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MORIN SINGULARITIES OF COLLECTIONS OF ONE-FORMS AND VECTOR FIELDS

CAMILA M. RUIZ

ABSTRACT. Inspired by the properties of a collection of n gradient vector fields $\nabla f_1, \ldots, \nabla f_n$ from a Morin map $f = (f_1, \ldots, f_n) : M \to \mathbb{R}^n$, with $\dim M \geq n$, we introduce the notion of Morin singularities in the context of collections of one-forms and collections of vector fields. We also study the singularities of generic one-forms which are related to specific collections (Morin collections) and we generalize a result of T. Fukuda on Euler characteristic ([5, Theorem 1]) for the case of collections of one-forms and vector fields.

1. INTRODUCTION

Morin maps are those which admit only Morin singularities. It is well known that these singularities are stable, and conversely, that corank one stable map-germs are Morin singularities. Thereby, Morin singularities are fundamental and frequently arise as singularities of maps from one manifold to another, as observed by K. Saji in [15]. These singularities have been studied by many authors in different contexts as [9, 1, 5, 12, 13], and more recently [7, 18, 21, 6, 3, 8, 2, 15, 16, 14, 11]. In particular, J.M. Èliašberg [4], J.R. Quine [10], T. Fukuda [5], O. Saeki [12] and N. Dutertre and T. Fukui [3] investigate relations between the topology of a manifold and the topology of the critical locus of maps with Morin singularities.

In this work, we introduce the notion of Morin singularities in the context of collections of one-forms that are not necessarily differential (Definition 2.26) and collections of vector fields that are not necessarily gradient (Definition 2.28). Our main result (Theorem 4.13) is a generalization of Fukuda's Theorem on Euler characteristic [5, Theorem 1] for the case of Morin collections of smooth one-forms: we show that if $\omega = \{\omega_i\}_{1 \leq i \leq n}$ is a Morin collection (Definition 2.26) defined on an m-dimensional compact manifold M then

$$\chi(M) \equiv \sum_{k=1}^{n} \chi(\overline{A_k(\omega)}) \mod 2,$$

where $\chi(M)$ denotes the Euler characteristic of M and $A_k(\omega)$ is the set given by the A_k-type singular points of ω.

Our original inspiration was provided by the following properties of a collection $\{\nabla f_1, \ldots, \nabla f_n\}$ of n gradient vector fields from a Morin map $f = (f_1, \ldots, f_n)$.

Let $f : M^m \to \mathbb{R}^n$ be a smooth Morin map defined on an m-dimensional Riemannian manifold M, with $m \geq n$. The singular points of $f = (f_1, \ldots, f_n)$ are the points $x \in M$ where the rank of the derivative $df(x)$ is equal to $n-1$. By taking the gradient of each coordinate function $f_1, \ldots, f_n$, we obtain a "singular collection" of n vector fields $\{\nabla f_1, \ldots, \nabla f_n\}$ defined on M whose singular locus Σ is given by

$$\Sigma = \{x \in M \mid \operatorname{rank}(\nabla f_1(x), \ldots, \nabla f_n(x)) = n-1\}.$$

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For any $k = 1,\dots,n$, it is known that the sets $A_k(f)$ and $\overline{A_k(f)}$, given by the A_k-type singular points of f and its topological closure, respectively, are $(n-k)$-dimensional smooth submanifolds of M satisfying

(i) $\Sigma = \overline{A_1(f)}$;

(ii) $\overline{A_k(f)} = \bigcup_{i=k}^{n} A_i(f)$;

(iii) For each $x \in \Sigma$,

$$\operatorname{rank} df|_{\overline{A_k(f)}}(x) = \begin{cases} n-k, & \text{if } x \in A_k(f), \\ n-k-1, & \text{if } x \in \overline{A_{k+1}(f)}; \end{cases}$$

(see [5], [9], [12] for Morin singularities). By item (iii), the intersection of the vector space spanned by $\nabla f_1(x),\dots,\nabla f_n(x)$ and the normal vector space to $\overline{A_k(f)}$ at x is a vector subspace whose dimension is given by

$$\dim(\langle \nabla f_1(x),\dots,\nabla f_n(x)\rangle \cap N_x\overline{A_k(f)}) = \begin{cases} k-1, & \text{if } x \in A_k(f), \\ k, & \text{if } x \in \overline{A_{k+1}(f)}. \end{cases}$$

Then, $\langle \nabla f_1(x),\dots,\nabla f_n(x)\rangle$ and $N_x\overline{A_k(f)}$ intersect transversally at x if and only if $x \in A_k(f)$. Otherwise, if $x \in \overline{A_{k+1}(f)}$ and $\{z_1(x),\dots,z_{n-k-1}(x)\}$ is a basis of a vector subspace complementary to $\langle \nabla f_1(x),\dots,\nabla f_n(x)\rangle \cap N_x\overline{A_k(f)}$ in $\langle \nabla f_1(x),\dots,\nabla f_n(x)\rangle$ then

$$\dim(\langle z_1(x),\dots,z_{n-k-1}(x)\rangle \cap N_x\overline{A_{k+1}(f)}) = \begin{cases} 0, & \text{if } x \in A_{k+1}(f), \\ 1, & \text{if } x \in \overline{A_{k+2}(f)}. \end{cases}$$

Therefore $\langle z_1(x),\dots,z_{n-k-1}(x)\rangle$ and $N_x\overline{A_{k+1}(f)}$ intersect transversally at x if and only if $x \in A_{k+1}(f)$, and A_{k+1}-type singular points of f can be distinguished from $\overline{A_{k+2}(f)}$ by this transversality or, equivalently, by the dimension of such intersection. We will follow this idea to define Morin singularities of collections.

This paper is organized as follows. In Section 2, we consider a non-degenerate collection of smooth one-forms $\omega = \{\omega_i\}_{1\le i\le n}$ (Definition 2.2) defined on a smooth m-dimensional manifold M, with $m \ge n$. Then, we define the A_k-type singularities of ω, for $k = 1,\dots,n$, in order to decompose the singular set $\Sigma^1(\omega)$ of ω into disjoint submanifolds according to the type of each singular point. To do that, we give an inductive definition of the singular subsets $\Sigma^k(\omega)$ and $A_k(\omega)$, in which we take successive transversality conditions (Definitions 2.3, 2.9, 2.10, 2.11, 2.18, 2.19, 2.25 and Remark 2.14). In particular, if the required transversality conditions hold, we show that the singular subsets $A_k(\omega)$ and $\Sigma^k(\omega) = \overline{A_k(\omega)}$ are $(n-k)$-dimensional smooth submanifolds of M (Lemmas 2.4, 2.12, 2.20 and Theorem 2.22) such that $\overline{A_k(\omega)} = \cup_{i\ge k} A_i(f)$ (Remark 2.24). Furthermore, in Proposition 2.23 (a) and Lemma 4.5 we provide equations that define the singular sets $\Sigma^k(\omega)$ locally.

We will say that $\omega = \{\omega_i\}_{1\le i\le n}$ is a Morin collection of one-forms (Definition 2.26) if it admits only Morin A_k-type singular points, for $k = 1,\dots,n$ (see Remark 2.27).

The definition of Morin singularities for collections of n one-forms can be analogously adapted to collections of n vector fields as follows. When considering a smooth manifold M, differential one-forms are naturally dual to vector fields, more specifically, if we fix a Riemannian metric on M then there exists an isomorphism between the tangent and cotangent bundles of M, such that vector fields and one-forms can be identified. To illustrate this notion, we give some examples of Morin collections of vector fields in the end of Section 2.

We remark that in the maximal case, that is, when we have a Morin collection of m vector fields defined on an m-dimensional manifold, our definition of A_k-type singularities is equivalent to that A_k-type singularities presented by Saji *et al.* [17].

Let $L \in \mathbb{R}P^{n-1}$ be a straight line in $\mathbb{R}^n$ and let $\pi_L : \mathbb{R}^n \to L$ be the orthogonal projection to L. In [5], T. Fukuda applied Morse theory and well known properties of the singular sets $A_k(f)$ of a Morin map $f : M \to \mathbb{R}^n$ to study critical points of mappings $\pi_L \circ f : M \to L$ and their restrictions to singular sets $\pi_L \circ f|_{A_k(f)}$ and $\pi_L \circ f|_{\overline{A_k(f)}}$. Similarly, in Sections 3 and 4, we investigate the zeros of a generic one-form

$$\xi(x) = \sum_{i=1}^{n} a_i \omega_i(x)$$

associated to a Morin collection of n smooth one-forms $\omega = \{\omega_i\}_{1\le i\le n}$. We verify that ξ, $\xi_{|_{A_k(\omega)}}$ and $\xi_{|_{\overline{A_k(\omega)}}}$ have properties that are similar to that of generic orthogonal projections $\pi_L \circ f(x)$ associated to Morin maps f.

More precisely, let $a = (a_1, \ldots, a_n) \in \mathbb{R}^n \smallsetminus \{\vec{0}\}$ and let $\omega = \{\omega_i\}_{1\le i\le n}$ be a Morin collection of smooth one-forms on M, in Section 3 we prove that the zero set of $\xi(x) = \sum_{i=1}^n a_i\omega_i(x)$ is contained in $\Sigma^1(\omega)$ (Lemma 3.1) and, for almost every $a \in \mathbb{R}^n \smallsetminus \{\vec{0}\}$, the zero set of $\xi_{|_{\Sigma^k(\omega)}}$ does not intercept $\Sigma^{k+2}(\omega)$, for $k = 0, \ldots, n-2$ (Lemmas 3.6 and 3.7). Moreover, we present necessary and sufficient conditions for a zero of $\xi_{|_{\Sigma^{k+1}(\omega)}}$ to be a zero of $\xi_{|_{\Sigma^k(\omega)}}$, for $k = 0, \ldots, n-1$ (Lemmas 3.2 and 3.3). In Section 4, we prove that generically the one-form $\xi(x)$ and its restrictions $\xi_{|_{\Sigma^k(\omega)}}$ and $\xi_{|_{A_k(\omega)}}$ admit only non-degenerate zeros (Lemmas 4.6, 4.7, 4.8 and 4.12). In Lemmas 4.9, 4.10 and 4.11, we give conditions for a non-degenerate zero of $\xi_{|_{\Sigma^{k+1}(\omega)}}$ to be a non-degenerate zero of $\xi_{|_{\Sigma^k(\omega)}}$, for $k = 0, \ldots, n-1$.

As a consequence of these results, we end the paper with Theorem 4.13 whose proof uses the classical Poincaré-Hopf Theorem for one-forms.

2. Morin singularities of collections of one-forms

Let $0 < n \le m$ be integer numbers and let M be an m-dimensional smooth manifold with cotangent space at $x \in M$ denoted by T_x^*M. We define the "n-cotangent bundle" of M by

$$T^*M^n = \{(x, \varphi_1, \ldots, \varphi_n) \mid x \in M;\ \varphi_i \in T_x^*M, i = 1, \ldots, n\},$$

which is an $m(n+1)$-dimensional smooth manifold locally diffeomorphic to $U \times M_{m,n}(\mathbb{R})$, where $U \subset \mathbb{R}^m$ is an open set and $M_{m,n}(\mathbb{R})$ denotes the set of real matrices of size $m \times n$.

Lemma 2.1. *Let $T^*M^{n,n-1} \subset T^*M^n$ be defined by*

$$T^*M^{n,n-1} = \{(x, \varphi_1, \ldots, \varphi_n) \in T^*M^n \mid \operatorname{rank}(\varphi_1, \ldots, \varphi_n) = n-1\}.$$

Then $T^*M^{n,n-1}$ is smooth a submanifold of T^*M^n of dimension $n(m+1)-1$.

Proof. Let $M_{m,n}^{n-1}(\mathbb{R})$ be the smooth submanifold of $M_{m,n}(\mathbb{R})$ of codimension $m-n+1$ consisting of the matrices with rank equal to $n-1$. The set $T^*M^{n,n-1}$ is locally diffeomorphic to $U \times M_{m,n}^{n-1}(\mathbb{R})$, where $U \subset \mathbb{R}^m$ is an open subset. Thus, $T^*M^{n,n-1}$ is a smooth submanifold of T^*M^n of dimension $n(m+1)-1$. $\square$

Let $\omega = \{\omega_i\}_{1\le i\le n}$ be a collection of n smooth one-forms on M, we will consider the smooth map $\omega : M \to T^*M^n$ defined by

$$\omega(x) = (x, \omega_1(x), \ldots, \omega_n(x)).$$

Definition 2.2. *We say that $\omega = \{\omega_i\}_{1\le i\le n}$ is a non-degenerate collection if the map $\omega : M \to T^*M^n$ as above satisfies the following conditions:*

(a) $\omega \pitchfork T^*M^{n,n-1}$ *in* T^*M^n,
(b) $\omega^{-1}(T^*M^{n,\leq n-2}) = \varnothing$,
where $T^*M^{n,\leq n-2} = \{(x,\varphi_1,\dots,\varphi_n) \in T^*M^n \mid \operatorname{rank}(\varphi_1,\dots,\varphi_n) \leq n-2\}$.

Notice that this definition implies that if $\omega = \{\omega_i\}_{1\leq i\leq n}$ is a non-degenerate collection on M, then for each $x \in M$ the rank of $\omega_1(x),\dots,\omega_n(x)$ is either equal to n or equal to $n-1$.

Definition 2.3. *Let $\omega = \{\omega_i\}_{1\leq i\leq n}$ be a non-degenerate collection on M. We define the singular set of the collection ω as the set $\Sigma^1(\omega)$ of points $x \in M$ where the rank of ω is not maximal, that is*

$$\Sigma^1(\omega) = \{x \in M \mid \operatorname{rank}(\omega_1(x),\dots,\omega_n(x)) = n-1\}.$$

Lemma 2.4. *Let $\omega = \{\omega_i\}_{1\leq i\leq n}$ be a non-degenerate collection on M. Then $\Sigma^1(\omega)$ is either the empty set or an $(n-1)$-dimensional smooth submanifold of M.*

Proof. Notice that $\Sigma^1(\omega) = \omega^{-1}(T^*M^{n,n-1})$ and that $\omega \pitchfork T^*M^{n,n-1}$. Thus, if $\Sigma^1(\omega) \neq \varnothing$ then $\Sigma^1(\omega)$ is a smooth submanifold of M of codimension $m-n+1$ and the result follows. □

Let $\omega = \{\omega_i\}_{1\leq i\leq n}$ be a non-degenerate collection of smooth one-forms defined on an m-dimensional smooth manifold M. If ω satisfies some transversality conditions, we will define the A_k-type singularities of ω, for $k = 1,\dots,n$, in order to decompose the singular set $\Sigma^1(\omega)$ into disjoint submanifolds according to the type of each singular point. Firstly, we define the A_1-type singular points in $\Sigma^1(\omega)$. We will denote by $\Sigma^2(\omega)$ the subset of $\Sigma^1(\omega)$ given by all singular points of ω that are not A_1-type. For each $k = 2,\dots,n$, we repeat this process defining the A_k-type singular points in $\Sigma^k(\omega)$ and denoting by $\Sigma^{k+1}(\omega)$ the subset of $\Sigma^k(\omega)$ given by all singular points of ω that are not A_k-type. To do that, we present in this section an inductive definition of A_k-type Morin singularities of ω.

Remark 2.5. *Let $S \subset M$ be a smooth submanifold of M. We will adopt the following notation*

$$N_x^*S = \{\psi \in T_x^*M \,|\, \psi(T_xS) = 0\}.$$

Definition 2.6. *Let $\omega = \{\omega_i\}_{1\leq i\leq n}$ be a non-degenerate collection on M. Given*

$$(x,\varphi) = (x,\varphi_1,\dots,\varphi_{n-1}),$$

we define the sets

$$T^*_{\Sigma^1}M^{n-1} = \{(x,\varphi) \mid x \in \Sigma^1(\omega); \varphi_1,\dots,\varphi_{n-1} \in T_x^*M\}$$

and

$$\begin{aligned} N^*_{\Sigma^1}M^{n-1} = \{(x,\varphi) \in T^*_{\Sigma^1}M^{n-1} \mid \operatorname{rank}(\varphi_1,\dots,\varphi_{n-1}) = n-1,\\ \dim(\langle\varphi_1,\dots,\varphi_{n-1}\rangle \cap N_x^*\Sigma^1(\omega)) = 1\}, \end{aligned}$$

where $\langle\varphi_1,\dots,\varphi_{n-1}\rangle$ denotes the subspace of T_x^*M spanned by $\{\varphi_1,\dots,\varphi_{n-1}\}$.

Lemma 2.7. *$T^*_{\Sigma^1}M^{n-1}$ is a smooth manifold of dimension $m(n-1)+n-1$.*

Proof. For a non-degenerate collection ω, we know that $\Sigma^1(\omega)$ is an $(n-1)$-dimensional smooth submanifold of M. Then, for each $(x,\varphi) \in T^*_{\Sigma^1}M^{n-1}$ there exists an open subset $V \subset \mathbb{R}^{n-1}$ such that $T^*_{\Sigma^1}M^{n-1}$ is locally diffeomorphic to $V \times M_{m,n-1}(\mathbb{R})$ near (x,φ) and the result follows. □

Lemma 2.8. *$N^*_{\Sigma^1}M^{n-1}$ is a smooth hypersurface of $T^*_{\Sigma^1}M^{n-1}$, that is, a smooth submanifold of dimension $m(n-1)+n-2$.*

Proof. Since ω is non-degenerate, it follows from Lemma 2.4 that $\Sigma^1(\omega)$ is a smooth submanifold of codimension $m-n+1$ of M. Then, for each $p \in \Sigma^1(\omega)$ there exist an open neighborhood $\mathcal{U}$ of p in M and smooth functions $F_1, \dots, F_{m-n+1} : \mathcal{U} \to \mathbb{R}$ such that

$$\mathcal{U} \cap \Sigma^1(\omega) = \{x \in \mathcal{U} \mid F_1(x) = \dots = F_{m-n+1}(x) = 0\}$$

with $\operatorname{rank}(dF_1(x), \dots, dF_{m-n+1}(x)) = m-n+1$, for each $x \in \mathcal{U} \cap \Sigma^1(\omega)$, and

$$N_p^* \Sigma^1(\omega) = \langle dF_1(p), \dots, dF_{m-n+1}(p) \rangle.$$

If $(p, \tilde{\varphi}) = (p, \tilde{\varphi}_1, \dots, \tilde{\varphi}_{n-1}) \in N_{\Sigma^1}^* M^{n-1}$ then

$$\operatorname{rank}(\tilde{\varphi}_1, \dots, \tilde{\varphi}_{n-1}, dF_1(p), \dots, dF_{m-n+1}(p)) = m-1,$$

since by the definition of $N_{\Sigma^1}^* M^{n-1}$, $\operatorname{rank}(\tilde{\varphi}_1, \dots, \tilde{\varphi}_{n-1}) = n-1$ and

$$\dim(\langle \tilde{\varphi}_1, \dots, \tilde{\varphi}_{n-1} \rangle \cap N_p^* \Sigma^1(\omega)) = 1.$$

In this way,

$$\det(dF_1(p), \dots, dF_{m-n+1}(p), \tilde{\varphi}_1, \dots, \tilde{\varphi}_{n-1}) = 0$$

and fixing the notation $\tilde{\varphi}_i = (\tilde{\varphi}_i^1, \dots, \tilde{\varphi}_i^m)$ for $i = 1, \dots, n-1$, we can assume that the minor

$$\begin{vmatrix} \dfrac{\partial F_1}{\partial x_1}(p) & \cdots & \dfrac{\partial F_{m-n+1}}{\partial x_1}(p) & \tilde{\varphi}_1^1 & \cdots & \tilde{\varphi}_{n-2}^1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial F_1}{\partial x_{m-1}}(p) & \cdots & \dfrac{\partial F_{m-n+1}}{\partial x_{m-1}}(p) & \tilde{\varphi}_1^{m-1} & \cdots & \tilde{\varphi}_{n-2}^{m-1} \end{vmatrix}$$

does not vanish and consequently, that

$$\begin{vmatrix} \dfrac{\partial F_1}{\partial x_1}(x) & \cdots & \dfrac{\partial F_{m-n+1}}{\partial x_1}(x) & \varphi_1^1 & \cdots & \varphi_{n-2}^1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial F_1}{\partial x_{m-1}}(x) & \cdots & \dfrac{\partial F_{m-n+1}}{\partial x_{m-1}}(x) & \varphi_1^{m-1} & \cdots & \varphi_{n-2}^{m-1} \end{vmatrix} \neq 0 \tag{1}$$

for all $(x, \varphi) \in (\Sigma^1(\omega) \cap \mathcal{U}) \times \mathcal{V}$, where $\mathcal{V} \subset \mathbb{R}^{m(n-1)}$ is an open neighborhood of $\tilde{\varphi}$. Thus, $N_{\Sigma^1}^* M^{n-1}$ can be locally given by

$$N_{\Sigma^1}^* M^{n-1} = \{(x, \varphi) \in \mathcal{U} \times \mathcal{V} \mid F_1 = \dots = F_{m-n+1} = \Delta = 0\},$$

where $\Delta(x, \varphi) = \det(dF_1(x), \dots, dF_{m-n+1}(x), \varphi_1, \dots, \varphi_{n-1})$. Let $B(x, \varphi)$ be the square matrix of order m whose columns are given by the coefficients of the one-forms $dF_1(x)$, …, $dF_{m-n+1}(x)$, φ_1, …, φ_{n-1}:

$$B(x, \varphi) = \begin{pmatrix} dF_1(x) & \cdots & dF_{m-n+1}(x) & \varphi_1 & \cdots & \varphi_{n-1} \end{pmatrix}.$$

By Laplace expansion along the last column of $B(x, \varphi)$, we have

$$\Delta(x, \varphi) = \sum_{i=1}^{m} \varphi_{n-1}^i \operatorname{cof}(\varphi_{n-1}^i, B),$$

where $\operatorname{cof}(\varphi_{n-1}^i, B)$ denotes the cofactor of φ_{n-1}^i in the matrix $B(x, \varphi)$. Thus

$$\frac{\partial \Delta}{\partial \varphi_{n-1}^m}(x, \varphi) = \sum_{i=1}^{m} \operatorname{cof}(\varphi_{n-1}^i, B) \frac{\partial \varphi_{n-1}^i}{\partial \varphi_{n-1}^m} + \varphi_{n-1}^i \frac{\partial \operatorname{cof}(\varphi_{n-1}^i, B)}{\partial \varphi_{n-1}^m}$$

and since $\operatorname{cof}(\varphi^i_{n-1}, B)$ does not depend on the variable φ^m_{n-1}, we have

$$\frac{\partial \operatorname{cof}(\varphi^i_{n-1}, B)}{\partial \varphi^m_{n-1}} = 0, \text{ for } i = 1, \dots, m.$$

Then,

$$\frac{\partial \Delta}{\partial \varphi^m_{n-1}}(x, \varphi) = \operatorname{cof}(\varphi^m_{n-1}, B) \overset{(1)}{\neq} 0,$$

and the derivative of $\Delta(x, \varphi)$ with respect to φ, denoted by $d_\varphi \Delta(x, \varphi)$, does not vanish. This implies that the matrix

$$\begin{bmatrix} dF_1(x) \\ \vdots \\ dF_{m-n+1}(x) \\ d\Delta(x,\varphi) \end{bmatrix} = \begin{bmatrix} d_x F_1(x) & \vdots & \\ \vdots & \vdots & O_{(m-n+1)\times(n-1)} \\ d_x F_{m-n+1}(x) & \vdots & \\ \cdots\cdots\cdots\cdots\cdots & \vdots & \cdots\cdots\cdots\cdots\cdots\cdots \\ d_x \Delta(x,\varphi) & \vdots & d_\varphi \Delta(x,\varphi) \end{bmatrix}$$

has rank $m - n + 2$, where $O_{(m-n+1)\times(n-1)}$ denotes a null matrix. Hence,

$$\operatorname{rank}(dF_1(x), \dots, dF_{m-n+1}(x), d\Delta(x, \varphi)) = m - n + 2,$$

for each $(x, \varphi) \in N^*_{\Sigma^1} M^{n-1} \cap (\mathcal{U} \times \mathcal{V})$. Therefore, $N^*_{\Sigma^1} M^{n-1}$ is a smooth submanifold of $T^*_{\Sigma^1} M^{n-1}$ of dimension $m + m(n-1) - (m - n + 2) = m(n-1) + n - 2$. $\square$

Let $\omega = \{\omega_i\}_{1 \le i \le n}$ be a non-degenerate collection on M and $\langle \omega_1(x), \dots, \omega_n(x) \rangle$ the subspace of $T^*_x M$ spanned by $\{\omega_1(x), \dots, \omega_n(x)\}$. Then for each $p \in \Sigma^1(\omega)$, $\dim\langle \omega_1(p), \dots, \omega_n(p) \rangle = n-1$, and there exist an open neighborhood $\mathcal{U}_p$ of p in M and a collection $\{\Omega_1, \dots, \Omega_{n-1}\}$ of $n-1$ smooth one-forms on $\mathcal{U}_p$ such that $\{\Omega_1(x), \dots, \Omega_{n-1}(x)\}$ is a basis of $\langle \omega_1(x), \dots, \omega_n(x) \rangle$ for each $x \in \mathcal{U}_p \cap \Sigma^1(\omega)$. Let $\Omega^1 : \mathcal{U}_p \cap \Sigma^1(\omega) \to T^*_{\Sigma^1} M^{n-1}$ be the map given by

$$\Omega^1(x) = (x, \Omega_1(x), \dots, \Omega_{n-1}(x));$$

we define:

Definition 2.9. *We say that collection $\omega = \{\omega_i\}_{1 \le i \le n}$ satisfies the "condition I_1" if for each $p \in \Sigma^1(\omega)$ there exist an open neighborhood $\mathcal{U}_p$ of p in M and a map $\Omega^1 : \mathcal{U}_p \cap \Sigma^1(\omega) \to T^*_{\Sigma^1} M^{n-1}$ as defined above, such that on $\mathcal{U}_p$ the following properties hold:*
(a) $\Omega^1 \pitchfork N^*_{\Sigma^1} M^{n-1}$ *in* $T^*_{\Sigma^1} M^{n-1}$,
(b) $(\Omega^1)^{-1}(N^*_{\Sigma^1} M^{n-1, \ge 2}) = \varnothing$,
where

$$N^*_{\Sigma^1} M^{n-1,\ge 2} = \{(x, \varphi) \in T^*_{\Sigma^1} M^{n-1} \mid \operatorname{rank}(\varphi_1, \dots, \varphi_{n-1}) = n-1, \dim(\langle \varphi_1, \dots, \varphi_{n-1} \rangle \cap N^*_x \Sigma^1(\omega)) \ge 2\}.$$

Notice that if ω satisfies the condition I_1, then for each $x \in \Sigma^1(\omega) \cap \mathcal{U}_p$,

$$\dim(\langle \Omega_1(x), \dots, \Omega_{n-1}(x) \rangle \cap N^*_x \Sigma^1(\omega))$$

is either equal to 0 or equal to 1. We will prove in Proposition 2.23 that this dimension and the condition I_1 do not depend on the choice of the basis $\{\Omega_1, \dots, \Omega_{n-1}\}$.

Definition 2.10. *Let $\omega = \{\omega_i\}_{1 \le i \le n}$ be a non-degenerate collection that satisfies the condition I_1. Given $p \in \Sigma^1(\omega)$, consider an open neighborhood $\mathcal{U}_p$ of p in M and a map*

$$\Omega^1(x) = (x, \Omega_1(x), \dots, \Omega_{n-1}(x))$$

as in Definition 2.9. We define the sets $A_1(\omega)$ and $\Sigma^2(\omega)$ as follows:

(a) *We say that* $x \in \mathcal{U}_p$ *belongs to* $A_1(\omega)$ *if* $x \in \Sigma^1(\omega)$ *and*

$$\dim(\langle \Omega_1(x), \dots, \Omega_{n-1}(x) \rangle \cap N_x^* \Sigma^1(\omega)) = 0.$$

(b) *We say that* $x \in \mathcal{U}_p$ *belongs to* $\Sigma^2(\omega)$ *if* $x \in \Sigma^1(\omega) \setminus A_1(\omega)$*, that is, if* $x \in \Sigma^1(\omega)$ *and*

$$\dim(\langle \Omega_1(x), \dots, \Omega_{n-1}(x) \rangle \cap N_x^* \Sigma^1(\omega)) = 1.$$

Then, for each $p \in \Sigma^1(\omega)$ *we may write*

$$A_1(\omega) \cap \mathcal{U}_p = \{x \in \Sigma^1(\omega) \cap \mathcal{U}_p \mid \dim(\langle \Omega_1(x), \dots, \Omega_{n-1}(x) \rangle \cap N_x^* \Sigma^1(\omega)) = 0\};$$
$$\Sigma^2(\omega) \cap \mathcal{U}_p = \{x \in \Sigma^1(\omega) \cap \mathcal{U}_p \mid \dim(\langle \Omega_1(x), \dots, \Omega_{n-1}(x) \rangle \cap N_x^* \Sigma^1(\omega)) = 1\};$$

and we have

$$A_1(\omega) = \bigcup_{p \in \Sigma^1(\omega)} (A_1(\omega) \cap \mathcal{U}_p) \quad \text{and} \quad \Sigma^2(\omega) = \bigcup_{p \in \Sigma^1(\omega)} \left(\Sigma^2(\omega) \cap \mathcal{U}_p\right).$$

Definition 2.11. *Let* $\omega = \{\omega_i\}_{1 \le i \le n}$ *be a non-degenerate collection on* M *that satisfies the condition* I_1*. We say that* $x \in M$ *is an* A_1*-type Morin singularity of* ω *if* $x \in A_1(\omega)$*.*

Lemma 2.12. *Let* $\omega = \{\omega_i\}_{1 \le i \le n}$ *be a non-degenerate collection on* M *that satisfies the condition* I_1*. Then* $\Sigma^2(\omega) \subset \Sigma^1(\omega)$ *and* $\Sigma^2(\omega)$ *is either the empty set or an* $(n-2)$*-dimensional smooth submanifold of* M*.*

Proof. Notice that, locally, $\Sigma^2(\omega) = (\Omega^1)^{-1}(N_{\Sigma^1}^* M^{n-1})$ and $\Omega^1 \pitchfork N_{\Sigma^1}^* M^{n-1}$. Thus, if $\Sigma^2(\omega) \neq \varnothing$ then $\Sigma^2(\omega)$ is a smooth submanifold of $\Sigma^1(\omega)$ of codimension 1 and the result follows. ☐

Lemma 2.13. *Let* $\omega = \{\omega_i\}_{1 \le i \le n}$ *be a non-degenerate collection on* M *that satisfies the condition* I_1*. For each* $p \in \Sigma^1(\omega)$*,*

$$p \in \Sigma^2(\omega) \Leftrightarrow \dim(\langle \omega_1(p), \dots, \omega_n(p) \rangle \cap N_p^* \Sigma^1(\omega)) = 1.$$

Proof. Given $p \in \Sigma^1(\omega)$, we can consider a neighborhood $\mathcal{U}_p$ of p in M and a map

$$\Omega^1(x) = (x, \Omega_1(x), \dots, \Omega_{n-1}(x)),$$

as in Definition 2.9, such that $\langle \Omega_1(p), \dots, \Omega_{n-1}(p) \rangle = \langle \omega_1(p), \dots, \omega_n(p) \rangle$. By Definition 2.10 (b), $p \in \Sigma^2(\omega)$ if and only if $\dim(\langle \Omega_1(p), \dots, \Omega_{n-1}(p) \rangle \cap N_p^* \Sigma^1(\omega)) = 1$. Thus, $p \in \Sigma^2(\omega)$ if and only if $\dim(\langle \omega_1(p), \dots, \omega_n(p) \rangle \cap N_p^* \Sigma^1(\omega)) = 1$. ☐

Remark 2.14. *The following results are used in the formulation of an inductive definition of* A_k*-type Morin singularities of* $\omega = \{\omega_i\}_{1 \le i \le n}$*, for* $k = 2, \dots, n$*.*

Let $3 \le k \le n$ *be an integer number and* $\omega = \{\omega_i\}_{1 \le i \le n}$ *a non-degenerate collection on* M *with singular set* $\Sigma^1(\omega)$*. Let us suppose that, for every* $i = 2, \dots, k-1$*,* $\Sigma^i(\omega)$ *is a smooth submanifold of* M *such that:*

(a) $\Sigma^i(\omega) \subset \Sigma^{i-1}(\omega) \subset \dots \subset \Sigma^1(\omega)$*;*
(b) $\Sigma^i(\omega)$ *is the empty set or an* $(n-i)$*-dimensional smooth submanifold of* M*;*
(c) *For each* $p \in \Sigma^{i-1}(\omega)$*, we have*

$$p \in \Sigma^i(\omega) \Leftrightarrow \dim(\langle \omega_1(p), \dots, \omega_n(p) \rangle \cap N_p^* \Sigma^{i-1}(\omega)) = i - 1.$$

Notice that in Lemmas 2.12 and 2.13 we have already proved that if $\omega = \{\omega_i\}_{1 \le i \le n}$ *satisfies the condition* I_1*, then the above hypothesis holds for* $k = 3$*, that is,* $\Sigma^2(\omega)$ *is a smooth submanifold of* M *satisfying* (a)*,* (b) *and* (c)*. Now, we assume that this hypothesis holds for every* $i = 2, \dots, k-1$*, with* $k > 3$*, and we will prove that it also holds for* $i = k$ *if* $\omega = \{\omega_i\}_{1 \le i \le n}$ *satisfies the "condition* I_{k-1}*" that will be given in Definition 2.18.*

Definition 2.15. *Let $r = n - k + 1$ and $(x,\varphi) = (x,\varphi_1,\dots,\varphi_r)$, we define the sets*

$$T^*_{\Sigma^{k-1}}M^r = \{(x,\varphi) \mid x \in \Sigma^{k-1}(\omega); \varphi_1,\dots,\varphi_r \in T^*_x M\}$$

and

$$\begin{aligned} N^*_{\Sigma^{k-1}}M^r = \{(x,\varphi) \in T^*_{\Sigma^{k-1}}M^r \mid \ & \operatorname{rank}(\varphi_1,\dots,\varphi_r) = r, \\ & \dim(\langle\varphi_1,\dots,\varphi_r\rangle \cap N^*_x\Sigma^{k-1}(\omega)) = 1\}, \end{aligned}$$

where $\langle\varphi_1,\dots,\varphi_r\rangle$ denotes the subspace of $T^*_x M$ spanned by $\{\varphi_1,\dots,\varphi_r\}$.

Lemma 2.16. *$T^*_{\Sigma^{k-1}}M^r$ is a smooth manifold of dimension $mr + r$.*

Proof. Analogously to the proof of Lemma 2.7. □

Lemma 2.17. *$N^*_{\Sigma^{k-1}}M^r$ is a smooth hypersurface of $T^*_{\Sigma^{k-1}}M^r$, that is, a smooth submanifold of dimension $mr + r - 1$.*

Proof. Analogously to the proof of Lemma 2.8. □

By hypothesis, for each $p \in \Sigma^{k-1}(\omega)$, we have that

$$\dim(\langle\omega_1(p),\dots,\omega_n(p)\rangle \cap N^*_p\Sigma^{k-2}(\omega)) = k - 2$$

and $\dim\langle\omega_1(p),\dots,\omega_n(p)\rangle = n-1$. Then, there exist an open neighborhood $\mathcal{U}_p$ of p in M and a collection $\{\Omega_1,\dots,\Omega_r\}$ of $r = n-k+1$ smooth one-forms on $\mathcal{U}_p$ such that $\{\Omega_1(x),\dots,\Omega_r(x)\}$ is a basis of a vector subspace complementary to $\langle\omega_1(x),\dots,\omega_n(x)\rangle \cap N^*_x\Sigma^{k-2}(\omega)$ in $\langle\omega_1(x),\dots,\omega_n(x)\rangle$ for each $x \in \mathcal{U}_p \cap \Sigma^{k-1}(\omega)$. That is, for each $x \in \mathcal{U}_p \cap \Sigma^{k-1}(\omega)$ we have that

$$\langle\Omega_1(x),\dots,\Omega_r(x)\rangle \oplus \big(\langle\omega_1(x),\dots,\omega_n(x)\rangle \cap N^*_x\Sigma^{k-2}(\omega)\big)$$

is equal to $\langle\omega_1(x),\dots,\omega_n(x)\rangle$. Let $\Omega^{k-1} : \mathcal{U}_p \cap \Sigma^{k-1}(\omega) \to T^*_{\Sigma^{k-1}}M^r$ be the map given by

$$\Omega^{k-1}(x) = (x,\Omega_1(x),\dots,\Omega_r(x)),$$

we define:

Definition 2.18. *We say that collection $\omega = \{\omega_i\}_{1\le i\le n}$ satisfies the "condition I_{k-1}", if for each $p \in \Sigma^{k-1}(\omega)$ there exist an open neighborhood $\mathcal{U}_p$ of p in M and a map*

$$\Omega^{k-1} : \mathcal{U}_p \cap \Sigma^{k-1}(\omega) \to T^*_{\Sigma^{k-1}}M^r$$

as defined above, such that on $\mathcal{U}_p$ the following properties hold:

(a) $\Omega^{k-1} \pitchfork N^*_{\Sigma^{k-1}}M^r$ *in* $T^*_{\Sigma^{k-1}}M^r$*;*
(b) $(\Omega^{k-1})^{-1}(N^*_{\Sigma^{k-1}}M^{r,\ge 2}) = \varnothing$*;*

where

$$N^*_{\Sigma^{k-1}}M^{r,\ge 2} = \{(x,\varphi) \in T^*_{\Sigma^{k-1}}M^r \mid \operatorname{rank}(\varphi_1,\dots,\varphi_r) = r, \dim(\langle\varphi_1,\dots,\varphi_r\rangle \cap N^*_x\Sigma^{k-1}(\omega)) \ge 2\}.$$

Notice that if ω satisfies the condition I_{k-1}, then for each $x \in \Sigma^{k-1}(\omega) \cap \mathcal{U}_p$,

$$\dim(\langle\Omega_1(x),\dots,\Omega_r(x)\rangle \cap N^*_x\Sigma^{k-1}(\omega))$$

is either equal to 0 or equal to 1. We will prove in Proposition 2.23 that this dimension and the condition I_{k-1} do not depend on the choice of the basis $\{\Omega_1,\dots,\Omega_r\}$.

Definition 2.19. *Let $\omega = \{\omega_i\}_{1\le i\le n}$ be a non-degenerate collection that satisfies the condition I_{k-1}. Given $p \in \Sigma^{k-1}(\omega)$, consider an open neighborhood $\mathcal{U}_p$ of p in M and a map*

$$\Omega^{k-1}(x) = (x,\Omega_1(x),\dots,\Omega_r(x))$$

as in Definition 2.18. We define the sets $A_{k-1}(\omega)$ and $\Sigma^k(\omega)$ as follows:

(a) *We say that* $x \in \mathcal{U}_p$ *belongs to* $A_{k-1}(\omega)$ *if* $x \in \Sigma^{k-1}(\omega)$ *and*

$$\dim(\langle \Omega_1(x), \dots, \Omega_r(x)\rangle \cap N_x^*\Sigma^{k-1}(\omega)) = 0.$$

(b) *We say that* $x \in \mathcal{U}_p$ *belongs to* $\Sigma^k(\omega)$ *if* $x \in \Sigma^{k-1}(\omega) \setminus A_{k-1}(\omega)$*, that is, if* $x \in \Sigma^{k-1}(\omega)$ *and*

$$\dim(\langle \Omega_1(x), \dots, \Omega_r(x)\rangle \cap N_x^*\Sigma^{k-1}(\omega)) = 1.$$

Then, for each $p \in \Sigma^{k-1}(\omega)$ *we may write*

$$\begin{aligned} A_{k-1}(\omega) \cap \mathcal{U}_p &= \{x \in \Sigma^{k-1}(\omega) \cap \mathcal{U}_p \mid \dim(\langle \Omega_1(x), \dots, \Omega_r(x)\rangle \cap N_x^*\Sigma^{k-1}(\omega)) = 0\}; \\ \Sigma^k(\omega) \cap \mathcal{U}_p &= \{x \in \Sigma^{k-1}(\omega) \cap \mathcal{U}_p \mid \dim(\langle \Omega_1(x), \dots, \Omega_r(x)\rangle \cap N_x^*\Sigma^{k-1}(\omega)) = 1\}; \end{aligned}$$

and we have

$$A_{k-1}(\omega) = \bigcup_{p \in \Sigma^{k-1}(\omega)} (A_{k-1}(\omega) \cap \mathcal{U}_p) \quad \textit{and} \quad \Sigma^k(\omega) = \bigcup_{p \in \Sigma^{k-1}(\omega)} \left(\Sigma^k(\omega) \cap \mathcal{U}_p\right).$$

Lemma 2.20. *Under the hypothesis of Remark 2.14, let* $\omega = \{\omega_i\}_{1 \le i \le n}$ *be a non-degenerate collection on* M *that satisfies the condition* I_{k-1}*. Then* $\Sigma^k(\omega) \subset \Sigma^{k-1}(\omega)$ *and* $\Sigma^k(\omega)$ *is either the empty set or an* $(n-k)$*-dimensional smooth submanifold of* M*.*

Proof. Analogously to the proof of Lemma 2.12. □

Lemma 2.21. *Under the hypothesis of Remark 2.14, let* $\omega = \{\omega_i\}_{1 \le i \le n}$ *be a non-degenerate collection on* M *that satisfies the condition* I_{k-1}*. For each* $p \in \Sigma^{k-1}(\omega)$*,*

$$p \in \Sigma^k(\omega) \Leftrightarrow \dim(\langle \omega_1(p), \dots, \omega_n(p)\rangle \cap N_p^*\Sigma^{k-1}(\omega)) = k-1.$$

Proof. We have that $\Sigma^{k-1}(\omega) \subset \Sigma^{k-2}(\omega)$ and for each $p \in \Sigma^{k-1}(\omega)$:

(i) $N_p^*\Sigma^{k-2}(\omega) \subset N_p^*\Sigma^{k-1}(\omega)$ (see Remark 2.5);

(ii) $\dim(\langle \omega_1(p), \dots, \omega_n(p)\rangle \cap N_p^*\Sigma^{k-2}(\omega)) = k-2$;

(iii) There exist an open neighborhood $\mathcal{U}_p$ of p in M and a collection $\{\Omega_1(x), \dots, \Omega_r(x)\}$ of $r = n-k+1$ smooth one-forms on $\mathcal{U}_p$ such that, for each $x \in \mathcal{U}_p \cap \Sigma^{k-1}(\omega)$, $\langle \omega_1(x), \dots, \omega_n(x)\rangle$ is equal to

$$\langle \Omega_1(x), \dots, \Omega_r(x)\rangle \oplus \left(\langle \omega_1(x), \dots, \omega_n(x)\rangle \cap N_x^*\Sigma^{k-2}(\omega)\right).$$

For clearer notations, let us denote

$$\langle \bar{\omega}(x)\rangle = \langle \omega_1(x), \dots, \omega_n(x)\rangle \quad \text{and} \quad \langle \bar{\Omega}^{k-1}(x)\rangle = \langle \Omega_1(x), \dots, \Omega_r(x)\rangle.$$

Then,

$$\begin{aligned} p \in \Sigma^k(\omega) \quad &\overset{\text{(Def. 2.19)}}{\Leftrightarrow} \quad \dim\left(\langle \bar{\Omega}^{k-1}(p)\rangle \cap N_p^*\Sigma^{k-1}(\omega)\right) = 1 \\ &\overset{(i),(iii)}{\Leftrightarrow} \quad \dim\left(\langle \bar{\omega}(p)\rangle \cap N_p^*\Sigma^{k-1}(\omega)\right) - \dim\left(\langle \bar{\omega}(p)\rangle \cap N_p^*\Sigma^{k-2}(\omega)\right) = 1 \\ &\overset{(ii)}{\Leftrightarrow} \quad \dim\left(\langle \bar{\omega}(p)\rangle \cap N_p^*\Sigma^{k-1}(\omega)\right) - (k-2) = 1 \\ &\Leftrightarrow \quad \dim\left(\langle \bar{\omega}(p)\rangle \cap N_p^*\Sigma^{k-1}(\omega)\right) = k-1. \end{aligned}$$

□

According to Lemmas 2.20 and 2.21, if the hypothesis of Remark 2.14 holds for every $i = 2, \dots, k-1$ and $\omega = \{\omega_i\}_{1 \le i \le n}$ satisfies the condition I_{k-1}, then this hypothesis will hold for $i = 2, \dots, k$. In other words, we can state the following result.

Theorem 2.22. *Let* $\omega = \{\omega_i\}_{1 \le i \le n}$ *be a non-degenerate collection on* M*. If* ω *satisfies the conditions* I_j*, for* $j = 1, \dots, n-1$*, then for every* $k = 1, \dots, n$ *we have that*

(a) $\Sigma^k(\omega) \subset \Sigma^{k-1}(\omega) \subset \dots \subset \Sigma^2(\omega) \subset \Sigma^1(\omega)$*;*

(*b*) *$\Sigma^k(\omega)$ is the empty set or an $(n-k)$-dimensional smooth submanifold of M;*
(*c*) *Let $k > 1$. For each $p \in \Sigma^{k-1}(\omega)$,*

$$p \in \Sigma^k(\omega) \Leftrightarrow \dim(\langle \omega_1(p), \dots, \omega_n(p) \rangle \cap N_p^* \Sigma^{k-1}(\omega)) = k-1.$$

The following proposition shows that Definitions 2.9, 2.10, 2.18 and 2.19 do not depend on the choice of the bases $\{\Omega_1(x), \dots, \Omega_{n-1}(x)\}$ and $\{\Omega_1(x), \dots, \Omega_r(x)\}$. The first part (a) provides equations that define the submanifolds $\Sigma^k(\omega)$ locally. We use these local equations to demonstrate part (b). The proof can be found in Appendix A.

Proposition 2.23.

(*a*) *Let $p \in \Sigma^{k-1}(\omega)$. There are an open neighborhood $\mathcal{U}$ of p in M and smooth functions $F_i : \mathcal{U} \to \mathbb{R}$, $i = 1, \dots, m-r$, such that*

$$\mathcal{U} \cap \Sigma^{k-1}(\omega) = \{x \in \mathcal{U} \mid F_1(x) = \ldots = F_{m-r}(x) = 0\}$$

with $\operatorname{rank}(dF_1(x), \dots, dF_{m-r}(x)) = m-r$ for $x \in \mathcal{U} \cap \Sigma^{k-1}(\omega)$, and there is a collection $\{\Omega_1(x), \dots, \Omega_r(x)\}$ of r smooth one-forms defined on $\mathcal{U}$ which is a basis of a vector subspace complementary to $\langle \bar{\omega}(x) \rangle \cap N_x^* \Sigma^{k-2}(\omega)$ in $\langle \bar{\omega}(x) \rangle$ for each $x \in \mathcal{U} \cap \Sigma^{k-1}(\omega)$. Let

$$\Delta_k(x) = \det(dF_1, \dots, dF_{m-r}, \Omega_1, \dots, \Omega_r)(x).$$

Then ω satisfies the condition I_{k-1} on $\mathcal{U}$ if and only if the following properties hold for each $x \in \mathcal{U} \cap \Sigma^{k-1}(\omega)$:

(*i*) $\dim\langle \Omega_1(x), \dots, \Omega_r(x) \rangle \cap N_x^* \Sigma^{k-1}(\omega) = 0$ *or* 1;

(*ii*) *if* $\dim\langle \Omega_1(x), \dots, \Omega_r(x) \rangle \cap N_x^* \Sigma^{k-1}(\omega) = 1$ *(or equivalently $\Delta_k(x) = 0$), then*

$$\operatorname{rank}(dF_1(x), \dots, dF_{m-r}(x), d\Delta_k(x)) = m-r+1.$$

In this case, $\Sigma^k(\omega)$ can be locally defined as

$$\mathcal{U} \cap \Sigma^k(\omega) = \{x \in \mathcal{U} \mid F_1(x) = \ldots = F_{m-r}(x) = \Delta_k(x) = 0\}.$$

(*b*) *The definitions of $\Sigma^1(\omega)$, $\Sigma^k(\omega)$ and $A_{k-1}(\omega)$ do not depend on the choice of the basis $\{\Omega_1, \dots, \Omega_{n-k+1}\}$, for every $k = 2, \dots, n$.*

Remark 2.24. *It is not difficult to see that, for every $k = 1, \dots, n$, $\Sigma^k(\omega)$ is a closed submanifold of M such that*

$$\Sigma^k(\omega) = A_k(\omega) \cup \Sigma^{k+1}(\omega) = \bigcup_{i=k}^{n} A_i(\omega).$$

Furthermore, $A_k(\omega) = \Sigma^k(\omega) \setminus \Sigma^{k+1}(\omega)$. Then, the singular sets $A_k(\omega)$ are $(n-k)$-dimensional submanifolds of M such that $\overline{A_k(\omega)} = \Sigma^k(\omega)$.

Finally, based on the previous considerations, we define:

Definition 2.25. *Let $\omega = \{\omega_i\}_{1 \le i \le n}$ be a non-degenerate collection on M that satisfies the condition I_j, for $j = 1, \dots, n-1$. For each $k \in \{1, \dots, n\}$, we say that $x \in M$ is an A_k-type Morin singularity of ω if $x \in A_k(\omega)$.*

Definition 2.26. *Let $\omega = \{\omega_i\}_{1 \le i \le n}$ be a collection of n smooth one-forms on M, with $0 < n \le m$. We call ω a Morin collection if ω is non-degenerate and it satisfies the condition I_j, for $j = 1, \dots, n-1$.*

Remark 2.27. *By Definition 2.26, if $\omega = \{\omega_i\}_{1 \le i \le n}$ is a Morin collection then ω admits only A_k-type singular points for $k = 1, \dots, n$.*

As we mentioned in Section 1, fixed a Riemannian metric on M, we can consider vector fields instead of one-forms and define the notion of Morin collection of n vector fields analogously to the definition of Morin collection of n one-forms:

Definition 2.28. *Let $V = \{V_i\}_{1\leq i\leq n}$ be a collection of n smooth vector fields on M, with $0 < n \leq m$. We call V a Morin collection if V is non-degenerate and it satisfies the condition I_j, for $j = 1,\dots,n-1$.*

Next, we present some examples of Morin collections of vector fields.

Example 2.29. *Let $f : M^m \to \mathbb{R}^n$ be a smooth Morin map defined on an m-dimensional Riemannian manifold M, with $m \geq n$. The collection of n vector fields $V(x) = \{\nabla f_1(x),\dots,\nabla f_n(x)\}$ given by the gradients of the coordinate functions of f is, clearly, a Morin collection of vector fields whose singular points are the same as the singular points of f. That is, $A_k(V) = A_k(f)$, for $k = 1,\dots,n$.*

Example 2.30. *Let $a \in \mathbb{R}$ be a regular value of a C^2 mapping $f : \mathbb{R}^3 \to \mathbb{R}$. Suppose that $M = f^{-1}(a)$ and consider $V = \{V_1,V_2\}$ be a collection of 2 vector fields on M, given by*

$$\begin{array}{rcl} V_1(x) & = & (-f_{x_2}(x), f_{x_1}(x), 0); \\ V_2(x) & = & (-f_{x_3}(x), 0, f_{x_1}(x)). \end{array}$$

Since a is a regular value of f, we have that $\nabla f(x) = (f_{x_1}(x), f_{x_2}(x), f_{x_3}(x)) \neq \vec{0}$, $\forall x \in M$. Thus, $\operatorname{rank}(V_1(x),V_2(x))$ is either equal to 2 or equal to 1 . The singular points of V are the points $x \in M$ where $\operatorname{rank}(V_1(x),V_2(x)) = 1$, that is,

$$\Sigma^1(V) = \{x \in M \mid f_{x_1}(x) = 0\}$$

and $V = \{V_1,V_2\}$ is non-degenerate if and only if $\operatorname{rank}(\nabla f(x),\nabla f_{x_1}(x)) = 2$ for each $x \in \Sigma^1(V)$. In this case, $\Sigma^1(V)$ is a submanifold of M of dimension 1. Let $x \in \Sigma^1(V)$ be a singular point of V, then the space $\langle V_1(x),V_2(x)\rangle$ is spanned by the vector $e_1 = (1,0,0)$ and $x \in A_2(V)$ if and only if

$$\operatorname{rank}(\nabla f(x),\nabla f_{x_1}(x), e_1) < 3,$$

that is, if and only if $\Delta_2 := f_{x_2} f_{x_1x_3} - f_{x_3} f_{x_1x_2}$ vanishes at x. Moreover, V satisfies the condition I_1 if and only if $\operatorname{rank}(\nabla f(x),\nabla f_{x_1}(x),\nabla\Delta_2(x)) = 3$ for $x \in A_2(V)$. In this case, $A_2(V)$ is a submanifold of M of dimension 0. Therefore, $V = \{V_1,V_2\}$ is a Morin collection of 2 vector fields if and only if $\operatorname{rank}(\nabla f(x),\nabla f_{x_1}(x)) = 2$ on the singular set $\Sigma^1(V) = \{x \in M \mid f_{x_1}(x) = 0\}$ and $\det(\nabla f(x),\nabla f_{x_1}(x),\nabla\Delta_2(x)) \neq 0$ on $A_2(V) = \{x \in M \mid f_{x_1}(x) = 0, \Delta_2(x) = 0\}$.

Example 2.31. *Let us apply Example 2.30 to the collection of 2 vector fields $V = \{V_1,V_2\}$ defined on the torus $\mathrm{T} := f^{-1}(R^2)$, where R^2 is a regular value of*

$$f(x_1,x_2,x_3) = (\sqrt{x_2^2 + x_3^2} - a)^2 + (x_1 + x_2)^2,$$

with $a > R$. Then, one can verify that $\Sigma^1(V) = \{x \in \mathrm{T} \,|\, x_1 + x_2 = 0\}$, that is,

$$\Sigma^1(V) = \{(x_1,x_2,x_3) \in \mathbb{R}^3 \mid \sqrt{x_2^2 + x_3^2} - a)^2 = R^2\}$$

and $\operatorname{rank}(\nabla f(x),\nabla f_{x_1}(x))$ is equal to

$$\operatorname{rank}\begin{bmatrix} 0 & \dfrac{2x_2(\sqrt{x_2^2+x_3^2}-a)}{\sqrt{x_2^2+x_3^2}} & \dfrac{2x_3(\sqrt{x_2^2+x_3^2}-a)}{\sqrt{x_2^2+x_3^2}} \\ 1 & 1 & 0 \end{bmatrix},$$

which is 2 *for all* $x \in \mathrm{T} \cap \Sigma^1(V)$*. Moreover,*

$$\Delta_2(x) = \frac{-4x_3(\sqrt{x_2^2 + x_3^2} - a)}{\sqrt{x_2^2 + x_3^2}},$$

such that

$$A_2(V) = \{x \in \mathrm{T} \,|\, x_1 + x_2 = 0; x_3 = 0\},$$

which is the set given by the points $(-a - R, a + R, 0)$*,* $(a + R, -a - R, 0)$*,* $(-a + R, a - R, 0)$ *and* $(a - R, -a + R, 0)$*. It is not difficult to see that* $\operatorname{rank}(\nabla f(x), \nabla f_{x_1}(x), \nabla \Delta_2(x)) = 3, \forall x \in \mathrm{T} \cap A_2(V)$*. Therefore, the collection* $V = \{V_1, V_2\}$ *given by*

$$\begin{aligned} V_1(x) &= \left(\frac{-2x_2(\sqrt{x_2^2+x_3^2}-a)}{\sqrt{x_2^2+x_3^2}} - 2(x_1 + x_2), 2(x_1 + x_2), 0 \right); \\ V_2(x) &= \left(\frac{-2x_3(\sqrt{x_2^2+x_3^2}-a)}{\sqrt{x_2^2+x_3^2}}, 0, 2(x_1 + x_2) \right), \end{aligned}$$

is a Morin collection of 2 *vector fields defined on the torus* T *which admits singular points of type* A_1 *and* A_2*.*

Example 2.32. *Let* $a \in \mathbb{R}$ *be a regular value of a* C^2 *mapping* $f : \mathbb{R}^3 \to \mathbb{R}$*. Suppose that* $M = f^{-1}(a)$ *and consider* $\overline{W_1}$ *and* $\overline{W_2}$ *be the orthogonal projections of* $e_2 = (0, 1, 0)$ *and* $e_3 = (0, 0, 1)$ *over* $T_x M$ *given by*

$$\begin{aligned} \overline{W_1} &= e_2 - \left\langle e_2, \tfrac{\nabla f}{|\nabla f|} \right\rangle \tfrac{\nabla f}{|\nabla f|}; \\ \overline{W_2} &= e_3 - \left\langle e_3, \tfrac{\nabla f}{|\nabla f|} \right\rangle \tfrac{\nabla f}{|\nabla f|}. \end{aligned}$$

Let $W = \{W_1, W_2\}$ *be the collection of* 2 *vector fields defined by* $W_1 = \|\nabla f\|^2 \overline{W_1}$ *and* $W_2 = \|\nabla f\|^2 \overline{W_2}$*, that is,*

$$\begin{aligned} W_1 &= (-f_{x_1} f_{x_2}, f_{x_1}^2 + f_{x_3}^2, -f_{x_2} f_{x_3}); \\ W_2 &= (-f_{x_1} f_{x_3}, -f_{x_2} f_{x_3}, f_{x_1}^2 + f_{x_2}^2). \end{aligned}$$

In this case, W_1 *and* W_2 *are gradients vector fields, that is,* W *is a collection of* 2 *gradient vector fields. It is not difficult to see that* $\operatorname{rank}(W_1(x), W_2(x))$ *is either equal to* 2 *or equal to* 1*, and the singular set of* W *is* $\Sigma^1(W) = \{x \in M \mid f_{x_1}(x) = 0\}$*. Let* $x \in \Sigma^1(W)$ *be a singular point of* W*, then the space* $\langle W_1(x), W_2(x) \rangle$ *is spanned by the vector* $(0, f_{x_3}, -f_{x_2})$*, such that* $A_2(W) = \{x \in M \mid f_{x_1}(x) = 0, f_{x_1 x_1}(x) = 0\}$*. Therefore,* $W = \{W_1, W_2\}$ *is a Morin collection of* 2 *vector fields if and only if* $\operatorname{rank}(\nabla f(x), \nabla f_{x_1}(x)) = 2$ *on the singular set* $\Sigma^1(W)$ *and* $\det(\nabla f(x), \nabla f_{x_1}(x), \nabla f_{x_1 x_1}(x)) \neq 0$ *on* $A_2(W)$*.*

Example 2.33. *Let us apply Example 2.32 to the collection of vector fields* $W = \{W_1, W_2\}$ *defined on the torus* $\mathrm{T} := f^{-1}(R^2)$ *of Example 2.31. In this situation, one can verify that* $\Sigma^1(W)$ *is the same singular set as* $\Sigma^1(V)$ *in the Example 2.31. Moreover,* $\operatorname{rank}(\nabla f(x), \nabla f_{x_1}(x)) = 2$ *for every* $x \in \Sigma^1(W)$*. However, since* $f_{x_1 x_1}(x) = 2$ *for every* $x \in \Sigma^1(W)$*,* W *does not admits singular points of type* A_2*. That is,* W *is Morin collection of* 2 *vector fields on* T *which admits only Morin singularities of type* A_1*.*

Example 2.34. *Let us consider the collections* $V = \{V_1, V_2\}$ *and* $W = \{W_1, W_2\}$ *from Examples 2.30 and 2.32 defined on the unit sphere* $M := f^{-1}(1)$*, where* $f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2$*. We know that the singular sets of* V *and* W *are the same, that is,* $\Sigma^1(V) = \Sigma^1(W) = \{x \in M \,|\, x_1 = 0\}$ *and* $\operatorname{rank}(\nabla f(x), \nabla f_{x_1}(x)) = 2$ *for all singular point* x*. However,* $\Delta_2(x) = 0, \forall x \in \Sigma^1(V)$*, such that* $\nabla \Delta_2 \equiv \vec{0}$*. On the other hand,* $f_{x_1 x_1}(x) \neq 0, \forall x \in \Sigma^1(W)$*, such that* $A_2(W) = \varnothing$*. Therefore,*

V is not a Morin collection and W is a Morin collection that admits only Morin singularities of type A_1.

Example 2.35. *In the Example 2.34, if we consider $f(x_1,x_2,x_3) = x_1^2 - x_1x_2 + x_3^2$ then one can verify that V and W are both Morin collections of 2 vector fields that admits only Morin singularities of type A_1. Let us consider the case where V of Example 2.30 is defined on $M := f^{-1}(-1)$ and $f(x_1,x_2,x_3) = x_1^2 - x_1x_2 + x_3^2$. It is easy to see that -1 is a regular value of f and $\Sigma^1(V) = \{x \in M \,|\, 2x_1 - x_2 = 0\}$. That is,*

$$\Sigma^1(V) = \{(x_1,x_2,x_3) \in \mathbb{R}^3 \,|\, x_1^2 - x_1x_2 + x_3^2 + 1 = 0; 2x_1 - x_2 = 0\}$$

and $\operatorname{rank}(\nabla f(x), \nabla f_{x_1}(x))$ *is equal to*

$$\operatorname{rank}\begin{bmatrix} (2x_1 - x_2) & -x_1 & 2x_3 \\ 2 & -1 & 0 \end{bmatrix}$$

which is 2, for all $x \in M \cap \Sigma^1(V)$. Moreover, $\Delta_2(x) = 2x_3$ and

$$A_2(V) = \{(x_1,x_2,x_3) \in \mathbb{R}^3 \,|\, x_1^2 - x_1x_2 + x_3^2 + 1 = 0; 2x_1 - x_2 = 0; x_3 = 0\}$$

which is the set given by the points $(1,2,0)$ and $(-1,-2,0)$. We also have that

$$\det(\nabla f(x), \nabla f_{x_1}(x), \nabla \Delta_2(x))$$

is equal to

$$\det\begin{bmatrix} (2x_1 - x_2) & -x_1 & 2x_3 \\ 2 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = 4x_1$$

which is equal to ± 4 for each $x \in A_2(V)$. That is, $\operatorname{rank}(\nabla f(x), \nabla f_{x_1}(x), \nabla \Delta_2(x)) = 3$, for all $x \in M \cap A_2(V)$. Therefore, the collection $V = \{V_1, V_2\}$ given by

$$\begin{aligned} V_1(x) &= (x_1, 2x_1 - x_2, 0); \\ V_2(x) &= (-2x_3, 0, 2x_1 - x_2). \end{aligned}$$

is a Morin collection of 2 vector fields defined on M which admits singular points of type A_1 and A_2.

3. Zeros of a generic one-form $\xi(x)$ associated to a Morin collection of one-forms

Let $a = (a_1, \dots, a_n) \in \mathbb{R}^n \setminus \{\vec{0}\}$ and let $\omega = \{\omega_i\}_{1 \le i \le n}$ be a Morin collection of n smooth one-forms defined on an m-dimensional manifold M. In this section, we will consider the one-form $\xi(x) = \sum_{i=1}^{n} a_i \omega_i(x)$ defined on M and we will prove some properties of the zeros of ξ and its restrictions to the singular sets of ω. We will consider the notation $\langle \bar{\omega}(x) \rangle = \langle \omega_1(x), \dots, \omega_n(x) \rangle$.

Lemma 3.1. *If p is a zero of the one-form ξ then $p \in \Sigma^1(\omega)$ and p is a zero of $\xi_{|_{\Sigma^1(\omega)}}$.*

Proof. Suppose that $\xi(p) = 0$. So $\operatorname{rank}(\omega_1(p), \dots, \omega_n(p)) \le n-1$, since $a \ne \vec{0}$. However, the collection ω is non-degenerate, thus $\operatorname{rank}(\omega_1(p), \dots, \omega_n(p)) = n-1$. That is, $p \in \Sigma^1(\omega)$. Moreover, $\xi(p) = 0$ implies that $T_pM \subset \ker(\xi(p))$ and since $T_p\Sigma_1(\omega) \subset T_pM$, we conclude that p is a zero of $\xi_{|_{\Sigma^1(\omega)}} = 0$. □

Lemma 3.2. *If $p \in A_{k+1}(\omega)$, then for each $k = 0, \dots, n-2$, p is a zero of $\xi_{|_{\Sigma^{k+1}(\omega)}}$ if and only if p is a zero of $\xi_{|_{\Sigma^k(\omega)}}$.*

Proof. Suppose that $p \in A_{k+1}(\omega)$ and that, locally, we have:

$$\begin{aligned} \mathcal{U} \cap \Sigma^k(\omega) &= \{x \in \mathcal{U} \mid F_1(x) = \ldots = F_{m-n+1}(x) = \Delta_2(x) = \ldots = \Delta_k(x) = 0\}; \\ \mathcal{U} \cap \Sigma^{k+1}(\omega) &= \{x \in \mathcal{U} \mid F_1(x) = \ldots = F_{m-n+1}(x) = \Delta_2(x) = \ldots = \Delta_{k+1}(x) = 0\}; \end{aligned}$$

for an open neighborhood $\mathcal{U}$ of p in M. If p is a zero of the restriction $\xi_{|_{\Sigma^k(\omega)}}$ then $\xi(p) \in N_p^*\Sigma^k(\omega) = \langle dF_1(p), \ldots, dF_{m-n+1}(p), d\Delta_2(p), \ldots, d\Delta_k(p)\rangle$. In particular, $\xi(p) \in N_p^*\Sigma^{k+1}(\omega)$, therefore p is a zero of $\xi_{|_{\Sigma^{k+1}(\omega)}}$.

On the other hand, if p is a zero of $\xi_{|_{\Sigma^{k+1}(\omega)}}$ then $\xi(p) \in N_p^*\Sigma^{k+1}(\omega) \cap \langle \bar{\omega}(p)\rangle$.

Since $p \in A_{k+1}(\omega)$, we have that $p \in \Sigma_{k+1}(\omega) \setminus \Sigma_{k+2}(\omega)$, thus

$$\begin{cases} \dim(\langle \bar{\omega}(p)\rangle \cap N_p^*\Sigma^k(\omega)) = k; \\ \dim(\langle \bar{\Omega}^{k+1}(p)\rangle \cap N_p^*\Sigma^{k+1}(\omega)) = 0; \end{cases}$$

where $\bar{\Omega}^{k+1}(p)$ represents a smooth basis for a vector subspace complementary to $\langle \bar{\omega}(p)\rangle \cap N_p^*\Sigma^k(\omega)$ in $\langle \bar{\omega}(p)\rangle$. Since $\dim(N_p^*\Sigma^k(\omega)) = m-n+k$, $\dim(N_p^*\Sigma^{k+1}(\omega)) = m-n+k+1$ and $N_p^*\Sigma^k(\omega) \subset N_p^*\Sigma^{k+1}(\omega)$, we have

$$\dim(\langle \bar{\omega}(p)\rangle \cap N_p^*\Sigma^{k+1}(\omega)) = \dim(\langle \bar{\omega}(p)\rangle \cap N_p^*\Sigma^k(\omega)) = k.$$

Thus, $\langle \bar{\omega}(p)\rangle \cap N_p^*\Sigma^k(\omega) = \langle \bar{\omega}(p)\rangle \cap N_p^*\Sigma^{k+1}(\omega)$. Therefore, $\xi(p) \in N_p^*\Sigma^k(\omega)$, that is, p is a zero of $\xi_{|_{\Sigma^k(\omega)}}$. ☐

Lemma 3.3. *If $p \in A_n(\omega)$ then p is a zero of the restriction $\xi_{|_{\Sigma^{n-1}(\omega)}}$.*

Proof. Analogously to Lemma 3.2, we consider local equations of $\Sigma^n(\omega)$:

$$\mathcal{U} \cap \Sigma^n(\omega) = \{x \in \mathcal{U} \mid F_1(x) = \ldots = F_{m-n+1}(x) = \Delta_2(x) = \ldots = \Delta_n(x) = 0\},$$

with $N_x^*\Sigma^n(\omega) = \langle dF_1(x), \ldots, dF_{m-n+1}(x), d\Delta_2(x), \ldots, d\Delta_n(x)\rangle$. Since $A_n(\omega) = \Sigma^n(\omega)$, if $p \in A_n(\omega)$ then

$$\dim(\langle \bar{\omega}(p)\rangle \cap N_p^*\Sigma^{n-1}(\omega)) = n-1.$$

Thus, $\langle \bar{\omega}(p)\rangle \subset N_p^*\Sigma^{n-1}(\omega)$ and consequently, $\xi(p) \in N_p^*\Sigma^{n-1}(\omega)$. Therefore, p is a zero of $\xi_{|_{\Sigma^{n-1}(\omega)}}$. ☐

Remark 3.4. *If $p \in \Sigma^1(\omega)$ then* $\operatorname{rank}(\omega_1(p), \ldots, \omega_n(p)) = n-1$ *and, writing $\omega_i = (\omega_i^1, \ldots, \omega_i^m)$, we can assume that*

$$\boldsymbol{M}(x) = \begin{vmatrix} \omega_1^1(x) & \omega_2^1(x) & \cdots & \omega_{n-1}^1(x) \\ \vdots & \vdots & \ddots & \vdots \\ \omega_1^{n-1}(x) & \omega_2^{n-1}(x) & \cdots & \omega_{n-1}^{n-1}(x) \end{vmatrix} \neq 0, \tag{2}$$

for all x in an open neighborhood $\mathcal{U}$ of p in M. In particular, if $p \in \mathcal{U}$ is a singular point of ξ then $a_n \neq 0$, otherwise, we would have $a_1 = \ldots = a_{n-1} = a_n = 0$. We will use this fact in next results.

Lemma 3.5. *Let $p \in \Sigma^1(\omega)$ such that $\boldsymbol{M}(p) \neq 0$. Then $\xi(p) = 0$ if and only if $\sum_{i=1}^n a_i \omega_i^j(p) = 0$, for every $j = 1, \ldots, n-1$.*

Proof. It follows easily from the definition of $\Sigma^1(\omega)$ and ξ. ☐

Lemma 3.6. *Let $Z(\xi)$ be the zero set of the one-form ξ. Then for almost every $a \in \mathbb{R}^n \smallsetminus \{\vec{0}\}$, $Z(\xi) \cap \Sigma^2(\omega) = \varnothing$.*

Proof. Let $\mathcal{U}$ be an open subset of M on which $\mathbf{M}(x) \neq 0$ and

$$\mathcal{U} \cap \Sigma^2(\omega) = \{x \in \mathcal{U} \,|\, F_1(x) = \ldots = F_{m-n+1}(x) = \Delta_2(x) = 0\},$$

with $\operatorname{rank}(dF_1(x), \ldots, dF_{m-n+1}(x), d\Delta_2(x)) = m-n+2$, for each $x \in \Sigma^2(\omega) \cap \mathcal{U}$. Let us consider $F : \mathcal{U} \times \mathbb{R}^n \smallsetminus \{\vec{0}\} \to \mathbb{R}^{m+1}$ the mapping defined by

$$F(x,a) = (F_1(x), \ldots, F_{m-n+1}(x), \Delta_2(x), \sum_{i=1}^n a_i \omega_i^1(x), \ldots, \sum_{i=1}^n a_i \omega_i^{n-1}(x)).$$

By Lemma 3.5, if $x \in \Sigma^1(\omega)$ then

$$\sum_{i=1}^n a_i \omega_i(x) = 0 \Leftrightarrow \sum_{i=1}^n a_i \omega_i^j(x) = 0, \forall j = 1, \ldots, n-1.$$

Thus, if $(x,a) \in F^{-1}(\vec{0})$ we have that $x \in Z(\xi) \cap \Sigma^2(\omega)$. Furthermore, the Jacobian matrix of F at a point $(x,a) \in F^{-1}(\vec{0})$:

$$\left[\begin{array}{c|cccc}
dF_1(x) & & & & \\
\vdots & & & & \\
dF_{m-n+1}(x) & & O_{(m-n+2)\times n} & & \\
d\Delta_2(x) & & & & \\
\cdots\cdots\cdots\cdots & \cdots\cdots\cdots & \cdots & \cdots\cdots\cdots & \cdots\cdots \\
 & \omega_1^1(x) & \cdots & \omega_{n-1}^1(x) & \omega_n^1(x) \\
(*) & \omega_1^2(x) & \cdots & \omega_{n-1}^2(x) & \omega_n^2(x) \\
 & \vdots & \ddots & \vdots & \vdots \\
 & \omega_1^{n-1}(x) & \cdots & \omega_{n-1}^{n-1}(x) & \omega_n^{n-1}(x)
\end{array}\right]$$

has rank $m+1$. That is, $\vec{0}$ is regular value of F and $F^{-1}(\vec{0})$ is a submanifold of dimension $n-1$. Let $\pi : F^{-1}(\vec{0}) \to \mathbb{R}^n \smallsetminus \{\vec{0}\}$ be the projection over $\mathbb{R}^n \smallsetminus \{\vec{0}\}$ given by $\pi(x,a) = a$, by Sard's Theorem, a is regular value of π for almost every $a \in \mathbb{R}^n \smallsetminus \{\vec{0}\}$. Therefore, $\pi^{-1}(a) \cap F^{-1}(\vec{0}) = \varnothing$ for almost every $a \in \mathbb{R}^n \smallsetminus \{\vec{0}\}$. However, $\pi^{-1}(a) \cap F^{-1}(\vec{0}) = \{(x,a) \in \mathcal{U} \times \{a\} : x \in Z(\xi) \cap \Sigma^2(\omega)\}$. Thus, $Z(\xi) \cap \Sigma^2(\omega) = \varnothing$ for almost every $a \in \mathbb{R}^n \smallsetminus \{\vec{0}\}$. □

Lemma 3.7. *Let $Z(\xi_{|_{\Sigma^k(\omega)}})$ be the zero set of the restriction of the one-form ξ to $\Sigma^k(\omega)$, with $k \geq 1$. Then for almost every $a \in \mathbb{R}^n \smallsetminus \{\vec{0}\}$, $Z(\xi_{|_{\Sigma^k(\omega)}}) \cap \Sigma^{k+2}(\omega) = \varnothing$.*

Proof. For each $k = 1, \ldots, n-2$, let $\mathcal{U}$ be an open subset of M on which

$$\mathcal{U} \cap \Sigma^k(\omega) = \{x \in \mathcal{U} \,|\, F_1(x) = \ldots = F_{m-n+k}(x) = 0\},$$

with $\operatorname{rank}(dF_1(x), \ldots, dF_{m-n+k}(x)) = m-n+k$, for all $x \in \mathcal{U} \cap \Sigma^k(\omega)$ and

$$\mathcal{U} \cap \Sigma^{k+2}(\omega) = \{x \in \mathcal{U} \,|\, F_1(x) = \ldots = F_{m-n+k+2}(x) = 0\},$$

with $\operatorname{rank}(dF_1(x), \ldots, dF_{m-n+k+2}(x)) = m-n+k+2$, for all $x \in \mathcal{U} \cap \Sigma^{k+2}(\omega)$.

By Szafraniec's characterization (see [19, p. 196]) adapted to one-forms, x is a zero of the restriction $\xi_{|_{\Sigma^k(\omega)}}$ if and only if there exists $(\lambda_1, \ldots, \lambda_{m-n+k}) \in \mathbb{R}^{m-n+k}$ such that

$$\xi(x) = \sum_{j=1}^{m-n+k} \lambda_j dF_j(x).$$

Let us write $\xi(x) = (\xi_1(x), \dots, \xi_m(x))$, where $\xi_s(x) = \sum_{i=1}^{n} a_i \omega_i^s(x)$, $s = 1, \dots, m$, we define

$$N_s(x, a, \lambda) := \xi_s(x) - \sum_{j=1}^{m-n+k} \lambda_j \frac{\partial F_j}{\partial x_s}(x),$$

such that $\xi_{|_{\Sigma^k(\omega)}}(x) = 0$ if and only if $N_s(x, a, \lambda) = 0$, for all $s = 1, \dots, m$.

Let $F : \mathcal{U} \times \mathbb{R}^n \setminus \{\vec{0}\} \times \mathbb{R}^{m-n+k} \to \mathbb{R}^{2m-n+k+2}$ be the mapping defined by

$$F(x, a, \lambda) = (F_1, \dots, F_{m-n+k+2}, N_1, \dots, N_m),$$

if $(x, a, \lambda) \in F^{-1}(\vec{0})$ then $x \in Z(\xi_{|_{\Sigma^k(\omega)}}) \cap \Sigma^{k+2}(\omega)$ and the Jacobian matrix of F at (x, a, λ):

$$\left[\begin{array}{c|cc} dF_1(x) & & \\ \vdots & \multicolumn{2}{c}{O_{(m-n+k+2)\times(m+k)}} \\ dF_{m-n+k+2}(x) & & \\ \hline d_x N_1(x, a, \lambda) & & \\ \vdots & B_{m\times n} & C_{m\times(m-n+k)} \\ d_x N_m(x, a, \lambda) & & \end{array}\right]$$

has rank $2m-n+k+1$, where $O_{(m-n+k+2)\times(m+k)}$ is a null matrix, $B_{m\times n}$ is a matrix whose columns vectors are given by the coefficients of the one-forms $\omega_1(x), \dots, \omega_n(x)$ of the collection ω:

$$B_{m\times n} = \begin{bmatrix} \omega_1^1(x) & \cdots & \omega_n^1(x) \\ \vdots & \ddots & \vdots \\ \omega_1^m(x) & \cdots & \omega_n^m(x) \end{bmatrix}$$

and $C_{m\times(m-n+k)}$ is the matrix whose columns vectors are, up to sign, the coefficients of the derivatives $dF_1, \dots, dF_{m-n+k}$ with respect to x:

$$C_{m\times(m-n+k)} = \begin{bmatrix} -\frac{\partial F_1}{\partial x_1}(x) & \cdots & -\frac{\partial F_{m-n+k}}{\partial x_1}(x) \\ \vdots & \ddots & \vdots \\ -\frac{\partial F_1}{\partial x_m}(x) & \cdots & -\frac{\partial F_{m-n+k}}{\partial x_m}(x) \end{bmatrix}.$$

Notice that, if $(x, a, \lambda) \in F^{-1}(\vec{0})$ then $x \in \Sigma^{k+1}(\omega)$ and, by Lemma 2.21,

$$\dim(\langle \bar{\omega}(x) \rangle \cap N_x^* \Sigma^k(\omega)) = k.$$

Thus, $\dim(\langle \bar{\omega}(x) \rangle + N_x^* \Sigma^k(\omega)) = m - 1$. Therefore,

$$\operatorname{rank} \left[\begin{array}{c|c} B_{m\times n} & C_{m\times(m-n+k)} \end{array}\right] = m - 1$$

and the Jacobian matrix of F at (x, a, λ) has rank $2m - n + k + 1$. That is, $F^{-1}(\vec{0})$ has dimension less or equal to $n - 1$. Let $\pi : F^{-1}(\vec{0}) \to \mathbb{R}^n \setminus \{\vec{0}\}$ be the projection over $\mathbb{R}^n \setminus \{\vec{0}\}$, that is, $\pi(x, a, \lambda) = a$. By Sard's Theorem, a is regular value of π for almost every $a \in \mathbb{R}^n \setminus \{\vec{0}\}$. Therefore, $\pi^{-1}(a) \cap F^{-1}(\vec{0}) = \varnothing$ for almost every $a \in \mathbb{R}^n \setminus \{\vec{0}\}$. However,

$$\pi^{-1}(a) \cap F^{-1}(\vec{0}) = \{(x, a, \lambda) \in \mathcal{U} \times \{a\} \times \mathbb{R}^{m-n+k} \mid x \in Z(\xi_{|_{\Sigma^k(\omega)}}) \cap \Sigma^{k+2}(\omega)\}.$$

Thus, $Z(\xi_{|_{\Sigma^k(\omega)}}) \cap \Sigma^{k+2}(\omega) = \varnothing$ for almost every $a \in \mathbb{R}^n \setminus \{\vec{0}\}$. $\square$

4. Non-degenerate zeros of a generic one-form $\xi(x)$ associated to a Morin collection of one-forms

In this section we will verify that, generically, the one-form $\xi(x)$ and its restrictions $\xi_{|_{\Sigma^k(\omega)}}$, $\xi_{|_{A_k(\omega)}}$ admit only non-degenerate zeros. Furthermore, we will see how these non-degenerate zeros can be related. Then, we end the paper with our main result (Theorem 4.13).

We start with some technical lemmas.

Lemma 4.1. *Let A be a square matrix of order m given by:*

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ a_{21} & \cdots & a_{2m} \\ \vdots & \cdots & \vdots \\ a_{m1} & \cdots & a_{mm} \end{bmatrix}.$$

If there exist $(\lambda_1, \dots, \lambda_m) \in \mathbb{R}^m \smallsetminus \{\vec{0}\}$ such that $\sum_{j=1}^{m} \lambda_j a_{ij} = 0$, $i = 1, \dots, m$, then

$$\lambda_j \operatorname{cof}(a_{ik}) - \lambda_k \operatorname{cof}(a_{ij}) = 0, \ \forall j, k = 1, \dots, m.$$

Lemma 4.2. *Let us consider the matrix*

$$M_i(x) = \begin{bmatrix} \omega_1^1(x) & \cdots & \omega_{n-1}^1(x) & \omega_n^1(x) \\ \vdots & \ddots & \vdots & \vdots \\ \omega_1^{n-1}(x) & \cdots & \omega_{n-1}^{n-1}(x) & \omega_n^{n-1}(x) \\ \omega_1^i(x) & \cdots & \omega_{n-1}^i(x) & \omega_n^i(x) \end{bmatrix}.$$

If x is a zero of ξ then for $\ell \in \{1, \dots, n-1\}$, $j \in \{1, \dots, n-1, i\}$ and $i \in \{n, \dots, m\}$, we have

$$a_n \operatorname{cof}(\omega_\ell^j, M_i) = a_\ell \operatorname{cof}(\omega_n^j, M_i).$$

Proof. This result is a consequence of Lemma 4.1 applied to the matrix $A = M_i(x)$, where $a_{\ell j} = \omega_j^\ell(x)$, for $j = 1, \dots, n$ and $\ell = 1, \dots, n-1, i$. It is enough to take $(\lambda_1, \dots, \lambda_n) = (a_1, \dots, a_n)$. □

Lemma 4.3. *Let $\mathcal{U} \subset \mathbb{R}^m$ be an open set and let $H : \mathcal{U} \times \mathbb{R}^n \smallsetminus \{\vec{0}\} \to \mathbb{R}^m$ be a smooth mapping given by $H(x,a) = (h_1(x,a), \dots, h_m(x,a))$. If*

$$\operatorname{rank}(dh_1(x,a), \dots, dh_m(x,a)) = m, \forall (x,a) \in H^{-1}(\vec{0})$$

then $\operatorname{rank}(d_x h_1(x,a), \dots, d_x h_m(x,a)) = m$ *for almost every $a \in \mathbb{R}^n \smallsetminus \{\vec{0}\}$.*

In the previous section we proved that every zero of ξ belongs to $\Sigma^1(\omega)$. Next, we will show that, generically, such zeros belong to $A_1(\omega)$ and they are non-degenerate. To do that, we must find explicit equations that define the manifolds $T^*M^{n,n-1}$ and $\Sigma^1(\omega)$ locally.

Lemma 4.4. *Let $(p, \tilde{\varphi}) \in T^*M^{n,n-1}$, it is possible to exhibit, explicitly, functions $\mathrm{m}_i(x,\varphi) : \tilde{\mathcal{U}} \to \mathbb{R}$, $i = n, \dots, m$, defined on an open neighborhood $\tilde{\mathcal{U}}$ of $(p, \tilde{\varphi})$ in T^*M^n, such that, locally*

$$T^*M^{n,n-1} = \left\{ (x,\varphi) \in \tilde{\mathcal{U}} \mid \mathrm{m}_n = \ldots = \mathrm{m}_m = 0 \right\}$$

with $\operatorname{rank}(d\mathrm{m}_n, \dots, d\mathrm{m}_m) = m - n + 1$, *for all $(x,\varphi) \in T^*M^{n,n-1} \cap \tilde{\mathcal{U}}$.*

Proof. Let $(p,\tilde{\varphi}) \in T^*M^{n,n-1}$, we may assume that

$$m(\varphi) = \begin{vmatrix} \varphi_1^1 & \varphi_2^1 & \cdots & \varphi_{n-1}^1 \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_1^{n-1} & \varphi_2^{n-1} & \cdots & \varphi_{n-1}^{n-1} \end{vmatrix} \neq 0$$

for (x,φ) in an open neighborhood $\tilde{\mathcal{U}}$ of $(p,\tilde{\varphi})$ in T^*M^n. In this situation, $T^*M^{n,n-1}$ can be locally defined as

$$T^*M^{n,n-1} = \left\{(x,\varphi) \in \tilde{\mathcal{U}} \mid m_n = \ldots = m_m = 0\right\},$$

where $m_i := m_i(\varphi)$ is the determinant

$$m_i(\varphi) = \begin{vmatrix} \varphi_1^1 & \varphi_2^1 & \cdots & \varphi_{n-1}^1 & \varphi_n^1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \varphi_1^{n-1} & \varphi_2^{n-1} & \cdots & \varphi_{n-1}^{n-1} & \varphi_n^{n-1} \\ \varphi_1^{i} & \varphi_2^{i} & \cdots & \varphi_{n-1}^{i} & \varphi_n^{i} \end{vmatrix}, \quad i = n,\ldots,m.$$

Let us verify that $\operatorname{rank}(dm_n,\ldots,dm_m) = m-n+1$ in $(T^*M^{n,n-1}) \cap \tilde{\mathcal{U}}$.

For clearer notations, consider $I = \{1,\ldots,n\}$ and $I_i = \{1,\ldots,n-1,i\}$ for each $i \in \{n,\ldots,m\}$. Then

$$dm_i(\varphi) = \sum_{j\in I,\ell\in I_i} \operatorname{cof}(\varphi_j^\ell, m_i)\, d\varphi_j^\ell, \tag{3}$$

where $\operatorname{cof}(\varphi_j^\ell, m_i)$ is the cofactor of φ_j^ℓ in the matrix

$$\begin{bmatrix} \varphi_1^1 & \varphi_2^1 & \cdots & \varphi_{n-1}^1 & \varphi_n^1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \varphi_1^{n-1} & \varphi_2^{n-1} & \cdots & \varphi_{n-1}^{n-1} & \varphi_n^{n-1} \\ \varphi_1^{i} & \varphi_2^{i} & \cdots & \varphi_{n-1}^{i} & \varphi_n^{i} \end{bmatrix}$$

and

$$d\varphi_j^\ell = \left(\frac{\partial\varphi_j^\ell}{\partial\varphi_1^1},\ldots,\frac{\partial\varphi_j^\ell}{\partial\varphi_1^m},\frac{\partial\varphi_j^\ell}{\partial\varphi_2^1},\ldots,\frac{\partial\varphi_j^\ell}{\partial\varphi_2^m},\ldots,\frac{\partial\varphi_j^\ell}{\partial\varphi_n^1},\ldots,\frac{\partial\varphi_j^\ell}{\partial\varphi_n^m}\right)$$

is the vector whose coordinate at the position $(j-1)m+\ell$ is equal to 1 and all the others are zero. In particular, since $i \in \{n,\ldots,m\}$,

$$d\varphi_n^i = (0,\ldots,0,\underbrace{0,\ldots,\overset{i}{1},\ldots,0}_{m-n+1}) \in \underbrace{(\mathbb{R}^m)^* \times \ldots \times (\mathbb{R}^m)^*}_{n \text{ times}}$$

and the $m-n+1$ last coordinates of $d\varphi_j^\ell$ are zero for all $j \neq n$ or $\ell \neq i$. Moreover,

$$\operatorname{cof}(\varphi_n^i, m_i) = m(\varphi) \neq 0, \quad \text{for} \quad i = n,\ldots,m.$$

Thus,

$$\frac{\partial(m_n,\ldots,m_m)}{\partial(\varphi_n^n,\ldots,\varphi_n^m)} = \begin{vmatrix} \operatorname{cof}(\varphi_n^n, m_n) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \operatorname{cof}(\varphi_n^m, m_m) \end{vmatrix}.$$

That is, for all $(x,\varphi) \in (T^*M^{n,n-1}) \cap \tilde{\mathcal{U}}$, we have

$$\frac{\partial(m_n,\ldots,m_m)}{\partial(\varphi_n^n,\ldots,\varphi_n^m)} = m(\varphi)^{(m-n+1)} \begin{vmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{vmatrix} \neq 0. \tag{4}$$

Therefore, $\operatorname{rank}(m_n,\ldots,m_m) = m-n+1$ for all $(x,\varphi) \in (T^*M^{n,n-1}) \cap \tilde{\mathcal{U}}$. □

Lemma 4.5. *Let $p \in \Sigma^1(\omega)$ be a singular point of ω, it is possible to exhibit, explicitly, functions $\mathbf{M}_i(x) : \mathcal{U} \to \mathbb{R}$, $i = n, \dots, m$, defined on an open neighborhood $\mathcal{U}$ of p in M, such that, locally*

$$\mathcal{U} \cap \Sigma^1(\omega) = \{x \in \mathcal{U} \mid \mathbf{M}_n(x) = \dots = \mathbf{M}_m(x) = 0\}$$

with $\operatorname{rank}(d\mathbf{M}_n(x), \dots, d\mathbf{M}_m(x)) = m - n + 1$, *for all $x \in \Sigma^1(\omega) \cap \mathcal{U}$.*

Proof. Let $\omega = \{\omega_i\}_{1 \le i \le n}$ be a Morin collection of one-forms and let $p \in \Sigma^1(\omega)$. By Remark 3.4, we can consider $\mathcal{U}$ an open neighborhood of p in M, where $\mathbf{M}(x) \neq 0$. Thus, in this neighborhood the set $\Sigma^1(\omega)$ can be defined as

$$\mathcal{U} \cap \Sigma^1(\omega) = \{x \in \mathcal{U} \mid \mathbf{M}_n = \dots = \mathbf{M}_m = 0\},$$

where $\mathbf{M}_i := \mathbf{M}_i(x)$ is the determinant

$$\mathbf{M}_i(x) = \begin{vmatrix} \omega_1^1(x) & \omega_2^1(x) & \cdots & \omega_{n-1}^1(x) & \omega_n^1(x) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \omega_1^{n-1}(x) & \omega_2^{n-1}(x) & \cdots & \omega_{n-1}^{n-1}(x) & \omega_n^{n-1}(x) \\ \omega_1^i(x) & \omega_2^i(x) & \cdots & \omega_{n-1}^i(x) & \omega_n^i(x) \end{vmatrix} \tag{5}$$

for $i = n, \dots, m$.

Let $G(\omega) = \{(x, \omega_1(x), \dots, \omega_n(x)) \mid x \in M\}$ be the graph of the collection ω. For each $x \in \Sigma^1(\omega) \cap \mathcal{U}$, we have that $G(\omega) \pitchfork T^*M^{n,n-1}$ at $(x, \omega(x))$. Then, the equations that define $G(\omega)$ and $T^*M^{n,n-1}$ locally are independent at $(x, \omega(x))$. By similar arguments to that used in the proof of Lemma 4.4, it follows that the functions $\mathbf{M}_n(x), \dots, \mathbf{M}_m(x)$ are independent at x, that is, for all $x \in \Sigma^1(\omega) \cap \mathcal{U}$, $\operatorname{rank}(d\mathbf{M}_n(x), \dots, d\mathbf{M}_m(x)) = m - n + 1$. ☐

Lemma 4.6. *For almost every $a \in \mathbb{R}^n \setminus \{\vec{0}\}$, the one-form $\xi(x) = \sum_{i=1}^n a_i \omega_i(x)$ admits only non-degenerate zeros. Moreover, such zeros belong to $A_1(\omega)$.*

Proof. Suppose that $p \in M$ is a zero of ξ. Then, by Lemmas 3.1 and 3.6, for almost every $a \in \mathbb{R}^n \setminus \{\vec{0}\}$ we have that $p \in \Sigma^1(\omega) \setminus \Sigma^2(\omega)$, that is, $p \in A_1(\omega)$. Assume that $\mathbf{M}(x) \neq 0$ in an open neighborhood $\mathcal{U}$ of p in M (see Remark 3.4) such that

$$\mathcal{U} \cap \Sigma^1(\omega) = \{x \in \mathcal{U} : \mathbf{M}_n(x) = \dots = \mathbf{M}_m(x) = 0\}.$$

Let us write

$$\xi_s(x) = \sum_{i=1}^n a_i \omega_i^s(x),\ s = 1, \dots, m$$

and let us consider the mapping $F : \mathcal{U} \times \mathbb{R}^n \setminus \{\vec{0}\} \to \mathbb{R}^m$ defined by

$$F(x, a) = (\mathbf{M}_n(x), \dots, \mathbf{M}_m(x), \xi_1(x), \dots, \xi_{n-1}(x)).$$

Its Jacobian matrix at a point (x, a) is given by:

$$\operatorname{Jac} F(x, a) = \left[\begin{array}{c|cccc} d_x\mathbf{M}_n(x) & & & & \\ \vdots & & O_{(m-n)\times n} & & \\ d_x\mathbf{M}_m(x) & & & & \\ \hline d_x\xi_1(x) & \omega_1^1(x) & \cdots & \omega_{n-1}^1(x) & \omega_n^1(x) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ d_x\xi_{n-1}(x) & \omega_1^{n-1}(x) & \cdots & \omega_{n-1}^{n-1}(x) & \omega_n^{n-1}(x) \end{array} \right].$$

Notice that, by Lemma 3.5, $F^{-1}(\vec{0})$ corresponds to the zeros of ξ on $\Sigma^1(\omega)\cap\mathcal{U}$. Since $\mathbf{M}(x)\neq 0$ and $\operatorname{rank}(d\mathbf{M}_n(x),\dots,d\mathbf{M}_m(x)) = m-n+1$ for all $x\in\Sigma^1(\omega)\cap\mathcal{U}$, then $\operatorname{rank}(\operatorname{Jac}F(x,a)) = m$ for all $(x,a)\in F^{-1}(\vec{0})$. Thus, $\dim F^{-1}(\vec{0}) = n$.

Let $\pi : F^{-1}(\vec{0})\to\mathbb{R}^n\smallsetminus\{\vec{0}\}$ be the projection $\pi(x,a) = a$, by Sard's Theorem, almost every $a\in\mathbb{R}^n\smallsetminus\{\vec{0}\}$ is a regular value of π and $\dim(\pi^{-1}(a)\cap F^{-1}(\vec{0})) = 0$. That is, for almost every a, the zeros of ξ are isolated in $\Sigma^1(\omega)$. Let us proof that, moreover, these zeros are non-degenerate.

Since $\operatorname{rank}(\operatorname{Jac}F(x,a)) = m$, for all $(x,a)\in F^{-1}(\vec{0})$, then by Lemma 4.3 we have that

$$\operatorname{rank}(d_x\mathbf{M}_n(p),\dots,d_x\mathbf{M}_m(p),d_x\xi_1(p),\dots,d_x\xi_{n-1}(p)) = m,$$

which happens if and only if $\operatorname{rank}(B) = m$, where B is the matrix

$$B = \begin{bmatrix} d_x\xi_1(p)\\ \vdots\\ d_x\xi_{n-1}(p)\\ a_n d_x\mathbf{M}_n(p)\\ \vdots\\ a_n d_x\mathbf{M}_m(p)\end{bmatrix}$$

whose row vectors we will denote by $R_i, i = 1,\dots,m$ (by Remark 3.4, $a_n\neq 0$).

Let us denote $I = \{1,\dots,n\}$ and $I_i = \{1,\dots,n-1,i\}$ for each $i\in\{n,\dots,m\}$. By Equation (5), we can write

$$d\mathbf{M}_i(x) = \sum_{\ell\in I, j\in I_i}\operatorname{cof}(\omega_\ell^j(x),M_i)\,d\omega_\ell^j(x)$$

and by Lemma 4.2,

$$d\mathbf{M}_i(p) = \sum_{\ell\in I, j\in I_i}\frac{a_\ell}{a_n}\operatorname{cof}(\omega_n^j(p),M_i)\,d\omega_\ell^j(p).$$

Thus,

$$\begin{aligned}
a_n d\mathbf{M}_i(p) &= \sum_{\ell\in I, j\in I_i} a_\ell\operatorname{cof}(\omega_n^j(p),M_i)\,d\omega_\ell^j(p)\\
&= \sum_{j\in I_i}\operatorname{cof}(\omega_n^j(p),M_i)\left[\sum_{\ell\in I}a_\ell\, d\omega_\ell^j(p)\right]\\
&= \sum_{j\in I_i}\operatorname{cof}(\omega_n^j(p),M_i)\left[d_x\xi_j(p)\right]\\
&= \operatorname{cof}(\omega_n^i(p),M_i)\left[d_x\xi_i(p)\right] + \sum_{j\in I_i\smallsetminus\{i\}}\operatorname{cof}(\omega_n^j(p),M_i)\left[d_x\xi_j(p)\right].
\end{aligned}$$

Notice that, $\operatorname{cof}(\omega_n^i(p),M_i) = \mathbf{M}(p)\neq 0$, for all $i = n,\dots,m$. Then, for each $i = n,\dots,m$, we replace the i^{th} row R_i of matrix B by

$$\frac{1}{\operatorname{cof}(\omega_n^i(p),M_i)}\left(R_i - \sum_{j=1}^{n-1}\operatorname{cof}(\omega_n^j(p),M_i)R_j\right)$$

such that we obtain the matrix of maximal rank:

$$\begin{bmatrix} d_x\xi_1(p)\\ \vdots\\ d_x\xi_{n-1}(p)\\ d_x\xi_n(p)\\ \vdots\\ d_x\xi_m(p)\end{bmatrix}.$$

Therefore, the zeros of $\xi(x)$ are non-degenerate. □

Lemma 4.7. *For almost every $a \in \mathbb{R}^n \setminus \{\vec{0}\}$, the one-form $\xi_{|_{A_k(\omega)}}$ admits only non-degenerate zeros, $k \geq 2$.*

Proof. Suppose that $\xi_{|_{A_k(\omega)}}(p) = 0$. By Proposition 2.23 (a) and Lemma 4.5, we can consider $\mathcal{U}$ an open neighborhood of p in M where $\mathbf{M}(x) \neq 0$ and on which the respective singular sets $(k = 2, \dots, n)$ can be locally defined as

$$\mathcal{U} \cap \Sigma^k(\omega) = \{x \in \mathcal{U} : \mathbf{M}_n(x) = \dots = \mathbf{M}_m(x) = \Delta_2(x) = \dots = \Delta_k(x) = 0\},$$

with $\operatorname{rank}(d\mathbf{M}_n, \dots, d\mathbf{M}_m, d\Delta_2, \dots, d\Delta_k) = m - n + k$, $\forall x \in \Sigma^k(\omega) \cap \mathcal{U}$.

Analogously to the proof of Lemma 3.7, by Szafraniec's characterization (see [19, p. 196]), x is a zero of the restriction $\xi_{|_{\Sigma^k(\omega)}}$ if and only if there exists $(\lambda_n, \dots, \lambda_m, \beta_2, \dots, \beta_k) \in \mathbb{R}^{m-n+k}$ such that

$$\xi(x) = \sum_{j=n}^{m} \lambda_j d\mathbf{M}_j(x) + \sum_{\ell=2}^{k} \beta_\ell d\Delta_\ell(x).$$

Let us consider the functions

$$N_s(x, a, \lambda, \beta) := \xi_s(x) - \sum_{j=n}^{m} \lambda_j \frac{\partial \mathbf{M}_j}{\partial x_s}(x) - \sum_{\ell=2}^{k} \beta_\ell \frac{\partial \Delta_\ell}{\partial x_s}(x), \;\; s = 1, \dots, m,$$

and let $G : \mathcal{U} \setminus \{\Delta_{k+1} = 0\} \times \mathbb{R}^n \setminus \{\vec{0}\} \times \mathbb{R}^{m-n+k} \to \mathbb{R}^{2m-n+k}$ be the mapping given by

$$G(x, a, \lambda, \beta) = (\mathbf{M}_n, \dots, \mathbf{M}_m, \Delta_2, \dots, \Delta_k, N_1, \dots, N_m).$$

Analogously to the proof of Lemma 4.6, if $(x, a, \lambda, \beta) \in G^{-1}(\vec{0})$ then $x \in A_k(\omega) \cap Z(\xi_{|_{\Sigma^k(\omega)}})$. On the other hand, if $x \in A_k(\omega)$ then

$$\dim(\langle \bar{\omega}(x) \rangle \cap N_x^* \Sigma^{k-1}(\omega)) = k - 1$$

and $\dim(\langle \bar{\omega}(x) \rangle \cap N_x^* \Sigma^k(\omega)) = k - 1$, such that $\dim(\langle \bar{\omega}(x) \rangle + N_x^* \Sigma^k(\omega)) = m$. This implies that the Jacobian matrix of G has maximal rank at every $(x, a, \lambda, \beta) \in G^{-1}(\vec{0})$. Thus $\dim G^{-1}(\vec{0}) = n$.

Let $\pi : G^{-1}(\vec{0}) \to \mathbb{R}^n \setminus \{\vec{0}\}$ be the projection $\pi(x, a, \lambda, \beta) = a$, then for almost every $a \in \mathbb{R}^n \setminus \{\vec{0}\}$, $\dim(\pi^{-1}(a) \cap G^{-1}(\vec{0})) = 0$ and $\pi^{-1}(a) \pitchfork G^{-1}(\vec{0})$. Therefore, the zeros of $\xi_{|_{A_k(\omega)}}$ are non-degenerate. □

Lemma 4.8. *For almost every $a \in \mathbb{R}^n \setminus \{\vec{0}\}$, the one-form $\xi_{|_{A_1(\omega)}}$ admits only non-degenerate zeros.*

Proof. This proof follows analogously the proof of Lemma 4.7. □

By Lemma 3.2, if $p \in A_{k+1}(\omega)$, then p is a zero of $\xi_{|_{\Sigma^{k+1}(\omega)}}$ if and only if p is a zero of $\xi_{|_{\Sigma^k(\omega)}}$. The next results state that this relation also holds for non-degenerate zeros.

Lemma 4.9. *Let $p \in A_1(\omega)$ be a zero of $\xi_{|_{\Sigma^1(\omega)}}$, then p is a non-degenerate zero of $\xi_{|_{\Sigma^1(\omega)}}$ if and only if p is a non-degenerate zero of ξ.*

Proof. Let $p \in A_1(\omega)$ be a zero of the restriction $\xi_{|_{\Sigma^1(\omega)}}$ and let $\mathcal{U}$ be an open neighborhood of p in M at which $\mathbf{M}(x) \neq 0$, $\forall x \in \mathcal{U}$ and $\mathcal{U} \cap \Sigma^1(\omega) = \{x \in \mathcal{U} : \mathbf{M}_n(x) = \dots = \mathbf{M}_m(x) = 0\}$. By Szafraniec's characterization ([19, p. 196]), $\exists!(\lambda_n, \dots, \lambda_m) \in \mathbb{R}^{m-n+1}$, such that

$$\xi(p) + \sum_{i=n}^{m} \lambda_i d\mathbf{M}_i(p) = 0.$$

Furthermore, p is a non-degenerate zero of $\xi|_{\Sigma^1(\omega)}$ if and only if the matrix

$$
\left[\begin{array}{c|cccc}
 & \frac{\partial \mathbf{M}_n}{\partial x_1}(p) & \cdots & \frac{\partial \mathbf{M}_m}{\partial x_1}(p) \\
\mathrm{Jac}\left(\xi + \sum_{i=n}^{m} \lambda_i d\mathbf{M}_i\right)(p) & \vdots & \ddots & \vdots \\
 & \frac{\partial \mathbf{M}_n}{\partial x_m}(p) & \cdots & \frac{\partial \mathbf{M}_m}{\partial x_m}(p) \\
\hline
d_x\mathbf{M}_n(p) & & & \\
\vdots & & O_{(m-n+1)} & \\
d_x\mathbf{M}_m(p) & & &
\end{array}\right] \tag{6}
$$

is non-singular. Since $\xi(p) = 0$, then $p \in \Sigma^1(\omega) \cap \mathcal{U}$ and $\sum_{i=n}^{m} \lambda_i d\mathbf{M}_i(p) = \vec{0}$. Thus,

$$\lambda_n = \ldots = \lambda_m = 0,$$

and writing $\xi = (\xi_1, \ldots, \xi_m)$ we have that the Matrix (6) is non-singular if and only if the matrix

$$
\left[\begin{array}{c|cccc}
d_x\xi_1(p) & \frac{\partial \mathbf{M}_n}{\partial x_1}(p) & \cdots & \frac{\partial \mathbf{M}_m}{\partial x_1}(p) \\
\vdots & \vdots & \ddots & \vdots \\
d_x\xi_m(p) & \frac{\partial \mathbf{M}_n}{\partial x_m}(p) & \cdots & \frac{\partial \mathbf{M}_m}{\partial x_m}(p) \\
\hline
a_n d_x\mathbf{M}_n(p) & & & \\
\vdots & & O_{(m-n+1)} & \\
a_n d_x\mathbf{M}_m(p) & & &
\end{array}\right] \tag{7}
$$

is non-singular (by Remark 3.4, $a_n \neq 0$). Moreover, by Equation (5) and Lemma 4.2, we can write

$$
\begin{aligned}
a_n d_x\mathbf{M}_i(p) &= a_n \sum_{\ell \in I, j \in I_i} \mathrm{cof}(\omega_\ell^j(p), M_i) d\omega_\ell^j(p) \\
&= \sum_{\ell \in I, j \in I_i} a_\ell \,\mathrm{cof}(\omega_n^j(p), M_i) d\omega_\ell^j(p) \\
&= \sum_{j \in I_i} \mathrm{cof}(\omega_n^j(p), M_i) \left[\sum_{\ell \in I} a_\ell d\omega_\ell^j(p)\right] \\
&= \sum_{j \in I_i} \mathrm{cof}(\omega_n^j(p), M_i) \left[d_x\xi_j(p)\right].
\end{aligned}
$$

Let us denote the m first row vectors of Matrix (7) by $L_j, j = 1, \ldots, m$, and let us denote the $m - n + 1$ last row vectors of Matrix (7) by $R_i, i = n, \ldots, m$:

$$\begin{aligned} L_j &= \left(d_x\xi_j(p), \frac{\partial \mathbf{M}_n}{\partial x_j}(p), \ldots, \frac{\partial \mathbf{M}_m}{\partial x_j}(p) \right); \\ R_i &= \left(a_n \frac{\partial \mathbf{M}_i}{\partial x_1}(p), \ldots, a_n \frac{\partial \mathbf{M}_i}{\partial x_m}(p), \vec{0} \right). \end{aligned}$$

Then, replacing each row vector R_i, $i = n, \ldots, m$, by $R_i - \sum_{j \in I_i} \operatorname{cof}(\omega_n^j, M_i) L_j$, we obtain

$$R_i = \left(\underbrace{0, \ldots\ 0}_{m \text{ times}}, -\sum_{j \in I_i} \operatorname{cof}(\omega_n^j, M_i) \frac{\partial \mathbf{M}_n}{\partial x_j}, \ldots, -\sum_{j \in I_i} \operatorname{cof}(\omega_n^j, M_i) \frac{\partial \mathbf{M}_m}{\partial x_j} \right)$$

and the Matrix (7) becomes:

$$\begin{bmatrix} d_x\xi_1(p) & \vdots & \frac{\partial \mathbf{M}_n}{\partial x_1}(p) & \cdots & \frac{\partial \mathbf{M}_m}{\partial x_1}(p) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ d_x\xi_m(p) & \vdots & \frac{\partial \mathbf{M}_n}{\partial x_m}(p) & \cdots & \frac{\partial \mathbf{M}_m}{\partial x_m}(p) \\ \cdots\ \cdots\ \cdots & \vdots & \cdots\ \cdots & \cdots\ \cdots & \cdots\ \cdots \\ & \vdots & & & \\ O_{(m-n+1)\times m} & \vdots & & \mathbf{M}'_{(m-n+1)} & \\ & \vdots & & & \end{bmatrix} \tag{8}$$

where $\mathbf{M}'_{(m-n+1)} = -\left(\ m_{ij}\ \right)_{n \le i,j \le m}$ is the matrix given by

$$m_{ij} = \sum_{k \in I_i} \operatorname{cof}(\omega_n^k, M_i) \frac{\partial \mathbf{M}_j}{\partial x_k},\ i, j = n, \ldots, m. \tag{9}$$

Next, we will verify that the matrix $\mathbf{M}'$ is non-singular. Since $p \in A_1(\omega)$, then

$$\dim(\langle \bar{\omega}(p) \rangle \cap N_p^* \Sigma^1(\omega)) = 0$$

and $\dim(\langle \bar{\omega}(p) \rangle \oplus N_p^* \Sigma^1(\omega)) = m$. Since $\mathbf{M}(p) \neq 0$, $\{\omega_1(p), \ldots, \omega_{n-1}(p)\}$ is a basis of the space $\langle \bar{\omega}(p) \rangle$ and, consequently, the matrix

$$\begin{bmatrix} \omega_1^1(p) & \cdots & \omega_1^{n-1}(p) & \omega_1^n(p) & \cdots & \omega_1^m(p) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \omega_{n-1}^1(p) & \cdots & \omega_{n-1}^{n-1}(p) & \omega_{n-1}^n(p) & \cdots & \omega_{n-1}^m(p) \\ \frac{\partial \mathbf{M}_n}{\partial x_1}(p) & \cdots & \frac{\partial \mathbf{M}_n}{\partial x_{n-1}}(p) & \frac{\partial \mathbf{M}_n}{\partial x_n}(p) & \cdots & \frac{\partial \mathbf{M}_n}{\partial x_m}(p) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \mathbf{M}_m}{\partial x_1}(p) & \cdots & \frac{\partial \mathbf{M}_m}{\partial x_{n-1}}(p) & \frac{\partial \mathbf{M}_m}{\partial x_n}(p) & \cdots & \frac{\partial \mathbf{M}_m}{\partial x_m}(p) \end{bmatrix} \tag{10}$$

has maximal rank. Let us denote the row vectors of Matrix (10) by L'_j, $j = 1, \dots, m$. Then, for $j = 1, \dots, n-1$, we replace L'_j by

$$\sum_{k=1}^{n-1} \operatorname{cof}(\omega_k^j, M) L'_k = \left(\sum_{k=1}^{n-1} \operatorname{cof}(\omega_k^j, M)\omega_k^1, \dots, \sum_{k=1}^{n-1} \operatorname{cof}(\omega_k^j, M)\omega_k^m \right). \tag{11}$$

It is not difficult to verify that

$$\sum_{k=1}^{n-1} \operatorname{cof}(\omega_k^j, M)\omega_k^\ell = \begin{cases} \mathbf{M}, & \ell = j; \\ 0 & \ell = 1, \dots, n-1 \text{ and } \ell \neq j; \\ -\operatorname{cof}(\omega_n^j, \mathbf{M}_\ell), & \ell = n, \dots, m. \end{cases}$$

Thus, Matrix (10) becomes

$$\left[\begin{array}{ccccccc} \mathbf{M} & \cdots & 0 & \vdots & -\operatorname{cof}(\omega_n^1, \mathbf{M}_n) & \cdots & -\operatorname{cof}(\omega_n^1, \mathbf{M}_m) \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbf{M} & \vdots & -\operatorname{cof}(\omega_n^{n-1}, \mathbf{M}_n) & \cdots & -\operatorname{cof}(\omega_n^{n-1}, \mathbf{M}_m) \\ \cdots & \cdots & \cdots & \vdots & \cdots\ \cdots\ \cdots & \cdots & \cdots\ \cdots\ \cdots \\ \dfrac{\partial \mathbf{M}_n}{\partial x_1} & \cdots & \dfrac{\partial \mathbf{M}_n}{\partial x_{n-1}} & \vdots & \dfrac{\partial \mathbf{M}_n}{\partial x_p} & \cdots & \dfrac{\partial \mathbf{M}_n}{\partial x_m} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial \mathbf{M}_m}{\partial x_1} & \cdots & \dfrac{\partial \mathbf{M}_m}{\partial x_{n-1}} & \vdots & \dfrac{\partial \mathbf{M}_m}{\partial x_p} & \cdots & \dfrac{\partial \mathbf{M}_m}{\partial x_m} \end{array} \right] \tag{12}$$

that still has maximal rank. Now, let us denote the first $n-1$ row vectors of Matrix (12) by L''_j, for $j = 1, \dots, n-1$, and let us consider the following expression for $j = n, \dots, m$,

$$\begin{aligned} &\mathbf{M}L'_j - \sum_{k=1}^{n-1} \frac{\partial \mathbf{M}_j}{\partial x_k} L''_k \\ &= \mathbf{M}\left(\frac{\partial \mathbf{M}_j}{\partial x_1}, \dots, \frac{\partial \mathbf{M}_j}{\partial x_{n-1}}, \frac{\partial \mathbf{M}_j}{\partial x_n}, \dots, \frac{\partial \mathbf{M}_j}{\partial x_m} \right) \\ &+ \left(-\mathbf{M}\frac{\partial \mathbf{M}_j}{\partial x_1}, \dots, -\mathbf{M}\frac{\partial \mathbf{M}_j}{\partial x_{n-1}}, \sum_{k=1}^{n-1} \frac{\partial \mathbf{M}_j}{\partial x_k} \operatorname{cof}(\omega_n^k, M_n), \dots, \sum_{k=1}^{n-1} \frac{\partial \mathbf{M}_j}{\partial x_k} \operatorname{cof}(\omega_n^k, M_m) \right) \\ &= \left(0, \dots, 0, \sum_{k=1}^{n-1} \frac{\partial \mathbf{M}_j}{\partial x_k} \operatorname{cof}(\omega_n^k, M_n) + \mathbf{M}\frac{\partial \mathbf{M}_j}{\partial x_n}, \dots, \sum_{k=1}^{n-1} \frac{\partial \mathbf{M}_j}{\partial x_k} \operatorname{cof}(\omega_n^k, M_m) + \mathbf{M}\frac{\partial \mathbf{M}_j}{\partial x_m} \right). \end{aligned}$$

Notice that $\mathbf{M} = \operatorname{cof}(\omega_n^i, \mathbf{M}_i)$, for $i = n, \dots, m$. Then the expression

$$\mathbf{M}L'_j - \sum_{k=1}^{n-1} \frac{\partial \mathbf{M}_j}{\partial x_k} L''_k \tag{13}$$

is equal to

$$\left(0, \dots, 0, \sum_{k \in I_n} \frac{\partial \mathbf{M}_j}{\partial x_k} \operatorname{cof}(\omega_n^k, M_n), \dots, \sum_{k \in I_m} \frac{\partial \mathbf{M}_j}{\partial x_k} \operatorname{cof}(\omega_n^k, M_m) \right).$$

Thus, by Equation (9), we obtain

$$\mathbf{M}L'_j - \sum_{k=1}^{n-1} \frac{\partial \mathbf{M}_j}{\partial x_k} L''_k = (0, \dots, 0, m_{nj}, \dots, m_{mj}).$$

In this way, we replace the row L'_j in Matrix (12) by (13) for $j = n, \dots, m$, and the matrix obtained

$$\left[\begin{array}{ccc:ccccccc}
\mathbf{M} & \cdots & 0 & -\operatorname{cof}(\omega_n^1, \mathbf{M}_n) & & & \cdots & & & -\operatorname{cof}(\omega_n^1, \mathbf{M}_m) \\
\vdots & \ddots & \vdots & \vdots & & & \ddots & & & \vdots \\
0 & \cdots & \mathbf{M} & -\operatorname{cof}(\omega_n^{n-1}, \mathbf{M}_n) & & & \cdots & & & -\operatorname{cof}(\omega_n^{n-1}, \mathbf{M}_m) \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
 & O_{(n-1)} & & & & & (-\mathbf{M}')^t & & &
\end{array}\right] \tag{14}$$

also is non-singular. Then, since $\mathbf{M} \neq 0$, we have that $\det \mathbf{M}' \neq 0$. Thus, we can conclude that Matrix (7) is non-singular if and only if Matrix (8) is non-singular, which occurs if and only if

$$\det \begin{bmatrix} d_x \xi_1(p) \\ \vdots \\ d_x \xi_m(p) \end{bmatrix} \neq 0.$$

In other words, p will be a non-degenerate zero of $\xi_{|_{\Sigma^1(\omega)}}$ if and only if p is a non-degenerate zero of ξ. □

Lemma 4.10. *Let $p \in A_{k+1}(\omega)$ be a zero of $\xi_{|_{\Sigma^{k+1}(\omega)}}$. Then, for almost every $a \in \mathbb{R}^n \setminus \{\vec{0}\}$, p is a non-degenerate zero of $\xi_{|_{\Sigma^{k+1}(\omega)}}$ if and only if p is a non-degenerate zero of $\xi_{|_{\Sigma^k(\omega)}}$.*

Proof. Let $p \in A_{k+1}(\omega)$ be a zero of $\xi_{|_{\Sigma^{k+1}(\omega)}}$ and let $\mathcal{U}$ be an open neighborhood of p in M at which $\mathbf{M}(x) \neq 0$, $\forall x \in \mathcal{U}$ and the singular sets $\Sigma^k(\omega)$ $(k = 2, \dots, n)$ are defined by $\mathcal{U} \cap \Sigma^k(\omega) = \{x \in \mathcal{U} : \mathbf{M}_n(x) = \dots = \mathbf{M}_m(x) = \Delta_2(x) = \dots = \Delta_k(x) = 0\}$. By Szafraniec's characterization ([19, p. 196]), p is a zero of the restriction $\xi_{|_{\Sigma^{k+1}(\omega)}}$ if and only if there exists a unique $(\lambda_n, \dots, \lambda_m, \beta_2, \dots, \beta_{k+1}) \in \mathbb{R}^{m-n+k+1}$ such that

$$\xi(p) + \sum_{i=n}^{m} \lambda_i d\mathbf{M}_i(p) + \sum_{j=2}^{k+1} \beta_j d\Delta_j(p) = 0. \tag{15}$$

Since p is a zero of $\xi_{|_{\Sigma^k(\omega)}}$, we have $\beta_{k+1} = 0$. Moreover, also by Szafraniec's characterization, for $\ell = k, k+1$, p is a non-degenerate zero of $\xi_{|_{\Sigma^\ell(\omega)}}$ if and only if the determinant of the following matrix does not vanish at p:

$$J_\ell = \left[\begin{array}{c:cccccc}
 & \frac{\partial \mathbf{M}_n}{\partial x_1} & \cdots & \frac{\partial \mathbf{M}_m}{\partial x_1} & \frac{\partial \Delta_2}{\partial x_1} & \cdots & \frac{\partial \Delta_\ell}{\partial x_1} \\
\operatorname{Jac}_x\left(\xi + \sum_{i=n}^{m} \lambda_i d\mathbf{M}_i + \sum_{j=2}^{k} \beta_j d\Delta_j\right) & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
 & \frac{\partial \mathbf{M}_n}{\partial x_m} & \cdots & \frac{\partial \mathbf{M}_m}{\partial x_m} & \frac{\partial \Delta_2}{\partial x_m} & \cdots & \frac{\partial \Delta_\ell}{\partial x_m} \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
d_x \mathbf{M}_n & & & & & & \\
\vdots & & & & & & \\
d_x \mathbf{M}_m & & & O_{(m-n+\ell)} & & & \\
d_x \Delta_2 & & & & & & \\
\vdots & & & & & & \\
d_x \Delta_\ell & & & & & &
\end{array}\right]. \tag{16}$$

Thus, to prove the lemma it is enough to show that the Matrix J_{k+1} is non-singular at p if and only if the Matrix J_k is non-singular at p.

Notice that the Jacobian matrix with respect to x

$$\mathrm{Jac}_x\left(\xi + \sum_{i=n}^{m} \lambda_i d\mathbf{M}_i + \sum_{j=2}^{k} \beta_j d\Delta_j\right) \tag{17}$$

is a submatrix of both Matrices J_{k+1} and J_k, and recall that, for x in an open neighborhood of p, $\Delta_{k+1} = \det(d\mathbf{M}_n, \dots, d\mathbf{M}_m, d\Delta_2, \dots, d\Delta_k, \Omega_1, \dots, \Omega_{n-k})$, where $\{\Omega_1(x), \dots, \Omega_{n-k}(x)\}$ is a basis of a vector subspace complementary to $\langle\bar{\omega}(x)\rangle \cap N_x^*\Sigma^{k-1}(\omega)$ in $\langle\bar{\omega}(x)\rangle$. That is,

$$\langle\bar{\omega}(x)\rangle = \langle\Omega_1(x), \dots, \Omega_{n-k}(x)\rangle \oplus (\langle\bar{\omega}(x)\rangle \cap N_x^*\Sigma^{k-1}(\omega)).$$

Since, for almost every a, $\xi_{|_{\Sigma^{k-1}(\omega)}}(p) \neq 0$ then $\xi(p) \in \langle\bar{\omega}(p)\rangle \smallsetminus N_p^*\Sigma^{k-1}(\omega)$ and there exists $(\mu_1, \dots, \mu_{n-k}) \in \mathbb{R}^{n-k} \smallsetminus \{\vec{0}\}$ such that $\xi(p) = \sum_{i=1}^{n-k} \mu_i \Omega_i(p) + \varphi(p)$, for some $\varphi(p) \in N_p^*\Sigma^{k-1}(\omega)$, where $\varphi(p) = \sum_{i=n}^{m} \tilde{\lambda}_i d\mathbf{M}_i(p) + \sum_{j=2}^{k-1} \tilde{\beta}_j d\Delta_j(p)$. Then, equation (15) can be written as:

$$\sum_{i=1}^{n-k} \mu_i \Omega_i(p) + \sum_{i=n}^{m} (\lambda_i + \tilde{\lambda}_i) d\mathbf{M}_i(p) + \sum_{j=2}^{k-1} (\beta_j + \tilde{\beta}_j) d\Delta_j(p) + \beta_k d\Delta_k(p) = 0. \tag{18}$$

Let us consider the mapping

$$H(x) = \sum_{i=1}^{n-k} \mu_i \Omega_i(x) + \sum_{i=n}^{m} (\lambda_i + \tilde{\lambda}_i) d\mathbf{M}_i(x) + \sum_{j=2}^{k-1} (\beta_j + \tilde{\beta}_j) d\Delta_j(x) + \beta_k d\Delta_k(x),$$

defined on $\mathcal{U}$. The Jacobian matrix of $H(x)$ is given by:

$$\begin{bmatrix} \sum_{i=1}^{n-k} \mu_i d_x \Omega_i^1 + \sum_{i=n}^{m} (\lambda_i + \tilde{\lambda}_i) d_x \dfrac{\partial \mathbf{M}_i}{\partial x_1} + \sum_{j=2}^{k-1} (\beta_j + \tilde{\beta}_j) d_x \dfrac{\partial \Delta_j}{\partial x_1} + \beta_k d_x \dfrac{\partial \Delta_k}{\partial x_1} \\ \vdots \\ \sum_{i=1}^{n-k} \mu_i d_x \Omega_i^m + \sum_{i=n}^{m} (\lambda_i + \tilde{\lambda}_i) d_x \dfrac{\partial \mathbf{M}_i}{\partial x_m} + \sum_{j=2}^{k-1} (\beta_j + \tilde{\beta}_j) d_x \dfrac{\partial \Delta_j}{\partial x_m} + \beta_k d_x \dfrac{\partial \Delta_k}{\partial x_m} \end{bmatrix}. \tag{19}$$

To apply Lemma 4.1, fix the notation: $A_i(x) = (a_{1i}(x), \dots, a_{mi}(x))$, where

$$A_i(x) := \begin{cases} \Omega_i(x), & i = 1, \dots, n-k; \\ d\mathbf{M}_i(x), & i = n, \dots, m; \end{cases}$$

$$A_{n-k+j-1}(x) := d\Delta_j(x), \quad j = 2, \dots, k;$$

$$\alpha_i := \begin{cases} \mu_i, & i = 1, \dots, n-k; \ \text{(we can suppose } \alpha_1 \neq 0, \text{ since } \xi(p) \neq \varphi(p)) \\ (\lambda_i + \tilde{\lambda}_i), & i = n, \dots, m; \end{cases}$$

$$\alpha_{n-k+j-1} := (\beta_j + \tilde{\beta}_j), \quad j = 2, \dots, k; \quad (\tilde{\beta}_k = 0).$$

In this way, equation (18) can be written as $\sum_{i=1}^{m} \alpha_i A_i(p) = 0$ which implies that

$$\sum_{i=1}^{m} \alpha_i a_{ji}(p) = 0, \ \forall j = 1, \dots, m.$$

We also have that

$$\begin{aligned} \Delta_{k+1} &= \det(A_n, \dots, A_m, A_{n-k+1}, \dots, A_{n-1}, A_1, \dots, A_{n-k}) \\ &= (-1)^{\varepsilon} \det(A_1, \dots, A_m) \end{aligned}$$

where ε is either equal to zero or equal to 1, depending on the number of required permutations between the columns of the matrix A to obtain Δ_{k+1}. Thus, by Lemma 4.1,

$$
\begin{aligned}
\alpha_1(-1)^{\varepsilon} d\Delta_{k+1} &\overset{\alpha_1\neq 0}{=} \alpha_1 \sum_{i,j=1}^{m} \operatorname{cof}(a_{ij}) da_{ij} \\
&= \sum_{i=1}^{m}\left(\alpha_1 \operatorname{cof}(a_{i1}) da_{i1} + \sum_{j=2}^{m} \alpha_j \operatorname{cof}(a_{i1}) da_{ij}\right) \\
&= \sum_{i=1}^{m} \operatorname{cof}(a_{i1}) \left[\sum_{j=1}^{m} \alpha_j da_{ij}\right] \\
&= \sum_{i=1}^{m} \operatorname{cof}(a_{i1}) \mathcal{L}_i
\end{aligned} \tag{20}
$$

where $\mathcal{L}_i, i = 1, \ldots, m$, denote the rows of the Jacobian matrix (19) at p. If we denote by $\tilde{L}_i, i = 1, \ldots, m$, the row vectors of Jacobian matrix (17) at p, then we can verify that

$$
\sum_{i=1}^{m} \operatorname{cof}(a_{i1}) \mathcal{L}_i = \sum_{i=1}^{m} \operatorname{cof}(a_{i1}) \tilde{L}_i. \tag{21}
$$

Let us denote the first m row vectors of Matrix J_{k+1} in (16) by $L_i, i = 1, \ldots, m$, and its last row vector by $L_{\Delta_{k+1}}$. By equations (20) at p and (21), if we replace $L_{\Delta_{k+1}}$ by

$$
(-1)^{\varepsilon} \alpha_1 L_{\Delta_{k+1}} - \sum_{i=1}^{m} \operatorname{cof}(a_{i1}) L_i, \tag{22}
$$

we obtain

$$
\left[\begin{array}{c:ccccccc:c}
 & \frac{\partial \mathbf{M}_n}{\partial x_1} & \cdots & \frac{\partial \mathbf{M}_m}{\partial x_1} & \frac{\partial \Delta_2}{\partial x_1} & \cdots & \frac{\partial \Delta_k}{\partial x_1} & & \frac{\partial \Delta_{k+1}}{\partial x_1} \\
\operatorname{Jac}\left(\xi + \sum_{i=n}^{m} \lambda_i d\mathbf{M}_i + \sum_{j=2}^{k} \beta_j d\Delta_j\right) & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & & \vdots \\
 & \frac{\partial \mathbf{M}_n}{\partial x_m} & \cdots & \frac{\partial \mathbf{M}_m}{\partial x_m} & \frac{\partial \Delta_2}{\partial x_m} & \cdots & \frac{\partial \Delta_k}{\partial x_m} & & \frac{\partial \Delta_{k+1}}{\partial x_m} \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \cdots & \cdots & \cdots\cdots & \cdots & \cdots & \cdots & & \cdots\cdots \\
d_x\mathbf{M}_n & & & & & & & & 0 \\
\vdots & & & & & & & & \vdots \\
d_x\mathbf{M}_m & & & & & & & & 0 \\
d_x\Delta_2 & & & & O_{(m-n+k)} & & & & 0 \\
\vdots & & & & & & & & \vdots \\
d_x\Delta_k & & & & & & & & 0 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \cdots & \cdots & \cdots\cdots & \cdots & \cdots & \cdots & & \cdots\cdots \\
\vec{0} & & & & \vec{0} & & & & \gamma_{\tilde{k+1}}
\end{array}\right]. \tag{23}
$$

Let us show that $\gamma_{\tilde{k+1}}(p) \neq 0$. We have

$$
\begin{aligned}
\gamma_{\tilde{k+1}} &\overset{(22)}{=} -\sum_{i=1}^{m} \operatorname{cof}(a_{i1}) \frac{\partial \Delta_{k+1}}{\partial x_i} \\
&= -\det(d\Delta_{k+1}, A_2, \ldots, A_m) \\
&= -\det(d\Delta_{k+1}, \Omega_2, \ldots, \Omega_{n-k}, d\Delta_2, \ldots, d\Delta_k, d\mathbf{M}_n, \ldots, d\mathbf{M}_m).
\end{aligned}
$$

Suppose that $\gamma_{\tilde{k+1}} = 0$. Since each one of the sets $\{\Omega_2(p), \ldots, \Omega_{n-k}(p)\}$ and

$$
\{d\Delta_{k+1}(p), d\Delta_2(p), \ldots, d\Delta_k(p), d\mathbf{M}_n(p), \ldots, d\mathbf{M}_m(p)\}
$$

consist of linearly independent vectors, there exists $j \in \{2,\dots,n-k\}$ such that $\Omega_j(p) \in N_p^*\Sigma^{k+1}(\omega)$. Suppose that $j = n-k$, that is,

$$\Omega_{n-k}(p) \in N_p^*\Sigma^{k+1}(\omega) = \langle d\mathbf{M}_n,\dots,d\mathbf{M}_m, d\Delta_2,\dots,d\Delta_k, d\Delta_{k+1}\rangle.$$

Since $\xi_{|_{\Sigma^{k+1}}}(p) = 0$, we have $\xi(p) \in N_p^*\Sigma^{k+1}(\omega)$. Then,

$$\sum_{i=1}^{n-k}\mu_i\Omega_i + \underbrace{\sum_{i=n}^{m}\tilde{\lambda}_i d\mathbf{M}_i + \sum_{j=2}^{k-1}\tilde{\beta}_j d\Delta_j}_{\in N_p^*\Sigma^{k+1}(\omega)} \in N_p^*\Sigma^{k+1}(\omega)$$

$$\Rightarrow \sum_{i=1}^{n-k-1}\mu_i\Omega_i = \sum_{i=1}^{n-k}\mu_i\Omega_i - \mu_{n-k}\Omega_{n-k} \in N_p^*\Sigma^{k+1}(\omega).$$

Thus, $\sum_{i=1}^{n-k-1}\mu_i\Omega_i$ and $\mu_{n-k}\Omega_{n-k}$ are linearly independent vectors in the vector subspace

$$\langle\Omega_1,\dots,\Omega_{n-k}\rangle \cap N_p^*\Sigma^{k+1}(\omega),$$

which implies that

$$\dim\big(\langle\Omega_1(p),\dots,\Omega_{n-k}(p)\rangle \cap N_p^*\Sigma^{k+1}(\omega)\big) \geq 2.$$

Consequently, since $\langle\bar{\omega}\rangle = \langle\Omega_1,\dots,\Omega_{n-k}\rangle \oplus \big(\langle\bar{\omega}\rangle \cap N_p^*\Sigma^{k-1}(\omega)\big)$ we have that

$$\dim\big(\langle\bar{\omega}(p)\rangle \cap N_p^*\Sigma^{k+1}(\omega)\big) \geq 2 + (k-1) = k+1,$$

which means that $p \in \Sigma^{k+2}(\omega)$. But this contradicts the hypothesis that $p \in A_{k+1}(\omega)$, since as we know $\Sigma^{k+2}(\omega) = \Sigma^{k+1}(\omega) \setminus A_{k+1}(\omega)$. Therefore $\tilde{\gamma_{k+1}}(p) \neq 0$, and we conclude that the Matrix J_{k+1} is non-singular at p if and only if the Matrix (23) is non-singular at p, which occurs if and only if the Matrix J_k is non-singular at the point p.

□

Lemma 4.11. *For almost every $a \in \mathbb{R}^n \setminus \{\vec{0}\}$, if $p \in A_n(\omega)$ then p is a non-degenerate zero of $\xi_{|_{\Sigma^{n-1}(\omega)}}$.*

Proof. We know that if $p \in A_n(\omega)$ then $\xi_{|_{\Sigma^{n-1}(\omega)}}(p) = 0$. By Szafraniec's characterization [20, p.149-151], p is a non-degenerate zero of $\xi_{|_{\Sigma^{n-1}(\omega)}}$ if and only if the following conditions hold:

(i) $\Delta(p) = \det(d\mathbf{M}_n,\dots,d\mathbf{M}_m, d\Delta_2,\dots,d\Delta_{n-1},\xi)(p) = 0$;

(ii) $\det(d\mathbf{M}_n,\dots,d\mathbf{M}_m, d\Delta_2,\dots,d\Delta_{n-1},d\Delta)(p) \neq 0$.

Condition (i) is clearly satisfied, since $\xi_{|_{\Sigma^{n-1}(\omega)}}(p) = 0$. Let us verify that condition (ii) also holds.

For each $x \in \Sigma^{n-1}(\omega)$ in an open neighborhood $\mathcal{U}$ of p in M, let $\{\Omega'(x)\}$ be a smooth basis for a vector subspace complementary to $\langle\bar{\omega}(x)\rangle \cap N_x^*\Sigma^{n-2}(\omega)$ in the vector space $\langle\bar{\omega}(x)\rangle$. Since $\xi(x) \in \langle\bar{\omega}(x)\rangle$, we have

$$\xi(x) = \lambda(x)\Omega'(x) + \varphi(x),$$

where $\lambda(x) \in \mathbb{R}$ and $\varphi(x) \in \langle\bar{\omega}(x)\rangle \cap N_x^*\Sigma^{n-2}(\omega)$, $\forall x \in \mathcal{U} \cap \Sigma^{n-1}(\omega)$.

In particular, if $x \in A_n(\omega)$, we know that, for almost every $a \in \mathbb{R}^n \setminus \{\vec{0}\}$, $\xi_{|_{\Sigma^{n-2}(\omega)}}(x) \neq 0$ and, consequently, $\xi(x) \notin N_x^*\Sigma^{n-2}(\omega)$. Thus $\lambda(p) \neq 0$. For all $x \in \mathcal{U} \cap \Sigma^{n-1}(\omega)$, we obtain

$$\begin{aligned}\Delta(x) &= \det(d\mathbf{M}_n,\dots,d\mathbf{M}_m, d\Delta_2,\dots,d\Delta_{n-1},\lambda\Omega' + \varphi)(x)\\ &= \lambda(x)\det(d\mathbf{M}_n,\dots,d\mathbf{M}_m, d\Delta_2,\dots,d\Delta_{n-1},\Omega')(x)\\ &= \lambda(x)\Delta_n(x),\end{aligned}$$

with $\Delta_n(p) = 0$ and $\lambda(p) \neq 0$. Then, we have

$$\begin{aligned}&\langle d\mathbf{M}_n(p), \dots, d\mathbf{M}_m(p), d\Delta_2(p), \dots, d\Delta_{n-1}(p), d\Delta(p)\rangle \\ &= \langle d\mathbf{M}_n(p), \dots, d\mathbf{M}_m(p), d\Delta_2(p), \dots, d\Delta_{n-1}(p), d(\lambda\Delta_n)(p)\rangle\end{aligned}$$

(see Lemma A.1). However, $d(\lambda\Delta_n)(x) = d\lambda(x)\Delta_n(x) + \lambda(x)d\Delta_n(x)$, $\Delta_n(p) = 0$ and $\lambda(p) \neq 0$. Thus,

$$\begin{aligned}&\langle d\mathbf{M}_n(p), \dots, d\mathbf{M}_m(p), d\Delta_2(p), \dots, d\Delta_{n-1}(p), d\Delta(p)\rangle \\ &= \langle d\mathbf{M}_n(p), \dots, d\mathbf{M}_m(p), d\Delta_2(p), \dots, d\Delta_{n-1}(p), d\Delta_n(p)\rangle.\end{aligned}$$

Therefore, $\det(d\mathbf{M}_n(p), \dots, d\mathbf{M}_m(p), d\Delta_2(p), \dots, d\Delta_{n-1}(p), d\Delta(p)) \neq 0$. □

Lemma 4.12. *For almost every $a \in \mathbb{R}^n \setminus \{\vec{0}\}$, the one-form $\xi_{|_{\Sigma^k(\omega)}}$ admits only non-degenerate zeros, $k \geq 1$.*

Proof. Suppose that $\xi_{|_{\Sigma^k(\omega)}}(p) = 0$. Then, for almost every $a \in \mathbb{R}^n \setminus \{\vec{0}\}$, $p \in A_k(\omega) \cup A_{k+1}(\omega)$ since $Z(\xi_{|_{\Sigma^k(\omega)}}) \cap \Sigma^{k+2}(\omega) = \varnothing$ by Lemma 3.7 and $\Sigma^k(\omega) = A_k(\omega) \cup A_{k+1}(\omega) \cup \Sigma^{k+2}(\omega)$.

If $p \in A_k(\omega)$ then $\xi_{|_{A_k(\omega)}}(p) = 0$. Since $\xi_{|_{A_k(\omega)}}$ admits only non-degenerate zeros and $A_k(\omega) \subset \Sigma^k(\omega)$ is an open subset, we conclude that p is a non-degenerate zero of $\xi_{|_{\Sigma^k(\omega)}}$.

If $p \in A_{k+1}(\omega)$ and $k < n-1$ then $\xi_{|_{\Sigma^{k+1}(\omega)}}(p) = 0$. In particular, since $A_{k+1}(\omega) \subset \Sigma^{k+1}(\omega)$ is an open subset then $\xi_{|_{A_{k+1}(\omega)}}(p) = 0$. By Lemmas 4.8 and 4.7, $\xi_{|_{A_{k+1}(\omega)}}$ admits only non-degenerate zeros, and since $A_{k+1}(\omega)$ is an open set of $\Sigma^{k+1}(\omega)$, we conclude that p is a non-degenerate zero of $\xi_{|_{\Sigma^{k+1}(\omega)}}$. Therefore, by Lemma 4.10, p is non-degenerate zero of $\xi_{|_{\Sigma^k(\omega)}}$. Finally, if $p \in A_n(\omega)$, by Lemma 4.11, p is a non-degenerate zero of $\xi_{|_{\Sigma^{n-1}(\omega)}}$. □

Theorem 4.13. *Let $\omega = \{\omega_i\}_{1\leq i\leq n}$ be a Morin collection of smooth one-forms defined on an m-dimensional compact manifold M. Then,*

$$\chi(M) \equiv \sum_{k=1}^{n} \chi(\overline{A_k(\omega)}) \mod 2.$$

Proof. Let us denote by $Z(\varphi)$ the set of zeros of a one-form φ and let us denote by $\#Z(\varphi)$ the number of elements of this set, whenever $Z(\varphi)$ is finite. Let

$$\xi(x) = \sum_{i=1}^{n} a_i\omega_i(x)$$

be a one-form with $a = (a_1, \dots, a_n) \in \mathbb{R}^n \setminus \{\vec{0}\}$ satisfying the generic conditions of the previous lemmas of Sections 3 and 4.

Since M is compact and the submanifolds $\Sigma^k(\omega)$ are closed in M, by the Poincaré-Hopf Theorem for one-forms we obtain

- $\chi(M) \equiv \#Z(\xi) \mod 2$;
- $\chi(\overline{A_k(\omega)}) = \chi(\Sigma^k(\omega)) \equiv \#Z(\xi_{|_{\Sigma^k(\omega)}}) \mod 2$, for $k = 1, \dots, n-1$;
- $\chi(\overline{A_n(\omega)}) = \chi(\Sigma^n(\omega)) \equiv \#Z(\xi_{|_{\Sigma^n(\omega)}}) \mod 2$.

By Lemma 3.1, if $p \in Z(\xi)$ then $p \in \Sigma^1(\omega)$ and $\xi_{|_{\Sigma^1(\omega)}}(p) = 0$. Moreover, by Lemma 3.6, $Z(\xi) \cap \Sigma^2(\omega) = \varnothing$. Thus $p \in A_1(\omega)$. On the other hand, Lemma 3.2 shows that if

$$p \in Z(\xi_{|_{\Sigma^1(\omega)}}) \cap A_1(\omega),$$

then p is also a zero of the one-form ξ. Thus,

$$\#Z(\xi) \equiv \#Z(\xi_{|_{\Sigma^1(\omega)}} \cap A_1(\omega)) \mod 2.$$

By Lemma 3.7, if $p \in Z(\xi_{|_{\Sigma^k(\omega)}})$ then $p \notin \Sigma^{k+2}(\omega)$. Thus, $p \in A_k(\omega) \cup A_{k+1}(\omega)$ and, for $k = 1, \ldots, n-1$, we have

$$\#Z(\xi_{|_{\Sigma^k(\omega)}}) \equiv \#Z(\xi_{|_{\Sigma^k(\omega)}} \cap A_k(\omega)) + \#Z(\xi_{|_{\Sigma^k(\omega)}} \cap A_{k+1}(\omega)) \mod 2.$$

By Lemma 3.2, we also have

$$\#Z(\xi_{|_{\Sigma^k(\omega)}} \cap A_{k+1}(\omega)) = \#Z(\xi_{|_{\Sigma^{k+1}(\omega)}} \cap A_{k+1}(\omega))$$

and by Lemma 3.3,

$$\#A_n(\omega) = \#Z(\xi_{|_{\Sigma^{n-1}(\omega)}} \cap A_n(\omega)).$$

Then,

- $\chi(M) \equiv \#Z(\xi_{|_{\Sigma^1(\omega)}} \cap A_1(\omega)) \mod 2$;
- For $k = 1, \ldots, n-1$,
$$\chi(\overline{A_k(\omega)}) \equiv \#Z(\xi_{|_{\Sigma^k(\omega)}} \cap A_k(\omega)) + \#Z(\xi_{|_{\Sigma^{k+1}(\omega)}} \cap A_{k+1}(\omega)) \mod 2;$$
- $\chi(\overline{A_n(\omega)}) = \#Z(\xi_{|_{\Sigma^{n-1}(\omega)}} \cap A_n(\omega))$.

Therefore,

$$\begin{aligned}
\chi(M) + \sum_{k=1}^{n} \chi(\overline{A_k(\omega)}) &\equiv 2\#Z(\xi_{|_{\Sigma^1(\omega)}} \cap A_1(\omega)) \\
&+ 2\#Z(\xi_{|_{\Sigma^2(\omega)}} \cap A_2(\omega)) + \ldots \\
&+ 2\#Z(\xi_{|_{\Sigma^{n-1}(\omega)}} \cap A_{n-1}(\omega)) \\
&+ 2\#Z(\xi_{|_{\Sigma^{n-1}(\omega)}} \cap A_n(\omega)) \mod 2 \\
&\equiv 0 \mod 2.
\end{aligned}$$

□

As for the definition of Morin collection of n one-forms, the results presented in Sections 3 and 4 of this paper also can be naturally adapted to the context of collections of n vector fields. In particular, the main theorems that have been used, as the Poincaré-Hopf Theorem and the Szafraniec's characterizations, have their respective versions for vector fields.

Finally, we end the paper with a very simple example. Let us verify that Theorem 4.13 indeed holds for the Morin collection of 2 vector fields $V = \{V_1, V_2\}$ presented in the Example 2.31. To do that, it is enough to see that the torus T is a compact manifold with $\chi(T) = 0$. Moreover, $\overline{A_1(V)} = \Sigma^1(V)$ is given by two circles in $\mathbb{R}^3$ and $\overline{A_2(V)}$ consists of four points, such that $\chi(\overline{A_1(V)}) = 0$ and $\chi(\overline{A_2(V)}) = 4$. Therefore,

$$\chi(T) \equiv \chi(\overline{A_1(V)}) + \chi(\overline{A_2(V)}) \mod 2.$$

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Appendix

A. Proof of Preposition 2.23

Proof of Proposition 2.23, part (a). Firstly, let us show that if $\bar{x} \in \mathcal{U} \cap \Sigma^{k-1}(\omega)$ such that $\Omega^{k-1}(\bar{x}) \in N^*_{\Sigma^{k-1}} M^r$, then the following conditions are equivalent:

(I) $\operatorname{rank}(dF_1(\bar{x}), \dots, dF_{m-r}(\bar{x}), d\Delta_k(\bar{x})) = m - r + 1$;
(II) $\Omega^{k-1} \pitchfork N^*_{\Sigma^{k-1}} M^r$ in $T^*_{\Sigma^{k-1}} M^r$ at $\bar{x}$.

Let $\Omega^{k-1}(\bar{x}) \in \mathcal{U} \times \mathcal{V}$. By the proof of Lemma 2.17, $N^*_{\Sigma^{k-1}} M^r$ can be locally given by independent equations as follows

$$N^*_{\Sigma^{k-1}} M^r = \{(x, \varphi) \in \mathcal{U} \times \mathcal{V} \mid F_1 = \dots = F_{m-r} = \Delta = 0\},$$

where $\Delta(x, \varphi) = \det(dF_1(x), \dots, dF_{m-r}(x), \varphi_1, \dots, \varphi_r)$and $\mathcal{V} \subset \mathbb{R}^{mr}$ is an open set. Let

$$G(\Omega^{k-1}) = \{(x, \Omega_1(x), \dots, \Omega_r(x)) \mid x \in \mathcal{U} \cap \Sigma^{k-1}(\omega)\}$$

be the restriction of the graph of $(\Omega_1(x), \dots, \Omega_r(x))$ to $\mathcal{U} \cap \Sigma^{k-1}(\omega)$, $G(\Omega^{k-1})$ can be locally given by

$$\begin{aligned} G(\Omega^{k-1}) = \{(x, \varphi) \in T^* M^r \mid & F_1(x) = \dots = F_{m-r}(x) = 0; \\ & \Omega_i^j(x) - \varphi_i^j = 0, i = 1, \dots, r \text{ and } j = 1, \dots, m\}, \end{aligned}$$

where T^*M^r denotes the r-cotangent bundle of M, $\Omega_i(x) = (\Omega_i^1(x), \dots, \Omega_i^m(x))$ and $\varphi_i = (\varphi_i^1, \dots, \varphi_i^m)$ for $i = 1, \dots, r$. In particular, the local equations of $G(\Omega^{k-1})$ are clearly independent and $\dim G(\Omega^{k-1}) = r$. Let (x, φ) be local coordinates in T^*M^r, with $x = (x_1, \dots, x_m)$ and

$$\varphi = (\varphi_1^1, \dots, \varphi_1^m, \varphi_2^1, \dots, \varphi_2^m, \dots, \varphi_r^1, \dots, \varphi_r^m),$$

let us consider the derivatives of the local equations of $N^*_{\Sigma^{k-1}} M^r$ and $G(\Omega^{k-1})$ with respect to (x, φ). We will denote the derivative with respect to x by d_x and the derivative with respect to φ by d_φ, then we have

$$d\left(\Omega_i^j(x) - \varphi_i^j\right) = \left(d_x \Omega_i^j(x)\ , -d_\varphi \varphi_i^j\right), \tag{24}$$

for $i = 1, \dots, r$ and $j = 1, \dots, m$, where $d_\varphi \varphi_i^j = (0, \dots, 0, 1, 0, \dots, 0)$ is the vector whose $m(i-1)+j^{th}$ entry is equal to 1 and the others are zero. By Lagrange's rules the determinant

$$\Delta(x, \varphi) = \det(dF_1(x), \dots, dF_{m-r}(x), \varphi_1, \dots, \varphi_r)$$

can be written as

$$\Delta(x, \varphi) = \sum_I F_I(x) N_I(\varphi)$$

for $I = \{i_1, \dots, i_r\} \subset \{1, \dots, m\}$, where

$$N_I(\varphi) = \begin{vmatrix} \varphi_1^{i_1} & \dots & \varphi_r^{i_1} \\ \vdots & \ddots & \vdots \\ \varphi_1^{i_r} & \dots & \varphi_r^{i_r} \end{vmatrix} \tag{25}$$

is the minor obtained from the matrix

$$\begin{bmatrix} \varphi_1^1 & \dots & \varphi_r^1 \\ \vdots & \ddots & \vdots \\ \varphi_1^m & \dots & \varphi_r^m \end{bmatrix}$$

taking the lines $i_1, \dots, i_r$, and

$$F_I(x) = \pm \begin{vmatrix} \dfrac{\partial F_1}{\partial x_{k_1}}(x) & \dots & \dfrac{\partial F_{m-r}}{\partial x_{k_1}}(x) \\ \vdots & \ddots & \vdots \\ \dfrac{\partial F_1}{\partial x_{k_{m-r}}}(x) & \dots & \dfrac{\partial F_{m-r}}{\partial x_{k_{m-r}}}(x) \end{vmatrix} \tag{26}$$

is, up to sign, the minor obtained from the matrix $(dF_1(x) \dots dF_{m-r}(x))$ removing the lines $i_1, \dots, i_r$, that is, $\{k_1, \dots, k_{m-r}\} = \{1, \dots, m\} \setminus I$. Therefore,

$$d\Delta(x, \varphi) = \Big(\sum_I N_I(\varphi) d_x F_I(x) \ , \ \sum_I F_I(x) d_\varphi N_I(\varphi) \ \Big).$$

Notice that $\Omega^{k-1} \pitchfork N^*_{\Sigma^{k-1}} M^r$ in $T^*_{\Sigma^{k-1}} M^r$ at the point $x \in \mathcal{U} \cap \Sigma^{k-1}(\omega)$ if and only if

$$G(\Omega^{k-1}) \pitchfork N^*_{\Sigma^{k-1}} M^r \quad \text{in} \quad T^*_{\Sigma^{k-1}} M^r \quad \text{at} \quad (x, \Omega_1(x), \dots, \Omega_r(x)).$$

Let π_1 be the projection of the cotangent space of $T^* M^r$ over the cotangent space of $T^*_{\Sigma^{k-1}} M^r$:

$$\begin{array}{rccc} \pi_1 : & T^*_{(x,\varphi)}(T^* M^r) & \longrightarrow & T^*_{(x,\varphi)}(T^*_{\Sigma^{k-1}} M^r) \\ & (\psi(x), \varphi_1, \dots, \varphi_r) & \longmapsto & (\pi(\psi(x)), \varphi_1, \dots, \varphi_r) \end{array}$$

where π denotes the restriction to $T_x \Sigma^{k-1}(\omega)$, that is, $\pi(\psi(x)) = \psi(x)|_{T_x \Sigma^{k-1}(\omega)}$. By Equation (24),

$$\pi_1 \left(d(\Omega_i^j(x) - \varphi_i^j) \right) = \left(\pi(d_x \Omega_i^j(x)) \ , -d_\varphi \varphi_i^j \right),$$

for $i = 1, \dots, r$ and $j = 1, \dots, m$. We also have that

$$\pi_1 \left(d\Delta(x, \varphi) \right) = \left(\ \pi \left(\sum_I N_I(\varphi) d_x F_I(x) \right) \ , \ \sum_I F_I(x) d_\varphi N_I(\varphi) \ \right).$$

Then, $G(\Omega^{k-1}) \pitchfork N^*_{\Sigma^{k-1}} M^r$ in $T^*_{\Sigma^{k-1}} M^r$ at $(x, \Omega_1(x), \dots, \Omega_r(x))$ such that

$$(x, \Omega_1(x), \dots, \Omega_r(x)) \in N^*_{\Sigma^{k-1}} M^r$$

if and only if the matrix

$$\begin{bmatrix} \pi(d_x \Omega_1^1(x)) & \vdots & \\ \vdots & \vdots & \\ \pi(d_x \Omega_1^m(x)) & \vdots & -Id_{mr} \\ \vdots & \vdots & \\ \pi(d_x \Omega_r^m(x)) & \vdots & \\ \cdots\cdots\cdots\cdots\cdots\cdots & \vdots & \cdots\cdots\cdots\cdots\cdots \\ \pi \left(\sum_I N_I(\varphi) d_x F_I(x) \right) & \vdots & \sum_I F_I(x) d_\varphi N_I(\varphi) \end{bmatrix} \tag{27}$$

has maximal rank at x. By the expression of $N_I(\varphi)$ in (25), we have

$$d_\varphi N_I(\varphi) = \sum_{i,j} \operatorname{cof}(\varphi_i^j) d_\varphi \varphi_i^j, \tag{28}$$

for $i = 1, \dots, r$, $j \in I$ and $\operatorname{cof}(\varphi_i^j)$ denoting the cofactor of φ_i^j in the matrix

$$\begin{bmatrix} \varphi_1^{i_1} & \dots & \varphi_r^{i_1} \\ \vdots & \ddots & \vdots \\ \varphi_1^{i_r} & \dots & \varphi_r^{i_r} \end{bmatrix}.$$

Let $d = C_{m,r} = \dfrac{m!}{r!(m-r)!}$, we will denote by $I_1, \dots, I_d$ the subsets of $\{1, \dots, m\}$ containing exactly r elements. By equation (28),

$$\sum_I F_I(x) d_\varphi N_I(\varphi) = \sum_{\ell=1}^d F_{I_\ell}(x) \left(\sum_{i=1}^r \sum_{j \in I_\ell} \operatorname{cof}(\varphi_i^j) d_\varphi \varphi_i^j \right)$$

and,

$$\begin{aligned}
&\sum_{\ell=1}^d F_{I_\ell}(x) \left(\sum_{i=1}^r \sum_{j \in I_\ell} \operatorname{cof}(\varphi_i^j) d_\varphi \varphi_i^j \right) \\
&= \sum_{i=1}^r \left[F_{I_1}(x) \left(\sum_{j \in I_1} \operatorname{cof}(\varphi_i^j) d_\varphi \varphi_i^j \right) + \dots + F_{I_d}(x) \left(\sum_{j \in I_d} \operatorname{cof}(\varphi_i^j) d_\varphi \varphi_i^j \right) \right] \\
&= \sum_{i=1}^r \left[\left(\sum_{I:\, 1 \in I} F_I(x) \right) \operatorname{cof}(\varphi_i^1) d_\varphi \varphi_i^1 + \dots + \left(\sum_{I:\, m \in I} F_I(x) \right) \operatorname{cof}(\varphi_i^m) d_\varphi \varphi_i^m \right] \\
&= \sum_{i=1}^r \left[\sum_{j=1}^m \left(\sum_{I:\, j \in I} F_I(x) \right) \operatorname{cof}(\varphi_i^j) d_\varphi \varphi_i^j \right].
\end{aligned}$$

Thus, for $i = 1, \dots, r$ and $j = 1, \dots, m$, we can write

$$\sum_I F_I(x) d_\varphi N_I(\varphi) = \sum_{i,j} \beta_i^j(x, \varphi) d_\varphi \varphi_i^j, \tag{29}$$

where

$$\beta_i^j(x, \varphi) = \left(\sum_{I:\, j \in I} F_I(x) \right) \operatorname{cof}(\varphi_i^j).$$

We will denote the rows of the Matrix (27) by $R_i^j = \left(\pi(d_x \Omega_i^j(x)) \;, -d_\varphi \varphi_i^j \right)$, for $i = 1, \dots, r$ and $j = 1, \dots, m$, and we denote the last row of the Matrix (27) by R_Δ. Replacing the row R_Δ by

$$R_\Delta + \sum_{i,j} \beta_i^j(x, \varphi) R_i^j$$

for $i = 1, \dots, r$ and $j = 1, \dots, m$, we obtain a new matrix

$$\begin{bmatrix}
\pi(d_x \Omega_1^1(x)) & \vdots & \\
\vdots & \vdots & -Id_{mr} \\
\pi(d_x \Omega_r^m(x)) & \vdots & \\
\cdots\cdots\cdots\cdots\cdots\cdots & \vdots & \cdots\cdots\cdots\cdots\cdots \\
R'_\Delta & \vdots & R''_\Delta
\end{bmatrix} \tag{30}$$

which has rank equal to the rank of the Matrix (27), where

$$R''_\Delta = \sum_I F_I(x) d_\varphi N_I(\varphi) + \sum_{i,j} \beta_i^j(x, \varphi)(-d_\varphi \varphi_i^j) \overset{(29)}{=} \vec{0}$$

and

$$\begin{aligned}
R'_\Delta &= \pi \left(\sum_I N_I(\varphi) d_x F_I(x) \right) + \sum_{i,j} \beta_i^j(x, \varphi) \pi \left(d_x \Omega_i^j(x) \right) \\
&= \pi \left(\sum_I N_I(\varphi) d_x F_I(x) + \sum_{i,j} \beta_i^j(x, \varphi) d_x \Omega_i^j(x) \right).
\end{aligned}$$

Notice that for each $\bar{x} \in \mathcal{U} \cap \Sigma^{k-1}(\omega)$, we have $\Omega_i^j(\bar{x}) = \varphi_i^j$. In this case, Equation (29) implies that

$$\sum_{i,j} \beta_i^j(\bar{x}, \varphi) d_x \Omega_i^j(\bar{x}) = \sum_{i,j} \beta_i^j(\bar{x}, \Omega^{k-1}(\bar{x})) d_x \Omega_i^j(\bar{x}) = \sum_I F_I(\bar{x}) d_x N_I(\Omega^{k-1}(\bar{x})).$$

Thus, at $\bar{x}$

$$R'_\Delta = \pi \left(\sum_I N_I(\Omega^{k-1}(\bar{x})) d_x F_I(\bar{x}) + \sum_I F_I(\bar{x}) d_x N_I(\Omega^{k-1}(\bar{x})) \right) = \pi(d\Delta_k(\bar{x}))$$

and the Matrix (30) is equal to

$$\left[\begin{array}{c|c} \pi(d_x \Omega_1^1(\bar{x})) & \\ \vdots & -Id_{mr} \\ \pi(d_x \Omega_r^m(\bar{x})) & \\ \hline \pi(d\Delta_k(\bar{x})) & \vec{0} \end{array} \right].$$

Thus, for each $\bar{x} \in \mathcal{U} \cap \Sigma^{k-1}(\omega)$ such that $\Omega^{k-1}(\bar{x}) \in N^*_{\Sigma^{k-1}} M^r$, $\Omega^{k-1} \pitchfork N^*_{\Sigma^{k-1}} M^r$ in $T^*_{\Sigma^{k-1}} M^r$ at $\bar{x}$ if and only if $\pi(d\Delta_k(\bar{x})) \neq 0$, that is, the restriction of $d\Delta_k(\bar{x})$ to $T_{\bar{x}} \Sigma^{k-1}(\omega)$ is not zero, which means that $d\Delta_k(\bar{x}) \notin \langle dF_1(\bar{x}), \ldots, dF_{m-r}(\bar{x}) \rangle$, or equivalently

$$\operatorname{rank}(dF_1(\bar{x}), \ldots, dF_{m-r}(\bar{x}), d\Delta_k(\bar{x})) = m - r + 1.$$

Now suppose that ω satisfies the condition I_{k-1} on $\mathcal{U}$. By property (b) of Definition 2.18, we have that $\dim \langle \Omega_1(x), \ldots, \Omega_r(x) \rangle \cap N_x^* \Sigma^{k-1}(\omega)$ is either equal to 0 or equal to 1 for each $x \in \mathcal{U} \cap \Sigma^{k-1}(\omega)$. If $\dim \langle \Omega_1(x), \ldots, \Omega_r(x) \rangle \cap N_x^* \Sigma^{k-1}(\omega) = 1$, then $x \in \mathcal{U} \cap \Sigma^k(\omega)$ and $\Delta_k(x) = 0$. In this case, the transversality given by property (a) of Definition 2.18 implies that

$$\operatorname{rank}(dF_1(x), \ldots, dF_{m-r}(x), d\Delta_k(x)) = m - r + 1.$$

On the other hand, we assume that properties (i) and (ii) hold for each $x \in \mathcal{U} \cap \Sigma^{k-1}(\omega)$. By property (i), the property (b) of Definition 2.18 holds on $\mathcal{U}$. If

$$\dim \langle \Omega_1(x), \ldots, \Omega_r(x) \rangle \cap N_x^* \Sigma^{k-1}(\omega) = 0,$$

then $\Omega^{k-1}(x)$ does not intersect $N^*_{\Sigma^{k-1}} M^r$, thus $\Omega^{k-1} \pitchfork N^*_{\Sigma^{k-1}} M^r$ in $T^*_{\Sigma^{k-1}} M^r$ at x. If

$$\dim \langle \Omega_1(x), \ldots, \Omega_r(x) \rangle \cap N_x^* \Sigma^{k-1}(\omega) = 1,$$

then $x \in \mathcal{U} \cap \Sigma^k(\omega)$ by Definition 2.19 and $\operatorname{rank}(dF_1(x), \ldots, dF_{m-r}(x), d\Delta_k(x)) = m - r + 1$ by property (ii). Thus $\Omega^{k-1} \pitchfork N^*_{\Sigma^{k-1}} M^r$ in $T^*_{\Sigma^{k-1}} M^r$ at x and ω satisfies the condition I_{k-1} on $\mathcal{U}$.

By the previous arguments and Definition 2.19, if ω satisfies the condition I_{k-1} on $\mathcal{U}$ then $\mathcal{U} \cap \Sigma^k(\omega) = \{x \in \mathcal{U} \mid F_1(x) = \ldots = F_{m-r}(x) = \Delta_k(x) = 0\}$. □

The following technical lemma will be used in the proof of Proposition 2.23, part (b).

Lemma A.1. *Let $f_i : \mathcal{V} \subset \mathbb{R}^\ell \to \mathbb{R}, i = 1, \ldots, s$ be smooth functions defined on an open subset of $\mathbb{R}^\ell$. Let $M \subset \mathbb{R}^\ell$ be a manifold locally given by $M = \{x \in \mathcal{V} \mid f_1(x) = \ldots = f_s(x) = 0\}$, with* $\operatorname{rank}(df_1(x), \ldots, df_s(x)) = s$, *for all $x \in M \cap \mathcal{V}$. If $g, h : \mathcal{V} \subset \mathbb{R}^\ell \to \mathbb{R}$ are smooth functions such that $g(x) = \lambda(x) h(x)$, for all $x \in M \cap \mathcal{V}$ and some smooth function $\lambda : \mathcal{V} \to \mathbb{R}$, then:*

(i) *If $\lambda(x) \neq 0$ and $x \in M$ then $g(x) = 0 \Leftrightarrow h(x) = 0$.*
(ii) *If $\lambda(x) \neq 0$, $x \in M$ and $h(x) = 0$ then*

$$\langle df_1(x), \ldots, df_s(x), dg(x) \rangle = \langle df_1(x), \ldots, df_s(x), dh(x) \rangle.$$

Proof of Proposition 2.23, part (b). Firstly, notice that the definition of $\Sigma^1(\omega)$ does not depend on the choice of any basis. Then, assume that the definition of $\Sigma^i(\omega)$ does not depend on the choice of the basis $\{\Omega_1(x),\dots,\Omega_{n-i+1}(x)\}$ for every $i=2,\dots,k-1$. As considered in part (a), for each $p\in\Sigma^{k-1}(\omega)$, there is an open neighborhood $\mathcal{U}$ of p in M such that

$$\begin{aligned}\mathcal{U}\cap\Sigma^1(\omega) &= \{x\in\mathcal{U}:F_1(x)=\ldots=F_{m-n+1}(x)=0\},\\ \mathcal{U}\cap\Sigma^{k-1}(\omega) &= \{x\in\mathcal{U}:F_1(x)=\ldots=F_{m-n+1}(x)=\Delta_2(x)=\ldots=\Delta_{k-1}(x)=0\},\\ \mathcal{U}\cap\Sigma^{k}(\omega) &= \{x\in\mathcal{U}:F_1(x)=\ldots=F_{m-n+1}(x)=\Delta_2(x)=\ldots=\Delta_{k}(x)=0\},\end{aligned}$$

with $\operatorname{rank}(dF_1(x),\dots,dF_{m-n+1}(x),d\Delta_2(x),\dots,d\Delta_{k-1}(x))=m-n+k-1$, for $x\in\mathcal{U}\cap\Sigma^{k-1}(\omega)$ and $\operatorname{rank}(dF_1(x),\dots,dF_{m-n+1}(x),d\Delta_2(x),\dots,d\Delta_{k}(x))=m-n+k$, for $x\in\mathcal{U}\cap\Sigma^{k}(\omega)$. Let us recall that

$$\Delta_k(x)=\det(dF_1,\dots,dF_{m-n+1},d\Delta_2,\dots,d\Delta_{k-1},\Omega_1,\dots,\Omega_{n-k+1})(x),$$

where $\{\Omega_1(x),\dots,\Omega_{n-k+1}(x)\}$ is a collection of $n-k+1$ smooth one-forms defined on $\mathcal{U}$ which is a basis of a vector subspace complementary to $\langle\bar\omega(x)\rangle\cap N_x^*\Sigma^{k-2}(\omega)$ in $\langle\bar\omega(x)\rangle$ for each $x\in\mathcal{U}\cap\Sigma^{k-1}(\omega)$.

Let us consider $\{\tilde\Omega_1(x),\dots,\tilde\Omega_{n-k+1}(x)\}$ a collection of $n-k+1$ smooth one-forms defined on $\mathcal{U}$ such that for each $x\in\mathcal{U}\cap\Sigma^{k-1}(\omega)$, $\{\tilde\Omega_1(x),\dots,\tilde\Omega_{n-k+1}(x)\}$ is another basis of a vector subspace complementary to $\langle\bar\omega(x)\rangle\cap N_x^*\Sigma^{k-2}(\omega)$ in $\langle\bar\omega(x)\rangle$. Then,

$$\langle\bar\omega(x)\rangle=\big(\langle\bar\omega(x)\rangle\cap N_x^*\Sigma^{k-2}(\omega)\big)\oplus\langle\tilde\Omega_1(x),\dots,\tilde\Omega_{n-k+1}(x)\rangle$$

and

$$\dim(\langle\tilde\Omega_1(x),\dots,\tilde\Omega_{n-k+1}(x)\rangle\cap N_x^*\Sigma^{k-1}(\omega))$$

is either equal to 0 or equal to 1, for $x\in\mathcal{U}\cap\Sigma^{k-1}(\omega)$. Moreover,

$$\begin{cases}\tilde\Omega_1(x)=\displaystyle\sum_{\ell=1}^{n-k+1}a_{\ell 1}(x)\Omega_\ell(x)+\varphi_1(x)\\ \tilde\Omega_2(x)=\displaystyle\sum_{\ell=1}^{n-k+1}a_{\ell 2}(x)\Omega_\ell(x)+\varphi_2(x)\\ \vdots\\ \tilde\Omega_{n-k+1}(x)=\displaystyle\sum_{\ell=1}^{n-k+1}a_{\ell(n-k+1)}(x)\Omega_\ell(x)+\varphi_{n-k+1}(x)\end{cases}$$

where $a_{ij}(x)\in\mathbb{R}$ and $\varphi_j(x)\in\langle\bar\omega(x)\rangle\cap N_x^*\Sigma^{k-2}(\omega)$, for $j=1,\dots,n-k+1$. We will show that for each $x\in\mathcal{U}\cap\Sigma^{k-1}(\omega)$,

$$\det(A(x))=\begin{vmatrix}a_{11}(x) & a_{12}(x) & \cdots & a_{1(n-k+1)}(x)\\ \vdots & \vdots & \ddots & \vdots\\ a_{(n-k+1)1}(x) & a_{(n-k+1)2}(x) & \cdots & a_{(n-k+1)(n-k+1)}(x)\end{vmatrix}\neq 0.$$

Suppose that the statement is false, that is, $\det(A(x))=0$. This means that the columns of matrix $A(x)$ are linearly dependent. So we can suppose without loss of generality that the first column of $A(x)$ can be written as a linear combination of the others columns:

$$(a_{11}(x),\dots,a_{(n-k+1)1}(x))=\sum_{s=2}^{n-k+1}\lambda_s(a_{1s}(x),\dots,a_{(n-k+1)s}(x)),$$

where $\lambda_s\in\mathbb{R}$, for $s=2,\dots,n-k+1$. Thus, removing x in the notation, we have

$$\tilde{\Omega}_1 = \sum_{\ell=1}^{n-k+1} a_{\ell 1}\Omega_\ell + \varphi_1 \quad \Rightarrow \tilde{\Omega}_1 = \sum_{\ell=1}^{n-k+1}\left(\sum_{s=2}^{n-k+1}\lambda_s a_{\ell s}\right)\Omega_\ell + \varphi_1$$

$$\Rightarrow \tilde{\Omega}_1 = \sum_{s=2}^{n-k+1}\lambda_s\left(\sum_{\ell=1}^{n-k+1} a_{\ell s}\Omega_\ell\right) + \varphi_1$$

then

$$\begin{aligned}\tilde{\Omega}_1 - \sum_{s=2}^{n-k+1}\lambda_s\tilde{\Omega}_s &= \left[\sum_{s=2}^{n-k+1}\lambda_s\left(\sum_{\ell=1}^{n-k+1} a_{\ell s}\Omega_\ell\right) + \varphi_1\right] - \sum_{s=2}^{n-k+1}\lambda_s\left(\sum_{\ell=1}^{n-k+1} a_{\ell s}\Omega_\ell + \varphi_s\right)\\ &= \varphi_1 - \sum_{s=2}^{n-k+1}\lambda_s\varphi_s.\end{aligned}$$

This means that

$$\tilde{\Omega}_1 - \sum_{s=2}^{n-k+1}\lambda_s\tilde{\Omega}_s \in \left(\langle\bar{\omega}\rangle \cap N_x^*\Sigma^{k-2}(\omega)\right) \cap \langle\tilde{\Omega}_1,\dots,\tilde{\Omega}_{n-k+1}\rangle = \{0\},$$

that is, $\tilde{\Omega}_1(x),\dots,\tilde{\Omega}_{n-k+1}(x)$ are linearly dependent. However, this contradicts the initial assumption that $\{\tilde{\Omega}_1(x),\dots,\tilde{\Omega}_{n-k+1}(x)\}$ is a basis of a vector subspace for each x in $\mathcal{U}\cap\Sigma^{k-1}(\omega)$. Therefore, $\det(A(x)) \neq 0$.

Let ${}^tA(x)$ be the transpose of matrix $A(x)$. For each $x \in \mathcal{U} \cap \Sigma^{k-1}(\omega)$, we have $\det({}^tA(x)) = \det(A(x)) \neq 0$ and, removing x in the notation,

$$\begin{aligned}&\det(dF_1,\dots,dF_{m-n+1},d\Delta_2,\dots,d\Delta_{k-1},\tilde{\Omega}_1,\dots,\tilde{\Omega}_{n-k+1})\\ &= \det(dF_1,\dots,dF_{m-n+1},d\Delta_2,\dots,d\Delta_{k-1},\sum_{\ell=1}^{n-k+1} a_{\ell 1}\Omega_\ell,\dots,\sum_{\ell=1}^{n-k+1} a_{\ell(n-k+1)}\Omega_\ell)\\ &= \det({}^tA)\det(dF_1,\dots,dF_{m-n+1},d\Delta_2,\dots,d\Delta_{k-1},\Omega_1,\dots,\Omega_{n-k+1}).\end{aligned} \tag{31}$$

Thus, for $x \in \mathcal{U}\cap\Sigma^{k-1}(\omega)$ we have that $\dim(\langle\tilde{\Omega}_1(x),\dots,\tilde{\Omega}_{n-k+1}(x)\rangle \cap N_x^*\Sigma^{k-1}(\omega))$ is equal to $\dim(\langle\Omega_1(x),\dots,\Omega_{n-k+1}(x)\rangle \cap N_x^*\Sigma^{k-1}(\omega))$. In particular, if $x \in \mathcal{U}\cap\Sigma^k(\omega)$ then $\Delta_k(x) = 0$ and

$$\tilde{\Delta}_k(x) = \det(dF_1,\dots,dF_{m-n+1},d\Delta_2,\dots,d\Delta_{k-1},\tilde{\Omega}_1,\dots,\tilde{\Omega}_{n-k+1}) = 0$$

such that, by statement (ii) of Lemma A.1,

$$\begin{aligned}&\langle dF_1(x),\dots,dF_{m-n+1}(x),d\Delta_2(x),\dots,d\Delta_{k-1}(x),d\Delta_k(x)\rangle\\ &= \langle dF_1(x),\dots,dF_{m-n+1}(x),d\Delta_2(x),\dots,d\Delta_{k-1}(x),d\tilde{\Delta}_k(x)\rangle,\end{aligned}$$

which implies that

$$\operatorname{rank}(dF_1(x),\dots,dF_{m-n+1}(x),d\Delta_2(x),\dots,d\Delta_{k-1}(x),d\tilde{\Delta}_k(x))$$

is equal to $m-n+k$. Therefore, the condition I_{k-1} and the definition of $\Sigma^k(\omega)$ do not depend on the choice of the basis $\{\Omega_1(x),\dots,\Omega_{n-k+1}(x)\}$.

Since $A_k(\omega) = \Sigma^k(\omega)\smallsetminus\Sigma^{k+1}(\omega)$ for $k = 1,\dots,n$, we conclude that $A_k(\omega)$ also does not depend on the choice of the basis. $\square$

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Camila M. Ruiz, Departamento de Matemática Aplicada, Instituto de Ciências Tecnológias e Exatas, Universidade Federal do Triângulo Mineiro, 38064-200 Uberaba - MG, Brasil
Email address: camila.ruiz@uftm.edu.br