2}, then the divisor E_2 at p_2 is locally given by $E_2 = \{x_2 = 0\}$. If we have T_2, then $E_2 = \{x_2 y_2 = 0\}$ at p_2, but $e_{p_2}(E^2) = 1$ hence we have only the T_{1,c_2} case. Recursively using the same argument, in every point p_i for $i = 1, 2, ..., \alpha_0$ we obtain coefficients $c_1, c_2, ..., c_{\alpha_0}$. At the point p_{α_0} we have the local coordinates $(x_{\alpha_0}, y_{\alpha_0})$ and we look at the blowing-up morphism in the coordinates given by the equation $T_{1,c_{\alpha_0}}(x_{\alpha_0}, y_{\alpha_0}) = (x_{\alpha_0}, x_{\alpha_0}(y_{\alpha_0} + c_{\alpha_0}))$. We consider the curve given by $\mathcal{L}' = \{y_{\alpha_0} + c_{\alpha_0} = 0\}$. Projecting this curve to the point p_0, we obtain the curve given by $\mathcal{L} = \left\{ y + \sum_{j=1}^{\alpha_0} c_j x^j = 0 \right\}$. Now if we consider the following change of coordinates given by $\tilde{x} = x$ and $\tilde{y} = y + \sum_{j=1}^{\alpha_0} c_j x^j$, we get the assertion of Lemma 2.8.

Throughout the present work we assume that the system of coordinates of $(\mathbb{C}^2, 0)$ is adapted to π.

Remark 2.9. Note that if the system of coordinates (x, y) of $(\mathbb{C}^2, 0)$ is adapted to π, then π is the resolution of singularities of the branch given by $y^n - x^m = 0$. If $\mathcal{C}$ is a branch of the equisingularity class (n, m) which is desingularized by π, then it is possible to show that its parameterization, up to reparameterization, is given by $\varphi(t) = (t^n, \varphi_2(t))$ with $\varphi_2(t) \in \mathbb{C}\{t\}$ and $\mathrm{ord}_0 \varphi_2(t) = m$.

Let us introduce the family $\mathfrak{F}_\pi$ of pseudo-cuspidal dicritical foliations.

Definition 2.10. (Pseudo-Cuspidal Dicritical Foliations) Let $\mathcal{F}$ be a holomorphic dicritical foliation with isolated singularity in $(\mathbb{C}^2, 0)$, let π be the finite sequence of blowing-up morphisms induced by the pair (n, m) and let $D = \pi^{-1}(0)$. We say that $\mathcal{F}$ belongs to the family of *pseudo-cuspidal dicritical foliations* $\mathfrak{F}_\pi$ if and only if the following assertions take place:

- (i) The induced foliation $\pi^*\mathcal{F}$ has a unique dicritical component E_{dic}. This dicritical component is the irreducible component of the exceptional divisor D that appears after the last blowing-up morphism of the sequence π, and there are not singular points or tangency points of $\pi^*\mathcal{F}$ in this component;
- (ii) $\mathcal{F}$ has, at least, two invariant smooth curves, $\mathcal{L}_0$, $\mathcal{L}_\infty$, such that $\mathcal{L}_0$ has π maximal-contact, and $\mathcal{L}_\infty$ has transverse intersection with $\mathcal{L}_0$.

We now introduce the notion of cuspidal dicritical foliations $\mathfrak{F}_\pi^C$, which is a subfamily of pseudo-cuspidal ones.

Definition 2.11. (Cuspidal Dicritical Foliations) Let $\mathcal{G}$ be a dicritical foliation in $\mathfrak{F}_\pi$. We say that $\mathcal{G}$ belongs to the family of (n,m)-*cuspidal dicritical foliations* or simply *cuspidal dicritical foliations* $\mathcal{F}_\pi^C$ if there exists a vector field v defining the foliation $\mathcal{G}$ such that its Jacobian matrix $\mathrm{D}_{(0,0)}\, v$ has two different eigenvalues $\lambda_1, \lambda_2 \in \mathbb{C}$, with $\lambda_1 \cdot \lambda_2 \neq 0$ and $\frac{l_1}{l_2} = \frac{n}{m} \in \mathbb{Q}_{>0}$.

Lemma 2.12. *Let $\mathcal{G} \in \mathfrak{F}_\pi^C$ be an (n,m)-cuspidal dicritical foliation. For every (n,m)-cuspidal dicritical foliation $\mathcal{G}$ there exists a suitable change of coordinates and a generator $\omega_\mathcal{G}$ of $\mathcal{G}$ such that in the aforementioned coordinates it has the form*

$$\omega_\mathcal{G} = (my + h_1(x,y))dx - (nx + cy + h_2(x,y))dy \quad \text{g.c.d.}(n,m) = 1, \tag{4}$$

where $c \in \mathbb{C}$, $h_i(x,y)$ is a convergent series for $i = 1,2$ and,

$$\mathrm{ord}_0(h_i) \geq 2 \quad \text{and} \quad \mathrm{ord}_x(h_1(x,0)) \geq \left\lfloor \frac{m}{n} \right\rfloor + 1.$$

Proof. Let $\mathcal{G}$ be an (n,m)-cuspidal dicritical foliation $\mathcal{G} \in \mathfrak{F}_\pi^C$. There exists a vector field v defining the foliation $\mathcal{G}$ such that its Jacobian matrix $\mathrm{D}_{(0,0)}\, v$ has two different eigenvalues $\lambda_1, \lambda_2 \in \mathbb{C}$, with $\lambda_1 \cdot \lambda_2 \neq 0$ and $\frac{\lambda}{\lambda_2} = \frac{n}{m}$. So, by Poincaré's Linearization Theorem (see [11]), the foliation is analytically equivalent to the foliation induced by the linear vector field $v_0 = nx\frac{\partial}{\partial x} + my\frac{\partial}{\partial y}$ or, equivalently, by the 1-form $\omega_0 = mydx - nxdy$. Since the coordinates are adapted to π, by conjugation of the linear part we have that the foliation is generated by

$$\hat{v} = (nx + cy + h_2(x,y))\,\frac{\partial}{\partial x} + (my + h_1(x,y))\,\frac{\partial}{\partial y},$$

with $\mathrm{ord} h_i \geq 2$ for $i = 1,2$ and $\mathrm{ord}_x h_1(x,0) \geq \lfloor \frac{m}{n} \rfloor + 1$. □

Remark 2.13. Let $\mathcal{G}$ be an (n,m)-cuspidal dicritical foliation. From the definition we know that there exists a vector field v generating the foliation $\mathcal{G}$, whose linear part has two eingenvalues $\lambda_1, \lambda_2 \neq 0$, such that $\frac{\lambda_1}{\lambda_2} = \frac{n}{m}$. Since $\frac{\lambda_1}{\lambda_2} = \frac{n}{m} \in \mathbb{R}_{>0}$ the eigenvalues are *non-resonant*. Moreover, the eigenvalues belong to the *Poincaré domain*, so it is follows from the Poincaré Linearization Theorem (see [23]) that $\mathcal{G}$ is analytically equivalent to the foliation $\mathcal{G}_0$ which is generated by the vector field $v_0 = nx\frac{\partial}{\partial x} + m\frac{\partial}{\partial y}$ or equivalently, by the 1-form $\omega_0 = mydx - nxdy$.

In order to preserve that the system of coordinates is adapted to π, the allowed change of coordinates are of the form

$$H(x,y) = (x + cy + H_1(x,y), y + H_2(x,y)), \tag{5}$$

with $c \in \mathbb{C}$, $\mathrm{ord}_0(H_i) \geq 2$, $i = 1,2$ and $\mathrm{ord}_0 H_2(x,0) \geq \lfloor \frac{m}{n} \rfloor + 1$.

2.1. The Zariski invariant and the Cuspidal Dicritical Foliations.

The principal goal of this section is to recover the Zariski invariant of a plane branch $\mathcal{C}$ in terms of the tangency order between $\mathcal{C}$ and a cuspidal dicritical foliation $\mathcal{G}$. In [35] it was proved that if a plane branch $\mathcal{C}$ of the equisingularity class (n,m) is not analytically equivalent to the plane branch given by the parameterization $\varphi(t) = (t^n, t^m)$, then $\mathcal{C}$ admits a parameterization of the following form

$$\varphi(t) = \left(t^n, t^m + t^\lambda + \sum_{j > \lambda} a_j t^j \right), \tag{6}$$

where λ is known as the Zariski invariant. We now recall some known facts about the analytic classification of plane branches.

Given any two plane branches, $\mathcal{C}, \mathcal{D}$, we say that $\mathcal{C}$ and $\mathcal{D}$ are analytically equivalent if there exists a germ of a biholomorphism H such that $H(\mathcal{C}) = \mathcal{D}$. The analytic classification of plane branches can be given in terms of the $\mathcal{A}$-equivalence of its parameterization.

Definition 2.14. Let $\mathcal{C}$ and $\mathcal{D}$ be plane branches and let φ, ψ be their respective parameterizations. We say that φ is $\mathcal{A}$-equivalent to ψ if there exists a germ of a biholomorphism $H : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ and an automorphism $\tau : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ such that the following equation takes place

$$\varphi = H \circ \psi \circ \tau^{-1}. \tag{7}$$

Lemma 2.15. *Let $\mathcal{C}$ and $\mathcal{D}$ be plane branches and let φ, ψ be their respective parameterizations. Then the branches $\mathcal{C}$ and $\mathcal{D}$ are analytically equivalent if and only if φ and ψ are $\mathcal{A}$-equivalent.*

The proof of the previous lemma can be found in [6]. We recall the definition of the tangency order between a plane branch $\mathcal{C}$ and a foliation $\mathcal{F}$ at the origin. (see [13]).

Definition 2.16. Let $\mathcal{F}$ be a singular holomorphic foliation with an isolated singularity in $(\mathbb{C}^2, 0)$. Let $\mathcal{C}$ be a plane branch that is not a separatrix of $\mathcal{F}$ and let φ be its parameterization. We define the *tangency order* of the branch $\mathcal{C}$ with $\mathcal{F}$ at 0, $\mathrm{Tan}_0(\mathcal{C}, \mathcal{F})$, as

$$\mathrm{Tan}_0(\mathcal{C}, \mathcal{F}) = \mathrm{ord}_t \varphi^* \omega_{\mathcal{F}}, \tag{8}$$

where $\{\omega_{\mathcal{F}} = 0\}$ is a local equation for $\mathcal{F}$.

We stress that Definition 2.16 does not depend of the parameterization of the branch. Namely, two different parameterizations of a given branch are related under an automorphism.

Remark 2.17. In the case where a plane branch $\mathcal{C}$ is a separatrix of $\mathcal{F}$ we say that $\mathrm{Tan}_0(\mathcal{C}, \mathcal{F}) = \infty$, because the pull-back of the generator $\omega_{\mathcal{F}}$ by the parameterization is 0.

Moreover, the tangency order has the following behavior under blowing-up morphisms.

Proposition 2.18. *Let $\mathcal{F}$ be a singular holomorphic foliation in $(\mathbb{C}^2, 0)$, let $\mathcal{C}$ be a plane branch that is not separatrix of $\mathcal{F}$, let $\pi_1 : (M_1, E_1) \to (\mathbb{C}^2, 0)$ be the blowing-up morphism of the origin and let $E_1 = \pi_1^{-1}(0)$ be the exceptional divisor. Then the following equalities take place:*

(i) $\mathrm{Tan}_{p_0}(\mathcal{C}, \mathcal{F}) = \mathrm{m}_{p_0}(\mathcal{F})\mathrm{m}_{p_0}(\mathcal{C}) + \mathrm{Tan}_{p_1}(\mathcal{C}^{(1)}, \mathcal{F}^{(1)})$*, when the divisor E_1 is not dicritical;*

(ii) $\mathrm{Tan}_{p_0}(\mathcal{C}, \mathcal{F}) = (\mathrm{m}_{p_0}(\mathcal{F}) + 1)\,\mathrm{m}_{p_0}(\mathcal{C}) + \mathrm{Tan}_{p_1}(\mathcal{C}^{(1)}, \mathcal{F}^{(1)})$*, when the divisor E_1 is dicritical,*

where $p_0 = 0$, $p_1 = E_1 \cap \mathcal{C}^{(1)}$ and $\mathcal{F}^{(1)}, \mathcal{C}^{(1)}$ are the strict transforms of $\mathcal{F}, \mathcal{C}$ by the morphism π_1.

Proof. Let $\mathcal{C}$ be a plane branch and let $\mathcal{F}$ be the foliation induced by $\{\omega_{\mathcal{F}} = 0\}$. Without loss of generality we assume that $\mathcal{C}$ is parametrized by $\varphi(t) = \left(t^n, \sum\limits_{j \geq n} a_j t^j\right)$ where $n = \mathrm{m}_{p_0}(\mathcal{C})$. The strict transform of $\mathcal{C}$ by π_1 is $\mathcal{C}^{(1)}$ and it is parametrized by $\varphi^{(1)} = \pi_1^{-1} \circ \varphi$. Then we have that

$$(\varphi^{(1)})^* \omega_{\mathcal{F}}^{(1)} = (\pi_1^{-1} \circ \varphi)^* \frac{\pi_1^* \omega_{\mathcal{F}}}{x_1^r}. \tag{9}$$

The following cases must be analyzed:

(i) $E_1 = \pi_1^{-1}(0)$ is non-dicritical. In this case $r = \mathrm{m}_{p_0}(\mathcal{F})$, and substituting in (9) we have that

$$(\varphi^{(1)})^* \omega_{\mathcal{F}}^{(1)} = (\pi_1^{-1} \circ \varphi)^* \frac{\pi_1^* \omega_{\mathcal{F}}}{x_1^{\mathrm{m}_{p_0}(\omega_{\mathcal{F}})}} = \frac{\varphi^* \omega_{\mathcal{F}}}{t^{n(\mathrm{m}_{p_0}(\omega_{\mathcal{F}}))}}.$$

This last equation implies that

$$\mathrm{ord}_t (\varphi^{(1)})^* \omega_{\mathcal{F}}^{(1)} = \mathrm{ord}_t \varphi^* \omega_{\mathcal{F}} - n\left(\mathrm{m}_{p_0}(\mathcal{F})\right),$$

or equivalently,

$$\mathrm{ord}_t \varphi^* \omega_{\mathcal{F}} = \mathrm{ord}_t (\varphi^{(1)})^* \omega_{\mathcal{F}}^{(1)} + \mathrm{m}_{p_0}(\mathcal{C})\left(\mathrm{m}_{p_0}(\mathcal{F})\right).$$

This proves the non-dicritical case.

(ii) $E_1 = \pi_1^{-1}(0)$ is dicritical for $\mathcal{F}$. In this case $r = \mathrm{m}_{p_0}(\mathcal{F}) + 1$ and with the same arguments as above we have that

$$\mathrm{ord}_t \varphi^* \omega_{\mathcal{F}} = \mathrm{ord}_t (\varphi^{(1)})^* \omega_{\mathcal{F}}^{(1)} + \mathrm{m}_{p_0}(\mathcal{C}) \left(\mathrm{m}_{p_0}(\mathcal{F}) + 1\right).$$

This last equality finishes the proof.

Definition 2.19. Let $\mathcal{C}$ be a plane branch of the equisingularity class (n, m). We say that $\mathcal{C}$ is (n, m)-*quasihomogenous* if it is analytically equivalent to the quasihomogenous cusp given by $\{y^n - x^m = 0\}$.

The following proposition characterizes the (n, m)-quasihomogeneous branch $\mathcal{C}$ of the equisingularity class (n, m) in terms of the cuspidal dicritical foliations.

Proposition 2.20. *Let $\mathcal{C}$ be a plane branch of the equisingularity class (n, m). The following statements are equivalent*

(i) $\mathcal{C}$ is (n, m)-quasihomogeneous,

(ii) There exists $\mathcal{G} \in \mathfrak{F}_\pi^C$ such that $\mathrm{Tan}_0(\mathcal{C}, \mathcal{G}) = \infty$

Proof. Let us suppose that $\mathcal{C}$ is analytically equivalent to the cusp $\{y^n - x^m = 0\}$. By Lemma 2.15 there exists a germ of a biholomorphism $H : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ and an automorphism $\tau : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ such that if φ is the parameterization of $\mathcal{C}$ and $\psi(t) = (t^n, t^m)$, then $\psi = H \circ \varphi \circ \tau^{-1}$. Let $\mathcal{G}$ be the foliation generated by $\omega_{\mathcal{G}} = H^* \omega_0$, where $\omega_0 = nxdy - mydx$. Then

$$\mathrm{Tan}_0(\mathcal{C}, \mathcal{G}) = \mathrm{ord}_t \varphi^* \omega_{\mathcal{G}} = \mathrm{ord}_t (\tau)^* \psi^* (H^{-1})^* \omega_{\mathcal{G}} = \mathrm{ord}_t \tau^* \psi^* \omega_0.$$

Since $\mathrm{ord}_t \psi^* \omega_0 = \infty$, then $\mathrm{Tan}_0(\mathcal{C}, \mathcal{G}) = \infty$.

Let us suppose now that there exists a cuspidal dicritical foliation $\mathcal{G} \in \mathcal{F}_\pi^C$ such that $\mathrm{Tan}_0(\mathcal{C}, \mathcal{G}) = \infty$. We know that there exists a biholomorphism $H : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$, $H(x, y) = (H_1(x, y), H_2(x, y))$ such that foliation $\mathcal{G}$ is analytically equivalent to the foliation given by the equation $\{\omega_0 = 0\}$, where $\omega_0 = nxdy - mydx$, that is, $H^* \omega_0 = \omega_{\mathcal{G}}$, with $\{\omega_{\mathcal{G}} = 0\}$ generating $\mathcal{G}$. Let $H_1(x, y) = x + cy + h_1(x, y)$ with $c \in \mathbb{C}$ and $\mathrm{ord}_0 h_1(x, y) \geq 2$, we consider $\rho := \varphi^* H_1(x, y) = H_1 \circ \varphi(t)$. We can choose $\tau : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ an automorphism such that $\tau^n := \rho$, and consider the composition $\psi = H \circ \varphi \circ \tau^{-1}$. Then,

$$\mathrm{ord}_t \psi^* \omega_0 = \mathrm{ord}_t (H \circ \varphi \circ \tau^{-1})^* \omega_0 = \mathrm{ord}_t (\tau^{-1})^* \varphi^* H^* \omega_0 = \mathrm{ord}_t \varphi^* \omega_{\mathcal{G}} = \mathrm{Tan}_0(\mathcal{C}, \mathcal{G}).$$

Since $\mathrm{Tan}_0(\mathcal{C}, \mathcal{G}) = \infty$, this last equation implies that $\psi(t) = (t^n, t^m)$ up to parameterization. Thus, $\mathcal{C}$ is analytically equivalent to $\{y^n - x^m = 0\}$ and we have finished the proof of the statement.

A straightforward consequence of the previous proposition is the following

Corollary 2.21. *Let $\mathcal{C}$ be a plane branch of the equisingularity class (n, m). If $\mathcal{C}$ is not analytically equivalent to $\{y^n - x^m = 0\}$ then for all (n, m)-cuspidal dicritical foliations $\mathcal{G} \in \mathfrak{F}_\pi^C$,* $\mathrm{Tan}_0(\mathcal{C}, \mathcal{G}) < \infty$.

As we stressed before, the Zariski invariant is an invariant for the analytic classification of plane branches. In [36] (see also [22]) it was shown that if $\mathcal{C}$ is a branch that is not analytically equivalent to $\{y^n - x^m = 0\}$ then it admits a Puiseux parameterization of the following type

$$\varphi(t) = \left(t^n, t^m + t^\lambda + \sum_{j > \lambda} a_j t^j\right), \tag{10}$$

with λ satisfying the following two conditions,

$$\lambda \notin S \quad \text{and} \quad \lambda + n - m \notin m\mathbb{Z}_{>0}, \tag{11}$$

where S represents the semigroup of the branch $\mathcal{C}$. Recall that given a plane branch $\mathcal{C}$ the equisingularity class of $\mathcal{C}$, which consists of all the plane branches with the same topological type, is completely determined by the corresponding semigroup. Given a plane branch $\mathcal{C}$ we define the intersection multiplicity of $\mathcal{C}$ with another analytic curve $\mathcal{D}$, that is, $\mathcal{D} = \{g = 0\}$, with $g \in \mathbb{C}\{x, y\}$ as

$$\iota_0 = \dim \frac{\mathbb{C}\{x, y\}}{\langle f, g\rangle},$$

where $\langle f, g\rangle$ is the ideal generated by f and g, $\mathcal{C} = \{f = 0\}$. The semigroup of the plane branch $\mathcal{C}$ is by definition the set of the intersection multiplicities between all the plane curves and the plane branch $\mathcal{C}$, that is

$$S = \{\iota_0(\mathcal{C}, \mathcal{D}) \;:\; \mathcal{C} \neq \mathcal{D}, \mathcal{D} = \{g = 0\}, g \in \mathbb{C}\{x, y\}\}.$$

In the case of plane branches of the equisingularity class (n, m) the semigroup is the set generated by the numbers n, m over $\mathbb{Z}_{\geq 0}$. That is,

$$S = \{\alpha n + \beta m \;:\; \alpha, \beta \in \mathbb{Z}_{\geq 0}\}\,. \tag{12}$$

Let $\mathcal{C}$ be a plane branch of the equisingularity class (n, m) and $\mathcal{G}^\star$ a cuspidal dicritical foliation $\mathcal{G}^\star \in \mathfrak{F}^c_\pi$. We say that $\mathrm{Tan}_0(\mathcal{C}, \mathcal{G}^\star)$ is maximal if for every cuspidal dicritical foliation $\mathcal{G} \in \mathfrak{F}^C_\pi$ we have

$$\mathrm{Tan}_0(\mathcal{C}, \mathcal{G}) \leq \mathrm{Tan}_0(\mathcal{C}, \mathcal{G}^\star).$$

The main result of this section is the following.

Theorem 2.22. *Let $\mathcal{C}$ be a plane branch of the equisingularity class (n, m) such that $\mathcal{C}$ is not (n, m)-quasihomogeneous, and let $\mathcal{G} \in \mathfrak{F}^C_\pi$ be a cuspidal dicritical foliation. Then the following statements are equivalent*

(i) The tangency order $\mathrm{Tan}_0(\mathcal{C}, \mathcal{G})$ is finite and maximal;
(ii) The tangency order satisfies

$$\mathrm{Tan}_0(\mathcal{C}, \mathcal{G}) + 1 - n \notin S \quad \textit{and} \quad \mathrm{Tan}_0(\mathcal{C}, \mathcal{G}) + 1 - m \notin m\mathbb{Z}_{>0}; \tag{13}$$

(iii) The value

$$\lambda = \mathrm{Tan}_0(\mathcal{C}, \mathcal{G}) + 1 - n \tag{14}$$

is the Zariski invariant of $\mathcal{C}$.

Remark 2.23. Note that by Remark 2.9 if we consider a branch $\mathcal{C}$ of the equisingularity class (n, m) then, up to reparameterization, it is possible to assume that the parameterization is given by $\varphi(t) = \left(t^n, t^m + \sum_{j>m} a_j t^j\right)$. Moreover, if we consider a cuspidal dicritical foliation $\mathcal{G} \in \mathfrak{F}^C_\pi$ with a generator $\omega_{\mathcal{G}}$ as in Lemma 2.12 then, after straightforward computations we have that $\mathrm{ord}_t \varphi^*\omega_{\mathcal{G}} + 1 - n > m$ or $\mathrm{ord}_t \varphi^*\omega_{\mathcal{G}} + 1 - n = ln$, with $l \geq \lfloor \frac{m}{n} \rfloor + 1$.

The following lemma characterizes the maximality of the tangency order of a given branch $\mathcal{C}$ (in the equisingularity class (n, m)) with the cuspidal dicritical foliations, in terms of the semigroup of the branch $\mathcal{C}$.

Lemma 2.24. *Let $\mathcal{C}$ be a plane branch in the equisingularity class (n,m) such that $\mathcal{C}$ is not (n,m)-quasihomogeneous. If there exists an (n,m)-cuspidal dicritical foliation $\mathcal{G} \in \mathcal{F}_\pi^C$ such that $\mathrm{Tan}_0(\mathcal{C},\mathcal{G})$ is maximal, then*

$$\mathrm{Tan}_0(\mathcal{C},\mathcal{G}) + 1 - n \notin S \quad \text{and} \quad \mathrm{Tan}_0(\mathcal{C},\mathcal{G}) + 1 - m \notin m\mathbb{Z}_{>0}. \tag{13}$$

Proof. Let us suppose that there exists a cuspidal dicritical foliation $\mathcal{G} \in \mathfrak{F}_\pi^C$ such that the order of tangency of $\mathcal{C}$ with $\mathcal{G}$ is finite and maximal. Let $\omega_\mathcal{G} = (my + h_1(x,y))\, dx - (nx + ky + h_2(x,y))\, dy$ be a generator of the foliation $\mathcal{G}$, where $\mathrm{ord}_0(h_i(x,y)) \geq 2$ for $i = 1,2$. We will prove that assuming $\mathrm{Tan}_0(\mathcal{C},\mathcal{G}) + 1 - n \in S$, leads to a contradiction. Namely, under such assumption there exist $\alpha, \beta \in \mathbb{Z}_{\geq 0}$ such that

$$\mathrm{Tan}_0(\mathcal{C},\mathcal{G}) + 1 - n = \alpha n + \beta m; \tag{15}$$

that is, $\varphi^*\omega_\mathcal{G} = \left(at^{(\alpha+1)n+\beta m-1} + \cdots\right) dt$, where the multiple dots denote higher order terms than $(\alpha+1)n + \beta m - 1$. Note by Remark 2.23 that if $\beta = 0$ then $\alpha \geq 2$ and if $\alpha = 0$ then $\beta \geq 2$. Let us consider the change of coordinates $H(x,y) = (x, y + cx^\alpha y^\beta) = (\tilde{x}, \tilde{y})$. We claim that there exists $c \in \mathbb{C}^*$ such that $\mathrm{Tan}_0(\mathcal{C},\tilde{\mathcal{G}}) > \mathrm{Tan}_0(\mathcal{C},\mathcal{G})$, $\tilde{\mathcal{G}}$ is the foliation given by $\{H^*\tilde{\omega}_\mathcal{G} = 0\}$ where $\tilde{\omega}_\mathcal{G}$ is the expresion of $\omega_\mathcal{G}$ in the coordinates $(\tilde{x}, \tilde{y})$,

$$\tilde{\omega}_\mathcal{G} = (m\tilde{y} + h_1(\tilde{x},\tilde{y}))\, d\tilde{x} - (n\tilde{x} + k\tilde{y} + h_2(\tilde{x},\tilde{y}))\, d\tilde{y}.$$

Namely,

$$\begin{aligned}
H^*\tilde{\omega}_\mathcal{G} &= \left(m(y + cx^\alpha y^\beta) + h_1(x, y + cx^\alpha y^\beta)\right) dx \\
&\quad - \left(\left(nx + k(y + cx^\alpha y^\beta)\right) d(y + cx^\alpha y^\beta) + \left(h_2(x, y + cx^\alpha y^\beta)\right) d(y + cx^\alpha y^\beta)\right) \\
&= \omega_\mathcal{G} + mcx^\alpha y^\beta dx + (cx^\alpha y^\beta \frac{\partial h_1}{\partial y} + \cdots)dx + \left(cx^\alpha y^\beta \frac{\partial h_2}{\partial y} + \cdots\right) dy \\
&\quad - \left(nx + ky + kcx^\alpha y^\beta + h_2(x, y + cx^\alpha y^\beta)\right) \left(c\alpha x^{\alpha-1} y^\beta dx + c\beta x^\alpha y^{\beta-1} dy\right) \\
&= \omega_\mathcal{G} + (mc - nc\alpha)\, x^\alpha y^\beta dx - nc\beta x^{\alpha+1} y^{\beta-1} dy + \left(cx^\alpha y^\beta \frac{\partial h_2}{\partial y} + \cdots\right) dy \\
&\quad + (cx^\alpha y^\beta \frac{\partial h_1}{\partial y} + \cdots)dx - \left(ky + kcx^\alpha y^\beta + h_2(x, y + cx^\alpha y^\beta)\right) c\beta x^\alpha y^{\beta-1} dy \\
&\quad - \left(k\left(y + cx^\alpha y^\beta\right) + h_2(x, y + cx^\alpha y^\beta)\right) \left(c\alpha x^{\alpha-1} y^\beta\right) dx.
\end{aligned}$$

Note that $H^*\tilde{\omega}_\mathcal{G}$ is given in the coordinates (x,y) which are adapted to π and we are considering the parameterization of $\mathcal{C}$ in these coordinates. Since the parameterization is

$$\varphi(t) = \left(t^n, t^m + \sum_{j>m} a_j t^j\right)$$

then, after straightforward computations, we have that

$$\varphi^* H^* \tilde{\omega}_\mathcal{G} = \left(at^{(\alpha+1)n+\beta m-1} + (mnc - cn^2\alpha - nm\beta c)t^{(\alpha+1)n+\beta m-1} \cdots\right) dt. \tag{16}$$

Recall that our goal is to show that there exists a constant $c \in \mathbb{C}^*$ such that $\mathrm{Tan}_0(\mathcal{C},\tilde{\mathcal{G}}) > \mathrm{Tan}_0(\mathcal{C},\mathcal{G})$. So, if we want to eliminate the monomial of degree $(\alpha+1)n + \beta m - 1$ we need to find c such that $a - cn^2\alpha - nm\beta c + mnc = 0$; equivalently,

$$\frac{a}{n} = c(n\alpha + m(\beta - 1)). \tag{17}$$

Equation (17) has one solution if and only if $n\alpha + m(\beta-1) \neq 0$. Since in Equation (15) we have several possibilities for α and β we analyze, thus, the following three possible cases:

(i) For $\beta > 1$, $n\alpha + m(\beta - 1) \neq 0$ and equation (17) may be solved.
(ii) For $\beta = 1$, if we suppose that $n\alpha = 0$, then this implies that $\alpha = 0$ and hence

$$\mathrm{Tan}_0(\mathcal{C}, \mathcal{G}) + 1 - n = m.$$

By Remark 2.23 we have that $\mathrm{Tan}_0(\mathcal{C}, \mathcal{G}) + 1 - n > m$ or $\mathrm{Tan}_0(\mathcal{C}, \mathcal{G}) + 1 - n = ln$ with $l \geq \lfloor \frac{m}{n} \rfloor + 1$. This implies that this case does not happen.
(iii) For $\beta = 0$, if we suppose that $n\alpha + m(\beta - 1) = 0$, then this implies that $n\alpha = m$ but this is impossible because $\mathrm{g.c.d}(n, m) = 1$; so, if $\beta = 0$ we have that $n\alpha - m \neq 0$ and again the equation (17) may be solved.

Therefore, equation (17) has always a solution $c \neq 0$. This finally implies that

$$\mathrm{Tan}_0(\mathcal{C}, \tilde{\mathcal{G}}) > \mathrm{Tan}_0(\mathcal{C}, \mathcal{G})$$

but this is a contradiction since by hypothesis $\mathrm{Tan}_0(\mathcal{C}, \mathcal{G})$ is maximal.

In a similar way as above, if we suppose that $\mathrm{Tan}_0(\mathcal{C}, \mathcal{G}) + 1 - m \in m\mathbb{Z}_{>0}$, then there exists $\tilde{\beta} \in \mathbb{Z}_{\geq 0}$ such that

$$\mathrm{Tan}_0(\mathcal{C}, \mathcal{G}) + 1 - m = \tilde{\beta} m.$$

Hence, $\mathrm{Tan}_0(\mathcal{C}, \mathcal{G}) = (\tilde{\beta} + 1)m - 1$. We consider the following change of coordinates

$$\hat{H}(x, y) = (x + \hat{c} y^{\tilde{\beta}}, y) = (\hat{x}, \hat{y}).$$

We claim that there exists $\hat{c} \in \mathbb{C}^*$ such that $\mathrm{Tan}_0(\mathcal{C}, \hat{\mathcal{G}}) > \mathrm{Tan}_0(\mathcal{C}, \mathcal{G})$ where $\hat{\mathcal{G}}$ is the foliation generated by $\left\{\hat{H}^* \hat{\omega}_{\mathcal{G}} = 0\right\}$ and $\hat{\omega}_{\mathcal{G}}$ is the expresion of $\omega_{\mathcal{G}}$ in the coordinates $(\hat{x}, \hat{y})$. Following the same reasoning and analogous computations as above we get that it is always possible to find $\hat{c} \neq 0$ such that $\mathrm{Tan}_0(\mathcal{C}, \hat{\mathcal{G}}) > \mathrm{Tan}_0(\mathcal{C}, \mathcal{G})$, but this again leds to a contradiction. The proof of the lemma is finished.

The following lemma shows the invariance of the tangency order under biholomorphisms. This invariance is a well known property.

Lemma 2.25. *Let $\mathcal{C}, \mathcal{D}$ be plane branches and let φ and ψ be their respective parameterizations. Let $\mathcal{F}$ be a foliation locally generated by the equation $\omega_{\mathcal{F}} = 0$. Let us suppose that $\mathcal{C}$ and $\mathcal{D}$ are analytically equivalent and consider the foliation $\mathcal{G}$ generated by the equation $\left\{(H^{-1})^* \omega_{\mathcal{F}} = 0\right\}$, where H is a biholomorphism of $(\mathbb{C}^2, 0)$ such that $H(\mathcal{C}) = \mathcal{D}$. Then*

$$\mathrm{Tan}_0(\mathcal{C}, \mathcal{F}) = \mathrm{Tan}_0(\mathcal{D}, \mathcal{G}) \tag{18}$$

With these lemmas we proceed to prove the theorem.

Proof of Theorem 2.22. First note that Corollary 2.21 implies $\mathrm{Tan}_0(\mathcal{C}, \mathcal{G}) < \infty$ for all cuspidal dicritical foliation $\mathcal{G} \in \mathfrak{F}^C_\pi$. The implication i) out of ii), is given by Lemma 2.24.

Now we prove that the assumption of the statement ii) implies iii). Let $\mathcal{G} \in \mathfrak{F}^C$ be a cuspidal dicritical foliation such that $\mathrm{Tan}_0(\mathcal{C}, \mathcal{G}) + 1 - n \notin S$ and $\mathrm{Tan}_0(\mathcal{C}, \mathcal{G}) + 1 - m \notin m\mathbb{Z}_{>0}$. We know that there exist $H : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ such that $\omega_{\mathcal{G}} = H^* \omega_0$. We denote by $\mathcal{G}_0$ the foliation locally generated by $\omega_0 = nxdy - mydx = 0$ and let $\psi_0 = H \circ \varphi \circ \tau^{-1}$, where τ is the automorphism introduced in the proof of Proposition 2.20 and $\mathcal{C}_0$ is the curve whose parameterization is given by ψ_0. Hence, by Lemma 2.25 we have that

$$\mathrm{Tan}_0(\mathcal{C}, \mathcal{G}) + 1 - n = \mathrm{Tan}_0(\mathcal{C}_0, \mathcal{G}_0) + 1 - n.$$

Therefore, $\mathrm{Tan}_0(\mathcal{C}_0, \mathcal{G}_0) + 1 - n \notin S$ and

$$\mathrm{Tan}_0(\mathcal{C}_0, \mathcal{G}_0) + 1 - n + n - m = \mathrm{Tan}_0(\mathcal{C}_0, \mathcal{G}_0) + 1 - m \notin m\mathbb{Z}_{>0}.$$

Note that if we consider a parameterization of the branch of the form

$$\varphi(t) = \left(t^n, t^m + \sum_{j>m} a_j t^j\right),$$

then if φ is $\mathcal{A}$-equivalent to ψ under a suitable change of coordinates H, then, up to reparameterization, $\psi(t) = \left(t^n, t^m + \sum_{j>m} \tilde{a}_j t^j\right)$. Hence up to reparameterization we have that $\psi_0(t) = \left(t^n, t^m + \sum_{j>m} \tilde{a}_j t^j\right)$. This implies that $\mathrm{ord}_\tau \psi_0^*(\omega_0) + 1 - n$ satisfies the conditions given in (11); it is, thus, the Zariski invariant.

Now we prove that the assumption of the statement iii) implies i). Let $\mathcal{G}$ be a cuspidal dicritical foliation, such that $\mathrm{Tan}_0(\mathcal{C}, \mathcal{G}) - n + 1$ is the Zariski invariant of $\mathcal{C}$, where $\mathcal{G}$ is given by $\{\omega_{\mathcal{G}} = 0\}$. Let $\tilde{\mathcal{G}}$ be another cuspidal dicritical foliation, we will prove that $\mathrm{Tan}_0(\mathcal{C}, \tilde{\mathcal{G}}) \leq \mathrm{Tan}_0(\mathcal{C}, \mathcal{G})$. Let $\psi_0 = H \circ \varphi \circ \tau^{-1}$ be a parameterization $\mathcal{A}$-equivalent to φ, with $H : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$, such that $\omega_{\mathcal{G}} = H^* \omega_0$ and τ as in the proof of Proposition 2.20. We denote by $\mathcal{C}_0$ the curve whose parameterization is given by ψ_0. Then, we have that $\mathrm{ord}_t \psi_0^* \omega_0 = \mathrm{ord}_t \varphi^* \omega_{\mathcal{G}}$. Thus, $\mathrm{ord}_t \psi_0^* \omega_0 - n + 1$ is the Zariski invariant of $\mathcal{C}$. Moreover, since the cuspidal dicritical foliation $\tilde{\mathcal{G}}$ is analytically equivalent to $\mathcal{G}_0$, there exists $\tilde{H} : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ biholomorphism,

$$\tilde{H}(x, y) = (\tilde{H}_1(x, y), \tilde{H}_2(x, y)),$$

such that $\omega_{\tilde{\mathcal{G}}} = \tilde{H}^* \omega_0$. Following the same ideas given in Proposition 2.20, let $\tilde{\tau} : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ be the automorphism such that $\tilde{\tau}^n = \tilde{H}_1 \circ \varphi(t)$ and let $\tilde{\psi}_0 := \tilde{H} \circ \varphi \circ \tilde{\tau}^{-1}$. Note that $\tilde{\psi}_0$ is a parameterization $\mathcal{A}$-equivalent to φ and this implies that $\tilde{\psi}_0$ is $\mathcal{A}$-equivalent to ψ_0.

$$\begin{tikzcd}
(\mathbb{C}^2,0) \arrow[r, "H"] & (\mathbb{C}^2,0) & (\mathbb{C}^2,0) \arrow[l, "\tilde{H}"'] \\
(\mathbb{C},0) \arrow[u, "\varphi"] \arrow[r, "\tau"'] & (\mathbb{C},0) \arrow[u, shift right, "\psi_0"] \arrow[u, shift left, "\tilde{\psi}_0"'] & (\mathbb{C},0) \arrow[u, "\varphi"'] \arrow[l, "\tilde{\tau}"]
\end{tikzcd}$$

Let us compute $\mathrm{ord}_t \varphi^* \omega_{\tilde{\mathcal{G}}}$,

$$\mathrm{ord}_t \varphi^* \omega_{\tilde{\mathcal{G}}} = \mathrm{ord}_t \tilde{\tau}^* \tilde{\psi}_0^* (\tilde{H}^{-1})^* \omega_{\tilde{\mathcal{G}}} = \mathrm{ord}_t \tilde{\tau}^* \tilde{\psi}_0^* \omega_0 = \mathrm{ord}_t \tilde{\psi}_0^* \omega_0$$

Since $\mathrm{ord}_t \psi^* \omega_0 - n + 1$ is the Zariski invariant of $\mathcal{C}$, then $\mathrm{ord}_t \tilde{\psi}_0^* \omega_0 - n + 1 \leq \mathrm{ord}_t \psi_0^*(\omega_0) - n + 1$ or, equivalently, $\mathrm{ord}_t \tilde{\psi}_0^* \omega_0 \leq \mathrm{ord}_t \psi_0^* \omega_0$. Hence $\mathrm{Tan}_0(\mathcal{C}, \tilde{\mathcal{G}}) \leq \mathrm{Tan}_0(\mathcal{C}, \mathcal{G})$. This proves Theorem 2.22.

Remark 2.26. Let $\mathcal{C}$ be a plane branch of the equisingularity class (n, m) and let φ be its parameterization. We can recover the Zariski invariant as follows. We consider the cuspidal dicritical foliation $\mathcal{G}_0$ defined by $\omega_0 = mydx - nxdy$. Let us compute $\mathrm{Tan}_0(\mathcal{C}, \mathcal{G}_0)$. If

$$\mathrm{Tan}_0(\mathcal{C}, \mathcal{G}_0) - n + 1 \notin S \quad \text{and} \quad \mathrm{Tan}_0(\mathcal{C}, \mathcal{G}_0) - m + 1 \notin m\mathbb{Z}_{>0}$$

then by Theorem 2.22 we are finished. If $\mathrm{Tan}_0(\mathcal{C}, \mathcal{G}_0) - n + 1 \in S$ or $\mathrm{Tan}_0(\mathcal{C}, \mathcal{G}_0) - m + 1 \in m\mathbb{Z}_{>0}$ then we apply one of the change of coordinates given in the proof of the Lemma 2.24 to find a new cuspidal dicritical foliation $\tilde{\mathcal{G}}$ with $\mathrm{Tan}_0(\mathcal{C}, \tilde{\mathcal{G}}) > \mathrm{Tan}_0(\mathcal{C}, \mathcal{G}_0)$. If the following two conditions $\mathrm{Tan}_0(\mathcal{C}, \tilde{\mathcal{G}}) - n + 1 \notin S$ and $\mathrm{Tan}_0(\mathcal{C}, \tilde{\mathcal{G}}) - m + 1 \in m \notin \mathbb{Z}_{>0}$ are satisfied, we stop the process. If on the contrary $\mathrm{Tan}_0(\mathcal{C}, \tilde{\mathcal{G}}) - n + 1 \in S$ or $\mathrm{Tan}_0(\mathcal{C}, \tilde{\mathcal{G}}) - m + 1 \in m\mathbb{Z}_{>0}$, then we apply again the change of coordinates given in Lemma 2.24. If the process does not stop, we will eventually find an (n,m)-cuspidal dicritical foliation $\hat{\mathcal{G}}$ such that $\mathrm{Tan}_0(\mathcal{C},\hat{\mathcal{G}})=\infty$.

3. Divisorial valuations and the tangency order

The main goal of this work is to show that the Zariski invariant of the non-isolated separatrices of each foliation $\mathcal{F}$ in $\mathfrak{F}_\pi$ that has polar transversality with $\mathfrak{F}_\pi^C$, coincides. In this section we show how to compute the tangency order between a non-isolated separatrix $\mathcal{C}_\mathcal{F}$ of a foliation $\mathcal{F}\in\mathfrak{F}_\pi$ and a foliation $\mathcal{G}\in\mathfrak{F}_\pi^C$ in terms of the divisorial valuation associated to the dicritical component.

Remark 3.1. Throughout this section the following notation will be used. As in previous sections π is the sequence of blowing-up morphisms determined by (n,m). Moreover, σ will denote another finite sequence of blowing-up morphisms; namely, if we decompose σ as $\sigma=\pi_1\circ\rho$, where

$$\pi_1:(M_1,E_1)\to(\mathbb{C}^2,0)$$

is the blowing-up morphism of the origin and $\rho:(M,D)\to(M_1,p_1)$ the rest of the sequence. Let $\mathcal{C}$ be a germ of plane curve of $(\mathbb{C}^2,0)$ and let $\mathcal{S}$ be a germ of plane curve of (M_1,p_1). Hence,

(1) $\tilde{\mathcal{C}}$ represents the strict transform of $\mathcal{C}$ by π.
(2) $\mathcal{C}^{(1)}$ represents the strict transform of $\mathcal{C}$ by π_1.
(3) $\hat{\mathcal{C}}$ represents the strict transform of $\mathcal{C}$ by σ.
(4) $\breve{\mathcal{S}}$ represents the strict transform of $\mathcal{S}$ by ρ.

Definition 3.2. Let $\sigma:(M,D)\to(\mathbb{C}^2,0)$ be a finite sequence of blowing-up morphisms. We denote the exceptional divisor by $D=\sigma^{-1}(0)$. Let E be an irreducible component of D and let $\{g=0\}$ be a local reduced equation of E at a point $p\in E$. The divisorial valuation with respect to E, that we denote by ν_E^σ, is the application given by

$$\begin{array}{cccc}\nu_E^\sigma & : & \mathbb{C}\{x,y\} & \to & \mathbb{Z}\\ & & h & \longmapsto & \nu_E^\sigma(h),\end{array}$$

where $\nu_E^\sigma(h)=\max\left\{k\in\mathbb{N}\;:\;g^k|\sigma^*(h)_p\right\}$ and $\sigma^*(h)_p$ is the pull-back of h by σ at the point $p\in E$.

Let

$$\Omega^1_{(\mathbb{C}^2,0)}=\{\omega=a(x,y)dx+b(x,y)dy\;:\;a(x,y),b(x,y)\in\mathbb{C}\{x,y\}\}$$

and

$$\Omega^2_{(\mathbb{C}^2,0)}=\{\eta=a(x,y)dx\wedge dy\;:\;a(x,y)\in\mathbb{C}\{x,y\}\}$$

the modules of holomorphic 1-forms and holomorphic 2-forms respectively. We define one application from $\Omega^1_{(\mathbb{C}^2,0)}$ in the integers numbers $\mathbb{Z}$ through π, in analogous way of the divisorial valuation for series. We call this application *infinitesimal divisorial valuation* for holomorphic 1-forms.

$$\begin{array}{cccc}\nu_{1,E}^\sigma & : & \Omega^1_{(\mathbb{C}^2,0)} & \to & \mathbb{Z}\\ & & \omega & \longmapsto & \nu_{1,E}^\sigma(\omega),\end{array}$$

where $\nu_{1,E}^\sigma(\omega)=\max\left\{k\in\mathbb{N}\;:\;g^k|a_1(\hat{x},\hat{y})\text{ and }g^k|b_1(\hat{x},\hat{y})\right\}$, $a_1(\hat{x},\hat{y})$, $b_1(\hat{x},\hat{y})$ are such that

$$\sigma^*\omega=a_1(\hat{x},\hat{y})d\hat{x}+b_1(\hat{x},\hat{y})d\hat{y},$$

$\hat{x}$, $\hat{y}$ are coordinates at one point $p\in E$ and $\{g=0\}$ is a reduced equation of E at the point p.

For the holomorphic 2-forms we define

$$\begin{array}{cccc}\nu_{2,E}^\sigma & : & \Omega^2_{(\mathbb{C}^2,0)} & \to & \mathbb{Z}\\ & & \eta & \longmapsto & \nu_{2,E}^\sigma(\eta),\end{array}$$

where $\nu^{\sigma}_{2,E}(\eta) = \max\left\{k \in \mathbb{N} \,:\, g^k | a_2(\hat{x},\hat{y})\right\}$ with $\sigma^*\eta = a_2(\hat{x},\hat{y}) d\hat{x} \wedge d\hat{y}$ where $\hat{x}$ and $\hat{y}$ are coordinates at one point $p \in E$ and $\{g = 0\}$ is a reduced equation of E at the point p. We call this application *infinitesimal divisorial valuation* for holomorphic 2-forms.

Throughout the present section we denote by ν^{σ}_E the infinitesimal divisorial valuation for series, 1-forms and 2-forms.

Remark 3.3. Let $h \in \mathbb{C}\{x,y\}$, $\omega \in \Omega^1_{(\mathbb{C}^2,0)}$ and $\eta \in \Omega^2_{(\mathbb{C}^2,0)}$, then we have that

$$\nu^{\sigma}_E(h\omega) = \nu^{\sigma}_E(h) + \nu^{\sigma}_E(\omega) \quad \text{and} \quad \nu^{\sigma}_E(h\eta) = \nu^{\sigma}_E(h) + \nu^{\sigma}_E(\eta).$$

Example 3.4. Let h be an analytic power series, $h \in \mathbb{C}\{x,y\}$, such that

$$h(x,y) = h_n(x,y) + h_{n+1}(x,y) + h_{n+2}(x,y) + \cdots, \tag{19}$$

where $h_i \in \mathbb{C}[x,y]$ are homogeneous polynomials of degree i and h_n is the first not identically vanishing polynomial. Let $\pi_1 : (M_1, E_1) \to (\mathbb{C}^2, 0)$ be the blowing-up morphism of the origin in $\mathbb{C}^2$, $E_1 = \pi_1^{-1}(0)$ and let $\pi_1(x_1,y_1) = (x_1, x_1y_1)$ be a local chart. In this chart we have

$$\begin{aligned}\pi^*(h) &= h \circ \pi(x_1,y_1) = h_n(x_1,x_1y_1) + h_{n+1}(x_1,x_1y_1) + \cdots \\ &= x_1^n \left(h_n(1,y_1) + x_1 h_{n+1}(1, y_1 + \cdots\right).\end{aligned}$$

Then, we have that $\nu^{\pi_1}_{E_1}(h) = n$, and it coincides with the multiplicity at the origin of h.

Remark 3.5. If $\mathcal{L}$ is a line through the origin of $\mathbb{C}^2$ such that the strict transform of $\mathcal{L}$ and the strict transform of the plane curve $\mathcal{H}$ given by the zero locus of h, $\mathcal{H} := \{h = 0\}$, by π_1 have no intersection, then the intersection multiplicity of $\mathcal{H}$ and $\mathcal{L}$ equals to the divisorial valuation of h with E_1, that is, $\iota_0(\mathcal{H}, \mathcal{L}) = \nu^{\pi_1}_{E_1}(h) = n$.

The previous remark can be generalized to a more general setting. Note that the previous remark shows that the divisorial valuation associated to the blowing-up morphism of the origin is like an intersection multiplicity at the origin for a suitable curve $\mathcal{L}$. We will show that this always happens for divisorial valuations. We denote by $\mathcal{E}^{\sigma}_E$ the set of equisingular plane branches $\mathcal{B}$ of $(\mathbb{C}^2, 0)$ such that the strict transform of $\mathcal{B}$ by σ is smooth and intersects transversally the component E at a point p that is not a corner point (this is the definition of *curvette*, see [33] page 53). We can decompose $\sigma = \pi_1 \circ \rho$, where $\pi_1 : (M_1, E_1) \to (\mathbb{C}^2, 0)$ is the blowing-up morphism of the origin and $\rho : (M, D) \to (M_1, p_1)$ is the rest of the sequence. We denote by $\mathcal{E}^{\rho}_E$ the set of germs of equisingular branches $\mathcal{S}$ of (M_1, p_1) such that the strict transform by ρ is smooth and intersects transversally the component E in a point p that is not a corner point. Note that the elements of $\mathcal{E}^{\rho}_E$ are the strict transforms by π_1 of the elements in $\mathcal{E}^{\sigma}_E$. In fact, let $\mathcal{B} \in \mathcal{E}^{\sigma}_E$, by definition, the strict transform of $\mathcal{B}$ by σ is smooth and it intersects transversaly the component E, then the strict transform of $\mathcal{S}$ by π_1 belongs to $\mathcal{E}^{\sigma}_E$. Moreover, the projection of every element in $\mathcal{E}^{\rho}_E$ belongs to $\mathcal{E}^{\sigma}_E$. The following theorem shows that the divisorial valuation of an element $h \in \mathbb{C}\{x,y\}$ can be interpreted as the intersection multiplicity between the locus zero of h and a suitable element of $\mathcal{E}^{\sigma}_E$.

Theorem 3.6. *Let $\sigma : (M, D) \to (\mathbb{C}^2, 0)$ be a finite sequence of blowing-up morphisms and let E be an irreducible component of $D = \sigma^{-1}(0)$. Let $h \in \mathbb{C}\{x,y\}$ and let $\mathcal{H} := \{h = 0\}$ be the curve given by the zero locus of h. Then*

$$\nu_E(h) = \min\left\{\iota_0(\mathcal{H}, \mathcal{B}) \,:\, \mathcal{B} \in \mathcal{E}^{\sigma}_E\right\}, \tag{20}$$

where $\iota_0(\mathcal{H}, \mathcal{B})$ denotes the intersection multiplicity at the origin between the curves $\mathcal{H}, \mathcal{B}$ (see section 2).

The proof of Theorem 3.6 can be found in [32] (page 131, Theorem 7.2).

The following lemma will be used in the proof of Theorem 3.8.

Lemma 3.7. *Let $\sigma : (M, D) \to (\mathbb{C}^2, 0)$ be a finite sequence of blowing-up morphisms, where $D = \sigma^{-1}(0)$ is the exceptional divisor and let E be an irreducible component of D. We write $\sigma = \pi_1 \circ \rho$ where $\pi_1 : (M_1, E_1) \to (\mathbb{C}^2, 0)$ is the blowing-up morphism of the origin and $\rho : (M, D) \to (M_1, p_1)$ is the rest of the sequence with p_1 the origin of the first coordinate chart, $\pi_1(x_1, y_1) = (x_1, x_1 y_1)$. We have*

$$\nu_E^\sigma(x) = \nu_E^\rho(x_1) = \chi(E),$$

where $\chi(E)$ is the multiplicity at the origin of a curve in $\mathcal{E}_E^\sigma$, that is, if $\mathcal{B} \in \mathcal{E}_E^\sigma$, $\mathrm{m}_0(\mathcal{B}) = \chi(E)$.

Proof. Let us compute $\nu_E^\sigma(x)$. By Theorem 3.6 we have that

$$\nu_E^\sigma(x) = \min\left\{\iota_0(\mathcal{L}_\infty, \mathcal{B}) \ : \ \mathcal{B} \in \mathcal{E}_E^\sigma\right\},$$

where $\mathcal{L}_\infty = \{x = 0\}$. By Noether's formula (see [14]) we have that

$$\iota_0(\mathcal{L}_\infty, \mathcal{B}) = \mathrm{m}_0(\mathcal{L}_\infty)\mathrm{m}_0(\mathcal{B}) + \iota_p(\mathcal{L}_\infty^{(1)}, \mathcal{B}^{(1)}); \tag{21}$$

since $\mathcal{L}_\infty^{(1)} \cap \mathcal{B}^{(1)} = \emptyset$ then we have from (21) that

$$\iota_0(\mathcal{L}_\infty, \mathcal{B}) = \mathrm{m}_0(\mathcal{L}_\infty)\mathrm{m}_0(\mathcal{B}) = \mathrm{m}_0(\mathcal{B}). \tag{22}$$

Therefore $\nu_E^\sigma(x) = \mathrm{m}_0(\mathcal{B})$. Since $\nu_E^\sigma(x) = \nu_E^\rho(x_1)$ we have proved Lemma 3.7.

3.1. **Tangency order of non-isolated separatrices of Pseudo-Cuspidal Dicritical Foliations and the Cuspidal Dicritical Foliations.** As we mentioned in the introduction of this work, our goal is to show under which conditions a family of pseudo-cuspidal dicritical foliations has the property that given a foliation in this family, all the non-isolated separatrices of this foliation have the same Zariski invariant. For this purpose we begin by looking at an auxiliary foliation (among the cuspidal ones) and the locus of tangencies between such a foliation and a foliation in the family. This locus of tangencies is known as the *Jacobian or polar curve.* The following theorem stresses that the order of tangency of a non-isolated separatrix of the fixed pseudo-cuspidal dicritical foliation and one cuspidal dicritical foliation is given in terms of its corresponding *Jacobian* curve.

Theorem 3.8. *Let $\mathcal{F}$ be a foliation in $\mathfrak{F}_\pi$ induced by the equation $\{\omega_\mathcal{F} = 0\}$ and let $\mathcal{G}$ be a foliation in $\mathfrak{F}_\pi^C$ induced by $\{\omega_\mathcal{G} = 0\}$. Let $\mathcal{C}_\mathcal{F}$ and $\mathcal{C}_\mathcal{G}$ be the non-isolated separatrices of $\mathcal{F}$ and $\mathcal{G}$, respectively, and let $\tilde{\mathcal{C}}_\mathcal{F}$ and $\tilde{\mathcal{C}}_\mathcal{G}$ be their corresponding strict transforms by π. If p is not a corner point, and p is the intersection point of $\tilde{\mathcal{C}}_\mathcal{F}$ and $\tilde{\mathcal{C}}_\mathcal{G}$ with the dicritical component E_{dic}, then*

$$\mathrm{Tan}_0(\mathcal{C}_\mathcal{F}, \mathcal{G}) = \nu_{E_{dic}}^\pi(\omega_\mathcal{F} \wedge \omega_\mathcal{G}) - \nu_{E_{dic}}^\pi(\omega_\mathcal{F}) + \iota_p(\tilde{\mathcal{C}}_\mathcal{G}, \tilde{\mathcal{J}}(\mathcal{F}, \mathcal{G})), \tag{23}$$

where $\tilde{\mathcal{J}}(\mathcal{F}, \mathcal{G})$ is the strict transform of the curve of tangencies (the Jacobian curve) of the foliations $\mathcal{F}$ and $\mathcal{G}$ by π, and ι_p is the corresponding multiplicity intersection at p.

Before proving the theorem we focus our attention on the polar curves of regular foliations.

Let $\mathcal{F}$ and $\mathcal{G}$ be holomorphic foliations of $(\mathbb{C}^2, 0)$. As we mentioned before, the *Jacobian* or *polar* curve between $\mathcal{F}$ and $\mathcal{G}$ is the locus of tangencies between $\mathcal{F}$ and $\mathcal{G}$. If $\mathcal{F}$ is given locally by the equation $\{\omega_\mathcal{F} = 0\}$ and $\mathcal{G}$ is generated by $\{\omega_\mathcal{G} = 0\}$ the tangency locus between $\mathcal{F}$ and $\mathcal{G}$ is the zero locus of the coefficient of the 2-form $\omega_\mathcal{F} \wedge \omega_\mathcal{G}$. We denote this Jacobian curve by $\mathcal{J}(\mathcal{F}, \mathcal{G})$. Since both foliations are regular, there exist unique invariant curves through the origin $\mathcal{C}_\mathcal{F}$ and $\mathcal{C}_\mathcal{G}$ for the foliations $\mathcal{F}$ and $\mathcal{G}$, respectively. Our aim is to describe the relation among the tangency order between the invariant curve $\mathcal{C}_\mathcal{F}$ and foliation $\mathcal{G}$ and the intersection multiplicity between the invariant curve $\mathcal{C}_\mathcal{G}$ and the Jacobian curve $\mathcal{J}(\mathcal{F}, \mathcal{G})$.

Given the curve $\mathcal{L} = \{x = 0\}$, we define the value

$$\nu_{\mathcal{L}}(\omega_{\mathcal{F}} \wedge \omega_{\mathcal{G}}) = \max\{k \in \mathbb{Z}_{\geq 0} \ : \ x^k | \omega_{\mathcal{F}} \wedge \omega_{\mathcal{F}}\}.$$

The following proposition will be useful in the proof of Theorem 3.8. Namely, the proof of that theorem will be achieved by induction, and the following proposition is the base step of the induction.

Proposition 3.9. *Let $\mathcal{F}$ and $\mathcal{G}$ be non-singular foliations of $(\mathbb{C}^2, 0)$, induced by $\{\omega_{\mathcal{F}} = 0\}$ and $\{\omega_{\mathcal{G}} = 0\}$ respectively, such that $\mathcal{F}$ and $\mathcal{G}$ are transverse to $\mathcal{L} = \{x = 0\}$. Let $\mathcal{C}_{\mathcal{F}}$ be the invariant curve of $\mathcal{F}$ through 0 and let $\mathcal{C}_{\mathcal{G}}$ be the invariant curve of $\mathcal{G}$ through 0. Then, for any parameterization φ of $\mathcal{C}_{\mathcal{F}}$, the following equality holds,*

$$\operatorname{ord}_t \varphi^* \omega_{\mathcal{G}} = \nu_{\mathcal{L}}(\omega_{\mathcal{F}} \wedge \omega_{\mathcal{G}}) + \iota_0(\mathcal{C}_{\mathcal{G}}, \bar{\mathcal{J}}(\mathcal{F}, \mathcal{G})), \tag{24}$$

where $\bar{\mathcal{J}}(\mathcal{F}, \mathcal{G}) = \{h_s = 0\}$, h_s satisfies $h = x^r h_s$ for $r = \nu_{\mathcal{L}}(\omega_{\mathcal{F}} \wedge \omega_{\mathcal{G}})$ and $\mathcal{J}(\mathcal{F}, \mathcal{G}) = \{h = 0\}$ is the Jacobian curve between $\mathcal{F}$ and $\mathcal{G}$.

Proof. Let $\mathcal{F}$ and $\mathcal{G}$ be non-singular foliations transversally intersecting $\mathcal{L}$. Since $\mathcal{F}$ and $\mathcal{G}$ are non-singular foliations, there exist $f, g \in \mathbb{C}\{x, y\}$ such that $\mathcal{F} = \{df = 0\}$ and $\mathcal{G} = \{dg = 0\}$. Up to a coordinate change of variables we may assume that $\mathcal{G} = \{dy = 0\}$ (this change of coordinates does not modify the transversality of $\mathcal{F}$ with $\mathcal{L}$). We write $f(x, y) = u(x, y)y + p(x)$ where $u(0, 0) \neq 0$. Let us compute $df \wedge dy$

$$df \wedge dy = \left(\left(\frac{\partial u}{\partial x} y + p'(x)\right) dx + \left(\frac{\partial u}{\partial y} y + u(x, y)\right) dy\right) \wedge dy \tag{25}$$

$$= (v(x, y)y + p'(x))\, dx \wedge dy = h(x, y) dx \wedge dy, \tag{26}$$

where $v(x, y) = \frac{\partial u(x,y)}{\partial x}$. We stress that the unique invariant curve of $\mathcal{F}$ through 0 is $\mathcal{C}_{\mathcal{F}} = \{f(x, y) = 0\}$, so a parameterization of $\mathcal{C}$ is given by $\varphi(t) = (t, ct^{\operatorname{ord}_x p(x)} + \cdots)$ with $c \in \mathbb{C}^*$. Therefore,

$$\varphi^* dy = c(\operatorname{ord}_x p(x)) t^{(\operatorname{ord}_x p(x) - 1)} + \cdots,$$

and $\operatorname{ord}_t \varphi^* dy = \operatorname{ord}_x p(x) - 1$. Let $r = \nu_{\mathcal{L}}(\omega_{\mathcal{F}} \wedge \omega_{\mathcal{G}})$ and $h_s(x, y) = \frac{v(x,y)}{x^r} y + \frac{p'(x)}{x^r}$. So we have that $\iota_0(\mathcal{C}_{\mathcal{G}}, \bar{\mathcal{J}}(\mathcal{F}, \mathcal{G})) = \operatorname{ord}_x p'(x) - r = \operatorname{ord}_x p(x) - 1 - r$. Thus,

$$\operatorname{ord}_t \varphi^* dy = \operatorname{ord}_x p(x) - 1 = r + \iota_0\left(\mathcal{C}_{\mathcal{G}}, \bar{\mathcal{J}}(\mathcal{F}, \mathcal{G})\right). \tag{27}$$

Finally,

$$\operatorname{ord}_t \varphi^* dy = \nu_{\mathcal{L}}(\omega_{\mathcal{F}} \wedge \omega_{\mathcal{G}}) + \iota_0\left(\mathcal{C}_{\mathcal{G}}, \bar{\mathcal{J}}(\mathcal{F}, \mathcal{G})\right).$$

This proves the proposition.

Lemma 3.10. *Let $\omega_{\mathcal{F}}$ and $\omega_{\mathcal{G}}$ be 1-forms that generate the foliations $\mathcal{F}$ and $\mathcal{G}$ respectively and let $\sigma : (M, D) \to (\mathbb{C}^2, 0)$ be a finite sequence of blowing-up morphisms of size n. Let $\pi_1 : (M_1, E_1) \to (\mathbb{C}^2, 0)$ be the blowing-up morphism of the origin and $\sigma = \pi_1 \circ \rho$. Given E an irreducible component of $D = \pi^{-1}(0)$, we have that*

$$\nu_E^{\rho}(\omega_{\mathcal{F}}^{(1)}) = \nu_E^{\sigma}(\omega_{\mathcal{F}}) - \chi(E)(\mathrm{m}_0(\omega_{\mathcal{F}}) + \delta_{\mathcal{F}}),$$

where $\omega_{\mathcal{F}}^{(1)}$ is the strict transform of $\omega_{\mathcal{F}}$ by π_1 and

$$\delta_{\mathcal{F}} := \begin{cases} 1, & \text{if } \pi_1 \text{ is dicritical for } \mathcal{F}, \\ 0, & \text{if } \pi_1 \text{ is non-dicritical for } \mathcal{F}. \end{cases}$$

For the case of 2-forms

$$\nu_E^{\rho}(\omega_{\mathcal{F}}^{(1)} \wedge \omega_{\mathcal{G}}^{(1)}) = \nu_E^{\sigma}(\omega_{\mathcal{F}} \wedge \omega_{\mathcal{G}}) - \chi(E)(\mathrm{m}_0(\omega_{\mathcal{F}}) + \mathrm{m}_0(\omega_{\mathcal{G}}) + \delta),$$

where $\omega_{\mathcal{F}}^{(1)}, \omega_{\mathcal{G}}^{(1)}$ are the strict transforms of $\omega_{\mathcal{F}}, \omega_{\mathcal{G}}$ by π_1 respectively and

$$\delta := \begin{cases} 2, & \textit{if } \pi_1 \textit{ is dicritical for } \mathcal{F} \textit{ and } \mathcal{G}, \\ 1, & \textit{if } \pi_1 \textit{ is non-dicritical for } \mathcal{F} \textit{ but not for } \mathcal{G}, \\ 0, & \textit{if } \pi_1 \textit{ is non-dicritical for } \mathcal{F} \textit{ neither } \mathcal{G}, \end{cases}$$

in both cases $\chi(E)$ is the multiplicity at the origin of a curve in $\mathcal{E}_E^{\pi}$, that is, if $\mathcal{C} \in \mathcal{E}_E^{\pi}$, then $\mathrm{m}_0(\mathcal{C}) = \chi(E)$.

Proof. We consider the decomposition $\sigma = \pi_1 \circ \rho$, with π_1 the blowing-up morphism of the origin. We compute

$$\sigma^*\omega_{\mathcal{F}} = (\pi_1 \circ \rho)^*\omega_{\mathcal{F}} = \rho^*\pi_1^*(\omega_{\mathcal{F}}) = \rho^*(x_1^{\mathrm{m}_0(\omega_{\mathcal{F}})+\delta_{\mathcal{F}}}\omega_{\mathcal{F}}^{(1)}),$$

where

$$\delta_{\mathcal{F}} := \begin{cases} 1, & \text{if } \pi_1 \text{ is dicritical for } \mathcal{F}, \\ 0, & \text{if } \pi_1 \text{ is non-dicritical for } \mathcal{F}. \end{cases}$$

From this equation we have that,

$$\nu_E^{\sigma}(\omega_{\mathcal{F}}) = \nu_E^{\rho}(x_1^{\mathrm{m}_0(\omega_{\mathcal{F}})+\delta_{\mathcal{F}}}\omega_{\mathcal{F}}^{(1)}) = (\mathrm{m}_0(\omega_{\mathcal{F}}) + \delta_{\mathcal{F}})\nu_E^{\rho}(x_1) + \nu_E^{\rho}(\omega_{\mathcal{F}}^{(1)}).$$

Since $\nu_E^{\rho}(x_1) = \chi(E)$, as a consequence of Lemma 3.7, then,

$$\nu_E^{\sigma}(\omega_{\mathcal{F}}) = \nu_E^{\rho}(x_1^{\mathrm{m}_0(\omega_{\mathcal{F}})+\delta_{\mathcal{F}}}\omega_{\mathcal{F}}^{(1)}) = (\mathrm{m}_0(\omega_{\mathcal{F}}) + \delta_{\mathcal{F}})\chi(E) + \nu_E^{\rho}(\omega_{\mathcal{F}}^{(1)}).$$

For 2-forms we have,

$$\sigma^*(\omega_{\mathcal{F}} \wedge \omega_{\mathcal{G}}) = (\pi_1 \circ \rho)^*(\omega_{\mathcal{F}} \wedge \omega_{\mathcal{G}}) = \rho^*\pi_1^*(\omega_{\mathcal{F}} \wedge \omega_{\mathcal{G}}) = \rho^*(x_1^{\mathrm{m}_0(\omega_{\mathcal{F}})+\mathrm{m}_0(\omega_{\mathcal{G}})+\delta}\omega_{\mathcal{F}}^{(1)} \wedge \omega_{\mathcal{G}}^{(1)}),$$

this last equation implies that

$$\begin{aligned} \nu_E^{\sigma}(\omega_{\mathcal{F}} \wedge \omega_{\mathcal{G}}) &= \nu_E^{\rho}(x_1^{\mathrm{m}_0(\omega_{\mathcal{F}})+\mathrm{m}_0(\omega_{\mathcal{G}})+\delta}\omega_{\mathcal{F}}^{(1)} \wedge \omega_{\mathcal{G}}^{(1)}) \\ &= (\mathrm{m}_0(\omega_{\mathcal{F}}) + \mathrm{m}_0(\omega_{\mathcal{G}}) + \delta)\nu_E^{\sigma}(x_1) + \nu_E^{\rho}(\omega_{\mathcal{F}}^{(1)} \wedge \omega_{\mathcal{G}}^{(1)}). \end{aligned}$$

Since $\nu_E^{\rho}(x_1) = \chi(E)$, as a consequence of Lemma 3.7, it follows that,

$$\nu_E^{\sigma}(\omega_{\mathcal{F}} \wedge \omega_{\mathcal{G}}) = (\mathrm{m}_0(\omega_{\mathcal{F}}) + \mathrm{m}_0(\omega_{\mathcal{G}}) + \delta)\chi(E) + \nu_E^{\sigma}(\omega_{\mathcal{F}}^{(1)} \wedge \omega_{\mathcal{G}}^{(1)}).$$

Note that in the case of $\pi_1 : (M, E_1) \to (\mathbb{C}^2, 0)$ the blowing-up morphism of the origin we have $\nu_{E_1}^{\pi_1}(\omega_{\mathcal{F}} \wedge \omega_{\mathcal{G}}) = (\mathrm{m}_0(\omega_{\mathcal{F}}) + \mathrm{m}_0(\omega_{\mathcal{G}}) + \delta)\chi(E) + \nu_{E_1}^{\pi_1}(\omega_{\mathcal{F}}^{(1)} \wedge \omega_{\mathcal{G}}^{(1)})$

Now we proceed to prove Theorem 3.8. First we prove the statement in the case when π is determined by $(n, n+1)$ and after in the general case.

Let us consider $\pi : (M, D) \to (\mathbb{C}^2, 0)$ a blowing-up morphism described by the Euclidean algorithm of the pair $(n, n+1)$, with $D = \pi^{-1}(0)$. In what follows we prove Theorem 3.8 for this case.

Lemma 3.11. *Let $\sigma : (M, D) \to (\mathbb{C}^2, 0)$ be a finite sequence of blowing-up morphisms of size n, where every center of the morphism σ at the point $p_i \in M_i$, for $i \in \{0, 1, 2, ..., n-1\}$, is the origin of the second coordinate chart of $\sigma_{i+1} : (M_{i+1}, \sigma_{i+1}^{-1}(p_i)) \to (M_i, \sigma_i^{-1}(p_i))$. Let $\mathcal{F}$ be a singular holomorphic foliation such that the last irreducible component of $D = \sigma^{-1}(0)$ is the unique dicritical component for $\mathcal{F}$ and $\sigma^*\mathcal{F}$ has no tangencies or singularities in this component. We denote by E_{dic} this irreducible component. Let $\mathcal{G}$ be a singular holomorphic foliation which is generated by a vector field v such that the linear part of v has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = n$, and the component E_{dic} is dicritical for $\mathcal{G}$. Let $\mathcal{C}_{\mathcal{F}}$ and $\mathcal{C}_{\mathcal{G}}$ be non-isolated separatrices of $\mathcal{F}$*

and $\mathcal{G}$, respectively, and let $\hat{\mathcal{C}}_{\mathcal{F}}$ and $\hat{\mathcal{C}}_{\mathcal{G}}$ be their corresponding strict transforms by σ. If p is the intersection point of $\hat{\mathcal{C}}_{\mathcal{F}}$ and $\hat{\mathcal{C}}_{\mathcal{G}}$ with the dicritical component E_{dic}, then

$$\operatorname{Tan}_0(\mathcal{C}_{\mathcal{F}},\mathcal{G}) = \nu^{\sigma}_{E_{dic}}(\omega_{\mathcal{F}}\wedge\omega_{\mathcal{G}}) - \nu^{\sigma}_{E_{dic}}(\omega_{\mathcal{F}}) + \iota_p(\hat{\mathcal{C}}_{\mathcal{G}},\hat{\mathcal{J}}(\mathcal{F},\mathcal{G})), \tag{28}$$

where $\hat{\mathcal{J}}(\mathcal{F},\mathcal{G})$ is the strict transform of the Jacobian curve of the foliations $\mathcal{F}$ and $\mathcal{G}$ by σ, and ι_p denotes the intersection multiplicity at p.

Proof. The proof is by induction over n. For $n=1$, let

$$\pi_1 : (M_1, E_1) \to (\mathbb{C}^2, 0)$$

be the blowing-up morphism of the origin, since $\pi_1^*\mathcal{F}$ does not have tangencies or singularities on $E_1 = \pi_1^{-1}(0) = E_{dic}$ then $\mathcal{F}$ is generated by

$$\omega_{\mathcal{F}} = (y+\cdots)\,dx - (x+\cdots)\,dy,$$

where the multiple dots represent higher order terms. Without loss of generality we assume that $\mathcal{G}$ is generated by $\omega_{\mathcal{G}} = ydx - xdy$. Let φ be a parameterization of $\mathcal{C}_{\mathcal{F}}$, by Proposition 2.18 for the dicritical case, we have

$$\operatorname{ord}_t\varphi^*\omega_{\mathcal{G}} = \mathrm{m}_0(\mathcal{C}_{\mathcal{F}})(\mathrm{m}_0(\omega_{\mathcal{G}})+1) + \operatorname{ord}_t(\varphi^{(1)})^*\omega^{(1)}_{\mathcal{G}}. \tag{29}$$

By Proposition 3.9, we have that

$$\operatorname{ord}_t(\varphi^{(1)})^*\omega^{(1)}_{\mathcal{G}} = \nu^{\pi_1}_{E_{dic}}(\omega^{(1)}_{\mathcal{F}}\wedge\omega^{(1)}_{\mathcal{G}}) + \iota_p(\mathcal{C}^{(1)}_{\mathcal{G}},\bar{\mathcal{J}}(\mathcal{F}^{(1)},\mathcal{G}^{(1)})), \tag{30}$$

where $\mathcal{F}^{(1)}$ and $\mathcal{G}^{(1)}$ are the foliations generated by $\omega^{(1)}_{\mathcal{F}}$ and $\omega^{(1)}_{\mathcal{F}}$, $\bar{\mathcal{J}}(\mathcal{F}^{(1)},\mathcal{G}^{(1)})$ is the curve given by the locus zero of $\{h_s = 0\}$ where $\omega^{(1)}_{\mathcal{F}}\wedge\omega^{(1)}_{\mathcal{G}} = x_1^r h_s$, with $E_{dic} = \{x_1 = 0\}$, $r = \nu^{\pi_1}_{E_{dic}}(\omega^{(1)}_{\mathcal{F}}\wedge\omega^{(1)}_{\mathcal{G}})$ and $\mathcal{C}^{(1)}_{\mathcal{G}}$ the strict transform of $\mathcal{C}_{\mathcal{G}}$ by π_1, as in Proposition 3.9.

Substituting (30) in (29) we have,

$$\begin{aligned}
\operatorname{ord}_t\varphi^*\omega_{\mathcal{G}} &= \mathrm{m}_0(\mathcal{C}_{\mathcal{F}})(\mathrm{m}_0(\omega_{\mathcal{G}})+1) + \nu^{\pi_1}_{E_{dic}}(\omega^{(1)}_{\mathcal{F}}\wedge\omega^{(1)}_{\mathcal{G}}) + \iota_p(\mathcal{C}^{(1)}_{\mathcal{G}},\bar{\mathcal{J}}(\mathcal{F}^{(1)},\mathcal{G}^{(1)}))\\
&= \nu^{\pi_1}_{E_{dic}}(\omega_{\mathcal{F}}\wedge\omega_{\mathcal{G}}) - (\mathrm{m}_0(\omega_{\mathcal{F}})+1) + \iota_p(\mathcal{C}^{(1)}_{\mathcal{G}},\bar{\mathcal{J}}(\mathcal{F}^{(1)},\mathcal{G}^{(1)})\\
&= \nu^{\pi_1}_{E_{dic}}(\omega_{\mathcal{F}}\wedge\omega_{\mathcal{G}}) - \nu^{\pi_1}_{E_{dic}}(\omega_{\mathcal{F}}) + \iota_p(\mathcal{C}^{(1)}_{\mathcal{G}},\bar{\mathcal{J}}(\mathcal{F}^{(1)},\mathcal{G}^{(1)})),
\end{aligned}$$

where the second equality is a consequence of Lemma 3.10.

Since the strict transform of the Jacobian curve $\mathcal{J}(\mathcal{F},\mathcal{G})$ by π_1 at the point p coincides with the curve $\bar{\mathcal{J}}(\mathcal{F}^{(1)},\mathcal{G}^{(1)})$ at the same point p, then we have,

$$\operatorname{ord}_t\varphi^*\omega_{\mathcal{G}} = \nu^{\pi_1}_{E_{dic}}(\omega_{\mathcal{F}}\wedge\omega_{\mathcal{G}}) - \nu^{\pi_1}_{E_{dic}}(\omega_{\mathcal{F}}) + \iota_p(\mathcal{C}^{(1)}_{\mathcal{G}},\mathcal{J}^{(1)}(\mathcal{F},\mathcal{G})).$$

This proves the assertion for $n=1$.

Now we assume that the assertion is true for n and we prove it for $n+1$. The sequence σ can be decomposed as $\sigma = \pi_1\circ\rho$, where π_1 is the blowing-up morphism of the origin. Let $\mathcal{F}$ be the foliation generated by $\{\omega_{\mathcal{F}} = 0\}$ and let $\mathcal{G}$ be the one generated by $\{\omega_{\mathcal{G}} = 0\}$. By Lemma 2.18 we have,

$$\operatorname{ord}_t\varphi^*\omega = \mathrm{m}_0(\mathcal{C}_{\mathcal{F}})\mathrm{m}_0(\omega_{\mathcal{G}}) + \operatorname{ord}_t(\varphi^{(1)})^*\omega^{(1)}_{\mathcal{G}}. \tag{31}$$

By the induction hypothesis, we have that

$$\operatorname{ord}_t(\varphi^{(1)})^*\omega^{(1)}_{\mathcal{G}} = \nu^{\rho}_{E_{dic}}(\omega^{(1)}_{\mathcal{F}}\wedge\omega^{(1)}_{\mathcal{G}}) - \nu^{\rho}_{E_{dic}}(\omega^{(1)}_{\mathcal{F}}) + \iota_p(\breve{\mathcal{C}}^{(1)}_{\mathcal{G}},\breve{\mathcal{J}}(\mathcal{F}^{(1)},\mathcal{G}^{(1)})), \tag{32}$$

where $\breve{\mathcal{J}}(\mathcal{F}^{(1)},\mathcal{G}^{(1)})$ and $\breve{\mathcal{C}}^{(1)}_{\mathcal{F}}$ represent the strict transform of the Jacobian curve $\mathcal{J}(\mathcal{F}^{(1)},\mathcal{G}^{(1)})$ and the curve $\mathcal{C}^{(1)}_{\mathcal{F}}$ by ρ at the point p.

Substituting (32) in (31) we get

$$\begin{aligned}\mathrm{ord}_t\varphi^*\omega_{\mathcal{G}} &= \mathrm{m}_0(\mathcal{C}_{\mathcal{F}})\mathrm{m}_0(\omega_{\mathcal{G}}) + \nu^{\sigma}_{E_{dic}}(\omega^{(1)}_{\mathcal{F}} \wedge \omega^{(1)}_{\mathcal{G}}) - \nu^{\sigma}_{E_{dic}}(\omega^{(1)}_{\mathcal{F}})\\ &\quad + \iota_p(\breve{\mathcal{C}}^{(1)}_{\mathcal{F}}, \breve{\mathcal{J}}(\mathcal{F}^{(1)}, \mathcal{G}^{(1)})).\end{aligned}$$

By Lemma 3.10

$$\nu^{\pi}_{E_{dic}}(\omega_{\mathcal{F}} \wedge \omega_{\mathcal{G}}) = \nu^{\sigma}_{E_{dic}}(\omega^{(1)}_{\mathcal{F}} \wedge \omega^{(1)}_{\mathcal{G}}) + \chi(E_{dic})\left(\mathrm{m}_0(\omega_{\mathcal{F}}) + \mathrm{m}_0(\omega_{\mathcal{G}})\right),$$

then

$$\begin{aligned}\mathrm{ord}_t\varphi^*\omega_{\mathcal{G}} &= \mathrm{m}_0(\mathcal{C}_{\mathcal{F}})\mathrm{m}_0(\omega_{\mathcal{G}}) + \nu^{\pi}_{E_{dic}}(\omega_{\mathcal{F}} \wedge \omega_{\mathcal{G}}) - \left(\mathrm{m}_0(\omega_{\mathcal{F}}) + \mathrm{m}_0(\omega_{\mathcal{G}})\right)\chi(E_{dic})\\ &\quad -\nu^{\pi}_{E_{dic}}(\omega^{(1)}_{\mathcal{F}}) + \iota_p(\breve{\mathcal{C}}^{(1)}_{\mathcal{F}}, \breve{\mathcal{J}}(\mathcal{F}^{(1)}, \mathcal{G}^{(1)}))\\ &= \left(\mathrm{m}_0(\mathcal{C}_{\mathcal{F}}) - \chi(E_{dic})\right)\mathrm{m}_0(\omega_{\mathcal{G}}) + \nu^{\pi}_{E_{dic}}(\omega_{\mathcal{F}} \wedge \omega_{\mathcal{G}}) - \mathrm{m}_0(\omega_{\mathcal{F}})\chi(E_{dic})\\ &\quad -\nu^{\pi}_{E_{dic}}(\omega^{(1)}_{\mathcal{F}}) + \iota_p(\breve{\mathcal{C}}^{(1)}_{\mathcal{F}}, \breve{\mathcal{J}}(\mathcal{F}^{(1)}, \mathcal{G}^{(1)})).\end{aligned}$$

Since $\mathrm{m}_0(\mathcal{C}_{\mathcal{F}}) = \chi(E_{dic})$, we have

$$\mathrm{ord}_t\varphi^*\omega = \nu^{\pi}_{E_{dic}}(\omega_{\mathcal{F}} \wedge \omega_{\mathcal{G}}) - \nu^{\pi}_{E_{dic}}(\omega_{\mathcal{F}}) + \iota_p(\breve{\mathcal{C}}^{(1)}_{\mathcal{F}}, \breve{\mathcal{J}}(\mathcal{F}^{(1)}, \mathcal{G}^{(1)})).$$

Since the strict transform of the Jacobian curve $\mathcal{J}(\mathcal{F},\mathcal{G})$ and the curve $\mathcal{C}_{\mathcal{G}}$ by π coincide with the curves $\breve{\mathcal{J}}(\mathcal{F}^{(1)}, \mathcal{G}^{(1)})$, $\breve{\mathcal{C}}^{(1)}$ at the point $p \in E_{dic}$ then we have,

$$\mathrm{ord}_t\varphi^*\omega_{\mathcal{G}} = \nu^{\pi}_{E_{dic}}(\omega_{\mathcal{F}} \wedge \omega_{\mathcal{G}}) - \nu^{\pi}_{E_{dic}}(\omega_{\mathcal{F}}) + \iota_p(\tilde{\mathcal{C}}_{\mathcal{G}}, \tilde{\mathcal{J}}(\mathcal{F}, \mathcal{G})).$$

This proves Lemma 3.11.

Now we prove Theorem 3.8 for the equisingularity class $(n, n+1)$.

Proof. Let $\pi : (M, D) \to (\mathbb{C}^2, 0)$ be a blowing-up morphism described by the Euclidean algorithm of the pair $(n, n+1)$, with $D = \pi^{-1}(0)$. We decompose $\pi = \pi_1 \circ \sigma$, with π_1 the blowing-up morphism of the origin and σ the rest of the sequence. Let $\mathcal{F}$, $\mathcal{G}$ be as in Theorem 3.8 and let $\mathcal{C}_{\mathcal{F}}$ be a non-isolated separatrix of $\mathcal{F}$ and let φ be its parameterization. By Proposition 2.18 for the case when the divisor is non-dicritical we have,

$$\mathrm{ord}_t\varphi^*\omega = \mathrm{m}_0(\mathcal{C}_{\mathcal{F}})\mathrm{m}_0(\omega_{\mathcal{G}}) + \mathrm{ord}_t(\varphi^{(1)})^*\omega^{(1)}_{\mathcal{G}}. \tag{33}$$

By Lemma 3.11 applied to the foliations $\omega^{(1)}_{\mathcal{F}}, \omega^{(1)}_{\mathcal{G}}$, we have that

$$\begin{aligned}\mathrm{ord}_t\varphi^*\omega_{\mathcal{G}} &= \mathrm{m}_0(\mathcal{C}_{\mathcal{F}})\mathrm{m}_0(\omega_{\mathcal{G}}) + \nu^{\sigma}_{E_{dic}}(\omega^{(1)}_{\mathcal{F}} \wedge \omega^{(1)}_{\mathcal{G}}) - \nu^{\sigma}_{E_{dic}}(\omega^{(1)}_{\mathcal{F}})\\ &\quad + \iota_p(\hat{\mathcal{C}}^{(1)}_{\mathcal{F}}, \hat{\mathcal{J}}(\mathcal{F}^{(1)}, \mathcal{G}^{(1)})).\end{aligned}$$

where $\hat{\mathcal{J}}(\mathcal{F}^{(1)}, \mathcal{G}^{(1)})$ represents the strict transform of the Jacobian curve of the foliations $\mathcal{F}^{(1)}$, $\mathcal{G}^{(1)}$ by σ. By Lemma 3.10,

$$\begin{aligned}\mathrm{ord}_t\varphi^*\omega_{\mathcal{G}} &= \mathrm{m}_0(\omega_{\mathcal{G}})\left(\mathrm{m}_0(\mathcal{C}_{\mathcal{F}}) - \chi(E_{dic})\right) + \nu^{\pi}_{E_{dic}}(\omega_{\mathcal{F}} \wedge \omega_{\mathcal{G}}) - \mathrm{m}_0(\omega_{\mathcal{F}})\chi(E_{dic})\\ &\quad -\nu^{\sigma}_{E_{dic}}(\omega^{(1)}_{\mathcal{F}}) + \iota_p(\hat{\mathcal{C}}^{(1)}_{\mathcal{F}}, \hat{\mathcal{J}}(\mathcal{F}^{(1)}, \mathcal{G}^{(1)}))\\ &= \mathrm{m}_0(\omega_{\mathcal{G}})(\mathrm{m}_0(\mathcal{C}_{\mathcal{F}}) - \chi(E_{dic})) + \nu^{\pi}_{E_{dic}}(\omega_{\mathcal{F}} \wedge \omega_{\mathcal{G}}) - \nu^{\pi}_{E_{dic}}(\omega_{\mathcal{F}})\\ &\quad + \iota_p(\hat{\mathcal{C}}^{(1)}_{\mathcal{F}}, \hat{\mathcal{J}}(\mathcal{F}^{(1)}, \mathcal{G}^{(1)})),\end{aligned}$$

Since $\chi(E_{dic}) = \mathrm{m}_0(\mathcal{C})$ with $\mathcal{C} \in \mathcal{E}^{\pi}_{E_{dic}}$, this implies the equality,

$$\mathrm{ord}_t\varphi^*\omega_{\mathcal{G}} = \nu^{\pi}_{E_{dic}}(\omega_{\mathcal{F}} \wedge \omega_{\mathcal{G}}) - \nu^{\pi}_{E_{dic}}(\omega_{\mathcal{F}}) + \iota_p(\hat{\mathcal{C}}^{(1)}_{\mathcal{F}}, \hat{\mathcal{J}}(\mathcal{F}^{(1)}, \mathcal{G}^{(1)})),$$

since the strict transforms of the Jacobian curve $\mathcal{J}(\mathcal{F},\mathcal{G})$ and the curve $\mathcal{C}_{\mathcal{F}}$ by π at the point p coincides with the curve $\hat{\mathcal{J}}(\mathcal{F}^{(1)},\mathcal{G}^{(1)})$ and the curve $\hat{\mathcal{C}}_{\mathcal{F}}^{(1)}$ respectively, then we have

$$\mathrm{ord}_t\varphi^*\omega_{\mathcal{G}} = \nu^{\pi}_{E_{dic}}(\omega_{\mathcal{F}}\wedge\omega_{\mathcal{G}}) - \nu^{\pi}_{E_{dic}}(\omega_{\mathcal{F}}) + \iota_p(\tilde{\mathcal{C}}_{\mathcal{F}},\tilde{\mathcal{J}}(\mathcal{F},\mathcal{G})).$$

We proceed to prove Theorem 3.8 in the general case.

Proof of Theorem 3.8. We recall that $n,m\in\mathbb{N}$ are such that $1<n<m$ and $\mathrm{g.c.d.}(n,m)=1$. The proof is by induction on the number $n+m$. The base step is when $(n,m)=(2,3)$, $n+m=5$, this is a direct consequence of the proof for the case $(n,n+1)$.

Let us suppose now that the theorem is true for any pair (n',m'), $1<n'<m'$ and $\mathrm{g.c.d}(n',m')=1$ with $n'+m'<n+m$. We consider the foliations $\mathcal{F}\in\mathfrak{F}_\pi$ and $\mathcal{G}\in\mathfrak{F}_\pi^C$ and we denote by $\mathcal{C}_{\mathcal{F}}$ a non-isolated separatrix of $\mathcal{F}$. Let φ be a parameterization of $\mathcal{C}_{\mathcal{F}}$ and $\{\omega_{\mathcal{G}}=0\}$ be an equation defining foliation $\mathcal{G}$. Again, the sequence π can be decomposed as $\pi=\pi_1\circ\sigma$, where π_1 is the blowing-up morphism of the origin. Then, by Proposition 2.18,

$$\mathrm{ord}_t\varphi^*\omega_{\mathcal{G}} = \mathrm{m}_0(\mathcal{C}_{\mathcal{F}})\mathrm{m}_0(\omega_{\mathcal{G}}) + \mathrm{ord}_t(\varphi^{(1)})^*\omega_{\mathcal{G}}^{(1)}. \tag{34}$$

The foliation $\mathcal{G}^{(1)}$ is induced by $\left\{\omega_{\mathcal{G}}^{(1)}=0\right\}$, that is, it is the strict transform of the foliation $\mathcal{G}$ by π_1. This foliation has a unique dicritical component that arises after perfoming the sequence of blowing-up morphisms described by the pair $(n,m-n)$. If $m-n>1$ then, by the induction assumption, we get

$$\begin{aligned}\mathrm{ord}_t\varphi^*\omega_{\mathcal{G}} &= \mathrm{m}_0(\mathcal{C}_{\mathcal{F}})\mathrm{m}_0(\omega_{\mathcal{G}}) + \mathrm{ord}_t(\varphi^{(1)})^*\omega_{\mathcal{G}}^{(1)}\\ &= \mathrm{m}_0(\mathcal{C}_{\mathcal{F}})\mathrm{m}_0(\omega_{\mathcal{G}}) + \nu^{\sigma}_{E_{dic}}(\omega_{\mathcal{F}}^{(1)}\wedge\omega_{\mathcal{G}}^{(1)}) - \nu^{\sigma}_{E_{dic}}(\omega_{\mathcal{F}}^{(1)})\\ &\quad+\iota_p(\hat{\mathcal{C}}_{\mathcal{G}}^{(1)},\hat{\mathcal{J}}(\mathcal{F}^{(1)},\mathcal{G}^{(1)})),\end{aligned} \tag{35}$$

where $\hat{\mathcal{J}}(\mathcal{F}^{(1)},\mathcal{G}^{(1)})$ and $\hat{\mathcal{C}}_{\mathcal{G}}^{(1)}$ represent the strict transforms of the Jacobian curve of the foliations $\mathcal{F}^{(1)},\mathcal{G}^{(1)}$ and the curve $\mathcal{C}_{\mathcal{G}}^{(1)}$ by σ, respectively. By Lemma 3.10 for 2-forms we have that,

$$\nu^{\pi}_{E_{dic}}(\omega_{\mathcal{F}}\wedge\omega_{\mathcal{G}}) = (\mathrm{m}_0(\omega_{\mathcal{F}})+\mathrm{m}_0(\omega_{\mathcal{G}}))\,\chi(E_{dic}) + \nu^{\sigma}_{E_{dic}}(\omega_{\mathcal{F}}^{(1)}\wedge\omega_{\mathcal{G}}^{(1)}). \tag{36}$$

By substituting (36) in (35) we have

$$\begin{aligned}\mathrm{ord}_t\varphi^*\omega_{\mathcal{G}} &= \mathrm{m}_0(\mathcal{C}_{\mathcal{F}})\mathrm{m}_0(\omega_{\mathcal{G}}) + \nu^{\pi}_{E_{dic}}(\omega_{\mathcal{F}}\wedge\omega_{\mathcal{G}}) - (\mathrm{m}_0(\omega_{\mathcal{F}})+\mathrm{m}_0(\omega_{\mathcal{G}}))\,\chi(E_{dic})\\ &\quad-\nu^{\sigma}_{E_{dic}}(\omega_{\mathcal{F}}^{(1)}) + \iota_p(\hat{\mathcal{C}}_{\mathcal{G}}^{(1)},\hat{\mathcal{J}}(\mathcal{F}^{(1)},\mathcal{G}^{(1)}))\\ &= (\mathrm{m}_0(\mathcal{C}_{\mathcal{F}})-\chi(E_{dic}))\,\mathrm{m}_0(\omega_{\mathcal{G}}) + \nu^{\pi}_{E_{dic}}(\omega_{\mathcal{F}}\wedge\omega_{\mathcal{G}}) - \mathrm{m}_0(\omega_{\mathcal{F}})\chi(E_{dic})\\ &\quad-\nu^{\sigma}_{E_{dic}}(\omega_{\mathcal{F}}^{(1)}) + \iota_p(\hat{\mathcal{C}}_{\mathcal{G}}^{(1)},\hat{\mathcal{J}}(\mathcal{F}^{(1)},\mathcal{G}^{(1)}))\\ &= (\mathrm{m}_0(\mathcal{C}_{\mathcal{F}})-\chi(E_{dic}))\,\mathrm{m}_0(\omega_{\mathcal{G}}) + \nu^{\pi}_{E_{dic}}(\omega_{\mathcal{F}}\wedge\omega_{\mathcal{G}}) - \nu^{\pi}_{E_{dic}}(\omega_{\mathcal{F}})\\ &\quad+\iota_p(\hat{\mathcal{C}}_{\mathcal{G}}^{(1)},\hat{\mathcal{J}}(\mathcal{F}^{(1)},\mathcal{G}^{(1)})).\end{aligned}$$

Since $\chi(E_{dic})=\mathrm{m}_0(\mathcal{C})$, $\mathcal{C}\in\mathcal{E}^{\pi}_{E}$, this implies,

$$\mathrm{ord}_t\varphi^*\omega_{\mathcal{G}} = \nu^{\pi}_{E_{dic}}(\omega_{\mathcal{F}}\wedge\omega_{\mathcal{G}}) - \nu^{\pi}_{E_{dic}}(\omega_{\mathcal{F}}) + \iota_p(\hat{\mathcal{C}}_{\mathcal{G}}^{(1)},\hat{\mathcal{J}}(\mathcal{F}^{(1)},\mathcal{G}^{(1)})).$$

Since the strict transform of the Jacobian curve $\mathcal{J}(\mathcal{F},\mathcal{G})$ and the curve $\mathcal{C}_{\mathcal{G}}$ by π at the point p coincides with the curves $\hat{\mathcal{J}}(\mathcal{F}^{(1)},\mathcal{G}^{(1)})$, $\hat{\mathcal{C}}_{\mathcal{G}}^{(1)}$ then we have,

$$\mathrm{Tan}_0(\mathcal{C}_{\mathcal{F}},\mathcal{G}) = \mathrm{ord}_t\varphi^*\omega_{\mathcal{G}} = \nu^{\pi}_{E_{dic}}(\omega_{\mathcal{F}}\wedge\omega_{\mathcal{G}}) - \nu^{\pi}_{E_{dic}}(\omega_{\mathcal{F}}) + \iota_p(\tilde{\mathcal{C}}_{\mathcal{G}},\tilde{\mathcal{J}}(\mathcal{F},\mathcal{G})).$$

The proof of Theorem 3.8 is finished.

4. Analytic Invariants of non-isolated separatrices

In this section we introduce the notion of *polar transversality* between a foliation $\mathcal{F} \in \mathfrak{F}_\pi$ and the family $\mathfrak{F}_\pi^C$. The property of *polar transversality* of a foliation $\mathcal{F} \in \mathfrak{F}_\pi$ with the family $\mathfrak{F}_\pi^C$ is given in terms of the Jacobian curve $\mathcal{J}(\mathcal{F},\mathcal{G})$ between the foliation $\mathcal{F}$ and every foliation $\mathcal{G} \in \mathfrak{F}_\pi^C$. It stresses that the strict transform of the *Jacobian* curve by π does not intersect the irreducible component E_{dic}. We show that every non-isolated separatrix of any element $\mathcal{F} \in \mathfrak{F}_\pi$ satisfiying the polar transversality property, has coinciding Zariski invariant.

Definition 4.1. We say that $\mathcal{F} \in \mathfrak{F}_\pi$ has *polar transversality* with $\mathfrak{F}_\pi^C$ if and only if for all $\mathcal{G} \in \mathfrak{F}_\pi^C$ the strict transform of the Jacobian curve $\mathcal{J}(\mathcal{F},\mathcal{G})$ does not intersect the irreducible component E_{dic}.

The family of foliations satisfying the *polar transversality* property will be denoted by $\mathfrak{F}_\pi^\star$. The following theorem tells us that the elements of $\mathfrak{F}_\pi^\star$ are foliations $\mathcal{F} \in \mathfrak{F}_\pi$ for which all the non-isolated separatrices of $\mathcal{F}$ have the same Zariski invariant.

Remark 4.2. Note that for every $\mathcal{G} \in \mathfrak{F}_\pi^C$ all the non-isolated separatrices of $\mathcal{G}$ are (n,m)-quasihomogeneous and for that reason they have the same Zariski invariant $\lambda = \infty$.

Theorem 4.3. *Let $\mathcal{F}$ be a pseudo-cuspidal dicritical foliation that has the polar transversality property with $\mathfrak{F}_\pi^C$, $\mathcal{F} \in \mathfrak{F}_\pi^\star$. Then all the non-isolated separatrices of $\mathcal{F}$ have coinciding Zariski invariant.*

Proof. Let $\mathcal{F} \in \mathfrak{F}_\pi^\star$. Let $\mathcal{C}_\mathcal{F}$ be a non-isolated separatrix of $\mathcal{F}$ and let φ be its parameterization. If there exists $\mathcal{G}$ such that $\mathrm{ord}_t \varphi^*\omega_\mathcal{G} = \infty$, then by Theorem 3.8

$$\infty = \mathrm{ord}_t \varphi^*\omega_\mathcal{G} = \nu^\pi_{E_{dic}}(\omega_\mathcal{F} \wedge \omega_\mathcal{G}) - \nu^\pi_{E_{dic}}(\omega_\mathcal{F}).$$

Since $\nu^\pi_{E_{dic}}(\omega_\mathcal{F}) < \infty$ then we have that $\nu^\pi_{E_{dic}}(\omega_\mathcal{F} \wedge \omega_\mathcal{G}) = \infty$, this happens if and only if $\omega_\mathcal{F} \wedge \omega_\mathcal{G} \equiv 0$ but this last equation implies that $\mathcal{F} = \mathcal{G}$. Since the cuspidal dicritical foliations do not have the property of polar transversality it implies that the foliation $\mathcal{F}$ does not have the property of polar transversality, but this is a contradiction. Hence we have that for all cuspidal dicritical foliations $\mathcal{G} \in \mathfrak{F}_\pi^C$, $\mathrm{ord}_t \varphi^*\omega_\mathcal{G} < \infty$. Let $\mathcal{G}^\star \in \mathfrak{F}_\pi^C$ be such that $\mathrm{ord}_t \varphi^*\omega_{\mathcal{G}^\star}$ is maximal.

By Theorem 3.8

$$\mathrm{ord}_t \varphi^*\omega_{\mathcal{G}^\star} = \nu^\pi_{E_{dic}}(\omega_\mathcal{F} \wedge \omega_{\mathcal{G}^\star}) - \nu^\pi_{E_{dic}}(\omega_\mathcal{F}).$$

Since $\mathrm{ord}_t \varphi^*\omega_{\mathcal{G}^\star}$ is maximal

$$\nu^\pi_{E_{dic}}(\omega_\mathcal{F} \wedge \omega_\mathcal{G}) - \nu^\pi_{E_{dic}}(\omega_\mathcal{F}) = \mathrm{ord}_t \varphi^*\omega_\mathcal{G} \le \mathrm{ord}_t \varphi^*\omega_{\mathcal{G}^\star} = \nu^\pi_{E_{dic}}(\omega_\mathcal{F} \wedge \omega_{\mathcal{G}^\star}) - \nu^\pi_{E_{dic}}(\omega_\mathcal{F}),$$

for every $\mathcal{G} \in \mathfrak{F}_\pi^C$. Therefore,

$$\nu^\pi_{E_{dic}}(\omega_\mathcal{F} \wedge \omega_\mathcal{G}) \le \nu^\pi_{E_{dic}}(\omega_\mathcal{F} \wedge \omega_{\mathcal{G}^\star}) \text{ for all } \mathcal{G} \in \mathfrak{F}_\pi^C. \tag{37}$$

This implies that $\nu^\pi_{E_{dic}}(\omega_\mathcal{F} \wedge \omega_{\mathcal{G}^\star})$ is maximal.
Moreover, we know by Theorem 2.22 that the Zariski invariant of $\mathcal{C}_\mathcal{F}$ is

$$\lambda = \mathrm{ord}_t \varphi^*\omega_{\mathcal{G}^\star} + 1 - n.$$

Therefore, if $\tilde{\mathcal{C}}_\mathcal{F}$ is another non-isolated separatrix and $\tilde{\varphi}$ its parameterization, then by Theorem 3.8,

$$\mathrm{ord}_t \tilde{\varphi}^*\omega_{\mathcal{G}^\star} = \nu^\pi_{E_{dic}}(\omega_\mathcal{F} \wedge \omega_{\mathcal{G}^\star}) - \nu^\pi_{E_{dic}}(\omega_\mathcal{F}).$$

This value is maximal by (37), hence,

$$\lambda = \mathrm{ord}_t \tilde{\varphi}^*\omega_{\tilde{\mathcal{G}}} + 1 - n = \nu^\pi_{E_{dic}}(\omega_\mathcal{F} \wedge \omega_{\mathcal{G}^\star}) - \nu^\pi_{E_{dic}}(\omega_\mathcal{F}) + 1 - n$$

is the Zariski invariant of $\tilde{\mathcal{C}}_{\mathcal{F}}$. This invariant coincides precisely with the invariant of $\mathcal{C}_{\mathcal{F}}$. Hence, all the non-isolated separatrices have coinciding Zariski invariant. This finishes the proof of Theorem 4.3.

The following example shows a foliation having the polar transversality property with the family $\mathfrak{F}_{\pi}^{\mathcal{C}}$.

Example 4.4. Let $\mathcal{F}$ be the foliation generated by

$$\omega_{\mathcal{F}} = \left(8y^3 + 5x^3y^2 + 2x^6y\right)dx - \left(3xy^2 + 2x^4y + x^7\right)dy.$$

The non-isolated separatrices of the foliation $\mathcal{F}$ are the curves given by

$$\mathcal{C}_k = \{y^3 - kx^8 + x^3y^2 + x^6y = 0\},$$

with $k \in \mathbb{C}^*$. These curves belong to the equisingularity class $(3, 8)$. Let $\mathcal{G}_0$ be the cuspidal dicritical foliation given by $\omega_{\mathcal{G}_0} = 8ydx - 3xdy$ and consider the product

$$\omega_{\mathcal{F}} \wedge \omega_{\mathcal{G}_0} = \left(x^4y^2 + 2x^7y\right)dx \wedge dy.$$

Note that the Jacobian curve is

$$\mathcal{J}(\mathcal{F}, \mathcal{G}_0) = \mathcal{J}_1(\mathcal{F}, \mathcal{G}_0) \cup \mathcal{J}_2(\mathcal{F}, \mathcal{G}_0) \cup \mathcal{J}_3(\mathcal{F}, \mathcal{G}_0),$$

where $\mathcal{J}_1(\mathcal{F}, \mathcal{G}_0) = \{x = 0\}$, $\mathcal{J}_2(\mathcal{F}, \mathcal{G}_0) = \{y = 0\}$ and $\mathcal{J}_3(\mathcal{F}, \mathcal{G}_0) = \{y + x^3 = 0\}$. Moreover, the strict transforms by π of these curves do not intersect the component E_{dic}.

After straightforward computations we have that $\nu^{\pi}_{E_{dic}}(\omega_{\mathcal{F}} \wedge \omega_{\mathcal{G}_0}) = 38$ and $\nu^{\pi}_{E_{dic}}(\omega_{\mathcal{F}}) = 27$. For fixed k, let $\mathcal{C}_k$ be a non-isolated separatix of $\mathcal{F}$. By the formula given in Theorem 3.8, we have

$$\begin{aligned}\operatorname{Tan}_0(\mathcal{C}_k, \mathcal{G}_0) &= \nu^{\pi}_{E_{dic}}(\omega_{\mathcal{F}} \wedge \omega_{\mathcal{G}_0}) - \nu^{\pi}_{E_{dic}}(\omega_{\mathcal{F}}) + \iota_p(\tilde{\mathcal{C}}_k, \tilde{\mathcal{J}}(\mathcal{F}, \mathcal{G}_0)) \\ &= 38 - 27 = 11.\end{aligned}$$

Note that $\operatorname{Tan}_0(\mathcal{C}_k, \mathcal{G}_0) + 1 - 3 = 9 \in S_{\mathcal{C}}$, and then 11 is not the Zariski invariant. We consider now the foliation $\mathcal{G}_\alpha$, induced by $\omega_{\mathcal{G}_\alpha} = \left(8y - \alpha x^3\right)dx - 3xdy$, $\alpha \in \mathbb{C}$. We compute

$$\omega_{\mathcal{F}} \wedge \omega_{\mathcal{G}_\alpha} = \left((1 - 3\alpha)\, x^4y^2 + (2 - 2\alpha)x^7y - \alpha x^{10}\right)dx \wedge dy$$

After straightforward computations we get that if $1 - 3\alpha \neq 0$ then $\nu^{\pi}_{E_{dic}}(\omega_{\mathcal{F}} \wedge \omega_{\mathcal{G}_\alpha}) = 38$. However, if $\alpha_0 = \frac{1}{3}$, then $1 - 3\alpha_0 = 0$ and then,

$$\omega_{\mathcal{F}} \wedge \omega_{\mathcal{G}_{\alpha_0}} = \left(\frac{4}{3}x^7y - \frac{1}{3}x^{10}\right)dx \wedge dy.$$

Note that the Jacobian curve is given by

$$\mathcal{J}(\mathcal{F}, \mathcal{G}_{\alpha_0}) = \mathcal{J}_1(\mathcal{F}, \mathcal{G}_{\alpha_0}) \cup \mathcal{J}_2(\mathcal{F}, \mathcal{G}_{\alpha_0}),$$

where $\mathcal{J}_1(\mathcal{F}, \mathcal{G}_{\alpha_0}) = \{x = 0\}$ and $\mathcal{J}_2(\mathcal{F}, \mathcal{G}_{\alpha_0}) = \{\frac{4}{3}y - \frac{1}{3}x^3 = 0\}$. Moreover the strict transform of the Jacobian curve does not intersect the componen E_{dic}. After explicit computations we have that $\nu^{\pi}_{E_{dic}}(\omega_{\mathcal{F}} \wedge \omega_{\mathcal{G}_{\alpha_0}}) = 39$. For fixed k, let $\mathcal{C}_k$ be a non-isolated separatrix of $\mathcal{F}$. By the formula given in Theorem 3.8 we have,

$$\begin{aligned}\operatorname{Tan}_0(\mathcal{C}_k, \mathcal{G}_{\alpha_0}) &= \nu^{\pi}_{E_{dic}}(\omega_{\mathcal{F}} \wedge \omega_{\mathcal{G}_{\alpha_0}}) - \nu^{\pi}_{E_{dic}}(\omega_{\mathcal{F}}) + \iota_p(\tilde{\mathcal{C}}_{\mathcal{G}_{\alpha_0}}, \tilde{\mathcal{J}}(\mathcal{F}, \mathcal{G}_0)) \\ &= 39 - 27 = 12.\end{aligned}$$

Since $\operatorname{Tan}_0(\mathcal{C}_k, \mathcal{G}_{\alpha_0}) + 1 - 3 = 12 + 1 - 3 = 10 \notin S_{\mathcal{C}}$ and $\operatorname{Tan}_0(\mathcal{C}_k, \mathcal{G}_{\alpha_0}) + 1 - 3 + 3 - 8 = 5 \notin 8\mathbb{Z}_{>0}$, from the Theorem 2.22 we have that 10 is the Zariski invariant. After forthright computations it is possible to show that the foliation $\mathcal{F}$ has the polar transversality property.

The following result gives us a condition under which all the non-isolated separatrices are (n, m)-quasihomogeneous.

Proposition 4.5. *Let $\mathcal{F} \in \mathfrak{F}_\pi$ be the foliation induced by $\{\omega_\mathcal{F} = 0\}$. If there exists $\mathcal{G}^\star \in \mathfrak{F}^C_\pi$ such that*

$$\nu^\pi_{E_{dic}}(\omega_\mathcal{F} \wedge \omega_{\mathcal{G}^\star}) - \nu^\pi_{E_{dic}}(\omega_\mathcal{F}) \geq m(n-1), \tag{38}$$

then all the non-isolated separatrices of $\mathcal{F}$ are $(n, m)-$quasihomogeneous.

Proof. Let $\mathcal{C}_\mathcal{F}$ be a non-isolated separatrix of $\mathcal{F}$ and let φ be a parameterization of $\mathcal{C}_\mathcal{F}$. By Theorem 3.8

$$\mathrm{ord}_t\varphi^*\omega_\mathcal{G} = \nu^\pi_{E_{dic}}(\omega_\mathcal{F} \wedge \omega_\mathcal{G}) - \nu^\pi_{E_{dic}}(\omega_\mathcal{F}) + \iota_p(\tilde{\mathcal{C}}_\mathcal{G}, \tilde{\mathcal{J}}(\mathcal{F}, \mathcal{G})).$$

Since, by hypothesis, there exists $\mathcal{G}^\star \in \mathfrak{F}^C_\pi$ such that $\nu^\pi_{E_{dic}}(\omega_\mathcal{F} \wedge \omega_{\mathcal{G}^\star}) - \nu^\pi_{E_{dic}}(\omega_\mathcal{F}) \geq m(n-1)$, then we have

$$\begin{aligned} \mathrm{ord}_t\varphi^*\omega_{\mathcal{G}^\star} &= \nu^\pi_{E_{dic}}(\omega_\mathcal{F} \wedge \omega_{\mathcal{G}^\star}) - \nu^\pi_{E_{dic}}(\omega_\mathcal{F}) + \iota_p(\tilde{\mathcal{C}}_{\mathcal{G}^\star}, \tilde{\mathcal{J}}(\mathcal{F}, \mathcal{G}^\star)) \\ &\geq m(n-1) + \iota_p(\mathcal{C}_{\tilde{\mathcal{G}}}, \tilde{\mathcal{J}}(\mathcal{F}, \mathcal{G}^\star)) \geq m(n-1). \end{aligned}$$

This implies that $\mathrm{ord}_t\varphi^*\omega_{\mathcal{G}^\star} + 1 - n \geq m(n-1) + 1 - n = (m-1)(n-1)$. Since the conductor of the semigroup of the plane branch is $c = (m-1)(n-1)$, then $\mathcal{C}_\mathcal{F}$ is analytically equivalent to the quasihomogeneous branch. Since $\mathcal{C}_\mathcal{F}$ is an arbitrary non-isolated separatrix of $\mathcal{F}$, the proposition is proved.

Theorem 4.3 shows that if $\mathcal{F}$ belongs to the family $\mathfrak{F}^\star_\pi$ all the non-isolated separatrices of $\mathcal{F}$ have the same Zariski invariant, however the Zariski invariant does not characterize the family $\mathfrak{F}^\star_\pi$. Namely, the following example shows the existence of a foliation in $\mathfrak{F}_\pi$ whose non-isolated separatrices have the same Zariski invariant even though this foliation does not belong to the subfamily $\mathfrak{F}^\star_\pi$. It is matter of a future study to provide the complete set of invariants characterizing the foliations having polar transversality with respect to the cuspidal dicritical foliations.

Example 4.6. Let $\mathcal{F}$ be the foliation induced by $\{\omega_\mathcal{F} = 0\}$ where

$$\omega_\mathcal{F} = (x^6y - 7y^6)dx + (6y^5x - x^7)dy.$$

The non-isolated separatrices are given by the equation $\mathcal{C}_k = \{y^6 - kx^7 - x^6y = 0\}$, for $k \in \mathbb{C}^*$. For each $k \in \mathbb{C}^*$ the branch $\mathcal{C}_k$ belongs to the equisingularity class $(6, 7)$. After some computations we can verify that the Zariski invariant of every $\mathcal{C}_k$ is 9. The foliation $\mathcal{F}$ belongs to the family $\mathfrak{F}_\pi$, however it does not belong to $\mathfrak{F}^\star_\pi$. Namely, let us consider the foliation $\mathcal{G}_\alpha$ induced by the equation $\{\omega_{\mathcal{G}_\alpha} = 0\}$, where $\omega_{\mathcal{G}_\alpha} = (6x + \alpha y)dy - 7ydx$, with $\alpha \in \mathbb{C}^*$. Then,

$$\begin{aligned} \omega_\mathcal{F} \wedge \omega_{\mathcal{G}_\alpha} &= \left[(x^6y - 7y^6)dx + (6y^5x - x^7)\right] dy \wedge [-(6x + \alpha y)dy - 7ydx] \\ &= \left[7\alpha y^7 + x^7y - \alpha x^6y^2\right] dx \wedge dy. \end{aligned}$$

Note that the Jacobian curve has two branches

$$\mathcal{J}(\mathcal{F}, \mathcal{G}_\alpha) = \mathcal{J}_1(\mathcal{F}, \mathcal{G}_\alpha) \cup \mathcal{J}_2(\mathcal{F}, \mathcal{G}_\alpha),$$

where $\mathcal{J}_1(\mathcal{F}, \mathcal{G}_\alpha) = \{y = 0\}$ and $\mathcal{J}_2(\mathcal{F}, \mathcal{G}_\alpha) = \{7\alpha y^6 - x^7 + \alpha x^6y = 0\}$. The strict transform of $\mathcal{J}_2(\mathcal{F}, \mathcal{G}_\alpha)$ by π intersects the component E_{dic} far from a corner point, for this reason $\mathcal{F}$ does not belong to $\mathfrak{F}^\star_\pi$. Now we recover the Zariski invariant of each non-isolated separatrix using the formula given in Theorem 3.8. Let $\mathcal{C}_k$ be a non-isolated separatrix of $\mathcal{F}$. For each fixed k, there exists $\alpha(k)$ such that the strict transform of the Jacobian curve $\mathcal{J}(\mathcal{F}, \mathcal{G}_{\alpha(k)})$ by π passes through the same point $p \in E_{dic}$ that the strict transform of the separatrix $\mathcal{C}_{\mathcal{G}_{\alpha(k)}}$ does. After forthright computations we have that

$$\begin{aligned}\mathrm{Tan}_0(\mathcal{C}_k, \mathcal{G}_{\alpha(k)}) &= \nu^{\pi}_{E_{dic}}(\omega_{\mathcal{F}} \wedge \omega_{\mathcal{G}_{\alpha(k)}}) - \nu^{\pi}_{E_{dic}}(\omega_{\mathcal{F}}) + \iota_p(\tilde{\mathcal{C}}_{G_{\alpha(k)}}, \tilde{\mathcal{J}}(\mathcal{F}, \mathcal{G}_{\alpha(k)}))\\ &= 61 - 48 + 1 = 14.\end{aligned}$$

Hence $\mathrm{Tan}_0(\mathcal{C}_k, \mathcal{G}_{\alpha(k)}) + 1 - 6 = 14 + 1 - 6 = 9 \notin S_{\mathcal{C}}$ and

$$\mathrm{Tan}_0(\mathcal{C}_k, \mathcal{G}_{\alpha(k)}) + 1 - 7 = 14 + 1 - 7 = 8 \notin 7\mathbb{Z}_{>0}.$$

From Theorem 2.22 it follows that 9 is the Zariski invariant.

The following example shows that there exist foliations for which their non-isolated separatrices have not necessarily coincident Zariski invariants.

Example 4.7. We consider the foliation $\mathcal{F}$ generated by

$$\omega_{\mathcal{F}} = \left(y^5 + 2x^9y - 2x^9y^2 - 9x^2y^4\right)dx + \left(4y^3x^3 - x^{10} + 2x^{10}y - 3xy^4 - x^8y^2\right)dy.$$

The non-isolated separatrices of the foliation are the branches given by

$$\mathcal{C}_k = \{y^4 - kx^9 + (k-1)x^7y + x^7y^2 = 0\},$$

$k \in \mathbb{C}^*$. For every $k \in \mathbb{C}^*$ except for $k = 1$ the Zariski invariant of the non-isolated separatrices coincides. Now we consider the foliation $\mathcal{G}_0$ given for $\omega_0 = 9ydx - 4xdy$, we have that

$$\omega_{\mathcal{F}} \wedge \omega_{\mathcal{G}_0} = -xy\left(y^4 - x^9 + 10x^9y - 9x^7y^2\right)dx \wedge dy.$$

After straightforward computations we have $\nu^{\pi}_{E_{dic}}(\omega_{\mathcal{F}} \wedge \omega_0) = 61$ y $\nu^{\pi}_{E_d}(\omega_{\mathcal{F}}) = 48$. Note that the Jacobian curve is $\mathcal{J}(\mathcal{F}, \mathcal{G}_0)$ which is the union of the curves $\mathcal{J}_1 = \{x = 0\}$, $\mathcal{J}_2 = \{y = 0\}$ and $\mathcal{J}_3 = \{y^4 - x^9 + 10x^9y - 9x^7y^2 = 0\}$. Moreover the strict transfom by π of the curve $\mathcal{J}_3$ intersects the dicritical component. Now we compute the Zariski invariant using the formula given in Theorem 3.8. Let $\mathcal{C}_k$ with fixed $k \neq 1$ be a non-isolated separatix of the foliation $\mathcal{F}$. By the formula given in Theorem 3.8 we have,

$$\begin{aligned}\mathrm{Tan}_0(\mathcal{C}_k, \mathcal{G}_0) &= \nu^{\pi}_{E_{dic}}(\omega_{\mathcal{F}} \wedge \omega_{\mathcal{G}_0}) - \nu^{\pi}_{E_{dic}}(\omega_{\mathcal{F}}) + \iota_p(\tilde{\mathcal{C}}_{G_0}, \tilde{\mathcal{J}}(\mathcal{F}, \mathcal{G}_0))\\ &= 61 - 48 = 13.\end{aligned}$$

Then, $\mathrm{Tan}_0(\mathcal{C}_k, \mathcal{G}_0) + 1 - 4 = 10 \notin S_{\mathcal{C}}$ and $\mathrm{Tan}_0(\mathcal{C}_k, \mathcal{G}_0) + 1 - 4 + 4 - 9 = 14 - 9 = 5 \notin 9\mathbb{Z}_{>0}$. This implies that 10 is the Zariski invariant.

For $k = 1$, after explicit computations we have that

$$\begin{aligned}\mathrm{Tan}_0(\mathcal{C}_1, \mathcal{G}_0) &= \nu^{\pi}_{E_{dic}}(\omega_{\mathcal{F}} \wedge \omega_{\mathcal{G}_0}) - \nu^{\pi}_{E_{dic}}(\omega_{\mathcal{F}}) + \iota_p(\tilde{\mathcal{C}}_{G_0}, \tilde{\mathcal{J}}(\mathcal{F}, \mathcal{G}_0))\\ &= 61 - 48 + 9 = 22.\end{aligned}$$

Then, $\mathrm{Tan}_0(\mathcal{C}_1, \mathcal{G}_0) + 1 - 4 = 23 - 4 = 19 \notin S_{\mathcal{C}}$ and $\mathrm{Tan}_0(\mathcal{C}_1, \mathcal{G}_0) + 1 - 4 + 4 - 9 = 23 - 9 = 14 \notin 9\mathbb{Z}_{>0}$. Therefore, the Zariski invariant of $\mathcal{C}_1$ is 19.

Thus, as we have seen in this example, there exist foliations for which their non-isolated separatrices have not necessarily coincident Zariski invariants.

Now let us show that the subfamily $\mathfrak{F}^{\star}_{\pi}$ is not empty. For this sake we rely on some ideas described in [30].

Let $n, m \in \mathbb{N}$, $1 < n < m$ and g.c.d$(n, m) = 1$. We define,

$$\mathrm{PZ} := \left\{(i,j) \in \mathbb{Z}^2_{\geq 0} \mid 0 \leq i \leq m-2,\ 0 \leq j \leq n-2 \text{ and } ni + mj > nm\right\}. \tag{39}$$

We define as well the set,

$$\mathrm{ZI} := \{s \in \mathbb{Z}_{\geq 0} \mid s + n \notin \langle n, m\rangle \text{ and } s + m \notin \langle n, m\rangle\}, \tag{40}$$

where $\langle n,m\rangle = \{ni+mj \mid i,j \in \mathbb{Z}_{\geq 0}\}$. Let $\mathcal{C}$ be a plane branch of the equisingularity class (n,m) and we assume that $\mathcal{C}$ admits a parameterization φ

$$\varphi(t) = \Big(t^n, t^m + a_\lambda t^\lambda + \sum_{j>\lambda} a_j t^j\Big), \tag{41}$$

where $a_\lambda \neq 0$ with λ satisfying $\lambda \notin S$ and $\lambda + n - m \notin m\mathbb{Z}_{>0}$. If we write $\lambda = m+s$ then $m+s \notin S$. Moreover, $\lambda + n - m = m+s+n-m = n+s \notin m\mathbb{Z}_{>0}$, hence, $n+s \notin S$. The set ZI is, thus, the set of Zariski invariants of the equisingularity class (n,m). The following lemma is proved in [30], page 26 Lemma 1.3.

Lemma 4.8. *There exists a bijection between the sets* PZ *and* ZI.

Moreover, in [30], page 27, Theorem 1.5 (see also [36], page 74, Proposition 2.1) the following theorem is proved.

Theorem 4.9. *Let $\mathcal{C}$ be a plane branch of the equisingularity class (n,m) that is not quasihomogeneous and let λ be its Zariski invariant. Then, after an analytic change of coordinates, $\mathcal{C}$ satisfies an equation of the following form*

$$y^n - x^m + x^p y^q + \sum_{\substack{(i,j)\in \mathrm{PZ} \\ ni+mj>nm}} a_{ij} x^i y^j = 0, \tag{42}$$

where $(p,q) \in$ PZ is the point associated to λ through the bijection beetwen PZ *and* ZI.

Using Lemma 4.8 it is possible to refine the bound given in Proposition 4.5.

Proposition 4.10. *Let $\mathcal{F} \in \mathfrak{F}_\pi$ be the foliation induced by $\{\omega_{\mathcal{F}} = 0\}$. If there exists $\mathcal{G}^\star \in \mathfrak{F}_\pi^C$ such that*

$$\nu^\pi_{E_{dic}}(\omega_{\mathcal{F}} \wedge \omega_{\mathcal{G}^\star}) - \nu^\pi_{E_{dic}}(\omega_{\mathcal{F}}) \geq c-1, \tag{43}$$

where c is the conductor of the semigroup, then all the non-isolated separatrices of $\mathcal{F}$ are (n,m)-quasihomogeneous.

Proof. Let $\mathcal{C}_{\mathcal{F}}$ be a non-isolated separatrix of $\mathcal{F}$ and let φ be a parameterization of $\mathcal{C}_{\mathcal{F}}$. By Theorem 3.8

$$\mathrm{ord}_t \varphi^* \omega_{\mathcal{G}} = \nu^\pi_{E_{dic}}(\omega_{\mathcal{F}} \wedge \omega_{\mathcal{G}}) - \nu^\pi_{E_{dic}}(\omega_{\mathcal{F}}) + \iota_p(\tilde{\mathcal{C}}_{\mathcal{G}}, \tilde{\mathcal{J}}(\mathcal{F},\mathcal{G})).$$

Since, by hypothesis there exists $\mathcal{G}^\star \in \mathfrak{F}_\pi^C$ such that $\nu^\pi_{E_{dic}}(\omega_{\mathcal{F}} \wedge \omega_{\mathcal{G}^\star}) - \nu^\pi_{E_{dic}}(\omega_{\mathcal{F}}) \geq c-1$, then we have

$$\begin{aligned} \mathrm{ord}_t \varphi^* \omega_{\mathcal{G}^\star} &= \nu^\pi_{E_{dic}}(\omega_{\mathcal{F}} \wedge \omega_{\tilde{\mathcal{G}}}) - \nu^\pi_{E_{dic}}(\omega_{\mathcal{F}}) + \iota_p(\tilde{\mathcal{C}}_{\mathcal{G}^\star}, \tilde{\mathcal{J}}(\mathcal{F},\mathcal{G}^\star)) \\ &\geq c - 1 + \iota_p(\mathcal{C}_{\tilde{\mathcal{G}}}, \tilde{\mathcal{J}}(\mathcal{F},\mathcal{G}^\star)) \geq c-1. \end{aligned}$$

This implies that $\mathrm{ord}_t \varphi^* \omega_{\mathcal{G}^\star} + 1 - n \geq c - 1 + 1 - n = c - n$. Since the conductor of the semigroup of a plane branch in the equisingularity class (n,m) is $c = (m-1)(n-1)$, then $c - n = mn - 2n - m + 1$. By Lemma 4.8, the Zariski invariant associated to the extreme point $(m-2, n-2)$ is $\lambda = n(m-2) + m(n-2) - mn + m = mn - 2n + m$ and this is the last invariant for the equisingularity class (n,m). Hence the separatrix $\mathcal{C}_{\mathcal{F}}$ is analytically equivalent to the quasi-homogeneous branch. Since $\mathcal{C}_{\mathcal{F}}$ is an arbitrary non-isolated separatrix of $\mathcal{F}$, the proposition is proved.

If we consider the foliations induced by the equations

$$\mathcal{F}_1 = \left\{ d\left(\frac{y^n - x^p y^q}{x^m}\right) = 0 \right\} \quad \text{and} \quad \mathcal{F}_2 = \left\{ d\left(\frac{y^n}{x^m - x^p y^q}\right) = 0 \right\}, \tag{44}$$

the non-isolated separatrices of the foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ are given by the branches

$$\mathcal{C}_{k,1} = \{y^n - x^p y^q - kx^m = 0\} \quad \text{and} \quad \mathcal{C}_{k,2} = \{y^n - k(x^m - x^p y^q) = 0\}.$$

According to Theorem 4.9, all the non-isolated separatrices of $\mathcal{F}_1$ and $\mathcal{F}_2$ have the same Zariski invariant λ, which is the associated to $(p,q) \in \mathrm{PZ}$. Moreover after straightforward computations it is possible to show that both foliations $\mathcal{F}_1, \mathcal{F}_2$ belong to the family $\mathfrak{F}^\star_\pi$.

The following proposition gives an explicit family of dicritical foliations that belong to $\mathfrak{F}_\pi$.

Proposition 4.11. *Let $(p,q) \in \mathrm{PZ}$. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be the foliations induced by $\{\omega_{\mathcal{F}_1} = 0\}$ and $\{\omega_{\mathcal{F}_2} = 0\}$ respectively, where*

$$\omega_{\mathcal{F}_1} = \left(a_1 y^{n-q+1} + a_2 x^p y + yA(x,y)\right) dx + \left(b_1 y^{n-q} x + b_2 x^{p+1} + xB(x,y)\right) dy,$$

with $na_1 + mb_1 = 0$, $na_2 + mb_2 \neq 0, \frac{a_2}{b_2} \in \mathbb{C} \setminus \mathbb{Q}$ and $A(x,y), B(x,y) \in \mathbb{C}\{x,y\}$ with order greater than or equal to $p+1$ and

$$\omega_{\mathcal{F}_2} = \left(\tilde{a}_1 x^{m-p} y + \tilde{a}_2 y^{q+1} + y\tilde{A}(x,y)\right) dx + \left(\tilde{b}_1 x^{m-p+1} + \tilde{b}_2 x y^q + x\tilde{B}(x,y))\right) dy,$$

with $m\tilde{a}_1 + n\tilde{b}_1 = 0$, $m\tilde{a}_2 + n\tilde{b}_2 \neq 0, \frac{\tilde{a}_2}{\tilde{b}_2} \in \mathbb{C} \setminus \mathbb{Q}$ and $\tilde{A}(x,y), \tilde{B}(x,y) \in \mathbb{C}\{x,y\}$ with order greater than or equal to $\max\{q+1, m-p\}$. Then the foliations $\mathcal{F}_1, \mathcal{F}_2$ belong to $\mathfrak{F}_\pi$. Moreover both foliations do not have saddle-node singularities.

Lemma 4.12. *Let $(p,q) \in \mathrm{PZ}$ and let $t, l \in \mathbb{N}^*$ with $t < m$, $l < n$ such that $tn - lm = 1$, then $\frac{p}{n-q} \geq \frac{t}{l}$.*

Proof. First we note that every other solution $(\tilde{t}, \tilde{l})$ of the equation $nx - my = 1$ is given by $\tilde{t} = t + k_0 m$ and $\tilde{l} = l + k_0 n$ for some $k_0 \in \mathbb{Z}$. Since (p,q) belongs to the set PZ, then $np - m(n-q) > 0$. Let $s \in \mathbb{N}^*$ such that $np - m(n-q) = s$. Since $s > 0$ then the solutions of the equation $nx - my = s$ are given by $x = st + k_0 m$ and $y = sl + k_0 n$ for $k_0 \in \mathbb{Z}$. Since (p,q) is a solution then there exists $k_0 \in \mathbb{Z}$ such that $p = st + k_0 m$ and $n - q = sl + k_0 n$.

If $k_0 \geq 1$ then $p > m$ and $n - q > n$ but this is a contradiction.

If $k_0 = 0$ then $\frac{p}{n-q} = \frac{st}{sl} = \frac{t}{l}$ and we finish the proof in this case.

Finally, if $k_0 \in \mathbb{Z}_{<0}$ then

$$pl - (n-q)t = lst + k_0 ml - (lst + k_0 nt) = k_0 (lm - nt) = -k_0 s.$$

Since $k_0 < 0$ then $-k_0 s > 0$ and this implies that $\frac{p}{n-q} > \frac{t}{l}$. This proves the lemma.

Proof of Proposition 4.11. We need to prove that $\mathcal{F}_1$ satisfies the two properties of the family $\mathfrak{F}_\pi$ (see Definition 2.10). Since the system of coordinates (x,y) is adapted to π, the property ii) in the definition is satisfied because the curves $\mathcal{L}_0 = \{y = 0\}$ and $\mathcal{L}_1 = \{x = 0\}$ are invariant for $\mathcal{F}_1$, and are pairwise transverse.

It remains to prove that $\mathcal{F}_1$ has one dicritical component that appears in the last blowing-up morphism of the sequence of blowing-up morphisms given by the Euclidean algorithm of the pair (n,m). As g.c.d$(n,m) = 1$, by Bezout's identity there exist $t, l \in \mathbb{N}^*$ such that $nt - lm = 1$. We consider $t, l \in \mathbb{N}^*$ such that $l < n$ and $t < m$. After straightforward computations we have that $n - l \leq m - t$ and $l < t$.

We consider the following morphism

$$\pi(u,v) = (u^{n-l} v^l, u^{m-t} v^t) = (x,y).$$

Let us compute $\pi^* \omega_{\mathcal{F}_1}$. For this sake, let us denote by $\alpha, \beta, \gamma, \delta$ the following expressions: $\alpha = (m-t)(n-q+1) + n - l$, $\beta = t(n-q+1) + l$, $\gamma = (n-l)(p+1) + m - t$, and $\delta = l(p+1) + t$. Therefore,

$$\pi^*\omega_{\mathcal{F}_1} = \Big(c_1 u^{\alpha-1}v^{\beta} + c_2 u^{\gamma-1}v^{\delta} + u^{\gamma+n-l-1+}v^{\delta+l}\Big((n-l)\hat{A}+(m-t)\hat{B}\Big)\Big)\,du$$
$$+\Big(d_1 u^{\alpha}v^{\beta-1} + d_2 u^{\gamma}v^{\delta-1} + u^{\gamma+n-l}v^{\delta+l-1}\Big(l\hat{A}+t\hat{B}\Big)\Big)\,dv,$$

where $c_1 = a_1(n-l)+b_1(m-t)$, $c_2 = a_2(n-l)+b_2(m-t)$, $d_1 = la_1+tb_1$, $d_2 = la_2+tb_2$, $\pi^*A = A\circ\pi = u^{(n-l)(p+1)}v^{l(p+1)}\hat{A}$, and $\pi^*B = B\circ\pi = u^{(n-l)(p+1)}v^{l(p+1)}\hat{B}$. Note that by the assumption $na_1+mb_1=0$ it follows that $c_1=-d_1$.

On the other hand we have that $\frac{p}{n-q} > \frac{m}{n} > \frac{m-t}{n-1}$, this implies that $p(n-l)-(m-t)(n-q)>0$, hence

$$\gamma-\alpha = p(n-l)-(m-t)(n-q) > 0.$$

So, by Lemma 4.12 we have that $pl-t(n-q)\geq 0$ and this implies that $\delta-\beta\geq 0$. By dividing by $u^{\alpha-1}v^{\beta-1}$ we have that

$$\tilde{\omega}_{\mathcal{F}_1} = \Big(-d_1 v + c_2 u^{\gamma-\alpha}v^{\delta-\beta+1} + u^{\gamma-\alpha+n-l}v^{\delta-\beta+l+1}\Big((n-l)\hat{A}+(m-t)\hat{B}\Big)\Big)\,du$$
$$+\Big(c_1 u + d_2 u^{\gamma-\alpha+1}v^{\delta-\beta} + u^{\gamma-\alpha+n-l+1}v^{\delta-\beta+l}\Big(l\hat{A}+t\hat{B}\Big)\Big)\,dv.$$

Note that the degree of the monomials $u^{\gamma-\alpha}v^{\delta-\beta+1}$, $u^{\gamma-\alpha+1}v^{\delta-\beta}$ is greater than or equal to 2, so this point is a dicritical singularity because $-c_1uv+c_1uv=0$ and we have finished the proof. The proof for the foliation $\mathcal{F}_2$ is analogous.

Under the assumptions, it is possible prove that $\mathcal{F}_1$ does not have saddle-node singularities. Let σ be the sequence of blowing-up morphisms determined by the Euclidean algorithm of the pair $(p,n-q)$. We will prove that there exists a finite number of singular points of $\sigma^*\mathcal{F}_1$ on the last irreducible component of $\sigma^{-1}(0)$, such that each of such points is a simple singularity, and except for at most one, there are no saddle-node singularities with the possible exception of one of these points. By Bezout's identity there exist $r,s\in\mathbb{N}^*$ such that $rp-s(n-q)=\varrho$ where ϱ:=g.c.d$(p,n-q)$. We assume that $r\leq s$, in the case that $r>s$ the computations are analogous. Let us consider the following morphism

$$\sigma(u,v) = (u^r v^{\frac{n-q}{\varrho}}, u^s v^{\frac{p}{\varrho}}),$$

and compute the induced form $\sigma^*\omega_{\mathcal{F}_1}$, let us denote by α, β, γ, δ the following expresions: $\alpha = s(n-q+1)+r-1$, $\beta = \frac{n-q}{\varrho}(p+1)+\frac{p}{\varrho}-1$, $\gamma = \varrho + r$ and $\delta = \frac{n-q}{\varrho}$

$$\sigma^*\omega_{\mathcal{F}_1} = u^{\alpha}v^{\beta}\left[\left(k_1v + k_2u^{\varrho}v + u^{\gamma}v^{\delta+1}C_1\right)du + \left(l_1u + l_2u^{\rho+1} + u^{\gamma+1}v^{\delta}D_1\right)dv\right],$$

where

$$k_1 = (ra_1+sb_1)\,,\quad k_2 = (ra_2+sb_2)\,,\quad l_1 = \left(\frac{n-q}{\rho}a_1 + \frac{p}{\rho}b_1\right),\quad l_2 = \left(\frac{n-q}{\varrho}a_2+\frac{p}{\varrho}b_2\right),$$

$$u^{r(p+1)}v^{\frac{n-q}{\varrho}(p+1)}A_1 = A\circ\sigma,\quad u^{r(p+1)}v^{\frac{n-q}{\varrho}(p+1)}B_1 = B\circ\sigma,$$

and $C_1 = rA_1+sB_1$, $D_1 = \frac{n-q}{\rho}A_1+\frac{p}{\rho}B_1$. We denote by $\hat{\omega}_{\mathcal{F}_1}$ the strict transform of $\omega_{\mathcal{F}_1}$ by σ.

The singular points for $\sigma^*\omega_{\mathcal{F}_1}$ are given by $v=0$ and the solutions of the equation

$$\left(\frac{n-q}{\varrho}a_1+\frac{p}{\varrho}b_1\right)u + \left(\frac{n-q}{\varrho}a_2+\frac{p}{\varrho}b_2\right)u^{\varrho+1} = 0.$$

Note that $u=0$ is a solution of the previous equation. If $u\neq 0$ then we have

$$u^{\varrho} = -\frac{(n-q)a_1+pb_1}{(n-q)a_2+pb_2}. \tag{45}$$

Let ξ_i, $i=1,2,...,\varrho$ be the solutions of equation (45) and let us consider the change of variables

$$u_i = u - \xi_i, \quad v_i = v,$$

for each $i \in \{1,2,...,\varrho\}$. We analyze the restriction of the form $\hat{\omega}_{\mathcal{F}_1}$ to each singular point $z_i=(\xi_i,0)$ and we prove, that under the conditions of the Proposition 4.11 the singular points are simple. Namely, at the point $z_i=(\xi_i,0)$ the equation (4) takes the form

$$\begin{aligned}\hat{\omega}_{\mathcal{F}_1}|_{z_i} &= (k_1+k_2\xi_i^{\varrho})\, v_i du_i + (l_1+(\varrho+1)l_2\xi_i^{\varrho})\, u_i dv_i + \left(v_i h_1 + v_i^{\frac{n-q}{\varrho}} g_1\right) du_i \\ &\quad + \left(h_2(u_i) + v_i^{\frac{n-q}{\varrho}+1} g_2\right) dv_i,\end{aligned}$$

where $h_1(u_i)=\sum_{j=1}^{\rho} c_j u_i^{\varrho-j}\xi_i^{j}$, $h_2(u_i)=\sum_{j=2}^{\rho} d_j u_i^{j}\xi_i^{\varrho-j}$, with $c_j, d_j \in \mathbb{C}$, and

$$g_1 = (u_i+\xi_i)^{\varrho+r}(rA_1(u_i+\xi_i,v_i)+sB_1(u_i+\xi_i,v_i)),$$

and

$$g_2 = (u_i+\xi_i)^{\varrho+r-1}\left(\frac{n-q}{\varrho}A_1(u_i+\xi_i,v_i)+\frac{p}{\rho}B_1(u_i+\xi_i,v_i)\right).$$

Let $X_{\mathcal{F}_1}$ be a vector field generating the same foliation at the point z_i. Its linear part is given by

$$DX_{\mathcal{F}_1}|_{z_i} = \begin{pmatrix} (n-q)a_1+pb_1 & \zeta \\ 0 & (ra_1+sb_1+(ra_2+sb_2)\,\xi_i^{\rho}) \end{pmatrix}.$$

Since $na_1+mb_1=0$, the quotient $\frac{\lambda_2}{\lambda_1}$ is,

$$\frac{\lambda_2}{\lambda_1} = \frac{sn-rm}{np-m(n-q)} - \frac{ra_2+sb_2}{(n-q)a_2+pb_2}.$$

If $\frac{ra_2+sb_2}{(n-q)a_2+pb_2}\in\mathbb{Q}$, then $\frac{a_2}{b_2}\in\mathbb{Q}$, which is a contradiction to the hypothesis $\frac{a_2}{b_2}\in\mathbb{C}\setminus\mathbb{Q}$. Thus, $\frac{ra_2+sb_2}{(n-q)a_2+pb_2}\notin\mathbb{Q}$; and in particular $\frac{ra_2+sb_2}{(n-q)a_2+pb_2}\neq 0$. This implies that $\frac{\lambda_1}{\lambda_2}\notin\mathbb{Q}_{>0}$ and this proves that every point $z_i=(\xi_i,0)$ for $i=1,2,...,\varrho$ is a simple singularity.

Now we analyze the singular point $(u,v)=(0,0)$. If we look for a vector field $X_{\mathcal{F}_1}$ generating the same foliation in a neighborhood of the point $(u,v)=(0,0)$, the linear part is given by

$$DX_{\mathcal{F}_1}|_{(0,0)} = \begin{pmatrix} \left(\frac{(n-q)}{\varrho}a_1+\frac{p}{\varrho}b_1\right) & 0 \\ 0 & -(ra_1+sb_1) \end{pmatrix}.$$

The quotient of the eigenvalues is

$$\frac{\lambda_1}{\lambda_2} = -\frac{1}{\varrho}\left(\frac{(n-q)a_1+pb_1}{ra_1+sb_1}\right).$$

Since $na_1+mb_1=0$ we have that $\frac{a_1}{b_1}=-\frac{m}{n}$ and then the quotient is

$$\frac{\lambda_1}{\lambda_2} = -\frac{1}{\rho}\left(\frac{np+mq-mn}{ns-rm}\right).$$

Since $(p,q)\in\mathrm{PZ}$, then $np+mq-mn>0$, so the posibility of the quotient $\frac{\lambda_1}{\lambda_2}$ to be a positive rational or not, depends on the sign of $ns-rm$. If $ns-rm>0$ then the point $(0,0)$ is a simple point and we have finished the proof. If $ns-rm<0$ then $\frac{\lambda_1}{\lambda_2}\in\mathbb{Q}_{>0}$ and this point leads to the dicritical component.

At the origin of the other chart $(\tilde{u},\tilde{v})$, where $\tilde{u}=\frac{1}{v}$, the unique singular point is the origin $(\tilde{u},\tilde{v})=(0,0)$. Namely, the equation at that point is given by

$$\begin{aligned}\hat{\omega}_{\mathcal{F}_1}|_{(0,0)} \;=\;& \left(l_1\tilde{v}^{\varrho+1}+l_2\tilde{v}\right)d\tilde{u}+\left(\left(l_1-k_1\right)\tilde{u}\tilde{v}^{\varrho}+\left(l_2-k_2\right)\tilde{u}\right)d\tilde{v}\\ &+\left(\tilde{u}^{\frac{n-q}{\varrho}}\tilde{v}^{\frac{n-q}{\varrho}-r+1}\left(\frac{n-q}{\rho}A_2+\frac{p}{\varrho}B_2\right)\right)d\tilde{u}\\ &+\left(\tilde{u}^{\frac{n-q}{\varrho}+1}\tilde{v}^{\frac{n-q}{\varrho}-r}\left(\left(\frac{n-q}{\varrho}-r\right)A_2+\left(\frac{p}{\varrho}-s\right)B_2\right)\right)d\tilde{v}.\end{aligned}$$

If we look for a vector field $X_{\mathcal{F}_1}$ generating the same foliation in the point $(\tilde{u},\tilde{v})=(0,0)$, the linear part is given by

$$DX_{\mathcal{F}_1}|_{(0,0)}=\begin{pmatrix}\left(\frac{n-q}{\varrho}a_2+\frac{p}{\varrho}b_2-ra_2-sb_2\right) & 0\\ 0 & -\left(\frac{n-q}{\varrho}a_2+\frac{p}{\varrho}b_2\right).\end{pmatrix}$$

The quotient of the eigenvalues is

$$\frac{\lambda_1}{\lambda_2}=-1+\rho\frac{ra_2+sb2}{(n-q)a_2+pb_2}.$$

If $\frac{ra_2+sb2}{(n-q)a_2+pb_2}\in\mathbb{Q}$ then $\frac{a_2}{b_2}\in\mathbb{Q}$, but this is a contradiction since by hypothesis $\frac{a_2}{b_2}\in\mathbb{C}\setminus\mathbb{Q}$; therefore, $\frac{\lambda_1}{\lambda_2}\notin\mathbb{Q}_{>0}$.

The following theorem shows that the family $\mathcal{F}_\pi^\star$ is not empty.

Theorem 4.13. *Let $(p,q)\in$ PZ. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be the foliations induced by $\{\omega_{\mathcal{F}_1}=0\}$ and $\{\omega_{\mathcal{F}_2}=0\}$ respectively, where*

$$\omega_{\mathcal{F}_1}=\left(a_1y^{n-q+1}+a_2x^py+yA(x,y)\right)dx+\left(b_1y^{n-q}x+b_2x^{p+1}+xB(x,y)\right)dy,$$

with $na_1+mb_1=0$, $na_2+mb_2\neq 0,\frac{a_2}{b_2}\in\mathbb{C}\setminus\mathbb{Q}$ and $A(x,y),B(x,y)\in\mathbb{C}\{x,y\}$ with order greater than or equal to $p+1$ and

$$\omega_{\mathcal{F}_2}=\left(\tilde{a}_1x^{m-p}y+\tilde{a}_2y^{q+1}+y\tilde{A}(x,y)\right)dx+\left(\tilde{b}_1x^{m-p+1}+\tilde{b}_2xy^q+x\tilde{B}(x,y))\right)dy,$$

with $m\tilde{a}_1+n\tilde{b}_1=0$, $m\tilde{a}_2+n\tilde{b}_2\neq 0,\frac{\tilde{a}_2}{\tilde{b}_2}\in\mathbb{C}\setminus\mathbb{Q}$ and $\tilde{A}(x,y),\tilde{B}(x,y)\in\mathbb{C}\{x,y\}$ with order greater than or equal to $\max\{q+1,m-p\}$. Then the foliations $\mathcal{F}_1$, $\mathcal{F}_2$ belong to the subfamily $\mathfrak{F}_\pi^\star$. Moreover, the Zariski invariant of all the non-isolated invariant branches of both foliations $\mathcal{F}_1,\mathcal{F}_2$ is the value λ associated to the point (p,q) in the set PZ.

The following lemma gives a procedure to determine the intersection of the polar curve of two foliations, one in $\mathfrak{F}_\pi$ and the other in $\mathfrak{F}_\pi^C$ and the corresponding irreducible component of the divisor, in terms of the *Newton polygon* of the polar curve.

We recall that, given $h\in\mathbb{C}\{x,y\}$, with $h=\sum h_{ij}x^iy^j$, the *support* of h is by definition

$$\operatorname{Supp}(h;x,y):=\left\{(i,j)\in\mathbb{Z}^2_{\geq 0}\mid h_{ij}\neq 0\right\}. \tag{46}$$

The *Newton polygon* of h, that we denote by $\mathcal{N}(h;x,y)$, with respect to the coordinates x,y is by definition the convex hull of the set $\operatorname{Supp}(h;x,y)+\mathbb{R}^2_{\geq 0}$. Given the curve $\mathcal{H}=\{h=0\}$ the Newton polygon of $\mathcal{H}$ is $\mathcal{N}(h;x,y)$.

Lemma 4.14. *Let $\pi:(M,D)\to(\mathbb{C}^2,0)$ be a finite sequence of blowing-up morphisms, described by the Euclidean algorithm of the pair (n,m), with the exceptional divisor $D=\pi^{-1}(0)$. Let E_{dic} be the irreducible component of the divisor $D=\pi^{-1}(0)$ that appears in the last blowing-up* *morphism of the sequence π. Let $\mathcal{J}(\mathcal{F},\mathcal{G})$ be the polar curve between the foliations $\mathcal{F}$ and $\mathcal{G}$, for $\mathcal{F}\in\mathfrak{F}_\pi$ and $\mathcal{G}\in\mathfrak{F}_\pi^C$. We denote by $\tilde{\mathcal{J}}(\mathcal{F},\mathcal{G})$ the strict transform of $\mathcal{J}(\mathcal{F},\mathcal{G})$ by π. Then the following statements are equivalent*

(i) $\tilde{\mathcal{J}}(\mathcal{F},\mathcal{G})\cap E_{dic}\neq\emptyset$ at a point p that is not a corner point.

(ii) There exists a coordinate system (x,y) adapted to π, such that the Newton polygon of the Jacobian curve $\mathcal{N}(\mathcal{J}(\mathcal{F},\mathcal{G});x,y)$ has exactly one side of slope $-\frac{n}{m}$.

(iii) For every coordinate systems (x,y) adapted to π, the Newton polygon of the Jacobian curve $\mathcal{N}(\mathcal{J}(\mathcal{F},\mathcal{G});x,y)$ has exactly one side of slope $-\frac{n}{m}$.

Proof of Lemma 4.14. That *iii*) implies *ii*) is immediate.

Now we prove the implication *i*) if and only if *ii*). Let (x,y) be a system of coordinates adapted to π. Let us recall that π is a finite sequence of blowing-up morphisms. We denote by k the length of the sequence of blowing-up morphisms. Let p_i, for $i=0,1,...,k$, be the center of $\pi_i:(M_i,\pi_i^{-1}(p_{i-1}))\to(M_{i-1},p_{i-1})$, where $p_0=(0,0)$ is the origin and $M_0=(\mathbb{C}^2,0)$. For each $l\in\{1,2,...,k\}$ we consider (x_l,y_l) a system of coordinates adapted to p_l. Note that the blowing-up morphism π_l is given at the point p_l by one the following equations

$$\sigma_{l,1}(x_l,y_l)=(x_l,x_ly_l),\quad\text{or}\quad \sigma_{l,2}(x_l,y_l)=(x_ly_l,y_l),\tag{47}$$

and associated with these equations we have two affine transformations $T_s:\mathbb{R}^2\to\mathbb{R}^2$, for $s=1,2$, given by,

$$T_1(i,j)=(i+j,j),\quad\text{or}\quad T_2(i,j)=(i,i+j).\tag{48}$$

For every $l=1,2,...,k$, the Newton polygon $\mathcal{N}_l(\mathcal{J}^{(l)}(\mathcal{F},\mathcal{G});x_l,y_l)$, where $\mathcal{J}^{(1)}(\mathcal{F},\mathcal{G})$ denotes the strict transform of $\mathcal{J}^{(l-1)}(\mathcal{F},\mathcal{G})$ by π_l. Note that this is the convex hull of the set

$$T_s(\mathrm{Supp}(\mathcal{J}^{l-1}(\mathcal{F},\mathcal{G});x_{l-1},y_{l-1})+\mathbb{R}^2_{\geq 0}$$

for $l=1,2,...,k$, $s\in\{1,2\}$, where $x_0=x$, $y_0=y$ and π_0 is the identity. After straightforward computations we have that the Newton polygon $\mathcal{N}(\mathcal{J}(\mathcal{F},\mathcal{G});x,y)$ has a side of slope $-\frac{n}{m}$ if and only if the Newton polygon $\mathcal{N}_{k-1}(\mathcal{J}^{(k-1)}(\mathcal{F},\mathcal{G});x_{k-1},y_{k-1})$ has a side of slope -1.

Let $\mathcal{J}(\mathcal{F},\mathcal{G})=\{h=0\}$ be the polar curve between $\mathcal{F}$ and $\mathcal{G}$. If we consider the composition $\sigma=\pi_1\circ\pi_2\circ\cdots\circ\pi_{k-1}$, then $h\circ\sigma=x_{k-1}^{\alpha}y_{k-1}^{\beta}\tilde{h}(x_{k-1},y_{k-1})$, where $\tilde{h}(0,y_{k-1})\neq 0\neq\tilde{h}(x_{k-1},0)$. We write $\tilde{h}=\tilde{h}_r+\tilde{h}_{r+1}+\cdots$, where $\tilde{h}_i\in\mathbb{C}[x_{k-1},y_{k-1}]$ are homogeneous polynomials of degree i and $\tilde{h}_r$ is the first not identically vanishing polynomial and the suspensive dots represent terms of higher order than $r+2$. We write $\tilde{h}_r(x_{k-1},y_{k-1})=x_{k-1}^{\gamma}y_{k-1}^{\delta}g(x_{k-1},y_{k-1})$ where g is a polynomial of degree $r-\gamma-\delta$.

Let π_k be the blowing-up morphism at the point p_{k-1}, where $\pi_k(x_k,y_k)=(x_k,x_ky_x)$. The transform of $\tilde{h}$ by π_k is,

$$\begin{aligned}\pi_k^*(\tilde{h})&=\tilde{h}_r(x_k,x_ky_k)+\tilde{h}_{r+1}(x_k,x_ky_k)+\cdots=x_k^ry_k^{\delta}g(1,y_k)+x_k^{r+1}\tilde{p}_{r+1}\cdots\\&=x_k^r\left(y_k^{\delta}g(1,y_k)+x_k\tilde{p}_{r+1}+\cdots\right),\end{aligned}$$

where the multiple dots represent terms of order greater or equal to $r+2$. Note that the exceptional divisor is given by $x_k=0$, thus except for corner points, we have $\tilde{\mathcal{J}}(\mathcal{F},\mathcal{G})\cap E_{dic}\neq\emptyset$ if and only if there exists $\xi\in\mathbb{C}^*$ such that $g(1,\xi)=0$. Let us recall that given a polynomial $q\in\mathbb{C}[T]$, there exists $\xi\in\mathbb{C}$ such that $q(\xi)=0$ if and only if the degree of q is greater than or equal to 1. We will show that the degree of g is greater than or equal to 1. If the degree of g is smaller than 1, that is to say, the polynomial g is constant, then the Newton polygon $\mathcal{N}_{k-1}(\mathcal{J}^{(k-1)}(\mathcal{F},\mathcal{G});x_{k-1},y_{k-1})$ consists of one point; this leads a contradiction because then the Newton polygon $\mathcal{N}(\mathcal{J}(\mathcal{F},\mathcal{G});x,y)$ does not have a side whose slope is $-\frac{n}{m}$. Hence, the degree of the polynomial g is greater than or equal to 1. Moreover, since the Newton polygon

$\mathcal{N}_{k-1}(\mathcal{J}^{(k-1)}(\mathcal{F},\mathcal{G});x_{k-1},y_{k-1})$ has a side whose slope is -1, then it is possible to find $\xi \in \mathbb{C}^*$ such that $g(1,\xi)=0$. That i) implies iii) follows from analogous arguments to the previous one. This finishes the proof of Lemma 4.14.

Proof of Theorem 4.13. Let $\mathcal{F}_1$ be the foliation induced by the equation $\{\omega_{\mathcal{F}_1}=0\}$, where

$$\omega_{\mathcal{F}_1}=\big(a_1y^{n-q+1}+a_2x^py+yA(x,y)\big)\,dx+\big(b_1y^{n-q}x+b_2x^{p+1}+xB(x,y)\big)\,dy,$$

and let $\mathcal{G}\in\mathfrak{F}_\pi^C$ be the foliation induced by the equation $\{\omega_{\mathcal{G}}=0\}$

$$\omega_{\mathcal{G}}=(my+h_1(x,y))\,dx-(nx+cy+h_2(x,y))\,dy.$$

We compute

$$\omega_{\mathcal{F}_1}\wedge\omega_{\mathcal{G}}=h(x,y)dx\wedge dy.$$

We will show that for every $\mathcal{G}\in\mathfrak{F}_\pi^C$, the Newton polygon of the polar curve $\mathcal{J}(\mathcal{G},\mathcal{F})=\{h=0\}$ does not have a side whose slope is $-\frac{n}{m}$. Thus, by Lemma 4.14, we conclude that $\mathcal{F}\in\mathfrak{F}_\pi^\star$.

After forthright computations we have that

$$\omega_{\mathcal{F}_1}\wedge\omega_G=\big[-kx^{p+1}y+ca_1y^{n-q+2}+H_0(x,y)+H_2(x,y)+H_1(x,y)\big]\,dx\wedge dy,$$

where $k=na_2+mb_2\neq 0$, and

$$\begin{aligned}H_0(x,y)&=xy\left(ca_2x^{p-1}y+nA(x,y)+mB(x,y)\right),\\ H_1(x,y)&=h_1(x,y)\left(b_1y^{n-q}x+b_2x^{p+1}+xB(x,y)\right),\\ H_2(x,y)&=h_2(x,y)\left(a_1y^{n-q+1}+a_2x^py+yA(x,y)\right).\end{aligned}$$

Now we recall that $\mathrm{ord}_0h_i(x,y)\geq 2$ for $i=1,2$ and $\mathrm{ord}_xh_1(x,0)\geq\lfloor\frac{m}{n}\rfloor+1=\alpha_0+1$. We write $\mathrm{ord}_xh_1(x,0)=\alpha_0+k$, with $k\geq 1$. We analyze the Newton polygon $\mathcal{N}(h;x,y)$; we need to focus on four cases. Namely,

(i) Assume that there exists a side ℓ of the Newton polygon $\mathcal{N}(h;x,y)$ whose slope is $-\frac{n}{m}$, and let (α,β), $(p+s,0)$ be the initial and final points, respectively, where $s=\alpha_0+k+1$. The equation of this side is given by

$$y=-\frac{\beta}{p+s-\alpha}x+\frac{\beta}{p+s-\alpha}(p+s).$$

We assume that for $x=p+1$ we have that $y\leq 1$. This implies that

$$-\frac{\beta}{p+s-\alpha}(p+1)+\frac{\beta}{p+s-\alpha}(p+s)\leq 1. \tag{49}$$

Since $-\frac{\beta}{p+s-\alpha}=-\frac{n}{m}$, then the equation (49) gives,

$$-\frac{n}{m}(p+1)+\frac{n}{m}(p+s)\leq 1$$

or equivalentely,

$$n(s-1)\leq m.$$

Moreover, since $s\geq\alpha_0+2$, then $s-1\geq\alpha_0+1$. Therefore,

$$m\geq n(s-1)\geq n(\alpha_0+1).$$

We stress that this is a contradiction because $m=\alpha_0n+r_0$ with $r_0<n$. Consequently, there is no side of the Newton polygon $\mathcal{N}(h;x,y)$ with slope $-\frac{n}{m}$ and extreme points (α,β), $(p+s,0)$.

(ii) Suppose that the Newton polygon $\mathcal{N}(h;x,y)$ has a side ℓ with extreme points $(p+1,1)$, $(p+s,0)$ and slope $-\frac{n}{m}$. Then the slope of this side is $-\frac{1}{s-1}$. This implies that $-\frac{n}{m}=-\frac{1}{s-1}$, or equivalently, $n(s-1)=m$, but this is a contradiction because g.c.d.$(n,m)=1$. Hence, the Newton polygon $\mathcal{N}(h;x,y)$ does not have a side whose slope is $-\frac{n}{m}$ and extreme points $(p+1,1)$, $(p+s,0)$.

(iii) Assume that the Newton polygon $\mathcal{N}(h;x,y)$ has a side ℓ with extreme points (α,β), $(p+1,1)$ whose slope is $-\frac{n}{m}$. Then the slope of such side is $-\frac{\beta-1}{p+1-\alpha}$. Thus $-\frac{\beta-1}{p+1-\alpha}=-\frac{n}{m}$. This implies that

$$m(\beta-1)=n(p+1-\alpha). \tag{50}$$

Moreover, since g.c.d.$(n,m)=1$ then there exists $t\in\mathbb{Z}$ such that $\beta-1=tn$. Substituting this equality in (50) we have

$$tnm=n(p+1-\alpha).$$

Hence, $p=tm+\alpha-1$. We have now two possible cases

(a) If $\alpha=0$ then $p=tm-1$. Since $t\geq 1$ then $p\geq m-1$, but this is a contradiction because $p\leq m-2$.

(b) If $\alpha\geq 1$ then $p\geq tm$. Since $t\geq 1$ then $p\geq m$, but $p\leq m-2$ and this is a contradiction.

Therefore there is not a side of $\mathcal{N}(h;x,y)$ with slope $-\frac{n}{m}$ and extreme points (α,β), $(p+1,1)$.

(iv) Assume that the Newton polygon $\mathcal{N}(h;x,y)$ has a side ℓ whose slope is $-\frac{n}{m}$ with extreme points (α_1,β_1) and (α_2,β_2), $\alpha_1,\alpha_2<p+1$. The slope of this side ℓ is given by $-\frac{\beta_1-\beta_2}{\alpha_2-\alpha_1}$. By hypothesis $-\frac{n}{m}=-\frac{\beta_1-\beta_2}{\alpha_2-\alpha_1}$ and this implies $n(\alpha_2-\alpha_1)=m(\beta_1-\beta_2)$. Since g.c.d.$(n,m)=1$ this implies that $n|\,(\beta_1-\beta_2)$, thus, $\beta_2-\beta_1=tn$ for some $t\in\mathbb{Z}_+$.

Therefore $n\,(\alpha_2-\alpha_1)=mtn$ or equivalently, $(\alpha_2-\alpha_1)=mt$. Moreover, since $t\geq 1$, then $(\alpha_2-\alpha_1)\geq m$. On the other hand, $(\alpha_2-\alpha_1)<p+1$. So

$$p+1>\alpha_2-\alpha_1\geq m.$$

This last equation implies that $p>m-1$ but $p\leq m-2$ and this is a contradiction again. This implies that the Newton polygon $\mathcal{N}(h;x,y)$ does not has a side whose slope is $-\frac{n}{m}$ and extreme points (α_1,β_1) and (α_2,β_2) with $\alpha_1,\alpha_2<p+1$.

Then after analyzing the cases (i)-(iv) we conclude, by Lemma 4.14, that there do not exist a Jacobian curve intersecting the component E_{dic}. This shows that $\mathcal{F}\in\mathfrak{F}^{\star}_{\pi}$. This finishes the proof of Theorem 4.13.

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Oziel Gómez-Martínez, Instituto de Matemáticas, Universidad Nacional Autónoma de México (UNAM) Área de la Investigación Científica, Circuito exterior, Ciudad Universitaria, 04510, Ciudad de México, México.

Email address: `ogomez@im.unam.mx`