Journal *of* Singularities
Volume 22 (2020), 92-103

Proc. of 15th International Workshop on Singularities, São Carlos, 2018
DOI: 10.5427/jsing.2020.22f

UNLINKING SINGULAR LOCI FROM REGULAR FIBERS AND ITS APPLICATION TO SUBMERSIONS

OSAMU SAEKI

Dedicated to Professor Maria Aparecida Soares Ruas on the occasion of her 70th birthday

ABSTRACT. Given a null-cobordant oriented framed link L in a closed oriented 3–manifold M, we study the condition for the existence of a generic smooth map of M to the plane that has L as an oriented framed regular fiber such that the singular point set is unlinked with L. As an application, we give a singularity theoretical proof to the theorem, originally proved by Hector, Peralta-Salas and Miyoshi, about the realization of a link in an open oriented 3–manifold as a regular fiber of a submersion to the plane.

1. INTRODUCTION

Let M be a smooth closed oriented 3–dimensional manifold and $f : M \to \mathbf{R}^2$ a smooth map. If $y \in f(M) \subset \mathbf{R}^2$ is a regular value, then $f^{-1}(y)$ is an oriented link in M and is naturally framed. Furthermore, if f is generic enough, then the singular point set $S(f)$ of f is an unoriented link in $M \smallsetminus f^{-1}(y)$. In our previous paper [19], for an oriented framed link L in M, we characterized those unoriented links in $M \smallsetminus L$ which arise as the singular point set of a generic map that has L as an oriented framed regular fiber. Such a characterization was given in terms of a relative Stiefel–Whitney class, or an obstruction to extending the trivialization of $TM|_L$ induced by the framing over the whole manifold M.

In this paper, we first study the obstruction class more in detail, and give a more practical characterization in terms of $\mathbf{Z}_2$ linking numbers. We also clarify the components of L which have non-trivial $\mathbf{Z}_2$ linking numbers with the singular point set. Then, as an application of such studies, we consider submersions of open oriented 3–manifolds to $\mathbf{R}^2$ that realize given oriented framed links as regular fibers. The idea is to consider a generic map f whose singular point set $S(f)$ is unlinked with a given oriented framed regular fiber and to delete a neighborhood of the singular point set $S(f)$ for obtaining a submersion. In this way, we get a singularity theoretical proof to the characterization theorem, originally due to Hector and Peralta-Salas [9] and Miyoshi [14], of those oriented (framed) links in $\mathbf{R}^3$ that arise as regular fibers of submersions. Recall that their proofs used the h-principle for submersions due to Phillips [16]. Instead, in this paper, we arrange the singular point set by using Levine's cusp elimination techniques [12] (see also [18, 19]) in a controlled way and push it to infinity, so that we get a submersion.

The paper is organized as follows. In §2, we recall several definitions and terminologies together with our main theorem in [19], which describes the characterization of singular point sets as unoriented links in terms of a certain obstruction class. In §3, we study the obstruction class more in detail, especially for closed oriented 3–manifolds M with $H_*(M; \mathbf{Z}) \cong H_*(S^3; \mathbf{Z})$. In such a case, we can identify the obstruction class in terms of $\mathbf{Z}_2$ linking numbers. Then, we can describe the condition for the obstruction class to vanish in terms of $\mathbf{Z}_2$ linking numbers. Finally

2000 *Mathematics Subject Classification.* Primary 57R45; Secondary 57R30, 58K30, 57M25, 57R20.

Key words and phrases. Submersion, link, 3–manifold, excellent map, singular point set, regular fiber, relative Stiefel–Whitney class, framing.

in §4, we apply these results to submersions of open oriented 3–manifolds to $\mathbf{R}^2$. We will see that our singularity theoretical proof works well for punctured 3–manifolds, i.e. open 3–manifolds of the form $M^\circ = M \smallsetminus D^3$ obtained from a closed 3–manifold M by removing a small closed 3–disk D^3 in M. For a general open oriented 3–manifold, we need to use an "absolute version" of the h-principle due to Phillips. Recall that the original proof due to Hector and Peralta-Salas [9] or Miyoshi [14] used the "relative version", stronger than the "absolute version", of the h-principle [7].

Throughout the paper, manifolds and maps are differentiable of class C^∞ unless otherwise indicated. All (co)homology groups are with $\mathbf{Z}_2$–coefficients unless otherwise indicated. The symbol "$\cong$" means an appropriate isomorphism between algebraic objects or a diffeomorphism between smooth manifolds.

2. Preliminaries

Let M (resp. N) be a closed 3–dimensional manifold (resp. a possibly noncompact surface) and consider a map $f : M \to N$. We denote by $S(f)$ the set of singular points of f. A point in $S(f)$ is a *fold singularity* (or a *cusp singularity*) of f if the map germ of f at that point is modeled on the map germ $(x, y, z) \mapsto (x, y^2 \pm z^2)$ (resp. $(x, y, z) \mapsto (x, y^3 + xy - z^2)$) at the origin. We say that a fold singularity is *definite* (resp. *indefinite*) if it is modeled on the map germ $(x, y, z) \mapsto (x, y^2 + z^2)$ (resp. $(x, y, z) \mapsto (x, y^2 - z^2)$). We say that f is *excellent* if $S(f)$ consists only of fold and cusp singularities. It is known that the set of excellent maps is always open and dense in the mapping space $C^\infty(M, N)$ endowed with the Whitney C^∞ topology (for example, see [6, 21]). If f is an excellent map, then $S(f)$ is an (unoriented) link in M, i.e. a finite disjoint union of smoothly embedded circles.

Let $f : M \to N$ be a map. For a regular value $y \in f(M) \subset N$, we call $L = f^{-1}(y)$ a *regular fiber*, which is a link in $M \smallsetminus S(f)$. Note that L is naturally *framed*: its framing is given as the pull-back of the trivial normal framing of the point y in N. Furthermore, when M and N are oriented, L is naturally oriented.

In the following, we fix an orientation for $\mathbf{R}^2$ once and for all. For excellent maps of closed oriented 3–manifolds into $\mathbf{R}^2$, we have the following (for details, see [17, Proposition 5.1] and [19]).

Lemma 2.1. *Let L be an oriented framed link in a closed oriented 3–manifold M. Then, it is realized as an oriented framed regular fiber of an excellent map $f : M \to \mathbf{R}^2$ if and only if it is framed null-cobordant: i.e. there exists a compact oriented normally framed surface V embedded in M whose framed boundary coincides with L.*

Remark 2.2. Let L be an oriented link in a closed oriented 3–manifold M. Then, we can easily show that it bounds a compact oriented surface in M if and only if L represents zero in $H_1(M; \mathbf{Z})$. This can be proved by considering a certain map $M \smallsetminus L \to S^1$. In particular, if $H_1(M; \mathbf{Z}) = 0$, then every oriented link bounds a compact oriented surface embedded in M.

Remark 2.3. It is known that every link in the 3–sphere is realized as a regular fiber of a restriction to S^3 of a certain polynomial map $\mathbf{R}^4 \to \mathbf{R}^2$ (see [1]). Furthermore, in [4], for a given link in the 3–sphere, the authors give an explicit algorithm to construct a quasi-holomorphic polynomial $\mathbf{C}^2 \to \mathbf{C}$ whose restriction to the unit sphere S^3 has the link as a regular fiber.

Now, let L be an oriented framed link in a closed oriented 3–manifold. If L is realized as a framed regular fiber of an excellent map $f : M \to \mathbf{R}^2$, then $S(f)$ is a link in $M \smallsetminus L$. Thus, it is natural to ask the following.

QUESTION 2.4. Which links in $M \smallsetminus L$ appear as the singular point set $S(f)$ of an excellent map $f : M \to \mathbf{R}^2$ such that $f^{-1}(y)$ coincides with L as oriented framed links for some regular value $y \in \mathbf{R}^2$?

In order to answer to the above question, let us prepare some notations and terminologies. For an (unoriented) link J in $M \smallsetminus L$, we denote by $[J]_2 \in H_1(M \smallsetminus L)$ the $\mathbf{Z}_2$–homology class represented by J. Let $N(L)$ be a small tubular neighborhood of L in M disjoint from J. Since L is a framed link, we have a natural trivialization of $TM|_{N(L)}$. The obstruction to extending it over M is the relative Stiefel–Whitney class (see [10]), denoted by $w_2(M, L)$, which is an element of the $\mathbf{Z}_2$–cohomology group $H^2(M, N(L)) \cong H^2(M, L)$. Note that by excision and Poincaré–Lefschetz duality, we have

$$H^2(M, N(L)) \cong H^2(M \smallsetminus \operatorname{Int} N(L), \partial N(L)) \cong H_1(M \smallsetminus \operatorname{Int} N(L)) \cong H_1(M \smallsetminus L).$$

The following characterization, which answers to Question 2.4, has been proved in [19]. Recall that the proof was singularity theoretical in the sense that we used a result of Thom [20] about the homology class represented by the singular locus, and a cusp elimination result by Levine [12] for arranging the singular locus of an excellent map.

THEOREM 2.5. *Let L be an oriented null-cobordant framed link in a closed oriented 3–manifold M, and J an unoriented link in $M \smallsetminus L$. Then, there exist an excellent map $f : M \to \mathbf{R}^2$ and a regular value $y \in \mathbf{R}^2$ such that $f^{-1}(y)$ coincides with L as oriented framed links and that $S(f) = J$ if and only if $[J]_2 \in H_1(M \smallsetminus L)$ is Poincaré dual to $w_2(M, L) \in H^2(M, L)$.*

3. CASE OF INTEGRAL HOMOLOGY 3–SPHERES

In this section, we mainly consider closed oriented 3–manifolds M with

$$H_*(M; \mathbf{Z}) \cong H_*(S^3; \mathbf{Z})$$

and replace the condition described by the obstruction class $w_2(M, L)$ in Theorem 2.5 with that of $\mathbf{Z}_2$ linking numbers.

First, let M be an arbitrary closed oriented 3–manifold and L an oriented framed link in M. For the inclusion $j : (M, \emptyset) \to (M, L)$, the induced homomorphism $j^* : H^2(M, L) \to H^2(M)$ sends $w_2(M, L)$ to the second Stiefel–Whitney class $w_2(M)$ of M, which vanishes. By the cohomology exact sequence

$$H^1(L) \xrightarrow{\ \delta\ } H^2(M, L) \xrightarrow{\ j^*\ } H^2(M),$$

we have that $w_2(M, L) = \delta(\alpha)$ for some $\alpha \in H^1(L)$, although such an α may not be unique. In fact, such a class can be explicitly given as follows.

Set $L = L_1 \cup L_2 \cup \cdots \cup L_\mu$, where L_s are the components of L, $s = 1, 2, \ldots, \mu$. It is known that the tangent bundle TM of a closed oriented 3–manifold M is always trivial. Once a trivialization τ of TM is fixed, we can compare it with the specific trivialization of $TM|_{L_s}$ associated with the framing given for each component L_s of the framed link L. (We consider the trivialization given by the ordered vector fields v_1, v_2 and v_3, where v_1 is tangent to L_s consistent with the orientation, and v_2, v_3 are consistent with the framing.) This defines a well-defined element a_s in $\pi_1(SO(3)) \cong \mathbf{Z}_2$ for each s. Then, we have proved the following in [19].

LEMMA 3.1. *Let $\alpha \in H^1(L)$ be the unique cohomology class such that the Kronecker product $\langle \alpha, [L_s]_2 \rangle \in \mathbf{Z}_2$ coincides with a_s for each component L_s of L. Then, we have $\delta(\alpha) = w_2(M, L)$.*

Note that the trivialization τ of TM may not be unique. The set of homotopy classes of such trivializations is in one-to-one correspondence with the homotopy set $[M, SO(3)]$. If we consider the set of homotopy classes of trivializations on the 2–skeleton of M, then each such trivialization up to homotopy defines a *spin structure* on M, and the set of spin structures is in one-to-one correspondence with $H^1(M)$ (see [13]).

By the cohomology exact sequence,

$$H^1(M) \xrightarrow{i^*} H^1(L) \xrightarrow{\delta} H^2(M,L) \xrightarrow{j^*} H^2(M), \tag{3.1}$$

we see that for an arbitrary element $\beta \in \operatorname{Im} i^*$, we could choose $\alpha + \beta$ instead of α, where $i : L \to M$ is the inclusion map. The observation in the previous paragraph shows that this corresponds to choosing another trivialization which is "twisted along β".

The following proposition has also been proved in [19].

LEMMA 3.2. *Let L be an oriented framed link which bounds a compact oriented surface V consistent with the framing. Let $\alpha \in H^1(L)$ be an element such that $\delta(\alpha) = w_2(M,L)$. Then, we have*

$$\begin{aligned} \langle w_2(M,L), [V, \partial V]_2 \rangle &= \langle \delta(\alpha), [V, \partial V]_2 \rangle \\ &= \langle \alpha, [L]_2 \rangle \\ &\equiv \sharp L \pmod 2, \end{aligned}$$

where $\langle \cdot\,, \cdot \rangle$ is the Kronecker product, $[V, \partial V]_2 \in H_2(M,L)$ is the fundamental class of V in $\mathbf{Z}_2$–coefficients, and $\sharp L$ denotes the number of components of L.

Note that the above lemma is applicable for an arbitrary null-cobordant framed link L and that the value $\langle \alpha, [L]_2 \rangle \in \mathbf{Z}_2$ does not depend on a particular choice of α. Furthermore, if L has an odd number of components, then the obstruction $w_2(M,L)$ never vanishes.

Let us now consider the case of a local knot component. Suppose that the oriented framed link L contains a component L_s that lies in the interior of a closed 3–disk D embedded in M. Set $U = \operatorname{Int} D$, which is an open set of M diffeomorphic to $\mathbf{R}^3$. In the following, let us identify U with $\mathbf{R}^3$. In this case, up to homotopy, we may assume that the trivialization τ of TM over U is given by the standard one of $T\mathbf{R}^3$.

Let $\pi : \mathbf{R}^3 \to H$ be the orthogonal projection onto a generic hyperplane $H \cong \mathbf{R}^2$ in the sense that $\pi|_{L_s}$ is an immersion with normal crossings. Recall that the first vector field defining the trivialization $TM|_{L_s}$ associated with the framing on L_s is tangent to L_s consistent with the orientation. Since $\pi|_{L_s}$ is an immersion, we may assume that at each point x of L_s the remaining two vector fields give a 2–framing that is a basis for a 2–plane $N_x \subset T_x\mathbf{R}^3$ transverse to T_xL_s containing the direction $H^\perp$ perpendicular to H. Then, we count the number of times modulo 2 the 2–framing rotates in N_x with respect to a fixed positive direction of $H^\perp$ while x goes once around L_s. This number is denoted by $t_v(L_s)$, which is an element in $\mathbf{Z}_2$. Then, we have proved the following in [19].

LEMMA 3.3. *Let $\alpha \in H^1(L)$ be an arbitrary element such that $\delta(\alpha) = w_2(M,L)$. Then, we have*

$$\langle \alpha, [L_s]_2 \rangle \equiv t_v(L_s) + c(L_s) + 1 \pmod 2,$$

where $c(L_s)$ denotes the number of crossings of the immersion $\pi|_{L_s} : L_s \to H$ with normal crossings.

From now on, we will consider integral homology 3–spheres for M in this section. Let us start with the following.

DEFINITION 3.4. For an oriented link L in a closed oriented 3–manifold M with $H_1(M;\mathbf{Z}) = 0$, we always have a *Seifert surface*, i.e. a compact oriented surface V embedded in M such that $\partial V = L$. Such a Seifert surface is not unique; however, it is known that the induced framing on L is uniquely determined (for example, see [9, §3.6.1]). In the following, such a framing is said to be *preferred*.

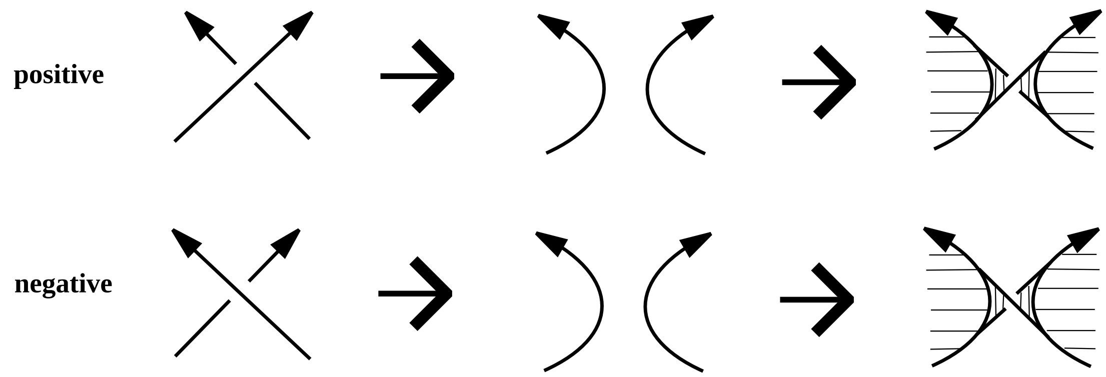

FIGURE 1. Seifert algorithm for positive and negative crossings

Then, for oriented links with preferred framings in the 3–sphere S^3, we have the following. In the following, we fix an orientation for S^3 once and for all.

PROPOSITION 3.5. *Let $L = L_1 \cup L_2 \cup \cdots \cup L_\mu$ be an oriented link in S^3, on which a preferred framing is given. Then $w_2(S^3, L) = 0$ if and only if for each s with $1 \le s \le \mu$, we have*

$$\sum_{t \neq s} \mathrm{lk}(L_s, L_t) \equiv 1 \pmod 2,$$

where lk *denotes the linking number.*

Proof. First, note that by the exact sequence (3.1) with $M = S^3$, we see that δ is injective and that $\alpha \in H^1(L)$ with $\delta(\alpha) = w_2(S^3, L)$ is uniquely determined. Therefore, $w_2(S^3, L) = 0$ if and only if $\langle \alpha, [L_s]_2 \rangle = 0$ for all s.

Now, we may assume that L is contained in $U \subset S^3$ as above, and let us consider the generic projection $\pi|_L : L \to H$. By the so-called Seifert algorithm, we can construct a compact oriented surface $V \subset S^3$ with $\partial V = L$ (see Fig. 1). Then, by construction, we see that when $\pi(x)$ goes once around $\pi(L_s)$, each time it goes through a positive (resp. negative) crossing point, it contributes $+1/2$ (resp. $-1/2$) to $t_v(L_s)$. Since the number of crossing points of $\pi(L_s)$ and $\pi(L_t)$ is even for each $t \neq s$, and $\pi(x)$ goes through each self-crossing point of $\pi(L_s)$ twice, we have

$$t_v(L_s) \equiv \frac{1}{2} \sum_{t \neq s} \widetilde{c}(L_s, L_t) + \widetilde{c}(L_s) \pmod 2$$

for each s, where $\widetilde{c}(L_s, L_t)$ is the sum of the signs of crossing points of $\pi(L_s)$ and $\pi(L_t)$, and $\widetilde{c}(L_s)$ is the sum of the signs of self-crossing points of $\pi(L_s)$. Then, since $\widetilde{c}(L_s) \equiv c(L_s) \pmod 2$, by Lemma 3.3, we have

$$\begin{aligned} \langle \alpha, [L_s]_2 \rangle &\equiv \frac{1}{2} \sum_{t \neq s} \widetilde{c}(L_s, L_t) + 1 \pmod 2 \\ &\equiv \sum_{t \neq s} \mathrm{lk}(L_s, L_t) + 1 \pmod 2, \end{aligned}$$

by the definition of linking numbers. Hence, the result follows. □

REMARK 3.6. The condition that appears in the statement of Proposition 3.5 is very similar to that in [9, Theorem 3.6.11]. In fact, in §4 we will prove the theorem obtained in [9] as an application of our Proposition 3.5.

In fact, we have the following more general result.

PROPOSITION 3.7. *Let M be a closed connected oriented 3–manifold with*

$$H_1(M; \mathbf{Z}) = 0$$

and $L = L_1 \cup L_2 \cup \cdots \cup L_\mu$ be an oriented link in M, on which a preferred framing is given. Then, $w_2(M, L) = 0$ if and only if for each s with $1 \leq s \leq \mu$, we have

$$\sum_{t \neq s} \mathrm{lk}(L_s, L_t) \equiv 1 \pmod 2. \tag{3.2}$$

Proof. Since $H_1(M; \mathbf{Z}) = 0$, there exists a Seifert surface V for L, which is a compact oriented surface embedded in M with $\partial V = L$. By definition, this is consistent with the framing of L. Set $V' = V \smallsetminus \mathrm{Int}\, N(L)$ and $\widetilde{L}_s = V' \cap N(L_s)$ for each s, where $N(L)$ is a small tubular neighborhood of L in M, $N(L_s)$ is the component of $N(L)$ containing L_s, $\partial N(L)$ intersects V transversely, and $V \cap N(L)$ is a collar neighborhood of ∂V in V. Note that $\widetilde{L}_s$ is a knot parallel to L_s, and we orient $\widetilde{L}_s$ consistently with L_s. Then, the oriented link $\widehat{L}_s = L \smallsetminus L_s$ is $\mathbf{Z}$–homologous to $-\widetilde{L}_s$ in $M \smallsetminus L_s$, where $-\widetilde{L}_s$ denotes $\widetilde{L}_s$ with the opposite orientation.

Now, suppose $w_2(M, L) = 0$. In this case, the given framing of L extends over M. Let us suppose that a Seifert surface V_s for L_s is consistent with the given framing of L_s for some s. Then, by Lemma 3.2 applied to L_s, $w_2(M, L_s) \in H^2(M, L_s)$ does not vanish, as we obviously have $\sharp L_s = 1$. This implies that $a_s \in \mathbf{Z}_2$ as appears in Lemma 3.1 does not vanish. This contradicts our assumption that the framing of L extends over M. Therefore, an arbitrary Seifert surface V_s for L_s is not consistent with the given framing of L_s for each s. Since V is consistent with the framing of L_s, the linking number of L_s and $\widetilde{L}_s$ must be an odd integer. Since $-\widetilde{L}_s$ is $\mathbf{Z}$–homologous to $\widehat{L}_s$ in $M \smallsetminus L_s$, we have the congruence (3.2).

Conversely, suppose (3.2) holds for each s. Then, by the above argument we see that $a_s = 0$ for each s. Hence, by Lemma 3.1, we have $w_2(M, L) = 0$. This completes the proof. □

In fact, the above argument implies the following.

PROPOSITION 3.8. *Let M be a closed connected oriented 3–manifold with*

$$H_1(M; \mathbf{Z}) = 0$$

and $L = L_1 \cup L_2 \cup \cdots \cup L_\mu$ be an oriented link in M, on which a preferred framing is given. For each s with $1 \leq s \leq \mu$, define $a_s \in \mathbf{Z}_2$ by

$$a_s = \sum_{t \neq s} \mathrm{lk}(L_s, L_t) + 1 \pmod 2.$$

Let $\alpha \in H^1(L)$ be the unique cohomology class such that $\langle \alpha, [L_s]_2 \rangle = a_s$ for all s. Then, we have $\delta(\alpha) = w_2(M, L)$.

When $H_1(M; \mathbf{Z}) = 0$, we have $H^1(M) = 0 = H^2(M)$, and hence the exact sequence (3.1) implies that we have the isomorphism $\delta : H^1(L) \to H^2(M, L)$. We easily see that its composition with the isomorphism $H^2(M, L) \to H_1(M \smallsetminus L)$ corresponds to the Alexander duality whose inverse isomorphism is given by taking $\mathbf{Z}_2$ linking numbers. This observation together with Theorem 2.5 leads to the following, which answers to Question 2.4 for oriented framed links in integral homology 3–spheres.

THEOREM 3.9. *Let M be a closed connected oriented 3–manifold with*

$$H_1(M;\mathbf{Z})=0,$$

$L=L_1\cup L_2\cup\cdots\cup L_\mu$ be an oriented link in M, and J be an unoriented link in $M\setminus L$. Then, there exists an excellent map $f:M\to\mathbf{R}^2$ such that $L=f^{-1}(y)$ for a regular value $y\in\mathbf{R}^2$ and $J=S(f)$ if and only if for each s with $1\le s\le\mu$, the $\mathbf{Z}_2$ linking number of J with L_s coincides with

$$\sum_{t\neq s}\mathrm{lk}(L_s,L_t)+1\pmod 2.$$

Proof. By the above observations, we see that $[J]_2\in H_1(M\setminus L)$ is Poincaré dual to

$$w_2(M,L)\in H^2(M,L)$$

if and only if it satisfies the condition on $\mathbf{Z}_2$ linking numbers in the theorem. Thus, the result follows from Theorem 2.5. □

Let us observe the following.

LEMMA 3.10. *If the congruence (3.2) holds, then the number of components of L must be even.*

Proof. Consider the sum of all linking numbers

$$\sum_{s=1}^{\mu}\sum_{t\neq s}\mathrm{lk}(L_s,L_t)\in\mathbf{Z}$$

over all s and t with $s\neq t$. Since $\mathrm{lk}(L_s,L_t)=\mathrm{lk}(L_t,L_s)$, the above sum must be even. On the other hand, the congruence (3.2) implies that the above sum has the same parity as the number of components of L. Thus the result follows. □

The above lemma together with Theorem 3.9 implies that for an integral homology 3–sphere M and an excellent map $f:M\to\mathbf{R}^2$, if $L=f^{-1}(y)$ has an odd number of components for a regular value $y\in\mathbf{R}^2$, then $S(f)$ has a non-trivial linking number with a component of L.

In order to get a more general result, let us introduce the following definition.

DEFINITION 3.11. Let M be a closed connected oriented 3–manifold and L, L' be non-empty disjoint closed sets in M. We say that L and L' are *not linked* if there exists an embedded 2–sphere in $M\setminus(L\cup L')$ which separates M into two components in such a way that one of them contains L and the other contains L'. If such a 2–sphere does not exist, then we say that L and L' are *linked*.

LEMMA 3.12. *Let M be a closed connected oriented 3–manifold containing an embedded 2–sphere S which separates M into two components M_1 and M_2, where M_1 and M_2 are the closures of the connected components of $M\setminus S$. If a framed link L is contained in $\mathrm{Int}\,M_1$ and is framed null-cobordant in M, then it is also framed null-cobordant in $\mathrm{Int}\,M_1$.*

Proof. Let V be a compact oriented normally framed surface in M which bounds L and is consistent with the framing of L. We may assume that V and S intersect each other transversely. Then, $V\cap S$ consists of a finite number of simple closed curves in the 2–sphere S. By considering $V\cap M_1$, adding 2–disks bounded by the simple closed curves in S, and by slightly translating the 2–disks in a parallel manner using the inner-most argument, we get a compact oriented surface embedded in $\mathrm{Int}\,M_1$. This gives a desired framed null-cobordism for L in $\mathrm{Int}\,M_1$. □

We have the following as a result of Lemma 3.12.

PROPOSITION 3.13. *Let M be a closed connected oriented 3–manifold and $f : M \to \mathbf{R}^2$ a smooth map. For a regular value $y \in \mathbf{R}^2$, if $L = f^{-1}(y)$ is non-empty and has an odd number of connected components, then L is necessarily linked with $S(f)$.*

Proof. Suppose that there exists a 2–sphere S that separates L and $S(f)$. Let M_1 and M_2 be the closures of the two components of $M \setminus S$ such that $L \subset \operatorname{Int} M_1$ and $S(f) \subset \operatorname{Int} M_2$. Since L is framed null-cobordant in M, it is also framed null-cobordant in $\operatorname{Int} M_1$ by Lemma 3.12. Therefore, there exists a compact oriented normally framed surface in $\operatorname{Int} M_1$ that bounds L. Let $\widehat{M_1}$ be the closed oriented 3–manifold obtained by attaching a 3–disk to M_1 along the boundary S. Then, since $f|_{M_1}$ is a submersion and $\pi_2(SO(3))$ vanishes, we see that the trivialization of $T\widehat{M_1}|_L$ extends to $\widehat{M_1}$, and hence $w_2(\widehat{M_1}, L)$ vanishes. Then, by Lemma 3.2 applied to $L \subset \widehat{M_1}$, this leads to a contradiction, since $\sharp L$ is odd by our assumption. Therefore, L and $S(f)$ are necessarily linked. This completes the proof. □

Note that the above proposition holds not only for excellent maps, but also for smooth maps. In the case of integral homology 3–spheres, by Theorem 3.9 we have the following.

PROPOSITION 3.14. *Let M be a closed connected oriented 3–manifold with*

$$H_1(M; \mathbf{Z}) = 0$$

and $L = L_1 \cup L_2 \cup \cdots \cup L_\mu$ be an oriented link in M. For an arbitrary excellent map $f : M \to \mathbf{R}^2$ such that $L = f^{-1}(y)$ for a regular value $y \in \mathbf{R}^2$, $S(f)$ necessarily links with each component L_s of L with

$$\sum_{t \neq s} \operatorname{lk}(L_s, L_t) \equiv 0 \pmod 2. \tag{3.3}$$

Compare the above proposition with [19, Problem 5.1]. For example, if the congruence (3.3) holds for all s, then for an excellent map $f : M \to \mathbf{R}^2$ such that $f^{-1}(y) = L$ for a regular value $y \in \mathbf{R}^2$, each component of L links with at least one component of $S(f)$.

We do not know if the results in this section for M with $H_1(M; \mathbf{Z}) = 0$ also hold for M with $H_1(M) = 0$ in $\mathbf{Z}_2$–coefficients.

REMARK 3.15. In fact, Proposition 3.14 holds not only for excellent maps, but also for smooth maps, which can be proved as follows. Suppose that there exists a smooth map $g : M \to \mathbf{R}^2$ such that $L = g^{-1}(y)$ for a regular value $y \in \mathbf{R}^2$ and that $S(g)$ does not link with L_s. Then, we can approximate g by an excellent map f such that $S(f) \subset N(S(g))$ and $f|_{M \setminus N(S(g))} = g|_{M \setminus N(S(g))}$ for a sufficiently small neighborhood $N(S(g))$ of $S(g)$. Then, such an f leads to a contradiction.

4. SUBMERSIONS OF OPEN 3–MANIFOLDS TO $\mathbf{R}^2$

In this section, as an application of our results in [19] and in the previous sections of the present paper, we consider submersions of open orientable 3–manifolds to $\mathbf{R}^2$.

First, let us recall the following fundamental theorem for submersions of $\mathbf{R}^3$ to $\mathbf{R}^2$ obtained in [9].

THEOREM 4.1 (Hector and Peralta-Salas, 2012). *Let $L = L_1 \cup L_2 \cup \cdots \cup L_\mu \subset \mathbf{R}^3$ be an oriented link in $\mathbf{R}^3$. Then, there exists a submersion $f : \mathbf{R}^3 \to \mathbf{R}^2$ such that $f^{-1}(y) = L$ for some $y \in \mathbf{R}^2$ if and only if for each s with $1 \leq s \leq \mu$, we have*

$$\sum_{t \neq s} \operatorname{lk}(L_s, L_t) \equiv 1 \pmod 2.$$

Recall that in [9], the authors used the h-principle for submersions [7, 16] for the proof. Here, we give a new proof to the above theorem using our singularity theoretical techniques.

Proof of Theorem 4.1. Let L be an oriented link in $\mathbf{R}^3$ which satisfies the condition about the linking numbers as in the theorem. By identifying the interior of an embedded 3–disk D in S^3 with $\mathbf{R}^3$, we may assume that $L \subset \operatorname{Int} D \subset S^3$. Then, by Proposition 3.5, we have $w_2(S^3, L) = 0$ with respect to the preferred framing on L. Therefore, for an arbitrary non-empty link J in $S^3 \setminus D$, there exists an excellent map $g : S^3 \to \mathbf{R}^2$ and a regular value $y \in \mathbf{R}^2$ such that $L = g^{-1}(y)$ and $J = S(g)$. By restricting g to $\mathbf{R}^3 = \operatorname{Int} D$, we get a submersion $f : \mathbf{R}^3 \to \mathbf{R}^2$ which has L as a regular fiber.

Conversely, suppose that we have a submersion $f : \mathbf{R}^3 \to \mathbf{R}^2$ and a regular value $y \in \mathbf{R}^2$ such that $f^{-1}(y) = L$. Then, we can find an embedded 3–disk $D \subset \mathbf{R}^3$ whose interior contains L. Note that $f|_D : D \to \mathbf{R}^2$ is a submersion which has L as a regular fiber. By embedding D into S^3, we can extend $f|_D$ to a smooth map $g_1 : S^3 \to \mathbf{R}^2$. Here, $f(\partial D)$ misses $y \in \mathbf{R}^2$, and since the second homotopy group of $\mathbf{R}^2 \setminus \{y\}$ is trivial, $f|_{\partial D}$ is null-homotopic inside $\mathbf{R}^2 \setminus \{y\}$. Therefore, we can arrange the smooth map g_1 in such a way that g_1 has $y \in \mathbf{R}^2$ as a regular value and that $g_1^{-1}(y) = L \subset \operatorname{Int} D$. Then, by slightly perturbing g_1 on a neighborhood of $S^3 \setminus \operatorname{Int} D$, we get an excellent map $g_2 : S^3 \to \mathbf{R}^2$ such that $y \in \mathbf{R}^2$ is a regular value, that $g_2^{-1}(y) = L$, and that $S(g_2)$ is contained in $S^3 \setminus \operatorname{Int} D$. In particular, $S(g_2)$ is $\mathbf{Z}_2$ null-homologous in $S^3 \setminus L$, and hence we have $w_2(S^3, L) = 0$. Then, by Proposition 3.5, we get the result. □

Remark 4.2. More generally, instead of $\mathbf{R}^3$, the above theorem holds also for an arbitrary open 3–manifold of the form $M \setminus D^3$ for a closed connected orientable 3–dimensional manifold M with $H_1(M; \mathbf{Z}) = 0$, where D^3 is a small closed 3–disk embedded in M.

In the case of a link with an odd number of components, we have the following.

Remark 4.3. Let $f : \mathbf{R}^3 \to \mathbf{R}^2$ be a smooth map, and suppose that $y \in \mathbf{R}^2$ is a regular value such that $L = f^{-1}(y)$ is compact and has an odd number of components. Then, by Proposition 3.13 together with an argument similar to the above, we see that the singular point set $S(f)$ necessarily links with L (see also the paragraph just after [15, Theorem 10]): in other words, we can find no 2–sphere embedded in $\mathbf{R}^3$ that separates L and $S(f)$. This implies, in particular, that such an f can never be a submersion.

In fact, we have the following.

Proposition 4.4. *Let M be a closed connected orientable 3–manifold with*

$$H_1(M; \mathbf{Z}) = 0$$

and set $M^\circ = M \setminus D^3$. Let $L = L_1 \cup L_2 \cup \cdots \cup L_\mu \subset M^\circ$ be an oriented link such that $f^{-1}(y) = L$ for some excellent map $f : M^\circ \to \mathbf{R}^2$ and a regular value $y \in \mathbf{R}^2$. Then, each component L_s of L with

$$\sum_{t \neq s} \operatorname{lk}(L_s, L_t) \equiv 0 \pmod 2 \tag{4.1}$$

links with at least one component of $S(f)$. In particular, such an f can never be a submersion.

Compare the above proposition with [19, Problem 5.1]. See also [2, 3, 5, 11] for related physical results.

Proof of Proposition 4.4. First note that each component of $S(f)$ is diffeomorphic to a circle or a real line. Furthermore, $S(f)$ is a closed submanifold of M° which may have infinitely many connected components.

Let V_s be a Seifert surface for L_s in M, where L_s satisfies (4.1). We may assume that $L_s \subset M^\circ$ and that $S(f)$ intersects V_s transversely at finitely many points. We have only to show that there are an odd number of intersection points.

Let $\widetilde{D}$ be a 3–disk in M such that $\operatorname{Int}\widetilde{D} \supset D^3$, $L \cap \widetilde{D} = \emptyset$, $V_s \cap \widetilde{D} = \emptyset$, and that $\partial\widetilde{D}$ intersects $S(f)$ transversely at finitely many points. Then, by an argument similar to that in the proof of Theorem 4.1, we can construct an excellent map $g : M \to \mathbf{R}^2$ such that $g|_{M \setminus \operatorname{Int}\widetilde{D}} = f|_{M \setminus \operatorname{Int}\widetilde{D}}$ and that $g^{-1}(y) = L$. By our assumption (4.1), we have that L_s has a non-trivial $\mathbf{Z}_2$ linking number with $S(g)$ by Theorem 3.9. Therefore, $S(g)$ intersects V_s transversely at an odd number of points. By construction of g, this implies that $S(f)$ also intersects V_s transversely at an odd number of points. This completes the proof. □

Remark 4.5. In fact, the above proposition holds not only for excellent maps, but also for smooth maps if we replace the statement "L_s links with at least one component of $S(f)$" by "L_s links with $S(f)$". This can be proved by an argument similar to that in Remark 3.15.

The following is a special case of a theorem proved by Miyoshi [14], who used a relative version of the h-principle for submersions [7]. Here, we use our singularity theoretical arguments in order to prove the theorem for punctured 3–manifolds.

Theorem 4.6. *Let M be a closed orientable 3–manifold and L a compact oriented framed link in $M^\circ = M \setminus D^3$. Then, there exists a submersion $f : M^\circ \to \mathbf{R}^2$ such that $f^{-1}(y)$ coincides with L as oriented framed links for some $y \in \mathbf{R}^2$ if and only if L bounds a proper normally framed surface in M° and the trivialization of $TM^\circ|_L$ induced by the framing of L extends over M°.*

Proof. If there exists a submersion f as in the theorem, then the inverse image by f of the half line $[y_1, \infty) \times \{y_2\} \subset \mathbf{R}^2$ is a proper normally framed surface in M° that bounds L, where $y = (y_1, y_2)$. Furthermore, since f is a submersion, we can pull-back the natural trivialization of $T\mathbf{R}^2$ to M° by f in such a way that the pull-back naturally extends the trivialization of $TM^\circ|_L$ induced by the framing of L.

Conversely, suppose that L bounds a proper normally framed surface V in M° and the trivialization of $TM^\circ|_L$ induced by the framing of L extends over M°. Let $\widetilde{D}$ be a small 3–disk neighborhood of D^3 whose interior contains D^3 such that $\widetilde{D} \subset M \setminus N(L)$ for a small tubular neighborhood $N(L)$ of L in M. Then, we may assume that V intersects $\partial\widetilde{D}$ transversely along finitely many embedded oriented circles. Note that then $V \cap \partial\widetilde{D}$ bounds a compact oriented surface V' in $\widetilde{D}$. Then, by replacing $V \cap \widetilde{D}$ by V', we see that L is framed null-cobordant in M. Furthermore, by our assumption, the trivialization of $TM^\circ|_L$ induced by the framing of L extends over M°. Since $\pi_2(SO(3))$ vanishes, this implies that it also extends over M. Therefore, we have that the obstruction $w_2(M, L)$ vanishes. Hence, by Theorem 2.5, there exists an excellent map $f : M \to \mathbf{R}^2$ and a regular value $y \in \mathbf{R}^2$ such that $f^{-1}(y)$ coincides with L as oriented framed links and that $S(f)$ is contained in $\operatorname{Int} D^3$. Then, f restricted to $M^\circ = M \setminus D^3$ is a desired submersion. □

In fact, if we use the "absolute version" of the h-principle [16] in order to treat the end of an open 3–manifold, we can prove the following. Note again that the following theorem was originally proved by Miyoshi [14] by using a "relative version" of the h-principle [7].

Theorem 4.7. *Let M be an open orientable 3–manifold and L a compact oriented framed link in M. Then, there exists a submersion $f : M \to \mathbf{R}^2$ such that $f^{-1}(y)$ coincides with L as oriented framed links for some $y \in \mathbf{R}^2$ if and only if L bounds a proper normally framed surface in M and the trivialization of $TM|_L$ induced by the framing of L extends over M.*

Proof. Necessity can be proved by the same argument as in the proof of Theorem 4.6.

Conversely, suppose that there exists a proper normally framed surface V in M that bounds L as described in the theorem. Let Q be a compact 3–dimensional submanifold of M with boundary such that $\operatorname{Int} Q \supset L$ and that ∂Q intersects V transversely along finitely many embedded circles.

Let us first construct a smooth map $g_1 : M \to \mathbf{R}^2$ as follows. Let $h : V \to [0, \infty)$ be a smooth function such that $h^{-1}(0) = \partial V = L$ and that h is non-singular near ∂V. Let $N(V) \cong V \times I$ be a tubular neighborhood of V in M, where $I = [-1, 1]$ and the I–factor is consistent with the normal orientation of V. Then, we define g_1 on $N(V)$ by

$$N(V) \cong V \times I \xrightarrow{h \times \mathrm{id}_I} [0, \infty) \times I \subset \mathbf{R}^2,$$

where id_I is the identity map of I. We can extend $g_1|_{N(V)}$ to $N(V) \cup N(L)$ in such a way that $g_1|_{N(L)}$ is a submersion, that the origin 0 is a regular value, and that the framed regular fiber $g_1^{-1}(0)$ coincides with L. Then, since $\mathbf{R}^2 \setminus g_1(N(V) \cup N(L))$ is contractible, we can extend g_1 to the whole manifold M in such a way that 0 is still a regular value and that the framed regular fiber $g_1^{-1}(0)$ coincides with L.

Set $Q' = Q \setminus \mathrm{Int}\, N(L)$, which is a compact 3–manifold with boundary $\partial Q \cup \partial N(L)$. Note that $g_1(Q') \subset \mathbf{R}^2 \setminus \mathrm{Int}\, D$, where D is a small 2–disk neighborhood of the origin.

By our assumption, the framing on L extends over M. Using such a framing, we can construct a bundle epimorphism $T(M \setminus \mathrm{Int}\, Q) \to T(\mathbf{R}^2 \setminus \mathrm{Int}\, D)$ covering $g_1|_{M \setminus \mathrm{Int}\, Q}$. Then, by the h-principle for submersions, g_1 is homotopic to a smooth map $g_2 : M \to \mathbf{R}^2$ such that

(1) g_2 is a submersion over $M \setminus \mathrm{Int}\, Q$,
(2) $g_2 = g_1$ over $N(L)$,
(3) $g_2(M \setminus \mathrm{Int}\, N(L)) \subset \mathbf{R}^2 \setminus \mathrm{Int}\, D$.

Then, we can approximate g_2 by an excellent map g_3 that enjoys the same properties as g_2 described above. Then, $S(g_3)$ is a closed subset of Q, which is compact. Therefore, $S(g_3)$ is an unoriented link in $Q \setminus \mathrm{Int}\, N(L)$. Furthermore, as we started with a framing that extends over M, the obstruction to extending the framing on $\partial(Q \setminus \mathrm{Int}\, N(L))$ induced by g_3 to the whole Q vanishes. This implies that the $\mathbf{Z}_2$–homology class represented by $S(g_3)$ vanishes in Q. Then, by our techniques developed in [19] using Levine's cusp eliminations (see [12, 18]), we can homotope g_3 to an excellent map g_4 that satisfies the properties described above such that $S(g_4)$ is unlinked from L: more precisely, there exists an embedded 3–disk $B \subset \mathrm{Int}\, Q \setminus N(L)$ such that $\mathrm{Int}\, B \supset S(g_4)$. Then, for an appropriate embedded arc $A \subset M \setminus N(L)$ that "connects" B to infinity, we see that M is diffeomorphic to $M \setminus (A \cup B)$ by a diffeomorphism that is the identity on $N(L)$ (for example, see [14]). Then, the restriction of g_4 to $M \setminus (A \cup B)$ gives the desired submersion. This completes the proof. □

Remark 4.8. It is known that there exist open 3–manifolds that cannot be embedded in compact 3–manifolds [8].

We finish this paper by posing an open problem.

Problem 4.9. Is there a polynomial map $\mathbf{R}^3 \to \mathbf{R}^2$ that is a submersion and has a compact regular fiber as in Theorem 4.1?

Compare the above problem with Remark 2.3.

One can find some relevant open problems in [9, §4] as well.

Acknowledgment

The author would like to express his sincere gratitude to Professor David Chillingworth for stimulating discussions and for indicating the works of Hector and Peralta-Salas [9, 15]. He would also like to thank the anonymous referee for invaluable comments that improved the presentation of the paper. The author has been supported in part by JSPS KAKENHI Grant Numbers JP16K13754, JP16H03936, JP17H06128.

References

[1] S. Akbulut and H. King, *All knots are algebraic*, Comment. Math. Helv. **56** (1981), 339–351.

[2] M.V. Berry and M.R. Dennis, *Knotted and linked phase singularities in monochromatic waves*, Proc. R. Soc. Lond. A **457** (2001), 2251–2263. DOI: 10.1098/rspa.2001.0826

[3] M.V Berry and M.R. Dennis, *Topological events on wave dislocation lines: birth and death of loops, and reconnection*, J. Physics A: Mathematical and Theoretical **40** (2007), 65–74. DOI: 10.1088/1751-8113/40/1/004

[4] B. Bode and M.R. Dennis, *Constructing a polynomial whose nodal set is any prescribed knot or link*, J. Knot Theory Ramifications **28**, (2019) 1850082 (31 pages). DOI: 10.1142/s0218216518500827

[5] M.R. Dennis, *Local phase structure of wave dislocation lines: twist and twirl*, J. Opt. A: Pure Appl. Opt. **6** (2004), S202–S208. DOI: 10.1088/1464-4258/6/5/011

[6] M. Golubitsky and V. Guillemin, *Stable mappings and their singularities*, Grad. Texts in Math., Vol. 14, Springer–Verlag, New York, Heidelberg, Berlin, 1973.

[7] A. Haefliger, *Lectures on the theorem of Gromov*, Proceedings of Liverpool Singularities Symposium, II (1969/1970) (Berlin), Lecture Notes in Math., vol. 209, Springer-Verlag, 1971, pp. 128–141. DOI: 10.1007/bfb0068900

[8] W. Haken, *Some results on surfaces in* 3*–manifolds*, in “1968 Studies in Modern Topology”, pp. 39–98, Math. Assoc. Amer. (distributed by Prentice-Hall, Englewood Cliffs, N.J.).

[9] G. Hector and D. Peralta-Salas, *Integrable embeddings and foliations*, Amer. J. Math. **134** (2012), 773–825. DOI: 10.1353/ajm.2012.0018

[10] M.A. Kervaire, *Relative characteristic classes*, Amer. J. Math. **79** (1957), 517–558.

[11] H. Larocque, D. Sugic, D. Mortimer, A.J. Taylor, R. Fickler, R.W. Boyd, M.R. Dennis, and E. Karimi , *Reconstructing the topology of optical polarization knots*, Nature Physics (2018), https://doi.org/10.1038/s41567-018-0229-2. DOI: 10.1038/s41567-018-0229-2

[12] H.I. Levine, *Elimination of cusps*, Topology **3**, Suppl. 2 (1965), 263–296.

[13] J. Milnor, *Spin structures on manifolds*, Enseignement Math. (2) **9** (1963), 198–203.

[14] S. Miyoshi, *Links and submersions to the plane on an open* 3*-manifold*, J. Topol. Anal. **8** (2016), 677–690. DOI: 10.1142/s1793525316500266

[15] D. Peralta-Salas, *Realization problems in the theory of foliations*, In: S. Ibáñez, J.S. Pérez del Río, A. Pumariño, J.A. Rodríguez (eds.), Progress and Challenges in Dynamical Systems, Springer Proceedings in Mathematics & Statistics, vol. 54, pp. 309–323, Springer, Berlin, Heidelberg, 2013. DOI: 10.1007/978-3-642-38830-9_19

[16] A. Phillips, *Submersions of open manifolds*, Topology **6** (1967), 171–206. DOI: 10.1016/0040-9383(67)90034-1

[17] O. Saeki, *Stable maps and links in* 3*–manifolds*, Kodai Math. J. **17** (1994), 518–529. DOI: 10.2996/kmj/1138040047

[18] O. Saeki, *Constructing generic smooth maps of a manifold into a surface with prescribed singular loci*, Ann. Inst. Fourier (Grenoble) **45** (1995), 1135–1162. DOI: 10.5802/aif.1489

[19] O. Saeki, *Linking between singular locus and regular fibers*, J. of Singularities, vol. 21 (2020), 234–248. DOI: 10.5427/jsing.2020.21n

[20] R. Thom, *Les singularités des applications différentiables*, Ann. Inst. Fourier, Grenoble **6** (1955–1956), 43–87.

[21] H. Whitney, *On singularities of mappings of Euclidean spaces. I. Mappings of the plane into the plane*, Ann. of Math. (2) **62** (1955), 374–410. DOI: 10.2307/1970070

Osamu Saeki, Institute of Mathematics for Industry, Kyushu University, Motooka 744, Nishi-ku, Fukuoka 819-0395, Japan

Email address: saeki@imi.kyushu-u.ac.jp